Homework Problem
OSE Lab, University of Chicago
Tony Smith, July 2019

The purpose of this problem is to solve numerically for the steady-state equilibrium in a version of the Aiyagari (1994) economy. In this economy there is a measure one of consumers and a typical consumer solves:

$$v(k, \epsilon; \bar{k}) = \max_{k'} \left[U(r(\bar{k}) + w(\bar{k})\epsilon + (1-\delta)k - k') + \beta E_{\epsilon'|\epsilon} v(k', \epsilon'; \bar{k})\right],$$

subject to $k' \geq 0$, where k is individual capital, $\bar{k}$ is aggregate capital, and ϵ is the consumer's employment status: $\epsilon \in \{\epsilon_1, \epsilon_2\}$, with $\epsilon_1 = 1$ (employed) and $\epsilon_2 = 0$ (unemployed). Calibrate the transition probabilities π_{ij} for ϵ so that: (i) the probability of remaining employed in the next period given that the consumer is employed today is 0.9; and (ii) the steady-state employment rate $u = 0.1$. The pricing functions $r(\bar{k})$ and $w(\bar{k})$ are the marginal products of a Cobb-Douglas production technology with coefficient $\alpha \in (0, 1)$: $r(\bar{k}) = \alpha \bar{k}^{\alpha-1}(1-u)^{1-\alpha}$ and $w(\bar{k}) = (1-\alpha)\bar{k}^{\alpha}(1-u)^{-\alpha}$, where aggregate labor supply is equal to $1 - u$.

In a steady-state equilibrium, (i) consumers optimize given $\bar{k}$; and (ii) the consumers' decisions, together with the law of motion for ϵ, determine an invariant distribution over (k, ϵ) with total capital holdings equal to $\bar{k}$. Your goal is calculate the steady-state equilibrium value of aggregate capital, $\bar{k}^*$.

Let a period be one year and set $u(c) = \log(c)$, $\beta = 0.96$, $\delta = 0.06$, and $\alpha = 0.36$. Proceed in steps: first, guess on $\bar{k}$ (perhaps start with the value that obtains in a representative-agent economy in steady state); second, restrict k to lie on a grid of $N = 200$ points in the interval $[0.001, 40]$ and then iterate on the Bellman equation to find $v(k, \epsilon; \bar{k})$ at each of the pairs in $\{k_1, \ldots, k_N\} \times \{\epsilon_1, \epsilon_2\}$; third, simulate a long time series for a typical consumer using the optimal decision rule (computed in the second step) and the law of motion for ϵ; fourth, calculate average capital holdings, $\hat{\bar{k}}$, using the simulated data and, if $\hat{\bar{k}}$ is not close (enough) to $\bar{k}$, update $\bar{k}$ and repeat steps two through four until it is! (Note: When simulating data, it is important to eliminate "jitter" by using the same seed in the random number generator in every iteration.)

Alternatively, you could accomplish steps three and four by explicitly obtaining the transition probability matrix across states (k_i, ϵ_j), $i = 1, \ldots, N$, $j = 1, 2$, and then either computing the eigenvector (normalized to sum to 1) associated with the eigenvalue equal to one or multiplying this matrix by itself repeatedly until its columns converge. Thus armed with the invariant distribution over (k_i, ϵ_j), it is straightforward to calculate total capital holdings across the population.

Your answer to this problem, in addition to the value of $\bar{k}^*$, should include: (i) a graph of the optimal decision rules, one for each value of ϵ; (ii) the coefficient of variation of capital holdings in the invariant distribution; and (iii) a copy of your code.

Finally, is the steady-state equilibrium interest rate higher or lower in this economy than in the usual neoclassical growth with complete markets and a representative agent? Try to explain why.