Solution of Nonlinear Equations $f(x) = 0$

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Outline: Chapter 2

Nonlinear Equations

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- A nonlinear equation is one that contains variables of degree greater or lower than 1.

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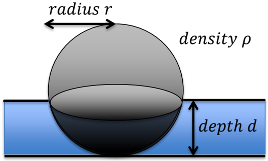
- Nonlinear equations are important due to they commonly appear in physical phenomenas.

Nonlinear Equations

Example: The problem of a spherical ball of radius r and density ρ that is submerged to a depth d in water.



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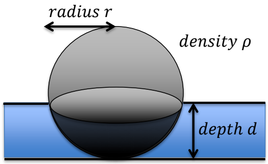

The Archimedes' law is applied to know the submerged depth d:

The mass of water displaced M_w is equal to the mass of the ball M_b.

$$M_w = M_b \tag{1}$$

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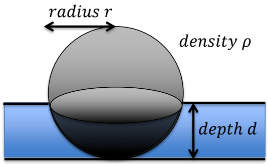

The Archimedes' law is applied to know the submerged depth d:

The mass of water displaced M_w is equal to the mass of the ball M_b.

$$M_w = M_b \tag{1}$$

The equality in (1) leads to the nonlinear equation:

$$f(d) = \frac{\pi(d^3 - 3d^2r + 4r^3\rho)}{3} = 0, \quad \text{where, } r \text{ and } d \text{ are known.}$$

The depth d is found by solving $f(d) = 0$.

SOLVING NONLINEAR EQUATION

Then, the goal is to develop a variety of methods for finding numerical approximations of the **roots** of a nonlinear equation.

A root is a value of the variable x that makes $f(x) = 0$.

Iteration approach for finding roots

- **Iteration methods** consist in repeating a process until to achieve an answer.

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- They need a **starting value** p_0 and a **rule** or function $g(x)$ for computing successive terms p_k:

General Sequence

$$
\begin{aligned}
&p_0\\
&p_1 = g(p_0)\\
&p_2 = g(p_1)\\
&\vdots\\
&p_k = g(p_{k-1})\\
&p_{k+1} = g(p_k)\\
&\vdots
\end{aligned}
$$

Function	Staring value	Iterative rule
$g(x)$	P_0	$P_{k+1} = g(P_k)$

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Function	Staring value	Iterative rule
$g(x)$	P_0	$P_{k+1} = g(P_k)$

Possible results:

✓Unending sequence of numbers
✓Diverge or periodic sequences
✓**Convergent sequences**

Iteration for Solving $x = g(x)$

Finding Fixed Points

Definition 1

A ***fixed point*** of a function $g(x)$ is a real number P such that $P = g(P)$.

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The iteration $p_{n+1} = g(p_n)$ for $n = 0, 1, \ldots$ is called ***fixed-point iteration.***

Theorem 1

Assume that g is a continous function and that $\{p_n\}_{n=0}^{\infty}$ is a sequence generated by fixed-point iteration. If $\lim_{n\to\infty} p_n = P$, then P is a fixed point of $g(x)$.

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p_3 &= e^{-p_2} = p_3 = e^{-0.545239} = 0.579703 \\
&\vdots \qquad\qquad \vdots \\
p_9 &= e^{-p_8} = p_9 = e^{-0.566409} = 0.567560 \\
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$$\boldsymbol{\lim_{n\to\infty} p_n = 0.567143\ldots.}$$

$$\Downarrow$$

Approximation for the fixed point of the function $y = e^{-x}$.

Finding Fixed Points

Theorem 2

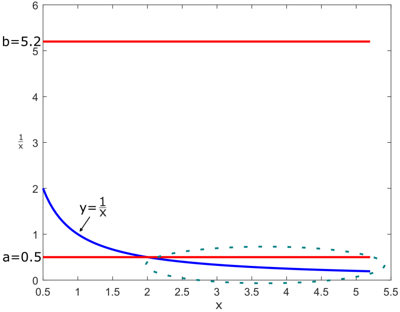
Assume that $g \in C[a, b]$.

1. If the range of the mapping $y = g(x)$ satisfies $y \in [a, b]$ for all $x \in [a, b]$, then g **has a fixed point** in $[a, b]$.

2. Furthermore, suppose that $g'(x)$ is defined over (a, b) and that a positive constant $K < 1$ exists with $|g'(x)| \leq K < 1$ for all $x \in (a, b)$; then g has a **unique** fixed point P in $[a, b]$.

Example:

Let $y = \dfrac{1}{x}$, x on $[0.5, 5.2]$. Determine if a fixed point exists according to literal 1 in Theorem 2.



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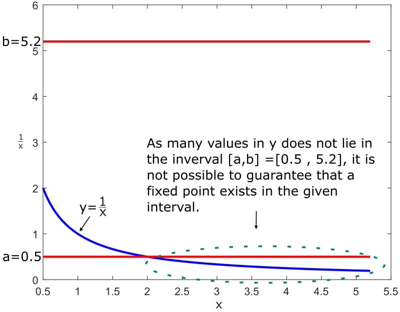
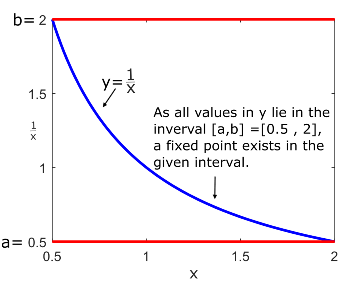


The range of the mapping $y = g(x)$ **does not** satisfy $y \in [0.5, 5.2]$ for all $x \in [0.5, 5.2]$.

Example:

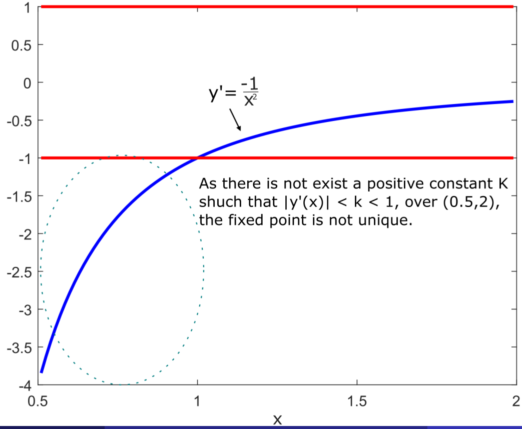
Let $y = \dfrac{1}{x}$, x on $[0.5, 2]$. Determine if a fixed point exists according to literal 1 in Theorem 2.



The range of the mapping $y = g(x)$ **satisfies** $y \in [0.5, 2]$ for all $x \in [0.5, 2]$.

Example:

Let $y = \dfrac{1}{x}$, x on $[0.5, 2]$. Determine if the fixed point is unique according to literal 2 in Theorem 2.



Iteration for Solving $x = g(x)$

Theorem 3: Fixed Point Theorem

Assume that (i) $g, g' \in C[a,b]$, (ii) K is a positive constant, (iii) $p_0 \in (a,b)$, and (iv) $g(x) \in [a,b]$ for all $x \in [a,b]$.

* If $|g'(x)| \leq K < 1$ for all $x \in [a,b]$, then the iteration $p_n = g(p_{n-1})$ will converge to the unique fixed point $P \in [a,b]$. In this case, P is said to be an **attractive fixed point.**

** If $|g'(x)| > 1$ for all $x \in [a,b]$, then the iteration $p_n = g(p_{n-1})$ will not converge to P. In this case, P is said to be a **repelling fixed point** and the iteration exhibits local divergence.

Remark 1. It is assumed that $p_0 \neq P$ in statement *.
Remark 2. Because g is continuous on an interval containing P, it is permissible to use the simpler criterion $|g'(P)| \leq K < 1$ and $|g'(P)| > 1$ in * and **, respectively.

Iteration for Solving $x = g(x)$

Graphical Interpretation of Fixed-Point Iteration

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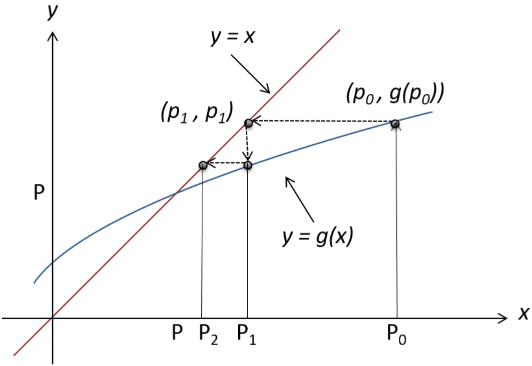
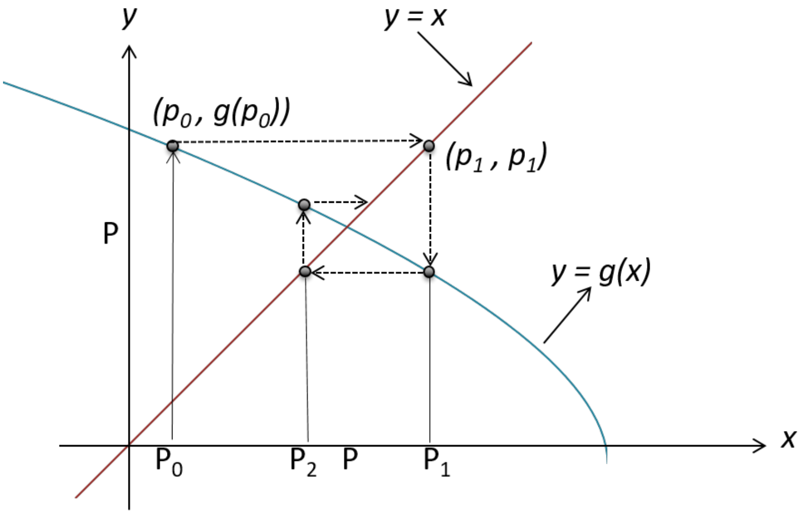
Possible cases of convergence.

- Monotone convergence when $0 < g'(P) < 1$.



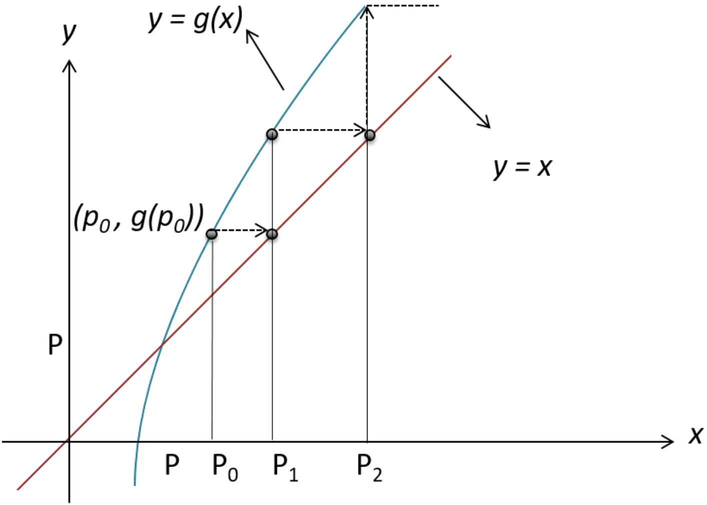
Iteration for Solving $x = g(x)$

- Oscillating convergence when $-1 < g'(P) < 0$.



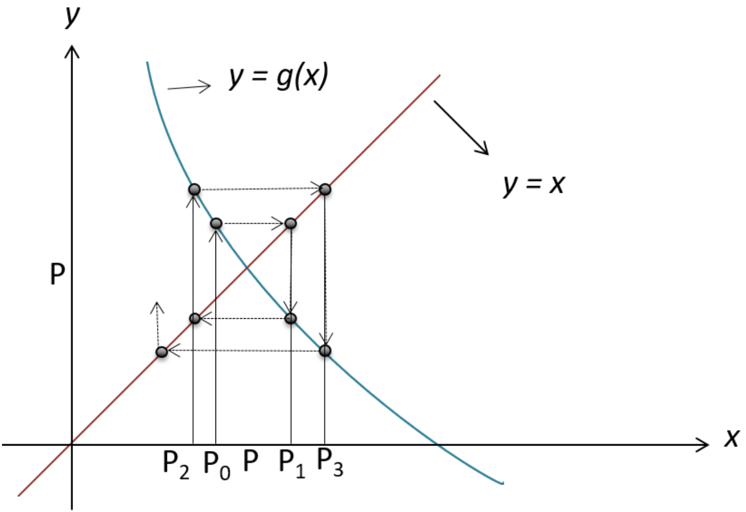
Iteration for Solving $x = g(x)$

- Monotone divergence when $1 < g'(P)$.



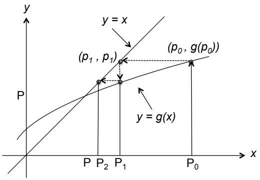
Iteration for Solving $x = g(x)$

- Divergent oscillation when $g'(P) < -1$.

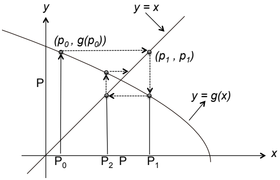


Summary

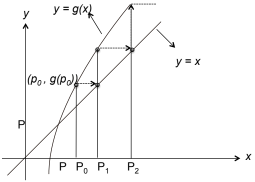
(a) Monotone convergence when $0 < g'(P) < 1$.



(b) Oscillating convergence when $-1 < g'(P) < 0$.



(c) Monotone divergence when $1 < g'(P)$.



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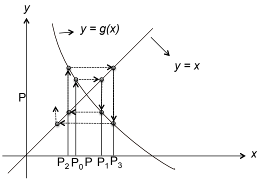

Example

Consider the iteration $p_{n+1} = g(p_n)$ when the function $g(x) = 1 + x - \dfrac{x^2}{4}$ is used.

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$$g'(x) = 1 - \frac{x}{2}$$

Numerical solution

case(i):	$P_1 = -2$	$P_2 = 2$
Start with	$p_0 = -2.05$	$p_0 = 1.6$
Sequence	$p_1 = -2.100625$	$p_1 = 1.96$
	$p_2 = -2.20378135$	$p_2 = 2.40789513$
	$p_3 = -2.41794441$	$p_3 = 1.99999996$
Conclusion	$\lim_{n\to\infty} p_n = -\infty$ Since $\lvert g'(x)\rvert > \frac{3}{2}$ on $[-3, -1]$, by Theorem 3, the sequence will not converge to $P = -2$	$\lim_{n\to\infty} p_n = 2$ Since $\lvert g'(x)\rvert < \frac{1}{2}$ on $[1, 3]$, by Theorem 3, the sequence will converge to $P = 2$

Example

Consider the iteration $p_{n+1} = g(p_n)$ when the function $g(x) = 2(x-1)^{1/2}$ for $x \geq 1$ is used.

Iteration for Solving $x = g(x)$

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Only one analytical fixed point $P = 2$ exists. The derivative is $g'(x) = 1/(x-1)^{1/2}$ and $g'(2) = 1$, so Theorem 3 **does not apply**. There are two cases to consider when the starting value lies to the left or right of $P = 2$.

Start with	$p_0 = 1.5$	$p_0 = 2.5$
Sequence	$p_1 = 1.41421356$	$p_1 = 2.44948974$
	$p_2 = 1.28718851$	$p_2 = 2.40789513$
	$p_3 = 1.07179943$	$p_3 = 2.37309514$
	$p_4 = 0.53590832$	$p_4 = 2.34358284$
Conclusion	$\vdots$ $p_5 = 2(-0.46409168)^{1/2}$ Since p_4 lies outside the domain of $g(x)$, the term p_5 cannot be computed	$\vdots$ $\lim_{n\to\infty} p_n = 2$ This sequence is converging too slowly to the value $P = 2$; indeed, $P_{1000} = 2.00398714$

Absolute and Relative Error Considerations

Consider the iteration $p_{n+1} = g(p_n)$ when the function $g(x) = 2(x-1)^{1/2}$ for $x \geq 1$ is used.

$$g'(x) = 1/(x-1)^{1/2} \text{ and } g'(2) = 1$$

$$p_{1000} = 2.00398714, \quad p_{1001} = 2.00398317 \quad \text{and} \quad p_{1002} = 2.00397921$$

WHAT ABOUT A CRITERION FOR STOPPING THE ITERATION?

1. Relative error

$$|p_{1001} - p_{1002}| = 0.00000396$$

2. Absolute error

$$|P - p_{1000}| = 0.00398714$$

Example:

If you save money by making regular monthly deposits P and the annual interest rate is I; the total amount A after N deposits is given by

$$A = P + P\left(1+\frac{I}{12}\right) + P\left(1+\frac{I}{12}\right)^2 + \cdots + P\left(1+\frac{I}{12}\right)^{N-1}.$$

In geometric series:

$$1 + r + r^2 + r^3 + \cdots + r^{N-1} = \frac{1-r^N}{1-r}.$$

Then, the annuity-due equation is,

$$A = P\frac{1-\left(1+\frac{I}{12}\right)^N}{1-\left(1+\frac{I}{12}\right)} = \frac{P}{I/12}\left(\left(1+\frac{I}{12}\right)^N - 1\right).$$

EXAMPLE

You save $\$250$ per month and you desire that the total value of all payments and interest be $\$250,000$ at the end of 20 years. What interest rate I is needed to achieve your goal?

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$N = 240$ months $\quad P = \$250 \quad I = ?$
$A = A(I) \quad A = \$250000$

$$A = \frac{P}{I/12}\left(\left(1 + \frac{I}{12}\right)^{N} - 1\right).$$

Bracketing Methods for Locating a Root

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You save $\$250$ per month and you desire that the total value of all payments and interest be $\$250,000$ at the end of 20 years. What interest rate I is needed to achieve your goal?

$N = 240$ months $P = \$250$ $I = ?$

$A = A(I)$ $A = \$250000$

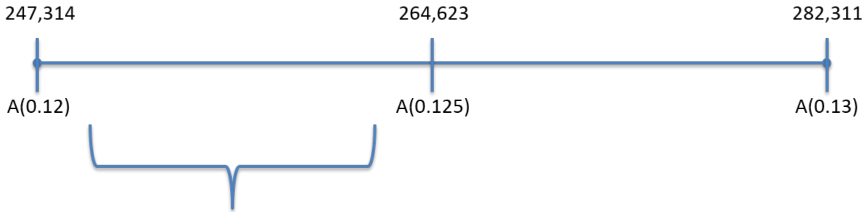
$$A = \frac{P}{I/12}\left(\left(1 + \frac{I}{12}\right)^N - 1\right).$$

To determine the value of I that satisfies our desire, we start with two guesses, $I_0 = 0.12$ and $I_1 = 0.13$, and we calculate the amount A for each case.

Desired $A = \$250000$

Bracketing Methods for Locating a Root

As the desired A is between the obtained values, we try a new value in the middle $I_2 = \frac{I_0+I_1}{2} = 0,125$



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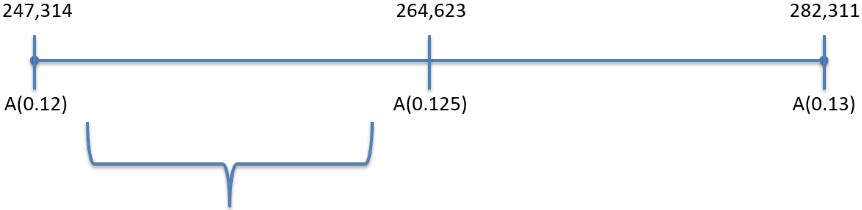

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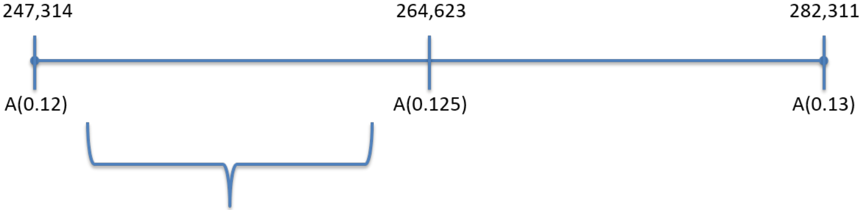

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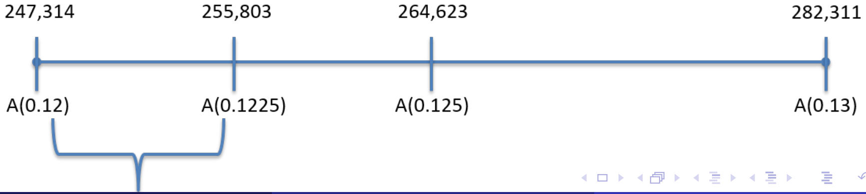
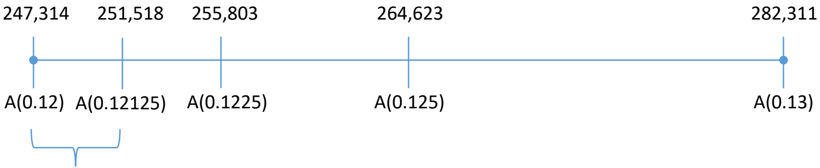


The interval is now narrowed to $[0.12, 0.1225]$. Our last calculation uses the midpoint $I_4 = 0.12125$



We could continue until finding the proper value for I that produces $A = 250000$.

Definition 3

Assume that $f(x)$ is a continuous function. Any number r for which $f(r) = 0$ is called a ***root of the equation*** $f(x)$. Also, we say that r is a ***zero of the function*** $f(x)$.

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Example

The equation $2x^2 + 5x - 3 = 0$ has two real roots $r_1 = 0.5$ and $r_2 = -3$, whereas the corresponding function $f(x) = 2x^2 + 5x - 3 = (2x - 1)(x + 3)$ has two real zeros, $r_1 = 0.5$ and $r_2 = -3$.

Bisection Method of Bolzano for finding roots

Details in the process:

Suppose that $[a_0, b_0]$ is a starting interval and $c_0 = (a_0 + b_0)/2$ is the midpoint.

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(1) If $f(a)$ and $f(c)$ have opposite signs, a zero lies in $[a, c]$.

(2) If $f(c)$ and $f(b)$ have opposite signs, a zero lies in $[c, b]$.

(3) If $f(c) = 0$, then the zero is c_0.

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$[a_1, b_1]$ is the second interval, which brackets the zero r, and c_1 is its midpoint; the interval $[a_1, b_1]$ is half as wide as $[a_0, b_0]$.

Bracketing Methods for Locating a Root

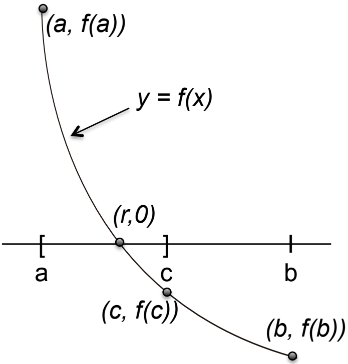
After arriving at the nth interval $[a_n, b_n]$, which brackets r and has midpoint c_n, the interval $[a_{n+1}, b_{n+1}]$ is constructed, which also brackets r and is half as wide as $[a_n, b_n]$.

(4) $$a_0 \leq a_1 \leq \ldots \leq a_n \leq \ldots \leq r \leq \ldots \leq b_n \leq \ldots \leq b_1 \leq b_0.$$

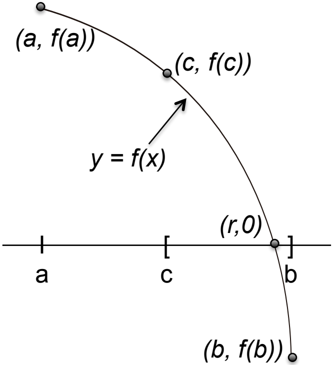
where $c_n = (a_n + b_n)/2$, and if $f(a_{n+1})f(b_{n+1}) < 0$, then

(5) $[a_{n+1}, b_{n+1}] = [a_n, c_n]$ or $[a_{n+1}, b_{n+1}] = [c_n, b_n]$ for all n.

Graphic interpretation of Bisection Method of Bolzano



(a) If $f(a)$ and $f(c)$ have opposite signs, then squeeze from the right.



(b) If $f(c)$ and $f(b)$ have opposite signs, then squeeze from the left.

Bisection Method of Bolzano

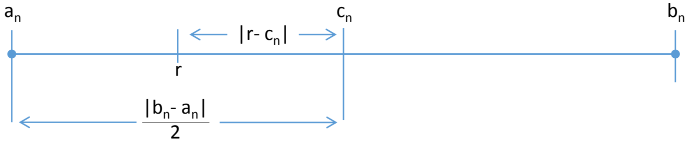
Theorem 4: Bisection Theorem

Assume that $f \in C[a,b]$ and that there exists a number $r \in [a,b]$ such that $f(r) = 0$. If $f(a)$ and $f(b)$ have opposite signs, and $\{c_n\}_{n=0}^{\infty}$ represents the sequence of midpoints generated by the bisection process of (1) and (2), then

$$|r - c_n| \leq \frac{b-a}{2^{n+1}} \quad for\ n = 0, 1, ...$$

and therefore the sequence $\{c_n\}_{n=0}^{\infty}$ converges to the zero $x = r$; that is

$$\lim_{n\to\infty} c_n = r.$$



Example

Find the value of x that lies in the interval $[0,2]$, where the function $h(x) = x\sin(x)$ takes the value $h(x) = 1$ (the function $\sin(x)$ is evaluated in radians).

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We have $h(x) = xsin(x) = 1 \rightarrow \boldsymbol{f(x) = xsin(x) - 1 = 0}$

Using the bisection method to find a zero of the function $f(x) = 0$:

- Starting with $a_0 = 0$ and $b_0 = 2$
- Calculate $f(a_0)$ and $f(b_0)$
  $f(0) = -1.000000$ and $f(2) = 0.818595$
- Verify that $f(0)f(2) < 0$
  $f(0)f(2) = -0.818595$

Example

- Determine the midpoint
  $c_0 = \frac{a_0+b_0}{2} = \frac{0+2}{2} = 1$
  $f(c_0) = -0.158529$
- Define the next interval, such that $f(a_1)f(b_1) < 0$
  $a_1 = c_0 = 1, b_1 = b_0 = 2$.
- Continue the process...

k	left endpoint, a_k	Midpoint, c_k	Right endpoint, b_k	Function value, $f(c_k)$
0	0	1	2	-0.158529
1	1.0	1.5	2	0.496242
2	1.00	1.25	1.50	0.186231
3	1.000	1.125	1.250	0.015051
4	1.0000	1.0625	1.1250	-0.071827
5	1.06250	1.09375	1.12500	-0.028362
6	1.093750	1.109375	1.125000	-0.006643
7	1.1093750	1.1171875	1.1250000	0.004208
8	1.10937500	1.11328125	1.11718750	-0.001216
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Example

How do you calculate the error bound in the thirty-first iteration?

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$$\text{Thus, } |E_{31}| \leq \frac{(2-0)}{2^{32}} \approx 4.656613 \times 10^{-10}$$

Bracketing Methods for Locating a Root

False Position Method for finding roots

Given the points a, and b to find the value c, do:

$$m = \frac{f(b) - f(a)}{b - a}$$

where the points $(a, f(a))$ and $(b, f(b))$ are used.

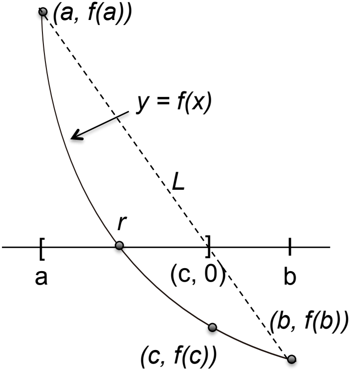
$$m = \frac{0 - f(b)}{c - b}$$

where the points $(c, 0)$ and $(b, f(b))$ are used.

Equating, we have,

$$\frac{f(b) - f(a)}{b - a} = \frac{0 - f(b)}{c - b} \quad \Longrightarrow \quad c = b - \frac{f(b)(b - a)}{f(b) - f(a)}$$

False Position Method for finding roots



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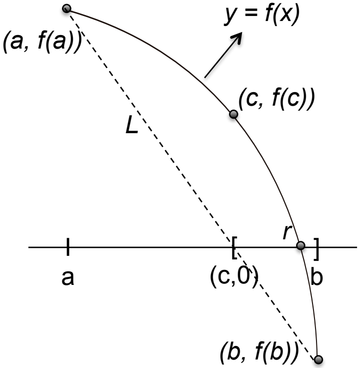


(b) If $f(c)$ and $f(b)$ have opposite signs, then squeeze from the left.

Convergence of the False Position Method

$$c = b - \frac{f(b)(b-a)}{f(b)-f(a)}$$

✓If $f(a)$ and $f(c)$ has opposite signs, a zero lies in [a, c].
✓If $f(c)$ and $f(b)$ have opposite signs, a zero lies in [c, b].
✓If $f(c) = 0$, then the zero is c.

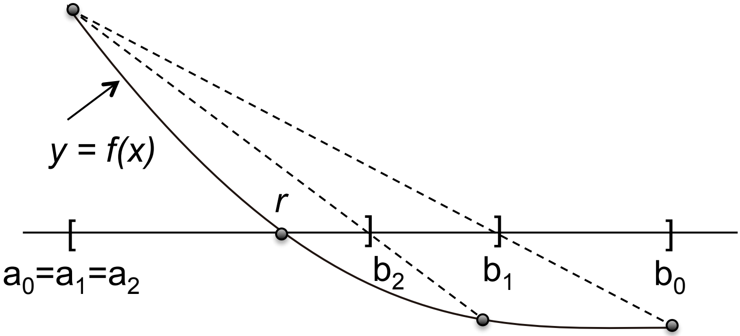
The decision process is used to construct a sequence of intervals $[a_n, b_n]$, each of which brackets the zero. At each step the approximation of the zero r is

$$c_n = b_n - \frac{f(b_n)(b_n-a_n)}{f(b_n)-f(a_n)}$$

and it can be proved that the sequence $\{c_n\}$ will converge to r.

Convergence of the False Position Method

If the graph of $y = f(x)$ is concave near $(r, 0)$, one of the endpoints becomes fixed and the other one marches into the solution.



Example: Using the false position method find the root of $x\sin(x)-1=0$ that is located in the interval $[0,2]$ ($\sin(x)$ is evaluated in radians).

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$a_0 = 0 \longrightarrow f(0) = -1.00$ and $b_0 = 2 \longrightarrow f(2) = 0.81859485$

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$$c_0 = 2 - \frac{0.81859485(2-0)}{0.81859485-(-1)} = 1.09975017 \quad \text{and} \quad f(c_0) = -0.02001921$$

4. Define the new interval:
New interval $[c_0, b_0] = [1.09975017, 2]$. Then $a_1 = c_0$ and $b_1 = b_0$.

$$c_1 = 2 - \frac{0.81859485(2-1.09975017)}{0.81859485-(-0.02001921)} = 1.12124074 \quad \text{and} \quad f(c_1) = 0.00983461$$

Example: Find the root of $x\sin(x) - 1 = 0$ that is located in the interval $[0, 2]$ (the function $\sin(x)$ is evaluated in radians).

$$a_0 = 0 \qquad c_0 = 1.09975017 \qquad b_0 = 2 \qquad f(c_0) = -0.02001921$$

5.Continue the process

k	left endpoint, a_k	Midpoint, c_k	Right endpoint, b_k	Function value, $f(c_k)$
0	0.00000000	1.09975017	2.00000000	-0.02001921
1	1.09975017	1.12124074	2.00000000	0.00983461
2	1.09975017	1.11416120	1.12124074	0.00000563
3	1.09975017	1.11415714	1.11416120	0.00000000

Slope Methods for Finding Roots

The ***Newton-Raphson*** method is one of the most useful and best known algorithms that relies on the continuity of $f'(x)$ and $f''(x)$.

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The slope of the line through $(p_1, 0)$ and $(p_0, f(p_0))$:

$$m = \frac{0 - f(p_0)}{p_1 - p_0},$$

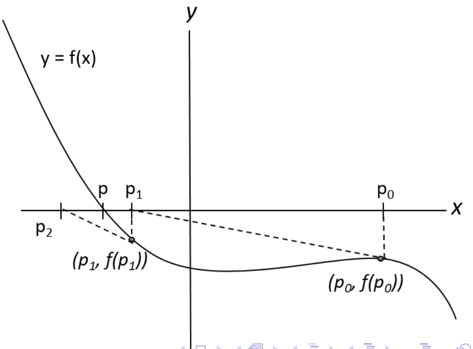
The slope at the point $(p_0, f(p_0))$:

$$m = f'(p_0),$$

Equating:

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}.$$

Graphical representation



Theorem 5: Newton-Raphson Theroem

Assume that $f \in C^2[a,b]$ and there exists a number $p \in [a,b]$, where $f(p) = 0$. If $f'(p) \neq 0$, then there exists a $\delta > 0$ such that the sequence $\{p_k\}_{k=0}^{\infty}$ defined by the iteration

$$p_k = g(p_{k-1}) = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})} \quad \text{for } k = 1, 2, \ldots$$

will converge to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

The function $g(x)$ is called the ***Newton-Raphson iteration function***:

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Corollary: Newton's Itetarion for Finding Square Roots

Assume that $A > 0$ is a real number and let $p_0 > 0$ be an initial approximation to $\sqrt{A}$. Define the sequence $\{p_k\}_{k=0}^{\infty}$ using the recursive rule

$$p_k = \frac{p_{k-1} + \dfrac{A}{p_{k-1}}}{2} \quad \text{for} \quad k = 1, 2, \ldots.$$

Then the sequence $\{p_k\}_{k=0}^{\infty}$ converges to $\sqrt{A}$; that is, $\lim_{n\to\infty} p_k = \sqrt{A}$.

Example: Use Newton's square-root algorithm to find $\sqrt{5}$.

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Starting with $p_0 = 2$ and using $p_k = \dfrac{p_{k-1} + \dfrac{A}{p_{k-1}}}{2}$, we compute

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Starting with $p_0 = 2$ and using $p_k = \dfrac{p_{k-1} + \dfrac{A}{p_{k-1}}}{2}$, we compute

$$p_1 = \frac{2 + 5/2}{2} = 2.25$$

$$p_2 = \frac{2.25 + 5/2.25}{2} = 2.236111111$$

$$p_3 = \frac{2.236111111 + 5/2.236111111}{2} = 2.236067978$$

$$p_4 = \frac{2.236067978 + 5/2.236067978}{2} = 2.236067978$$

Newton-Raphson and Secant Methods

Example: Motion of a projectile

Suppose that a projectile is fired with an angle of elevation b_0 and an initial velocity v_0. Air resistance is neglected and the height $y = y(t)$ and the distance traveled $x = x(t)$, obey the rules

$$y = v_y t - 16t^2 \quad \text{and} \quad x = v_x t,$$

where the horizontal and vertical components of the initial velocity are $v_x = v_0 \cos(b_0)$ and $v_y = v_0 \sin(b_0)$. If the air resistance is proportional to the velocity, the equations of motion become

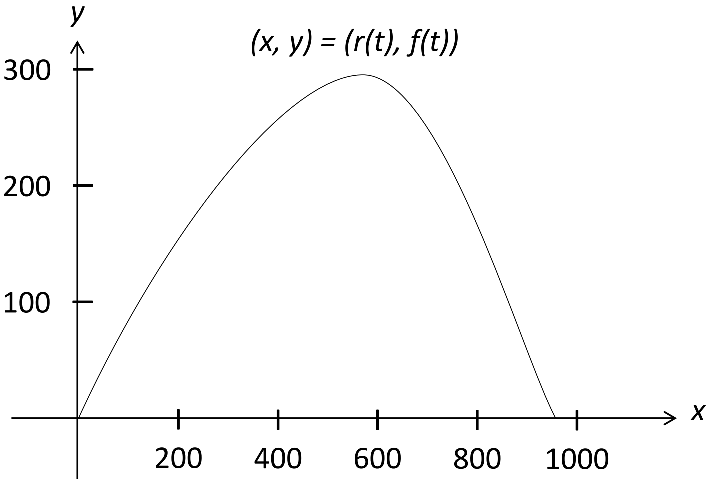
$$y = f(t) = (Cv_y + 32C^2)(1 - e^{-t/C}) - 32Ct \tag{2}$$

$$x = r(t) = Cv_x(1 - e^{-t/C}) \tag{3}$$

where $C = m/k$ and k is the coefficient of air resistance and m is the mass of the projectile.

Newton-Raphson and Secant Methods

The path of a projectile with air resistance.



Example: A projectile is fired with an angle of elevation $b_0 = 45$, $v_y = v_x = 160$ft/sec, and $C = 10$. Find the elapsed time t until the projectile impacts and finds the range. Using equations (2) and (3).

The elapsed time is the laps during the projectile is in the air. The range is the distance from the final to the initial point.

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The equations of motion are

$$y = f(t) = 4800(1 - e^{-t/10}) - 320t \quad \text{and} \quad x = r(t) = 1600(1 - e^{-t/10})$$

To find the elapsed time we need to find the root of the equation $f(t)$

- We will use the initial guess $p_0 = 8$, and $f(8) = 83.22097200$.

Newton-Raphson and Secant Methods

- We will use the initial guess $p_0 = 8$, and $f(8) = 83.22097200$.
- The derivative is $f'(t) = 480e^{t/10} - 320$, then, $f'(p_0) = f'(8) = -104.3220972$.

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- Using theorem 5 calculate the next point

$$p_k = g(p_{k-1}) = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})} \qquad \text{for} \qquad k = 1, 2, ...$$

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$$p_1 = 8 - \frac{83.22097200}{-104.3220972} = 8.797731010$$

...Continue the process

Finding the Time When the Height $f(t)$ is zero

k	Time, p_k	$p_{k+1} - p_k$	Height, $f(p_k)$
0	8.00000000	0.79773101	83.22097200
1	8.79773101	-0.05530160	-6.68369700
2	8.74242941	-0.00025475	-0.03050700
3	8.74217467	-0.00000001	-0.00000100
4	8.74217466	0.00000000	0.00000000

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0	8.00000000	0.79773101	83.22097200
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2	8.74242941	-0.00025475	-0.03050700
3	8.74217467	-0.00000001	-0.00000100
4	8.74217466	0.00000000	0.00000000

The value p_4 has eight decimal places of accuracy, and the time until impact is $t \approx 8.74217466$ seconds. The range can now be computed using $r(t)$, and we get

$$r(8.74217466) = 1600(1 - e^{-0.874217466}) = 932.4986302 \text{ ft.}$$

Division By Zero Error

Definition 4: Order of a Root

Assume that $f(x)$ and its derivatives $f'(x), \ldots, f^{(M)}(x)$ are defined and continuous on an interval about $x = p$. We say that $f(x) = 0$ has a root of order M at $x = p$ if and only if

$$f(p) = 0, \quad f'(p) = 0, \quad \ldots, \quad f^{(M-1)}(p) = 0, \quad \text{and} \quad f^{(M)}(p) \neq 0.$$

A root of order $M = 1$ is often called a ***simple root***, and if $M > I$, it is called a ***multiple root***. A root of order $M = 2$ is sometimes called a ***double root***, and so on.

Lemma 1

If the equation $f(x) = 0$ has root of order M at $x = p$, there exists a continuous function $h(x)$ so that $f(x)$ can be expressed as the product

$$f(x) = (x - p)^M h(x), \qquad \text{where} \qquad h(p) \neq 0.$$

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The function $f(x) = x^3 - 3x + 2$ has a simple root at $p = -2$ and a double root at $p = 1$:

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For $p = 1, f(1) = 0, f'(1) = 0$ and $f''(1) = 6$, so $M = 2$; hence $p = 1$ is a double root.

Speed of Convergence

- If p is a simple root of $f(x) = 0$, Newton's method will converge rapidly, and the number of accurate decimal places doubles with each iteration.
- If p is a multiple root, the error in each successive approximation is a fraction of the previous error. To make this precise, we define the ***order of convergence***. This is a measure of how rapidly a sequence converges.

Speed of Convergence

Definition 5

Assume that $\{p_n\}_{n=0}^{\infty}$ converges to p and set $E_n = p - p_n$ for $n \geq 0$. If two positive constants $A \neq 0$ and $R > 0$ exist, and

$$\lim_{n\to\infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = A, \tag{4}$$

then, the sequence is said to converge to p with ***order of convergence*** R. The number A is called the *asymptotic error constant*.

$$\text{If } R = 1, \text{ the convergence of } \{p_n\}_{n=0}^{\infty} \text{ is called } \textbf{linear}. \tag{5}$$

$$\text{If } R = 2, \text{ the convergence of} \{p_n\}_{n=0}^{\infty} \text{ is called } \textbf{quadratic}. \tag{6}$$

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If R is large, the sequence $\{p_n\}$ converges rapidly to p; that is, relation (4) implies that for large values of n we have the approximation $|E_{n+1}| \approx A|E_n|^R$.

Example: Quadratic Convergence at a Simple Root

Start with $p_0 = -2.4$ and use Newton-Raphson iteration to find the root $p = -2$ of the polynomial $f(x) = x^3 - 3x + 2$. The iteration formula for computing $\{p_k\}$ is

$$p_{k+1} = p_k - \frac{x^3-3x+2}{3x^2-3}.$$

Using formula (4) with $R = 2$ to check for quadratic convergence, we get the values:

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k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^2}$
0	-2.400000000	0.323809524	0.400000000	0.476190475
1	-2.076190476	0.072594465	0.076190476	0.619469086
2	-2.003596011	0.003587422	0.003596011	0.664202613
3	-2.000008589	0.000008589	0.000008589	
4	-2.000000000	0.000000000	0.000000000	

Example: Linear Convergence at a Double Root

Start with $p_0 = 1.2$ and use Newton-Raphson iteration to find the double root $p = 1$ of the polynomial $f(x) = x^3 - 3x + 2$. Using formula (4) to check for linear convergence, we get the values.

Example: Linear Convergence at a Double Root

Start with $p_0 = 1.2$ and use Newton-Raphson iteration to find the double root $p = 1$ of the polynomial $f(x) = x^3 - 3x + 2$. Using formula (4) to check for linear convergence, we get the values.

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert}$
0	1.200000000	-0.096969697	-0.200000000	0.515151515
1	1.103030303	-0.050673883	-0.103030303	0.508165253
2	1.052356420	-0.025955609	-0.052356420	0.496751115
3	1.026400811	-0.013143081	-0.026400811	0.509753688
4	1.013257730	-0.006614311	-0.013257730	0.501097775
5	1.006643419	-0.003318055	-0.006643419	0.500550093
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Theorem 6: Convergence Rate for Newton-Raphson Iteration

Assume that Newton-Raphson iteration produces a sequence $\{p_n\}_{n=0}^{\infty}$ that converges to the root p of the function $f(x)$. If p is a simple root, convergence is quadratic and

$$|E_{n+1}| \approx \frac{|f''(p)|}{2|f'(p)|}|E_n|^2 \qquad \text{for } n \text{ sufficiently large.}$$

If p is a multiple root of order M, convergence is linear and

$$|E_{n+1}| \approx \frac{M-1}{M}|E_n| \qquad \text{for } n \text{ sufficiently large.}$$

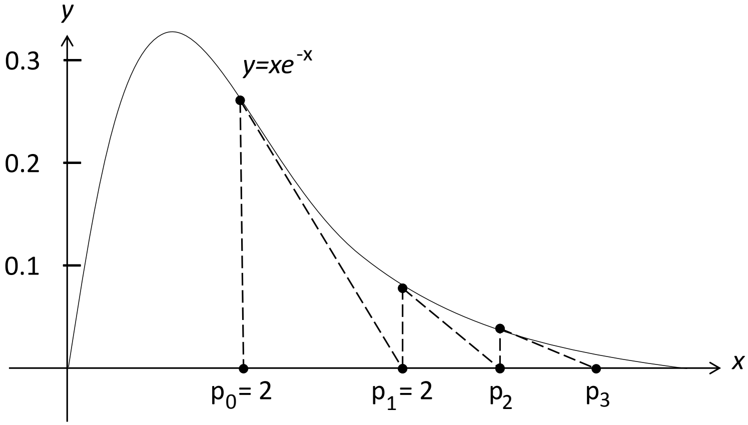
Pitfalls

Suppose that the function is $f(x) = x^2 - 4x + 5$; then the sequence $\{p_k\}$ of real numbers generated by formula

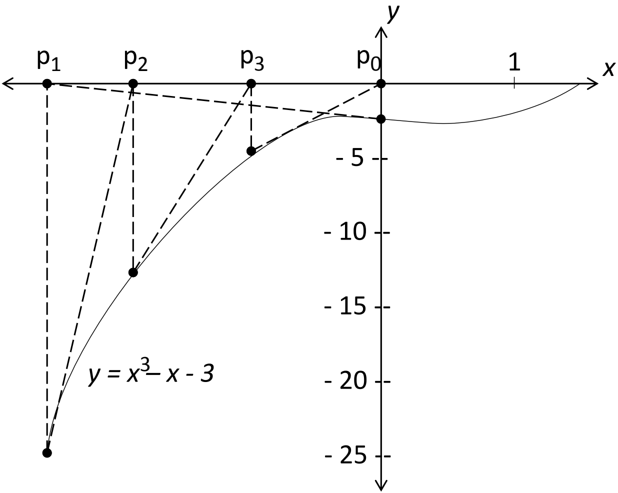
$$p_k = g(p_{k-1}) = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})} \qquad \text{for } k = 1, 2, ...$$

will wander back and forth from left to right and not converge. A simple analysis of the situation reveals that $f(x) > 0$ and has no real roots.

Newton-Raphson iteration for $f(x) = xe^{-x}$ can produce a divergent sequence.

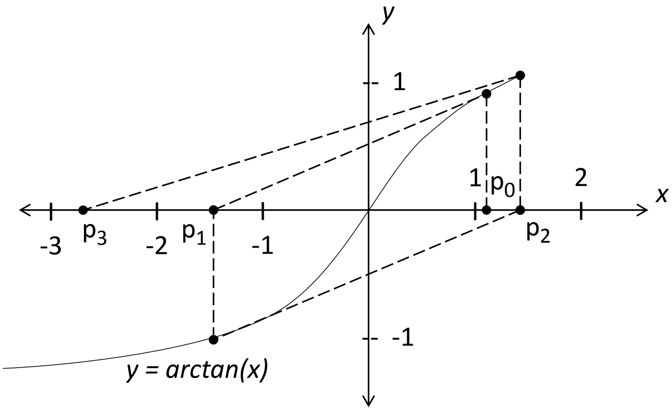


Newton-Raphson iteration for $f(x) = x^3 - x - 3$ can produce a cyclic sequence.



Newton-Raphson and Secant Methods

Newton-Raphson iteration for $f(x) = arctan(x)$ can produce a divergent oscillating sequence.



Secant Method for finding roots

The secant method will require only one evaluation of $f(x)$ per step and at a simple root has an order of convergence $R = 1.618033989$. It is almost as fast as Newton's method, which has order 2.

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Two initial points $(p_0, f(p_0))$ and $(p_1, f(p_1))$ near the point $(p, 0)$ are needed. Define p_2 to be the abscissa of the point of intersection of the line through these two points and the x-axis. The equation relating p_2, p_1, and p_0 is found by considering the slope

$$m = \frac{f(p_1) - f(p_0)}{p_1 - p_0} \text{ and } m = \frac{0 - f(p_1)}{p_2 - p_1}.$$

The values of m are the slope of the secant line through the first two approximations and the slope of the line through $(p_1, f(p_1))$ and $(p_2, 0)$, respectively.

Newton-Raphson and Secant Methods

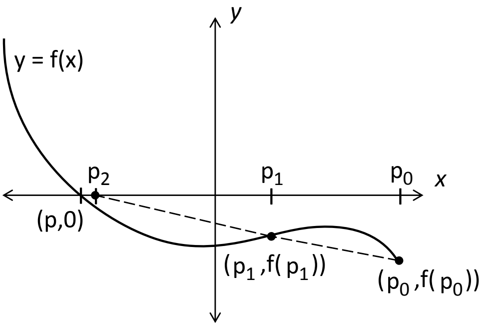
Set the right-hand sides equal and solve for $p_2 = g(p_1, p_0)$ and get

$$p_2 = g(p_1, p_0) = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}.$$

The general term is given by the two-point iteration formula

$$p_{k+1} = g(p_k, p_{k-1}) = p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})}.$$

The geometric construction of p_2 for the secant method



Example: Secant Method at a Simple Root

Start with $p_0 = -2.6$ and -2.4 and use the secant method to find the root $p = -2$ of the polynomial function $f(x) = x^3 - 3x + 2$. In this case the iteration formula is.

$$p_{k+1} = g(p_k, p_{k-1}) = p_k - \frac{(p_k^3 - 3p_k + 2)(p_k - p_{k-1})}{p_k^3 - p_{k-1}^3 - 3p_k + 3p_{k-1}}.$$

This can be algebraically manipulated to obtain

$$p_{k+1} = g(p_k, p_{k-1}) = \frac{p_k^2 p_{k-1} + p_k p_{k-1}^2 - 2}{p_k^2 + p_k p_{k-1} + p_{k-1}^2 - 3}.$$

The sequence of iterates is given in the table of convergence of the secant method at a simple root.

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^{1.618}}$
0	-2.600000000	0.200000000	0.600000000	0.914152831
1	-2.400000000	0.293401015	0.400000000	0.469497765
2	-2.106598985	0.083957573	0.106598985	0.847290012
3	-2.022641412	0.021130314	0.022641412	0.693608922
4	-2.001511098	0.001488561	0.001511098	0.825841116
5	-2.000022537	0.000022515	0.000022537	0.727100987
6	-2.000000022	0.000000022	0.000000022	
7	-2.000000000	0.000000000	0.000000000	

Newton-Raphson and Secant Methods

This is a relationship between the secant method and Newton's method. For a polynomial function $f(x)$, the secant method two-point formula $p_{k-1} = g(p_k, p_{k-1})$ will reduce to Newton's one-point formula $p_{k+1} = g(p_k)$ if p_k is replaced by p_{k-1}. Indeed, if we replace p_k by p_{k-1}

$$|E_{k+1}| \approx |E_k|^{1.618} \left| \frac{f''(p)}{2f'(p)} \right|^{0.618}.$$

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$$|E_{k+1}| \approx |E_k|^{1.618} \left| \frac{f''(p)}{2f'(p)} \right|^{0.618}.$$

Where the order of convergence is $R = (1+\sqrt{5})/2 \approx 1.618$

Accelerated Convergence

Theorem 7: Acceleration of Newton-Raphson Iteration

Suppose that the Newton-Raphson algorithm produces a sequence that converges linearly to the root $x = p$ of order $M > 1$. Then the Newton-Raphson iteration formula

$$p_k = p_{k-1} - \frac{Mf(p_{k-1})}{f'(p_{k-1})},$$

will produce a sequence $\{p_k\}_{k=0}^{\infty}$ that converges quadratically to p.

Example: Acceleration of convergence at a Double Root

Start with $p_0 = 1.2$ and use accelerated Newton-Raphson iteration to find the double root $p = 1$ of $f(x) = x^3 - 3x + 2$. Since $M = 2$, the acceleration formula becomes

$$p_k = p_{k-1} - 2\frac{f(p_{k-1})}{f'(p_{k-1})} = \frac{p_{k-1}^3 + 3p_{k-1} - 4}{3p_{k-1}^2 - 3},$$

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Start with $p_0 = 1.2$ and use accelerated Newton-Raphson iteration to find the double root $p = 1$ of $f(x) = x^3 - 3x + 2$. Since $M = 2$, the acceleration formula becomes

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and we obtain the values

k	p_k	$p_{k+1} - p_k$	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^2}$
0	1.200000000	-0.193939394	-0.200000000	0.151515150
1	1.006060606	-0.006054519	-0.006060606	0.165718578
2	1.000006087	-0.000006087	-0.000006087	
3	1.000000000	0.000000000	0.000000000	

Newton-Raphson and Secant Methods

Comparison of the Speed of Convergence for the methods described before:

Method	Special considerations	Relation between successive error terms
Bisection		$E_{k+1} \approx \frac{1}{2}\|E_k\|$
Regula falsi		$E_{k+1} \approx A\|E_k\|$
Secant method	Multiple root	$E_{k+1} \approx A\|E_k\|$
Newton-Raphson	Multiple root	$E_{k+1} \approx A\|E_k\|$
Secant method	Simple root	$E_{k+1} \approx A\|E_k\|^{1.618}$
Newton-Raphson	Simple root	$E_{k+1} \approx A\|E_k\|^2$
Accelerated Newton-Raphson	Multiple root	$E_{k+1} \approx A\|E_k\|^2$

The table compares the speed of convergence of the various root-finding methods that we have studied so far. The value of the constant A is different for each method.