# Assignment 2

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## Question 1

### (1a)

We will show that the process $X_t = \frac{2}{5}X_{t-1} + \frac{1}{4}X_{t_2} - \frac{1}{10}X_{t-3} + Z_t$ is stationary by showing that the roots of the characteristic polynomial lie outside the unit circle in the complex plane. We can rearrange to get

$$\begin{aligned} Z_t &= X_t - \frac{2}{5}X_{t-1} - \frac{1}{4}X_{t-2} + \frac{1}{10}X_{t-3} \\ &= (1 - \frac{2}{5}B - \frac{1}{4}B^2 + \frac{1}{10}B^3)X_t. \end{aligned}$$

Now we have to find $b$ such that

$$1 - \frac{2}{5}b - \frac{1}{4}b^2 + \frac{1}{10}b^3 = 0.$$

We multiply both sides by 20 to work with integers, and group the terms like so:

$$(2b^3 - 5b^2) - (8b - 20) = 0$$

$$b^2(2b-5) - 4(2b-5) = 0$$

$$(b^2 - 4)(2b - 5) = 0$$

Now we can see that this is satisfied when $b = -2, 2, \frac{5}{2}$. And since all of these lie outside the unit circle on the complex plane, $\{X_t\}_{t\in\mathbb{N}}$ is stationary.

### (1b)

First we multiply both sides of $X_t = \frac{2}{5}X_{t-1} + \frac{1}{4}X_{t_2} - \frac{1}{10}X_{t-3} + Z_t$ by $X_{t-k}$ for $k = 1, 2, 3$ to get 3 equations

$$X_tX_{t-1} = \frac{2}{5}X_{t-1}X_{t-1} + \frac{1}{4}X_{t-2}X_{t-1} - \frac{1}{10}X_{t-3}X_{t-1} + Z_tX_{t-1} \tag{1}$$

$$X_tX_{t-2} = \frac{2}{5}X_{t-1}X_{t-2} + \frac{1}{4}X_{t-2}X_{t-2} - \frac{1}{10}X_{t-3}X_{t-2} + Z_tX_{t-2} \tag{2}$$

$$X_tX_{t-3} = \frac{2}{5}X_{t-1}X_{t-3} + \frac{1}{4}X_{t-2}X_{t-3} - \frac{1}{10}X_{t-3}X_{t-3} + Z_tX_{t-3} \tag{3}$$

Next we take expectation of both sides of each equation to get equations 4-6

$$\gamma(1) = \frac{2}{5}\gamma(0) + \frac{1}{4}\gamma(1) - \frac{1}{10}\gamma(2) \tag{4}$$

$$\gamma(2) = \frac{2}{5}\gamma(1) + \frac{1}{4}\gamma(0) - \frac{1}{10}\gamma(1)$$

$$= \frac{1}{4}\gamma(0) + \frac{3}{10}\gamma(1) \tag{5}$$

$$\gamma(3) = \frac{2}{5}\gamma(2) + \frac{1}{4}\gamma(1) - \frac{1}{10}\gamma(0) \tag{6}$$

Substituting (5) into (4) yields

$$\gamma(1) = \frac{2}{5}\gamma(0) + \frac{1}{4}\gamma(1) - \frac{1}{10}[\frac{1}{4}\gamma(0) + \frac{3}{10}\gamma(1)]$$

and after some simple algebraic manipulations we see

$$\gamma(1) = \frac{150}{312}\gamma(0).$$

Dividing $\gamma(1)$, as well as equations (5) and (6), by $\gamma(0)$ yields

$$\rho(1) = \frac{\frac{150}{312}\gamma(0)}{\gamma(0)}$$

$$= \frac{150}{312} = 0.4807692 \tag{7}$$

$$\rho(2) = \frac{3}{10}\rho(1) + \frac{\frac{1}{4}\gamma(0)}{\gamma(0)}$$

$$= \frac{3}{10}(\frac{150}{312}) + \frac{1}{4} = 0.3942308 \tag{8}$$

$$\rho(3) = \frac{2}{5}\rho(2) + \frac{1}{4}\rho(1) - \frac{\frac{1}{10}\gamma(0)}{\gamma(0)}$$

$$= \frac{2}{5}[\frac{3}{10}(\frac{150}{312}) + \frac{1}{4}] + \frac{1}{4}(\frac{150}{312}) - \frac{1}{10} = 0.1778846 \tag{9}$$

Now we wish to find the roots d such that $d^3 - \frac{2}{5}d^2 - \frac{1}{4}d + \frac{1}{10} = 0$. We find the roots in the same fashion as in question (1a):

$$20d^3 - 8d^2 - 5d + 2 = 0$$

$$4d^2(5d - 2) - (5d - 2) = 0$$

$$(4d^2 - 1)(5d - 2) = 0$$

We see that this is satisfied when $d = -\frac{1}{2}, \frac{2}{5}, \frac{1}{2}$. Armed with the values of $\rho(1), \rho(2), \rho(3), d_1, d_2, d_3$ we can solve for $A_1, A_2, A_3$ in the following system of equations:

$$\rho(1) = A_1(\frac{-1}{2}) + A_2(\frac{2}{5}) + A_3(\frac{1}{2})$$

$$\rho(2) = A_1(\frac{-1}{2})^2 + A_2(\frac{2}{5})^2 + A_3(\frac{1}{2})^2$$

$$\rho(3) = A_1(\frac{-1}{2})^3 + A_2(\frac{2}{5})^3 + A_3(\frac{1}{2})^3$$

We can represent this system of equations as a matrix and reduce it to identify $A_1, A_2$, and $A_3$:

```r
library(pracma)
d <- c(-1/2, 2/5, 1/2)
rho <- c(0.4807692, 0.3942308, 0.1778846)
mat <- matrix(c(d, rho[1], d^2, rho[2], d^3, rho[3]),
              nrow = 3, ncol = 4, byrow = TRUE)
A <- rref(mat)[,4]
A
```

```
## [1]  0.1794873 -1.6025639  2.4230768
```

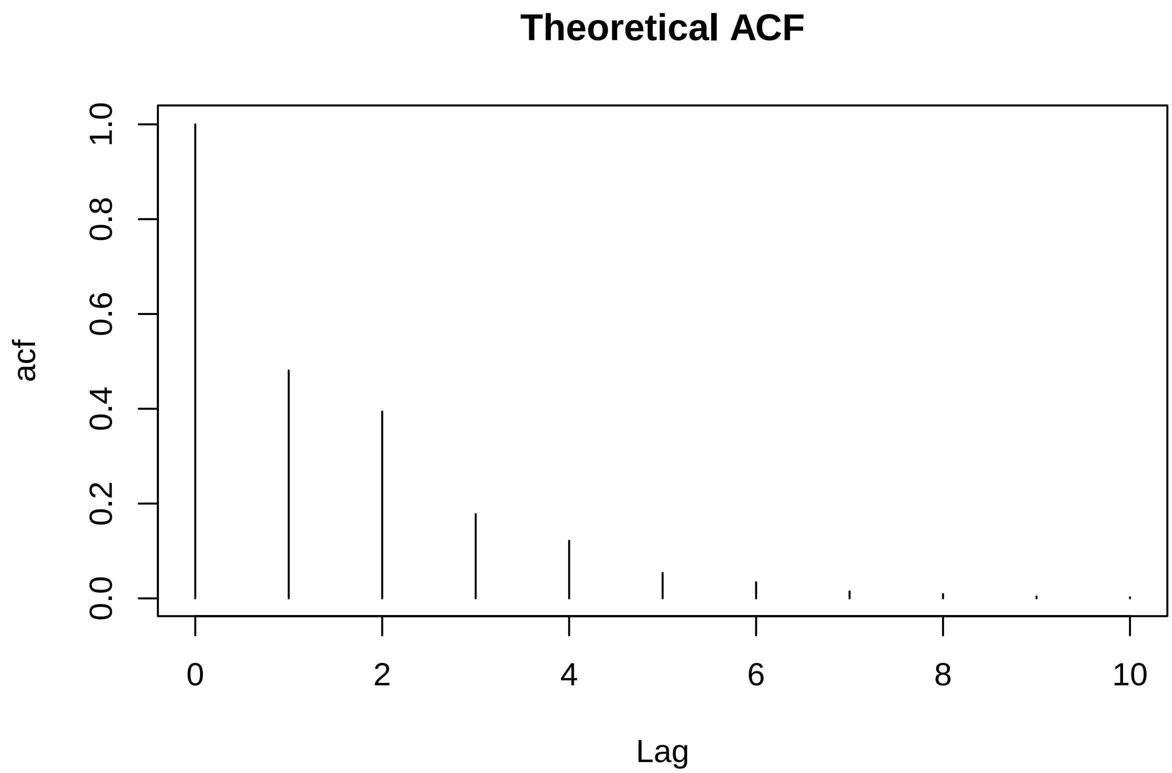
We see that $A_1 = 0.1794873$, $A_2 = -1.6025639$, and $A_3 = 2.4230768$ so that

$$\rho(h) = 0.1794873(\frac{-1}{2})^{|h|} - 1.6025639(\frac{2}{5})^{|h|} + 2.4230768(\frac{1}{2})^{|h|}.$$

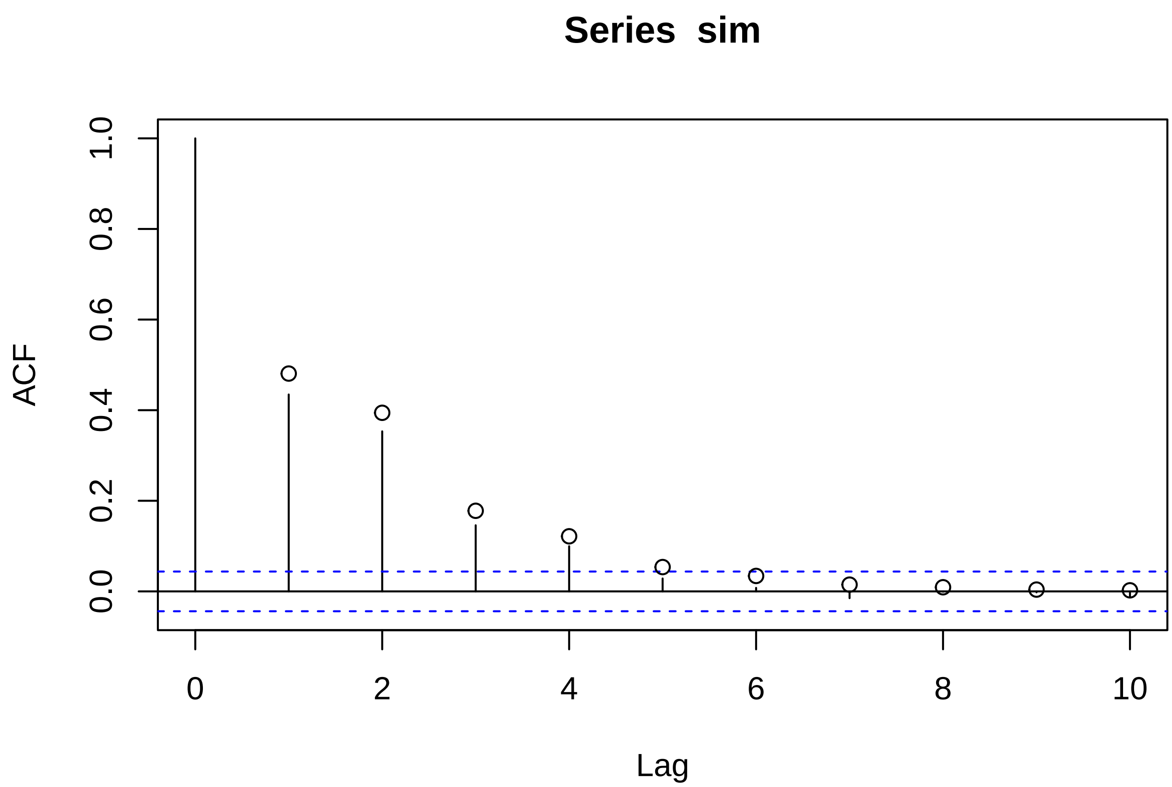
**(1c)**

```r
# simulate Xt
set.seed(123)
sim <- arima.sim(list(ar = c(2/5, 1/4, -1/10)), n = 2000, sd = sqrt(1.96))

# plot just the theoretical acf
rho_h <- function(h) {
  A[1] * d[1] ^ h + A[2] * d[2] ^ h + A[3] * d[3] ^ h
}
acf_theory <- c(1)
for (i in 2:11) {
  acf_theory[i] <- rho_h(i-1)
}
plot(x = c(0:10), y = acf_theory, type = "h", xlab = "Lag", ylab = "acf",
     main = "Theoretical ACF")
```




```
# plot the sample acf with theoretical acf overlayed on top
acf(sim, lag.max = 10)
points(acf_theory[2:11])
```




### Question 2

**(2a)**

We can rewrite the process as

$$(1 - \frac{7}{10}B)X_t = (1 - \frac{1}{10}B)Z_t$$

where $(1 - \frac{7}{10}B) = \phi(B)$ and $(1 - \frac{1}{10}B) = \theta(B)$. Then the roots $b_1$ and $b_2$ of $\phi(B)$ and $\theta(B)$ are $\frac{10}{7}$ and 10, both of which have modulus greater than unity, so the process is stationary and invertible.

**(2b)**

We first identify the polynomial $\psi(B) = \frac{\theta(B)}{\phi(B)} = \frac{1-\frac{1}{10}B}{1-\frac{7}{10}B}$ which can be expanded to the geometric sum

$$\psi(B) = (1 - \frac{1}{10}B)[1 + \frac{7}{10}B + (\frac{7}{10})^2 B^2 + ...]$$

$$= [1 + \frac{7}{10}B + (\frac{7}{10})^2 B^2 + ...] - \frac{1}{10}[B + \frac{7}{10}B^2 + (\frac{7}{10})^2 B^3 + ...]$$

$$= 1 + \sum_{j=1}^{\infty}[0.7^j - 0.1(0.7)^{j-1}]B^j$$

$$= 1 + \sum_{j=1}^{\infty}(0.7)^{j-1}(0.6)B^j$$

Then the model can be expressed as a pure MA process $X_t = \psi(B)Z_t$ or

$$X_t = [1 + \sum_{j=1}^{\infty}(0.7)^{j-1}(0.6)B^j]Z_t.$$

**(2c)**

We first identify the polynomial $\pi(B) = \frac{\phi(B)}{\theta(B)} = \frac{1-0.7B}{1-0.1B}$ which can be expanded to the geometric sum

$$\pi(B) = (1-0.7B)(1+0.1B+0.1^2B^2+...)$$
$$= 1+0.1B+0.1^2B^2+...) - 0.7(B+0.1B^2+0.1^2B^3+...)$$
$$= 1+\sum_{i=1}^{\infty}[0.1^i - 0.7(0.1)^{i-1}]B^i$$
$$= 1+\sum_{i=1}^{\infty}(0.1)^{i-1}(-0.6)B^i$$

Then the model can be expressed as a pure AR process $Z_t = \pi(B)X_t$ or

$$Z_t = [1+\sum_{i=1}^{\infty}(0.1)^{i-1}(-0.6)B^i]X_t.$$

**(2d)**

To derive the acf will need $Var(X_t)$.

$$Var(X_t) = Var(Z_t) + 0.6^2 Var(Z_{t-1} + 0.7Z_{t-2} + 0.7^2 Z_{t-3} + ...)$$
$$= \sigma^2 + 0.6^2(\sigma^2 + 0.7^2\sigma^2 + 0.7^4\sigma^2 + ...)$$
$$= \sigma^2 + \frac{0.6^2\sigma^2}{1-0.7^2}$$
$$= \sigma^2(1+\frac{0.6^2}{1-0.7^2}).$$

Now we find $Cov(X_t, X_{t+1})$. Recall $X_t = \psi(B)Z_t = \sum_{i=0}^{\infty}\psi_i B^i Z_t = \sum_{i=0}^{\infty}\psi_i Z_{t-i}$ where $Y_i = 0.6(0.7)^{j-1}$ for $j = 1, 2, ...$ and $Y_0 = 1$.

$$Cov(X_t, X_{t+1}) = E(X_t X_{t+1})$$
$$= E(\sum_{i=0}^{\infty}\psi_i Z_{t-i}\sum_{j=0}^{\infty}\psi_j Z_{t-j+1})$$

Since $E(Z_{t-a}Z_{t-b}) = \sigma^2$ when $a = b$ and 0 otherwise, we have

$$Cov(X_t, X_{t+1}) = \sigma^2(\sum_{i=0}^{\infty}\psi_i\psi_{i+1})$$

$$= \sigma^2(0.6 + \sum_{i=1}^{\infty}\psi_i\psi_{i+1})$$

$$= \sigma^2(0.6 + \sum_{i=1}^{\infty} 0.6^2(0.7)^{2i+1})$$

$$= \sigma^2(0.6 + \frac{0.6^2(0.7)}{1-0.7^2})$$

Next we will need $Cov(X_t, X_{t+h})$.

$$Cov(X_t, X_{t+h}) = E(X_t X_{t+h})$$

$$= E(\sum_{i=0}^{\infty} \psi_i Z_{t-i} \sum_{j=0}^{\infty} \psi_j Z_{t-j+h})$$

Since $E(Z_{t-a}Z_{t-b}) = \sigma^2$ when $a = b$ and 0 otherwise, we have

$$Cov(X_t, X_{t+h}) = \sigma^2(\sum_{i=0}^{\infty} \psi_i \psi_{i+h})$$

$$= \sigma^2[0.6(0.7^{|h|-1}) + 0.6^2(0.7^{|h|}) + 0.6^2(0.7^{|h|+2})) + 0.6^2(0.7^{|h|+4}) + ...]$$

$$= \sigma^2[0.6(0.7^{|h|-1}) + \frac{0.6^2(0.7^{|h|})}{1-0.7^2}]$$

$$= 0.7^{|h|-1} Cov(X_t, X_{t+1}).$$

Then

$$\rho(1) = \frac{Cov(X_t, X_{t+1})}{Var(X_t)} = \frac{\sigma^2(0.6 + \frac{0.6^2(0.7)}{1-0.7^2})}{\sigma^2((1 + \frac{0.62}{1-0.7^2}))}$$

$$= 0.6413793,$$

$$\rho(h) = \frac{Cov(X_t, X_{t+h})}{Var(X_t)} = \frac{0.7^{|h|-1} Cov(X_t, X_{t+1})}{Var(X_t)}$$

$$= 0.7^{|h|-1}\rho(1)$$

$$= 0.7^{|h|-1} * 0.6413793$$

for $h > 0$, and $\rho(0) = \frac{Var(X_t)}{Var(X_t)} = 1$.

## Question 3

Let $\{X_t\}$ be a SARIMA$(2, 1, 0) \times (0, 1, 2)_{12}$ process:

$$(1 - \alpha_1 B^1 - \alpha_2 B^2)W_t = (1 + \beta_1 B^{12} + \beta_2 B^{24})Z_t$$

where $W_t = (1-B^{12})[(1-B)X_t]$. We will use this fact later but will keep the $W_t$ term for now. Distributing the terms gives

$$W_t - \alpha_1 W_{t-1} - \alpha_2 W_{t-2} = Z_t + \beta_1 Z_{t-12} + \beta_2 Z_{t-24}$$

and now writing $W_t$ in terms of $X_t$ gives

$$X_t - X_{t-1} - X_{t-12} + X_{t-13} - \alpha_1(X_{t-1} - X_{t-2} - X_{t-13} + X_{t-14}) - \alpha_2(X_{t-2} - X_{t-3} - X_{t-14} + X_{t-15}) = Z_t + \beta_1 Z_{t-12} + \beta_2 Z_{t-24}.$$

Without a need to simplify further we can see that this is an ARMA$(15, 24)$ process.