ECE-302
PROJECT 4

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April 19, 2021

Gaussian MAP Rule

$$p_0(r) = p_Y(y \mid H = H_0) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$
$$p_1(r) = p_Y(y \mid H = H_1) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)$$

The MAP Rule chooses H_0 if:

$$P(y \mid H_0) * P_0 > P(y \mid H_1) * (1 - P_0)$$
$$\frac{P_0}{1-P_0} * P(y \mid H_0) > P(y \mid H_1)$$
$$\frac{P_0}{1-P_0} * \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) > \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)$$
$$\frac{P_0}{1-P_0} * \exp\left(-\frac{x^2}{2\sigma^2}\right) > \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)$$
$$\ln\left(\frac{P_0}{1-P_0}\right) + \left(-\frac{x^2}{2\sigma^2}\right) > -\frac{(x-A)^2}{2\sigma^2}$$
$$-1 * \ln\left(\frac{P_0}{1-P_0}\right) * 2\sigma^2 + x^2 < (x-A)^2$$
$$-1 * \ln\left(\frac{P_0}{1-P_0}\right) * 2\sigma^2 + x^2 < x^2 - 2Ax + A^2$$
$$-1 * \ln\left(\frac{P_0}{1-P_0}\right) * 2\sigma^2 < A^2 - 2Ax$$
$$2Ax < A^2 + \ln\left(\frac{P_0}{1-P_0}\right) * 2\sigma^2$$
$$x < \frac{A}{2} + \frac{\sigma^2}{A} \ln\left(\frac{P_0}{1-P_0}\right) = \eta$$

Theoretical Error

$$
\begin{aligned}
P_{err} &= P(\hat{H} = A \cap H_0) + P(\hat{H} = 0 \cap H_1) \\
&= \left[\int_{\eta}^{\infty} P(y \mid H_0) dx * P_0\right] + \left[\int_{-\infty}^{\eta} P(y \mid H_1) dx * (1 - P_0)\right] \\
&= \left[\int_{\eta}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx * P_0\right] + \left[\int_{-\infty}^{\eta} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right) dx * (1 - P_0)\right] \\
&= \left[\int_{\eta}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx * P_0\right] + \left[\int_{-\eta}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right) dx * (1 - P_0)\right] \\
&= Q\left(\frac{\eta}{\sigma}\right) * P_0 + Q\left(\frac{A-\eta}{\sigma}\right) * (1 - P_0)
\end{aligned}
$$

Gaussian MAP Rule #2

$$
\begin{aligned}
p_0(r) = p_Y(y \mid H = H_0) &= \frac{1}{\sigma_z\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma_z^2}\right) \\
p_1(r) = p_Y(y \mid H = H_1) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)
\end{aligned}
$$

The MAP Rule chooses H_0 if:

$$P(y \mid H_0) * P_0 > P(y \mid H_1) * (1 - P_0)$$
$$\frac{P_0}{1-P_0} * P(y \mid H_0) > P(y \mid H_1)$$
$$\frac{P_0}{1-P_0} * \frac{1}{\sigma_z\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma_z^2}\right) > \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)$$
$$\frac{P_0}{1-P_0} * \frac{\sigma}{\sigma_z} \exp\left(-\frac{(x-A)^2}{2\sigma_z^2}\right) > \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)$$
$$\ln\left(\frac{\sigma P_0}{\sigma_z(1-P_0)}\right) + \left(-\frac{(x-A)^2}{2\sigma_z^2}\right) > -\frac{(x-A)^2}{2\sigma^2}$$
$$\ln\left(\frac{\sigma P_0}{\sigma_z(1-P_0)}\right) > (x-A)^2\left[\frac{1}{2\sigma_z^2} - \frac{1}{2\sigma^2}\right]$$
$$\ln\left(\frac{\sigma P_0}{\sigma_z(1-P_0)}\right) > (x-A)^2\left[\frac{2\sigma^2 - 2\sigma_z^2}{4\sigma^2\sigma_z^2}\right]$$
$$\ln\left(\frac{\sigma P_0}{\sigma_z(1-P_0)}\right) > (x-A)^2\left[\frac{\sigma^2 - \sigma_z^2}{2\sigma^2\sigma_z^2}\right]$$
$$(x-A)^2 < \ln\left(\frac{\sigma P_0}{\sigma_z(1-P_0)}\right)\left[\frac{2\sigma^2\sigma_z^2}{\sigma^2 - \sigma_z^2}\right]$$

Theoretical Error #2

$$P_{err} = P(\hat{H} = A \cap H_0) + P(\hat{H} = 0 \cap H_1)$$
$$= \left[\int_{\eta}^{\infty} P(y \mid H_0)dx * P_0\right] + \left[\int_{-\infty}^{\eta} P(y \mid H_1)dx * (1-P_0)\right]$$
$$= \left[\int_{\eta}^{\infty} \frac{1}{\sigma_z\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma_z^2}\right)dx * P_0\right] + \left[\int_{-\infty}^{\eta} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)dx * (1-P_0)\right]$$
$$= \left[\int_{\eta}^{\infty} \frac{1}{\sigma_z\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma_z^2}\right)dx * P_0\right] + \left[\int_{-\eta}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)dx * (1-P_0)\right]$$
$$= Q\left(\frac{A+\eta}{\sigma_z}\right) * P_0 + Q\left(\frac{A-\eta}{\sigma}\right) * (1-P_0)$$