

Math 122 Assignment 3 Solutions

Due: Wednesday, October 22, 2025 at 23:59. Please submit on Crowdmark: <https://app.crowdmark.com/sign-in/university-of-victoria>.

There are five questions of equal value (worth a total of 40 marks). There are also 4 bonus marks available, as indicated below. Please feel free to discuss these problems with each other. You may not access any “tutoring” or “help” website or use AI assistance in any way. Suspected violations of any relevant policies on Academic Integrity will be treated seriously. Each person must write up their own solution, in their own words, in a way that reflects their own understanding. Complete solutions are those which are coherently written, and include appropriate justifications.

Two bonus marks are available for the bonus question, and two more bonus marks are available if the answers to all questions are typeset using the mathematical typesetting system LaTeX. Information on obtaining and using LaTeX is on the cross-listed Brightspace page.

1. Let A, B and C be sets. Consider the equality $(A \setminus B)^c \cup (B \cup C)^c = A^c \cup (B \cup C^c)$.

- (a) Prove the given equality using the Laws of Set Theory.
- (b) Prove the given equality by describing the LHS and RHS using set-builder notation and then using the Laws of Logic to demonstrate that the sets are defined by logically equivalent statements.
- (c) Prove the given equality by arguing that the LHS is a subset of the RHS, and the RHS is a subset of the LHS.

Solution. Part (a):

$$\begin{aligned} & (A \setminus B)^c \cup (B \cup C)^c \\ &= (A \cap B^c)^c \cup (B \cup C)^c && \text{By definition of set difference} \\ &= (A^c \cup B) \cup (B^c \cap C^c) && \text{DeMorgan } \times 2 \\ &= A^c \cup (B \cup (B^c \cap C^c)) && \text{Associativity} \\ &= A^c \cup ((B \cup B^c) \cap (B \cup C^c)) && \text{Distributivity} \\ &= A^c \cup (\mathcal{U} \cap (B \cup C^c)) && \text{Known Set Equality} \\ &= A^c \cup (B \cup C^c) && \text{Known Set Equality} \end{aligned}$$

Part (b): This follows almost the same steps as part (a). We have

$$\begin{aligned} & (A \setminus B)^c \cup (B \cup C)^c \\ &= \{x : x \in (A \setminus B)^c \cup (B \cup C)^c\} \\ &= \{x : \neg((x \in A) \wedge \neg(x \in B)) \vee \neg((x \in B) \vee (x \in C))\} && \text{By definition of the set operations} \\ &= \{x : (\neg(x \in A) \vee (x \in B)) \vee (\neg(x \in B) \wedge \neg(x \in C))\} && \text{DeMorgan } \times 2 \\ &= \{x : \neg(x \in A) \vee ((x \in B) \vee (\neg(x \in B) \wedge \neg(x \in C)))\} && \text{Associativity} \\ &= \{x : \neg(x \in A) \vee (((x \in B) \vee \neg(x \in B)) \wedge ((x \in B) \vee \neg(x \in C)))\} && \text{Distributivity} \\ &= \{x : \neg(x \in A) \vee (\mathbf{1} \wedge ((x \in B) \vee \neg(x \in C)))\} && \text{Known Tautology} \\ &= \{x : \neg(x \in A) \vee ((x \in B) \vee \neg(x \in C))\} && \text{Known LE} \\ &= A^c \cup (B \cup C^c) \end{aligned}$$

Part (c): First, we prove $(A \setminus B)^c \cup (B \cup C)^c \subseteq A^c \cup (B \cup C^c)$. Take $x \in (A \setminus B)^c \cup (B \cup C)^c$. Then either $x \in (A \setminus B)^c$ or $x \in (B \cup C)^c$. We divide the proof into cases.

Case 1: Suppose $x \in (A \setminus B)^c$.

Then Since $(A \setminus B)^c = (A \cap B^c)^c = A^c \cup B$ by the definition of set difference and DeMorgan, we have $x \in A^c$ or $x \in B$. If $x \in A^c$, then $x \in A^c$ or $x \in (B \cup C^c)$ and so we get $x \in A^c \cup (B \cup C^c)$. If $x \in B$, then $x \in B$ or $x \in C^c$ and so $x \in B \cup C^c$. So, $x \in A^c$ or $x \in B \cup C^c$ and so $x \in A^c \cup (B \cup C^c)$.

Case 2: Suppose $x \in (B \cup C)^c$.

Then, by DeMorgan, $x \in B^c \cap C^c$. So, $x \in B^c$ and $x \in C^c$. So, $x \in C^c$. Thus, $x \in B$ or $x \in C^c$ and so $x \in B \cup C^c$. So, $x \in A^c$ or $x \in B \cup C^c$ and so $x \in A^c \cup (B \cup C^c)$.

This completes the proof that $(A \setminus B)^c \cup (B \cup C)^c \subseteq A^c \cup (B \cup C^c)$.

Next, we prove $A^c \cup (B \cup C^c) \subseteq (A \setminus B)^c \cup (B \cup C)^c$. Let $x \in A^c \cup (B \cup C^c)$. Then $x \in A^c$ or $x \in B$ or $x \in C^c$. So, we divide the proof into three cases.

Case 1: Suppose $x \in A^c$.

Then $x \in A^c$ or $x \in B$. Thus, $x \in A^c \cup B$ which is equal to $(A \setminus B)^c$ by the definition of set difference and DeMorgan. So, $x \in (A \setminus B)^c$ or $x \in (B \cup C)^c$ and thus $x \in (A \setminus B)^c \cup (B \cup C)^c$.

Case 2: Suppose $x \in B$.

Then $x \in A^c$ or $x \in B$. Thus, $x \in A^c \cup B$ which is equal to $(A \setminus B)^c$ by the definition of set difference and DeMorgan. So, $x \in (A \setminus B)^c$ or $x \in (B \cup C)^c$ and thus $x \in (A \setminus B)^c \cup (B \cup C)^c$.

Case 3: Suppose $x \in C^c$.

This last case is a bit tricky. We note that $C^c = C^c \cap \mathcal{U} = C^c \cap (B \cup B^c) = (C^c \cap B) \cup (C^c \cap B^c)$ by various known set equalities and laws of set theory. Thus, we can divide the proof further into cases depending on whether $x \in C^c \cap B$ or $x \in C^c \cap B^c$.

Case 3.1: $x \in C^c \cap B$.

In this case, we have $x \in B$. So, $x \in A^c$ or $x \in B$. Thus, $x \in A^c \cup B$ which is equal to $(A \setminus B)^c$ by the definition of set difference and DeMorgan. So, $x \in (A \setminus B)^c$ or $x \in (B \cup C)^c$ and thus $x \in (A \setminus B)^c \cup (B \cup C)^c$.

Case 3.2: $x \in C^c \cap B^c$.

Note that $C^c \cap B^c = (B \cup C)^c$. So, $x \in (B \cup C)^c$ and therefore $x \in (A \setminus B)^c \cup (B \cup C)^c$.

□

2. Let A and B be sets. *Hint:* For some parts of this question, proof by cases or contrapositive are helpful.

- (a) Prove that if $A \cap B = \emptyset$, then $B \setminus A = B$.
- (b) Prove that if $B \setminus A = B$, then $B \subseteq A \oplus B$.
- (c) Prove that if $B \subseteq A \oplus B$, then $B \subseteq A^c$.
- (d) Prove that if $B \subseteq A^c$, then $A \cap B = \emptyset$.
- (e) Explain why parts (a), (b), (c) and (d) together imply that the statements $B \subseteq A^c$, $A \cap B = \emptyset$, $B \setminus A = B$, and $B \subseteq A \oplus B$ are all logically equivalent.

Solution. Part (a): Suppose that $A \cap B = \emptyset$. We prove that $B \setminus A = B$ by showing that both of these sets are subsets of each other.

First, take $x \in B \setminus A$. Then, since $B \setminus A = B \cap A^c$, we have $x \in B$ and $x \in A^c$. So, in particular, $x \in B$. Therefore $B \setminus A \subseteq B$.

Now, take $x \in B$. If $x \in A$, then we would have $x \in A$ and $x \in B$ which would mean that $x \in A \cap B$. However, we know that $A \cap B = \emptyset$ and so this cannot be the case. Therefore, since $x \in A$ leads to a contradiction, we must have that $x \notin A$ which means that $x \in A^c$. So, $x \in B \cap A^c = B \setminus A$. Therefore, $B \subseteq B \setminus A$.

Part (b): Suppose that $B \setminus A = B$. We prove that $B \subseteq A \oplus B$. Let $x \in B$. Then, since $B = B \setminus A$, we know that $x \in B \setminus A$. Recall that $A \oplus B = (A \setminus B) \cup (B \setminus A)$. So, since $x \in B \setminus A$, we have that either $x \in A \setminus B$ or $x \in B \setminus A$ and so $x \in (A \setminus B) \cup (B \setminus A) = A \oplus B$.

Part (c): Suppose that $B \subseteq A \oplus B$. We show that $B \subseteq A^c$. Let $x \in B$. Then, since $B \subseteq A \oplus B$, we have $x \in A \oplus B$. Recall that $A \oplus B = (A \setminus B) \cup (B \setminus A)$. So, either $x \in A \setminus B$ or $x \in B \setminus A$. We divide into cases.

Case 1: $x \in A \setminus B$.

In this case, $x \in A$ and $x \notin B$. However, since we already know that $x \in B$, this is a contradiction. So, this case cannot occur.

Case 2: $x \in B \setminus A$.

In this case, $x \in B$ and $x \notin A$. Therefore, $x \in A^c$.

We have proven that $(x \in B) \Rightarrow (x \in A^c)$ and so $B \subseteq A^c$.

Part (d): We prove the contrapositive. Suppose that $A \cap B \neq \emptyset$. Then we can choose an element $x \in A \cap B$. Therefore, $x \in A$ and $x \in B$. This is the same as saying that $x \in B$ and $x \notin A^c$. So, since there exists an element x of the universe such that $x \in B$ and $x \notin A^c$, we have that $x \in B$ does not logically imply $x \in A^c$. So, B is not a subset of A^c .

Part (e): Recall that $p \Leftrightarrow q$ means the same thing as $p \Rightarrow q$ and $q \Rightarrow p$. We have shown that

$$A \cap B = \emptyset \Rightarrow B \setminus A = B \Rightarrow B \subseteq A \oplus B \Rightarrow B \subseteq A^c \Rightarrow A \cap B = \emptyset.$$

The fact that all of these statements are logically equivalent comes from the Chain Rule and the fact that this chain of implications starts and ends with the same statement. For example, to prove that $B \subseteq A \oplus B \Rightarrow A \cap B = \emptyset$, we simply “chain together” the implications $B \subseteq A \oplus B \Rightarrow B \subseteq A^c$ and $B \subseteq A^c \Rightarrow A \cap B = \emptyset$. \square

3. (a) Let $A = \{1, 2, \dots, 99\}$. Determine the number of subsets of A that satisfy each condition, and explain your reasoning. You can write your final answer as a formula like $2^{10} + 2^{15} - 6$ rather than computing a number. In fact, that's preferred.
- The subset does not contain any numbers that are larger than 50.
 - Either all numbers in the subset are odd, or all numbers in the subset are even. *Be Careful! Are there any subsets which satisfy both conditions?*
 - The subset contains 5 or 6 but not 7.
 - The subset has cardinality at least 50. *Hint:* Think about complements.
- (b) A survey of app usage at a local high school shows that 83% of students use Tiktok, 82% use Instagram, 73% use YouTube, 72% use Tiktok and Instagram, 68% use Tiktok and YouTube, 65% use YouTube and Instagram and 60% use all three. What percentage of students use YouTube but not Tiktok nor Instagram? Explain your reasoning. *Hint:* You can assume that the universe consists of 100 students.

Solution. Part (a): i. Every such set is nothing more than a subset of $\{1, \dots, 50\}$. So, there are 2^{50} such sets.

ii. Let A be the set of all subsets in which all of the numbers are odd and B be the set of all subsets in which all of the numbers are even. There are 50 odd numbers and 49 even numbers between 1 and 99, and so have $|A| = 2^{50}$ and $|B| = 2^{49}$. Also, the only set in $A \cap B$ is the empty set. So, by inclusion-exclusion, $|A \cup B| = |A| + |B| - |A \cap B| = 2^{50} + 2^{49} - 1$.

iii. Let A be the collections of sets containing 5, B be the collection containing 6 and C be the collection containing 7. We want to compute $|(A \cup B) \cap C|$ which is the same thing as $|(A \cap C) \cup (B \cap C)|$ by distributivity. By inclusion-exclusion,

$$\begin{aligned} |(A \cap C) \cup (B \cap C)| &= |A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)| \\ &= |A \cap C| + |B \cap C| - |A \cap B \cap C|. \end{aligned}$$

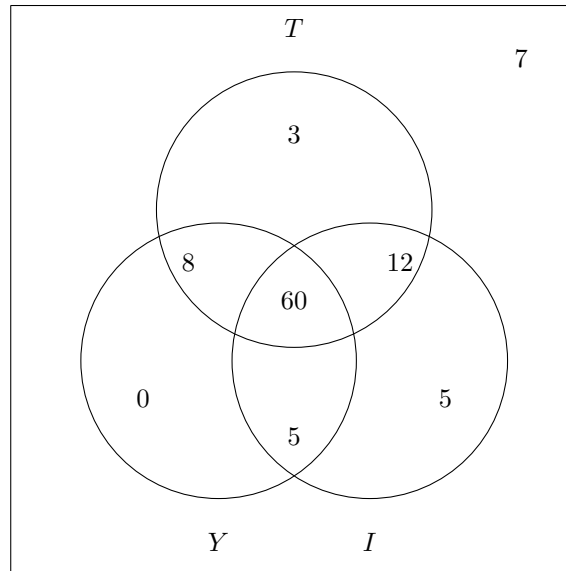
Now, $|A \cap C| = |B \cap C| = 2^{97}$ and $|A \cap B \cap C| = 2^{96}$ and so we get $2^{97} + 2^{97} - 2^{96}$.

iv. A subset S of $\{1, \dots, 99\}$ has cardinality at least 50 if and only if its complement has cardinality at most 49. So, half of the sets satisfy each condition and therefore the number of such subsets is $\frac{1}{2}2^{99} = 2^{98}$.

Part (b): We can assume that the number of students is 100. Let T be the set of students who use Tiktok, I be the set who use Instagram and Y be the set who use YouTube. We are given that

$$\begin{aligned} |T| &= 83 \\ |I| &= 82 \\ |Y| &= 73 \\ |T \cap I| &= 72 \\ |T \cap Y| &= 68 \\ |Y \cap I| &= 65 \\ |T \cap Y \cap I| &= 60 \\ |T \cup Y \cup I| &= 100. \end{aligned}$$

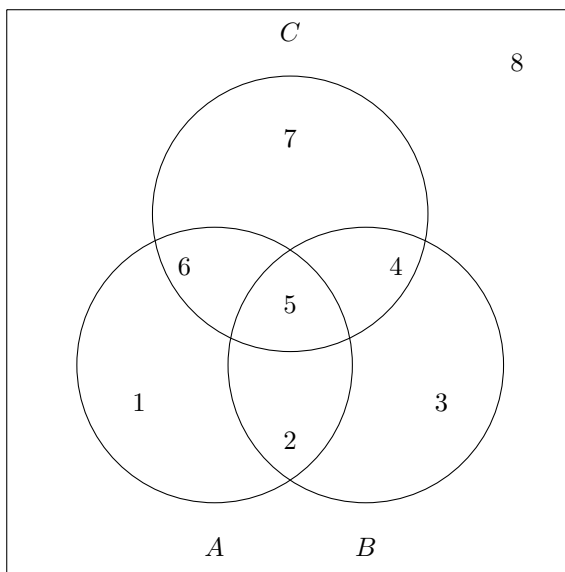
So, we can use this to populate the Venn diagram. For example, using the information that $|Y \cap I| = 65$ and $|T \cap Y \cap I| = 60$, we know that $|Y \cap I \cap T^c| = 5$, which allows us to fill in the segment corresponding to the set of students who use YouTube and Instagram but not Tiktok. Continuing in this way, we get the following:



So, the answer is that 0% of students use YouTube but not Tiktok or Instagram.

□

4. Let A, B, C be sets. Disprove each of the following statements with reference to the Venn diagram below. In each case, if the information from the Venn diagram suggests that the set on one side of the equals sign is a subset of the set on the other side, then state the suggested subset relationship and prove it.



(a) $A^c \cup (B \setminus C) = (A^c \setminus C) \cup B$.

(b) $A^c \cup (B \setminus C) = (A^c \cup B) \setminus C$.

Solution. Part (a): As suggested by the Venn diagram, we let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 5, 6\}$, $B = \{2, 3, 4, 5\}$ and $C = \{4, 5, 6, 7\}$. Then $A^c = \{3, 4, 7, 8\}$ and $B \setminus C = \{2, 3\}$. So,

$$A^c \cup (B \setminus C) = \{2, 3, 4, 7, 8\}.$$

Also, $A^c \setminus C = \{3, 8\}$. So,

$$(A^c \setminus C) \cup B = \{2, 3, 4, 5, 8\}.$$

The set $A^c \cup (B \setminus C)$ contains the element 7 but $(A^c \setminus C) \cup B$ does not. Also, $(A^c \setminus C) \cup B$ contains 5 but $A^c \setminus C$ does not. So, the two sets are not equal, and neither of them are subsets of the other.

Part (b): The sets \mathcal{U}, A, B, C are defined as above. As before,

$$A^c \cup (B \setminus C) = \{2, 3, 4, 7, 8\}.$$

We have $A^c \cup B = \{2, 3, 4, 5, 7, 8\}$. So,

$$(A^c \cup B) \setminus C = \{2, 3, 8\}.$$

Thus, the sets are not equal. However, this suggests that the second set may be a subset of the first. Let's prove this in general.

Take $x \in (A^c \cup B) \setminus C$. Then $x \in A^c \cup B$ and $x \notin C$. We divide the proof into cases.

Case 1: $x \in A^c$.

In this case, $x \in A^c$ easily implies that either $x \in A^c$ or $x \in B \setminus C$ and so $x \in A^c \cup (B \setminus C)$.

Case 2: $x \in B$.

In this case, we have $x \in B$ and $x \notin C$. So, $x \in B \setminus C$ which gives us that $x \in A^c$ or $x \in B \setminus C$, and therefore $x \in A^c \cup (B \setminus C)$.

So, $x \in (A^c \cup B) \setminus C \Rightarrow x \in A^c \cup (B \setminus C)$. Therefore, $(A^c \cup B) \setminus C \subseteq A^c \cup (B \setminus C)$. □

5. Consider the alphabet $A = \{a, b\}$. A *word* is a finite sequence of symbols in A . For example, *aaabab* is a word of length six. A *palindrome* is a word that is the same if written forwards and backwards. For example, *bab* and *abaaba* are palindromes but *abbab* is not.

- (a) For $n \geq 1$, let P_n be the set of all palindromes of length n . Write out all of the elements of P_1 and P_2 .
- (b) Give a recursive definition of the sequence $P_1, P_2, P_3, P_4, \dots$. Your definition should specify the sets P_1 and P_2 found in part (a) and a “rule” for generating each set P_n for $n \geq 3$ from the previous sets in the sequence. Briefly justify your answer. *Hint*: Think about how the palindromes of length 3 or 4 can be obtained from palindromes of length 1 or 2.
- (c) Using your solutions to the previous parts, explain why $|P_1| = 2, |P_2| = 2$ and $|P_n| = 2|P_{n-2}|$ for all $n \geq 3$. Using this, come up with a non-recursive formula for $|P_n|$ (depending only on n , not on the previous sets in the sequence) and give a short justification for it. *Hint*: The formula may be different depending on whether n is even or odd.

Solution. Part (a): $P_1 = \{a, b\}$ and $P_2 = \{aa, bb\}$.

Part (b): $P_1 = \{a, b\}$, $P_2 = \{aa, bb\}$ and, for, $n \geq 2$,

$$P_n = \{axa : x \in P_{n-2}\} \cup \{bxb : x \in P_{n-2}\}.$$

Here is a brief justification. Every palindrome must start and end with the same letter. So, they either start and end with a or they start and end with b . If we remove the first and last letter, the remaining word is still a palindrome of length $n-2$. So, the palindromes of length n can be constructed recursively from palindromes of length $n-2$ by adding the same letter at the start and end.

Part (c): We have $|P_1| = |P_2| = 2$ simply by definition. By the recursive definition and inclusion-exclusion,

$$\begin{aligned} |P_n| &= |\{axa : x \in P_{n-2}\} \cup \{bxb : x \in P_{n-2}\}| \\ &= |\{axa : x \in P_{n-2}\}| + |\{bxb : x \in P_{n-2}\}| - |\{axa : x \in P_{n-2}\} \cap \{bxb : x \in P_{n-2}\}| \\ &= |P_{n-2}| + |P_{n-2}| - 0 \\ &= 2|P_{n-2}| \end{aligned}$$

for $n \geq 3$.

The non-recursive formula is

$$|P_n| = \begin{cases} 2^{\frac{n+1}{2}} & \text{if } n \text{ is odd,} \\ 2^{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$$

The reason is as follows. For $n = 1$, the formula is true because there are 2 palindromes and $2^{\frac{1+1}{2}} = 2$. Similarly, it is correct for $n = 2$. Now, for $n \geq 3$, the recursive formula gives us

$$\begin{aligned} |P_n| &= 2|P_{n-2}| \\ &= \begin{cases} 2 \cdot 2^{\frac{(n-2)+1}{2}} & \text{if } n \text{ is odd,} \\ 2 \cdot 2^{\frac{n-2}{2}} & \text{if } n \text{ is even,} \end{cases} \quad (\text{by induction}) \\ &= \begin{cases} 2^{\frac{n+1}{2}} & \text{if } n \text{ is odd,} \\ 2^{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

□

6. (Bonus question; 2 bonus marks) There are 250 Math 122 students standing in the quad and each of them is holding a frisbee. Assume that all of the pairwise distances between the students are distinct. In other words, if x, y, z, w are students with $x \neq y$ and $z \neq w$ such that the distance from x to y is equal to the distance from z to w , then $\{x, y\} = \{z, w\}$. All at once, every student throws their frisbee to the other student who is closest to them. Prove that there exists a student who does not receive any frisbees.