Counting

**Basic Combinatorial Results:**

The number of **permutations** (arrangements) of $n$ distinct items is $n!$, which we read as "n factorial".

For positive integers, $n! = n(n-1)(n-2)\ldots(2)(1)$. We define $0!$ to be equal to $1$.

**Example:** The number of different ways to arrange $4$ people for a photograph is $4! = 24$.

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The number of arrangements of $r$ items taken from a collection of $n$ distinct items is:

$$P(n,r) = n\,P\,r = n^{(r)} = \frac{n!}{(n-r)!}$$

**Example:** Suppose I have a class of $20$ students. The number of ways I can select $4$ of these students and arrange them for a photograph is:

$$20\,P\,4 = \frac{20!}{16!} = 116280$$

The number of **combinations** (selections) of $r$ items taken from a collection of $n$ distinct items is:

$$C(n, r) = n\,C\,r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

**Example:** Suppose I have a class of $20$ students. The number of ways I can select (but not arrange) $4$ of these students is:

$$\binom{20}{4} = \frac{20!}{4!16!} = 4845$$

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**Example:** Suppose I have a box containing slips of paper, numbered $1, 2, \ldots 30$. If I select three of the thirty slips at random, what is the probability that all three slips show a number which is $9$ or less?

**Bernoulli Process:** An experiment consisting of one or more trials, each having the following properties.

1. Each trial has exactly two outcomes, which we call **success** and **failure**.
2. The trials are independent of each other.
3. For all trials the probability of success, $p$, is a constant.

A **binomial experiment** is a Bernoulli process where $n$, the number of trials, is fixed in advance.

Let $X$ count the number of successes in a binomial experiment Then $X$ is a **binomial random variable**, and we write $X \sim Bin(n, p)$, where $n$ is the number of trials, and $p$ is the probability of successes. For a binomial random variable, $n$ and $p$ are its parameters.

**Example:** In a manufacturing process, each item has a probability of $0.05$ of being defective, independent of all other items. Suppose $12$ items are selected at random, and we let $W$ denote the number of defective items.

**Binomial Probability Distribution:**

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \text{ for } x = 0, 1, 2, \ldots, n$$

**Example:** On a multiple choice test, there are $10$ questions, each with $8$ possible responses. I will complete the test by randomly selecting answers. What is the probability that I will get $1$ question correct?

**Example:** In the manufacture of lithium batteries, is is found that $7\%$ of all batteries are defective. Suppose that we test $6$ randomly selected batteries. What is the probability that at least two batteries are defective?

**Expected Value and Variance:** If $X \sim Bin(n, p)$, then:

$$E(X) = np \text{ and } V(X) = np(1-p)$$

**Example:** What is the expected number of defective lithium batteries per batch of $6$? What is the variance?

**Cumulative Distribution Tables:** These tables give $P(X \leq x)$ for "nice" values of $n$ and $p$.

**Example:** It is known that $20\%$ of all tablet computers will need the touch-screen repaired within the first two years of use. Suppose we select $15$ tablet computers at random.

What is the probability that no more than $6$ tablets will need repairs to the touch-screen within the first two years of use?

**Example:** What is the probability that exactly $5$ tablets will need touch-screen repairs?

**Example:** What is the probability that at least $2$ tablets will need touch-screen repairs?

**Example:** It is known that $30\%$ of all laptops of a certain brand experience hard-drive failure within $3$ years of purchase. Suppose that $20$ laptops are selected at random. Let the random variable $X$ denote the number of laptops which have experienced hard-drive failure within $3$ years of purchase.

If it is known that at least $3$ laptops experience hard-drive failure, what is the probability that no more than $6$ laptops will experience hard-drive failure?

Set 12

**Poisson Experiment:** An experiment having the following properties.

1. The number of successes that occur in any interval is independent of the number of successes occurring in any other interval.

2. The probability of success in an interval is proportional to the size of the interval.

3. If two intervals have the same size, then the probability of a success is the same for both intervals.

**Poisson Random Variable:** If in a Poisson experiment, $X$ counts the number of successes that occur in *one* interval of time/space, then $X$ is a Poisson random variable. We write $X \sim Poisson(\lambda)$, where $\lambda$ is the average number of successes per region/interval.

Note: Some books will use $\mu$ rather than $\lambda$ for the parameter of the Poisson random variable.

**Example:** At a bank, customers use the bank machine at an average rate of 40 customers per hour. Let $X$ count the number of customers that use the machine in a 30-minute interval.

**Example:** At a busy intersection, it is noted that on average $5$ cars pass through the intersection per minute. Let $X$ count the number of cars which pass through the intersection in an hour.

**Example:** Suppose that a typist makes on average 10 errors while typing 300 pages of text. Let $X$ count the number of errors on one page of text.

**Example:** We examine ten pages of text. Let $Y$ count the number of pages with at least one error. The random variable $Y$ is **not** Poisson. Why?

**Poisson Probability Distribution:**

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

**Example:** Suppose a machine makes defective items at an average rate of $5$ defective items per hour. What is the probability that the machine will make exactly $4$ defective items in an hour?

**Expected Value and Variance:** If $X \sim Poisson(\lambda)$, then:

$$E(X) = \lambda \text{ and } V(X) = \lambda$$

**Example:** What is the expected number of defective items made by the machine in an hour? What is the variance?

**Cumulative Distribution Tables:** These tables give $P(X \leq x)$ for "nice" values of $\lambda$

**Example:** Suppose the machine is watched for three hours. What is the probability that it will make no more than $12$ defective items?

(Recall that the machine makes on average $5$ defective items per hour)

**Example:** What is the probability that at least $6$ defective items will be made?

**Example:** What is the probability that exactly $13$ defective items will be made?

**Example:** Suppose that a typist makes on average of 2 errors per page. Suppose the typist is creating a ten-page document. What is the probability that exactly three of the pages do not contain any errors?

**Poisson approximation to Binomial:** If $X$ is a binomial random variable where $n$ is very large and $p$ is very small then $X$ can be approximated with a Poisson distribution with $\lambda = np$.

Provided $n \geq 100$ and $np \leq 10$, the approximation will be quite good. It will still be reasonably good when $n \geq 20$, as long as $p \leq 0.05$.

**Example:** Brugada syndrome is a rare disease which afflicts $0.02\%$ of the population. Suppose 10,000 people are selected at random and tested for Brugada syndrome. What is the probability that no more than $3$ of the tested people will have Brugada syndrome?

**Continuous Random Variable:** A random variable which can assume an uncountable number of values (i.e. some interval of real numbers).

For a random variable, the **probability distribution** or **probability density function** (pdf) is a function $f(x)$ satisfying

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

for any two numbers $a$ and $b$ (with $a \leq b$).

Some immediate consequences:

1. $f(x) \geq 0$ for all $x$.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

**Note:** Since a valid pdf must never be below the $x$ axis, we can interpret $P(a \le X \le b)$ as the area under $f(x)$ on the interval $[a, b]$.

Some further consequences for any valid pdf:

1. $P(X = a) = 0$ for any $a$.

2. $P(X \ge a) = P(X > a)$ and $P(X \le a) = P(X < a)$

3. $P(X \ge a) = 1 - P(X \le a)$

4. $P(a \le X \le b) = P(x \le b) - P(X \le a)$ (provided $a \le b$)

**Uniform Probability Distribution:** For a uniform probability distribution, the pdf is:

$$f(x; a, b) = \frac{1}{b-a} \text{ where } a \leq x \leq b$$

The graph of $f(x)$ is a horizontal line segment from $a$ to $b$ with height $1/(b-a)$.

$$P(x_1 \leq X \leq x_2) = (height) \times (width) = \left(\frac{1}{b-a}\right)(x_2 - x_1)$$

**Example:** Suppose that the continuous rv $X$ has the following pdf:

$$f(x) = \begin{cases} \dfrac{4}{609}x^3 & 2 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(3 \leq X \leq 4)$.

**Example:** Find an expression for $P(X \leq b)$, where $b$ is some number in $[2, 5]$.

**Note:** We can conclude that:

- $P(X \leq x) = x^4/609 - 16/609$ for all $x$ in the interval $[2, 5]$.
- If $x$ is less than $2$, $P(X \leq x) = 0$.
- If $x$ is greater than $5$, $P(X \leq x) = 1$.

Using this, we can write the **cumulative distribution function**, $F(x)$, for the given pdf.

$$F(x) = \begin{cases} 0, & x < 2 \\ \dfrac{x^4}{609} - \dfrac{16}{609}, & 2 \leq x \leq 5 \\ 1, & x > 5 \end{cases}$$

**Note:** The fundamental theorem of calculus tells us that for every $x$ at which $F'(x)$ exists, that $F'(x) = f(x)$.

**Example:** Suppose the random variable $X$ has the following cdf:

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{x+1}, & x \geq 0 \end{cases}$$

Find the pdf for the random variable $X$.

Let $p$ be a number between $0$ and $1$. The $100p^{th}$ **percentile** of a continuous random variable is the value $\alpha$ such that $F(\alpha) = p$.

**Example:** For the random variable from the previous example, find the $90^{th}$ percentile.

**Example:** Suppose the random variable $X$ has pdf

$$f(x) = \begin{cases} 2e^{-2x} & 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the median of the distribution.

**Note:** The **median**, $\widetilde{\mu}$ of a continuous random variable is the $50^{th}$ percentile.

The **expected value** or **mean** of a continuous random variable $X$ with pdf $f(x)$ is:

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$$

(provided this integral converges)

The **variance** of a continuous random variable $X$ with pdf $f(x)$ is:

$$V(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

(provided this integral converges)

and the standard deviation, $\sigma = \sqrt{\sigma^2}$.

As with discrete random variables, we have the following:

- $V(X) = E(X^2) - \mu^2$
- $E(aX + b) = aE(X) + b$
- $V(aX + b) = a^2 V(X)$

**Example:** Suppose the random variable $X$ has pdf

$$f(x) = \begin{cases} 2e^{-2x} & 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and the variance.

**Normal Density Function:** If $X$ is normally distributed with mean $\mu$ and standard deviation $\sigma$, then we write $X \sim N(\mu, \sigma)$. The pdf of $X$ is:

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

**Properties of Normal Curves:**

- All normal curves are defined on $(-\infty, \infty)$ and is bell-shaped.
- There is a single peak at $x = \mu$ and the curve is symmetric about this peak.
- The mean, median, and mode are all $\mu$; the variance is $\sigma^2$.
- There are points of inflection at $\mu - \sigma$ and $\mu + \sigma$
- As $\mu$ increases, the peak moves further to the right. As $\mu$ decreases, the peak moves further to the left. ($\mu$ is a **location parameter**)
- As $\sigma$ increases, the peak becomes lower, and the curve becomes flatter. As $\sigma$ decreases, the curve becomes more abruptly peaked, and the peak becomes taller. ($\sigma$ is a **scale** parameter).

**Standard Normal Distribution:** The standard normal random variable has mean $0$ and standard deviation $1$. We use the letter $Z$ to denote the standard normal distribution.

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

The standard normal curve is:

- has its peak at $0$, and is symmetric about the $y$-axis
- has points of inflection at $1$ and $-1$.

If our random variable is $Z$, then we denote the cdf $P(Z \leq z)$ by $\Phi(z)$.

**Symmetry Property:** Since the random variable $Z$ is symmetric about $Z = 0$, then for any $\alpha$:

$$P(Z \leq \alpha) = P(Z \geq -\alpha)$$

**Example:** Find $P(Z \leq 2.56)$

**Example:** Calculate $P(Z \geq 0.16)$.

Select the closest to your unrounded answer from the following:

(A) $0.2$ (B) $0.4$ (C) $0.6$ (D) $0.8$

**Example:** Calculate $P(-1.22 \leq Z \leq 1.73)$.

Select the closest to your unrounded answer from the following:

(A) $0.2$ (B) $0.4$ (C) $0.6$ (D) $0.8$

**Example:** Suppose that the heights of Andean flamingos are normally distributed with a mean of $105\ cm$ and a standard deviation of $2\ cm$. Let the random variable $X$ denote the height of a randomly selected Andean flamingo

What is the **median** Andean flamingo height?

Select the closest to your unrounded answer:

(A) $105\ cm$

(B) Not enough information to answer this question.

**Example:** Is $P(X \geq 100) = P(X \leq -100)$?

(A) Yes

(B) No

**Example:** What is $P(X = 105)$?

Select the closest answer:

(A) $0$

(B) $0.5$

(C) $1$

(D) Not enough information available.

Notation: $z_\alpha$ is the number such that $P(Z > z_\alpha) = \alpha$. Alternately, $z_\alpha$ is the $100(1 - \alpha)$ percentile of the standard normal distribution.

**Example:** Find the $97.5^{th}$ percentile of the standard normal distribution.

**Standardizing a Normal Random Variable:** If $X$ is normally distributed with mean $\mu$ and standard deviation $\sigma$, then:

$$Z = \frac{X - \mu}{\sigma}$$

**Example:** The masses of a certain type of bolt is approximately normally distributed with $\mu = 15$ g, and $\sigma = 2$ g. What is the probability that a randomly selected bolt has a mass between $14.3$ g and $17.1$ g?

**Example:** What is the probability that at a randomly selected bolt will have a mass of at least $20$ g?

**Example:** What is the minimum mass of the heaviest 5% of bolts?

The **empirical rule** states that if the distribution of a variable is approximately normal, then:

1. About 68% of values lie within $\sigma$ of the mean.
2. About 95% of values lie within $2\sigma$ of the mean.
3. About 99.7% of values lie within $3\sigma$ of the mean.

From this, we can conclude that almost all bolts will have a mass within $6$ g of the mean $15$ (i.e. about 99.7% will have a a mass between $9$ g and $21$ g).

**Approximating the Binomial Distribution with the Normal Distribution:** Suppose $X \sim Bin(n, p)$ where $np$ and $n(1-p)$ are both at least 5.
Then $X \approx N(\mu = np, \sigma^2 = np(1-p))$.

This means that:

$$P(X \leq x) \approx P\left(Z \leq \frac{x - np}{\sqrt{np(1-p)}}\right)$$

Since we are using a continuous distribution to approximate a discrete one, this approximation will be slightly off. If we wish to get a better approximation, use the following, with a **continuity correction**:

$$P(X \leq x) \approx P(X \leq x + 0.5)$$

$$\approx P\left(Z \leq \frac{x - np + 0.5}{\sqrt{np(1-p)}}\right)$$

**Example:** Suppose it is known that $20\%$ of batteries have a lifespan shorter than the advertised lifespan. Suppose that $100$ batteries are selected at random. What is the approximate probability (using the continuity correction) that at least $10$ batteries will have a short lifespan?

**Example:** Suppose it is known that the reaction time of type of voice-activated robot is normally distributed with $\mu = 6.3$ microseconds, and $\sigma = 2$ microseconds.

Suppose I select one voice-activated robot at random. What is the probability that its reaction time is between $5$ and $7$ microseconds? Report your answer to three decimal places.

**Example:** Suppose that I select five robots and test each of them. Assume the reaction time of each robot is independent of the other robot reaction-times. What is the probability that exactly three of the robots will have a reaction time between $5$ and $7$ microseconds? Report your answer to three decimal places.