Math 122 Assignment 5

Due: Wednesday, November 26, 2025 at 23:59. Please submit on Crowdmark: `https://app.crowdmark.com/sign-in/university-of-victoria`.

There are five questions of equal value (worth a total of 40 marks). There are also 4 bonus marks avaiable, as indicated below. Please feel free to discuss these problems with each other. You may not access any "tutoring" or "help" website or use AI assistance in any way. Suspected violations of any relevant policies on Academic Integrity will be treated seriously. Each person must write up their own solution, in their own words, in a way that reflects their own understanding. Complete solutions are those which are coherently written, and include appropriate justifications.

Two bonus marks are available for the bonus question, and two more bonus marks are available if the answers to all questions are typeset using the mathematical typesetting system LaTeX. Information on obtaining and using LaTeX is on the cross-listed Brightspace page.

---

1. (a) Let $n = (d_5d_4d_3d_2d_1d_0)_8$ be a base eight number with six digits. Leading zeros are allowed (for example, d_5 may be 0). Use the fact that $8^2 \equiv -1 \pmod{13}$ to prove that $13 \mid n$ if and only if $13 \mid ((d_5d_4)_8 - (d_3d_2)_8 + (d_1d_0)_8)$.

   (b) Let b and n be integers such that $b > 1$ and $n \geq 1$. Suppose that the base b representation of n consists of two identical digits. Prove that $n \leq b^2 - 1$ and that $(b + 1) \mid n$.

2. Let a, b and c be positive integers.

   (a) Prove that if a and b are relatively prime and $c \mid b$, then a and c are relatively prime.

   (b) Suppose that $a \geq 2$ and let p be a prime factor of a. Prove that if $pa^3 \mid b$ then $p^6 \mid b^2c$.

   (c) Let n be an integer with $n \geq 2$ such that every divisor of n, except for n itself, is strictly less than $\sqrt{n}$. Prove that n is prime.

3. (a) Suppose $gcd(a, b) = 3$. What are the possibilities for $gcd(a, 15b)$? Explain.

   (b) Find the prime power decomposition (also called the prime factorization) of the smallest positive integer that is divisible by 35 and can be written as the 12th power of an integer and the 15th power of a different integer.

   (c) Suppose $k \equiv 5 \pmod{13}$. What is the remainder when $26^{k^2} + k^{19} - 3k^5 + 40$ is divided by 13?

4. For each of the following relations, determine whether it is reflexive, whether it is symmetric and whether it is transitive. After doing so, clearly state whether it is an equivalence relation. If it is an equivalence relation, then determine its equivalence classes (with proof).

   (a) Let $\mathcal{R}$ be the relation on $\mathcal{P}(\{1, 2, \ldots, 1000\})$ where $A\mathcal{R}B$ if and only if $|A \cap B|$ is even.

   (b) Let ? be the relation on $\mathbb{Z}$ where $a?b$ if and only if $4 \mid a^2 - b^2$.

   (c) Let X be the set of all locations in the universe. Define $\sim$ to be the relation on X where two locations x and y satisfy $x \sim y$ if x and y are within 1 meter of each other.

5. Consider the following property of a relation $\mathcal{R}$ on a set A:

   $(*)$ for all $(a, b) \in \mathcal{R}$ there exists c such that $b\mathcal{R}c$ and $c\mathcal{R}a$.

   (a) Let $\mathcal{R}$ be a transitive relation on a set A. Prove that if $\mathcal{R}$ satisfies $(*)$, then $\mathcal{R}$ is symmetric.

   (b) Let $\mathcal{R}$ be a transitive relation on a set A. Prove that if $\mathcal{R}$ is symmetric, then $\mathcal{R}$ satisfies $(*)$.

   (c) Give an example of a relation $\mathcal{R}$ on $\{1, 2, 3, 4\}$ which is transitive, not symmetric and not anti-symmetric. Explain why it has these three properties.

6. (Bonus question, 2 bonus marks) Show that for any positive integer n, there is a multiple of n, other than zero, whose base 10 representation consists of some non-zero number of sevens followed by some (possibly zero) number of zeros. For example, the base 10 representation looks like 77700000, 7777770 or 777. *Hint:* Look up the "pigeonhole principle." It might help.