"Open Source Macroeconomics Laboratory Boot Camp Dynamic Stochastic General Equilibrium Modeling"

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Outline

Definition of General Equilibrium

Definition

A *competitive equilibrium* is a set of allocations, $\{x_i\}_{i=1}^{I}$, and prices, $\{p_i\}_{i=1}^{I}$, for each factor of production and consumable such that:

1. households optimize utility,
2. firms optimize profits,
3. the government meets its budget constraints, and
4. all factor and goods markets clear

Walras' Law

Walras Law - If an economy has N markets and $N-1$ of those markets are in equilibrium, then the remaining market must also be in equilibrium.

DGE and DSGE models

1. DGE - Dynamic General Equilibrium
2. DSGE - Dynamic Stochastic General Equilibrium

Captivating Dynamic Fan Videos

Outline

Household's Problem

$$\max_{\{x_{it},k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t E_t\{u(\{x_{it}\}_{i=1}^{I})\}$$

$$\text{s.t.} \quad \sum_{i=1}^{I} p_{it}x_{it} + \sum_{j=1}^{J}(1 + r_{jt} - \delta_j)k_{jt} - k_{j,t+1} = 0 \quad \forall t \qquad (1)$$

Household's Problem

$$V(\{k_{jt}\};\theta_t) = \max_{\{x_{it}\},\{k_{j,t+1}\}} u(\{x_{it}\}) + \beta E_t\{V(\{k_{j,t+1}\},\theta_{t+1})\}$$

$$\text{s.t. } \sum_{i=1}^{I} p_{it}x_{it} + \sum_{j=1}^{J}(1 + r_{jt} - \delta_j)k_{jt} - k_{j,t+1} = 0 \qquad (2)$$

Household's Problem Solution

$$u_{x_i}(\{x_{it}\}_{i=1}^{I}) + \lambda_t p_{it} = 0 \quad \forall i \tag{3}$$

$$-\lambda_t + \beta E_t\{V_{k_j}(\{k_{j,t+1}\}, \theta_{t+1})\} = 0 \quad \forall j \tag{4}$$

The condition on each λ_t recovers the budget constraint.
Envelope condition:

$$V_{k_j}(\{k_t\}; \theta_t) = \lambda_t(1 + r_{jt} - \delta_j) \quad \forall j \tag{5}$$

Euler Equations

Euler equations:

$$\lambda_t = \beta E_t\{\lambda_{t+1}(1 + r_{j,t+1} - \delta_j)\} \quad \forall j \tag{6}$$

or

$$\frac{u_{x_i}(\{x_{it}\}_{i=1}^{I})}{p_{it}} = \beta E_t\left\{\frac{u_{x_i}(\{x_{i,t+1}\}_{i=1}^{I})}{p_{i,t+1}}(1 + r_{j,t+1} - \delta)\right\} \quad \forall i, j \tag{7}$$

Firm's Problem

$$\max_{\{K_{jt}\},\{X_{it}\}} \Pi_t = \sum_{i=1}^{I} P_{it}X_{it} - \sum_{j=1}^{J} R_{jt}K_{jt} \text{ s.t. } F(\{X_{it}\},\{K_{jt}\},\{z_{nt}\}) = 0$$

Maximization yields the following first-order conditions:

$$P_{it} + \Lambda_t F_{Xi}(\{X_{it}\},\{K_{jt}\},\{z_{nt}\}) = 0 \quad \forall i \tag{8}$$

$$-R_{jt} + \Lambda_t F_{Kj}(\{X_{it}\},\{K_{jt}\},\{z_{nt}\}) = 0 \quad \forall j \tag{9}$$

$$F(\{X_{it}\},\{K_{jt}\},\{z_{nt}\}) = 0 \tag{10}$$

Adding-Up and Market Clearing

Market clearing conditions for period are given by:

$$x_{it} = X_{it} \quad \forall i \tag{11}$$

$$k_{jt} = K_{jt} \quad \forall i \tag{12}$$

We also must impose equivalences between the prices faced by households and those faced by firms to prevent infinite arbitrage opportunities.

$$p_{it} = P_{it} \quad \forall i \tag{13}$$

$$r_{jt} = R_{jt} \quad \forall j \tag{14}$$

Exogenous Laws of Motion

Finally, we need to specify a stochastic process for the technology shocks. Let Z_t denote the vector of $\{z_{nt}\}$'s each period.

$$Z_t = (I - P)\bar{Z} + PZ_{t-1} + H_t; \quad H_t \sim \text{i.i.d.}(0, \Sigma) \tag{15}$$

where P is a square matrix of VAR coefficients, H_t is a vector of random variables, and Σ is a variance/covanriance matrix.

Equilibrium

Categorize Variables

$$\begin{aligned} X_t &= \{k_{j,t-1}\}_{j=1}^{J} \\ Y_t &= \left\{\{x_{it}, p_{it}\}_{i=1}^{I}, \{r_{jt}\}_{j=1}^{J}\right\} \\ Z_t &= \{z_{nt}\}_{n=1}^{N} \end{aligned} \tag{16}$$

Our goal is to find a policy function, Φ, that maps the values of the current state into the values for the endogenous state variables next period.

$$X_{t+1} = \Phi(X_t, Z_t) \tag{17}$$

Outline

Household's Problem

$$V(k_t;\theta_t) = \max_{\ell_t, k_{t+1}} u(c_t, \ell_t) + \beta E_t\{V(k_{t+1}, \theta_{t+1})\} \tag{18}$$

$$\text{s.t. } (1-\tau)\left[w_t \ell_t + (r_t - \delta)k_t\right] + k_t + T_t = c_t + k_{t+1} \tag{19}$$

We can dispense with the Lagrangian if we rewrite (19) as

$$c_t = (1-\tau)\left[w_t\ell_t + (r_t-\delta)k_t\right] + k_t + T_t - k_{t+1} \tag{20}$$

and substitute it into the utility function of (18).
The first order conditons from the problem are:

$$u_\ell(c_t,\ell_t) + u_c(c_t,\ell_t)w_t(1-\tau) = 0 \tag{21}$$

$$-u_c(c_t,\ell_t) + \beta E_t\{V_k(k_{t+1},\theta_{t+1})\} = 0 \tag{22}$$

The envelope condition is :

$$V_k(k_t;\theta_t) = u_c(c_t,\ell_t)[(r_t-\delta)(1-\tau)+1] \tag{23}$$

Combining (22) and (23) and rearranging terms gives us the intertemporal Euler equation.

$$u_c(c_t, \ell_t) = \beta E_t \left\{ u_c(c_{t+1}, \ell_{t+1})[(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \tag{24}$$

Rewriting (21) gives the a consumption-leisure Euler equation.

$$-u_\ell(c_t, \ell_t) = u_c(c_t, \ell_t) w_t (1 - \tau) \tag{25}$$

Firm's Problem

$$\max_{K_t, L_t} \Pi_t = f(K_t, L_t, z_t) - W_t L_t - R_t K_t$$

where K_t and L_t are capital and labor hired, R_t and W_t are the factor prices, and $f(.)$ is the production function.
It yields the following first-order conditions:

$$R_t = f_K(K_t, L_t, z_t) \tag{26}$$

$$W_t = f_L(K_t, L_t, z_t) \tag{27}$$

Government

The government collects distortionary taxes and refunds these to the households lump-sum:

$$\tau\left[w_t \ell_t + (r_t - \delta) k_t\right] = T_t \tag{28}$$

Adding-Up and Market Clearing

Market clearing is:

$$\ell_t = L_t \tag{29}$$

$$k_t = K_t \tag{30}$$

Price equivalences are:

$$w_t = W_t \tag{31}$$

$$r_t = R_t \tag{32}$$

Exogenous Laws of Motion

The stochastic process for the technology is:

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.}(0, \sigma_z^2) \qquad (33)$$

The Equilibrium

The dynamic equilibrium for the model is defined by (20) and (24) – (33), a system of eleven equations in eleven unknowns. We can simplify this, however, by using (31) and (32) as definitions to eliminate the variables W_t and R_t. Similarly, (29) and (30) eliminate L_t and K_t. This leaves us with seven equations in seven unknowns, $\{c_t, k_t, \ell_t, w_t, r_t, T_t \;\&\; z_t\}$.

The equations are:

$$c_t = (1-\tau)\left[w_t\ell_t + (r_t-\delta)k_t\right] + k_t + T_t - k_{t+1} \tag{34}$$

$$u_c(c_t,\ell_t) = \beta E_t\left\{u_c(c_{t+1},\ell_{t+1})[(r_{t+1}-\delta)(1-\tau)+1]\right\} \tag{35}$$

$$-u_\ell(c_t,\ell_t) = u_c(c_t,\ell_t)w_t(1-\tau) \tag{36}$$

$$r_t = f_K(k_t,\ell_t,z_t) \tag{37}$$

$$w_t = f_L(k_t,\ell_t,z_t) \tag{38}$$

$$\tau\left[w_t\ell_t + (r_t-\delta)k_t\right] = T_t \tag{39}$$

$$z_t = (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{i.i.d.}(0,\sigma_z^2) \tag{40}$$

The state of the economy is defined by z_t and k_{t-1}. All other variables are jump variables. This gives us the following classifications:

$$\begin{aligned} X_t &= \{k_{t-1}\} \\ Y_t &= \{c_t, \ell_t, w_t, r_t, T_t\} \\ Z_t &= \{z_t\} \end{aligned} \tag{41}$$

Our goal is the policy function:

$$X_{t+1} = \Phi(X_t, Z_t) \tag{42}$$

Outline

The Solow Model

A familiar example is the simple Solow growth model. In this model households save a constant portion of their income, savings and investment are equal and capital accumulates with depreciation.

$$
\begin{aligned}
S_t &= sY_t & Y_t &= K_t^{\alpha} L_t^{1-\alpha} \\
L_t &= \bar{L} = 1 & S_t &= I_t \\
I_t &= K_{t+1} - (1-\delta)K_t & &
\end{aligned}
$$

Iterative subsitution yields the following policy function:

$$K_{t+1} = (1-\delta)K_t + sK_t^{\alpha} \tag{43}$$

The solution for $\bar{K}$ is:

$$\bar{K} = \left(\frac{se^{(1-\alpha)\bar{z}}}{\delta} \right)^{\frac{1}{1-\alpha}} \tag{44}$$

Brock and Mirman's Model

Households solve the followng dynamic program.

$$V(K_t, z_t) = \max_{K_{t+1}} \ln(e^{z_t} K_t^{\alpha} - K_{t+1}) + \beta E_t\{V(K_{t+1}, z_{t+1})\} \quad (45)$$

The associated Euler equation is:

$$\frac{1}{e^{z_t} K_t^{\alpha} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} {K_{t+1}}^{\alpha-1}}{e^{z_{t+1}} {K_{t+1}}^{\alpha} - K_{t+2}} \right\} \quad (46)$$

The law of motion for z is:

$$z_{t+1} = \rho z_t + \varepsilon_t; \quad \varepsilon_t \sim i.i.d(0, \sigma^2) \quad (47)$$

Brock and Mirman's Model

You be we asked in an exercise at the end of this chapter to verify that the policy function takes the following form:

$$K_{t+1} = Ae^{z_t}K_t^{\alpha} \tag{48}$$

You will need to find the value of A as a function of the model's parameters.
The steady state value of K is defined by using (48) with $K_t = K_{t+1}$ and $z_t = 0$.

$$\bar{K} = A^{\frac{1}{1-\alpha}} \tag{49}$$

Steady State of the Baseline Model

To find the steady state in our DSGE model we need to proceed similarly and replace the time-period-specific values of$\{c_t, k_t, \ell_t, w_t, r_t, \tau_t \ \& \ z_t\}$ with their steady state values as below:

$$\bar{c} = (1-\tau)\left[\bar{w}\bar{\ell} + (\bar{r}-\delta)\bar{k}\right] + \bar{T} \tag{50}$$

$$u_c(\bar{c},\bar{\ell}) = \beta E_t\left\{u_c(\bar{c},\bar{\ell})[(\bar{r}-\delta)(1-\tau)+1]\right\} \tag{51}$$

$$-u_\ell(\bar{c},\bar{\ell}) = u_c(\bar{c},\bar{\ell})\bar{w}(1-\tau) \tag{52}$$

$$\bar{r} = f_K(\bar{k},\bar{\ell},\bar{z}) \tag{53}$$

$$\bar{w} = f_L(\bar{k},\bar{\ell},\bar{z}) \tag{54}$$

$$\tau\left[\bar{w}\bar{\ell} + (\bar{r}-\delta)\bar{k}\right] = \bar{T} \tag{55}$$

Additional Issues

GDP isgiven by:

$$y_t = f(k_t, \ell_t, z_t) \tag{56}$$

Investment is given by:

$$i_t = k_{t+1} - (1 - \delta)k_t \tag{57}$$

In addition, while r_t is the rental rate on capital, it is not the same as the real interest rate that would be offered on financial assets. The closest analog is the *user cost of capital*, which is defined as:

$$u_t = r_t - \delta \tag{58}$$

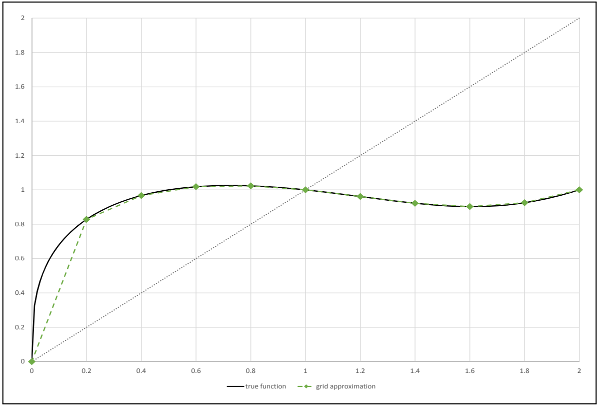
Outline

Solution Methods

- discrete grid approximations
- spline function approximations
- linear-quadratic approximations objective functions and constraints
- log-linearization of the characterizing equations
- higher-order approximation of the characterizing equations

Grid Method

Figure: **Grid Approximation to the True Policy Function**



Polynomial Methods

Figure: **Polynomial Approximations to the True Policy Function**