Lecture 21: Dynamic Discrete Choice I

ResEcon 703: Topics in Advanced Econometrics

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Agenda

Last time

- Individual-Specific Parameters Example in R

Today

- Static Models with Panel Data
- Dynamic Discrete Choice Examples
- Dynamic Discrete Choice

Upcoming

- Reading for next time
  ▶ Rust (1987)
- Problem sets
  ▶ Problem Set 4 is posted, due November 21

Static Vs. Dynamic Models

Static structural models

- An agent maximizes their objective function within the current time period without considering the effect on future time periods
- Some simple "dynamics" can be incorporated by including
  - Lagged or future explanatory variables
  - State dependence through lagged outcome variables
- All the models we have discussed have been static
- Panel data models can be (and usually are) static

Dynamic structural models

- An agent maximizes their objective function while explicitly considering the effect on future time periods
  - A choice in one period may change the choice set in future periods
  - A choice in one period may change the utility of future choices
- Modeling this dynamic behavior requires a more complex framework

Static Models with Panel Data

Logit Model with Panel Data

We can use the logit model to model discrete choices with panel data

$$U_{njt} = \beta' x_{njt} + \varepsilon_{njt} \quad \Rightarrow \quad P_{nit} = \frac{e^{\beta' x_{nit}}}{\sum_j e^{\beta' x_{njt}}}$$

The logit assumption has to hold

$$\varepsilon_{njt} \sim \text{i.i.d. type I extreme value (Gumbel) with } Var(\varepsilon_{njt}) = \frac{\pi^2}{6}$$

- But the unobserved characteristics of a decision maker that affect choice are unlikely to be independent over time

The logit model with panel data is a sequence of static choices, not a fully dynamic model

- We assume the decision maker maximizes utility in each time period
- But we do not represent how a decision can affect future choices

Mixed Logit Model with Panel Data

We can better represent a sequence of choices over time using the mixed logit model

$$U_{njt} = \beta_n' x_{njt} + \varepsilon_{njt} \quad \Rightarrow \quad P_{ni} = \int \prod_{t=1}^{T} \frac{e^{\beta' x_{n i_t t}}}{\sum_{j=1}^{J} e^{\beta' x_{n j_t t}}} f(\beta \mid \theta) d\beta$$

The individual-specific coefficients represent unobserved preferences

- We model these coefficients as random coefficients and estimate their distributions
- These individual coefficients yield unobserved correlations in choices over time periods

Although the mixed logit model better represents choices over multiple time periods, it is still inherently a sequence of static choices

Simple "Dynamics" in Static Models

When using these static models, we can include lagged outcome variables to model

- Habit formation
- Variety-seeking behavior
- Switching costs
- Brand loyalty

These are all examples of how past choices affect the decision maker's utility in the current time period

But if past choices affect utility in the current time period, then the current choice affects future utility

- A rational decision maker will consider these effects on future utility when making the current choice
- To fully capture this discrete choice framework, we need to use a dynamic discrete choice model

Dynamic Discrete Choice Examples

Dynamic Discrete Choice Example

Why do people attend college? (Or graduate school?)

- Because four years in college provides more utility than anything else the individual could have done in those four years?
- Or because college opens up a new set of jobs and higher salaries compared to not attending college?

A static model implicitly assumes that the first answer is correct

- The static model has no way to represent that the future choice set and salaries will be different after attending to college

To consider the second answer, we need to use a dynamic discrete choice model

- This model will explicitly represent how the decision to attend college affects the future choice set and salaries

Decision Maker with Two-Period Dynamics

A decision maker thinking about college considers two time periods

1. College (C) or work (W) for four years
   - U_{1C}: utility in period 1 from four years in college
   - U_{1W}: utility in period 1 from four years working
2. A set of J possible jobs for a career over many future years
   - U_{2j}^C: utility in period 2 from job j after attending college
   - U_{2j}^W: utility in period 2 from job j after working

The total utility obtained from attending college or working in period 1 is

$$TU_C = U_{1C} + \lambda \max_j (U_{2j}^C)$$
$$TU_W = U_{1W} + \lambda \max_j (U_{2j}^W)$$

where λ reflects the relative weighting of the two periods

The decision maker attends college if and only if $TU_C > TU_W$

Econometrician with Two-Period Dynamics

We decompose the utility of each alternative into an observed and an unobserved component

$$U_{1C} = V_{1C} + \varepsilon_{1C} \qquad U_{2j}^C = V_{2j}^C + \varepsilon_{2j}^C$$
$$U_{1W} = V_{1W} + \varepsilon_{1W} \qquad U_{2j}^W = V_{2j}^W + \varepsilon_{2j}^W$$

and define $\varepsilon = \{\varepsilon_{1C}, \varepsilon_{1W}, \varepsilon_{2j}^C, \varepsilon_{2j}^W\}$ with joint density $f(\varepsilon)$

The probability that the decision maker chooses to attend college is

$$
\begin{aligned}
P_C &= \Pr(TU_C > TU_W) \\
&= \int I\left[V_{1C} + \varepsilon_{1C} + \lambda \max_j (V_{2j}^C + \varepsilon_{2j}^C) \right. \\
&\qquad\qquad \left. > V_{1W} + \varepsilon_{1W} + \lambda \max_j (V_{2j}^W + \varepsilon_{2j}^W) \right] f(\varepsilon) d\varepsilon
\end{aligned}
$$

We have to approximate this integral using simulation

Simplifications for Two-Period Dynamics

If we assume that unobserved utility in the first period, ε_{1C} and ε_{1W}, is i.i.d. extreme value, the choice probability of attending college simplifies to

$$P_C = \int \frac{e^{V_{1C} + \lambda \max_j(V_{2j}^C + \varepsilon_{2j}^C)}}{e^{V_{1C} + \lambda \max_j(V_{2j}^C + \varepsilon_{2j}^C)} + e^{V_{1W} + \lambda \max_j(V_{2j}^W + \varepsilon_{2j}^W)}} g(\varepsilon_2) d\varepsilon_2$$

which is a simpler integral to simulate

If we also assume that unobserved utility in the second period, ε_{2j}^C and ε_{2j}^W, is i.i.d. extreme value, we get the joint probability of attending college in period 1 and then taking job i in period 2 as

$$P_{Ci} = P_C \times \frac{e^{V_{2i}^C}}{\sum_{j=1}^{J} e^{V_{2j}^C}}$$

where P_C is the probability of attending college given above

Example With Three Periods

After many years working job j, the decision maker reaches retirement age and has a new choice to make

- Continue working full time
- Work part time and spend retirement funds
- Retire and collect social security and/or pension funds

The retirement plan, social security payout, etc. differs for each of the possible jobs in period 2

- In period 2, the decision maker will consider how their job will affect utility in period 3
- In period 1, the decision maker will consider how the decision to attend college will affect all periods

Decision Maker with Three-Period Dynamics

A decision maker thinking about college considers three time periods

1. College (C) or work (W) for four years
2. A set of J possible jobs for a career over many future years
3. A set of S possible retirement plans
   - U_{3s}^{Cj}: utility in period 3 from retirement plan s after attending college in period 1 and working job j in period 2
   - U_{3s}^{Wj}: utility in period 3 from retirement plan s after working in period 1 and working job j in period 2

The total utility obtained from attending college or working in period 1 is

$$TU_C = U_{1C} + \lambda \max_j \left[U_{2j}^C + \theta \max_s (U_{3s}^{Cj}) \right]$$

$$TU_W = U_{1W} + \lambda \max_j \left[U_{2j}^W + \theta \max_s (U_{3s}^{Wj}) \right]$$

where λ and θ reflect the relative weighting of the three periods

The decision maker goes to college if and only if $TU_C > TU_W$

Econometrician with Three-Period Dynamics

We decompose the utility of each alternative into an observed and an unobserved component

$$U_{1C} = V_{1C} + \varepsilon_{1C} \qquad U_{2j}^C = V_{2j}^C + \varepsilon_{2j}^C \qquad U_{3s}^{Cj} = V_{3s}^{Cj} + \varepsilon_{3s}^{Cj}$$
$$U_{1W} = V_{1W} + \varepsilon_{1W} \qquad U_{2j}^W = V_{2j}^W + \varepsilon_{2j}^W \qquad U_{3s}^{Wj} = V_{3s}^{Wj} + \varepsilon_{3s}^{Wj}$$

and define $\varepsilon = \{\varepsilon_{1C}, \varepsilon_{1W}, \varepsilon_{2j}^C, \varepsilon_{2j}^W, \varepsilon_{3s}^{Cj}, \varepsilon_{3s}^{Wj}\}$ with joint density $f(\varepsilon)$

The probability that the decision maker chooses to attend college is

$$
\begin{aligned}
P_C &= \Pr(TU_C > TU_W) \\
&= \int I \left\{ V_{1C} + \varepsilon_{1C} + \lambda \max_j \left[V_{2j}^C + \varepsilon_{2j}^C + \theta \max_s (V_{3s}^{Cj} + \varepsilon_{3s}^{Cj}) \right] \right. \\
&\qquad\qquad \left. > V_{1W} + \varepsilon_{1W} + \lambda \max_j \left[V_{2j}^W + \varepsilon_{2j}^W + \theta \max_s (V_{3s}^{Wj} + \varepsilon_{3s}^{Wj}) \right] \right\} \\
&\quad \times f(\varepsilon) d\varepsilon
\end{aligned}
$$

Dynamic Discrete Choice

Dynamic Optimization Notation and Terminology

Some common notation and terminology for dynamic structural models

- $\{i_1, i_2, \ldots, i_t\}$: sequence of choices up to and including period t
- $U_{tj}(i_1, i_2, \ldots, i_{t-1})$: utility obtained in period t from alternative j, which depends on all previous choices
- $TU_{tj}(i_1, i_2, \ldots, i_{t-1})$: total utility (current and all future time periods) obtained from choosing alternative j in period t
  - Known as the "conditional value function"
- $TU_t(i_1, i_2, \ldots, i_{t-1})$: total utility obtained from the optimal choice in period t
  - $TU_t(i_1, i_2, \ldots, i_{t-1}) = \max_j TU_{tj}(i_1, i_2, \ldots, i_{t-1})$
  - Known as the "value function" or "valuation function" at time t

We need to calculate all possible $TU_{tj}(i_1, \ldots, i_{t-1})$ in order to express the optimal choice in each time period

Bellman Equation for Dynamic Discrete Choice

The decision maker chooses optimally (maximizes utility) in the current period knowing they will also choose optimally in every future period (and discounting the future with discount rate δ), which yields an expression for the value function at time t

$$TU_t(i_1, \ldots, i_{t-1}) = \max_j \left[U_{tj}(i_1, \ldots, i_{t-1}) + \delta TU_{t+1}(i_1, \ldots, i_t = j) \right]$$

This relation is the Bellman equation for dynamic discrete choice

We can also write down a Bellman equation for the conditional valuation function, $TU_{tj}(i_1, \ldots, i_{t-1})$

$$TU_{tj}(i_1, \ldots, i_{t-1}) = U_{tj}(i_1, \ldots, i_{t-1}) + \delta \max_k \left[TU_{t+1,k}(i_1, \ldots, i_t = j) \right]$$

Applying the Bellman Equation

If the number of time periods is finite, we can apply the Bellman equation through backward recursion to calculate all possible $TU_{tj}(i_1, \ldots, i_{t-1})$

1. Start in the last time period, $t = T$, with

$$TU_{Tj}(i_1, \ldots, i_{T-1}) = U_{Tj}(i_1, \ldots, i_{T-1})$$

2. Calculate total utility in period $T - 1$, $TU_{T-1,j}(i_1, \ldots, i_{T-2})$, as a function of the values of $TU_{Tj}(i_1, \ldots, i_{T-1})$ from step 1

3. Continue working backward until you reach period 1

We have to calculate $U_{tj}(i_1, \ldots, i_{t-1})$ for each t, each j, and each $\{i_1, i_2, \ldots, i_{t-1}\}$

- If there are J alternatives in each of T time periods, we have to calculate $(J^T)T$ utilities

- This computational burden is known as the "curse of dimensionality"

Choice Probabilities in Dynamic Discrete Choice

We will ultimately use these conditional value functions, $TU_{tj}(i_1, \ldots, i_{t-1})$, to crate an expression for choice probabilities

For example, the probability of choosing alternative i in period 1 is

$$P_{1i} = \Pr(TU_{1i} > TU_{1j} \; \forall j \neq i)$$
$$= \int I[TU_{1i}(\varepsilon) > TU_{1j}(\varepsilon) \; \forall j \neq i] f(\varepsilon) d\varepsilon$$

We have to simulate this choice probability by taking random draws from an assumed joint density of all unobserved utilities, $f(\varepsilon)$

See Chapter 7.7.3 in the Train textbook for more details

Uncertainty in Dynamic Discrete Choice Models

So far we have assumed that the decision maker has perfect information about the future

- Utility of each alternative in each future time period
- How every possible sequence of choices affects this future utility
- But this is unlikely to be true!

We can model utility as being a function factors that are unknown in previous periods

- The decision maker maximizes total expected utility with the expectation taken over the density of the unknown factors
- This expectation adds another integral that has to be simulated, adding yet another layer of complexity and dimensionality to a problem that already suffers from the curse of dimensionality

See Chapter 7.7.3 in the Train textbook for more details

Simplifications for Dynamic Discrete Choice Models

Two ways to simplify dynamic discrete choice models

1. Use as few time periods as possible
   - We could model the college-job-retirement sequence of choices annually (or monthly, weekly, daily) instead of three broad time periods
   - Estimation is feasible with three time periods, but it becomes (at least) an order of magnitude more difficult with 60 individual year

2. Assume the factors that the decision maker does not observe are the same factors that the econometrician does not observe, and these factors are i.i.d. extreme value
   - Choice probabilities have closed-form expressions that are easy to calculate
   - This assumption is unrealistic, but it may be the only way to make the model tractable

See Chapter 7.7.3 in the Train textbook for more details

Announcements

Reading for next time

- Rust (1987)

Office hours

- Reminder: 2:00–3:00 on Tuesdays in 218 Stockbridge

Upcoming

- Problem Set 4 is posted, due November 21