Harmonic Algebra: Mathematical Foundations

Appendix A: Rigorous Framework Based on Wiener Algebra

A.1 Definition and Basic Properties

Definition A.1 (Wiener Algebra). The *Wiener algebra* $A(\mathbb{T})$ consists of all continuous periodic functions $f : \mathbb{R} \to \mathbb{C}$ with absolutely convergent Fourier series:

$$f(t) = \sum_{k \in \mathbb{Z}} c_k e^{ik\omega_0 t}, \quad \text{where } \sum_{k \in \mathbb{Z}} |c_k| < \infty$$

The algebra is equipped with the norm $\|f\|_1 = \sum_{k \in \mathbb{Z}} |c_k|$.

Theorem A.1 (Banach Algebra Structure). $(A(\mathbb{T}), +, \cdot, \| \cdot \|_1)$ is a commutative Banach algebra where:

- **Addition**: $(f + g)(t) = \sum_k (c_k + d_k) e^{ik\omega_0 t}$
- **Multiplication**: $(f \cdot g)(t) = \sum_k \left(\sum_{j+\ell=k} c_j d_\ell \right) e^{ik\omega_0 t}$
- **Submultiplicativity**: $\|fg\|_1 \leq \|f\|_1 \|g\|_1$

Proof Sketch: The key is that multiplication corresponds to convolution of coefficient sequences, and $\ell^1 * \ell^1 \subseteq \ell^1$ with submultiplicative norm. □

A.2 Corrected Multiplication Formula

For the specific case of finite harmonic sums:

$$F(t) = A_1 \cos(\omega_1 t) + A_2 \sin(\omega_1 t), \quad G(t) = B_1 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)$$

The product is:

$$F(t)G(t) = \frac{A_1 B_1}{2} [\cos((\omega_1 - \omega_2)t) + \cos((\omega_1 + \omega_2)t)]$$

$$+ \frac{A_1 B_2}{2} [\sin((\omega_1 + \omega_2)t) - \sin((\omega_1 - \omega_2)t)]$$

$$+ \frac{A_2 B_1}{2} [\sin((\omega_1 + \omega_2)t) + \sin((\omega_1 - \omega_2)t)]$$

$$+\frac{A_2B_2}{2}[\cos((\omega_1-\omega_2)t)-\cos((\omega_1+\omega_2)t)]$$

Note: The coefficients are $\frac{1}{2}$, not 2 as in the original formulation.

A.3 Division via Wiener's Lemma

Wiener's Lemma (Invertibility Criterion). Let $f \in A(\mathbb{T})$. Then f is invertible in $A(\mathbb{T})$ if and only if $f(t) \neq 0$ for all $t \in \mathbb{R}$. Moreover, if $\inf_t |f(t)| \geq \delta > 0$, then $\|f^{-1}\|_1 \leq C(\delta, \|f\|_1)$ for some constant C.

Computational Algorithm (Neumann Series). If $f = a_0(1+h)$ where $a_0 \neq 0$ and $\|h\|_1 < 1$, then:

$f^{-1} = \frac{1}{a_0}\sum_{n=0}^{\infty}(-h)^{*n}$

where h^{*n} denotes the n-fold convolution of h with itself.

Algorithm A.1 (Stable Division).

```
Input: f, g ∈ A(T) with g non-vanishing
Output: f/g ∈ A(T)

1. Extract DC component: c₀ = g₀ (coefficient of e⁰)
2. If |c₀| > δ (threshold):
   a. Form h = (g/c₀) - 1
   b. Check convergence: ||h||₁ < 1
   c. Compute g⁻¹ = (1/c₀) Σₙ₌₀^∞ (-h)^*n
   d. Return f * g⁻¹
3. Else: Use FFT regularization (Algorithm A.2)
```

A.4 FFT-Based Regularized Division

Algorithm A.2 (FFT Regularization).

```
Input: f, g ∈ A(T), regularization λ > 0
Output: Approximation to f/g

1. Sample on uniform grid: tⱼ = 2πj/N, j = 0,...,N-1
2. Evaluate: f̃ⱼ = f(tⱼ), g̃ⱼ = g(tⱼ)
3. Regularized division: q̃ⱼ = f̃ⱼ/(g̃ⱼ + λ)
4. IFFT to get coefficients of q
5. Prune modes below tolerance τ
```

A.5 Extension to Manifolds

Definition A.2 (Harmonic Field Algebra). On a Riemannian manifold $(M, \mathfrak{g})$, let $\{\phi_\ell\}_{\ell\geq 0}$ be the eigenfunctions of the Laplace-Beltrami operator: $-\Delta\phi_\ell = \lambda_\ell\phi_\ell$

The *harmonic field algebra* $\mathcal{H}(M)$ consists of finite linear combinations: $f = \sum_{\ell=0}^{L} a_\ell\phi_\ell$

Definition A.3 (Projected Multiplication). For $f, g \in \mathcal{H}(M)$, define: $f \odot g := \Pi_\Lambda(fg)$where Π_Λ is orthogonal projection onto $\text{span}\{\phi_\ell : \ell \leq L\}$.

Structure Constants. The multiplication is determined by: $\phi_\ell \odot \phi_m = \sum_{n=0}^{L} C_{\ell m}^{n}\phi_n$where $C_{\ell m}^{n} = \int_M \phi_\ell\phi_m\phi_n \, d\mu$.

A.6 Complexity and Convergence

Theorem A.2 (Sparse Spectral Complexity). For harmonic functions with K non-zero modes:

- **Multiplication**: $O(K^2)$ operations (sparse convolution)
- **Division** (Neumann): $O(K^2 \log(1/\epsilon))$ for accuracy ϵ
- **FFT path**: $O(N \log N)$ for N-point grid

Theorem A.3 (Approximation Error). Let $f \in A(\mathbb{T})$ with $\|f\|_1 = M$. The truncation error for keeping the largest K Fourier modes satisfies: $\left\| f - \sum_{|k|\leq K} c_k e^{ik\omega_0 t} \right\|_\infty \leq \sum_{|k|>K} |c_k| \leq \frac{M}{K^\alpha}$for functions with polynomial decay $|c_k| \lesssim |k|^{-\alpha-1}$.

A.7 Associativity Under Projection

Warning: The projected multiplication $\odot$ is generally **not associative** when $L < \infty$.

Theorem A.4 (Associativity Condition). The operation $\odot$ is associative if and only if the frequency set $\Omega = \{\omega_\ell\}$ is closed under addition modulo the projection.

Practical Consequence: For approximate associativity, choose the cutoff L such that most triple products $\phi_\ell\phi_m\phi_n$ with $\ell, m, n \leq L$ have negligible components beyond mode L.

A.8 Computational Recipes

Recipe 1 (Time Series → HA):

```python
```

```python
def time_series_to_ha(t, values, max_harmonics=50):
    fft_coeffs = np.fft.fft(values) / len(values)
    freqs = 2*π * np.fft.fftfreq(len(t), t[1]-t[0])
    spectrum = {freq: coeff for freq, coeff in zip(freqs, fft_coeffs)
                if abs(coeff) > tolerance}
    return prune_to_top_k(spectrum, max_harmonics)
```

Recipe 2 (Multiplication):

```python
def ha_multiply(f_spectrum, g_spectrum):
    result = defaultdict(complex)
    for ωf, cf in f_spectrum.items():
        for ωg, cg in g_spectrum.items():
            result[ωf + ωg] += cf * cg
    return dict(result)
```

Recipe 3 (Safe Division Check):

```python
def can_divide_safely(g_spectrum, delta=1e-10):
    # Check non-vanishing condition
    t_test = np.linspace(0, 2*π, 1000)
    g_vals = evaluate_spectrum(g_spectrum, t_test)
    return np.min(np.abs(g_vals)) > delta
```

A.9 Applications Summary

1. **Signal Processing**: HA provides closed-form operations on multi-component signals with preserved phase relationships.
2. **Dynamical Systems**: Koopman eigenfunctions form a harmonic basis; HA enables algebraic manipulation of observables.
3. **Partial Differential Equations**: Solution operators on harmonic functions can be computed via spectral methods in the HA framework.
4. **Machine Learning**: HA features preserve interpretable frequency structure while enabling nonlinear combinations.
5. **Cryptanalysis**: Structural invariants in periodic/quasi-periodic text can be detected via HA spectral analysis.

References

[1] N. Wiener, "Tauberian theorems," *Annals of Mathematics* (1932)

[2] Y. Katznelson, *An Introduction to Harmonic Analysis*, Cambridge University Press (2004)

[3] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer (2003)

[4] A. Zygmund, *Trigonometric Series*, Cambridge University Press (2002)

Computational Implementation: The complete framework is implemented in the accompanying Python module `harmonic_algebra.py` with optimized sparse spectral dictionaries and FFT acceleration.