Math 122 Assignment 1 SOLUTIONS

Due: Wednesday, September 24, 2025 at 23:59. Please submit to Crowdmark:
`https://app.crowdmark.com/sign-in/university-of-victoria`.

There are five questions of equal value (worth a total of 40 marks), and one bonus question (worth 4 marks). Please feel free to discuss these problems with each other. You may not access any "tutoring" or "help" website or use AI assistance in any way. Suspected violations of any relevant policies on Academic Integrity will be treated seriously. Each person must write up their own solution, in their own words, in a way that reflects their own understanding. Complete solutions are those which are coherently written, and include appropriate justifications.

1. (a) The *Absorption Law* is the logical equivalence $p \vee (p \wedge q) \Leftrightarrow p$. Use a truth table to prove that the Absorption Law is true. Explain your reasoning.

   (b) Prove that $p \wedge (p \vee q)$ is logically equivalent to p by starting with the statement $p \wedge (p \vee q)$ and using the other Laws of Logic and Known Logical Equivalences to get p. You are allowed to use Absorption Law from part (a) in your proof. Do not use the idea of "duals" from the course notes. Justify each step.

   (c) Use Laws of Logic and Known Logical Equivalences to prove that $\neg(p \wedge (r \rightarrow \neg(q \rightarrow r)))$ is logically equivalent to $p \rightarrow r$. You are allowed to use the first Absorption Law that you proved in part (a) or the second Absorption Law that you proved in part (b).

Solution. Part (a): Here is the truth table.

p	q	$p \wedge q$	$p \vee (p \wedge q)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

The column corresponding to p has the same truth values as the column corresponding to $p \vee (p \wedge q)$. Therefore, they are logically equivalent.

Part (b): We have

$$\begin{aligned} p \wedge (p \vee q) &\Leftrightarrow (p \wedge p) \vee (p \wedge q) && \text{Distributivity} \\ &\Leftrightarrow p \vee (p \wedge q) && \text{Known LE (Idempotence)} \\ &\Leftrightarrow p && \text{Known LE (Absorption)} \end{aligned}$$

Another way of doing it is like this

$$\begin{aligned} p \wedge (p \vee q) &\Leftrightarrow \neg\neg\,(p \wedge (p \vee q)) && \text{Double Negation} \\ &\neg\,(\neg p \vee \neg(p \vee q)) && \text{DeMorgan} \\ &\neg\,(\neg p \vee (\neg p \wedge \neg q)) && \text{DeMorgan} \\ &\Leftrightarrow \neg(\neg p) && \text{Known LE (Absorption)} \\ &\Leftrightarrow p && \text{Double Negation} \end{aligned}$$

There are probably other proofs as well.

Part (c): We have

$$
\begin{aligned}
\neg(p \wedge (r \to \neg(q \to r))) &\Leftrightarrow \neg(p \wedge (r \to \neg(\neg q \vee r))) && \text{Known LE (Implication)}\\
&\Leftrightarrow \neg(p \wedge (\neg r \vee \neg(\neg q \vee r))) && \text{Known LE (Implication)}\\
&\Leftrightarrow \neg p \vee \neg(\neg r \vee \neg(\neg q \vee r)) && \text{DeMorgan}\\
&\Leftrightarrow \neg p \vee (r \wedge (\neg q \vee r)) && \text{DeMorgan}\\
&\Leftrightarrow \neg p \vee r && \text{Absorption (the second one)}\\
&\Leftrightarrow p \to r && \text{Known LE (Implication)}
\end{aligned}
$$

$\square$

2. (a) Recall that the Associativity Law says that, for any statements p, q and r, the following two logical equivalences hold

- A1: $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
- A2: $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$.

Prove that the Associativity Law A2 is true by starting with the statement $p \wedge (q \wedge r)$ and using the other Laws of Logic and Known Logical Equivalences to get $(p \wedge q) \wedge r$. You are allowed to use Associativity Law A1 in your proof but are not allowed to use Associativity Law A2. Do not use the idea of "duals" from the course notes. Justify each step.

(b) Let s be a statement whose truth value depends on statements p, q and r according to the following truth table.

p	q	r	s
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Find an expression in disjunctive normal form (see Section 1.10) that is logically equivalent to s

(c) Use Laws of Logic and Known Logical Equivalences to show that the expression for s found in part (b) is logically equivalent to $p \wedge \neg(\neg q \wedge r)$. Justify each step. *Hint:* It may help to apply the Distributivity Law in the "reverse" direction.

Solution. Part (a): We have

$$\begin{aligned}
p \wedge (q \wedge r) &\Leftrightarrow \neg\neg(p \wedge (q \wedge r)) && \text{Double Negation}\\
&\Leftrightarrow \neg(\neg p \vee \neg(q \wedge r)) && \text{DeMorgan}\\
&\Leftrightarrow \neg(\neg p \vee (\neg q \vee \neg r)) && \text{DeMorgan}\\
&\Leftrightarrow \neg((\neg p \vee \neg q) \vee \neg r) && \text{Associativity Law A1}\\
&\Leftrightarrow \neg(\neg p \vee \neg q) \wedge r && \text{DeMorgan, Double Negation}\\
&\Leftrightarrow (p \wedge q) \wedge r && \text{DeMorgan, Double Negation}
\end{aligned}$$

Part (b): It is as follows:

$$(p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r).$$

Part (c):

$$\begin{aligned}
&(p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)\\
&\Leftrightarrow (p \wedge ((\neg q \wedge \neg r) \vee (q \wedge \neg r))) \vee (p \wedge q \wedge r) && \text{Distributivity}\\
&\Leftrightarrow p \wedge ((\neg q \wedge \neg r) \vee (q \wedge \neg r) \vee (q \wedge r)) && \text{Distributivity}\\
&\Leftrightarrow p \wedge ((\neg q \wedge \neg r) \vee (q \wedge (\neg r \vee r))) && \text{Distributivity}\\
&\Leftrightarrow p \wedge ((\neg q \wedge \neg r) \vee (q \wedge \mathbf{1})) && \text{Known Tautology}\\
&\Leftrightarrow p \wedge ((\neg q \wedge \neg r) \vee q) && \text{Known LE (Identity)}\\
&\Leftrightarrow p \wedge ((\neg q \vee q) \wedge (\neg r \vee q)) && \text{Distributivity}\\
&\Leftrightarrow p \wedge (\mathbf{1} \wedge (\neg r \vee q)) && \text{Known Tautology}\\
&\Leftrightarrow p \wedge (\neg r \vee q) && \text{Known LE (Identity)}\\
&\Leftrightarrow p \wedge \neg(r \wedge \neg q) && \text{DeMorgan, Double Negation}\\
&\Leftrightarrow p \wedge \neg(\neg q \wedge r) && \text{Commutativity}
\end{aligned}$$

$\square$

3. The logical connective “exclusive or,” also known as “XOR,” is written $p \veebar q$. The statement $p \veebar q$ is true if exactly one of p or q is true.[1]

   (a) Using a truth table, show that $p \veebar q$ is logically equivalent to $(p \vee q) \wedge (\neg p \vee \neg q)$. Using this logical equivalence and the Laws of Logic, explain why $\veebar$ is commutative. In other words, explain why $p \veebar q \Leftrightarrow q \veebar p$ for any statements p and q.

   (b) Find an expression that is logically equivalent to $p \veebar q$ which only uses the symbols $p, q, \rightarrow, \neg$ and brackets. Use Laws of Logic and Known Logical Equivalences to justify that your expression is logically equivalent to the expression for $p \veebar q$ from Part (a).

   (c) Prove that XOR is associative.

Solution. Part (a):

p	q	$\neg p$	$\neg q$	$p \veebar q$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \wedge (\neg p \vee \neg q)$
0	0	1	1	0	0	1	0
0	1	1	0	1	1	1	1
1	0	0	1	1	1	1	1
1	1	0	0	0	1	0	0

The fifth and eighth columns are identical. Therefore, $p \veebar q \Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q)$.

Now, let us show that XOR is commutative. We have

$$\begin{aligned} p \veebar q &\Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q) && \text{As shown in the truth table} \\ &\Leftrightarrow (q \vee p) \wedge (\neg q \vee \neg p) && \text{Commutativity of } \vee\ (\times 2) \\ &\Leftrightarrow q \veebar p && \text{As shown in the truth table} \end{aligned}$$

Note: Some students may simply say that XOR is commutative simply because the definition is symmetric (if exactly one of p or q is true, then exactly one of q or p is true). They aren't wrong. They can have the points for that.

Part (b): There are several possible expressions. Probably the simplest such example is $\neg((\neg p \rightarrow q) \rightarrow \neg(p \rightarrow \neg q))$. We have

$$\begin{aligned} \neg((\neg p \rightarrow q) \rightarrow \neg(p \rightarrow \neg q)) &\Leftrightarrow \neg(\neg(\neg p \rightarrow q) \vee \neg(p \rightarrow \neg q)) && \text{Known LE} \\ &\Leftrightarrow (\neg p \rightarrow q) \wedge (p \rightarrow \neg q) && \text{DeMorgan, Double Negation} \\ &\Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q) && \text{Known LE } (\times 2) \\ &\Leftrightarrow p \veebar q && \text{Part (a)} \end{aligned}$$

Part (c): For this, we need to show that $(p \veebar q) \veebar r \Leftrightarrow p \veebar (q \veebar r)$. There are many ways of doing this. In these solutions, we will show two different solutions, but to get full marks, you only need to include one of them.

First, let's show how we can prove it using a truth table. We have

[1] The logical connective $\veebar$ is only being used in this assignment. You should not use it in solutions to problems that you encounter in quizzes, other assignments or the final exam.

p	q	r	$p \veebar q$	$(p \veebar q) \veebar r$	$q \veebar r$	$p \veebar (q \veebar r)$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	0	0	0
1	0	0	1	1	0	1
1	0	1	1	0	1	0
1	1	0	0	0	1	0
1	1	1	0	1	0	1

The fifth and seventh columns are the same, and so $(p \veebar q) \veebar r \Leftrightarrow p \veebar (q \veebar r)$.

For a second proof, let's use the LE from part (a) together with the Laws of Logic and other Known LEs. We have

$$
\begin{aligned}
(p \veebar q) \veebar r &\Leftrightarrow ((p \veebar q) \vee r) \wedge (\neg(p \veebar q) \vee \neg r) && \text{Part (a)} \\
&\Leftrightarrow (((p \vee q) \wedge (\neg p \vee \neg q)) \vee r) \wedge (\neg((p \vee q) \wedge (\neg p \vee \neg q)) \vee \neg r) && \text{Part (a) } (\times 2) \\
&\Leftrightarrow (((p \vee q) \vee r) \wedge ((\neg p \vee \neg q) \vee r)) \wedge (\neg((p \vee q) \wedge (\neg p \vee \neg q)) \vee \neg r) && \text{Distributivity} \\
&\Leftrightarrow (((p \vee q) \vee r) \wedge ((\neg p \vee \neg q) \vee r)) \wedge ((\neg(p \vee q) \vee \neg(\neg p \vee \neg q)) \vee \neg r) && \text{DeMorgan} \\
&\Leftrightarrow (((p \vee q) \vee r) \wedge ((\neg p \vee \neg q) \vee r)) \wedge (((\neg p \wedge \neg q) \vee (p \wedge q)) \vee \neg r) && \text{DeMorgan } (\times 2), \\
& && \text{Double Negation } (\times 2) \\
&\Leftrightarrow (((p \vee q) \vee r) \wedge ((\neg p \vee \neg q) \vee r)) \wedge (((\neg p \vee (p \wedge q)) \wedge (\neg q \vee (p \wedge q))) \vee \neg r) && \text{Distributivity}
\end{aligned}
$$

$$
\begin{aligned}
&\Leftrightarrow (((p \vee q) \vee r) \wedge ((\neg p \vee \neg q) \vee r)) \wedge ((((\neg p \vee p) \wedge (\neg p \vee q)) \wedge ((\neg q \vee p) \wedge (\neg q \vee q))) \vee \neg r) && \text{Distributivity } (\times 2) \\
&\Leftrightarrow (((p \vee q) \vee r) \wedge ((\neg p \vee \neg q) \vee r)) \wedge (((\mathbf{1} \wedge (\neg p \vee q)) \wedge ((\neg q \vee p) \wedge \mathbf{1})) \vee \neg r) && \text{Known Tautology } (\times 2) \\
&\Leftrightarrow (((p \vee q) \vee r) \wedge ((\neg p \vee \neg q) \vee r)) \wedge (((\neg p \vee q) \wedge (\neg q \vee p)) \vee \neg r) && \text{Known LE } (\times 2) \\
&\Leftrightarrow (((p \vee q) \vee r) \wedge ((\neg p \vee \neg q) \vee r)) \wedge (((\neg p \vee q) \vee \neg r) \wedge ((\neg q \vee p) \vee \neg r)) && \text{Distributivity} \\
&\Leftrightarrow (((p \vee q) \vee r) \wedge ((p \vee \neg q) \vee \neg r)) \wedge (((\neg p \vee \neg q) \vee r) \wedge ((\neg p \vee q) \vee \neg r)) && \text{Associativity,} \\
& && \text{Commutativity} \\
&\Leftrightarrow ((p \vee (q \vee r)) \wedge (p \vee (\neg q \vee \neg r))) \wedge ((\neg p \vee (\neg q \vee r)) \wedge (\neg p \vee (q \vee \neg r))) && \text{Associativity,} \\
& && \text{Commutativity} \\
&\Leftrightarrow (p \vee ((q \vee r) \wedge (\neg q \vee \neg r))) \wedge (\neg p \vee ((\neg q \vee r) \wedge (q \vee \neg r))) && \text{Distributivity } (\times 2) \\
&\Leftrightarrow (p \vee ((q \vee r) \wedge (\neg q \vee \neg r))) \wedge (\neg p \vee ((\neg q \wedge (q \vee \neg r)) \vee (r \wedge (q \vee \neg r)))) && \text{Distributivity} \\
&\Leftrightarrow (p \vee ((q \vee r) \wedge (\neg q \vee \neg r))) \wedge (\neg p \vee (((\neg q \wedge q) \vee (\neg q \wedge \neg r)) \vee ((r \wedge q) \vee (r \wedge \neg r)))) && \text{Distributivity} \\
&\Leftrightarrow (p \vee ((q \vee r) \wedge (\neg q \vee \neg r))) \wedge (\neg p \vee ((\mathbf{0} \vee (\neg q \wedge \neg r)) \vee ((r \wedge q) \vee \mathbf{0}))) && \text{Known Contradiction} \\
&\Leftrightarrow (p \vee ((q \vee r) \wedge (\neg q \vee \neg r))) \wedge (\neg p \vee ((\neg q \wedge \neg r) \vee (r \wedge q))) && \text{Known LE} \\
&\Leftrightarrow (p \vee ((q \vee r) \wedge (\neg q \vee \neg r))) \wedge (\neg p \vee \neg((q \vee r) \wedge (\neg q \vee \neg r))) && \text{DeMorgan } (\times 3) \\
& && \text{Commutativity} \\
&\Leftrightarrow (p \vee (q \veebar r)) \wedge (\neg p \vee \neg(q \veebar r)) && \text{Part (a) } (\times 2) \\
&\Leftrightarrow p \veebar (q \veebar r)) && \text{Part (a)}
\end{aligned}
$$

Phew! That was long and complicated. The truth table is clearly the simpler approach here! $\square$

4. Determine whether each pair of statements below is logically equivalent, and justify your answer. It may help to translate each of the statements into symbolic form. Remember to say which letters correspond to which statements.

   (a) "If you write with pen, then you cannot erase" and "It is not the case that you write with pen and you can erase".

   (b) "You can go cycling if it is not raining" and "If it is raining, then you cannot go cycling".

   (c) "Chris is happy only if he is eating chocolate cake or he is playing Mario Kart" and "If Chris is not eating chocolate cake and he is not playing Mario Kart, then he is not happy."

Solution. Part (a): The first statement is $p \to \neg e$ where p is the statement "you write in pen" and e is the statement "you can erase." The second statement is $\neg(p \wedge e)$ which is logically equivalent to $\neg p \vee \neg e$ by DeMorgan and $p \to \neg e$ by Implication. So, they are logically equivalent.

Part (b): The first statement is $\neg r \to c$ where r is the statement "it is raining" and c is the statement "you can go cycling." The second statement is $r \to \neg c$. These are not logically equivalent. To see this, note that, if r is false and c is false, then the first statement is false but the second statement is true.

Part (c): The first statement is $h \to (c \vee m)$ where h is "Chris is happy," c is "Chris is eating chocolate cake" and m is "Chris is playing Mario Kart." The second statement is $(\neg c \wedge \neg m) \to \neg h$. The contrapositive is $h \to \neg(\neg c \wedge \neg m)$ which, by DeMorgan, is $h \to (c \vee m)$. So, they are logically equivalent. $\square$

5. For each of the following, first write the argument in symbolic form, and then determine if it is valid or invalid. If the argument is valid, prove it using Laws of Logic, Known Logical Equivalences and Inference Rules, and justify each step. If it is invalid, demonstrate that by giving a counterexample and explain why your counterexample shows the argument is invalid. It may help to translate each of the statements into symbolic form. Remember to say which letters correspond to which statements.

   (a) If Stacey is happy and she does not paint a picture, then Felix is happy and he eats beans for lunch. If Stacey paints a picture, then Felix is happy. Felix does not eat beans for lunch. Therefore, if Stacey is happy, then Felix is happy

   (b) You live in Victoria or New York, but not both. If you live in Victoria, then you can recycle your pizza box if and only if the pizza box is not greasy. If your pizza box is greasy, then you can compost your pizza box. If you live in New York, then there are too many pests and you cannot compost your pizza box. Therefore, if you can compost your pizza box, then you cannot recycle your pizza box or your pizza box is greasy.

Solution. Part (a): Define the following statements:

$$s : \text{“Stacey is happy”}$$

$$p : \text{“Stacey paints a picture”}$$

$$f : \text{“Felix is happy”}$$

$$b : \text{“Felix eats beans for lunch”}.$$

The argument described in the question is

$$\begin{array}{rl} & (s \wedge \neg p) \to (f \wedge b) \\ & p \to f \\ & \neg b \\ \hline \therefore & s \to f \end{array}$$

The argument is valid. Here is the proof.

1.	$\neg b$	Premise
2.	$(s \wedge \neg p) \to (f \wedge b)$	Premise
3.	$\neg(f \wedge b) \to \neg(s \wedge \neg p)$	2., Contrapositive
4.	$(\neg f \vee \neg b) \to \neg(s \wedge \neg p)$	3., DeMorgan
5.	$\neg f \vee \neg b$	1., Disjunctive Amplification
6.	$\neg(s \wedge \neg p)$	4., 5., Modus Ponens
7.	$\neg s \vee p$	6., DeMorgan
8.	$s \to p$	7., Known LE (Implication)
9.	$p \to f$	Premise
10.	$s \to f$	8., 9., Chain Rule

Part (b): Part (a): Define the following statements:

$$n : \text{“You live in New York”}$$

$$v : \text{“You live in Victoria”}$$

$$r : \text{“You can recycle your pizza box”}$$

$$g: \text{“Your pizza box is greasy”}.$$

$$c: \text{“You can compost your pizza box”}.$$

$$p: \text{“There are too many pests”}.$$

The argument described in the question is

$$\begin{array}{rl} & (n \wedge \neg v) \vee (\neg n \vee v) \\ & v \rightarrow (r \leftrightarrow \neg g) \\ & g \rightarrow c \\ & n \rightarrow (p \wedge \neg c) \\ \hline \therefore & c \rightarrow (\neg r \vee g) \end{array}$$

The argument is invalid. To show this, we need to find an assignment of truth values so that the premises are all true and the conclusion is false (i.e. a counterexample).

Let's explain how we come up with a counterexample. This part is not necessary to write down in your solution, but it may help you learn how to solve these problems. To make the conclusion false, you need c to be true, r to be true and g to be false. The fact that c is true makes the third premise true automatically. Since c is true, the only way to make the fourth premise true is to make n false. So, to make the first premise true, v must be true. Given that v is true and g is false, the only way to make the second premise true is to make r true. We can make p true or false; it doesn't matter. So, either of the following two assignments of truth values are counterexamples:

n	v	r	g	c	p
0	1	1	0	1	0 or 1

Let's double-check that this is a counterexample. The first premise is definitely true because exactly one of n or v is true. Since r and g have different truth values, the second premise is true. Since c is false, the third premise is true. Since n is false, the fourth premise is true. However, the conclusion is false because c is true, r is true and g is false. □

6. (Bonus Question, 4 Bonus Marks) At Boolean High School, every student must participate in exactly one sports team and play exactly one instrument and records their choices in a log book, one line per student. The school has a rule that members of the basketball team are not allowed to play flute or piano. One of the teachers spilled coffee on the log book. As a result, for the first six students, only the following information is readable:

   - Bob: guitar
   - Jenny: volleyball
   - Sam: basketball
   - Gwen: piano
   - Hao: soccer
   - Karina: flute

   The principal would like to make sure that the rule is being followed and so they need to ask some of these six students about the information that is missing from the log book. What is the minimum number of students that the principal needs to talk to in order to determine whether the rule is being followed?