HW2 Q1

a) Given $x \sim \text{Uniform}(0, w)$:

$$f_1(x|w) = \frac{1}{w} \mathbb{1}[x \le w]$$

$$f_n(x_1, \dots x_n | w) = \frac{1}{w^n} \prod_{i=1}^{n} \mathbb{1}[x_i \le w]$$

$$= \frac{1}{w^n} \mathbb{1}[\max_i x_i \le w]$$

b) $\text{Lik}(w) = \frac{1}{w^n} \mathbb{1}[\max_i x_i \le w]$

$\max_i x_i > w$ $\quad \text{Lik}(x_1 \dots x_n; w) = 0$

$\max_i x_i \le w$ $\quad \text{Lik}(x_1 \dots x_n; w) = \frac{1}{w^n} \Rightarrow$ Mono-decreasing function

So maximum $w = \max_i x_i$

c) $p(w | x_1 \dots x_n) = \dfrac{f_w(x_1 \dots x_n | w) p(w)}{\int_{w'} f_n(x_1 \dots x_n | w') p(w')}$

Numerator: $f_w(x_1 \dots x_n | w) p(w) = \frac{1}{w^n} \mathbb{1}[\max_i x_i \le w] \cdot \frac{\alpha \beta^\alpha}{w^{\alpha+1}} \mathbb{1}(w > \beta)$

$$= \frac{\alpha \beta^\alpha}{w^{n+\alpha+1}} \mathbb{1}[\max_i x_i \le w] \mathbb{1}(w > \beta)$$

$$= \frac{\alpha \beta^\alpha}{w^{n+\alpha+1}} \mathbb{1}[w \ge \max(x_i, \beta)]$$

$$\int_{w'} f_n(x_1 \dots x_n | w') p(w') = \int_0^\infty \frac{\alpha \beta^\alpha}{w'^{n+\alpha+1}} \mathbb{1}[w' \ge \max[\max_i x_i, \beta]] \, dw'$$

$\alpha \beta^\alpha$ is constant

$$\hookrightarrow \alpha \beta^\alpha \int \frac{\mathbb{1}[w' \ge \max[\max_i x_i, \beta]]}{w'^{n+\alpha+1}} dw'$$

Let $m = \max(\max_i x_i, \beta) \Rightarrow \alpha \beta^\alpha \left[\dfrac{w'^{-\alpha-n}}{-\alpha-n} \right] \Big|_m^\infty = \alpha \beta^\alpha \dfrac{m^{-\alpha-n}}{\alpha+n}$

$$P'(w \mid x_1 \dots x_n) = \frac{\frac{\alpha \beta^\alpha}{w^{n+\alpha+1}} m}{\alpha \beta^\alpha \frac{m^{-\alpha-n}}{\alpha+n}} = \frac{(\alpha+n)\, m^{(\alpha+n)}}{w^{(\alpha+n+1)}} \mathbb{1}[w > m]$$

set $\alpha+n = \delta \Rightarrow \frac{\delta\, m^\delta}{w^{\delta+1}} \mathbb{1}[w > m]$

↳ Pareto(δ, m)

d) prior: $w \sim$ Pareto(α, β)

posterior $\sim$ Pareto(δ, m)

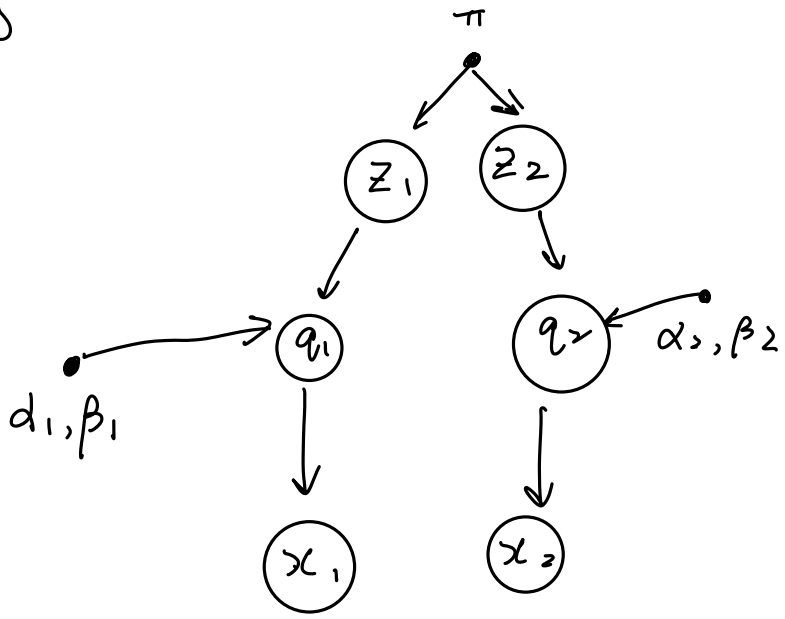
where $\delta = \alpha + n$ $\quad m = \max(\max_i x_i, \beta)$

α: track # of samples seen

m: record/update maximum Length seen

Question 2

a)



c) i)

$$P[Bin(50, 0.45) \leq 50]$$

$$\Rightarrow \sum_{i=0}^{50} \binom{100}{i} (0.45)^{i} (0.55)^{100-i} = 0.87$$

d)

$$q_0 \sim \text{Beta}\left(\alpha_0 + n_0,\ \beta_0 - n_0 + \sum_{i:z_i=0} x_i\right)$$

$$q_1 \sim \text{Beta}\left(\alpha_1 + n_1,\ \beta_1 - n_1 + \sum_{i:z_i=1} x_i\right)$$

$$n_0 = \sum_{i=0}^{n} \mathbb{1}_{z_i=0}$$

$$n_1 = \sum_{i=0}^{N} \mathbb{1}_{z_i=1}$$