(a) Sensitivity: $TP = P(D=1 \mid R=1) = 99\% \Rightarrow FN = 1\% = P(D=0 \mid R=1)$

Specificity: $TN = P(D=0 \mid R=0) = 98\% \Rightarrow FP = 2\% = P(D=1 \mid R=0)$

(i) $P(D=1 \mid H_0) = P(D=1 \mid R=0) = FP = 2\%$

(ii) $$P(R=0 \mid D=1) = \frac{P(R=0 \cap D=1)}{P(D=1)} = \frac{P(D=1 \mid R=0) \cdot P(R=0)}{P(D=1 \mid R=0) \cdot P(R=0) + P(D=1 \mid R=1) P(R=1)}$$

$$= \frac{FP \cdot \pi}{FP \cdot \pi + TP \cdot (1-\pi)}$$

$$= \frac{0.02 \cdot 0.995}{0.02 \cdot 0.995 + 0.005 \cdot 0.99} = \frac{0.0199}{0.02485} = 0.801$$

(iii)

WANT $P(D_2 = 1 \mid D_1 = 1)$

Independence:

$$P(D_1 \cap D_2 \mid R) = P(D_1 \mid R) P(D_2 \mid R)$$

$$P(D_2 = 1 \mid D_1 = 1) = \frac{P(D_1 = 1 \cap D_2 = 1)}{P(D_1 = 1)}$$

$$= \frac{P(D_1=1 \cap D_2=1 \mid R=0) P(R=0) + P(D_1=1 \cap D_2=1 \mid R=1) P(R=1)}{P(D_1=1)}$$

$$= \frac{\left[[P(D_1=1 \mid R=0) P(D_2=1 \mid R=0)] P(R=0) + [P(D_1=1 \mid R=1) P(D_1=1 \mid R=1)] P(R=1) \right]}{P(D=1 \mid R=0) \cdot P(R=0) + P(D=1 \mid R=1) P(R=1)}$$

$$= \frac{FP^2 \cdot \pi + TP^2 \cdot (1-\pi)}{FP \cdot \pi + TP \cdot (1-\pi)}$$

$$= \frac{(0.02)^2 \cdot 0.995 + 0.99^2 \times (0.005)}{0.002 \cdot 0.995 + 0.99 \times 0.005}$$

$$= 0.213$$

b) (i) $LR(T) = \dfrac{f_1(T)}{f_0(T)} = \dfrac{2c\, e^{-2ct}}{c\, e^{-ct}} = 2\, e^{-ct}$

(ii) $2e^{-ct} < \eta \qquad T < -\frac{1}{c}\log\left(\frac{\eta}{2}\right)$

$FPR = P(D=1 \mid R=0) = P(D=1 \mid H_0) = \alpha$

$= P\left(T < -\frac{1}{c}\log\left(\frac{\eta}{2}\right) \,\middle|\, H_0\right)$

$$= \int_0^{-\frac{1}{c}\log\left(\frac{\eta}{2}\right)} c\, e^{-ct}\, dt \qquad = 1 - \frac{\eta}{2}$$

$1 - \frac{\eta}{2} = \alpha \qquad \eta = 2 - 2\alpha$

(iii) $P\left(T < -\frac{1}{c}\log\left(\frac{\eta}{2}\right) \,\middle|\, H_1\right)$

$$= \int_0^{-\frac{1}{c}\log(\eta/2)} f_1(t)\, dt \qquad = 1 - e^{-2c\cdot\left(-\frac{1}{c}\log\left(\frac{\eta}{2}\right)\right)}$$

$$= 1 - \frac{\eta^2}{4}$$

$$= 1 - (1-\alpha)^2$$