Laboratorio 2

Taller 3

1. a $\left(P - \frac{a}{V^2}\right)(V - b) = RT$

$$\left(P - \frac{a}{V^2}\right)(V - b) - RT = 0 \Rightarrow \left(5 - \frac{0.245}{V^2}\right)(V - 0.0266) - 28.721 = 0$$

2. b) $V_{k+1} = V_k - \frac{F(k)(X_{k-1} - x)}{F(V_k) - F(k-1)}$ $V_0 = 35$ $V_1 = 30$ $|V_k - V_{k-1}| = 5$

$E_K = 29.23766$
$E_K = 21.23766$

$$V_2 = V_1 - \frac{F(V_1)(V_1 - V_0)}{F(V_1) - F(V_0)} = 30 - \frac{F(30)(30-35)}{F(30) - F(35)} =$$

$E_K = 5.69\times10^{-3}$

$$V_2 = 30 - \frac{121.15416(30-35)}{121.15416 - 146.15290} = 5.76603$$ $|V_k - V_{k-1}| = 21.23196$

$E_K = 0$

$$V_3 = 5.76803 - \left[\frac{0.02845(5.76803 - 30)}{0.02845 - 121.15416}\right] = 5.76234$$ $|V_k - V_{k-1}| = 0.00569$

$E_K = 0$

$$V_4 = 5.76234 - \left[\frac{0.00003(5.76234 - 5.76803)}{0.00003 - 0.02845}\right] = 5.76234$$ $|V_k - V_{k-1}| = 0$

c) $R = 1.618$

$E_{k+1} = A|E_k|^{1.618}$

$$A = \frac{E_{k+1}}{E_k} = \frac{29.23766}{21.23766} = 1.20627$$

2. a) $F(x) = x^3 - 3x - 2$

$$P = P_0 - \frac{F(x)}{F'(x)} = P_0 - \frac{x^3 - 3x - 2}{3x^2 - 3}$$

b) $P_0 = 2.1$

$$P_1 = 2.1 - \frac{0.961}{10.23} = 2.00606$$

$$P_2 = 2.00606 - \frac{0.05477}{9.07264} = 2.00002 \qquad |P_K - P_{K-1}| = 0.09394$$

$$P_3 = 2.00002 - \frac{0.00022}{9.00079} = 2 \qquad |P_K - P_{K-1}| = 0.00604$$

$$P_4 = 2 - \frac{0}{8.99999} = 2 \qquad |P_K - P_{K-1}| = 0$$

c) Su orden de convergencia es cuadratico.

3. $N(t) = N_0 e^{\lambda t} + \nu \dfrac{e^{\lambda t} - 1}{\lambda}$; $N_0 = N(t_0)$

$N(t_0) = 1000$; $\nu = 435$ y $N(t_1) = 1564$

a) $N(\lambda) = 1000e^{\lambda} + 435\left[\dfrac{e^{\lambda} - 1}{\lambda}\right] - 1564$

b) $\lambda_0 = 0.5$ y precisión de $|\lambda_k - \lambda_{k-1}| < 10^{-6}$

$$P_1 = P_0 - \frac{F(x)}{F'(x)} = P_0 - \left[\frac{1000e^{\lambda} + 435\left[\frac{e^{\lambda}-1}{\lambda}\right] - 1564}{1000e^{\lambda} + 435\left[\frac{\lambda e^{\lambda} - e^{\lambda} + 1}{\lambda^2}\right]}\right]$$

$$P_2 = 0.5 - \frac{649.10878}{1954.33377} = 0.16786$$

$$P_3 = 0.16786 - \frac{92.41424}{1426.71539} = 0.10306 \qquad |\lambda_k - \lambda_{k-1}| = 0.33214$$

$$P_4 = 0.10306 - \frac{2.77011}{1341.60135} = 0.10100 \qquad |\lambda_k - \lambda_{k-1}| = 0.0648$$

$P_6 = 0.10100 - \frac{0.0027}{1338.99151} = 0.10100 \quad |P_n - P_{n-1}| = 0.00266$

1\. a) $g(x) = \frac{x^2 - 8x + 25}{3}$, $x \in [1.7]$

a.1) el teorema dice que:

$y = g(x)$ tiene al menos un punto fijo se se cumple que $y \in [a,b]$ para todos los $x \in [a,b]$

$P = g(P)$

$P = g(P) = \frac{1}{3}(P^2 - 8P + 25)$

$3P = P^2 - 8P + 25$

$P^2 - 11P + 25 = 0$

$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$P_1 = 7.79128$

$P_2 = 3.2087$ → esta raiz se encuentra acotada entre $[a,b]$, por lo tanto se garantiza al menos un punto fijo.

$F(a) > a$ y $f(b) < b$.

$g(1) = \frac{1^2 - 8(1) + 25}{3} = 6$

$g(7) = \frac{49 - 56 + 25}{3} = 6$

a.2) el segundo teorema indica

Si $g'(x)$ existe en (a,b) y existe una constante positiva $k < 1$ con

$|g'(x)| \leq k$ para toda $x \in (a,b)$

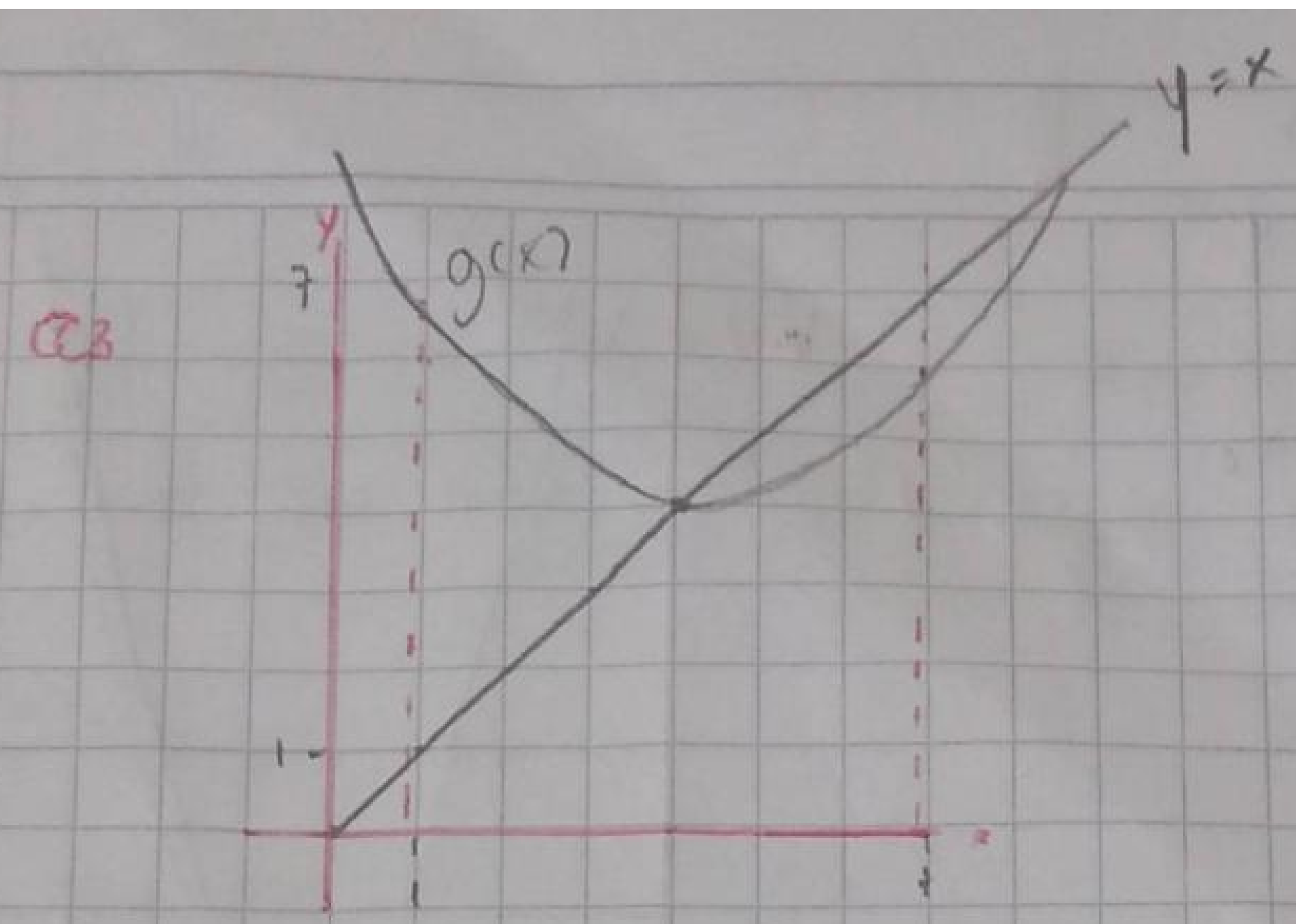
entonces hay exactamente un fijo en $[a,b]$.

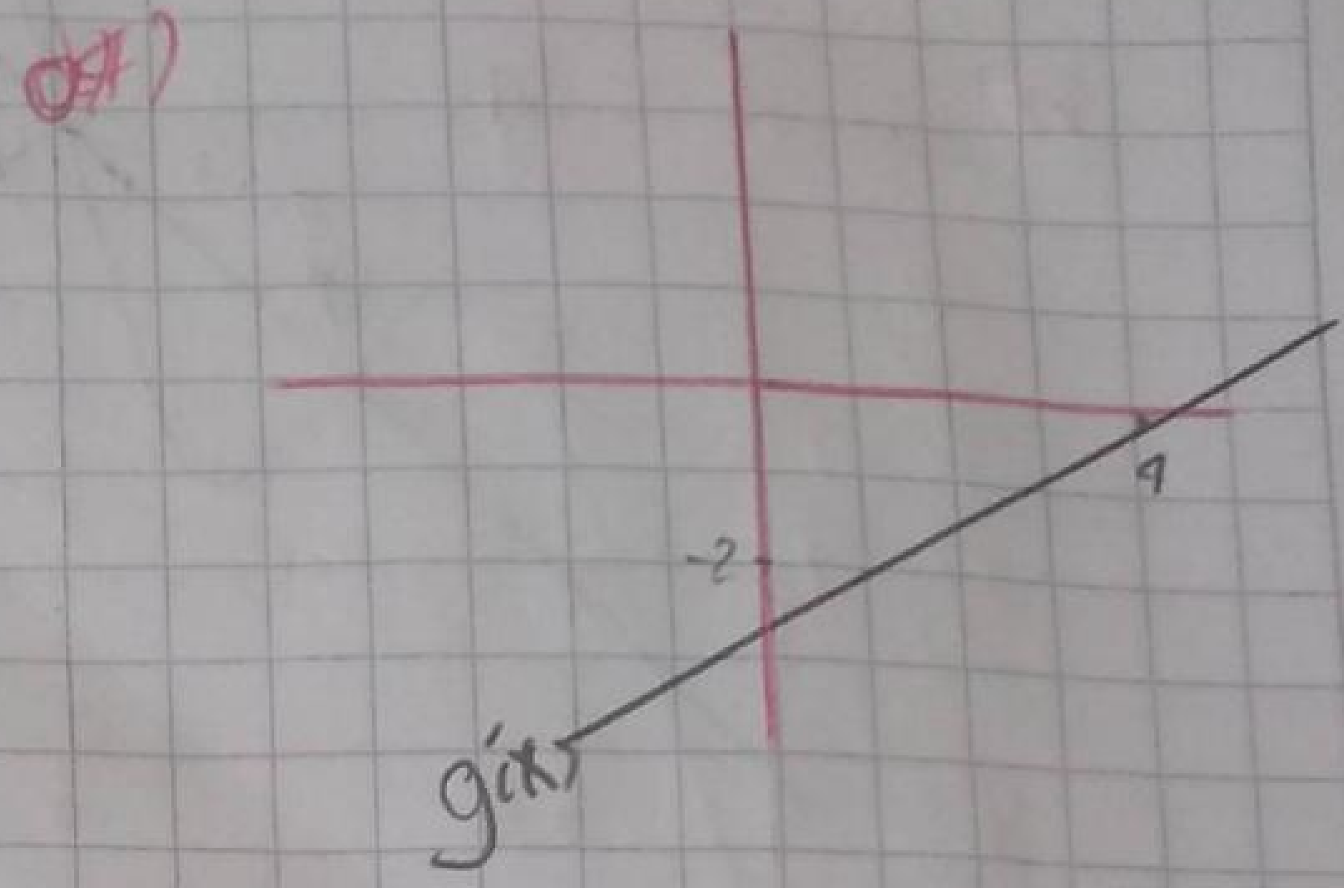
$g'(x) = \frac{1}{3}(2P - 8)$

$g(1) = -8$

$g(7) = 2$

$|g'(x)|$ es mayor que k, no se cumple el segundo teorema.





Aplicando Método de punto fijo

$P_0 = 3.15$

$P_1 = F(P_0) = 3.240833333$

$P_2 = F(P_1) = 3.192111313$

$P_3 = F(P_2) = 3.217561361$

$P_4 = 3.204070075$

$P_5 = 3.211168149$

$P_0 = 3.25$

$P_1 = F(P_0) = 3.1875$

$P_2 = 3.220052083$

$P_3 = 3.202772918$

$P_4 = 3.211857007$

$P_5 = 3.207056459$

b. b1) $g(x) = \dfrac{11x^3 - 141x^2 + 556x - 546}{30}$ $x \in [2, 7]$

$g(2) = 3$
$g(7) = 7$

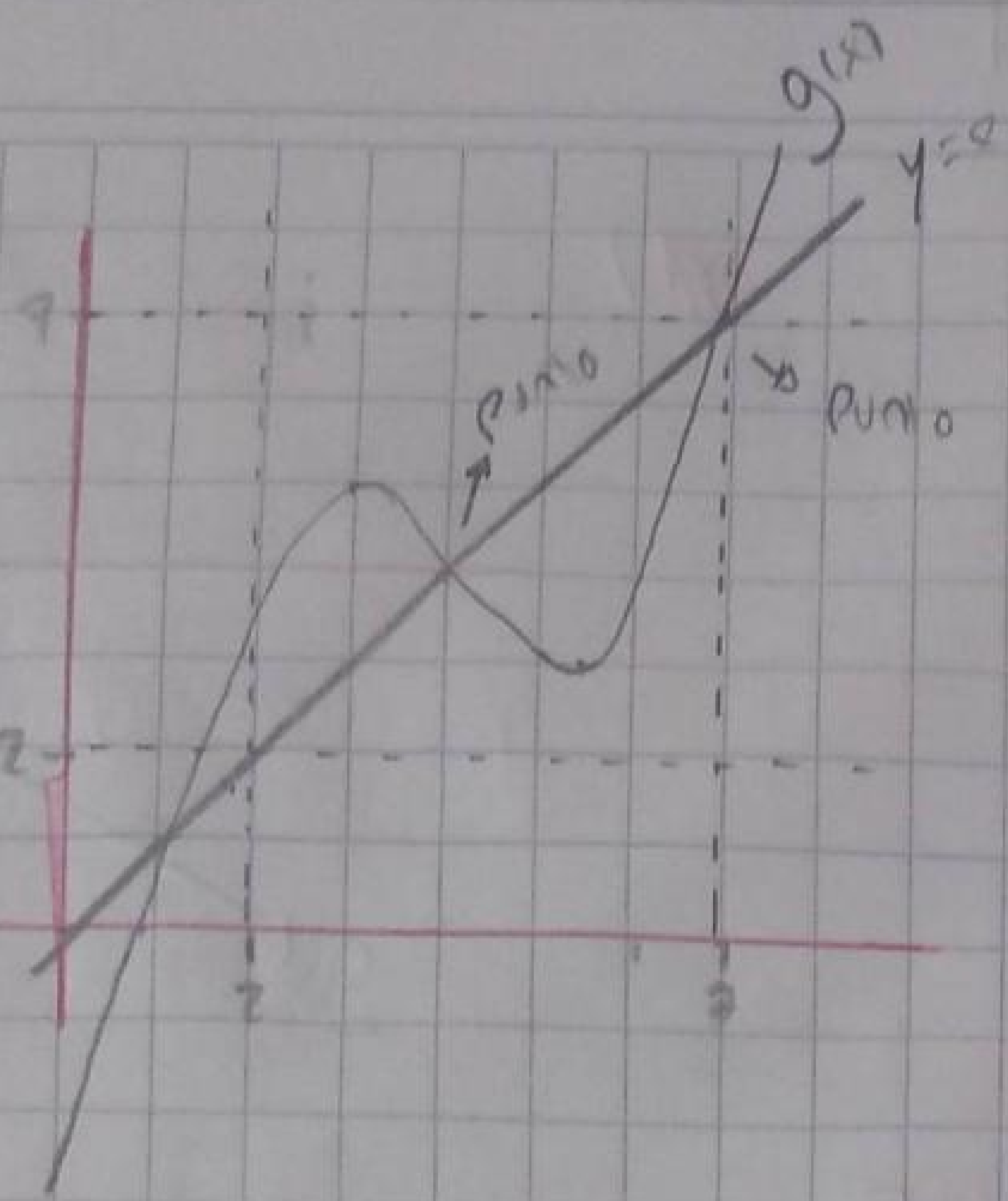
$f(a) > a$ $f(b) < b$ → Hay un punto fijo en este extremo
$3 > 2$ $7 < 7$

b2) $g'(x) = \dfrac{33x^2 - 282x + 556}{30}$

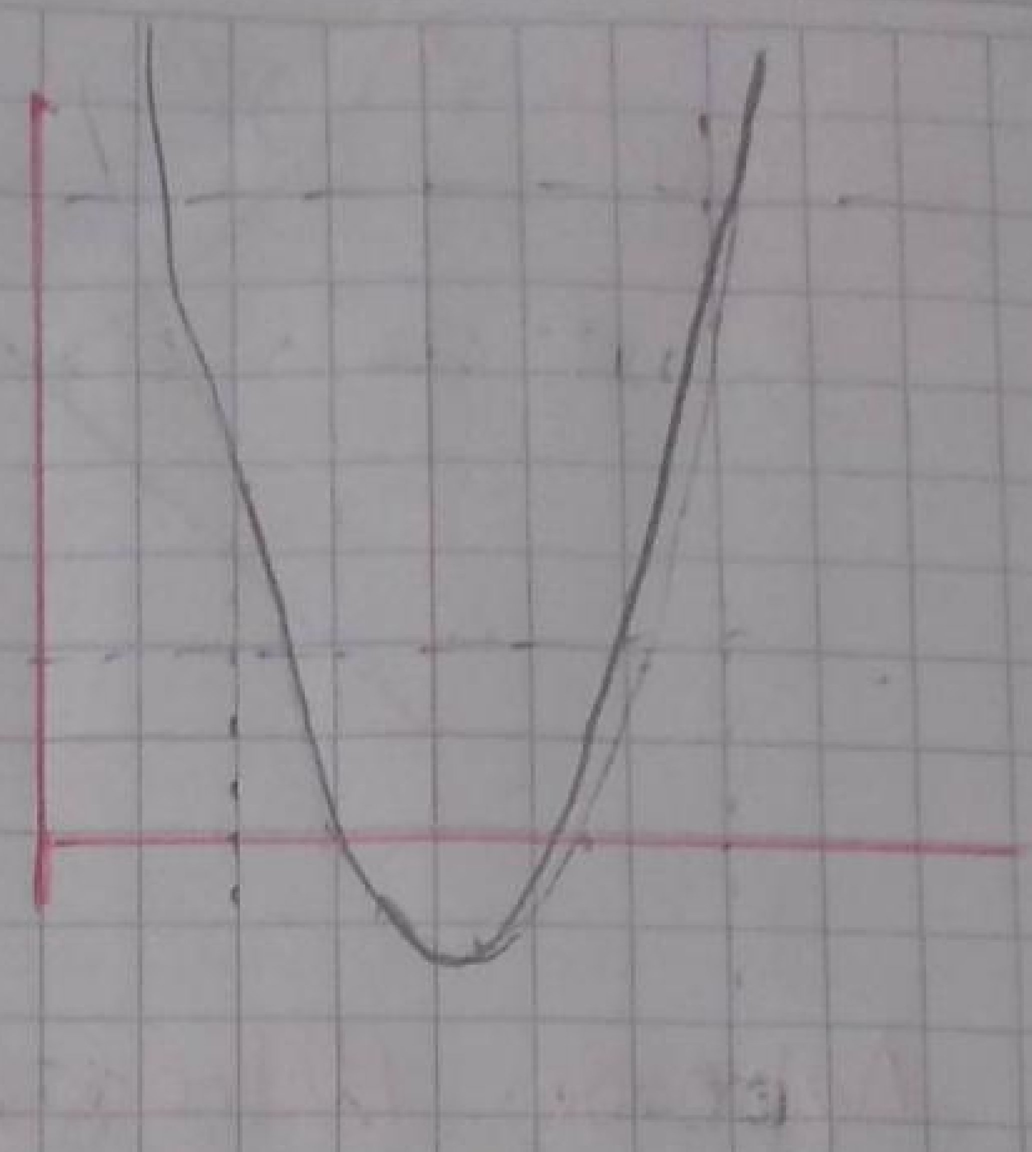
$g'(2) = 4.133333333$
$g'(7) = 6.633333333$] > k

No se garantiza un punto fijo único.

b3)



b4)



Calculo del Punto Fijo

$P_0 = 4.1$

$P_1 = (4.1) = 4.0507$

$P_2 = (4.0507) = 4.124916638$

$P_3 = f(P_2) = 4.012822374$

$P_4 = 4.18114527$

$P_5 = 3.976655665$

$P_0 = 6.95$

$P_1 = f(P_0) = 6.6757675$

$P_2 = f(P_1) = 5.152235919$

$P_3 = f(P_2) = 2.672672679$

$P_4 = f(P_3) = 4.760764107$

$P_5 = f(P_4) = 3.072027873$

5. a) $M_w = M_b$

$$M_w = \frac{\pi d^2 (3r - d)}{3} \quad ; \quad M_b = \frac{4\pi r^3 \rho}{3}$$

$$M_w - M_b = 0$$

$$\frac{\pi d^2 (3r - d)}{3} = \frac{4\pi r^3 \rho}{3} = 0$$

$$d^2 (3r - d) - 4r^3 \rho = 0$$

$$3rd^2 - d^3 - 4r^3 \rho = 0$$

$$-d^3 + 45d^2 - 8613 = 0$$

b) $|C_k - C_{k-1}| < 1 \times 10^{3}$ $a_0 = 17.6$; $b_0 = 18$

$$r = a + \frac{(b-a)}{2}$$

Iteración 0

$a_0 = 17.6$ $c_0 = \frac{a+b}{2} = 17.8$

$b_0 = 18$

$F(x) = -x^3 + 45x^2 - 8613$

$F(a_0) = -175.576$

$F(b_0) = 85$

$F(c_0) = -44.952$

Iteración 1.

$a_1 = 17.8$ $c_1 = \frac{a+b}{2} = 17.9$ $|c_1 - c_0| = 0.1 > 1 \times 10^{-3}$

$b_1 = 18$

$f(a_1) = -44.952$

$f(b_1) = 85$

$F(c_1) = 20.111$

Iteración 2.

$a_2 = 17.8$ $c_2 = \frac{a+b}{2} = 17.85$ $|c_2 - c_1| = 0.05 > 1 \times 10^{-3}$

$b_2 = 17.9$

$F(a_2) = -44.952$

$f(b_2) = 20.111$

$f(c_2) = -12.39917$

Iteración 3

$a_3 = 17.85$ $c_3 = \frac{a+b}{2} = 17.875$ $|c_3 - c_2| = 0.025$

$b_3 = 17.9$

$f(a_3) = -12.39917$

$f(b_3) = 20.111$

$f(c_3) = 3.86133$

Iteración 4

$a_4 = 17.85$
$b_4 = 17.875$
$F(a_4) = -17.39512$
$F(b_4) = 3.86133$
$F(c_4) = -4.26756$

$c_4 = \frac{a+b}{2} = 17.8625$ $|c_4 - c_3| = 0.025 > 1\times10^{-3}$

Iteración 5

$a_5 = 17.8625$
$b_5 = 17.875$

$F(a_5) = -4.26756$
$F(b_5) = 3.86133$
$F(c_5) = -0.20298$

$c_5 = \frac{a+b}{2} = 17.86875$ $|c_5 - c_4| = 6.25\times10^{-3} > 1\times10^{-3}$

Iteración 6

$a_6 = 17.86875$
$b_6 = 17.875$

$F(a_6) = -0.20298$
$F(b_6) = 3.86133$
$F(c_6) = 1.82936$

$c_6 = \frac{a+b}{2} = 17.87187$ $|c_6 - c_5| = 3.12\times10^{-3} > 1\times10^{-3}$

Iteración 7

$a_7 = 17.86875$
$b_7 = 17.87187$

$F(a_7) = -0.20298$
$F(b_7) = 1.82936$
$F(c_7) = 0.81331$

$c_7 = \frac{a+b}{2} = 17.87031$ $|c_7 - c_6| = 1.56\times10^{-3} > 1\times10^{-3}$

Iteración 8

$a_8 = 17.86875$
$b_8 = 17.87031$

$F(a_8) = -0.20298$
$F(b_8) = 0.81331$
$F(c_8) = 0.30527$

$c_8 = \frac{a+b}{2} = \underline{17.86957}$ $|c_8 - c_7| = \underline{7.8\times10^{-4} < 1\times10^{-3}}$

c) Número requerido de iteraciones para garantizar que un punto medio es una aproximación e.

$\delta = 1 \times 10^{-15}$
$a_0 = 17$
$b_0 = 18$

$$N \geq \frac{\ln(b-a) - \ln(\varepsilon)}{\ln(2)} - 1$$

$$N \geq \frac{\ln(18-17) - \ln(1 \times 10^{-15})}{\ln(2)} - 1$$

$N = \underline{48.82892}$

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Undergraduate Studies

Numerical Analysis

Homework #2

The solution of Nonlinear Equations $f(x) = 0$

DATE: 9th June 2020 DUE : **21th June 2020**

1 INDICATIONS

1. You **must** fill this sheet with just the answer for each problem and return it to the professor. However, you have to present the process on a separate exam sheet.
2. Answers with no process are **not valid**.
3. Make all calculations with 5 decimal places of precision.

2 THE SOLUTION OF NONLINEAR EQUATIONS

1. (1.0 point) The Van der Waals equation relates the density of fluids to the pressure P, volume v, and temperature T conditions. Thus, it is a thermodynamic equation of state given by,

$$\left(P + \frac{a}{v^2}\right)(v - b) = RT, \tag{2.1}$$

where, a, b and R are constasts that depends on the gas.

If $P = 5$, $a = 0.245$, $b = 0.0266$, $R = 0.08206$ and $T = 350$,

a) Determine a nonlinear equation $f(v) = 0$ that allows to calculate the volume v by finding its root.

$f(v) =$

b) Use the Secant method to find the root of the nonlinear equation in literal a). Use $v_0 = 35$, $v_1 = 30$ and iterate until achieving a precision of $|v - v_{k-1}| < 1 \times 10^{-5}$ *Make all calculations with 5 decimal places*.

Iteration rule: $v_{k+1} =$

k	v_k	$f(v_k)$	$\lvert v_k - v_{k-1}\rvert$	$E_k = \lvert 5.76234 - v_k\rvert$
0	35	146.15299	$-----$	29.23766
1	30	121.15416	5	24.23766
2	5.76803	0.02845	24.23197	5.69e-3
3	5.76234	0.00003	0.00569	0
4	5.76234	0.00003	1e-05	0
5	5.76234	0.00003	0.0	0
6				
7				
8				
9				
10				

c) The rate of convergence for the secant method is given by, $R \approx 1.618$. Thus, the relation between successive error terms is $E_{k+1} = A|E_k|^{1.618}$. Use information found in table of literal b), to calculate the value of A.

$A =$ 1.20629

2. (1.0 point) Let $f(x) = x^3 - 3x - 2$

a) Find the Newton-Rapshon formula $p_k = g(p_{k-1})$

Iteration rule: $p_k =$

b) Start with $p_0 = 2.1$ and find p_1, p_2, p_3, p_4 and p_5.

k	p_k	$f(p_k)$	$f'(p_k)$	$\|p_k - p_{k-1}\|$
0	2.1	0.96100	10.23000	$-----$
1	2.00606	0.05477	9.07284	0.09394
2	2.00002	0.00022	9.00029	0.00604
3	2.00000	0.00000	8.99999	2e-05
4	2.00000	0.00000	8.99999	0
5	2.00000	0.00000	8.99999	0

c) Is the sequence converging quadratically or linearly?

Answer: Quadratically

3. (1.0 point) The world population N can be simulated by a function that grows in proportion to the number of individuals in a given time t. This is called the *logistic function* and it follows the equation (2.2),

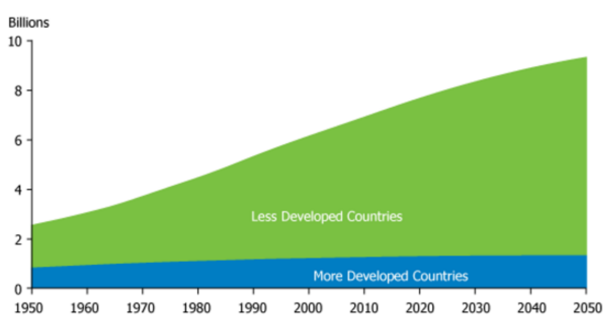

Figure 2.1: World population growth

$$N(t) = N_0 e^{\lambda t} + \mu \frac{e^{\lambda t} - 1}{\lambda}, \qquad (2.2)$$

where, $N_0 = N(t_0)$ is the amount of individuals at the beginning of the simulation period, λ is the growth rate and μ simulates the immigration rate.

Suppose that $N(t_0) = 1000$, $\mu = 435$ and $N(t_1) = 1564$.

a) Determine a nonlinear equation $g(\lambda) = 0$ that allows to calculate the growth rate λ by finding its root.

$g(\lambda) =$

b) Use the Newton method to find the root of the nonlinear equation in literal a). Use $\lambda_0 = 0.5$ and iterate until achieving a precision of $|\lambda_k - \lambda_{k-1}| < 1 \times 10^{-6}$ *Make all calculations with 5 decimal places.*

Iteration rule: $\lambda_{k+1} =$

k	λ_k	$g(\lambda_k)$	$g'(\lambda_k)$	$\lvert\lambda_k - \lambda_{k-1}\rvert$
0	0.5	649.10878	1954.33377	$-----$
1	0.16786	92.41474	1426.21639	0.33214
2	0.10306	2.77011	1341.60135	0.0648
3	0.10100	0.00270	1338.99151	0.00206
4	0.10100	0.00270	1338.99151	0.0
5				
6				
7				
8				
9				
10				

4. (1.0 point) Determine rigorously if each function has an unique fixed point.

a) $g(x) = \dfrac{x^2 - 8x + 25}{3}$, $x \in [1, 7]$

fixed point existence theorem verification:	unique fixed point theorem verification:
Sketch $g(x)$ and $y = x$	Sketch $g'(x)$

Use the starting value $p_0 = 3.15$ and compute p_1, p_2, p_3, p_4 and p_5.

p_0	p_1	p_2	p_3	p_4	p_5
3.15	3.24083	3.19211	3.21756	3.20407	3.21117
Do the sequence converge? Si.					

Use the starting value $p_0 = 3.25$ and compute p_1, p_2, p_3, p_4 and p_5.

p_0	p_1	p_2	p_3	p_4	p_5
3.25	3.18750	3.22005	3.20277	3.20277	3.20706
Do the sequence converge? Si.					

b) $g(x) = \dfrac{11x^3 - 141x^2 + 556x - 546}{30}$, $x \in [2, 7]$

fixed point existence theorem verification:	unique fixed point theorem verification:
Sketch $g(x)$ and $y = x$	Sketch $g'(x)$

Use the starting value $p_0 = 4.1$ and compute p_1, p_2, p_3, p_4 and p_5.

p_0	p_1	p_2	p_3	p_4	p_5
4.1	4.0507	4.12492	4.01282	4.18115	3.92266
Do the sequence converge? Si.					

Use the starting value $p_0 = 6.95$ and compute p_1, p_2, p_3, p_4 and p_5.

p_0	p_1	p_2	p_3	p_4	p_5
6.95	6.67579	5.12224	2.67727	4.76076	3.07028
Do the sequence converge? No.					

5. (1.0 point) Archimedes' principle indicates that the upward buoyant force that is exerted on a body immersed in a fluid, is equal to the weight of the fluid that the body displaces, and it acts in the upward direction at the centre of mass of the displaced fluid. Suppose that a sphere of radius $r = 15$ constructed with a material of density $\rho = 0.638$ is submerged in water to a depth d as shown in Fig. 2.2. According to Archimedes' principle, the mass of water displaced M_w is equal to the mass of the ball M_b, thus,

$$M_w = M_b \quad \text{where,} \tag{2.3}$$

$$M_w = \frac{\pi d^2 (3r - d)}{3}, \tag{2.4}$$

$$M_b = \frac{4\pi r^3 \rho}{3}. \tag{2.5}$$

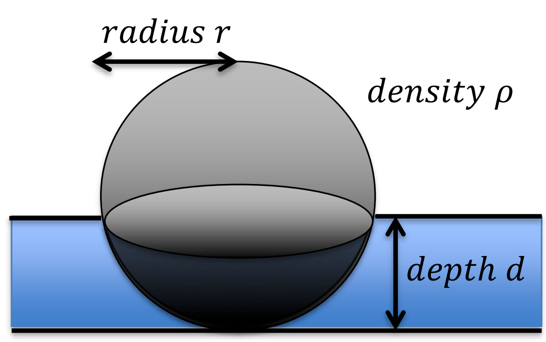

Figure 2.2: Sphere submerged in water

a) Use equations (2.3), (2.4) and (2.5) to find a nonlinear equation of the form $f(d) = 0$, that allows to determine the depth d.

$f(d) =$

b) Use the bisection method of Bolzano to calculate the roots of the nonlinear equation in literal a). Use $a_0 = 17.6$ and $b_0 = 18$ and iterate until achieving a precision of $|c_k - c_{k-1}| < 1 \times 10^{-3}$. *Make all calculations with 5 decimal places.*

k	a_k	b_k	c_k	$f(a_k)$	$f(b_k)$	$f(c_k)$	$\lvert c_k - c_{k-1}\rvert$
0	17.6	18	17.8	-175.576	85	-44.952	-------
1	17.8	18	17.9	-44.952	85	20.111	0.1
2	17.8	17.9	17.85	-44.952	20.111	-12.39912	0.05
3	17.85	17.9	17.875	-12.39912	20.111	3.86133	0.025
4	17.85	17.875	17.8625	-12.39912	3.86133	-4.26756	0.0125
5	17.8625	17.875	17.86875	-4.26756	3.86133	-0.20278	6.25e-3
6	17.86875	17.875	17.87187	-0.20278	3.86133	1.82936	3.12e-3
7	17.86875	17.871875	17.87031	-0.20278	1.82936	0.81331	1.56e-3
8	17.86875	17.87031	17.86953	-0.20278	0.81331	0.30527	7.8e-4
9							
10							

c) **Without doing iterations**, determine the required number of iterations N to guarantee that the midpoint c_N is an aproximation to a zero of the nonlinear equation in literal a) with an error less than $\delta = 1 \times 10^{-15}$, where $a_0 = 17$ and $b_0 = 18$.

$N =$ 48.82892