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Assignment 3 - COMP 1805

1. (8pts) Define each set using set-builder notation. If the set is finite, state its cardinality; otherwise, indicate that it is infinite.

   (a) $\{10, 20, 30, 40, \ldots, 1000\}$
   Solution:
   $S = \{x | (x \in \mathbb{N}) \wedge (10 \leq x \leq 1000) \wedge (10|x)\}$
   $|S| = |\{10, 20, 30, 40, \ldots, 1000\}| = 100$

   (b) $\{-6, -4, -2, 0, 2, 4, 6\}$
   Solution:
   $S = \{x | (x \in \mathbb{Z}) \wedge (|x| \leq 6) \wedge (2|x)\}$
   $|S| = |\{-6, -4, -2, 0, 2, 4, 6\}| = 7$

   (c) $\{0, 3, 6, 9, 12, \ldots\}$
   Solution:
   $S = \{x | (x \in \mathbb{N}) \wedge (3|x)\}$
   $|S| = |\{0, 3, 6, 9, 12, \ldots\}| = \infty$ Cardinality is infinite

   (d) $\{0.5, 1, 2, 4, 8, 16, \ldots\}$
   Solution:
   $S = \{x | x = 0.5 \cdot 2^n, n \in \mathbb{N}\}$
   $|S| = |\{0.5, 1, 2, 4, 8, 16, \ldots\}| = \infty$ Cardinality is infinite

2. (16pts) Determine whether each statement is true or false for all sets A and B. Provide a justification for your answer. Note that the difference between two sets A and B can be denoted $A \setminus B$ or $A - B$.

   (a) If $A \subset B$ then $A \subseteq B$.
   Solution:
   TRUE. If A is a proper subset of B, then all elements of A are also in B. This is enough for A to be a subset of B.

   (b) If $A \subseteq B$ then $A \subset B$.
   Solution:
   FALSE. We only know A is a subset of B. However it is possible that B is also a subset of A. If so, then $A = B$, thus $A \not\subset B$.

   (c) If $A = B$ then $A \subseteq B$.
   Solution:
   TRUE, because we know that $(A = B) \leftrightarrow (A \subseteq B \wedge B \subseteq A)$, so if $A = B$ then $A \subseteq B$.

   (d) If $A = B$ then $B \subset A$.
   Solution:
   FALSE. $B \subset A$ is true only if $A \nsubseteq B$. However since $A = B$, then $A \subseteq B$. Thus, $B \not\subset A$.

   (e) If $A \subset B$ then $A \neq B$ and $B \neq \emptyset$.
   Solution:
   TRUE. If $A \subset B$, then we know $B \nsubseteq A$. Thus, it's impossible for A and B to be equal (since there exists element in B that is not in A). Additionally, $B \neq \emptyset$ because otherwise B would be a subset of A since empty sets are a subset of all sets.

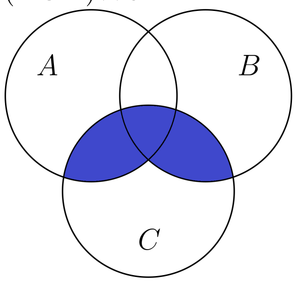
   (f) If $A \subseteq B$ then $A \cup B = B$ and $A \cap B = A$.
   Solution:
   TRUE. If $A \subseteq B$, then $A \cup B$ is simply B since all A is included in B, and $A \cap B = A$ since all and only all elements in A intersects with B.

   (g) If $A \cap (B - A) = \emptyset$ then $A = \emptyset$.
   Solution:
   FALSE. $A \cap (B - A) = \emptyset$ is true for any A and B, since it is taking the intersection of A and B AFTER any intersection with A was removed from B, thus resulting in no intersection regardless of elements in the set.

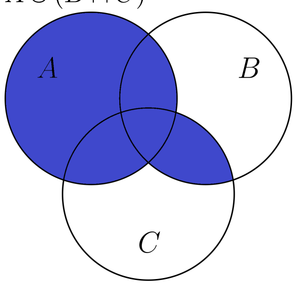
   (h) If $A - (B - A) = \emptyset$ then $A = \emptyset$.
   Solution:
   TRUE. This statement simplifies to A if $A \neq \emptyset$, since taking the difference of $(B - A)$ from A does nothing since any intersection between B and A has already been removed beforehand.
   As such, $A - (B - A) = A = \emptyset$ if and only if $A = \emptyset$.

3. (10pts) Using the provided template, draw a Venn Diagram to represent the following sets:

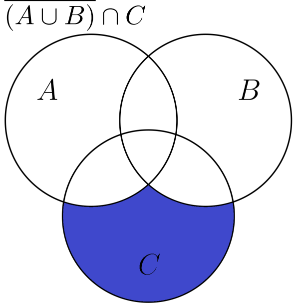
   (a) $(A \cup B) \cap C$



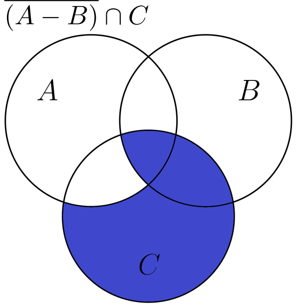
   (b) $A \cup (B \cap C)$



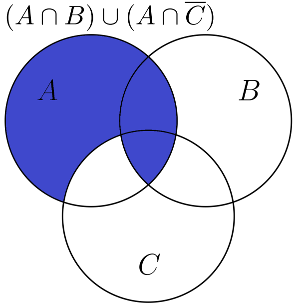
   (c) $\overline{(A \cup B)} \cap C$



   (d) $\overline{(A - B)} \cap C$



   (e) $(A \cap B) \cup (A \cap \overline{C})$



4. (18pts) Let A, B and C be sets. Determine the validity of each statement below. Justify your answer using set identities or membership tables. If two sets are not equivalent, provide a counterexample. A sample solution is provided for part (a).

   (a) $A \cap (B - C) = (A - C) \cap B$
   Solution: The equation is TRUE. We will show this using set identities.

$$\begin{aligned} A \cap (B - C) &= A \cap (B \cap \overline{C}) && \text{Difference Equivalence} \\ &= (A \cap \overline{C}) \cap B && \text{Commutative and Associative Laws} \\ &= (A - C) \cap B && \text{Difference Equivalence} \end{aligned}$$

   (b) $(A - \overline{B}) \cup (B - \overline{A}) = (A \cap B)$
   Solution: The equation is TRUE. We will show this using set identities.

$$\begin{aligned} (A - \overline{B}) \cup (B - \overline{A}) &= (A \cap B) \cup (B \cap A) && \text{Difference Equivalence} \\ &= (A \cap B) \cup (A \cap B) && \text{Commutative Law} \\ &= A \cap B && \text{Idempotent Law} \end{aligned}$$

   (c) $\overline{(A - B) \cup (B - A)} = \overline{A} \cup B$
   Solution: The equation is FALSE. We will show this by using a counterexample to find an element that is in one set but not the other.

$$\begin{aligned} \overline{(A - B) \cup (B - A)} &= \overline{(A \cap \overline{B}) \cup (B \cap \overline{A})} && \text{Difference Equivalence} \\ &= \overline{(A \cap \overline{B})} \cap \overline{(\overline{A} \cap B)} && \text{De Morgan's Law, Commutative Law} \\ &= (\overline{A} \cup B) \cap (A \cup \overline{B}) && \text{De Morgan's Law} \end{aligned}$$

   let x be an element; $x \in B$
   then $x \in \overline{A} \cup B$ [def'n of $\cup$]
   since $x \in \overline{A}$, then $x \notin A$ [def'n of complement]
   since $x \in B$, then $x \notin \overline{B}$ [def'n of complement]

   $\therefore x \notin (\overline{A} \cup B) \cap (A \cup \overline{B})$, because $x \notin A \cap \overline{B}$ [def'n of $\cap$]

   (d) $((A - B) \cup (A \cap B)) \cap (\overline{(B \cap \overline{B})} - A) = \emptyset$
   Solution: The equation is TRUE. We will show this using set identities.

$$\begin{aligned} ((A - B) \cup (A \cap B)) \cap (\overline{(B \cap \overline{B})} - A) &= ((A \cap \overline{B}) \cup (A \cap B)) \cap ((\overline{B} \cup B) \cap \overline{A}) && \text{De Morgan's Laws} \\ &= (A \cap (\overline{B} \cup B)) \cap ((\overline{B} \cup B) \cap \overline{A}) && \text{Distributive Laws} \\ &= (\overline{B} \cup B) \cap (A \cap \overline{A}) && \text{Distributive Laws} \\ &= U \cap \emptyset && \text{Complement Laws} \\ &= \emptyset && \text{Domination Law} \end{aligned}$$

(e) $A - (B - C) = (A - B) - C$
Solution: The equation is FALSE. We will show this by using a counterexample to find an element that is in one set but not the other.
Left side:

$$\begin{aligned} A - (B - C) &= A \cap \overline{(B \cap \overline{C})} && \text{Difference Equivalence} \\ &= A \cap (\overline{B} \cup C) && \text{Morgan's Law} \end{aligned}$$

Right side:

$$\begin{aligned} (A - B) - C &= (A \cap \overline{B}) \cap \overline{C} && \text{Difference Equivalence} \\ &= A \cap \overline{B} \cap \overline{C} && \text{Associative Law} \end{aligned}$$

let x be an element; $x \in A, B, C$
then $x \in A \cap (\overline{B} \cup C)$ [def'n of $\cup$, def'n of $\cap$]
since $x \in B$, then $x \notin \overline{B}$ [def'n of complement]
since $x \in C$, then $x \notin \overline{C}$ [def'n of complement]

$\therefore x \notin A \cap \overline{B} \cap \overline{C}$, because $x \notin \overline{B} \cap \overline{C}$ [def'n of $\cap$]

(f) $(B \cup C) - (A \cup \overline{C}) = \overline{A} \cap C = \overline{A} \cap (A \cup C)$
Solution: The equation is TRUE. We will show this using set identities.
Let $L.S. = (B \cup C) - (A \cup \overline{C})$
Let $R.S. = \overline{A} \cap (A \cup C)$
Let $M.S. = \overline{A} \cap C$

L.S.:

$$\begin{aligned} (B \cup C) - (A \cup \overline{C}) &= (B \cup C) \cap \overline{(A \cup \overline{C})} && \text{Difference Equivalence} \\ &= (B \cup C) \cap (\overline{A} \cap C) && \text{De Morgan's Law} \\ &= \overline{A} \cap (C \cap (B \cup C)) && \text{Associative Law, Commutative Law} \\ &= \overline{A} \cap C && \text{Absorption Law} \end{aligned}$$

R.S.:

$$\begin{aligned} \overline{A} \cap (A \cup C) &= (\overline{A} \cap A) \cup (\overline{A} \cap C) && \text{Distributive Law} \\ &= \emptyset \cup (\overline{A} \cap C) && \text{Complement Law} \\ &= \overline{A} \cap C && \text{Identity Law} \end{aligned}$$

$\therefore L.S. = M.S. = R.S. = \overline{A} \cap C$; The equation is true.

(g) $(A \cup \overline{C}) - (B \cup C) = A \cap B \cap \overline{C}$

Solution: The equation is FALSE. We will show this by using a counterexample to find an element that is in one set but not the other.

Left side:

$$\begin{aligned}
(A \cup \overline{C}) - (B \cup C) &= (A \cup \overline{C}) \cap \overline{(B \cup C)} && \text{Difference Equivalence} \\
&= (A \cup \overline{C}) \cap (\overline{B} \cap \overline{C}) && \text{Morgan's Law} \\
&= \overline{B} \cap \overline{C} \cap (A \cup \overline{C}) && \text{Associative Law, Commutative Law} \\
&= \overline{B} \cap \overline{C} && \text{Absorption Law}
\end{aligned}$$

let x be an element; $x \in (A - (B \cup C))$
then $x \in \overline{B} \cap \overline{C}$ [def'n of $\cap$]
since $x \in \overline{B}$, then $x \notin B$ [def'n of complement]
since $x \in \overline{C}$, then $x \notin C$ [def'n of complement]

$\therefore x \notin A \cap B \cap \overline{C}$, because $x \notin B$ [def'n of $\cap$]

5. (10pts) Determine whether f is a function or not. Justify your answer. Recall, that $\mathbb{R}$ is the set of all real numbers, $\mathbb{Z}$ is the set of all integers.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{3-x}$.
   Solution: NO, because $f(3) = \frac{1}{0} \notin \mathbb{R}$.

   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \pm\sqrt{x^2+5}$.
   Solution: NO, because every element in the domain has up to 2 images.
   E.g. $f(0) = \begin{cases} \sqrt{5} \\ -\sqrt{5} \end{cases}$.

   (c) $f : \mathbb{Z} \to \mathbb{R}$, $f(x) = \sqrt{x^2+8}$.
   Solution: YES, because every element in the domain maps to one element in the codomain. $\sqrt{x^2+8}$ is always a real number for any integer input x.

   (d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{x+1}$.
   Solution: NO, because there exists elements in the domain that do not map to an element in the codomain. E.g. $f(-2) = \sqrt{-1} \notin \mathbb{R}$

   (e) $f : \mathbb{R} \to \mathbb{Z}$, $f(x) = \lceil x \rceil$. Note that f is the ceiling function.
   Solution: YES, because every element in the domain maps to one element in the codomain. $\lceil x \rceil$ is always an integer for any real number input x.

6. (21pts) For each of the following functions $f : \mathbb{R} \to \mathbb{R}$ prove or disprove the following:
   - The function is injective (one-to-one).
   - The function is surjective (onto).
   - The function is bijective (both injective and surjective). If the function is a bijection, find its inverse.

   (a) $f(x) = \lfloor 3x \rfloor$. Note that $\lfloor x \rfloor$ is the **floor** function.
   Solution: This function is injective, not surjective, not bijective.

   Injective?
   Assume that $f(x_1) = f(x_2)$. Show that $x_1 = x_2$.

$$\begin{aligned} \lfloor 3x_1 \rfloor &= \lfloor 3x_2 \rfloor \\ 3x_1 &= 3x_2 \\ x_1 &= x_2 \end{aligned}$$

   $\therefore f(x)$ is injective.

   Surjective?
   Let $y = 1.4$
   There is no $x \in \mathbb{R}$ such that $f(x) = \lfloor 3x \rfloor = 1.4 = y$, because $\lfloor 3x \rfloor \in \mathbb{Z}$ and $y \notin \mathbb{Z}$.
   $\therefore f(x)$ is NOT surjective.

   Bijective?
   $f(x)$ is not surjective.
   $\therefore f(x)$ is NOT bijective.

   (b) $f(x) = 8 - 3x$
   Solution: This function is injective, surjective, bijective.

   Injective?
   Assume that $f(x_1) = f(x_2)$. Show that $x_1 = x_2$.

$$\begin{aligned} 8 - 3x_1 &= 8 - 3x_2 \\ -3x_1 &= -3x_2 \\ x_1 &= x_2 \end{aligned}$$

   $\therefore f(x)$ is injective.

   Surjective?
   Let y be an arbitrary image.

$$\begin{aligned} y &= 8 - 3x \\ y - 8 &= -3x \\ \frac{y-8}{-3} &= x \end{aligned}$$

   For any $y \in \mathbb{R}$: $\frac{y-8}{-3} \in \mathbb{R}$
   We found an element $x = \frac{y-8}{-3} \in \mathbb{R}$ such that $f(x) = f(\frac{y-8}{-3}) = 8 - 3(\frac{y-8}{-3}) = y$
   $\therefore f(x)$ is surjective.

Bijective?
$f(x)$ is injective and surjective.
$\therefore f(x)$ is bijective.
Inverse:
$f^{-1}(y) = \frac{y-8}{-3}$

(c) $f(x) = |8 - 3x|$
Solution: This function is not injective, not surjective, not bijective.

Injective?
Assume that $f(x_1) = f(x_2)$. Show that $x_1 = x_2$.

$$\begin{aligned}
|8 - 3x_1| &= |8 - 3x_2| \\
8 - 3x_1 &= \pm(8 - 3x_2) \\
-3x_1 &= \pm(8 - 3x_2) - 8 \\
x_1 &= \frac{\pm(8 - 3x_2) - 8}{-3} \\
x_1 &= \begin{cases} \frac{(8-3x_2)-8}{-3} = x_2 \\ \frac{-(8-3x_2)-8}{-3} = \frac{16}{3} - x_2 & x_1 \neq x_2 \end{cases}
\end{aligned}$$

$\therefore f(x)$ is NOT injective.

Surjective?
Let $y = -1$
There is no $x \in \mathbb{R}$ such that $f(x) = |8 - 3x| = -1 = y$, because $|8 - 3x|$ does not have images in codomain $\mathbb{R} < 0$.
$\therefore f(x)$ is NOT surjective.

Bijective?
$f(x)$ is not injective or surjective.
$\therefore f(x)$ is NOT bijective.

(d) $f(x) = 4x^3 + 5$
Solution: This function is injective, surjective, bijective.

Injective?
Assume that $f(x_1) = f(x_2)$. Show that $x_1 = x_2$.

$$\begin{aligned}
4{x_1}^3 + 5 &= 4{x_2}^3 + 5 \\
4{x_1}^3 &= 4{x_2}^3 \\
{x_1}^3 &= {x_2}^3 \\
x_1 &= x_2
\end{aligned}$$

$\therefore f(x)$ is injective.

Surjective?

Let y be an arbitrary image.

$$\begin{aligned} y &= 4x^3 + 5 \\ y - 5 &= 4x^3 \\ \frac{y-5}{4} &= x^3 \\ \sqrt[3]{\frac{y-5}{4}} &= x \end{aligned}$$

For any $y \in \mathbb{R}$: $\sqrt[3]{\frac{y-5}{4}} \in \mathbb{R}$
We found an element $x = \sqrt[3]{\frac{y-5}{4}} \in \mathbb{R}$ such that $f(x) = f(\sqrt[3]{\frac{y-5}{4}}) = 4(\sqrt[3]{\frac{y-5}{4}})^3 + 5 = y$
$\therefore f(x)$ is surjective.

Bijective?
$f(x)$ is injective and surjective.
$\therefore f(x)$ is bijective.
Inverse:
$f^{-1}(y) = \sqrt[3]{\frac{y-5}{4}} = x$

(e) $f(x) = 2x^2 - 3$
Solution: This function is not injective, not surjective, not bijective.

Injective?
Assume that $f(x_1) = f(x_2)$. Show that $x_1 = x_2$.

$$\begin{aligned} 2{x_1}^2 - 3 &= 2{x_2}^2 - 3 \\ 2{x_1}^2 &= 2{x_2}^2 \\ {x_1}^2 &= {x_2}^2 \\ x_1 &= \pm\sqrt{{x_2}^2} \\ x_1 &= \pm x_2 \\ x_1 &= \begin{cases} x_2 \\ -x_2 & x_1 \neq x_2 \end{cases} \end{aligned}$$

$\therefore f(x)$ is NOT injective.

Surjective?
Let $y = -4$
There is no $x \in \mathbb{R}$ such that $f(x) = 2x^2 - 3 = -4 = y$, because $2x^2 - 3$ does not have images in codomain $\mathbb{R} < -3$.
$\therefore f(x)$ is NOT surjective.

Bijective?
$f(x)$ is not injective or surjective.
$\therefore f(x)$ is NOT bijective.

7. (9pts) Let f and g both be functions from real numbers to real numbers. Let $f(x) = 3x^2 - 8$ and $g(x) = x - 2$. Define each of the following, simplify, and then evaluate. You need to show your work.

   (a) $(f \circ g)(x = -1)$
   Solution:

$$\begin{aligned}
(f \circ g)(x = -1) &= f(g(x)) \\
&= 3(x-2)^2 - 8 \\
&= 3((-1)-2)^2 - 8 \\
&= 3(-3)^2 - 8 \\
&= 3(9) - 8 \\
&= 27 - 8 \\
&= 19
\end{aligned}$$

   (b) $(g \circ f)(x = 2)$
   Solution:

$$\begin{aligned}
(g \circ f)(x = 2) &= g(f(x)) \\
&= (3x^2 - 8) - 2 \\
&= (3(2)^2 - 8) - 2 \\
&= (3(4) - 8) - 2 \\
&= (4) - 2 \\
&= 2
\end{aligned}$$

   (c) $((g \circ f) \circ g)(x = 1)$ Reuse solution to part (b).
   Solution:

$$\begin{aligned}
(g \circ f) &= (3x^2 - 8) - 2 && \text{from (b)} \\
((g \circ f) \circ g)(x = 1) &= (3(g(x))^2 - 8) - 2 \\
&= (3(x-2)^2 - 8) - 2 \\
&= 3(x-2)^2 - 10 \\
&= 3((1)-2)^2 - 10 \\
&= 3(-1)^2 - 10 \\
&= 3(1) - 10 \\
&= -7
\end{aligned}$$

8. (8 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Prove or disprove the followings:

   (a) if f and g are both bijections, then $f(x) + g(x)$ is also a bijection.
   Solution: FALSE. We will disprove using a counterexample:
   let f(x)=x; f is a bijection
   let g(x)=-x; g is a bijection

$$\begin{aligned} f(x) + g(x) &= x + (-x) \\ &= 0 \end{aligned}$$

   Let $y = 1$
   There is no $x \in \mathbb{R}$ such that $f(x) + g(x) = x + (-x) = 1 = y$, because $x + (-x) = 0$; it does not have images in codomain $\mathbb{R} \neq 0$.
   $\therefore f(x) + g(x)$ is NOT surjective.
   $\therefore f(x) + g(x)$ is NOT bijective.

   (b) if f is a bijection, then for any real number $c \neq 0$, $c \cdot f(x)$ is also a bijection.
   Solution: TRUE. $c \cdot f(x)$ is bijective.

   Injective?
   Assume that $c \cdot f(x_1) = c \cdot f(x_2)$, $c \neq 0$. Show that $x_1 = x_2$.

$$\begin{aligned} c \cdot f(x_1) &= c \cdot f(x_2) \\ f(x_1) &= f(x_2) \\ x_1 &= x_2 && f(x) \text{ has 1 and only 1 preimage for all } f(x) \in \mathbb{R} \end{aligned}$$

   $\therefore c \cdot f(x)$ is injective.

   Surjective?
   Let y be an arbitrary image.

$$\begin{aligned} y &= c \cdot f(x) \\ \frac{y}{c} &= f(x) \\ f^{-1}(\frac{y}{c}) &= x \end{aligned}$$

   For any $y \in \mathbb{R}$: $f^{-1}(\frac{y}{c}) \in \mathbb{R}, c \neq 0$
   We found an element $x = f^{-1}(\frac{y}{c}) \in \mathbb{R}$ such that $c \cdot f(x) = c \cdot f(f^{-1}(\frac{y}{c})) = c \cdot f(f^{-1}(\frac{y}{c})) = y$
   $\therefore c \cdot f(x)$ is surjective.

   Bijective?
   $c \cdot f(x)$ is injective and surjective.
   $\therefore c \cdot f(x)$ is bijective.