

Math 122 Assignment 2 SOLUTIONS

Due: Wednesday, October 8, 2025 at 23:59. Please submit on Crowdmark: <https://app.crowdmark.com/sign-in/university-of-victoria>.

There are five questions of equal value (worth a total of 40 marks). There are also 4 bonus marks available, as indicated below. Please feel free to discuss these problems with each other. You may not access any “tutoring” or “help” website or use AI assistance in any way. Suspected violations of any relevant policies on Academic Integrity will be treated seriously. Each person must write up their own solution, in their own words, in a way that reflects their own understanding. Complete solutions are those which are coherently written, and include appropriate justifications.

Two bonus marks are available for the bonus question, and two more bonus marks are available if the answers to questions 3 and 4 are typeset using the mathematical typesetting system LaTeX. Information on obtaining and using LaTeX is on the cross-listed Brightspace page.

1. Let x and y be variables where the universe for x consists of four people, Ben, Gary, Jon and Michelle, and the universe for y consists of six things, hot sauce, cycling, photography, calculus, IPA and pizza. Let $\text{enjoys}(x, y)$ be the open statement “ x enjoys y .” So, for example, $\text{enjoys}(\text{Gary}, \text{cycling})$ is the statement “Gary enjoys cycling.”
 - (a) Express the statement “There is someone who enjoys everything” using quantifiers and $\text{enjoys}(x, y)$.
 - (b) Write $\exists x, \exists y, \forall w, \neg \text{enjoys}(x, w) \leftrightarrow (w = y)$ in plain English. (The universe for the variable w is clear from context).
 - (c) Express the statement “Everything is enjoyed by at least two people” using quantifiers, logical connectives, negation, the $=$ symbol, and $\text{enjoys}(x, y)$. Do not use \neq , do not use $=$ between two statements and do not use \Leftrightarrow or \Rightarrow .

Solution. Part (a): $\exists x, \forall y, \text{enjoys}(x, y)$.

Part (b): Probably the simplest way to express this statement is “There is a person who enjoys all but exactly one thing.”

Part (c) $\forall y, \exists x, \exists z, \neg(x = z) \wedge (\text{enjoys}(x, y) \wedge \text{enjoys}(z, y))$. □

2. The universe for all variables in this question is the real numbers.

- (a) Write the statement below in plain English. Your sentence may include the variables x and y but it should not mention the variable w . It may help to use the word “unique” in your sentence (but only if you use it correctly).

$$\forall x, \exists y, \forall w, ((xy = 1) \wedge ((w \neq y) \rightarrow (xw \neq 1))) \vee (x = 0).$$

- (b) Write a statement that is logically equivalent to the negation of the statement in part (a) which does not use the negation symbol \neg and does not contain any negated mathematical symbols such as \neq . (*Hint:* How can you re-write $a \neq b$?). Your answer should be written with symbols, not in plain English.
- (c) Use the open statement $p(x)$: “ $x > 0$ ” and $q(x)$: “ $x < 0$ ” to explain why $(\exists x, p(x)) \wedge (\exists x, q(x))$ is not logically equivalent to $\exists x, p(x) \wedge q(x)$.
- (d) Is $(\forall x, p(x)) \wedge (\forall x, q(x))$ logically equivalent to $\forall x, p(x) \wedge q(x)$? Explain why or why not.

Solution. Part (a): For every non-zero real number x , there is a unique real number y such that $xy = 1$.

Part (b): We start with

$$\neg \forall x, \exists y, \forall w, ((xy = 1) \wedge ((w \neq y) \rightarrow (xw \neq 1))) \vee (x = 0)$$

and move the negation inside the quantifiers to get

$$\exists x, \forall y, \exists w, \neg((xy = 1) \wedge ((w \neq y) \rightarrow (xw \neq 1))) \vee (x = 0).$$

We need to get rid of the negation symbol and the \neq symbols. Here is how we do that:

$$\begin{aligned} & \neg((xy = 1) \wedge ((w \neq y) \rightarrow (xw \neq 1))) \vee (x = 0) \\ \Leftrightarrow & \neg(xy = 1) \wedge \neg((w \neq y) \rightarrow (xw \neq 1)) \wedge \neg(x = 0) && \text{DeMorgan} \\ \Leftrightarrow & \neg(xy = 1) \vee \neg((w \neq y) \rightarrow (xw \neq 1)) \wedge \neg(x = 0) && \text{DeMorgan} \\ \Leftrightarrow & \neg(xy = 1) \vee \neg(\neg(w \neq y) \vee (xw \neq 1)) \wedge \neg(x = 0) && \text{Known LE} \\ \Leftrightarrow & \neg(xy = 1) \vee ((w \neq y) \wedge \neg(xw \neq 1)) \wedge \neg(x = 0) && \text{DeMorgan, Double Negation} \\ \Leftrightarrow & ((xy \neq 1) \vee ((w \neq y) \wedge (xw = 1))) \wedge (x \neq 0) && \text{By definition of } \neq, \text{ Double Negation} \end{aligned}$$

The final trick is to notice that $a \neq b$ is the same as $(a > b) \vee (a < b)$. Making this substitution yields

$$\exists x, \forall y, \exists w, (((xy > 1) \vee (xy < 1)) \vee (((w > y) \vee (w < y)) \wedge (xw = 1))) \wedge ((x > 0) \vee (x < 0)).$$

Part (c): The statement $\exists x, p(x)$ is true because $p(5)$ is true and the statement $\exists x, q(x)$ is true because $q(-10)$ is true. Therefore, $(\exists x, p(x)) \wedge (\exists x, q(x))$ is true. However, for each real number x , the statements $p(x)$ and $q(x)$ cannot both be true (because a number can't be positive and negative). So, $\exists x, p(x) \wedge q(x)$ is false. Therefore, $(\exists x, p(x)) \wedge (\exists x, q(x))$ and $\exists x, p(x) \wedge q(x)$ can have different truth values, and so they are not logically equivalent.

Part (d): Yes, they are logically equivalent. The first statement is true exactly when $p(x)$ is true for all x and $q(x)$ is true for all x . This happens if and only if $p(x) \wedge q(x)$ is true for all x , which is precisely the condition for the second statement to be true. \square

3. (a) Prove that if n is odd, then n^3 is odd.
 (b) Prove that if n^7 is odd, then n is odd.
 (c) Prove that $n(n+1)$ is even for any integer n . *Hint:* Proof by cases.

Solution. Part (a): Suppose that n is odd. Then there exists an integer k such that $n = 2k + 1$. So,

$$\begin{aligned} n^3 &= (2k+1)^3 = (2k+1)(2k+1)^2 \\ &= 2k(2k+1)^2 + (2k+1)^2 \\ &= 2k(2k+1)^2 + (2k+1)(2k+1) \\ &= 2k(2k+1)^2 + 2k(2k+1) + 1 \\ &= 2(k(2k+1)^2 + k(2k+1)) + 1. \end{aligned}$$

Since k is an integer, so is $k(2k+1)^2 + k(2k+1)$. Therefore, n^3 is odd.

Part (b): We prove the contrapositive. That is, we prove that, if n is even, then n^7 is even. Suppose that n is even. Then there exists an integer k such that $n = 2k$. We have

$$n^7 = (2k)^7 = 2 \cdot 2^6 \cdot k^7.$$

Since k is an integer, $2^6 \cdot k^7$ is an integer. Therefore, n^7 is even.

Part (c): We divide the proof into cases based on whether n is even or odd.

Case 1: n is even.

In this case, there exists an integer k such that $n = 2k$. We have

$$n(n+1) = 2k(2k+1) = 2(k(2k+1)).$$

Since k is an integer, so is $k(2k+1)$. Thus, $n(n+1)$ is even.

Case 2: n is odd.

In this case, there exists an integer k such that $n = 2k+1$. We have

$$n(n+1) = (2k+1)(2k+2) = 2((2k+1)(k+1)).$$

Since k is an integer, so is $((2k+1)(k+1))$. Thus, $n(n+1)$ is even. □

4. In this question, the universe for all variables is the real numbers.

- (a) Prove that the sum of two rational numbers is rational.
- (b) Prove that the sum of a rational number and an irrational number is irrational. *Hint:* Proof by Contradiction. Part (a) may be useful.
- (c) Prove that if xy is irrational, then either x is irrational or y is irrational.
- (d) Is the converse of the statement in Part (c) true? Explain.

Solution. Part (a): Suppose that x and y are rational numbers. Then there exist integers p, q, s, t with $q \neq 0$ and $t \neq 0$ such that $x = p/q$ and $y = s/t$. We have

$$x + y = \frac{p}{q} + \frac{s}{t} = \frac{pt + sq}{qt}.$$

Since p, q, s, t are integers, so are $pt + sq$ and qt . Also, since $q \neq 0$ and $t \neq 0$, we have $qt \neq 0$. Therefore, $x + y$ is rational.

Part (b): We prove this by contradiction. Let x be rational and y be irrational. Suppose, for the sake of deriving a contradiction, that $x + y$ is rational. Then we can write y as

$$y = (x + y) - x = (x + y) + (-x).$$

So, y is a sum of $x + y$, which is rational, and $-x$, which is rational (because x is rational). By Part (a), this implies that y is rational. However, this contradicts the fact that y is irrational.

Part (c): We prove the contrapositive. That is, we assume that x and y are both rational and prove that xy is rational. Suppose that x and y are rational numbers. Then there exist integers p, q, s, t with $q \neq 0$ and $t \neq 0$ such that $x = p/q$ and $y = s/t$. We have

$$xy = \frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt}.$$

Since p, q, s, t are integers, so are $pt + sq$ and qt . Also, since $q \neq 0$ and $t \neq 0$, we have $qt \neq 0$. Therefore, xy is rational.

Part (d): The converse is false. Let $x = y = \sqrt{2}$. Then the statement that x or y is irrational is true. However, $xy = 2$ which is rational. \square

5. Answer each of the following true or false and justify your answer:

- (a) $\{x/2 : x \in \mathbb{Z}\} = \mathbb{Q}$.
- (b) $\{\sqrt{n} : n \in \mathbb{N}\} \subseteq \mathbb{Q}$. *Typo: x should be n here!*
- (c) $|\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}| \in \{0, 4\}$.
- (d) $a \in \{\{a\}, b, \{a, b\}\}$.
- (e) $\{a\} \in \{\{a\}, b, \{a, b\}\}$.
- (f) $\{a\} \subseteq \{\{a\}, b, \{a, b\}\}$.
- (g) $A \cap B \in \mathcal{P}(A) \cap \mathcal{P}(B)$ for any sets A and B .
- (h) Every set A satisfies $A \subsetneq A$.

Solution. (a) False. For example, $1/3 \in \mathbb{Q}$ but $1/3 \notin \{x/2 : x \in \mathbb{Z}\}$.

(b) False. $\sqrt{2} \in \{\sqrt{n} : n \in \mathbb{N}\}$ but $\sqrt{2}$ is irrational (and so $\sqrt{2} \notin \mathbb{Q}$).

(c) False. The cardinality is three. The three elements are \emptyset , $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$. Therefore, the cardinality is not 0 or 4.

(d) False. The elements of the set are $\{a\}$, b and $\{a, b\}$ and so a is not an element of the set.

(e) True.

(f) False because a is an element of $\{a\}$ but not an element of $\{\{a\}, b, \{a, b\}\}$. So, it is not true that every element of $\{a\}$ is an element of $\{\{a\}, b, \{a, b\}\}$.

(g) True. We have $A \cap B \subseteq A$ and so $A \cap B \in \mathcal{P}(A)$. Also, $A \cap B \subseteq B$ and so $A \cap B \in \mathcal{P}(B)$. Therefore, $A \cap B \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

(h) False. The statement $A \subsetneq B$ means $A \subseteq B$ and $A \neq B$. Every set A satisfies $A \subseteq A$ (that is, A is a subset of itself). However, no set satisfies $A \subsetneq A$ because, in order for A to be a proper subset of A , it would need to be the case that $A \neq A$. \square

6. (Bonus Question, 2 Bonus Marks) There are four Math 122 students who got a final grade of 49% and they are asking their instructor to round their grade up to 50%. The instructor agrees that if the students can win a game, then each of them can get the extra 1% that they need. The game goes like this. The instructor puts either a yellow hat or a green hat on each of their heads. The students cannot see their own hat, but they can see the hats of all other students. Each student writes down either “green” or “yellow” on a slip of paper and passes it to the instructor. They cannot see what other students wrote and cannot communicate in any way during the game. The students win if either (a) all of the students write the colour of their own hat or (b) none of the students write the colour of their own hat. The students were told the rules of the game in advance and they can agree on a strategy together. Show that it is possible for the students to guarantee that they win the game. *Hint:* Odds and evens. It may be useful (and fun) to play with the cases of one, two and three students first.