

"Open Source Macroeconomics Laboratory Boot Camp Perturbation Methods"

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July 19, 2019

Outline

- 1 Solution with "Jump" Variables (Y_t)
- 2 Introduction and Motivation
- 3 Perturbation Methods in General
- 4 Brock and Mirman Model
- 5 Perturbation in DSGE Models

Types of Variables

- Z_t is a vector of n_Z exogenous state variables.
- X_{t-1} is a vector of n_X endogenous state variables.
- Y_t is a vector of n_Y implicitly-defined non-state or "jump" variables.
- D_t is a vector of n_D explicitly-defined non-state or "jump" variables.

Note we can lump Y_t and D_t into X_t if we like. This may increase computational cost, but is otherwise sound logically. We are searching for a policy function $X_t = \Phi(X_{t-1}, Z_t)$, and perhaps a jump function $Y_t = \Psi(X_{t-1}, Z_t)$.

Dynamic Behavior

We will take the characterizing equations for the model and write them as a vector of two sets of functions stacked in the following form:

$$\Gamma_Y(X_t, X_{t-1}, Y_t, Z_{t+1}, Z_t) = 0 \quad (1)$$

$$E_t\{\Gamma_X(X_{t+1}, X_t, X_{t-1}, Y_{t+1}, Y_t, Z_{t+1}, Z_t)\} = 0 \quad (2)$$

X_{t+1} , X_t and X_{t-1} are $n_X \times 1$ vectors. Y_{t+1} and Y_t are $n_Y \times 1$ vectors. Z_{t+1} and Z_t are $n_Z \times 1$ vectors and Γ outputs a $n_X \times 1$ vector.

Dynamic Behavior

We can use a first-order Taylor-series approximation of these equations to get a linear approximation of the characterizing equations.

$$A\tilde{X}_t + B\tilde{X}_{t-1} + C\tilde{Y}_t + D\tilde{Z}_t = 0 \quad (3)$$

$$E_t \left\{ F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + J\tilde{Y}_{t+1} + K\tilde{Y}_t + L\tilde{Z}_{t+1} + M\tilde{Z}_t \right\} = 0 \quad (4)$$

where \tilde{X}_t denotes $X_t - \bar{X}$

Linear approximations of policy and jump functions are:

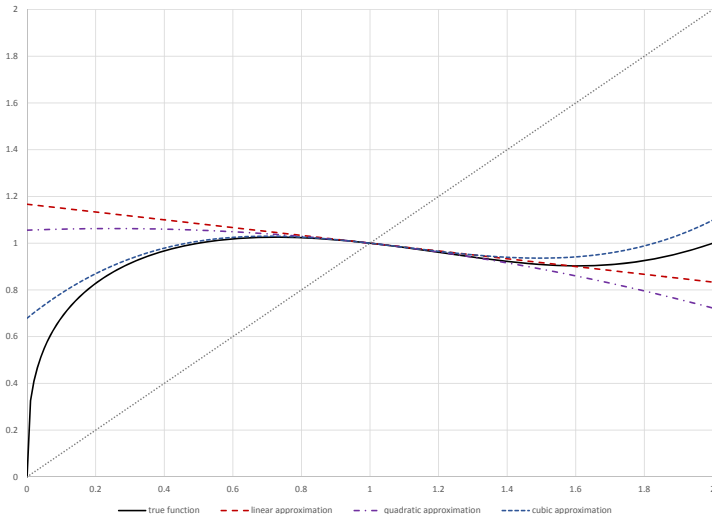
$$\tilde{X}_t = U + P\tilde{X}_{t-1} + Q\tilde{Z}_t \quad (5)$$

$$\tilde{Y}_t = V + R\tilde{X}_{t-1} + S\tilde{Z}_t \quad (6)$$

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Illustration of Polynomial Approximations



Linear approximation is:

$$\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t$$

Quadratic approximation is:

$$\tilde{X}_t = H_X\tilde{X}_{t-1} + H_Z\tilde{Z}_t + \frac{1}{2} \left[H_{XX}\tilde{X}_{t-1}^2 + H_{ZZ}\tilde{Z}_t^2 + 2H_{XZ}\tilde{X}_{t-1}\tilde{Z}_t \right]$$

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First-Order Peturbations

Suppose we have a condition on a potentially nonlinear bivariate function:

$$F(x, u) = 0 \tag{7}$$

Assume u is an exogenously given variable, and x will be chosen to satisfy (7). Denote the solution to this condition as $x(u)$ and assume that the value of $x(u_0)$ is known.

First-Order Peturbations

Taking the derivative of (7) with respect to u gives:

$$F_x\{x(u), u\}x_u(u) + F_u\{x(u), u\} = 0 \quad (8)$$

If we evaluate this at $u = u_0$ and solve for the first derivative of $x(u)$, we have:

$$x_u(u_0) = -\frac{F_u\{x(u_0), u_0\}}{F_x\{x(u_0), u_0\}}$$

First-Order Perturbations

Since $x(u_0)$ is known, as long as $F_x\{x(u_0), u_0\} \neq 0$ we can find the value for the first derivative. The first-order (linear) Taylor-series approximation of $x(u)$ will be:

$$x(u) = x(u_0) + x_u(u_0)(u - u_0)$$

Second-Order Perturbations

To find the second-order terms we differentiate (8) again with respect to u .

$$\begin{aligned} & F_{xx}\{x(u), u\}x_u(u)x_u(u) + F_{xu}\{x(u), u\}x_u(u) \\ & + F_x\{x(u), u\}x_{uu}(u) + F_{xu}\{x(u), u\}x_u(u) \\ & + F_{uu}\{x(u), u\} = 0 \end{aligned}$$

Second-Order Perturbations

Again evaluating at $u = u_0$ and solving this time for the second derivative of $x(u)$, we have:

$$x_{uu}(u_0) = -\frac{F_{xx}\{x(u_0), u_0\}[x_u(u_0)]^2 + 2F_{xu}\{x(u_0), u_0\}x_u(u_0) + F_{uu}}{F_x\{x(u_0), u_0\}}$$

Hence, the second-order (quadratic) Taylor-series approximation of $x(u)$ will be:

$$x(u) = x(u_0) + x_u(u_0)(u - u_0) + \frac{1}{2}x_{uu}(u_0)(u - u_0)^2$$

Higher-Order Perturbations

Higher order terms can be obtained by successive differentiation, setting $u = u_0$ and solving for the appropriate derivative.

Each will be a function of the various derivatives of $F(x, u)$ and the lower-order derivatives of $x(u)$ obtained from previous iterations.

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Brock and Mirman

Recall the Euler equation for the non-stochastic version of the model is:

$$\frac{1}{K_t^\alpha - K_{t+1}} - \beta \frac{\alpha K_{t+1}^{\alpha-1}}{K_{t+1}^\alpha - K_{t+2}} = 0$$

In terms of our notation from the previous section we have:

$$u = K_t$$

$$x = x(u) = K_{t+1}$$

$$y = x(x) = K_{t+2}$$

$$F(y, x, u) = F(y(x(u)), x(u), u)$$

$$= \frac{1}{K_t^\alpha - K_{t+1}} - \beta \frac{\alpha K_{t+1}^{\alpha-1}}{K_{t+1}^\alpha - K_{t+2}} = 0 \quad (9)$$

Differentiate

Take the derivative of (9) with respect to $u = K_t$:

$$F_y(y(x(u)), x(u), \bar{u})x_u(x(u))x_u(u) + F_x(y(x(u)), x(u), u)x_u(u) + F_u(y(x(u)), x(u), u) = 0 \quad (10)$$

Differentiate

Evaluating (10) at $u = \bar{u} = \bar{K}$ and noting that $x(\bar{u}) = \bar{u}$:

$$\begin{aligned}
 &F_y(y(x(\bar{u})), x(\bar{u}), \bar{u})x_u(x(\bar{u}))x_u(\bar{u}) \\
 &+ F_x(y(x(\bar{u})), x(\bar{u}), \bar{u})x_u(\bar{u}) + F_u(y(x(\bar{u})), x(\bar{u}), \bar{u}) = 0 \\
 &F_y(\bar{u}, \bar{u}, \bar{u})x_u(\bar{u})^2 + F_x(\bar{u}, \bar{u}, \bar{u})x_u(\bar{u}) + F_u(\bar{u}, \bar{u}, \bar{u}) = 0
 \end{aligned}$$

Note that $F_y(\bar{u}, \bar{u}, \bar{u})$ is the same as F from the linearization notes. Similarly, $F_x(\bar{u}, \bar{u}, \bar{u})$ is G , and $F_u(\bar{u}, \bar{u}, \bar{u})$ is H . Also note that $x_u(\bar{u})$ is P . As in those notes the value of $x_u(\bar{u})$ comes from solving a quadratic.

Differentiate Again

$$\begin{aligned}
 & F_{yy}(\bar{u}, \bar{u}, \bar{u}) x_u(\bar{u})^4 + F_{yx}(\bar{u}, \bar{u}, \bar{u}) x_u(\bar{u})^3 \\
 & + F_{yu}(\bar{u}, \bar{u}, \bar{u}) x_u(\bar{u})^2 + F_y(\bar{u}, \bar{u}, \bar{u}) x_{uu}(\bar{u}) x_u(\bar{u})^2 \\
 & + F_y(\bar{u}, \bar{u}, \bar{u}) x_u(\bar{u}) x_{uu}(\bar{u}) + F_{yx}(\bar{u}, \bar{u}, \bar{u}) x_u(\bar{u})^3 \\
 & + F_{xx}(\bar{u}, \bar{u}, \bar{u}) x_u(\bar{u})^2 + F_{xu}(\bar{u}, \bar{u}, \bar{u}) x_u(\bar{u}) \\
 & + F_x(\bar{u}, \bar{u}, \bar{u}) x_{uu}(\bar{u}) + F_{yu}(\bar{u}, \bar{u}, \bar{u}) x_u(\bar{u})^2 \\
 & + F_{xu}(\bar{u}, \bar{u}, \bar{u}) x_u(\bar{u}) + F_{uu}(\bar{u}, \bar{u}, \bar{u}) = 0
 \end{aligned}$$

Differentiate Again

Supressing the function arguments for the sake of clarity we can rewrite (??) as below.

$$(F_{yy} x_u^4 + 2F_{yx} x_u^3 + 2F_{yu} x_u^2 + F_{xx} x_u^2 + 2F_{xu} x_u + F_{uu}) + (F_y x_u^2 + F_y x_u + F_x)x_{uu} = 0$$

Note the F_{ij} are all second-derivatives evaluated at the steady state. Since x_u has already been solved we can solve this for x_{uu} .

Differentiate Again

$$x_{uu} = - \frac{F_{yy} x_u^4 + 2F_{yx} x_u^3 + 2F_{yu} x_u^2 + F_{xx} x_u^2 + 2F_{xu} x_u + F_{uu}}{(F_y x_u^2 + F_y x_u + F_x)}$$

The quadratic approximation to the policy function is given by:

$$\tilde{K}_{t+1} = x_u \tilde{K}_t + \frac{1}{2} x_{uu} \tilde{K}_t^2 \quad (11)$$

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DSGE Perturbation

Write our stacked dynamic equations as:

$$E_t\{\Gamma(X_{t+1}, X_t, X_{t-1}, Y_{t+1}, Y_t, Z_{t+1}, Z_t)\} = 0 \quad (12)$$

Recall the exogenous law of motion:

$$Z_t = NZ_{t-1} + v\Omega\varepsilon_t; \varepsilon_t \sim (0, I_{n_Z}) \quad (13)$$

where v is a scalar, and Ω is a matrix that determines correlations of the elements in ε_t .

The policy function and jump functions are:

$$X_t = H(X_{t-1}, Z_t, v) \quad (14)$$

$$Y_t = G(X_{t-1}, Z_t, v) \quad (15)$$

DSGE Perturbation

For notational ease define the following.

$$A_t \equiv [X_{t+1} \quad X_t \quad X_{t-1} \quad Y_{t+1} \quad Y_t \quad Z_{t+1} \quad Z_t]^T$$

$$S_t \equiv [X_{t-1} \quad Z_t \quad v]$$

$$n_A \equiv 3n_X + 2n_Y + 2n_Z$$

$$n_S \equiv n_X + n_Z + 1$$

DSGE Perturbation

The Taylor-series approximation of Γ with second-order terms for the variance is:

$$\begin{aligned} \Gamma(A_t) \doteq & \Gamma(\bar{X}, \dots, \bar{Z}) + [\Gamma_1 \quad \dots \quad \Gamma_7] \begin{bmatrix} \tilde{X}_{t+1} \\ \vdots \\ \tilde{Z}_t \end{bmatrix} \\ & + \frac{1}{2} \left(I_{n_A} \otimes \begin{bmatrix} \tilde{X}_{t+1} & \vdots & \tilde{Z}_t \end{bmatrix} \right) \begin{bmatrix} \Gamma_{11} & \dots & \Gamma_{17} \\ \vdots & \ddots & \vdots \\ \Gamma_{71} & \dots & \Gamma_{77} \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ \vdots \\ \tilde{Z}_t \end{bmatrix} \end{aligned} \quad (16)$$

Γ_1 through Γ_7 are combinations of the A through M matrices in Uhlig's notation. Γ_{11} through Γ_{77} are all Magnus and Neudecker matrices of second-order coefficients.

Using 3-Dimensional Tensors

We need to get a matrices of first and second derivatives for the functions Γ , H , and G . All of these are vector-valued functions of vectors.

- The outputs are 1-dimensional
- The Jacobians are 2-dimensional
- The Hessians are 3-dimensional

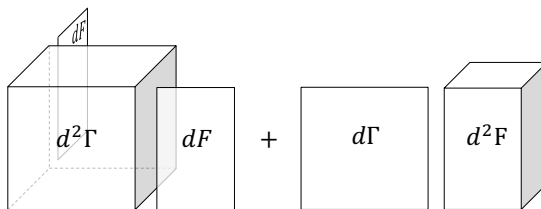
We can either work with matrix algebra extended beyond two dimensions - tensors. Or we can restack 3-dimensional matrices into two dimensions. The latter is what Magnus and Neudecker (1999) discuss.

Using Tensors

$$\Gamma(A) = 0; \quad A = F(S)$$

$$\Gamma_F(F(S))F_S(S) = 0$$

$$F_S(S)^T \Gamma_{FF} F_S(S) + \Gamma_F(F(S)) F_{SS}(S)$$



Let's NOT do this.

Magnus and Neudecker Matrices

If $Y = F(X)$ is a vector-valued function of a vector input.
Then the Magnus and Neudecker Jacobian is:

$$\begin{bmatrix} \frac{\partial Y_1}{\partial X_1} & \cdots & \frac{\partial Y_1}{\partial X_{n_X}} \\ \vdots & \ddots & \vdots \\ \frac{\partial Y_{n_Y}}{\partial X_1} & \cdots & \frac{\partial Y_{n_Y}}{\partial X_{n_X}} \end{bmatrix}$$

The Hessian is...

Magnus and Neudecker Matrices

$$\begin{bmatrix}
 \frac{\partial^2 Y_1}{\partial X_1 \partial X_1} & \cdots & \frac{\partial^2 Y_1}{\partial X_{n_X} \partial X_1} \\
 \vdots & \ddots & \vdots \\
 \frac{\partial^2 Y_{n_Y}}{\partial X_1 \partial X_1} & \cdots & \frac{\partial^2 Y_{n_Y}}{\partial X_{n_X} \partial X_1} \\
 \frac{\partial^2 Y_1}{\partial X_1 \partial X_2} & \cdots & \frac{\partial^2 Y_1}{\partial X_{n_X} \partial X_2} \\
 \vdots & \ddots & \vdots \\
 \frac{\partial^2 Y_{n_Y}}{\partial X_1 \partial X_2} & \cdots & \frac{\partial^2 Y_{n_Y}}{\partial X_{n_X} \partial X_2} \\
 \vdots & & \vdots \\
 \frac{\partial^2 Y_1}{\partial X_1 \partial X_{n_X}} & \cdots & \frac{\partial^2 Y_1}{\partial X_{n_X} \partial X_{n_X}} \\
 \vdots & \ddots & \vdots \\
 \frac{\partial^2 Y_{n_Y}}{\partial X_1 \partial X_{n_X}} & \cdots & \frac{\partial^2 Y_{n_Y}}{\partial X_{n_X} \partial X_{n_X}}
 \end{bmatrix}$$

DSGE Perturbation

The Taylor-series approximation of H with second-order terms for the variance is:

$$\begin{aligned}
 H(X_{t-1}, Z_t, v) \doteq H(\bar{X}, \bar{Z}, \bar{v}) + [H_X \quad H_Z \quad H_v] \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \\ \tilde{v} \end{bmatrix} \\
 + \frac{1}{2} \left(I_{n_Y+n_X} \otimes \begin{bmatrix} \tilde{X}_{t-1}^T & \tilde{Z}_t^T & \tilde{v} \end{bmatrix} \right) \begin{bmatrix} H_{XX} & H_{XZ} & 0 \\ H_{ZX} & H_{ZZ} & 0 \\ 0 & 0 & H_{vv} \end{bmatrix} \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \\ \tilde{v} \end{bmatrix}
 \end{aligned}
 \tag{17}$$

H_X and H_Z terms are the P and Q matrices in Uhlig's notation. H_{XX} , H_{ZZ} , H_{ZX} , H_{XZ}^T and H_{vv} are all Magnus and Neudecker matrices of second-order coefficients.

DSGE Perturbation

A similar setup is used for the approximation of the G function.

$$\begin{aligned}
 G^k(X_{t-1}, Z_t, v) \doteq G(\bar{X}, \bar{Z}, \bar{v}) &+ [G_X \quad G_Z \quad G_v] \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \\ \tilde{v} \end{bmatrix} \\
 &+ \frac{1}{2} \left(I_{n_Y+n_X} \otimes \begin{bmatrix} \tilde{X}_{t-1}^T & \tilde{Z}_t^T & \tilde{v} \end{bmatrix} \right) \begin{bmatrix} G_{XX} & G_{XZ} & 0 \\ G_{ZX} & G_{ZZ} & 0 \\ 0 & 0 & G_{vv} \end{bmatrix} \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \\ \tilde{v} \end{bmatrix}
 \end{aligned}
 \tag{18}$$

G_X and G_Z terms are the R and S matrices in Uhlig's notation.

DSGE Perturbation

We can substitute (14), (15) and (13) into our definition of A_t to get the following function:

$$A_t = F(S_t) = \begin{bmatrix} H(H(X_{t-1}, Z_t, v), NZ_t + v\Omega\varepsilon_{t+1}, v) \\ H(X_{t-1}, Z_t, v) \\ X_{t-1} \\ G(H(X_{t-1}, Z_t, v), NZ_t + v\Omega\varepsilon_{t+1}, v) \\ G(X_{t-1}, Z_t, v) \\ NZ_{t-1} + v\Omega\varepsilon_t \\ Z_t \end{bmatrix} \quad (19)$$

See the chapter handout for the Jacobian and Hessian matrices, $F_S(S_t)$ and $F_{SS}(S_t)$.

DSGE Perturbation

Using (19) in (12) we get $\Delta(S_t) \equiv \Gamma(F(S_t)) = 0$. Magnus and Neudecker (1999) show that the chain-rule for this function with our setup for the organization of the Jacobian and Hessian matrices is as follows.

$$\Delta_{SS} = (F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S + (I_{n_S} \otimes \Gamma_A) F_{SS} \quad (20)$$

Solving for Linear Terms

We can solve for the linear terms as we did in the linearization chapter. This will generate the coefficient matrices H_X , H_Z , G_X and G_Z . (Uhlig's P , Q , R and S .) Discussion in the chapter shows that H_V and G_V are zero.

Solving for Quadratic Terms

With the first-order coefficients for the H and G functions known, we can use the expectation of (20) to solve for the second-order coefficients. We note that F_S is a function of the first-order coefficients as shown in the chapter handout. Similarly, we know that F_{SS} a function of both the first and second-order coefficients.

Solving for Quadratic Terms

Before taking expectations, we need to multiply out the term $\Lambda \equiv (F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S$. Examine the F_S matrix and note that terms with ε_{t+1} appear only in the third column.

$$F_S = \begin{bmatrix} H_X H_X & H_X H_Z + H_Z N & H_X H_V + H_Z \Omega \varepsilon_{t+1} + H_V \\ H_X & H_Z & H_V \\ 1 & 0 & 0 \\ G_X H_X & G_X H_Z + G_Z N & G_X H_V + G_Z \Omega \varepsilon_{t+1} + G_V \\ G_X & G_Z & G_V \\ 0 & N & \Omega \varepsilon_{t+1} \\ 0 & 1 & 0 \end{bmatrix}$$

Solving for Quadratic Terms

If we take the expectation of F_S the ε_{t+1} terms disappear.

$$E\{F_S\} = \begin{bmatrix} H_X H_X & H_X H_Z + H_Z N & H_X H_V + H_V \\ H_X & H_Z & H_V \\ 1 & 0 & 0 \\ G_X H_X & G_X H_Z + G_Z N & G_X H_V + G_V \\ G_X & G_Z & G_V \\ 0 & N & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Solving for Quadratic Terms

Let's look at the $(3, 3)$ block in Λ . Recall that H_v and G_v are zeros.

$$\Lambda(3, 3) = \varepsilon_{t+1}^T (\Omega H_Z^T \Gamma_{11} H_Z \Omega + \Omega G_Z^T \Gamma_{44} G_Z \Omega) \varepsilon_{t+1}$$

This is the only term where the quadratic form of ε_{t+1} appears. In every other term it is either absent or appears as a linear term. Hence when expectations are taken the terms with ε_{t+1} disappear. This means we can use

$E\{\Lambda\} = (E\{F_S\} \otimes I_{n_X+n_Y})^T \Gamma_{AA} E\{F_S\}$ and then replace the $(3, 3)$ term.

Solving for Quadratic Terms

To take expectations of $\Lambda(3, 3)$ it is useful to know that if the elements of a column vector of random variables $\varepsilon \sim iid(0, I)$, then $E\{\varepsilon^T A \varepsilon\} = \text{tr}(A)$.

So we replace the zero in the $(3, 3)$ block with $\text{tr}(\Omega[H_Z^T \Gamma_{11} H_Z] + [G_Z^T \Gamma_{44} G_Z] \Omega)$.

Solving for Quadratic Terms

Recall equation (20):

$$\Delta_{SS} = (F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S + (I_{n_S} \otimes \Gamma_A) F_{SS}$$

Unfortunately, $I_{n_A} \otimes \Gamma_A$ is not a square matrix and therefore not invertable. However, we can solve for the second-order coefficients numerically.

Solving for Quadratic Terms

The coefficients we need to find are

$\Theta = \{H_{XX}, H_{XZ}, H_{ZZ}, H_{VV}, G_{XX}, G_{XZ}, G_{ZZ}, G_{VV}\}$. $E\{F_S\}$, Γ_A and Γ_{AA} are known. We can therefore write a Δ_{SS} function as shown below and numerically solve for the values of Θ that set it equal to zero. We note that Δ_{SS} will return a matrix of size $n_S(n_X + n_Y) \times n_S$. This will be $n_X + n_Y$ blocks of symmetric $n_S \times n_S$ matrices.

$$\Delta_{SS}(\Theta) = (E\{\Lambda\} + (I_{n_S} \otimes \Gamma_A)E\{F_{SS}(\Theta)\}) = 0$$

$$\Lambda = (F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S$$

Solving for Quadratic Terms

The symmetric blocks in the Δ_{SS} matrix will be denoted Δ_{SS}^i for $i \in \{1, 2, \dots, n_X + n_Y\}$ and can be decomposed into nine parts.

$$\Delta_{SS}^i = \begin{bmatrix} \Delta_{XX}^i & \Delta_{XZ}^i & 0 \\ (\Delta_{XZ}^i)^T & \Delta_{ZZ}^i & 0 \\ 0 & 0 & \Delta_{VV}^i \end{bmatrix} \quad (21)$$

Hence we have $n_X^2 + n_Z^2 + n_X n_Z + 1$ unique values for each i , for a total of $(n_X^2 + n_Z^2 + n_X n_Z + 1)(n_X + n_Y)$. We have $(n_X^2 + n_X n_Z + n_Z + 1)n_X$ terms in the H_{SS} coefficients and $(n_X^2 + n_X n_Z + n_Z + 1)n_Y$ terms in the G_{SS} coefficients. Hence the $\Delta_{SS} = 0$ condition will exactly identify Θ .

To summarize. We get the quadratic terms by:

- Taking first and second derivatives of the Γ function at the steady state: Γ_A and Γ_{AA} .
- Finding the first order terms: H_X and H_Z .
- These allow us to get $E\{F_S\}$.
- We then use a numerical equation solver to solve for $\Theta = \{H_{XX}, H_{XZ}, H_{ZZ}, H_{ww}, G_{XX}, G_{XZ}, G_{ZZ}, G_{ww}\}$, which are inputs into the F_{SS} function in the equation below.
- The (3.3) element of Λ is assigned as discussed.

$$\Delta_{SS}(\Theta) = 0$$

$$E\{\Lambda\} + (I_{n_S} \otimes \Gamma_A)E\{F_{SS}(\Theta)\} = 0$$

$$E\{(F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S\} + (I_{n_S} \otimes \Gamma_A)E\{F_{SS}(\Theta)\} = 0$$