Gauge Theory of Shallow Water

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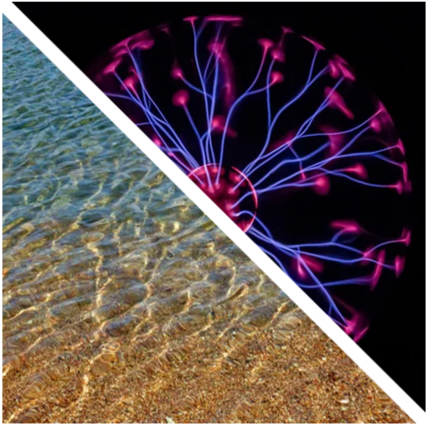
"The Largest and The Smallest"
SJTU International Summer School on Physics

July 17, 2025

Outline

Introduction



In the following presentation, we will show striking similarities between shallow water currents and E&M fields, and cover how some of the conclusions of gauge theory are naturally derived into ocean physics, such as the physical relation beneath chiral mode of Kelvin wave and global ocean current patterns.

Shallow Water Field Equations

The field equations of shallow water are:

$$\frac{\mathrm{D}h}{\mathrm{D}t} = -h\nabla \cdot \vec{u}, \quad \frac{\mathrm{D}u_i}{\mathrm{D}t} = f\epsilon_{ij}u_j - g\frac{\partial h}{\partial x_i} \tag{1}$$

where $h(x, y, t)$ is the height of fluid and $\vec{u}(x, y, t)$ the horizontal velocity, g the gravitational acceleration, f the Coriolis parameter.

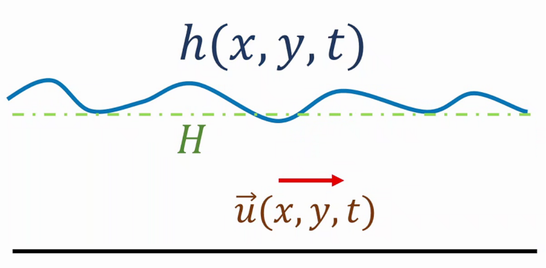


Figure 1: Variables of shallow water

Conserved Currents

It is obvious that there are two conserved currents [Tong, 2023]: the first is about conservation of mass of the fluid.

$$J_0 = h, \quad J_i = hu_i \tag{2}$$

which leads to the first part of Eq.1:

$$0 = \partial_\mu J_\mu = \frac{\partial h}{\partial t} + \nabla \cdot (h\vec{u}) = \frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla \cdot \vec{u} \tag{3}$$

Conserved Current

The second is the conservation of vorticity.

Definition

The vorticity of shallow water is given by:

$$\zeta = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \tag{4}$$

Thus we can define the conserved current:

$$\tilde{J}_0 = \zeta + f, \quad \tilde{J}_i = (\zeta + f)u_i \tag{5}$$

and correspondingly,

$$0 = \cdots = \partial_\mu \tilde{J}_\mu \tag{6}$$

The Action of Shallow Water Field

As the shallow-water system can be easily defined as a vector field, it is natural to draw an analogy between the shallow-water field and the 2+1d electromagnetic field.

Definition

The 'magnetic' field and 'electric' field are given by:

$$B = h, \quad E_i = \epsilon_{ij} h u_j \tag{7}$$

We thus obtain the Lagrange density (without vorticity):

$$\mathcal{L}_0 = \frac{1}{2}h|u|^2 - \frac{1}{2}gh^2 = \frac{|E|^2}{2B} - \frac{1}{2}gB^2 \tag{8}$$

Then, to include vorticity, we introduce two supplimentary scalar fields α and β , and rewrite $\tilde{J}_\mu$ as:

$$\tilde{J}_\mu = -\epsilon_{\mu\nu\rho}\partial_\nu \beta \partial_\rho \alpha \tag{9}$$

The Action of Shallow Water Field

We construct the vorticity term:

$$\mathcal{L}_{\mathrm{V}} = -A_\mu \tilde{J}_\mu = -\epsilon_{\mu\nu\rho} A_\mu \partial_\nu \beta \partial_\rho \alpha \tag{10}$$

At the same time, we introduce Coriolis parameter f and couple it with A_0:

$$\mathcal{L}_{\mathrm{C}} = f A_0 \tag{11}$$

Ultimately, the action of shallow water field is:

$$S = \int \mathrm{d}t\mathrm{d}^2x \Big(\frac{|E|^2}{2B} - \frac{1}{2} g B^2 + f A_0 - \epsilon_{\mu\nu\rho} A_\mu \partial_\nu \beta \partial_\rho \alpha \Big) \tag{12}$$

and corresponding Lagrange density is:

$$\mathcal{L} = \frac{|E|^2}{2B} - \frac{1}{2} g B^2 + f A_0 - \epsilon_{\mu\nu\rho} A_\mu \partial_\nu \beta \partial_\rho \alpha \tag{13}$$

E.O.M. of Shallow Water Field

The term $\epsilon_{\mu\nu\rho} A_\mu \partial_\nu \beta \partial_\rho \alpha$ is also called the Chern-Simons term. By introducing $\tilde{A}_\mu$, we can rewrite it as:

$$\epsilon_{\mu\nu\rho} A_\mu \partial_\nu \tilde{A}_\rho \quad \text{with} \quad \tilde{A}_\mu = \partial_\mu \chi + \beta \partial_\mu \alpha \tag{14}$$

Substituting Eq.13 to the Euler-Lagrange equations, we obtain that:

1. For α and β:

$$\epsilon_{\mu\nu\rho} F_{\mu\nu} \partial_\rho \beta = 0, \quad \epsilon_{\mu\nu\rho} F_{\mu\nu} \partial_\rho \alpha = 0 \tag{15}$$

2. For A_0:

$$\frac{\partial}{\partial x_i}\left(\frac{E_i}{B}\right) + f = \epsilon_{ij} \partial_i \beta \partial_j \alpha = \zeta + f \tag{16}$$

3. For A_i:

$$\frac{\partial}{\partial t}\left(\frac{E_i}{B}\right) + \frac{1}{2}\epsilon_{ij}\partial_j\left(\frac{|E|^2}{B^2}\right) + g\epsilon_{ij}\partial_j B - \epsilon_{ij}(\dot{\beta}\partial_j \alpha - \dot{\alpha}\partial_j \beta) = 0 \tag{17}$$

Linearized Gauge Theory

We linearize the field as below, where $H = \text{Const.}$ and $\eta \ll H$:

$$B = H + \eta, \quad E_i = \epsilon_{ij} h u_j \approx \epsilon_{ij} H u_j \tag{18}$$

and rewrite the 2 scalars α, β in terms of a new dummy gauge field:

$$\begin{aligned} \tilde{A}_\mu &:= \partial_\mu \chi + \beta\, \partial_\mu \alpha, & \tilde{E}_i &:= \tilde{F}_{0i}, \\ \tilde{F}_{\mu\nu} &:= \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu, & \epsilon_{ij} \tilde{B} &:= \tilde{F}_{ij}. \end{aligned} \tag{19}$$

Note that the potential vorticity Q is time-independent:

$$Q := H\zeta - f\eta, \quad \partial_t Q = 0 \tag{20}$$

Linearization and Maxwell-Chern-Simons Action

If we further assume that $Q = 0$ and exploiting the E.O.M of A_0 (Eq.16), then $\tilde{A}_\mu = fA_\mu/H$. Therefore the action Eq.12 turns into:

Maxwell-Chern-Simons Action

$$S_{\text{M-CS}} = \int_\Sigma \mathrm{d}t\mathrm{d}^2x \frac{1}{2H}\left(F_{\mu\nu}F^{\mu\nu} - f\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\right) \tag{21}$$

There is an interesting episode where we found an error in D. Tong's paper. We cannot derive the action (using his method), though the result is correct.

Next, we turn to the Poincaré waves. Here, life is simpler if we work in the gauge $A_0 = 0$. We treat p in (3.9) as a Lagrange multiplier, imposing the equation (3.12) for q. The linearised action (3.9) then becomes

$$S = \int dt\, d^2x\; \frac{1}{2H}\left(\dot{A}_i^2 - c^2B^2 + f\epsilon^{ij}A_i\dot{A}_j\right) \tag{3.16}$$

where we have used the condition $\epsilon_{ij}\partial_i\hat{\beta}\,\partial_j\hat{\alpha} = f$. This should be accompanied by the Gauss' law constraint (3.10).

Figure 2: The error in D. Tong's paper, p.11

What is a Memory Effect

Definition

A **memory effect** is a permanent change in a physical observable caused by a transient wave or fluctuation [Sheikh-Jabbari et al., 2023].

Even after the wave has passed, the system retains a residual imprint:

$$\Delta\mathcal{O} = \mathcal{O}(t \to +\infty) - \mathcal{O}(t \to -\infty) \neq 0$$

In the **linearized shallow water system**, the memory field ΔA_i induces a permanent change in the circulation current:

$$\Delta\gamma_i = \frac{g}{f}\left(\frac{1}{2}\Delta A_i - \partial_i(\partial_j \Delta A_j)\right)$$

Relevant modes that contribute to the memory effect are typically long-wavelength and low-frequency:

$$\omega \sim \frac{1}{T}, \quad \lambda \sim \frac{v}{\omega} \sim R$$

Memory Effect from Probe Trajectories

Definition

A **probe memory effect** refers to a permanent shift in the position of a fluid particle due to the passage of a transient wave:

$$\Delta x_i \equiv \lim_{t \to +\infty} x_i(t) - x_i(-\infty) \neq 0 \tag{22}$$

The trajectory of a particle is determined by the local velocity field through:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = u_i(x(t), t)$$

The total displacement can be expanded perturbatively in terms of the velocity amplitude:

$$\Delta x_i = \Delta x_{\mathrm{E},i} + \Delta x_{\mathrm{S},i} + \mathcal{O}(u^3)$$

Stokes Drift as Fluid Memory

Definition

The **Stokes drift** is the second-order displacement caused by wave nonlinearity:

$$\Delta x_{\mathrm{S},i} = \int_{-T}^{+T} \mathrm{d}t \int_{-T}^{t} \mathrm{d}t' \, u_j(y,t') \, \partial_j u_i(y,t) \tag{23}$$

The full displacement of a fluid particle consists of the first-order drift and the Stokes drift:

$$\Delta x_i = \int_{-T}^{+T} u_i(y,t) \, \mathrm{d}t + \Delta x_{\mathrm{S},i} + \cdots$$

The first-order term is known as the Eulerian drift:

$$\Delta x_{\mathrm{E},i} = \int_{-T}^{+T} u_i(y,t) \, \mathrm{d}t$$

Maxwell-Chern-Simons Action

Definition

The Chern-Simons action on 3-fold $\mathcal{M} \in \mathrm{Mor}_0(\mathbf{Fold})$ is

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{tr}\left(A_\alpha \wedge \mathrm{d}A_\beta\right) \tag{24}$$

- The Maxwell-Chern-Simons action Eq.21 ($\Sigma = \mathbb{R}^2 \times \mathbb{R}^+$):

$$\begin{aligned} S_{\text{M-CS}} &= S_{\text{Maxwell}} + S_{\text{CS}} \\ &= \int_\Sigma \mathrm{d}t\mathrm{d}^2x \frac{1}{2H}\left(F_{\mu\nu}F^{\mu\nu} - f\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\right) \end{aligned}$$

- The Gauge transformation on $\partial\Sigma$ is given by $A_\mu \mapsto A_\mu + \partial_\mu\theta$ and

$$S_{\mathrm{CS}} \mapsto S_{\mathrm{CS}} + \int_{\partial\Sigma} \mathrm{d}t\mathrm{d}y\theta E_2$$

by restricting $\theta\big|_{\partial\Sigma} = 0$ the additional term vanishes [Tong, 2023].

Equation of Motion

By simply taking the variation at Σ with the boundary $x = 0$, we have:

$$\delta S_{\text{M-CS}} = \delta S_{\text{bulk}} + \frac{1}{2H}\int_{\partial\Sigma} \mathrm{d}t\mathrm{d}^2x[-(2E_1 - fA_2)\delta A_0 - (2c^2B - fA_0)\delta A_2]$$

$$\begin{aligned}\delta S_{\text{bulk}} &= \int_{\Sigma} \mathrm{d}t\mathrm{d}^2x\frac{1}{H}\delta A_0\left(-\partial_0 E_i - c^2\epsilon_{ij}\partial_j B + f\epsilon_{ij}E_j\right)\\ &\quad + \int_{\Sigma} \mathrm{d}t\mathrm{d}^2x\frac{1}{H}\delta A_i(-\partial_i E_i + fB)\end{aligned}$$

The bulk variation gives the equation of motion

$$\partial_0 E_i = -c^2\epsilon_{ij}\partial_j B + f\epsilon_{ij}E_j \tag{25}$$

$$\partial_i E_i = fB \tag{26}$$

Coastal Kelvin Wave

With the boundary condition of $A_0 = A_2 = 0$ and considering the single mode $A_1 = A(x)\mathrm{e}^{\mathrm{i}(\omega t - ky)}$, the equation of motion Eq.25 and 26 gives:

$$\partial_0 E_i = -c^2 \partial_2 B \implies \omega = \pm ck$$

$$\partial_1 E_1 = fB \implies \omega \partial_1 A = kfA \implies A' = \pm \frac{f}{c} A$$

For $f > 0$, corresponding to the Northern hemisphere, the restriction $x > 0$ gives the convergent solution only if $\omega = -ck$, which means

$$A \sim \mathrm{e}^{-fx/c} \tag{27}$$

This is the chiral coastal Kelvin wave, first found in [Thomson, 1880].

Dynamics in Frequency Space

In the section, take $\mathbf{k} = (k_1, k_2)$ and $k = \|\mathbf{k}\|$, with gauge choice $A_0 = 0$, that is, $E_i = \mathrm{i}\omega A_i$ and $B = \mathrm{i}\epsilon_{ij} k_i A_j$ (take $\omega > 0$, the negative mode will automatically give the conjugate). The dynamics in frequency space gives

$$-\mathrm{i}\omega E_i = -c^2 \epsilon_{ij} \partial_j B + f \epsilon_{ij} E_j$$
$$\mathrm{i}\omega k_i A_i = \mathrm{i}\epsilon_{ij} f \omega A_j$$

The polerzation degree of freedom can be summerize as follows:

- The dynamic equations above reduce a degree of freedom
- Defined $\mathcal{V}_{\mathbf{k}} := \{\mathbf{A} \in \mathbb{C}^2 : \text{EOM, Dymanics Constraint}\}$ and $\mathcal{G}_{\mathbf{k}} := \{\mathbf{A} = i\mathbf{k}\lambda, \lambda \in \mathbb{C}\}$. The polorization state can be reduced to the fiber $\mathcal{H}_{\mathbf{k}} = \mathcal{V}_{\mathbf{k}} / \mathcal{G}_{\mathbf{k}} \cong \mathbb{C}$ with the base manifold $\mathbb{R}^2 \backslash \{0\}$. The first Chern number calculated for this complex line bundle is the topological index we want.

Berry Phase in This Problem

To construct the polarization, take $\hat{\mathbf{k}} = \frac{1}{k}(k_1, k_2)$ and $\hat{\mathbf{k}}_\perp = \frac{1}{k}(-k_1, k_2)$. The orthogonal decomposition gives $\mathbf{A} = A_L\hat{\mathbf{k}} + A_T\hat{\mathbf{k}}_\perp$. With some simple calculation, the polarization vector reads:

$$\mathbf{e}(\mathbf{k}) = A_T \mathrm{e}^{\mathrm{i}\varphi(\mathbf{k})}\sqrt{1+\frac{f^2}{\omega^2}}, \quad \varphi(\mathbf{k}) = \arg(k_1 + \mathrm{i}k_2) \tag{28}$$

To show the topology transparently, the polarization is written as a two-component complex state

$$|u(\mathbf{k})\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1+\frac{f^2}{\omega^2}}e^{-i\varphi(\mathbf{k})/2} \\ \sqrt{1-\frac{f^2}{\omega^2}}e^{i\varphi(\mathbf{k})/2}\end{pmatrix} \tag{29}$$

It is sufficient to get the Berry connection $\mathcal{A}_i = \langle u(\mathbf{k})|\partial_{k_i} u(\mathbf{k})\rangle$ from this definition.

Compactification

The Berry curvature is given by (the dispersion relation given in Tong's paper is $\omega^2 = c^2k^2 + f^2$, [Tong, 2023].)

$$\mathcal{F}_{12} = \partial_{k_1}\mathcal{A}_2 - \partial_{k_2}\mathcal{A}_1 = \frac{c^2 f}{2(c^2k^2 + f^2)^{3/2}} \tag{30}$$

The space we are considering is not compact, a method is to consider a single point compactification to ensure $\mathbb{R}^2 \cup \{\infty\} \cong S^2$. However, with $\omega(\mathbf{k}) \sim ck$ in the asytotic case $k \to \infty$, the vector

$$\mathbf{n}(\mathbf{k}) = \frac{1}{\omega}(ck_1, ck_2, f) \to (\hat{k}_1, \hat{k}_2, 0)$$

which is obviously pathological. Thus, single-point compactification is illegal. To deal with this, we work on an effective low-energy theory, trusting only $k \lesssim k_\Lambda$, and $\mathbf{n}(\mathbf{k}) \to (0, 0, +1)$ asytotically.

Chern Number

With the compactness of the space, the Chern number is given by

$$\begin{aligned} C &= C_{\text{bare}} - C_{\text{ref}} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}k_1 \mathrm{d}k_2 \mathcal{F}_{12} - \frac{1}{2} = \frac{1}{2}\mathrm{sgn} f - \frac{1}{2} \\ &= \begin{cases} 0, & \text{in North Hemisphere} \\ -1, & \text{in South Hemisphere} \end{cases} \end{aligned} \tag{31}$$

Thus, a critical result is that the difference in Chern number $\Delta C = 1$.

Physical Result

In the northern hemisphere, when $f > 0$, the coast is at $x = 0$ and the ocean is at $x > 0$, and the Kelvin wave requires $\omega = -ck$ (phase velocity along $-y$, that is, clockwise around the land); in the southern hemisphere, when $f < 0$, $\omega = +ck$ (counterclockwise), as shown in figure Fig.3 in the next page.

Chern Number and Ocean Current

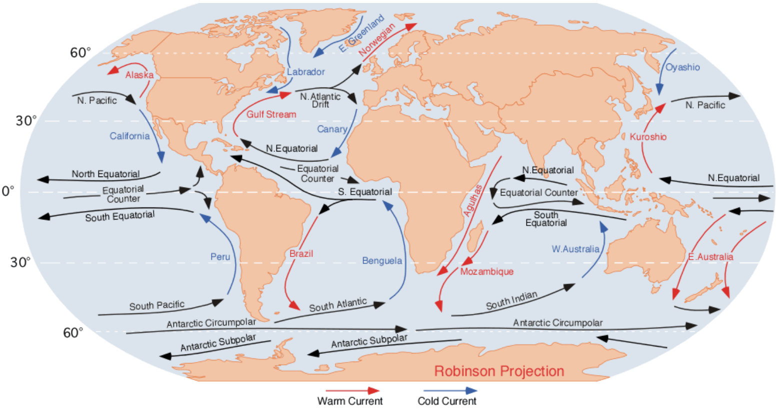


Figure 3: The ocean current direaction causes by the chiral mode difference. The figure is from the Wikipeida page of the ocean current [Wikipedia, 2025].

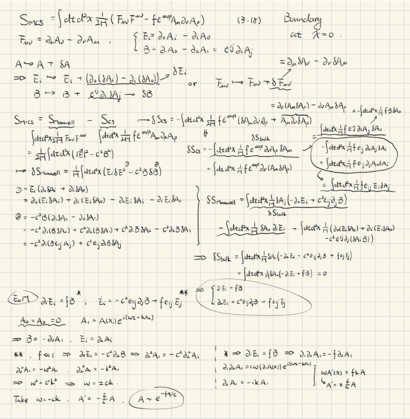
Conclusion

To summerize: We

- Introduced a gauge-theoretic formulation of the shallow-water system, analogous to a U(1) electromagnetic field.
- After linearization, showed a correspondence with Maxwell–Chern–Simons (MCS) dynamics.
- Identified and analyzed the memory effect predicted by the gauge formulation.
- Computed the Chern number using a topological method with the restriction described in Tong section 3.3

We hope this framework proves useful for future studies of fluid-dynamical phenomena.

Q&A



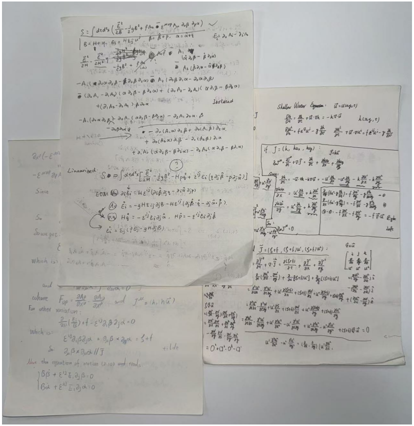


Figure 4: Handwritten derivations of the Maxwell-Chern-Simons theory, written by Siyuan Wei (R), Lunyi Xie (R), and Zeyu Zeng (L)

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THE END!

Thank you! :)