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Amid the Cloud-Sea

Notes on Geometry and Topology

从微分流形开始的异世界魔法笔记

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"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

John von Neumann

Introduction

Historical Survey and Scope

Modern geometry and topology can be read as a long dialogue between two complementary impulses: the urge to *forget* geometry and keep only qualitative structure, and the urge to *measure* geometry intrinsically. Euler's treatment of the bridges of Königsberg is a canonical example of the first impulse: one erases lengths and angles and keeps only connectivity, arriving at invariants of *geometric situation* (*geometria situs*) [16]. His later study of polyhedra isolates a purely combinatorial shadow of a surface in the integer

$$\chi = V - E + F,$$

the Euler characteristic [17]. Gauss represents the second impulse: his *Disquisitiones* insist that curvature is an intrinsic invariant of a surface, independent of how it sits in Euclidean space [19]. The conceptual synthesis is that local differential data can control global topological information, a philosophy epitomized by Gauss–Bonnet–Chern and Chern's intrinsic approach to curvature integrals [9].

Riemann's 1854 habilitation lecture pushes Gauss's intrinsic viewpoint from surfaces to abstract manifolds by introducing a smoothly varying inner product on tangent spaces [40]. In this step, "space" becomes a field of local data whose global behavior must be extracted by analysis and topology. Levi-Civita's connection formalism supplies the analytic mechanism: parallel transport and covariant differentiation make curvature computable and make variational questions on manifolds precise [29]. Cartan's moving frames and the theory of symmetric spaces then reorganize curvature and symmetry into differential forms on principal bundles [7, 8], preparing the modern language of connections, curvature, and characteristic invariants.

The first recognizably modern topology appears in Poincaré's *Analysis Situs* [35]. Fundamental groups, homology, and the homotopy-theoretic viewpoint formalize the idea that spaces should be probed by maps into and out of them. Hurewicz clarifies how homotopy and homology interact [25], while Hopf's study of maps between spheres shows how subtle global information can be detected by algebraic invariants (linking phenomena and early fibration ideas) [23]. A decisive analytic turn comes with de Rham's reinterpretation of cohomology in terms of differential forms [39]: topological invariants become computable by solving differential equations and integrating local data over global cycles.

At roughly the same time, the language that now underlies much of geometry and topology was forged. Leray's sheaf-theoretic and spectral-sequence viewpoint reorganizes cohomology as a local-to-global machine [28]. Eilenberg and Mac Lane's functorial perspective initiates category theory [14], later systematized by Mac Lane [31], while Eilenberg–Steenrod axiomatize homology and cohomology theories [15]. From this point onward, the boundaries between geometry, topology, and analysis are less mathematical than linguistic: the same constructions reappear as functors, natural transformations, and invariants extracted from local structure.

The mid-twentieth century also assembles the differential-topological apparatus that makes smooth manifolds workable objects. Whitney establishes the foundational theory of smooth manifolds and embeddings [49, 50], Sard explains why generic smooth maps behave well at the level of critical values [41], and Thom's transversality philosophy turns "genericity" into a systematic tool, stabilizing intersection theory and cobordism-style arguments [47]. Ehresmann's viewpoint on bundles and connections supplies a geometric backbone for the bundle calculus that permeates modern geometry [13].

Two additional themes shape the landscape of these notes. First, Riemannian geometry grows a global theory: Hopf–Rinow links metric completeness to geodesic completeness [24], Myers ties curvature bounds to global topological constraints [34], and comparison theorems of Rauch type connect Jacobi fields to rigidity phenomena [38]. Second, "global topology from local curvature" becomes a guiding principle: Chern–Weil theory shows how invariant polynomials applied to curvature forms produce cohomology classes [10], and the works of Pontryagin and Hirzebruch crystallize the idea that characteristic numbers govern deep classification problems in topology and complex geometry [22, 36, 37].

At the interface of analysis and topology lies Morse theory. Morse relates critical points of a function to the topology of its sublevel sets [33], and Milnor reframes the theory in the language of handle decompositions [32]. This viewpoint foreshadows infinite-dimensional analogues such as Floer's Morse theory for Lagrangian intersections [18], and it also feeds into high-dimensional manifold topology via Smale's use of transversality and handle calculus in the generalized Poincaré conjecture and related structure theorems [45, 46].

In four dimensions, elliptic operators and gauge-theoretic ideas fundamentally change the set of available invariants. Yang–Mills introduces non-abelian gauge fields [52]; Atiyah–Bott analyze the Yang–Mills functional via infinite-dimensional Morse theory [3]; Donaldson extracts smooth invariants from anti-self-dual moduli spaces [12]; and Seiberg–Witten theory provides a more flexible analytic package with striking topological consequences [43]. Parallel threads in complex and symplectic geometry—rooted in classical mechanics and complex analysis—are reshaped by Kodaira's foundations [26], Calabi's program [6], Yau's solution of the complex Monge–Ampère equation [53], Arnold's symplectic viewpoint [1], Gromov's pseudo-holomorphic curves [21], and Kontsevich's deformation quantization [27]. Beyond the smooth category, Gromov's metric viewpoint studies limits and rigidity [20], while Connes' noncommutative geometry replaces spaces by operator algebras and recovers topological information through cohomological and index-theoretic tools [11]. Topology then returns to physics through topological quantum field theory (TQFT): Atiyah frames TQFT as symmetric monoidal functors out of bordism categories [2], Witten's models connect path integrals to classical invariants [51], Segal refines the functorial perspective in the conformal setting [42], and the higher-categorical viewpoints of Baez–Dolan and Lurie place field theories within the modern landscape of higher categories and homotopy theory [4, 30].

These notes are organized to follow a corresponding conceptual arc:

- **Part I (Chapters 1–3): Smooth Manifold and Fundamental Structures.** Chapter 1 develops the basic language of topological and smooth manifolds, submanifolds, tangent/cotangent constructions (with induced linear maps), transversality, Lie groups, and manifolds with boundary. Chapter 2 collects core tools about smooth functions on manifolds: partitions of unity and their applications, critical points and Sard's theorem, Whitney embedding ideas, Morse-function preliminaries, and degree mod 2 considerations. Chapter 3 develops the calculus toolkit on manifolds: tangent bundles and vector fields, normal bundles and the tubular neighborhood theorem, vector fields and the Euler characteristic, distributions and integrability, tensors, differential forms and exterior algebra, orientation and integration, and Stokes' formula.

- **Part II (Chapters 4–7): A First Step to Geometrical Analysis on Manifolds.** Chapter 4 is a first course in Riemannian geometry from a geometric-analysis viewpoint: Riemannian manifolds as metric spaces, connections, geodesics and Jacobi fields, curvature, geometry of submanifolds, and homogeneous/symmetric spaces. Chapter 5 develops de Rham cohomology and Hodge theory in the Riemannian setting. Chapter 6 introduces comparison theory and "geometry in the large" via curvature bounds. Chapter 7 develops elliptic operators on manifolds (Sobolev spaces, analytic proofs of Hodge theory, spinors and Dirac operators, heat kernels, index-theoretic ideas) and ends with an introduction to Seiberg–Witten theory.

- **Part III (Chapters 8–12): Tools of Studying Global Topology.** Chapter 8 develops homotopy theory through fundamental groups, van Kampen, coverings, CW complexes, and higher homotopy groups. Chapter 9 develops (co)homology: singular theories, Mayer–Vietoris, cellular/axiomatic frameworks, cup products and naturality, universal coefficient and Künneth theorems, the de Rham theorem, Poincaré duality, Thom classes, and intersection numbers. Chapter 10 studies fibrations/cofibrations, obstruction theory, Eilenberg–MacLane spaces, the Leray–Serre spectral sequence and transgression, and classifying spaces/universal bundles. Chapter 11 develops connections and curvature on principal bundles, Chern–Weil theory and characteristic classes (including the stable classification viewpoint via topological K-theory), culminating in applications such as Gauss–Bonnet–Chern. Chapter 12 gives an entry point to Morse theory from the modern viewpoint.

- **Part IV (Chapters 13–20): Introduction to Further Topics.** Chapters 13–16 survey Riemann surfaces, sheaves and ringed-space viewpoints on smooth manifolds, a first look at schemes, and foundational complex geometry. Chapter 17 introduces noncommutative algebras as geometric objects. Chapters 18–19 begin symplectic geometry, culminating in the first appearance of J-holomorphic curves. Chapter 20 gives a brief overview of cobordisms and TQFT (cobordisms, Pontryagin–Thom, functorial axioms, and low-dimensional models).

- **Part V (Appendices A–E)** collects background in general topology, linear algebra, real/functional analysis, elliptic regularity, and categorical language.

The aim is not to reproduce the original sources cited above, but to carve out a coherent path through the landscape they opened: from Euler's combinatorial shadows and Gauss's intrinsic curvature to the homotopical, cohomological, and analytic mechanisms that now dominate geometry and topology. These notes are closer to a working document than a polished textbook: they record derivations, intuitions, and a modular toolkit meant to be revised and discussed.

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Finally, I thank all the authors of the textbooks and papers cited in the bibliography. These notes are, at best, a small and very incomplete reflection of the ideas developed in their work. Any errors, inaccuracies, or misunderstandings that remain are entirely my own responsibility.

Contents

Part I

Smooth Manifold and Fundamental Structures

Chapter 1

Smooth Manifolds and Submanifolds

1.1 Topological Manifold and Smooth Structure

The goal of Chapter 1 is to define the central object of modern geometry, the smooth manifold. To define a smooth manifold, we first need to study a more basic case called a topological manifold, which is a special type of topological space that locally resembles Euclidean space.

The key to defining a topological manifold is the following property: locally Euclidean.

Definition 1.1 (Locally Euclidean)**.** A topological space X is said to be locally Euclidean iff $\forall x \in X$, there is some open neighborhood $x \in U \subseteq X$, such that we can find a map $\varphi : U \to V \subseteq \mathbb{R}^n$ that is a homeomorphism.

Definition 1.2 (Topological Manifolds)**.** A topological manifold is an n-dimensional topological space M iff M is second countable, Hausdorff, and locally Euclidean to some fixed $\mathbb{R}^n$. A chart is a pair $(U_\alpha, \varphi_\alpha)$ that $U_\alpha \subseteq M$ is a open subset and $\varphi_\alpha : U_\alpha \to \phi(U_\alpha) \subset \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. The set of charts that can cover the entire manifold M is called an atlas.

From my previous experiences, it seems that it is often more important to have a taste of what is not a topological manifold, so here are some examples:

Example 1.1. *If we consider the set $S = (\mathbb{R} \setminus \{0\}) \cup \{A, B\}$, with the topology defined by the following laws:*

- *In $\mathbb{R} \setminus \{0\}$, take the subset topology inherit from $\mathbb{R}$.*
- *The open set that contains A (or B), for some $c, d \in \mathbb{R}_{\geq 0}$ consider the set $I_A(c, d) = (-c, 0) \cup (0, d) \cup \{A\}$ and the basis is given by $\mathcal{B} = \{I_A(c, d) : c, d \in \mathbb{R}_{\geq 0}\} \cup \{I_B(c, d) : c, d \in \mathbb{R}_{\geq 0}\} \cup \{(a, b) \subseteq \mathbb{R} : 0 \notin (a, b)\}$*

This set is locally Euclidean, second countable, but not Hausdorff.

Proof. It is quite obvious that the set is locally Euclidean, since if we consider $I \subseteq S$ such that $A, B \notin I$, then $I \subseteq \mathbb{R}$. For some open set containing A or B, we just take $(I \setminus \{A, B\}) \subseteq \mathbb{R}$ and take $A, B \mapsto 0$. The continuity of this map is obvious. In addition, the second countability of the topological space S is also easy to prove. However, consider any open neighborhood of A and B, $I_A(a, b)$ and $I_B(c, d)$, then there will always intersect. Let $\alpha := \max\{a, c\}$ and $\beta := \min\{b, d\}$, then $I_A(a, b) \cap I_B(c, d) = (\alpha, 0) \cup (0, \beta)$ always nonempty. Thus, points A and B are not disjoint, and the space S is not Hausdorff. $\square$

Example 1.2. *As a counterexample of a topological space that is locally Euclidean, Hausdorff, but not second-countable, consider the uncountable index set I and the topological space*

$$X = \coprod_{i \in I} S_i := \{(x, i) \mid x \in S^1, i \in I\}$$

with the coproduct topology. This space is not second-countable. Recall that the coproduct has a universal property that the following diagram commutes with $f : Y \to X_1 \sqcup X_2$ is unique:

$$\begin{tikzcd}
& Y & \\
X_1 \arrow[ur, "f_1"] \arrow[r, "i_1"'] & X_1 \sqcup X_2 \arrow[u, dashed, "\exists! f"'] & X_2 \arrow[ul, "f_2"'] \arrow[l, "i_2"]
\end{tikzcd}$$

and the coproduct topology is defined to be the coarsest topology such that i_1, i_2 are continuous and f is continuous if f_1, f_2 are continuous. Then, $U \in X$ is open if $i_k^{-1}(U)$ is open in S^1.

Proof. Suppose X has a topological basis $\exists \mathcal{B}$. For any $k \in I$, S^1_k denotes the i-th 1-sphere in the coproduct. Then, since $S^1_k \subseteq X$ such that $p_k \in B_k \subseteq S^1_k$. Notice that $\forall k \neq l : S^1_k \cap S^1_l = \emptyset$, thus, for any S^1_i, we can take B_i and each two of the B_i with different indices are disjoint. Then we can take the map $I \to \mathcal{B}$ sends each i to B_i, which means $|B| \geq |I|$ is uncountable. Thus X is not second countable. $\square$

The topological manifold only allows the continuity (C^0) to be defined, which is not enough for calculus. To define calculus on a smooth manifold, which is a generalization of smooth curves and surfaces in Euclidean space, we need to define a smooth structure.

Definition 1.3 (C^k-Structure)**.** A C^k-structure on a manifold is an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ for some indexes set I, and satisfies the following properties:

- $\{U_\alpha \mid \alpha \in I\}$ covers M, $\bigcup_\alpha U_\alpha = M$.
- For any $\alpha, \beta \in I$, $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.
- The collection $\mathcal{A}$ is maximal: $\forall \alpha \in I$, if charts (U, φ) and $(U_\alpha, \varphi_\alpha)$ are either $U \cap U_\alpha = \emptyset$, or $\varphi \circ \varphi_\alpha$ is a diffeomorphism, then $(U, \varphi) \in \mathcal{A}$, i.e., $\mathcal{A}$ does not been properly contained in any other C^k-atlas.

Remark. *An alternative way is that we defined the equivalence relation between C^k-atlas $\mathcal{A}_1$, $\mathcal{A}_2$ such that*

$$\mathcal{A}_1 \sim \mathcal{A}_2 \iff \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \text{ is still a } C^k\text{-atlas.}$$

The smooth structure can be represented by the equivalence class of C^k-atlas $[\mathcal{A}]$.

A smooth manifold $(M, \mathcal{O}_M, \mathcal{A})$ is a topological manifold $(M, \mathcal{O}_M)$ with a smooth structure $\mathcal{A}$ on it. However, a potential problem in constructing the smooth structure requires maximal, which is extremely hard to prove. To deal with this, we have the following proposition.

Proposition 1.1. *Every C^k-atlas is contained in a unique maximal C^k-atlas.*

Proof. Consider the topological manifold $(M, \mathcal{O}_M)$ and a smooth atlas $\mathcal{A}_1 = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. Then, we have to apply the following lemma

Lemma. *Given smoothly compatible charts (U_1, φ_1) and (U_2, φ_2), if a chart (V, ψ) is compatible with one of these two charts, then it is also compatible with the other one.*

Without loss of generality, we can assume the open sets U_1, U_2, and V are not disjoint. The proof of the lemma is simply to consider that the compatible condition implies that $\varphi_2 \circ \varphi_1^{-1}$ and $\varphi_1 \circ \varphi_2^{-1}$ are both smooth. Without loss of generality, consider (V, ψ) smoothly compatible with (U_1, φ_1), then check the transition maps

$$\varphi_2 \circ \psi^{-1} = \varphi_2 \circ (\varphi_1^{-1} \circ \varphi_1) \circ \psi^{-1} = (\varphi_2 \circ \varphi_1^{-1}) \circ (\varphi_1 \circ \psi^{-1})$$

Since both $\varphi_2 \circ \varphi_1^{-1}$, $\varphi_1 \circ \psi^{-1}$ are smooth, $\varphi_2 \circ \psi^{-1}$ is smooth. Same argument also apply for $\psi \circ \varphi_2^{-1}$. Thus, the lemma was proved.
Then, to prove the proposition, take $(U_\alpha, \varphi_\alpha) \in \mathcal{A}_1$, we can defined the atlas $\mathcal{A}$ such that

$$(V, \psi) \in \mathcal{A} \iff \exists (U_\alpha, \varphi_\alpha) \in \mathcal{A}_1 : (U_\alpha, \varphi_\alpha) \text{ smoothly compatible with } (V, \psi)$$

We claim that $\mathcal{A}$ is unique and is a maximum atlas. □

A corollary is that the equivalence relation of the atlas that has been used to define the C^k-structure can also be written as

$$\mathcal{A}_1 \sim \mathcal{A}_2 \iff \mathcal{A}_1, \mathcal{A}_2 \text{ is contained in the same maximum atlas } \mathcal{A}$$

and also, the proposition also shows that to prove the topological manifold M has a C^k-structure, it is enough to find a single smooth atlas $\mathcal{A}$ on M.

As examples of topological and smooth manifolds, consider the following sets:

Example 1.3. *As a topological manifold,* $\mathbb{R}^n$ *can be constructed by a single chart* $(\mathbb{R}^n, \mathrm{id})$*, and the smoothness is trivial. However, as a critical counterexample to the uniqueness of the smooth structure, we can take another chart* $(\mathbb{R}^n, \varphi)$*, where* $\varphi(u) = u^3$ *is indeed a diffeomorphism. The two smooth structures above are not compatible.*

Proof. Consider the map $\varphi \circ \mathrm{id} = \varphi$ is smooth. However, the inverse map $(\varphi \circ \mathrm{id})^{-1} = \mathrm{id}^{-1} \circ \varphi^{-1} = \varphi^{-1}$, which is defined by $\varphi^{-1}(v) = v^{1/3}$ is not globally smooth since $(\varphi^{-1})'(v) = v^{-2/3}/3$, which is undefined at the origin. □

Example 1.4 (Unite Circle S^1)**.** *A classical example of a smooth manifold is the unit circle* S^1*, take*

$$S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \cong \{\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in [0, 2\pi]\}$$

As a topological subset of $\mathbb{R}^2$*, consider the following chats:*

$$\begin{aligned} U_1 &= S^1 \setminus \{(1, 0)\} \cong \{\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in (0, 2\pi)\} \\ U_2 &= S^1 \setminus \{(-1, 0)\} \cong \{\mathrm{e}^{\mathrm{i}\eta} \mid \eta \in (\pi, 3\pi)\} \end{aligned}$$

Then, a defined homeomorphism

$$\varphi_1 : U_1 \to (0, 2\pi), \quad \varphi_2 : U_2 \to (\pi, 3\pi)$$

where φ_1 *and* φ_2 *are homeomorphism. The transformation map is given by*

$$\varphi_2 \circ \varphi_1^{-1} : (0, \pi) \cup (\pi, 2\pi) \to (\pi, 2\pi) \cup (2\pi, 3\pi)$$

which is defined by

$$\varphi_2 \circ \varphi_1^{-1}(\theta) = \begin{cases} \theta + 2\pi, & \theta \in (0, \pi) \\ \theta, & \theta \in (\pi, 2\pi) \end{cases}$$

is smooth, and also we can also check $\varphi_1 \circ \varphi_2^{-1}$ *in the same way. Thus,* $\mathcal{A} = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ *is a smooth structure, and* S^1 *is a smooth manifold.*

Another important example of a smooth manifold is the product manifold, which is given by the following proposition:

Proposition 1.2 (Product Manifold). *M and N are smooth manifolds with dimension m and n $\iff$ $M \times N$ is smooth manifold, and $\dim M \times N = m + n$.*

Proof. The Hausdorff and second countable properties are preserved in the Cartesian product, and the Hausdorff and second countable properties on a superset can be directly extended to a subset; the proof is in Appendix A.
($\Rightarrow$) If M and N are orientable C^∞-manifolds with dimension m and n, then we can take the orientable smooth structures $\mathcal{A}_M = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ and $\mathcal{A}_N = \{(V_\beta, \psi_\beta) \mid \beta \in J\}$. Consider the inherent smooth structure $\mathcal{A}_{M\times N} = \{(U \times V, \varphi) \mid U \in \mathcal{A}_M, V \in \mathcal{A}_N\}$ and the coordinate map is given by

$$\forall (p, q) \in M \times N : \varphi(p, q) = (\phi^1(p), \dots, \phi^m(p), \psi^1(q), \dots, \psi^n(q))$$

Since $\phi : M \to \mathbb{R}^m$ and $\psi : N \to \mathbb{R}^n$ are homomorphisms onto their image, the coordinate map given above on $M \times N$ is a homomorphism onto its image $\varphi(U \times V) = \phi(U) \times \psi(V) \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Then we have to check the smoothness of the manifold, and the transition map is given by

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x) = \left(\phi_\alpha \circ \phi_\beta^{-1}(x^1, \dots, x^m); \psi_\alpha \circ \psi_\beta^{-1}(x^{m+1}, \dots, x^{m+n})\right)$$

By the smoothness of ϕ and ψ, the transition map $\varphi_\alpha \circ \varphi_\beta^{-1}$ is smooth. Thus, $M \times N$ is a smooth manifold with dimension $m + n$.
($\Leftarrow$) In the similar way, $\varphi_\alpha \circ \varphi_\beta^{-1}(x) = (y^1(x), \dots, y^{m+n}(x)) \in C^\infty(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$ indicates that each of its component is a smooth function on $\mathbb{R}^{m+n}$, which leads to the smooth structure on M and N separately. □

The smooth manifold is the generalization of a well-behaved subset of $\mathbb{R}^n$, so it is natural to talk about the smoothness of the map.

Definition 1.4 (C^k-Map). For some smooth manifold M and N of dimension m and n, a continuous map $f : M \to N$ is said to be C^k iff the local coordinate representation of chart (U, φ) on M and (V, ψ) on N

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is C^k as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.

We shall check that the smoothness is well-defined, i.e., independent of the choice of charts in the given smooth structure

Proposition 1.3 (Well-Definedness of Smooth Map). *Given $f \in C^\infty(M, N)$, given smooth atlas $\mathcal{A}$ and $\mathcal{B}$ on M and N. If $\mathcal{A}_1$ and $\mathcal{B}_1$ compatible with $\mathcal{A}$ and $\mathcal{B}$, then f is smooth also under the smooth atlas $\mathcal{A}_1$, $\mathcal{B}_1$.*

The proof of this proposition is to simply notice that $\forall (U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{A}_1$ such that $U \cap V \neq \emptyset$, the compatibility implies that both $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are diffeomorphism. A similar argument also applies to $\mathcal{B}$.

Proposition 1.4 (Composition of C^k Maps). *The composition of finite C^k maps $f_1, \dots, f_n$ is still a C^k-map.*

Proof. Because continuity and differentiability are defined chart-wise and composition is a local operation, it suffices to consider maps between open subsets of Euclidean spaces. Let

$$f : V \subset \mathbb{R}^n \to \mathbb{R}^m, \qquad g : U \subset \mathbb{R}^p \to V$$

1. If f and g are continuous, then for any open set $O \subset \mathbb{R}^m$,

$$(f \circ g)^{-1}(O) = g^{-1}\big(f^{-1}(O)\big),$$

which is open because $f^{-1}(O)$ is open and g is continuous. Thus $f \circ g$ is continuous, i.e., lies in C^0.

2. Fix $k \geq 1$ and assume that if $f, g \in C^{k-1}$, then $f \circ g \in C^{k-1}$.
3. Since $f, g \in C^k$, they are C^1. For each $x \in U$,

$$D(f \circ g)(x) = Df\big(g(x)\big) \circ Dg(x)$$

The map $x \mapsto Df(g(x))$ is the composition of g (class C^k) with Df (class C^{k-1}), hence is C^{k-1} by the induction hypothesis. Likewise $x \mapsto Dg(x)$ is C^{k-1}. Since both Dg and Df are C^{k-1}, by the hypothesis, $D(f \circ g)(x) = D_{g(x)}f(g(x)) \circ D_x g(x)$ is a C^{k-1} map.

Therefore $f \circ g \in C^k$. Together with the base case, induction on k completes the proof. □

Definition 1.5 (Diffeomorphism)**.** A map $f : M \to N$ is said to be a diffeomorphism between smooth manifold M and N iff

1. f is bijective
2. f and f^{-1} are both smooth

Consider smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ on a n-dimensional manifold M, the coordinate transformation is given by $\varphi_\beta \circ \varphi_\alpha^{-1}(x) = (y^1, \ldots, y^m)$, where $x = (x^1, \ldots, x^n) \in \varphi_\alpha(U_\alpha \cap U_\beta)$, the Jacobi matrix is given by

$$J(\varphi_\beta \circ \varphi_\alpha^{-1}) = \left(\frac{\partial y^i}{\partial x^j}\right)_{1 \leq i,j \leq n}$$

And the Jacobian is

$$\det J(\varphi_\beta \circ \varphi_\alpha^{-1})(p) = \frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)}(\varphi_\alpha(p))$$

With the Jacobian, the orientability of a smooth manifold can be defined as follows.

Definition 1.6 (Orientability of Manifolds)**.** Let M be a smooth manifold with (at least C^1) cover $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$, M is orientable iff

$$\forall \alpha, \beta \in I : \big(U_\alpha \cap U_\beta \neq \emptyset \implies J(\varphi_\alpha \circ \varphi_\beta^{-1}) > 0\big)$$

And $\mathcal{A}$ is said to be an orientable cover. If orientable covers do not exist, then M is not orientable.

An obvious fact about the orientation is the following proposition:

Proposition 1.5. *Smooth manifolds M and N orientable $\iff$ smooth manifold $M \times N$ orientable.*

Proof. We defined the smooth structure on $M \times N$ to be induced by the smooth structure on M and N; the smooth structure and transition map are given by the following equations:

$$\mathcal{A}_{M \times N} = \{(U_\alpha \times V_\beta, \varphi_{(\alpha,\beta)} = (\phi_\alpha, \psi_\beta)) \mid (\alpha, \beta) \in I \times J\}$$
$$\varphi_{(\alpha_1,\beta_1)} \circ \varphi_{(\alpha_2,\beta_2)}^{-1}(x) = \big(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1}(x^1, \ldots, x^m), \psi_{\beta_1} \circ \psi_{\beta_2}^{-1}(x^{m+1}, \ldots, x^{m+n})\big)$$

And the Jacobi matrix is defined by

$$J\big(\varphi_{(\alpha_1,\beta_1)} \circ \varphi_{(\alpha_2,\beta_2)}^{-1}\big)(x) = \begin{pmatrix} \partial_j(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})^i(x) & \\ & \partial_j(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})^i(x) \end{pmatrix}$$

By the orientability of M and N, we can always choose the open cover to make the following Jacobian positive

$$\det J(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})(x) = \det \left(\frac{\partial(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})^i}{\partial x^j}(x) \right)_{m \times m} > 0$$
$$\det J(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})(x) = \det \left(\frac{\partial(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})^i}{\partial x^j}(x) \right)_{n \times n} > 0$$

Thus, the Jacobi matrix of the corresponding chart transition map on $M \times N$

$$\det J\big(\varphi_{(\alpha_1,\beta_1)} \circ \varphi^{-1}_{(\alpha_2,\beta_2)}\big)(x) = \det J(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})(x) \cdot \det J(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})(x) > 0$$

Thus, there exists an orientable atlas on $M \times N$; $M \times N$ is orientable. □

Example 1.5 (Real Projective Space $\mathbb{RP}^n$)**.** *The real projective space is given by* $\mathbb{RP}^n := (\mathbb{R}^{n+1} - \{0\})/\sim$, *where the equivalence relation is given by*

$$\forall x, y \in (\mathbb{R}^{n+1} \setminus \{0\}) : \exists \lambda \in \mathbb{R} \setminus \{0\} : (x \sim y \iff x = \lambda y)$$

Consider the quotient map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$, *the topology on* $\mathbb{RP}^n$ *is induced by the quotient map, i.e.*

$$V \subseteq \mathbb{RP}^n \text{ open} \iff \pi^{-1}(V) \text{ is open in } \mathbb{R}^{n+1}$$

It is obvious that the quotient topology ensures that π *is continuous and surjective, and thus* $\mathbb{RP}^n$ *is second countable and Hausdorff. Real projective space can also be written as a quotient space of* $S^n/\sim$, *where* $\forall x, y \in S^n$,

$$x \sim y \iff x = -y$$

The chart on $\mathbb{RP}^n$ *is given by*

$$U_k = \{[x] \in \mathbb{RP}^n \mid x = (x^1, \dots, x^{n+1}), x^k \neq 0\}, \quad \text{where } [x^1 : \cdots : x^k] \sim \left[\frac{x^1}{x^{n+1}} : \cdots : \frac{x^{k-1}}{x^k} : 1 : \frac{x^{k+1}}{x^k} : \cdots : \frac{x^{n+1}}{x^k} \right]$$

The overlapping region of two charts with $k \neq l$ *is just*

$$U_k \cap U_l = \{[x] = [x^1 : \cdots : x^{n+1}] \mid x^k, x^l \neq 0\}$$

By definition $\mathbb{RP}^n = \bigcup_{k=1}^{n+1} U_k$, *the coordinate map is given by* $\varphi_k : U_k \to \mathbb{R}^n$

$$\varphi_k([x]) = (\eta^1, \eta^2, \dots, \eta^n)$$

where $\eta^i = x^i/x^k$ *if* $i < k$ *and* $\eta^i = x^{i+1}/x^k$ *if* $i \geq k$. *To compute the transition map, denote* ${}_j\xi^i = x^i/x^j$, *then*

$$\varphi_l \circ \varphi_k^{-1}({}_k\xi^1, \dots, {}_k\xi^{k-1}, {}_k\xi^{k+1}, \dots, {}_k\xi^{n+1}) = ({}_l\xi^1, \dots, {}_l\xi^{l-1}, {}_l\xi^{l+1}, \dots, {}_l\xi^{n+1})$$

Since the coordinate has the relation

$$\begin{aligned} {}_l\xi^h &= x^h/x^l = \left(\frac{x^h}{x^k}\right) / \left(\frac{x^l}{x^k}\right) = {}_k\xi^h / {}_k\xi^l, \quad h \neq l, k \\ {}_l\xi^k &= x^k/x^l = ({}_k\xi^l)^{-1} \end{aligned}$$

By the fact that x^l *and* x^k *are nonzero, then it is obvious that* $\varphi_l \circ \varphi_k^{-1} : \varphi_k(U_k \cap U_l) \to \varphi_l(U_l \cap U_k)$ *is a smooth map. Since this statement is general,* U_k *and* U_l *are smoothly compatible* $\forall k, l$. *Thus,* $\mathbb{RP}^n$ *is a smooth manifold.*

The generalization of the projective space is called the Grassmannian, which is also a smooth manifold.

Example 1.6 (Grassmannian)**.** *Given an finite dimensional vector space V such that $\dim V = n$, $n \geq k$, then the Grassmannian on V is given by*

$$\mathrm{Gr}_k(V) := \{W \subseteq V \textit{ linear subspace} \mid \dim W = k\}$$

Remark. *The Grassmannian can reduce to projective space* $\mathrm{Gr}_1(\mathbb{R}^{n+1}) \cong \mathrm{Gr}_n(\mathbb{R}^{n+1}) \cong \mathbb{RP}^n$.

Theorem 1.1. Grassmannian $\mathrm{Gr}_k(V)$ wirh V being n-dimensional $\mathbb{R}$-vector space is a $(n-k)k$ dimensional C^∞-manifold.

Proof. WOLG, take $V = \mathbb{R}^n$. It is an obvious fact that the Grassmannian is a quotient manifold from the set of k-frames in V, denoted as

$$\mathrm{Fr}_k(\mathbb{R}^n) := \{(v_1, \dots, v_k) \in \mathbb{R}^{nk} \mid (v_1, \dots, v_k) \text{ linear independent}\} \cong \{F \in M_{n\times k}(\mathbb{R}) \mid \operatorname{rank} F = k\}$$

Grassmannian has the topology as a quotient topology from $\mathrm{Fr}_k(\mathbb{R}^n)$ with the equivland class $\forall F_1, F_2 \in \mathrm{Fr}_k(\mathbb{R}^n)$

$$F_1 \sim F_2 \iff \exists M \in \mathrm{GL}_k(\mathbb{R}) : F_1 = F_2 M$$

i.e., $\mathrm{Gr}_k(\mathbb{R}^n) \cong \mathrm{Fr}_k(\mathbb{R}^n)/\,\mathrm{GL}_k(\mathbb{R}^n)$. Thus, the topology on the Grassmannian is defined by the quotient map $\pi : \mathrm{Fr}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$

$$U \subseteq \mathrm{Gr}_k(\mathbb{R}^n) \text{ is open} \iff \pi^{-1}(U) \text{ is open.}$$

Open sets in the k-frame are given by $U_I = \big\{F \in \mathrm{Fr}_k(\mathbb{R}^n) \mid \det F_I \neq 0\big\}$. Where $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ such that $i_a \neq i_b$ $\forall a \neq b$, and $F_I \in M_k(\mathbb{R})$ is defined by $(F_I)_{ab} = F_{i_a b}$, also, since U_I is invariant under $\mathrm{GL}_k(\mathbb{R})$-right action $\rho : \mathrm{Fr}_k(\mathbb{R}^n) \times \mathrm{GL}_k(\mathbb{R}) \to \mathrm{Fr}_k(\mathbb{R}^n)$, $\rho(A, M) = AM$. Thus, U_I can be viewed as the open set in $\mathrm{Gr}_k(\mathbb{R}^n)$.
Take the local coordinate of each chart U_I be

$$\begin{aligned} \phi_I : U_I &\to M_{(n-k)\times k}(\mathbb{R}) \cong \mathbb{R}^{(n-k)k} \\ F_I &\to F_{I^c} \end{aligned}$$

As an example, consider $n = 4$, $k = 2$. Take $I = (2, 3)$ (i.e., $I^c = (1, 4)$), then the frame and the coordinate are given by

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \\ f_{41} & f_{42} \end{pmatrix} \sim A = FM = \begin{pmatrix} * & * \\ 1 & 0 \\ 0 & 1 \\ * & * \end{pmatrix}, \quad \begin{aligned} &F_I = \begin{pmatrix} f_{21} & f_{22} \\ f_{31} & f_{32} \end{pmatrix},\ M = (F_I)^{-1} \\ &\phi_I(F) = F_{I^c} = \begin{pmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{pmatrix} \end{aligned}$$

Where $\phi_I(F) = F_{I^c}$ up to the fixed identification $M_{(n-k)\times k}(\mathbb{R}) \cong \mathbb{R}^{(n-k)k}$. For $M \in M_{(n-k)\times k}(\mathbb{R})$, let $A^{(J)}(M) \in M_{n\times k}(\mathbb{R})$ be the matrix whose J-rows equal I_k and whose J^c-rows equal M (in order). Set

$$B_{I\leftarrow J}(M) := \big(A^{(J)}(M)\big)_I, \qquad C_{I\leftarrow J}(M) := \big(A^{(J)}(M)\big)_{I^c}$$

Then, on the intersection domain

$$U_I \cap U_J = \{\, M \in M_{(n-k)\times k}(\mathbb{R}) \mid \det B_{I\leftarrow J}(M) \neq 0 \,\}$$

The transition map is then given by

$$\phi_I \circ \phi_J^{-1}(M) = C_{I\leftarrow J}(M)\big(B_{I\leftarrow J}(M)\big)^{-1}$$

Since the transition map is linear, the Grassmannian is a C^∞-map. ☐

For the next proposition about orientability, we need to introduce a topological operation first.

Definition 1.7 (The Connected Sum)**.** Let M_1 and M_2 be connected n-dimensional smooth manifolds, take points $p_1 \in M_1$ and $p_2 \in M_2$. Take local coordinate systems on M_1 and M_2, denotes as (U_1, φ_1) and (U_2, φ_2) such that $\varphi_1(p_1) = \varphi_2(p_2) = 0 \in \mathbb{R}^n$, and

$$\varphi_1(U_1) = \varphi_2(U_2) = B_2(0) = \Big\{x \in \mathbb{R}^n \mid \sum_{i=1}^n (x^i)^2 < 4\Big\}$$

Denote the set

$$A(1/2, 2) = B_2(0) - \overline{B_{1/2}(0)} = \Big\{x \in \mathbb{R}^n \mid \frac{1}{4} < \sum_{i=1}^n (x^i)^2 < 4\Big\}$$

and its preimage are $V_1 = \varphi_1^{-1}(A(1/2, 2))$, $V_2 = \varphi_2^{-1}(A(1/2, 2))$. Consider the map

$$\phi : A(1/2, 2) \to A(1/2, 2), \phi(x) = x\Big(\sum_{i=1}^n (x^i)^2\Big)^{-1}$$

The following lemma is significant:

Lemma. *ϕ is a diffeomorphism.*

Proof. By the smoothness and bijectivity of $f(x) = x$ and $f(x) = 1/|x|$ when $x > 0$, the lemma is obvious. □

By the lemma above, by the smoothness of the manifolds M_1 and M_2, the map $\varphi_2^{-1} \circ \phi \circ \varphi_1 : V_1 \to V_2$ is also a diffeomorphism. Consider the quotient space:

$$M_1 \# M_2 = \big(M_1 - \varphi_1^{-1}(\overline{B_{1/2}(0)})\big) \sqcup \big(M_2 - \varphi_2^{-1}(\overline{B_{1/2}(0)})\big)/\sim$$

Where the quotient is being defined based on the map $\varphi_2^{-1} \circ \phi \circ \varphi_1$

$$\forall x \in V_1 : \forall y \in V_2 : \big(x \sim y \iff y = \varphi_2^{-1} \circ \phi \circ \varphi_1(x)\big)$$

The smooth manifold $M_1 \# M_2$ is called the connected sum of M_1 and M_2.

Proposition 1.6. *The connected sum of two orientable n-dimensional C^∞-manifolds is still orientable.*

Proof. Consider connected smooth manifolds M_1 and M_2 in the definition of connected sum above, suppose M_1 and M_2 are both orientable. By the given definition of the connected sum of manifolds, the only place that needs to be examined is the open set that includes the quotient part. For arbitrary open sets V_1 and V_2 that contain the quotient part of the manifold, the chart transition map is given by

$$\varphi_2 \circ (\varphi_2^{-1} \circ \phi \circ \varphi_1) \circ \varphi_1^{-1} = \phi$$

which we already know is a diffeomorphism, since M_2 is given to be orientable, it is safe to change a coordinate map by composition with a reflection transformation

$$\hat{\varphi}_2 = \mathcal{P} \circ \varphi_2(x), \quad \mathcal{P}x = (-x^1, \ldots, x^n)$$

Where the chart transition map can be expanded in the new chart as $\hat{\varphi}_2 \circ (\varphi_2^{-1} \circ \phi \circ \varphi_1) \circ \varphi_1^{-1}$, the Jacobian of the transition map is then given by

$$\det J(\hat{\varphi}_2 \circ \varphi_2^{-1} \circ \phi \circ \varphi_1 \circ \varphi_1^{-1}) = \det J(\mathcal{P} \circ \phi) = -\det J(\phi)$$

Take $r^2 = \|x\|^2 = \sum_i (x_i)^2$, then $\phi(x) = x/r^2$, then we can calculate the partial derivative

$$J_{ij}(\phi) = \frac{\partial \phi^i}{\partial x^j} = \delta^i{}_j r^{-2} + x^i \frac{\partial r^{-2}}{\partial x^j} = \delta^i{}_j r^{-2} - 2x^i x_j r^{-4}$$

which, in matrix notation, is

$$J(\phi) = (r^{-2}I - 2r^{-4}xx^T)$$

And the Jacobian is just the determinant of this matrix

$$\det J(\phi) = (r^{-2})^n \det\bigl(I - 2r^{-2}xx^T\bigr)$$

To compute the determinant, we will need a lemma

Lemma (Sylvester Identity of Determinants)**.** *If $A \in M_{m\times n}(\mathbb{R})$, and $B \in M_{n\times m}(\mathbb{R})$, then*

$$\det(I + AB) = \det(I + BA)$$

Consider the following block matrix in $(m+k)\times(m+k)$,

$$M = \begin{pmatrix} I_m & A \\ B & I_k \end{pmatrix}$$

By elementary transformation, we can get

$$\det M = \det \begin{pmatrix} I_m - AB & O \\ B & I_k \end{pmatrix} = \det \begin{pmatrix} I_m & A \\ O & I_k - BA \end{pmatrix}$$

Which proves the lemma $\det(I_m - AB) = \det(I_k - BA)$

With the lemma, $\det J(\phi) = r^{-2n}(1 - 2r^{-2}\|x\|^2) = -r^{-2n} < 0$. Then, $\det(\mathcal{P}\circ\phi) = r^{-2n} > 0$. This means $M_1\#M_2$ is orientable. □

The orientation of smooth manifolds can also be defined by differential forms. In Chapter 2, we will prove the equivalence of the two definitions. In the last part of this section, we will introduce the connectivity of manifolds and their relation with orientation.

Proposition 1.7. *If M is a connected topological manifold, then M is path-connected.*

Proof. Note that M is connected, which indicates that there do not exist any open sets $U, V \in \mathcal{O}_M$ like $M = U \sqcup V$. Locally, take $p \in M$ and the chart (U, φ) contains p. By the definition of a chart,

$$\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$$

By the definition of open sets, we can take $B_r(\varphi(p)) \subseteq \varphi(U)$. Let $V = \varphi^{-1}\bigl(B_r(\varphi(p))\bigr)$. By the given coordinate map, V is open and path-connected as an inherent property of Euclidean space. Thus, M is locally path-connected.
Thus, for some $p \in M$, it is sufficient to take a branch connected to a path not empty $C_p \subseteq M$. By path-connectivity, $\forall q \in C_p : \exists V \in \mathcal{O}_M$ such that $q \in V \subseteq C_p$, makes C_p open.

Lemma. *Let X be a locally path-connected topological space, and the path-connected branch containing $x \in X$ is given by*

$$C_x = \bigl\{y \in X \mid \exists\gamma \in C^0\bigl([0,1], X\bigr), \gamma(0) = x, \gamma(1) = y\bigr\}$$

Then, $\forall x, y \in X$ either $C_x = C_y$ or $C_x \cup C_y = \emptyset$ and $\bigcup_{x\in X} C_x = X$, i.e., the locally path-connected branch constructs an equivalence class.

To prove the lemma, we need to check reflexivity, symmetry, and transitivity to prove that the relation above is an equivalence relation. First, define the relation as

$$\forall x, y \in X : \Bigl(x \sim y \iff \exists\gamma \in C^0\bigl([0,1], X\bigr) : \gamma(0) = x, \gamma(1) = y\Bigr)$$

1. (Reflexivity) $x \sim x$ by the constant map $\forall a \in [0,1] : \gamma(a) = x$.

2. (Symmetry) If $x \sim y$, i.e., $\gamma : [0,1] \to X$ connected x and y, then $\forall t \in [0,1] : \bar{\gamma}(t) = \gamma(1-t)$ ensures that $y \sim x$.
3. (Transitivity) Suppose for $x, y, z \in X$, $x \sim y$ by path γ_1 and $y \sim z$ by path γ_2, then consider
$$\gamma(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le 1/2 \\ \gamma_2(2t-1), & 1/2 \le t \le 1 \end{cases}$$
thus, $(x \sim y) \land (y \sim z) \implies (x \sim z)$.

Thus, $\sim$ is an equivalence relation, and the lemma has been proved.

By the lemma, if $C_p \neq M$, then $\exists p \neq q \in M : C_p \sqcup C_q = M$, which disobeys the connectivity of M. Thus, $C_p = M$, M is a path-connected set. □

Definition 1.8 (Orientation)**.** Let M be an orientable smooth manifold, and $\mathcal{D}$ be an orientable cover. If every chart that is compatible (in the sense of orientation) with $(U, \varphi) \in \mathcal{D}$ is in $\mathcal{D}$, then $\mathcal{D}$ is said to be an orientation.

Proposition 1.8. *Connected orientable smooth manifolds always have exactly two orientations.*

Proof. Let M be the connected orientable smooth manifold, take an orientable atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta) : \beta \in J\}$. We take $(U_\alpha, \phi_\alpha) \in \mathcal{A}$ and $(V_\beta, \psi_\beta) \in \mathcal{B}$, such that $U_\alpha \cap V_\beta \neq \emptyset$, and $\forall p \in U_\alpha \cap V_\beta$:
$$f(p) = \frac{J(\phi_\alpha \circ \psi_\beta^{-1})}{|J(\phi_\alpha \circ \psi_\beta^{-1})|}(p)$$
Since A and B are both orientable atlas, f can be either 1 or -1, which indicates $\mathcal{B} = \mathcal{A}$ or $\mathcal{B} = \mathcal{A}^-$ □

Example 1.7 (Orientability of Real Projective Space)**.** *By the previous example 1.5 of projective spaces, the transition map of $\mathbb{RP}^n$ is given by*
$$\varphi_l \circ \varphi_k^{-1}({}_k\xi^1, \dots, {}_k\xi^{k-1}, {}_k\xi^{k+1}, \dots, {}_k\xi^{n+1}) = ({}_l\xi^1, \dots, {}_l\xi^{l-1}, {}_l\xi^{l+1}, \dots, {}_l\xi^{n+1})$$
where ${}_j\xi^i = x^i/x^j$. Thus, for $h \neq k, l$
$$\frac{\partial_l \xi^h}{\partial_k \xi^\beta} = \begin{cases} \frac{1}{{}_k\xi^l}, & \beta = h \\ -\frac{{}_k\xi^h}{({}_k\xi^l)^2}, & \beta = l \\ 0, & \textit{otherwise} \end{cases}$$
And for $h = k$
$$\frac{\partial_l \xi^h}{\partial_k \xi^\beta} = \begin{cases} -\frac{1}{({}_k\xi^l)^2}, & \beta = l \\ 0, & \textit{otherwise} \end{cases}$$
With proper coordinate transformation (exchange the order of coordinates) that moves $\beta = l$ to the last column and $h = k$ to the last row, the upper-triangular form of the Jacobi matrix is given by
$$J(\varphi_l \circ \varphi_k^{-1}) = \begin{pmatrix} \frac{1}{{}_k\xi^l} I_{n-1} & \left[\frac{-{}_k\xi^h}{({}_k\xi^l)^2}\right]_{h\neq k,l} \\ 0 & -\frac{1}{({}_k\xi^l)^2} \end{pmatrix}$$
Thus, the determinant is given by
$$\det J(\varphi_l \circ \varphi_k^{-1}) = ({}_k\xi^l)^{-n-1}$$
Notice that if $n+1$ (i.e. n is odd) is even, then $({}_k\xi^l)^{-n-1}$ is always positive, and when n is even, the $({}_k\xi^l)^{-n-1}$ does not have a certain sign. Thus, $\mathbb{RP}^n$ orientable is n is odd.

Moreover, we can define the category of C^p-manifolds:

Definition 1.9 (Category of C^p-Manifolds)**.** The category of C^p-manifolds, Diff^p, is the category given by

1. $\mathrm{Obj}(\mathsf{Diff}^k) = $ All C^k-manifolds
2. $\forall M, N \in \mathsf{Diff}^k : \mathrm{Hom}^k_{\mathsf{Diff}}(M, N) = C^k(M, N)$
3. The composition of morphisms is given by the composition of maps.

1.2 Submanifolds

The study of submanifolds is naturally introduced from the concept of a subset and the rank of a map.

Example 1.8 (Invertible Linear Map)**.** *Consider the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, from undergraduate linear algebra, A is invertible iff $\det A \neq 0$; we also say that the linear map is "full rank". From the perspective of smooth manifolds, we can extend the concept of full rank to the map on manifolds.*
Any linear map on $\mathbb{R}^n$ can be written in matrix form, which can also be viewed as the elements in $\mathbb{R}^{n^2}$, on which we can induce a norm. For $A : \mathbb{R}^n \to \mathbb{R}^n$, the L^2-norm on $\mathbb{R}$ can be defined by

$$\begin{aligned} \|\cdot\| :& \mathbb{R}^n \to [0, \infty) \\ & x \mapsto \|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \end{aligned}$$

Then, we can induce the operator norm in the following way:

Definition 1.10 ((Linear) Operator Norm)**.** With given vector norm $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$, the linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ has norm:

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

Then we have the following proposition

Proposition 1.9. *The linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ satisfies that $\|B\| < 1$, then $I_n - B$ is invertible.*

Proof. To prove the bijectivity, in this case, we only need to check the kernel of the linear map $I_n - B$. Consider the equation $(I_n - B)x = 0$, then $x = Bx$. Thus, the 2-norm is given by

$$\|x\| = \|Bx\| \leq \|B\| \|x\|$$

Which indicates $(1 - \|B\|)\|x\| \leq 0$. Known that $1 - \|B\| > 0$, then

$$(I_n - B)x = 0 \iff x = 0$$

Which means $\ker(I_n - B) = \{0\}$. Since $B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$, B is bijective. □

A significant observation from the example above is that the identity matrix is still nondegenerate after a small perturbation. Thus, we have the following generalization.

Definition 1.11 (Rank of C^k-Maps)**.** Let $f : M \to N$ to be a C^k-map ($k \geq 1$) between smooth manifolds, let $p \in M$ and $q = f(p) \in N$. Taking local maps (U, φ) on M and (V, ψ) on N, the rank of map f at point p is given by

$$\mathrm{rank}_p f := \mathrm{rank}\, J(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}$$

Proposition 1.10. *The definition of rank does not depend on the choice of coordinates.*

Proof. Consider the chart transition map, $\Theta := \tilde{\varphi}\circ\varphi^{-1}$ and $\Phi : \tilde{\psi}\circ\psi^{-1}$, by definition, $\Theta : \varphi(\tilde{U}\cap U) \to \tilde{\varphi}(\tilde{U}\cap U)$ and $\Phi : \psi(\tilde{V}\cap V) \to \tilde{\psi}(\tilde{V}\cap V)$ are C^k-diffeomorphisms ($k \geq 1$). Thus, consider $U, \tilde{U} \subseteq M$, $V, \tilde{V} \subseteq N$ and $\dim M = m$, $\dim N = n$.

$$P := J(\Theta) = \Big(\frac{\partial(\tilde{\varphi}\circ\varphi)^i}{\partial x^j}\Big)_{1\leq i,j\leq m}, \quad Q := J(\Phi) = \Big(\frac{\partial(\tilde{\psi}\circ\psi)^i}{\partial y^j}\Big)_{1\leq i,j\leq n}$$

are invertible matrices. Then, consider the local coordinate representation: $\tilde{f} = \psi\circ f\circ\varphi$ and $\tilde{\psi}\circ f\circ\tilde{\varphi} = \Phi\circ\tilde{f}\circ\Theta^{-1}$, we can compute the rank of the Jacobi matrix:

$$\operatorname{rank} J(\Phi\circ\tilde{f}\circ\Theta^{-1})_{\varphi(p)} = \operatorname{rank}(PJ(\tilde{f})Q)_{\varphi(p)} = \operatorname{rank} J(\psi\circ f\circ\varphi)_{\varphi(p)} = \operatorname{rank}_p f$$

Thus, the rank of a function at a point $\operatorname{rank}_p f$ is well defined under coordinate transformations. $\square$

An important way to define a submanifold is to consider the image (preimage) of a smooth map, formally via the inverse function theorem (IFT). The IFT on smooth manifolds is directly induced by the IFT on Euclidean spaces.

Theorem 1.2 (Inverse Function Theorem)**.** Let $f : M^n \to N^n$ be the C^k-map between C^∞-manifold M and N, if $\operatorname{rank}_p f = n$, then there exists an open neighborhood $U \subseteq M$ that contains p and $V \subseteq N$ contains $q = f(p)$, such that $f|_U : U \to V$ is a C^k-diffeomorphism.

Proof. As a local constraint to differentiable functions on C^∞-manifolds, it is sufficient to take $M = N = \mathbb{R}^n$, and take $p = 0 \in \mathbb{R}^m$. Since $\operatorname{rank}_0 F = n$, it is sufficient to consider the Jacobi matrix of the map $J(f)_0 = I_n$ by the composition of a linear map, which will not change the result. Thus, at the origin $p = 0$, the map F is a perturbation of the identity map. Let

$$\forall x \in \mathbb{R}^n : \ g = f(x) - x$$

Then, $J(g)_0 = J(f)_0 - I_n = 0$, then $\exists\epsilon > 0$,

$$\forall x \in \overline{B_\epsilon(0)} : \ \|J(g)_x\| \leq \frac{1}{2}$$

And by the mean value theorem

$$\forall x_1, x_2 \in \overline{B_\epsilon(0)} : \ \|g(x_1) - g(x_2)\| \leq \|J(g)_\xi\|\|x_1 - x_2\| \leq \frac{1}{2}\|x_1 - x_2\|$$

Which indicates that g is 1/2-Lipschitz continuous, $\forall\xi \in B_\epsilon(0)$, solving the equation $y = f(x)$ is just finding the fixed point of $g_y(x) = x - (f(x) - y)$. The function is also 1/2-Lipschitz, since $g_y(x_1) - g_y(x_2) = g(x_1) - g(x_2)$. By the Banach fixed-point theorem, $\exists x_y \in B_\epsilon(0) : g_y(x_y) = x_y + y - f(x_y) = x_y$, i.e., $f(x_y) = y$ has a unique solution $x_y = h(y)$ such that

$$f(h(y)) = y \iff y - g(h(y)) = h(y)$$

Given $V = B_{\epsilon/2}(0)$ and $U = f^{-1}(V) \cap B_\epsilon(0)$, to prove that $h : V \to U$ is a C^k-diffeomorphism, we only need to check the continuity and C^k of the map.

- h is continuous. Since $\forall y_1, y_2 \in B_\epsilon(0)$

$$\begin{aligned}\|h(y_1) - h(y_2)\| &\leq \|y_1 - y_2\| - \|g(h(y_1)) - g(h(y_2))\| \\ &\leq \|y_1 - y_2\| - \frac{1}{2}\|h(y_1) - h(y_2)\|\end{aligned}$$

  Thus, $\|h(y_1) - h(y_2)\| \leq \frac{2}{3}\|y_1 - y_2\|$ is Lipschitz. Thus, h is continuous.

- h is C^k. Firx a base point $y_0 \in V$, given any $y \in V$,

$$\begin{aligned} h(y) - h(y_0) &= (y - y_0) + \big(g(h(y)) - g(h(y_0))\big) \\ &= (y - y_0) + J(g)_{h(y_0)}\big(h(y) - h(y_0)\big) + O(\|h(y) - h(y_0)\|) \\ \implies y - y_0 &= \big[I_n - J(g)_{h(y_0)}\big](h(y) - h(y_0)) + O(\|h(y) - h(y_0)\|) \end{aligned}$$

  Which implies that the Jacobi matrix exists and $Jh(y_0) = I_n - J(g)_{h(y_0)} = (Jf(h(y)_0))^{-1}$. Repeat this procedure, by the C^k-differentiability of f, h is a C^k-map.

Thus, we proved that f is locally a C^k-diffeomorphism. □

Based on the IFT, it is sufficient to define the following behavior of maps, which provides us with a nice hint on how to study the submanifold.

Definition 1.12 (Immersion, embedding, and submersion)**.** Let $f : M^m \to N^n$ be a C^k-map between differentiable manifolds. If $\operatorname{rank}_p f = m \ \forall p \in M$, f is said to be a C^k-immersion. If $f : M \to f(M)$ is a C^k-immersion and also a homeomorphism ($f(M) \subseteq N$ has the subset topology of N), then f is a C^k-embedding. If $\operatorname{rank}_p f = n \ \forall p \in M$, then f is a C^k-submersion.

Here are some examples:

Example 1.9 (An immersion that is not embedding (not injective))**.** *Consider the map* $f : \mathbb{R} \to \mathbb{R}^2$ *such that*

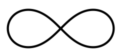
$$f(\theta) = (\cos\theta, \sin\theta)$$

It is easy to check the smoothness and $\operatorname{rank} f = 1$*. However, since* $f(\theta) = f(\theta + 2n\pi)$*,* f *is not injective, and thus, not an embedding.*

Example 1.10 (An injective immersion that is not embedding)**.** *Consider the map* $f : \mathbb{R} \to \mathbb{R}^2$ *defined by*

$$f(t) = \left(\frac{t^3 + t}{t^4 + 1}, \frac{t^3 - t}{t^4 + 1}\right) \quad \forall t \in \mathbb{R}$$

f *is injective and* $\operatorname{rank} f = 1$*. However, it is not a submersion since the image of the map is the lemniscate shown in the following figure:*



Which is clearly not homeomorphic to $\mathbb{R}$ *and thus, not diffeomrorphic to* $\mathbb{R}$

Example 1.11 (Embedding of torus)**.** *The torus is given by* $T^2 = S^1 \times S^1$*. Consider the map* $f : T^2 \to \mathbb{R}^3$ *defined by taking* $R > r > 0$ *and*

$$f(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\phi}) = \big((R - r\cos\phi)\cos\theta,\ (R - r\cos\phi)\sin\theta,\ r\sin\theta\big), \quad \forall \theta, \phi \in [0, 2\pi]$$

is a smooth embedding of T^2 *into* $\mathbb{R}^3$*.*

Theorem 1.3 (Local Form of Immersion)**.** Let $f : M^m \to N^n$ be an immersion between differentiable manifolds, $f(p) = q$. Then, exist charts (U, φ) on M contains p and (V, ψ) on N contains q such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ given by

$$\psi \circ f \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0)$$

Proof. Since the result is local, we can consider only the case that $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ $(m < n)$. Then, let the component form of the immersion be given by $f(x) = (f_1(x), \dots, f_n(x))$ $\forall x \in \mathbb{R}^m$, consider the Jacobi matrix of the immersion (which is a $n \times m$ matrix with rank m)

$$J(f)_x = \frac{\partial(f_1, \dots, f_n)}{\partial(x^1, \dots, x^m)}(x)$$

By Gauss reduction, we can take the $m \times m$ component nondegenerate near $x = 0$

$$\left|\frac{\partial(f_1, \dots, f_m)}{\partial(x^1, \dots, x^m)}\right|(x) \neq 0$$

And the map defined by $g(x^1, \dots, x^n) = (f_1, \dots, f_m, f_{m+1} - x^{m+1}, \dots, f_n - x^n)$ has Jacobi matrix

$$J(g) = \begin{pmatrix} \left[\frac{\partial f_i}{\partial x^j}\right]_{m \times m} & 0 \\ * & I_{n-m} \end{pmatrix}$$

It is clear that this Jacobi matrix is nondegenerate near $x = 0$. By IFT (Theorem 1.2), $\exists U', V$ charts such that $g|_{U'} : U \to V$, then $\exists \psi : V \to U'$ be the inverse of $g|_{U'} \circ \psi = \mathrm{id}_V$. Let

$$U = \{(x^1, \dots, x^m) \in \mathbb{R}^m \mid (x^1, \dots, x^m, 0, \dots, 0) \in U'\}$$

Then, $f|_U : U \to V$ has the local coordinate representation with the local coordinate (U, φ)

$$\psi \circ f(x^1, \dots, x^m) = \psi(f_1(x), \dots, f_n(x)) = (x^1, \dots, x^m, 0, \dots, 0)$$

which proves the theorem. □

Corollary. *Any immersion is locally embedded.*

The corollary is directly from the local form. Since the results show that immersions can be expressed (using proper coordinates) in local level sets of coordinate functions.

Theorem 1.4 (Local Form of Submersion)**.** Let $f : M^m \to N^n$ be a submersion between differentiable manifolds, $f(p) = q$. Then, exist charts (U, φ) on M contains p and (V, ψ) on N contains q such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ given by

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n)$$

Proof. It is still sufficient to take $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ $(m > n)$, and $f(x) = (f_1(x), \dots, f_n(x))$. Since f is a submersion, $\operatorname{rank} f(x) = n$. Thus,

$$J(f)_x = \frac{\partial(f_1, \dots, f_n)}{\partial(x^1, \dots, x^m)}(x) \text{ is a } n \times m \text{ matrix with rank } n$$

That means, using Gauss reduction, the full rank block Jacobi matrix can be written as

$$\left|\frac{\partial(f_1, \dots, f_n)}{\partial(x^1, \dots, x^n)}\right| \neq 0$$

Then, take the following construction: consider the function defined by $g(x^1, \dots, x^m) = (f_1, \dots, f_n, x^{n+1}, \dots, x^m)$

$$J(g) = \begin{pmatrix} \left[\frac{\partial f_i}{\partial x^j}\right]_{n \times n} & * \\ 0 & I_{m-n} \end{pmatrix}$$

Thus, by IFT (Theorem 1.2), the function g is a local diffeomorphism. Let U' be the chart that $g|_{U'} : U' \to V$ invertible, $\psi(f_1, \dots, f_n, x^{n+1}, x^m) = (x^1, \dots, x^m)$ be the local inverse. Then, take $\forall x \in V$ and ψ as the local coordinate map on V

$$\psi \circ f(x^1, \dots, x^m) = (x^1, \dots, x^n)$$

Which proves the theorem. □

Similarly, we have the following corollary

Corollary. *Submersion on manifolds without boundary must be an open map (and thus, a quotient map).*

Definition 1.13 (Immersion/Embedded (Regular) Submanifold)**.** Consider M^m, N^n be smooth manifolds. If $i : M \hookrightarrow N$ is an injective immersion, then M (with the induced topology by i) is the immersion submanifold of N; if i is an embedding, then M is a embedded submanifold (or regular submanifold) of N.

In the folloing example, let M^m, N^n be smooth manifolds, $(U, \varphi = (x^1, \ldots, x^m))$ and $(V, \psi = (y^1, \ldots, y^n))$ be coodrdinate maps.

Example 1.12 (Graphs are embedded submanifolds)**.** *Consider the smooth map $f : M^m \to N^n$, the graph of f is defined by*

$$\Gamma_f := \{(m, n) \in M \times N \mid n = f(m)\}$$

The Graph of f is an m-dimensional embedded submanifold of $M \times N$. Let the local coordinate representation of f be $g = \psi \circ f \circ \varphi^{-1}$, the Jacobi matrix of the inclusion map $\Gamma_f \overset{i}{\hookrightarrow} M \times N$ in local coordinate should be a $(m + n) \times m$ dimensional matrix

$$J(i) = \begin{pmatrix} I_m \\ J(g) \end{pmatrix}$$

Where $J(g)$ is the Jacobi matrix of g in dimension $n \times m$. It is obvious that $\operatorname{rank} i = m$, which means the inclusion map is an immersion. To prove it is an embedding, it is sufficient to prove that the projection map $p_M : M \times N \to M$ that sends $(p, q) \mapsto p$ is the continuous inverse from Γ_f to M, which means that i is an embedding.

Remark. *The immersion submanifolds in N do not require a subset topology from N. A counterexample is that the Example 1.10 is an immersion submanifold of $\mathbb{R}^2$, but is not a manifold if it contains a subset topology from $\mathbb{R}^2$.*

Example 1.13 (Immersion Submanifold that Dense in T^2)**.** *Consider the inclusion map $i : \mathbb{R} \to T^2 = S^1 \times S^1$ defines by*

$$i(t) = (e^{2\pi i t}, e^{2\pi \alpha i t})$$

for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The image $i(\mathbb{R})$ is dense in T^2 and thus, the inclusion map is not proper when the image is in the subset topology induced from T^2. That means the image cannot be a regular manifold in T^2.

If $\pi : M \to N$ is some continuous map, a smooth local section $\sigma : U \to M$ of π is a right inverse of π in some open neighborhood U of $\pi(p)$ in N, i.e., $\pi \circ \sigma = \mathrm{id}|_U$. Based on the local section, the submersion also has the following property:

Theorem 1.5 (Local Section Theorem)**.** Suppose M and N are smooth manifolds and $\pi : M \to N$ is a smooth map. Then it is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.

Proof. First, suppose $\pi : M \to N$ is a submersion, then, by the local coordinate representation of the submersion (Theorem 1.4), there exists a local coordinate (U, φ) such that

$$\psi \circ \pi \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$$

Then, we can find the local inverse $\sigma : V \subseteq N \to M$ with the local coordinate representation

$$\sigma(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$$

Since the choice of $p \in M$ is arbitrary, and we can always find a coordinate φ that maps p to $(0, \ldots, 0)$ and $\varphi(U) = B_\epsilon(0)$, every point of M is in the image of a smooth local section of π.

Conversely, suppose $\forall p \in N : V \subseteq N$ is an open neighborhood of p, there exists $\sigma : V \subseteq N \to M$ such that $\pi \circ \sigma = \mathrm{id}|_V$. Then the Jacobi matrix of the map is

$$\begin{aligned} J(\psi \circ \pi \circ \sigma \circ \psi^{-1}) &= J(\psi \circ \mathrm{id}|_V \circ \psi^{-1}) \\ &= I_n, \quad \text{which is in rank } n \end{aligned}$$

Then, $J(\psi \circ \pi \circ \varphi^{-1})$ is in rank n, and thus, $\forall p \in M : \mathrm{rank}_p\, \pi = n$, which means π is a immersion. ☐

We will see an important example of submersion in this type, called the fiber bundle.

Theorem 1.6 (The Chateceristic of Embedded Submanifold)**.** Consider smooth manifolds M^m and N^n. M is the embedded submanifold of $N \iff M$ is the topological subspace of N and $\forall p \in M$, exists a local chart U contains p with coordinate map $\varphi = (x^1, \dots, x^n)$ such that

$$M \cap U = \{q \in U \mid x^i(q) = 0 \ \forall i = m+1, \dots, n\}$$

Proof. ($\Rightarrow$) Suppose M is the embedded submanifold of N. Then, the inclusion map $M \overset{i}{\hookrightarrow} N$ is an embedding. By the local coordinate form of immersion (Theorem 1.3), we can find the local coordinate (U, φ) and (V, ψ) on M and N

$$\begin{aligned} \psi \circ i \circ \varphi^{-1} &: \varphi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n \\ \varphi(p) &= (x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0) \end{aligned}$$

Then it is natural to defined the chart $(\tilde{U}, \tilde{\varphi})$ on N that

$$\tilde{U} := \{q \in V \mid (x^1(q), \dots, x^m(q)) \in \varphi(U)\}$$

Where $\psi(q) = (x^1(q), \dots, x^n(q))$. Then we can check the coordinate representation on $M \cap \tilde{U}$. Let $q = i(p)$ for some $p \in U$. Thus

$$\psi(q) = \psi \circ i(p) = (x^1(q), \dots, x^m(q), 0, \dots, 0)$$

And thus $x^{m+1}(q) = \cdots = x^n(q) = 0$. Then, if we take $q \in \tilde{U}$ and $x^{m+1}(q) = \cdots = x^n(q) = 0$. Denote

$$u = (u^1, \dots, u^m) := (x^1(q), \dots, x^m(q)) \in \varphi(U)$$

Then $p = \varphi^{-1}(u^1, \dots, u^m) \in U$, and thus

$$\psi(q) = \psi \circ i \circ \varphi^{-1}(u^1, \dots, u^m) = (u^1, \dots, u^m, 0, \dots, 0)$$

Thus, $q \in M$, and $M \cap \tilde{U} = \{q \in \tilde{U} \mid x^i(q) = 0, m+1 \le i \le n\}$.
($\Leftarrow$) It is easy to show that the inclusion map satisfies the condition above in the theorem is an embedding. ☐

With the characteristics of a embedded submanifold, we can prove the following theorem.

Theorem 1.7 (Constant Rank Theorem)**.** Let M^m and N^n be smooth manifolds. $f : M^m \to N^n$ is C^∞-map, if $\mathrm{rank}_p\, f = k \ \forall p \in M$. Then, $\forall p, q \in M, N$, there is a pair of locoal coordinates (U, φ) and (V, ψ) such that f has local coordinate form

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

Proof. Since the claim is local, we can let $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ and f be smooth map with constant rank k, written as

$$f(x^1, \dots, x^m) = (f_1(x), \dots, f_n(x))$$

And similar to the previous proofs, we can assume the matrix

$$M_f = \left[\frac{\partial f_i}{\partial x}\right]_{1\le i,j\le k}$$

has ranl k, i.e. M_f is the largest invertible block in $J(f)$. The IFT inspires us to define the function

$$\varphi(x^1,\dots,x^m) = (f_1,\dots,f_k(x),x^{k+1},\dots,x^m)$$

The Jacobi matrix of φ is given by

$$J(\varphi) = \begin{pmatrix} M_f & * \\ 0 & I_{m-k} \end{pmatrix}$$

which is nondegenerate at the origin, and implies that we can use IFT (Theorem 1.2) in an open neighborhood. Consider open neighborhood $U, V \subseteq \mathbb{R}^m$ of $0 \in \mathbb{R}^m$ such that $\varphi|_U : U \to V$ has local inverse $\varphi^{-1} : V \to U$. Then, take φ as a local coordinate map and

$$F(x) = f\circ\varphi^{-1}(x^1,\dots,x^m) = (x^1,\dots,x^k,F^{k+1},\dots,F^n)$$

Since $\operatorname{rank} f \equiv k$,

$$\frac{\partial F^i}{\partial x^j} = 0, \quad k+1\le i\le n,\ k+1\le j\le m$$

Which implies the function $F^i = F^i(x^1,\dots,x^k)$. Then, defined $x = (x^1,\dots,x^k)$, consider the coordinate transformation such that

$$\psi(x^1,\dots,x^n) = \big(x^1,\dots,x^k,x^{k+1}-F^{k+1}(x),\dots,x^n-F^n(x)\big)$$

With this coordinate map

$$\begin{aligned}\psi\circ f\circ\varphi^{-1}(x^1,\dots,x^m) &= \big(x^1,\dots,x^k,(F^{k+1}-F^{k+1})(x),\dots,(F^n-F^n)(x)\big)\\ &= \big(x^1,\dots,x^k,0,\dots,0\big)\end{aligned}$$

In this way, we proved the theorem. □

An application of the constant rank theorem (Theorem 1.7) can provide us with the construction of a class of embedded submanifolds:

Theorem 1.8 (Level Set Theorem)**.** Let $f : M^m \to N^n$ be a smooth map with constant rank $\operatorname{rank} f = l$, then $\forall q \in N$

$$f^{-1}(q) := \{p\in M \mid f(p) = q\}$$

is either empty, or a $m-l$ dimensional submanifold of M.

Proof. Let $S = f^{-1}(q)$ for some $q \in N$. Suppose S is nonempty, by the constant rank theorem (Theroem 1.7), $\forall p \in S$, there exists chart (U,φ) contains p and chart (V,ψ) on N contains q such that the local coordinate representation of f take the form

$$\psi\circ f\circ\varphi^{-1}(x^1,\dots,x^m) = (x^1,\dots,x^l,0,\dots,0)$$

Without the loss of generality, we can let $\psi(q) = 0 \in \mathbb{R}^n$, then

$$\psi\circ f\circ\varphi^{-1}(x^1,\dots,x^m) = (x^1,\dots,x^l,0,\dots,0)$$

which always be 0 $\forall p \in S$. In this way, we shall claim that there $\forall p \in S$, there exists chart (U,φ) such that for coordinate map $\varphi = (x^1,\dots,x^m)$

$$S\cap U = \{p\in U \mid x^1(p) = \cdots = x^l(p) = 0\}$$

By the characteristic of the embedded submanifold (Theorem 1.6), S is a embedded submanifold. □

1.3 Tangent and Cotangent Spaces with the Induced Linear Maps

1.3.1 Tangent Space and Pushforward

As a locally Euclidean topological space, the open coordinate charts are homeomorphic to open sets in $\mathbb{R}^n$. Recall that in multivariable calculus, the key property is that every (differentiable) map $f : \mathbb{R}^m \to \mathbb{R}^n$ is locally linearized at a small neighborhood p as a linear transformation, or "matrix". More precisely, a map $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be differentable at $a \in \mathbb{R}^m$ if

$$\exists! L \in \mathrm{Hom}_{\mathsf{Vect}}(\mathbb{R}^m, \mathbb{R}^n) : \lim_{h \to 0} \frac{\|f(a+h) - f(a) - Lh\|}{\|h\|} = 0$$

Which means the global differentiability is just saying $f(x+h) - f(x) = Lh + \alpha(x,h)$, where $\forall x \in \mathbb{R}^n$: $\alpha(x,h) = o(h)$. To construct a similar (locally) linear structure on C^∞-manifolds, we need the tangent space associated to each point on the manifold. In the following passage, $C^\infty(M)$ is the global smooth function on M and C_p^∞ is the function germ.

Definition 1.14 (C^∞-Function Germ at p)**.** The function germ is defined by $C_p^\infty = C^\infty(M)/\sim_p$, where the equivlance relation is given by

$$\forall f, g \in C^\infty(M) : f \sim_p g \iff \exists U_p \subseteq M : f|_{U_p} = g|_{U_p}$$

Where U_p is an open neighborhood of p.

It is easy to show that C_p^∞ is an $\mathbb{R}$-module.

Definition 1.15 (Tangent Vector)**.** Let M be a smooth manifold, fix a point $p \in M$. A tangent vector at point p is a $\mathbb{R}$-linear map $X_p : C_p^\infty \to \mathbb{R}$ such that $\forall f, g \in C_p^\infty$

$$X_p(f \cdot g) = f(p)X_p(g) + X_p(f)g(p)$$

The space of all vector spaces at p is called the tangent space at p, denoted as T_pM.

Remark. *A more abstract way to define a derivation is that, given a ring R, let A be an R-algebra and B be an A-bimodule. A derivation is a R-linear map $D : A \to B$ satisfies the Leibniz rule: $\forall a, b \in A$*

$$D(ab) = aD(b) + D(a)b$$

The purpose of this definition is to define the differential of a smooth function on a manifold. In Euclidean space

$$\boxed{\text{Vectors } v_p = (v^1, \dots, v^n)_p} \xleftrightarrow{\text{One to one correspondence}} \boxed{\text{derivations } D_p = v^1\partial_1|_p + \cdots + v^n\partial_n|_p}$$

each vector v has a corresponding directional derivativor $v \cdot \nabla$. The derivativor (algebraic) definition of a tangent vector is simply to use the correspondence relation above, which uses the linearity and Leibniz law to define the tangent vector.

From the definition, the following properties are easy to check:

Proposition 1.11 (Properties of Tangent Vectors)**.** *Let M be a smooth manifold, $p \in M$. $\forall v_p \in T_pM$, the following statements are true*

1. $\forall f \in C^\infty(M) : f = \text{const} \implies v_p(f) = 0$
2. $\forall f, g \in C^\infty(M) : f(p) = g(p) = 0 \implies v_p(f \cdot g) = 0$

The proof is simply to apply the linearity and Leibniz law.

Other than the algebraic definition, there is a more geometrical way to define a tangent vector using smooth curves on a manifold $\gamma : I \to M$, where $I = (-\epsilon, \epsilon)$ for some $\epsilon > 0$.

Definition 1.16 (Tangent Vector as Velocity of Curves)**.** Let M be a smooth manifolds, $\gamma : I \to M$, a tangent vector along γ is defined by

$$X(f) := \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0} \in T_pM$$

Remark. *The equivalent form of this definition is that given a chart (U, φ) on M contains p, $\varphi = (x^1, \dots, x^m)$, $T_pM = \{$Curves cross $p \in M\}/\sim$ where the equivalence relation is defined by $\forall \gamma_1, \gamma_2 : I \to \mathbb{R}$.*

$$\gamma_1 \sim \gamma_2 \iff \frac{\mathrm{d}}{\mathrm{d}t}\big(x^i \circ \gamma_1(t) - x^i \circ \gamma_2(t)\big)\Big|_{t=0} = 0$$

The definition using curves is significant since this is the only way that one can define the tangent space and, more importantly, the chart-induced basis on a C^1-manifold.

It is easy to check that X is linear and Leibniz. We also need to show that the tangent vector in the curve definition is well defined on function germs, i.e.

Proposition 1.12 (Well-Definedness of Tangent Vectors)**.** *Let M be a smooth manifold, $p \in M$. $f, g \in C^\infty(M)$ where U_p is an open neighborhood of p. Then $\forall X_p \in T_pM$*

$$f|_{U_p} = g|_{U_p} \implies X_p(f) = X_p(g)$$

Proof. By the definition of tangent vector using curves, take $I_\epsilon := (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and the curve $\gamma : I_\epsilon \to M$ such that $\gamma(0) = p$

$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0}$$

Since $f|_{U_p} = g|_{U_p}$, it is sufficient to take $0 < \epsilon' < \epsilon$ such that $\gamma(I_{\epsilon'}) \subseteq U_p$ and thus

$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} g \circ \gamma(t)\Big|_{t=0} = X_p(g)$$

This proves the well-definedness. □

This interpretation of tangent vectors also provides us with a canonical choice of the basis in the tangent space.

Theorem 1.9 (Chart-Induced Basis)**.** Let M be a smooth manifold and $p \in M$. Consider the chart (U, φ) contains p, where $\varphi : U \to \mathbb{R}^m$ has coordinatte forms $\varphi = (x^1, \dots, x^m)$, then the chart induced basis is given by the curve $\gamma(t) := \varphi^{-1}(\varphi(p) + te^i)$, where e^i is the i-th basis of $\mathbb{R}^n$

$$\frac{\partial}{\partial x^i}\Big|_p f := \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0}, \quad i = 1, \dots, m$$

In simpler words, $\varphi(p) = 0 \in \mathbb{R}^m$

$$\frac{\partial}{\partial x^i}\Big|_p f := \frac{\partial f \circ \varphi^{-1}}{\partial x^i}\big(\varphi(p)\big)$$

It is easy to check that the two definitions given above are equivalent.

Proof. With the local chart (U,φ) that $\varphi = (x^1,\dots,x^m)$ on smooth manifold M, it is easy to check that

$$\frac{\partial}{\partial x^j}\Big|_p x^i = \delta^i{}_j$$

Thus, the set $\{\partial/\partial x^i|_p \mid i = 1,\dots,m\}$ is a linear independent set.
(Proof of generation with the algebraic definition). It follows the well-definedness proposition above that this construction satisfies the Leibniz rule. Then, $\forall f \in C^\infty(M)$, take $p, q \in U \subseteq M$ which $\varphi = (x^1,\dots,x^m) : U \to \varphi(U) \subseteq \mathbb{R}^m$ be the coordinate map, $x := \varphi(q)$ and $a := \varphi(p)$

$$\begin{aligned} f(q) = f\circ\varphi^{-1}(x) &= f\circ\varphi^{-1}(a) + \int_0^1 \mathrm{d}t \Big[\frac{\mathrm{d}}{\mathrm{d}t} f\circ\varphi^{-1}\big(a - t(x-a)\big)\Big] \\ &= f\circ\varphi^{-1}(a) + \sum_{i=1}^m (x^i - a^i) g_i(x), \quad g_i(x) = \int_0^1 \mathrm{d}t \frac{\partial f\circ\varphi^{-1}}{\partial x^i}(a - t(x-a)) \end{aligned}$$

By the definition,

$$g(a) = \frac{\partial f\circ\varphi^{-1}}{\partial x^i}\big(\varphi(p)\big) = \frac{\partial}{\partial x^i}\Big|_p f$$

Then, given a tangent vector $X_p \in T_pM$

$$\begin{aligned} X_p f &= X_p\Big(f\circ\varphi^{-1}(a) + \sum_{i=1}^m (x^i - a^i) g_i(x)\Big) \\ &= \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}\Big|_p f, \quad \text{where } a^i = X_p(x^i) \end{aligned}$$

Which proves the theorem.
(Proof of generation using the velocity of curves). For any curve $\gamma : I \to M$ with $\gamma(0) = p$, the velocity is given by

$$\begin{aligned} \dot\gamma(f) &= \frac{\mathrm{d}}{\mathrm{d}t} f\circ\gamma(t)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\tilde f\circ\tilde\gamma(t)\Big|_{t=0}, \quad \text{where } \tilde f = f\circ\varphi^{-1} \text{ and } \tilde\gamma = \varphi\circ\gamma \\ &= \sum_{i=1}^m a_i \frac{\partial f\circ\varphi^{-1}}{\partial x^i}\big(\varphi(p)\big) = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}\Big|_p f, \quad \text{where } a^i = \frac{\mathrm{d}x^i\circ\gamma}{\mathrm{d}t}\Big|_{t=0} \end{aligned}$$

Which proves the theorem. □

The proof above also shows that the two definitions of tangent spaces are equivalent, and $\mathcal{D}_pM \cong T_pM = \mathrm{Span}\{\partial_i|_p \mid i = 1,\dots,n\}$ (without ambiguity, we write $\partial_i|_p := \partial/\partial x^i|_p$ when there is only one coordinate), and for any differentiable n-dimensional manifolds, the tangent space is n-dimensional vector space. Another thing that needs to be checked is the behavior of tangent vectors under a change of charts:

Proposition 1.13. *Given charts $\big(U,\varphi = (x^1,\dots,x^m)\big)$ and $\big(V,\psi = (y^1,\dots,y^m)\big)$ be charts on smooth manifold M contains p. Then $\forall X_p \in T_pM$*

$$X_p = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^m b^i \frac{\partial}{\partial y^i}\Big|_p$$

where the coefficient $\mathbf{a} = (a^1,\dots,a^n)^T$ and $\mathbf{b} = (b^1,\dots,b^n)^T$ satisfies $\mathbf{a} = J(\varphi\circ\psi^{-1})\mathbf{b}$, or, in component

$$a^i = \sum_{j=1}^m \frac{\partial(x^i\circ\psi^{-1})}{\partial y^j}\big(\psi(p)\big) b^j$$

Proof. By the definition of a chart-induced basis, one can write

$$X_p = \sum_{i=1}^{m} (X_p x^i) \frac{\partial}{\partial x^i}|_p$$

Then, let $f_j = (\varphi \circ \psi^{-1})_j$

$$\frac{\partial}{\partial y^i}\Big|_p = \sum_{j=1}^{m} \Big(\frac{\partial}{\partial y^i}\Big|_p x^j \Big) \frac{\partial}{\partial x^j}\Big|_p = \sum_{j=1}^{m} \frac{\partial f_j}{\partial y^i}(\psi(p)) \frac{\partial}{\partial x^j}\Big|_p$$

Where $\partial f_j / \partial y^i$ is the (i,j)-th entry of the Jacobi matrix $J(\varphi \circ \psi^{-1})$. Thus, the proposition was proved. □

As the final topic of the "space" level illustration of linearization, here is a quick overview of differential topology that shows how a certain decomposition of the tangent space gives a submanifold.

Theorem 1.10. Let M be a smooth manifold, given $p \in M$ and $H_p \subseteq T_pM$ be a linear subspace. Then exists a embedded submanifold $S \subseteq T_pM$ that $p \in S$ and

$$T_pS = H_p \subseteq T_pM$$

The key to the proof is to use the level set theorem (Theorem 1.8).

Proof. Suppose $\dim M = m$, $\dim H_p = n$, one shall define the submersion F that for some chart $(U, \psi : U \to \mathbb{R}^m)$ contains p with $\psi(p) = 0 \in \mathbb{R}^m$, and consider the linear subspace

$$W = \psi_{*,p}(H_p) \subseteq \mathbb{R}^m$$

We shall give a linear map (for example, a projection) $L : \mathbb{R}^m \to \mathbb{R}^{m-n}$ such that $\ker L = W$. Then, there exists a natural submersion

$$F := L \circ \psi : U \to \mathbb{R}^{m-n}$$

(it is clearly a submersion since ψ is a difeomorphism and linear map L always have constant rank.) such that the pushforward of F is given by

$$F_{*,p} = L \circ \psi_{*,p} : T_pM \to \mathbb{R}^{m-n}$$

The kernel of the linear map above is given by

$$\ker F_{*,p} = (\psi_{*,p})^{-1}(\ker L) = (\psi_{*,p})^{-1}(W) = H_p$$

Using the level set theorem (Theorem 1.8), since $F : M \to \mathbb{R}^{m-n}$ has constant rank, $S = F^{-1}(0) \subseteq M$ is an regular submanfold. Since $F(p) = L \circ \psi(p) = L(0) = 0$, $p \in S$. By the constant rank theorem (Theorem 1.7), one shall pick local coordinates such that F hasa local coordinate representation

$$\tilde{F}(u^1, \ldots, u^m) = (u^1, \ldots, u^{m-n})$$

and thus $\tilde{F}^{-1}(0) \cong \{0\}^{m-n} \times \mathbb{R}^n$ with $F^{-1}(0) \cap V \cong \tilde{F}^{-1}(0) \cap \varphi(V)$ for some chart (V, φ) in M. Defined

$$\tilde{S} := \tilde{F}^{-1}(0) \cap \varphi(V) = \{(u^1, \ldots, u^m) \in \varphi(V) \mid u^1 = \cdots = u^{m-n} = 0\} \cong S$$

Then $T_{\varphi(p)}\tilde{S} \cong T_pS$ with explicitly coordinate representation that

$$T_{\varphi(p)}\tilde{S} = \{(v^1, \ldots, v^m) \in \mathbb{R}^m \mid v^1 = \cdots = v^{m-n} = 0\}$$

On the other hand, since $\tilde{F}_{*,p}(v^1, \ldots, v^m) = (v^1, \ldots, v^{m-n})$

$$\ker F_{*,p} \cong \ker \tilde{F}_{*,\varphi(p)} = \{(v^1, \ldots, v^m) \in \mathbb{R}^m \mid v^1 = \cdots = v^{m-n} = 0\} = T_{\varphi(p)}\tilde{S} \cong T_pS$$

Which shows that $T_pS \cong \ker F_{*,p}$. □

After stating the space level linearization, there must be some mapping level linearization locally, which, in more precise terms, is a local induced map on tangent spaces from a differentiable map between manifolds.

Definition 1.17 (Pushforwards / Tangent Maps)**.** Given smooth manifolds M^m and N^n and a differentiable map $f : M \to N$. Then, the pushforward corresponding to f is defined by $\forall X_p \in T_pM : \forall g \in C^\infty_{f(p)}$

$$f_{*,p}X_p(g) = X_p(g \circ f)$$

In this way, we obtain a map $f_{*,p} : T_pM \to T_{f(p)}N$.

Proposition 1.14. *For smooth map $f : M \to N$, its pushforwards $f_{*,p}$ has following properties:*

1. *$f_{*,p}$ is a linear map.*
2. $(\mathrm{id}_M)_{*,p} = \mathrm{id}_{T_pM}$
3. *With the curve definition, $\forall g \in C^\infty_p : \forall h \in C^\infty_{f(p)}$*

$$X_p(g) = \frac{\mathrm{d}}{\mathrm{d}t} g \circ \gamma(t)\Big|_{t=0} \iff f_{*,p}X_p(g) = \frac{\mathrm{d}}{\mathrm{d}t} g \circ f \circ \gamma(t)\Big|_{t=0}$$

The proof is simple; we shall only prove the linearity.

Proof. Suppose $X_p = a_1X_1 + a_2X_2$, directly by the definition $\forall g \in C^\infty_{f(p)}$

$$\begin{aligned} f_{*,p}X_p(g) &= X_p(g \circ f) = a_1X_1(g \circ f) + a_2X_2(g \circ f) \\ &= a_1(f_{*,p}X_1)(g) + a_2(f_{*,p}X_2)(g) \end{aligned}$$

This shows the linearity. □

As we have defined above, the tangent vector can be written as the velocity of smooth curves. Now, with the pushforward of smooth maps, we can say that for smooth curves $\gamma : I \to M$ such that $\gamma(t_0) = p$, then the velocity vector $\gamma'(t_0) \in T_pM$

$$\gamma'(t_0) := \gamma_*\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0}\Big), \quad \text{where } \mathrm{d}/\mathrm{d}t|_{t_0} \text{ is just the unit vector on } T_{t_0}\mathbb{R}.$$

Similar to the differential in Euclidean spaces, the chain rule also applies to the pushforward on smooth manifolds

Theorem 1.11 (Chain Rule)**.** Let M, N, and P be smooth manifolds. If $f : M \to N$ and $g : N \to P$ be smooth maps between manifolds, $\forall p \in M$

$$(g \circ f)_{*,p} = g_{*,f(p)} \circ f_{*,p}$$

Proof. Given arbitary $X_p \in T_pM$, and $\forall h \in C^\infty_{g\circ f(p)}$. Then

$$(g \circ f)_{*,p}X_p(h) = X_p(h \circ g \circ f)$$

and

$$(g_{*,f(p)} \circ f_{*,p})X_p(h) = f_{*,p}X_p(h \circ g) = X_p(h \circ g \circ f)$$

which completes the proof. □

The pushforward of a differentiable map $f : M \to N$ works similarly to the Jacobi matrix of the differentiable map in Euclidean space (actually, at the end of this subsection, one can see that it is the same as the Jacobi matrix), so we shall also define the regular and critical points using the pushforwards:

Definition 1.18 (Regular/Critical Points)**.** For C^∞-map $f : M \to N$, $p \in M$ is a regular point if $f_{*,p}$ is surjective. The point $q \in N$ is said to be a regular value if $f^{-1}(q)$ contains only regular points. Otherwise, p is a critical point and $f(p)$ is a critical value.

Note that the image of regular points may not be regular value.

In the following passages, we will find that under the chart-induced basis, the pushforward can be written in matrix form as a Jacobi matrix, and in this case, the chain rule directly follows from the chain rule of the Jacobi matrix on Euclidean space.

Recall that the pushforward is a generalization of differentiation in Euclidean space, which means the pushforward is expected to have the form of a Jacobi matrix in local coordinates:

Theorem 1.12 (Local Coordinate Expression of Pushforwards)**.** Consider chart $(U, \varphi = (x^1, \ldots, x^m))$ and $(V, \psi = (y^1, \ldots, y^n))$ on M and N where $f : M \to N$ has local coordinate representation

$$\psi \circ f \circ \varphi^{-1} = \big(f^1(x), \ldots, f^n(x)\big)$$

With the chart-induced basis, $\forall X_p \in T_pM$, $X_p = \sum_{i=1}^m a^i \partial/\partial x^i|_p$

$$f_{*,p}X_p = \sum_{j=1}^n b^j \frac{\partial}{\partial y^j}\Big|_{f(p)}, \quad b^j = \sum_{i=1}^m a^i \frac{\partial f^j}{\partial x^i}\big(\varphi(p)\big)$$

The proof is direct by computation.

Proof. By the definition of a chart-induced basis

$$f_{*,p}\frac{\partial}{\partial x^i}\Big|_p = \sum_{j=1}^n \Big(f_{*,p}\frac{\partial}{\partial x^i}\Big|_p\Big)(f^i)\frac{\partial}{\partial y^j}\Big|_{f(p)}$$

To compute the coefficient, it is sufficient to show the transformation law on the basis of:

$$\begin{aligned} f_{*,p}\frac{\partial}{\partial x^i}\Big|_p(f^j) &= \frac{\partial}{\partial x^i}\Big|_p(y^j \circ f) = \frac{\partial}{\partial x^i}(y^j \circ f \circ \varphi^{-1})\big(\varphi(p)\big) \\ &= \frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})_j\big(\varphi(p)\big) = \frac{\partial f^j}{\partial x^i}\big(\varphi(p)\big) \end{aligned}$$

Thus, the pushforward of a general vector $X_p \in T_pM$ has coordinate representation given by

$$\begin{aligned} f_{*,p}X_p &= \sum_{i=1}^m a^i f_{*,p}\frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^m a^i \sum_{j=1}^m \frac{\partial f^j}{\partial x^i}\big(\varphi(p)\big)\frac{\partial}{\partial y^j}\Big|_{f(p)} \\ &= \sum_{j=1}^m \Big(\sum_{i=1}^m a^i \frac{\partial f^j}{\partial x^i}\big(\varphi(p)\big)\Big)\frac{\partial}{\partial y^j}\Big|_{f(p)} = \sum_{j=1}^m b^j \frac{\partial}{\partial y^j}\Big|_{f(p)} \end{aligned}$$

Where $(a^1, \ldots, a^m)^T = J(\psi \circ f \circ \varphi^{-1})(b^1, \ldots, b^m)^T$ gives the transformation on coefficients. $\square$

The theorem above also shows that the matrix form we obtain from the chart induced basis $\{\partial/\partial x^i \mid i = 1, \ldots m\}$ and $\{\partial/\partial y^j \mid j = 1, \ldots n\}$ is the Jacobi matrix of the map f in local coordinate:

$$f_{*,p} = J(\psi \circ f \circ \varphi^{-1})\big(\varphi(p)\big)$$

The idea of the chart induced form of the push forward also helps us prove the following theorem: the preimage of a regular value of smooth maps gives a embedded submanifolds on the source manifold:

Theorem 1.13 (Preimage of Regular Value)**.** Let $f : M^m \to N^n$ with regular value $q \in N$, then

- $f^{-1}(q)$ is a $(m-n)$-dimensional embedded submanifold of M.
- $\forall p \in f^{-1}(q)$, the tangent space satisfies

$$T_p f^{-1}(q) = \ker f_{*,p}$$

Proof. Consider charts (U, φ) on M and (V, ψ) contains q on N such that $\varphi(p) = \psi(q) = 0$,

$$\tilde{f} := \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n, \quad \tilde{f}(0) = 0$$

q is a regular value of $\tilde{f}$ if and only if the Jacobi matrix $J(\tilde{f})$ has rank n at $0 \in \mathbb{R}^m$. With a permutation of coordinates, we shall let $\tilde{f} = (f_1, \ldots, f_n)$ with $f_i : \varphi(U) \to \mathbb{R}^n$

$$\left| \frac{\partial(f_1, \ldots, f_n)}{\partial(x^1, \ldots, x^n)}(0) \right| \neq 0$$

By inverse function theorem (Theorem 1.2), we shall take $F : \varphi(U) \to \mathbb{R}^m$ define by

$$F(x^1, \ldots, x^m) := (f_1(x), \ldots, f_n(x), x^{n+1}, \ldots, x^m)$$

and this gives an local diffeomorphism $\varphi(U) \supseteq W \cong W'$ and one shall defined a new chat $(U' := \varphi^{-1}(W), \phi)$ with $\phi = (y^1, \ldots, y^m)$ defines by

$$\phi := (F|_W)^{-1} \circ \varphi : U' \to W', \text{ which explicitly just } \begin{cases} \big(y^1(x), \ldots, y^n(x)\big) = \tilde{f}(x) \\ \big(y^{n+1}(x), \ldots, y^m(x)\big) = (x^{n+1}, \ldots, x^m) \end{cases}$$

where $f^{-1}(q) \cap U' \cong \tilde{f}^{-1}(0) \cap \psi(U')$ (since $\psi(q) = 0$) has coordinate representation

$$f^{-1}(q) \cap U' := \{p \in U' \mid y^1(p) = \cdots = y^n(p) = 0\}$$

By Theorem 1.6, this shows that $f^{-1}(q)$ is an $(m-n)$-dimensional submanifold.
Furthermore, $f^{-1}(q)$ clearly has tangent space given by

$$T_p f^{-1}(q) := \text{Span}\left\{ \frac{\partial}{\partial y^{n+1}}\Big|_p, \ldots, \frac{\partial}{\partial y^m}\Big|_p \right\}$$

Since in local coordinate $F(y) = (y^1, \ldots, y^n)$,

$$\ker f_{*,p} \cong \ker(JF) = \{v \in \mathbb{R}^m \mid v^1 = \cdots = v^n = 0\}$$

which shows that $\ker f_{*,p} = \text{Span}\{\partial_{y^{n+1}}, \ldots, \partial_{y^m}\} = T_p f^{-1}(q)$. $\square$

With the local coordinate representation of pushforward, it is obvious that the rank of the smooth map $f : M \to N$ at $p \in M$ can be represented by the rank of pushforward $f_{*,p} : T_pM \to T_{f(p)}N$ as a linear map. Thus, we have the following lemma of differentiable maps with constant rank:

Proposition 1.15. *Let M and N be smooth manifolds. The smooth map $f : M \to N$ is said to be an immersion if $\forall p \in M : f_{*,p} : T_pM \to T_{f(p)}N$ is injective, and a submmersion if $\forall p \in M : f_{*,p} : T_pM \to T_{f(p)}N$ is surjective (i.e., every points $p \in M$ is a regular point).*

The proof is simply by definition.

After this section, we are going to slightly abuse a notation. In the following passages, without ambiguity, for any $f \in C^\infty(M)$, we are going to use the notation under the chart-induced basis

$$\frac{\partial f}{\partial x^i}\Big|_p := \frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(p))$$

1.3.2 Transversality

With the previous introduction to submanifolds and the tangent map, we can now answer a natural question: Suppose $f \in C^\infty(M, N)$ and a embedded submanifold $S \subseteq N$, when does the preimage $f^{-1}(S)$ become a embedded submanifold in M?

The answer is the tansversality:

Definition 1.19 (Transversality)**.** Let $f \in C^\infty(M, N)$ and $Z \subseteq N$ be an embedded submanidfold, then f intersect with Z transversely if $\forall p \in M$ with $q = f(p) \in Z$

$$f_{*,p}(T_pM) + T_qZ = T_qN$$

denote as $f \pitchfork Z$. Sometimes we say f intersect with Z transversely also when $f(M) \cap Z = \emptyset$.

Remark. *To have a better understanding of the definition, consider the following lemma:*

Lemma. *Suppose $L \in \mathrm{Hom}_{\mathsf{Vect}}(V, W)$ is surjective, then $L(V_1) = W$ for linear subspace $V_1 \subseteq V$ if and only if*

$$V_1 + \ker L = V$$

The proof of the lemma is simple. The lemma actually gives the Proposition 1.16 in the following text, which shows the connection between regular points and transversality.

in particular, if $M \subseteq N$ is also an embedded manifold, the $M \pitchfork N$ if $\forall p \in M \cap Z$

$$T_pM + T_pZ = T_pN$$

and if we take $Z = \{q\}$ be a 0-dimensional embedded submanifold, by Theorem 1.13 $f \pitchfork \{q\}$ if and only if q is a regular value.

Proposition 1.16. *Let M and N be differentiable manifolds, $0 \in \mathbb{R}^k$ is a regular value of the smooth function $g : N \to \mathbb{R}^k$, and $Z := g^{-1}(0)$. Then $f \pitchfork Z$ if and only if 0 is a regular value of $g \circ f$.*

Proof. ($\Rightarrow$) Suppose $f \pitchfork Z$, then

$$f_{*,p}(T_pM) + T_{f(p)}Z = T_{f(p)}N$$

We shall compute

$$(g \circ f)_{*,p}(T_pM) = g_{*,f(p)}(f_{*,p}T_pM) = g_{*,f(p)}(T_{f(p)}N - T_{f(p)}Z)$$

By the fact $Z = g^{-1}(0)$, $\ker(g_{*,f(p)}) = T_{f(p)}Z$. Thus, by the lemma above

$$(g \circ f)_{*,p}(T_pM) = T_0\mathbb{R}^k \cong \mathbb{R}^k$$

Which shows that 0 is a regular value of $g \circ f$.
($\Leftarrow$) Follows the same logic, if 0 is a regular value of $g \circ f$,

$$(g \circ f)_{*,p}(T_pM) = g_{*,f(p)}(f_{*,p}T_pM) = T_0\mathbb{R}^k$$

By rank-nullity theorem, since $\ker(g_{*,f(p)}) = T_{f(p)}Z$ and $g_{*,f(p)} : T_{f(p)}N \to T_0\mathbb{R}^k$

$$f_{*,p}T_pM + T_{f(p)}Z = T_pN$$

which completes the proof of the proposition. $\square$

A more general notion of dimension that appears in the discussion of transversality and foliation (it will appear in Section 3.4) is the codimension.

Definition 1.20 (Codimension)**.** Let N be a differentiable manifold, and let $Z \subseteq N$ be an embedded submanifold. The codimension of Z is given by

$$\operatorname{codim} Z = \dim N - \dim Z$$

A better motivation on introducing the codimension instead of the dimension is the following theorem. Then, we have the following result that connects the codimension of submanifolds and transversality:

Theorem 1.14. Let M^m and N^n be smooth manifolds, $f \in C^\infty(M, N)$. Then, for embedded submanifold $Z \subseteq N$, $f \pitchfork Z$ implies $f^{-1}(Z) \subseteq M$ is a embedded submanifold and

$$\operatorname{codim} f^{-1}(Z) = \operatorname{codim} Z$$

Proof. Let $p \in f^{-1}(Z)$, $q = f(p) \in Z$, by Theorem 1.6, one shall find $U_q \subseteq N$ be an open neighborhood of q with a submersion $g : U_q \to \mathbb{R}^k$ ($k = \dim N - \dim Z$), $g = (x^{n-k+1}, \dots, x^n)$ such that

$$Z \cap U_q := g^{-1}(0) = \{q' \in U_q \mid x^{n-k+1}(q') = \cdots = x^n(q') = 0\}$$

By Proposition 1.16, $f \pitchfork Z$ implies that 0 is a regular value of $g \circ f$, thus, by Theorem 1.8

$$f^{-1}(Z) = (g \circ f)^{-1}(0) \subseteq f^{-1}(U_q) =: V_p$$

is a $\dim M - k$ dimensional embedded submanifold. Zince $k = \dim N - \dim Z$

$$\dim M - \dim f^{-1}(Z) = \dim M - (\dim M - \dim N + \dim Z) = \dim N - \dim Z$$

Which completes the proof of the theorem. □

As the final theorem of this subsection about transversality, we shall show that this property has stability. More precisely, the transversality is preserved in a small perturbation.

Theorem 1.15 (Stability of Transversality)**.** Let M, P, and N be differentiable manifolds, where M compact, $Z \subseteq N$ is an (topologically) closed embedded submanifold. Let $f : M \times P \to N$ be a smooth map, $\forall p \in P$, we shall defined

$$f_p(x) := f(x, p) \quad \forall x \in M$$

Then $\Omega := \{p \in P \mid f_p \pitchfork Z\} \subseteq P$ is open.

Proof. Suppose $p_0 \in \Omega$, we shall prove by contradiction. Suppose Ω is not open, then there exists sequence $\{p_i\}_{i=1}^\infty$ such that $p_i \to p_0$ but f_{p_i} $\forall i$ does not transverse with Z. Then we shall take $\{x_i\}_{i=0}^\infty \subseteq M$ such that $f(x_i, p_i) = y_i \in Z$ $\forall i$ and by the definition of transversality

$$f_{*,(x_i,p_i)}(T_{p_i}M) + T_{y_i}Z \neq T_{y_i}N$$

By the compactness of M, we shall take a convergent subsequence of $\{x_i\}$. Without loss of generality, we shall take $x_i \to x_0$, i.e., $(x_i, p_i) \to (x_0, p_0)$. Since Z is closed in N, $y_i \to y_0 = f(x_0, p_0)$. By Theorem 1.6, we shall take open set $V \subseteq N$ contains y_0 and $\psi : V \to \mathbb{R}^k$, such that

$$Z \cap V = \psi^{-1}(0)$$

then by Proposition 1.16, $0 \in \mathbb{R}^k$ is the regular value of $g_{p_0} := \psi \circ f_{p_0}$, i.e., $(g_{p_0})_{*,x_0}$ surjective. We know that subjectivity implies the existence of a k-minor with non-zero determinant. Since the determinant is continuous, there must exist a small neighborhood $U \times V$ of (x_0, p_0) such that $\forall (x_i, p_i) \in U \times V$, the pushforward $(g_{p_i})_{*,x_i}$ is surjective, which contradicts the setting that for any (x_i, p_i) in the sequence, f_{p_i} not transverse with Z at x_i. Thus, the theorem was proved. □

The following theorem is the main theorem of this subsection:

Theorem 1.16 (Transversality Theorem)**.** Let $f : M \times P \to N$ be a smooth map with $f_p := f(-, p)$ for $p \in P$, and $f \pitchfork Z$ for some embedded submanifold $Z \subseteq N$. Then, $f_p \pitchfork Z$ almost everywhere on N.

We cannot prove the theorem yet, since it requires Sard's theorem (Theorem 2.6), the proof will be in section 2.3. As a result of Theorem 1.16, we know that the transversality is generic:

Corollary. *For any smooth map $f : M \to \mathbb{R}^k$ with embedded submanifold $N \subseteq \mathbb{R}^k$, for almost every $v \in \mathbb{R}^k$,*

$$f_v : M \to \mathbb{R}^k, \quad p \mapsto f_v(p) = f(p) + v$$

transverse with N.

Proof. Take smooth map $F : M \times \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$F(p, v) := f(p) + v$$

it is easy to show that for fixed $p \in M$, $F(p, -)$ is a diffeomorphism on $\mathbb{R}^k$. Thus, F is a submersion from $M \times \mathbb{R}^k$ to $\mathbb{R}^k$. By the definition of submersion, since $f_{*,p}$ is surjective, the condition required for transversality has automatically been satisfied. □

Recall that two maps $f, g : M \to N$ are homotopy if $\exists H \in C(I \times M, N)$ with $I := [0, 1]$, such that $\forall x \in M$

$$H(0, x) = f(x), \quad H(1, x) = g(x)$$

The homotopy is an equivalent relation on $C(M, N)$, denoted as $H : f \sim g$.

Using the tubular neighborhood theorem (Theorem 3.1), we can also show the following theorem:

Theorem 1.17 (Homotopy Transversality Theorem)**.** Let $f \in C^\infty(M, N)$, and $Y \subseteq N$ be a embedded submanifold. Then exists $g : M \to N$ such that $f \sim g$ and $g \pitchfork Y$. Also, if $X \subseteq M$ is a (topologically) closed embedded submanifold, and $\forall p \in f^{-1}(Y) \cap X$, $f \pitchfork Y$, then we shall choose g such that $g|_X = f|_X$.

We will revisit this theorem and show more detail about this in Section 3.2.

1.3.3 Cotangent Space and Pullback

Definition 1.21 (Cotangent Space)**.** The cotangent space is defined to be the dual space of the tangent space, i.e., let M be a differentiable manifold, $\forall p \in M : T_p^*M = (T_pM)^*$

There is a class of cotangent vectors induced from the smooth function on the manifold M,

Definition 1.22 (Differential of $\mathbb{R}$-Valued Functions)**.** Let $f \in C^\infty(M)$, then $\forall p \in M : \exists \mathrm{d}f_p \in T_p^*M$ defined by

$$\mathrm{d}f_p(X_p) := X_p(f)$$

In the previous passage, we have defined the pushforward of a general smooth map between smooth manifolds. We shall show that the above definition of the differential of a real-valued function coincides with the previous definition.

Proposition 1.17. *Let $f \in C^\infty(M)$, then $\forall p \in M : \forall X_p \in T_pM$*

$$f_{*,p}X_p = \mathrm{d}f_p(X_p)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{f(p)}$$

where $\mathrm{d}/\mathrm{d}t|_{f(p)}$ is the unit vector on $T_{f(p)}\mathbb{R}$.

Proof. Since $T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is a 1-dimensional vector space, one can write the tangent vector as

$$f_{*.p}X_p = a\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{f(p)}$$

To evaluate the coefficient a, recall the definition of the chart-induced basis of the tangent space, the coordinate map of $\mathbb{R}$ canonically is given by the identity $\varphi(t) = t$, and thus, the coefficient is given by

$$a = f_{*,p}X_p(t) = X_p(t \circ f) = X_p(f) = \mathrm{d}f_p(X_p)$$

Thus, the proof is complete. □

Recall the definition of dual space in linear algebra, the cotangent space (on a real manifold) is just $T_p^*M := \mathrm{Hom}_{\mathsf{Vect}}(T_pM, \mathbb{R})$, the collection of all linear maps from the tangent space at $p \in M$ to real numbers. Then, similar to vectors, there should be a chart-induced basis of cotangent spaces:

Theorem 1.18 (Local-Coordinate Expression of Cotangent Vectors)**.** Let M be a smooth manifold and (U, φ) be a chart on M, where $\varphi = (x^1, \ldots, x^m)$. The cotangent space has the dual basis $T_p^*M = \langle(\mathrm{d}x^1)_p, \ldots, (\mathrm{d}x^m)_p\rangle$ such that

$$(\mathrm{d}x^i)_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \delta^i{}_j$$

Proof. The key point is to show that the cotangent vector $\mathrm{d}x^i$ is a dual basis, and it automatically generates the cotangent space and is linearly independent.
By the previous definition of the differential of a real-valued function,

$$(\mathrm{d}x^i)_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \frac{\partial}{\partial x^j}\Big|_p x^i = \frac{\partial(x^i \circ \psi^{-1})}{\partial x^j}\big(\psi(p)\big) = \delta^i{}_j$$

Thus, $\{\mathrm{d}x^i\}_{i=1}^m$ is a dual basis. □

Thus, $\forall \omega_p \in T_pM$, the cotangent vector can be written in the chart-induced basis

$$\omega_p = \sum_{i=1}^m a_i \mathrm{d}x^i|_p$$

Where the coefficient is given by $a_i = \omega_p(\partial/\partial x^i|_p)$ since $\mathrm{d}x^i|_p(\partial/\partial x^j|_p) = \delta^i{}_j$.

Also, recall that in real analysis, with some $U \subseteq \mathbb{R}^m$, $\forall f \in C^\infty(U) : \mathrm{d}f = \sum_{i=1}^n (\partial f/\partial x^i)\mathrm{d}x^i$. An analogous local expression holds for smooth manifolds. However, while this formula is globally valid in $\mathbb{R}^m$, its globalization to an arbitrary smooth manifold requires the machinery of the tangent and cotangent bundles. Suppose

$$\mathrm{d}f_p = \sum_{i=1}^m a_i(\mathrm{d}x^i)_p$$

By the expression of the coefficient we get above,

$$\mathrm{d}f_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \sum_{i=1}^m a_i(\mathrm{d}x^i)_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \sum_{i=1}^m a_i\delta^i{}_j = a_j$$

and by the definition of $\mathrm{d}f_p(\partial/\partial x^j|_p) = \partial f/\partial x^j|_p$ (this is slightly a notation abuse since the derivation is in the sense of a tangent vector acting on a smooth function). Thus, $a_i = \partial f/\partial x^i|_p$, and

$$\mathrm{d}f_p = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i}\Big|_p (\mathrm{d}x^i)_p$$

Similar to the pushforward induced from the smooth map, the smooth map also induces a linear map between the cotangent spaces.

Definition 1.23 (Codifferential/Pullback of Smooth Maps)**.** Consider M and N be differential manifolds, $f \in C^\infty(M,N)$. $\forall p \in M$, the pullback induced by f is the dual of pushforward

$$f_p^* := (f_{*,p})^\vee : T_{f(p)}^* N \to T_p^* M$$

which $\forall \omega_{f(p)} \in T_{f(p)}^* N$ and $X_{f(p)} \in T_p M$, the following construction has is the explicit illustration of the term "dual", denotes as $(f_{*,p})^\vee$

$$f_p^* \omega_{f(p)}(X_p) := \big((f_{*,p})^\vee \omega_{f(p)}\big)(X_p) = \omega_{f(p)}(f_{*,p} X_p)$$

Remark. *One can also pullback the function on manifolds via $\forall g \in C^\infty(N) : f^* g = g \circ f \in C^\infty(M)$. This map is linear and distributive on the multiplication of smooth functions. As the dual of pullback, the pushforward of $X_p \in T_p M$ can then be written as*

$$\forall g \in C^\infty(N) : f_{*,p} X_p(g) = X_p(f_p^* g)$$

The generalization (globalization) will be useful in the following discussion of tangent and cotangent vector fields.

Similar to pushforwards, the pullback has the following properties:

Proposition 1.18. *For smooth map $f : M \to N$, its pullback f_p^* has the following properties:*

1. *f_p^* is a linear map.*
2. *$(\mathrm{id}_M)_p^* = \mathrm{id}_{T_p^* M}$*

The proof simply follows the properties of pushforwards.

As the dual of pushforward/differential of the smooth map, the pullback/codifferential should also have a local coordinate representation:

Theorem 1.19 (Local Coordinate Expression of Pullbacks)**.** Consider chart $(U, \varphi = (x^1, \ldots, x^m))$ and $(V, \psi = (y^1, \ldots, y^n))$ on M and N where $f : M \to N$ has local coordinate representation

$$\psi \circ f \circ \varphi^{-1} = \big(f^1(x), \ldots, f^n(x)\big)$$

With the chart-induced basis, $\forall \omega_{f(p)} \in T_{f(p)}^* N$, $\omega_{f(p)} = \sum_{i=1}^n a_i (\mathrm{d}y^i)_{f(p)}$

$$f_p^* \omega_{f(p)} = \sum_{j=1}^{m} b_j (\mathrm{d}x^j)_p, \quad b_j = \sum_{i=1}^{n} a_i \frac{\partial f^i}{\partial x^j}\big(\varphi(p)\big)$$

Proof. We can prove directly by definition, and it is sufficient to prove the transformation rule on the chart-induced basis. Consider the chart-induced basis $\{(\mathrm{d}y^i)_{f(p)} \mid i = 1, \ldots, n\}$, $\forall X_p \in T_p M$, by the definition of pullback

$$f_p^* (\mathrm{d}y^i)_{f(p)}(X_p) = (\mathrm{d}y^i)_{f(p)}(f_{*,p} X_p)$$

Given that $X_p = \partial/\partial x^i|_p$ the base vector of T_pM,

$$f_{*,p}\frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^{n}\frac{\partial f^j}{\partial x^i}(\varphi(p))\frac{\partial}{\partial y^j}\Big|_{f(p)}$$

Then, it is sufficient to apply the cotangent vector on the basis

$$\begin{aligned}
f_p^*(\mathrm{d}y^i)_{f(p)}\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) &= (\mathrm{d}y^i)_{f(p)}\Big(f_{*,p}\frac{\partial}{\partial x^j}\Big|_p\Big) \\
&= \sum_{k=1}^{n}\frac{\partial f^k}{\partial x^j}(\varphi(p))(\mathrm{d}y^i)_{f(p)}\Big(\frac{\partial}{\partial y^k}\Big|_{f(p)}\Big) = \frac{\partial f^i}{\partial x^j}(\varphi(p)) \\
\implies f_p^*(\mathrm{d}y^i)_{f(p)} &= \sum_{j=1}^{m} f_p^*(\mathrm{d}y^i)_{f(p)}\Big(\frac{\partial}{\partial x^j}\Big|_p\Big)(\mathrm{d}x^j)_p = \sum_{j=1}^{m}\frac{\partial f^i}{\partial x^j}(\varphi(p))(\mathrm{d}x^j)_p
\end{aligned}$$

Thus, for a general cotangent vector $\omega_{f(p)} = \sum_{i=1}^n a_i(\mathrm{d}y^i)_{f(p)} \in T^*_{f(p)}N$, the pullback is given by

$$\begin{aligned}
f_p^*\omega_{f(p)} &= \sum_{i=1}^{n} a_i f_p^*(\mathrm{d}y^i)_{f(p)} = \sum_{i=1}^{n} a_i \sum_{j=1}^{m}\frac{\partial f^i}{\partial x^j}(\varphi(p))(\mathrm{d}x^j)_p \\
&= \sum_{j=1}^{m}\Big(\sum_{i=1}^{n} a_i\frac{\partial f^i}{\partial x^j}(\varphi(p))\Big)(\mathrm{d}x^j)_p = \sum_{j=1}^{m} b_j(\mathrm{d}x^j)_p
\end{aligned}$$

Thus, the theorem was proved. □

Before we start the nest topic, it is important to briefly talk about tangent and cotangent bundle in a more intuitive way. The tangent bundle, on set level, is just the disjoint union of all tangent spaces:

$$TM := \coprod_{p\in M} T_pM$$

which allow us to (smoothly) assign each point a vector (in the corresponding vector space), and defines (smooth) vector fields. Similarly, the cotangent bundle is

$$T^*M := \coprod_{p\in M} T_p^*M$$

which allow us to defined covector field, or, differentiable 1-forms. The formal definition of this object will be in Chapter 3.

We have observed that in the formalism of tangent and cotangent spaces, not only has the base space been changed between manifolds and vector spaces, but each (smooth) map is also being mapped to a corresponding induced map. Such a phenomenon can be formalized in the language of category theory and is known as the functorial. In the following section, our goal is to show that the relation of tangent and cotangent spaces is a nice example of a widely existed phenomena called the dual of a category. The use of categorical language is not particularly beneficial here, but more as a concise example for the reader to get familiar with the concepts in category theory.

A Functor Approach of Tangent and Cotangent Spaces*

In this section, we admit the Axiom of Choice and fix a Grothendieck universe $\mathscr{U}$; throughout, all categories are $\mathscr{U}$-categories. We assume the reader is familiar with the basic notions of categories and functors; background relevant to this section is collected in Appendix E. For further details on category theory, as well as rigorous

*This part lies somewhat out of the main line of development of the chapter, and may be omitted in a first reading set-theoretic foundations that avoid size issues, see [31, 54]. The purpose of this section is to familiarize the reader with categorical language; it is not strictly necessary for the topics discussed here. A more algebraic (categorical) treatment of differential geometry requires the use of sheaves; see [48].

The most classical way of describing the tangent and cotangent spaces as functors is to use the category of C^k-manifolds Diff_k and the category of vector bundles VBdl. However, at this point, we have not yet introduce the vector bundle as the nontrivial globalization of point-wise attached vector spaces (like tangent spaces). Thus, in this section, the category we are considering is the category of C^k-manifolds with a marked base point, $\mathsf{Mani}^k_\bullet$; here is the precise definition:

Definition 1.24 (Manifold with a Base Point)**.** The category $\mathsf{Mani}^k_\bullet$ consists of the following data as a category:

1. $\mathrm{Obj}(\mathsf{Mani}^k_\bullet)$ = the pair (M,p) such that M is a C^k-manifolds and $p \in M$ is the marked point.
2. The morphism is given by $\forall (M,p),(N,q) \in \mathrm{Obj}(\mathsf{Mani}^k_\bullet)$, the morphism between them is given by

$$\mathrm{Hom}_{\mathsf{Mani}^k_\bullet}\big((M,p),(N,q)\big) := \{f \in C^k(M,N) \mid f(p) = q\}$$

   and the composition of the morphism is the composition of maps (as in Set).

As we have defined in the previous sections, in this category, there are two most direct (contra/co)variant functors, valued only in the "tangent/cotangent space":

Definition 1.25 (Tangent Functor)**.** Then tangent function $T : \mathsf{Mani}^k_\bullet \to \mathsf{Vect}_{\mathbb{R}}$ is a covariant functor that defined by

$$(M,p) \mapsto T_pM, \quad \big(f : (M,p) \to (N,q)\big) \mapsto (f_{*,p} : T_pM \to T_qN)$$

The functorial is given by the properties of pushforwards that

$$(\mathrm{id}_M)_{*,p} = \mathrm{id}_{T_pM}, \quad (f \circ g)_{*,f(p)} = f_{*,p} \circ g_{*,p}$$

Which, in our new language,

$$T(\mathrm{id}_{(M,p)}) = \mathrm{id}_{T_pM}, \quad T(f \circ g) = T(f) \circ T(g)$$

This shows the covariant functorial.

To provide a dual description of the category, we need to define the opposite category.

Definition 1.26 (Opposite (Dual) Category)**.** Given a category $\mathcal{C}$, the dual of $\mathcal{C}$ is named the opposite category, denoted as $\mathcal{C}^\vee$, such that

1. $\mathrm{Obj}(\mathcal{C}) = \mathrm{Obj}(\mathcal{C}^\vee)$.
2. $\forall X, Y \in \mathrm{Obj}(\mathcal{C}^\vee) : \big(f \in \mathrm{Hom}_{\mathcal{C}^\vee}(X,Y) \iff f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)\big)$.
3. The composition $f \circ g$ in $\mathcal{C}^\vee$ is just $g \circ f$ in $\mathcal{C}$.

The same trick also applies to cotangent cases, which we just replace the category $\mathsf{Mani}^k_\bullet$ by the opposite category (reverse all morphisms), $(\mathsf{Mani}^k_\bullet)^\vee$.

Definition 1.27 (Cotangent Functor)**.** The cotangent functor is the dual of the tangent functor. Explicitly, $T^* = (T)^\vee : (\mathsf{Mani}^k_\bullet)^\vee \to \mathsf{Vect}_\mathbb{R}$ is a covariant functor on the opposite category, defined by

$$(M,p) \mapsto T_p^* M, \quad \big(f : (M,p) \to (N,q)\big) \mapsto (f_p^* : T_q N \to T_p M)$$

Remark. *Note that for categories $\mathcal{C}, \mathcal{D}$, the contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is the same as the covariant functor on the opposite category $F : \mathcal{C}^\vee \to \mathcal{D}$, since the opposite category is just the category with the same objects but with reversed direction on all morphisms.*

In the same way, the cotangent functor is indeed the contravariant functorial on $\mathsf{Mani}^k_\bullet$

$$T^*(\mathrm{id}_{(M,p)}) = \mathrm{id}_{T_p^* M}, \quad T^*(f \circ g) = T^*(g) \circ T^*(g)$$

The final result of this section is to consider the behavior of the tangent functor T under (Cartesian) product (Note that we are not allowed to take infinite product on manifold). Consider the product of differentiable manifolds as

$$\forall (M,p),(N,q) \in \mathsf{Mani}^k_\bullet : (M,p) \times (N,q) := \big(M \times N, (p,q)\big)$$

It is sufficient to show that this satisfies the universal property of the product, such that the following diagram commutes

$$\begin{tikzcd}
 & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[d, "f"] \arrow[dr, "\pi_2"] & \\
X_1 \arrow[r, "f_1"] & Y & X_2 \arrow[l, "f_2"']
\end{tikzcd}$$

Then, by the functoriality of the tangent/cotangent functor, we know that $T_{(p,q)}(M \times N) = T_p M \times T_q N$ and $T^*_{(p,q)}(M \times N) = T_p^* M \oplus T_q^* N$. As finite-dimensional vector spaces, the product $\times$ and coproduct $\oplus$ are equivalent (called biproduct), we can also write

$$\begin{aligned} T_{(p,q)}(M \times N) &= T_p M \oplus T_q N \\ T^*_{(p,q)}(M \times N) &= T_p^* M \oplus T_q^* N \end{aligned}$$

A more algebraic geometrical taste of the smooth manifold is in Appendix **??**, which introduces the C^∞-ring and C^∞-Scheme.

1.4 The Essential Fact of Topological Groups and Lie Groups

The final topic of this chapter concerns the action of transformation groups on manifolds. In general topology, it is important to understand under what conditions a group action (or the corresponding quotient space) preserves Hausdorffness. In geometry, our focus shifts to smooth actions of Lie groups on fiber bundles, as these play a central role in the study of topological invariants known as characteristic classes, which will be introduced in Section 4.6 in Chapter 4 about homogeneous spaces and Section 11.1 in Chapter 11 about the Lie groups and Lie algebras prerequisite of the study of principle bundles.

In particular, we are interested in topological groups that also possess a smooth manifold structure—these are the Lie groups. Lie groups constitute fundamental examples of smooth manifolds, and they provide the mathematical framework for describing most geometric symmetries.

1.4.1 Topological Groups

Definition 1.28 (Topological Groups/Lie Groups)**.** The group G is a topological group if G (with some topology) is a topological space such that the multiplication $m : G \times G \to G$ and inverse map $i : G \to G$ defineds by $\forall g \in G : i(g) = g^{-1}$ are both continuous map.
Furthermore, if G is a C^r-manifolds and μ, i are C^r-maps, then G is a C^r-Lie group.

Without other examination of the differentiability of the Lie group, then the group is a C^∞-Lie group.

Remark. *Note that the product and any embedded submanifold of Lie groups are reasonably expected to be Lie groups, but not every smooth manifold can be given a Lie group structure.*

We are going to just list some basic properties of topological groups without proof, since most of the proofs use nothing but basic group theory and general topology. More detailed study of topological group can be find in [44].

Topological groups have remarkable separation properties:

Proposition 1.19. *For a topological group G, the following properties are equivalent:*

1. *G is T_0.*
2. *G is T_1.*
3. *G is Hausdorff.*

Moreover, every topological group is T_3.

Also, the multiplication on a topological group gives a homeomorphism (automorphism) of the group G, more precisely:

Proposition 1.20. *Let G be a topological group, the the following maps are all homeomorphisms:*

1. *The left and right translation $\forall g \in G : L_g, R_g : G \to G$ defines by*
$$\forall h \in G : L_g(h) = gh, \quad R_g(h) = hg$$
2. *The inverse map $x \mapsto x^{-1}$.*
3. *The conjugate map $\forall g, h \in G : c_g(h) = ghg^{-1}$.*

When we consider subgroups,

Proposition 1.21. *Let G be a topological group, then*

1. *If $H \leq G$ is a subgroup, the so is $\overline{H}$*
2. *If $H \trianglelefteq G$, then so is $\overline{H}$*

And also,

Proposition 1.22. *The center $Z(G)$ of a Hausdorff topological group G is a closed normal subgroup.*

Proposition 1.23. *The connected component contains e of a topological group G is a normal subgroup.*

We are going to use the following two propositions in the discussion of Lie groups:

Proposition 1.24. *Let G be a connected topological group, $e \in U \subseteq G$ is open. Then*

$$\bigcup_{n\geq 1} U^n = G$$

where $U^n = \{g_1 g_2 \cdots g_n \mid g_i \in U \ \forall i\}$

Proof. Defined $\forall U \subseteq G : U^{-1} := \{g^{-1} \mid g \in U\}$. Let $V = U \cap U^{-1}$, then V is also an open neighborhood of e and $V = V^{-1}$. Let

$$H = \bigcup_{n\geq 1} V^n \subseteq \bigcup_{n\geq 1} U^n$$

We claim that H is a subgroup of G. To show that $H = G$, we shall prove that H is both open and closed.

To show that H is a closed subgroup, note that since H is open, $\forall g \in G : gH$ is open (left translation is an homeomorphism.). Thus

$$H = G \setminus \Big(\bigcup_{g \notin H} gH \Big)$$

is closed. Since H is nonempty clopen in a connected topological group G, $H = G$. Thus,

$$H \subseteq \bigcup_{n\geq 1} U^n \subseteq G$$

i.e. $\bigcup_{n\geq 1} U^n = G$. □

Proposition 1.25. *Let H and G be topological groups, $H \subseteq G$. If H is locally closed (i.e., $\forall h \in H : \exists U \subseteq G$ open and contains h such that $U \cap H$ is closed in U), then $H \subseteq G$ is a closed subset.*

To prove the proposition, we need the following lemma:

Lemma. *If $H \leq G$ is a Lie subgroup, then $K = \overline{H}$ is also a subgroup of G.*

Proof. Since inverse map i and multiplication m are both continuous,

$$i(K) \subseteq \overline{i(H)} = K, \quad m(K \times K) \subseteq \overline{m(H \times H)} = K$$

Thus, K is a subgroup of G. □

Now we can prove the proposition.

Proof of Proposition 1.25. Let $H \leq G$ be locally closed, and $K = \overline{H}$. Let the open set that associate to the element $h \in H$ be $U_h \subseteq G$ such that $U_h \cap H$ closed in U_h. Let $V_h = K \cap U_h$, then V_h is open in K (with subset topology). Also $H \cap V_h = H \cap U_h$, which close in U_h (since it is closed in U_h). Suppose $y \in V_h \setminus H$, then $\exists O \subseteq K$ open that

$$\emptyset \neq O \subseteq V_h \setminus H$$

Which contradict to $K = \overline{H}$, i.e. $V_h \subseteq H$. Thus,

$$H = \bigcup_{h \in H} V_h$$

is open in K. Then, $\forall k \in K$, the left coset kH is open (since left translation is homeomorphism), and

$$K = \coprod_{k \in K/H} H$$

Thus,

$$K \setminus H = \coprod_{kH \neq H} kH$$

is open, i.e. H is closed in K. Since $K = \overline{H}$ is closed in G, $H \leq G$ is a closed subgroup. □

Finally, we defined the topological group action:

Definition 1.29 (Topological Group Action)**.** Given the topological group G and topological space X, G act on X continuously if there exists a continuous map

$$\begin{aligned} \rho : G \times X &\to X \\ (g, x) &\mapsto g \cdot x \in M \end{aligned}$$

satisfies the following conditions:

1. $\forall x \in X : e \cdot x = x$
2. $\forall g_1, g_2 \in G : g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

The action of topological groups on spaces does not always preserve the Hausdorffness of the topological space (under the quotient).

Definition 1.30 (Proper Action)**.** A topological group action $\rho : G \times X \to X$ on a topological space X is said to be proper if

$$\tilde{\rho} : G \times X \to X \times X \tag{1.4.1}$$

$$(g, x) \mapsto \big(\rho(g, x), x\big) \tag{1.4.2}$$

is a proper map.

Then, we shall state a fact about the action os topological groups.

Theorem 1.20. Consider a topological group G and a space X that are both locally compact and Hausdorff groups. If $G \times X \to X$ is a proper action, then X/G is Hausdorff.

1.4.2 Lie Groups

Similar to topological groups, the most fundamental structure on a Lie group is the multiplication:

Proposition 1.26 (Left/Right Translation)**.** *Let G be a Lie group, then we can define the following diffeomorphisms:*

1. *The left/right translation $\forall g \in G : L_g, R_g : G \to G$ defined by $\forall h \in G$*

$$L_g(h) = gh, \quad R_g(h) = hg$$

2. *The conjugate map $\forall g \in G : c_g : G \to G$ defined by*

$$\forall h \in G : c_g(h) = ghg^{-1}$$

Also, the pushforward corresponding to the multiplication, $m_{*,(g,h)} : T_{(g,h)}(G \times G) \cong T_g G \times T_h G \to T_{gh} G$ satisfies the relation

$$m_{*,(g,h)}(X_g, Y_h) = (R_h)_{*,g} X_g + (L_g)_{*,h} Y_h$$

Proof. To show $L_g : G \to G$ is a diffeomorphism, notice that L_g has inverse $L_g^{-1} = L_{g^{-1}}$, thus, L_g is bijective. Since

$$L_g : G \xrightarrow{h \mapsto (g,h)} G \times G \xrightarrow{m} G$$

is the composition of smooth map, $\forall g \in G : L_g$ smooth. In the same way, we shall prove that R_g is smooth, and $c_g = L_g \circ R_{g^{-1}}$ is smooth diffeomorphism.
To show the property of the multiplication, we apply the linearity of the pushforward, since $T_{(g,h)}(G \times G) = T_g G \times T_h G$

$$m_{*,(g,h)}(X_g, Y_h) = m_{*,(g,h)}(X_g, 0) + m_{*,(g,h)}(0, Y_h)$$

Which we claim that $m_{*,(g,h)}(X_g, 0) = (R_h)_{*,g} X_g$ and $m_{*,(g,h)}(0, Y_h) = (L_g)_{*,h} Y_h$. □

Proposition 1.27. *Let G be a topological group as well as a C^∞-manifold, then the smoothness of multiplication implies the smoothness of the inverse.*

Proof. The key idea of the proof is to construct the inverse map i by the composition of diffeomorphisms. Consider the map

$$\begin{aligned} f : G \times G &\to G \times G \\ (g, h) &\mapsto (g, gh) \end{aligned}$$

This is clearly a bijection. Since the pushforward of f in any neighborhood of arbitrary $(g, h) \in G \times G$ is given by proposition 1.26:

$$\begin{aligned} f_{*,(g,h)} : T_g G \oplus T_h G &\to T_g G \oplus T_{gh} G \\ (X_g, Y_h) &\mapsto \left(X_g, (R_h)_{*,g} X_g + (L_g)_{*,h} Y_h\right) \end{aligned}$$

Since both L_g and R_h are diffeomorphism, $f_{*,(g,h)}$ non-degenerate. By IFT (Theorem 1.2), $\forall (g, h) \in G \times G$, there is a neighborhood in which f is a diffeomorphism. By the bijectivity, f is a global diffeomorphism. Then, the inverse of f

$$\begin{aligned} f^{-1} : G \times G &\to G \times G \\ (g, h) &\mapsto (g, g^{-1} h) \end{aligned}$$

is a smooth map. Then we shall write

$$\begin{aligned} i : G \hookrightarrow G \times G &\xrightarrow{f^{-1}} G \times G \xrightarrow{\pi_2} G \\ g \longmapsto (g, e) &\longmapsto (g, g^{-1}) \longmapsto g^{-1} \end{aligned}$$

is a diffeomorphism. □

Note that the same proposition does NOT apply in general topological groups.

Example 1.14 (Galilean Group)**.** *The Galilean group is the spacetime symmetry in classical physical systems, which includes the spacetime translation, rotation, and Galilean boost:*

$$\mathbf{x} \mapsto R\mathbf{x} + \mathbf{v}t + \mathbf{a}, \quad t \mapsto t + s$$

Such that $R \in \mathrm{SO}\,3$, $\mathbf{v}, \mathbf{a} \in \mathbb{R}^3$, and $s \in \mathbb{R}$. Thus, the Galilean group has a semidirect product structure

$$\mathrm{Ga} = (\mathrm{SO}(3) \ltimes \mathbb{R}^3) \ltimes \mathbb{R}^4$$

with multiplication given by $(R_2, \mathbf{v}_2, \mathbf{a}_2, s_2) \cdot (R_1, \mathbf{v}_1, \mathbf{a}_1, s_1) = (R_2 R_1, \mathbf{v}_2 + R_2 \mathbf{v}_1, \mathbf{a}_2 + R_2 \mathbf{a}_1, s_2 + s_1)$.

Another extremely important example of Lie groups is the Heisenberg group.

Example 1.15 (Heisenberg Group)**.** *The Heisenberg group (in a narrow sense) is defined by the matrix group defined by the following 3×3 matrices:*

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad \forall x, y, z \in \mathbb{R}$$

Obviously, as a smooth manifold, $H \cong \mathbb{R}^3$, and $H \leq \mathrm{GL}(3, \mathbb{R})$. Thus, H is a Lie group.

Another fact is that:

$$S^n \text{ is a Lie group } \iff n = 1, 3.$$

Then, we defined the map between groups:

Definition 1.31 (Lie Group Homomorphism). The homomorphism between Lie groups is a smooth group homomorphism $\varphi : G \to H$, i.e. $\forall g_q, g_2 \in G$

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$$

Example 1.16 (Determinant). *The determinant* $\det : \mathrm{GL}(n, \mathbb{F}) \to \mathbb{F}^\times$ *is a Lie group homomorphism, where* $\mathbb{F}^\times = (\mathbb{F} \setminus \{0\}, \times)$ *is the multiplication group over field* $\mathbb{F}$*.*

The most remarkable property about Lie group homomorphisms is:

Proposition 1.28. *Any Lie group homomorphism has constant rank.*

Proof. Let G, H be Lie groups, and $\varphi : G \to H$ be a homomorphism. Then, consider the left translation $L_g : G \to G$

$$\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$$

So if we compute the pushforward of the homomorphism, one can obtain

$$\varphi_{*,g} \circ (L_g)_{*,e} = (L_{\varphi(g)})_{*,\tilde{e}} \circ \varphi_{*,e}$$

where $\tilde{e} \in H$ is the identity in H. Since $(L_g)_{*,e}$ and $(L_{\varphi(g)})_{*,\tilde{e}}$ both are linear isomorphisms (by Proposition 1.26), the homomorphism has constant rank since $\operatorname{rank} \varphi_{*,g} = \operatorname{rank} \varphi_{*,e}$. □

Corollary. *Let* $\varphi : G \to H$ *be a Lie group homomorphism. Let* $h \in H$*, if* $\varphi^{-1}(h) \neq \emptyset$*, then it is a* $\dim G - \dim \ker \varphi$ *dimensional embedded submanifold of* G*. In particular,* $\ker \varphi$ *is a Lie subgroup.*

The proof of the corollary uses the fact that Lie group homomorphisms always have constant rank, and simply applies Theorem 1.8. Another corollary is that:

Corollary. *Injective Lie group homomorphisms must be immersions, surjective Lie group homomorphisms must be submersions, and Lie group isomorphisms must be diffeomorphisms.*

The proof needs Sard's theorem (Theorem 2.6) to show that constant rank surjection is a submersion.

Definition 1.32 (Lie Subgroup). Let H, G be Lie groups, $i : H \hookrightarrow G$ be a Lie group homomorphism, if i is injective, then H is a Lie subgroup of G. Moreover, if i is a embedding, then H is a closed Lie subgroup of G

It is easy to check that:

Proposition 1.29. *A closed Lie subgroup is a closed set.*

Proof. By Theorem 1.6, for all $h \in H$, one can find a chart $(U, \varphi = (x^1, \ldots, x^n))$ such that

$$H \cap U = \{q \in U \mid x^j(q) = 0 \ \forall j > \dim H\}$$

Thus, we know that H is locally closed. By Proposition 1.25, $H \subset G$ is a closed set. □

Next, we are going to see that the pushforward of the Lie group homomorphism can strongly determine the map itself, which also provides us a hint that one can actually study a Lie group via studying its tangent space, which we will introduce in Subsection 4.6.1

Proposition 1.30. *Let $\varphi : G \to H$ be a Lie group homomorphism and H be a connected Lie group, if $\varphi_{*,e}$ is surjective, then φ is surjective.*

Proof. By Proposition 1.28, since φ has constant rank, φ is a submersion, and thus, is an open map. In particular, $\exists V \subseteq G$ contains e such that $\varphi(V) \subseteq H$ open. Since H is connected, by Proposition 1.24

$$H = \bigcup_{n\in\mathbb{N}} \varphi(V)^n \subseteq \varphi(G)$$

i.e. φ is surjective. □

In the following part, we are going to study the further properties that appear on Lie group homomorphisms if its pushforward is a linear isomorphism. To do that, we will need to know about Lie algebras.

1.4.3 Lie Algebra and Covering

This subsection is aimed at answering a question: How much does an isomorphism between tangent spaces (or Lie algebras) determine the property of the homomorphism between Lie groups?

Definition 1.33 (Lie Algebra)**.** A Lie algebra over an $\mathbb{R}$-vector space[a] V is a pair $(V, [-,-])$ where the Lie bracket is given by

$$\begin{aligned} [-,-] : V \times V &\to V \\ (v, w) &\mapsto [v, w] \end{aligned}$$

such that $\forall u, v, w \in V$ and $\forall a, b \in \mathbb{R}$,

1. (Antisymmetry) $[u, v] = -[v, u]$
2. (Bilinear) $[au + bv, w] = a[u, w] + b[v, w]$
3. (Jacobi Identity) $\big[u, [v, w]\big] + \big[v, [w, u]\big] + \big[w, [u, v]\big] = 0$

[a] The Lie algebra can be defined in a general field, we only consider $\mathbb{R}$ here.

Remark. *The Jacobi identity can be obtained by antisymmetry; we use it as a part of the definition to emphasize its importance.*

We are going to show that the structure of finite dimensional Lie algebra is naturally exists on the tangent space from multiplication.

Definition 1.34 (Lie Algebra Associate to Lie Group)**.** The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ associated to a Lie group G is the Lie algebra $(T_eG, [-,-])$ with the Lie bracket inherent from the group structure.

More precisely, the Lie bracket is the local linearization of the adjoint action. Consider the adjoint map $\mathrm{Ad} : G \times G \to G$ defined by $\forall g, h \in G : \mathrm{Ad}_g(h) = ghg^{-1}$ the conjugate action. To induce a map on the Lie algebra, we linearize the map in the following way: let $g(t) = \exp(tX)$ and $h(s) = \exp(sY)$ for some $X, Y \in \mathfrak{g}$, then, the induced map on Lie algebra $\mathfrak{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is

$$[X, Y] := \mathfrak{ad}_X Y := \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{Ad}_{g(t)}\big(h(s)\big)\Big|_{s=0}\right)\Big|_{t=0}$$

It is easy to check that it satisfies the following properties:

1. $[X, Y] = -[Y, X]$.

2. $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear.
3. The *Jacobi identity*: $\forall X, Y, Z \in \mathfrak{g}$, $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$.

Often, at least in matrix groups, $[X, Y]_{\mathfrak{g}} = XY - YX$ (follows the Lie bracket in differential geometry, which will be introduced in Chapter 3). A more formal statement is that

Theorem 1.21 (Ado)**.** Any finite-dimensional Lie algebra $\mathfrak{g}$ over a field K of characteristic zero has a faithful (injective) representation that

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

where V is a K-vector space.

Note that $\mathfrak{gl}(V) \leq \mathrm{End}(V)$ is a subalgebra of associative algebra $\mathrm{End}(V)$, with Lie bracket $[X, Y]_{\mathfrak{gl}(V)}$. With the basic knowledge of Lie algebra, we also need to introduce the covering sp;ce, modetailsail will be introduced in Chapter **??**. Continuous surjection between topological spaces $p : Y \to X$ is a covering map if $\forall x \in X$: $\exists V_x \subseteq X$ open neighborhood of X such that

$$p^{-1}(V_x) = \coprod_{\alpha} U_\alpha$$

where U_α are disjoint open sets in Y and $p|_{U_\alpha} : U_\alpha \cong V_x$ $\forall \alpha$ (i.e., exists a discrete set F, $p^{-1}(V_x) \cong V_x \times F$). If Y is simply connected, then Y is the universal covering of X.

Example 1.17 (Universal Cover of S^1)**.** *The Lie group homomorphism* $\exp : \mathbb{R} \to S^1$ *defined by* $\exp(t) = \mathrm{e}^{2\pi \mathrm{i} t}$ *shows that* $\mathbb{R}$ *is the universal cover of* S^1.

We can finally prove the proposition:

Proposition 1.31. *Let* $\varphi : G \to H$ *be a homomorphism between connected Lie groups with Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$. *Then* φ *is a covering map if and only if* $\varphi_{*,e} : \mathfrak{g} \to \mathfrak{h}$ *is a linear isomorphism.*

Proof. ($\Rightarrow$) If $\varphi : G \to H$ is a covering map as well as a Lie group homomorphism, then φ is surjective, and thus, and submersion (second corollary of Proposition 1.28). By the definition of a covering map, for any open neighborhood V of e, $\varphi^{-1}(V) \cong V \times F$, such that F is discrete. Thus, $\ker \varphi = \varphi^{-1}(e) \cong F$ is a discrete subgroup of G, i.e.

$$\operatorname{rank} \varphi = \dim G - \dim \ker \varphi = \dim G$$

Which means $\varphi_{*,e}$ is also injective and thus, a linear isomorphism.
($\Leftarrow$) Since $\varphi_{*,e}$ is a linear isomorphism, Proposition 1.28, the constant rank map (Lie group homomorphism) $\varphi : G \to H$ is both immersion and submersion, i.e.,

$$\operatorname{rank} \varphi = \dim G = \dim H$$

Since φ is both an immersion and a submersion with constant rank, using Sard's theorem (Theorem 2.6), one can show that it is surjective, and also an open map. Since

$$\dim \ker \varphi = \dim T_e G - \operatorname{rank} \varphi_{*,e} = 0$$

and $\ker \varphi$ is an embedded submanifold, by Theorem 1.6, we can find open neighborhood $W \subseteq G$ of the identity e, such that $W \cap \ker \varphi = \{e\}$ and $\varphi|_W : W \to \varphi(W)$ is an diffeomorphism. Then, one shall take $V \subseteq W$ contains e such that $V^{-1}V \subseteq W$. Let $U = \varphi(V)$, clearly, $g_1 \cdot V \cap g_2 \cdot V = \emptyset$ for any $g_1 \neq g_2 \in \ker \varphi$. Since $\forall g \in \ker \varphi$, $\varphi(g \cdot V) = U$ is diffeomorphism, then,

$$\varphi^{-1}(U) = \bigcup_{g \in \ker \varphi} g \cdot V$$

Which proves the statement. $\square$

The statement above is really showing that the isomorphism of Lie algebra somehow provided us with the global information of the topology on the Lie group. In the further discussion about Lie group in Section 4.6, we will see that the reason for this phenomenon is the group structure.

1.4.4 Basic Facts about the Action of Lie Groups

In this section, we denote the smooth action of G on differentiable manifolds M as

$$\rho_M : G \times M \to M$$

We need to clarify certain terms we used about group action:

1. The orbit of point $p \in M$ is
$$G \cdot p := \{g \cdot p \in M \mid \forall g \in G\}$$
2. The stabilizer of $p \in M$ is
$$G_p = \{g \in G \mid g \cdot p = p\}$$
3. The action of G on M is a transitive action if $\forall p, q \in M : \exists g \in G$
$$p = g \cdot q$$
i.e., there is only one orbit for the group action.
4. The action of G on M is effective (or faithful) if $\forall p \in M : g \cdot p = p$ implies $g = e$.
5. The action is free if $\forall g \neq e : \forall p \in M : g \cdot p \neq p$, i.e., the stabilizer $G_p = \{e\}$ $\forall p \in M$.

Proposition 1.32. *Suppose $\rho_M : G \times M \to M$ is an smooth action of Lie group G, then*

$$\rho_g^M := \rho_M(g, -) : M \to M$$

is a diffeomorphism.

Proof. Since ρ_g^M is smooth with smooth inverse $\rho_{g^{-1}}^M$. □

Definition 1.35 (Equivariant). $f \in C^\infty(M, N)$ is said to be equivariant under group action ρ_M, ρ_N of G on M and N if $\forall g \in G$, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M(g,-)"'] & N \arrow[d, "\rho_N(g,-)"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Proposition 1.33. *If $f \in C^\infty(M, N)$ is equivariant under smooth action ρ_M, ρ_N of Lie group G, then f has constant rank on each orbit $G \cdot p$ of the group action.*

Proof. By the Proposition 1.35, we already know that the map $\rho_g^M : M \to M$ and $\rho_g^N : N \to N$ $\forall g \in G$ are both diffeomorphism, thus $\forall p \in M$ and $\forall q \in N$

$$(\rho_g^M)_{*,p} : T_pM \to T_{g \cdot p}M, \quad (\rho_g^N)_{*,q} : T_qN \to T_{g \cdot q}N$$

are both linear isomorphisms. Thus, consider the equivariant smooth map f, we shall consider the pushforward given by the commutative diagram, which gives $\forall g \in G$, $\forall p \in M$, and $q = f(p) \in N$:

$$(\rho_g^N)_{*,q} \circ f_{*,p} = f_{*,g \cdot p} \circ (\rho_g^M)_{*,p}$$

Since composition with linear isomorphism does not change the rank, $\forall \tilde{p} \in G \cdot p$

$$\operatorname{rank}_{\tilde{p}} f = \operatorname{rank}_p f$$

i.e. f has constant rank on each orbit of group action. $\square$

If the group action is transitive, since one can always find $g \in G$ such that $\forall p_1, p_2 \in M : \rho_M(g, p_1) = p_2$, f has constant rank.

Next, we study the property of orbit map $\rho^{(p)} : G \to M$, $\rho^{(p)}(g) = g \cdot p \in M$, which gives $G \cdot p$ a unique smooth structure such that $\rho^{(p)} : G \to G \cdot p$ be a surjective submersion. More precisely, consider the stabilizer G_p of p and its Lie subalgebra $\mathfrak{g}_p = T_e G_p \subseteq T_e G$, $\forall X \in \mathfrak{g}_p$

$$\rho^{(p)}_{*,e}(X) = \frac{\mathrm{d}}{\mathrm{d}t} \rho\big(\exp(tX), p\big)\Big|_{t=0} = 0$$

since $\rho\big(\exp(tX), p\big) = p$ is a constant path. Thus, $\dim G \cdot p = \dim G - \dim G_p$. The following theorem shows how group action orbits on smooth manifold M (and the Lie group itself) be a submanifold of M.

Theorem 1.22. Let G be a Lie group, $\rho : G \times M \to M$ defines a smooth action of G on a smooth manifold M, and $g \cdot p := \rho(g, p)$ $\forall p \in M$. Then any orbits $G \cdot p$ for $p \in M$ are immersion submanifolds on M. Furthermore, if the action ρ is freely and properly on M, then $G \cdot p$ are embedded submanifolds.

Proof. We shall first show that the inclusion map $i : G \cdot p \hookrightarrow M$ has constant rank. Notice that the following diagram commutes by the definition of group action:

$$\begin{tikzcd} G \cdot p \arrow[r, hook, "i"] \arrow[d, "\rho(g,-)"'] & M \arrow[d, "\rho(g,-)"] \\ G \cdot p \arrow[r, hook, "i"] & M \end{tikzcd}$$

i.e., i is equivariant under the action. By Proposition 1.33, i has constant rank on $G \cdot p$. To explicitly compute the rank of the inclusion map, note that one can defined the submersion $\alpha : G \to G \cdot p$, and $i \circ \alpha = \rho^{(p)}$. The tangent map is then given by

$$\rho^{(p)}_{*,g} = i_{*,g\cdot p} \circ \alpha_{*,g} : T_g G \to T_{g\cdot p} M, \quad \operatorname{rank} \rho^{(p)}_{*,e} = \dim G - \dim G_p = \dim G \cdot p$$

Since α is a submersion, $i = \mathrm{id}_{G\cdot p} : G \cdot p \to G \cdot p \subseteq M$ is bijective onto its image,

$$\operatorname{rank}(i_{*,g\cdot p} \circ \alpha_{*,g}) = \operatorname{rank} \alpha_{*,g} = \dim G \cdot p$$

Which shows that $i : G \cdot p \hookrightarrow M$ is an immersion.
Furthermore, if the group action $\rho : G \times M \to M$ is free and proper. By definition of free action, since the stabilizer $G_p = \{e\}$ $\forall p \in G$, $\mathfrak{g}_p = \{0\}$ and if $\rho^{(p)}(g_1) = \rho^{g_2}(p)$, then

$$\rho(g_2^{-1} g_1, p) = p \implies g_1 = g_2$$

which shows $\rho^{(p)} : G \to M$ is an injective immersion with image $G \cdot p$, and thus, $i : G \cdot p \hookrightarrow M$ is an injective submersion. Also, since the action is proper, i.e.

$$\tilde{\rho} : G \times M \to M \times M, \quad \tilde{\rho}(g, p) := (g \cdot p, p)$$

is a proper map. Then, for any compact set $C \subseteq M$, consider $C \times \{p\} \subseteq M \times M$

$$\tilde{\rho}(C \times \{p\}) = \{(g, p) \in G \times M \mid g \cdot p \in C\} = (\rho^{(p)})^{-1}(C) \times \{p\} \subseteq G \times M$$

is a compact set. Thus, $\rho^{(p)}$ is also a proper map so that (i is also a proper map). Since manifolds are Hausdorff, proper maps are closed. The closeness together with the fact that i is an injective smooth immersion shows that i is an embedding. $\square$

Then, we can introduce the basic theorem about the action of Lie groups on smooth manifolds.

We shall use the following proposition:

Proposition 1.34 (Univerdsal Property of Quotient)**.** *Let $\pi : M \to N$ be a smooth surjective submersion (quotient map), then consider the following diagram:*

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "f\circ\pi"'] & N \arrow[d, "f"] \\ & P \end{tikzcd}$$

Then the smoothness of $f \circ \pi$ implies the smoothness of f.

Proof. We shall use Theorem 1.5 to prove this proposition. Suppose π is a quotient map (and thus, immersion) and $f \circ \pi$ is a smooth map, $\forall q \in N : \exists U \subseteq N$ be an open neighborhood of q. There is a smooth local section

$$\sigma : V \to U \subseteq M, \quad \pi \circ \sigma = \mathrm{id}_V$$

Then $f|_U = (f \circ \pi) \circ \sigma : U \to P$ is smooth since it is the composition of smooth maps. Since the point $q \in N$ is arbitrary, f is smooth on N. □

Theorem 1.23. Let G be a Lie group. If the smooth action of G acts on a smooth manifold M freely and properly, then M/G is a topological manifold and admits a unique smooth structure such that the quotient map is a submersion.

Proof. (Uniqueness) Suppose there are two smooth structures $\mathcal{A}_1$, $\mathcal{A}_2$ on M. We shall just prove that id : $M/G \to M/G$ is a diffeomorphism. To prove diffeomorphism, we only need to prove the smoothness. But the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & (M/G, \mathcal{A}_1) \arrow[d, "\mathrm{id}"] \\ & (M/G, \mathcal{A}_2) \end{tikzcd}$$

Since π_1 and π_2 are both smooth quotient maps (i.e., surjective submersions), by the Proposition 1.34 id is smooth (so it is a diffeomorphism).

(Existence) The proof inculuse two parts: First, we shall prove that M/G is a differentiable manifold; Second, the quotient map induces a smooth structure. To prove M/G is a topological manifold, the Hausdorffness is preserved by Theorem 1.20. Consider the map

$$\begin{aligned} \Theta : G \times M &\to M \times M \\ (g, x) &\mapsto (g \cdot x, x) \end{aligned}$$

This map is clearly smooth. We shall construct a countable basis of M to prove second countable, take $\{U_i\}_{i\in I}$ to be a countable basis of M, then $\{\pi(U_i)\}_{i\in I}$ is countable, and since π is open, it is still a basis. Finally, we shall prove that it is locally Euclidean. Let M be an n-dimensional smooth manifold. By Theorem 1.22, orbits of free and proper action of Lie groups are embedded submanifolds in M. Thus, one shall define $\mathcal{O}_p := G \cdot p$ with $T_p\mathcal{O}_p \subseteq T_pM$ such that $\dim T_p\mathcal{O}_p = \dim G =: k$. Then, one shall pick the complement H_p of $T_p\mathcal{O}_p$ such that

$$T_pM = T_p\mathcal{O}_p \oplus H_p, \quad \dim H_p = \dim T_pM - k = n - k$$

Using Theorem 1.10, we shall find an embedded submanifold S_p such that $T_pS_p \cong H_p$. We shall conclude that in some open neighborhood of p, S_p always gives a direct sum decomposition $T_pM = T_p\mathcal{O}_p \oplus T_pS_p$.
With the restriction of group action on S_p

$$\rho|_{G\times S_p} : G \times S_p \to M$$

For any $(\xi, v) \in \mathfrak{g} \times T_pS_p$, the pushforward $\rho_{*,(e,p)} : T_p\mathcal{O}_p \times T_pS_p \to T_pM$ is given by

$$\rho_{*,(e,p)}(\xi, v) = X_\xi(p) + v \in T_pM$$

where $X_\xi(p) := \dot{\rho}\big(\exp(t\xi), p\big)|_{t=0}$, and $\mathrm{Span}\{X_\xi(p) \mid \xi \in \mathfrak{g}\} = T_p\mathcal{O}_p$. Thus, $\mathrm{im}(\rho_{*,(e,p)}) = T_p\mathcal{O}_p \oplus T_pS_p$, i.e., $\rho_{*,(e,p)}$ is linear isomorphism. By inverse function theorem (Theorem 1.2), exists neighborhood $e \in U_G \subseteq G$, $p \in U_S \subseteq S_p$, and $p \in U \subseteq M$ such that

$$\rho|_{U_G \times U_S} : U_G \times U_S \xrightarrow{\sim} U$$

is a local diffeomorphism. Then, we need a lemma

Lemma. *We shall find U_G and U_S such that $\forall g \in G \setminus \{e\}$*

$$g \cdot U_S \cap U_S = \emptyset$$

Which means every orbit $G \cdot p$ $\forall p \in U_S$ can only intersect U_S once.

To prove the lemma, we already know that $\rho|_{U_G \times U_S} : U_G \times U_S \xrightarrow{\sim} U$ is diffeomorphism. Then, $\forall s_1, s_2 \in U_S$, $g \cdot s_2 = s_1$, then

$$s_1 = \rho|_{U_G \times U_S}(e, s_1) = \rho|_{U_G \times U_S}(g, s_2)$$

by the bijectivity, $(e, s_1) = (g, s_2)$, and thus, $g = e$, $s_1 = s_2$. That completes the proof of the lemma.
By the lemma, one shall find U_S such that $g \cdot U_S \cap U_S = \emptyset$ $\forall g \neq e$. Let

$$W_p := G \cdot U_S = \bigcup_{g \in G} g \cdot U_S$$

Hence, we have $\forall q \in W_p$, $\exists! g \in G : \exists! s \in U_S : q = g \cdot s$. Thus, the restriction of group action $\rho|_{G \times U_S} : G \times U_S \xrightarrow{\sim} W_p$ is also a diffeomorphism, i.e., we have constructed the local open set $W_p \subseteq M$ that contains p so that $W_p \cong G \times U_S$. We exam the quotient map $\pi : M \to M/G$, we shall restrict the map to U_S, and

$$\pi|_{U_S} : U_S \xrightarrow{\sim} \pi(U_S)$$

is a homeomorphism. To see this, we know that

1. Continuity and openness are from the definition of quotient topology.
2. Bijective since $\forall s \in U_S$, $G \cdot s$ intersect with U_S exactly once when the group element is e.
3. The inverse map $(\pi|_{U_S})^{-1} : \pi(U_S) \to U_S$ is also continuous by the universal property of the quotient topology.

Thus, on the given neighborhood $U_S^p \subseteq S_p$ contains $p \in M$ with a local chart $\kappa_p : U_S^p \xrightarrow{\sim} V_p \subseteq \mathbb{R}^{n-k}$, defined the chart on $\pi(U_S^p)$ as

$$\varphi_p := \kappa_p \circ (\pi|_{U_S^p})^{-1} : \pi(U_S^p) \xrightarrow{\sim} V_p \subseteq \mathbb{R}^{n-1}$$

which gives an atlas $\{\pi(U_S^p), \varphi_p\}_{p \in M}$ on M/G. In conclusion, M/G is a topological manifold. To show the smoothness, consider $U_{pq} := \pi(U_S^p) \cap \pi(U_S^q) \neq \emptyset$, to prove the transition map

$$\varphi_q \circ \varphi_p^{-1} = \kappa_q \circ (\pi|_{U_S^q})^{-1} \circ (\pi|_{U_S^p}) \circ \kappa_p : \varphi_p(U_{pq}) \to \varphi_q(U_{pq})$$

is smooth. This statement is equivalent to $H := \sigma_q \circ \sigma_p^{-1}$ is smooth. We shall define the following maps:

$$\Psi_p := \rho|_{G \times U_S^p} : G \times U_S^p \xrightarrow{\sim} W_p, \quad \Psi_q := \rho|_{G \times U_S^q} : G \times U_S^q \xrightarrow{\sim} W_q$$

Where W_p and W_q are open sets in M defined previously. Now we consider the following map on $W_{pq} := W_p \cap W_q \neq \emptyset$

$$\Psi_q^{-1} \circ \Psi_p : \Psi_p^{-1}(W_{pq}) \subset (G \times U_S^p) \xrightarrow{\sim} \Psi_q^{-1}(W_{pq}) \subset (G \times U_S^q)$$

It is also a diffeomorphism, thus, one shall write $\Psi_q^{-1} \circ \Psi_p = (G_{pq}(g,s), H_{pq}(g,s))$ where G_{pq} and H_{pq} are both smooth maps. In particular, with $g = e$, $H_{pq}(e,s) \in U_S^q$ and $\pi(H_{pq}(e,s)) = \pi(s)$. Thus, H_{pq} is actually the map

$$H_{pq} = (\pi|_{U_S^1})^{-1} \circ \pi|_{U_S^p} : (\pi|_{U_S^p})^{-1}(U_{pq}) \to U_S^q$$

since the projection π is a diffeomorphism on U_S, H_{pq} is the composition of smooth maps, and thus is a smooth map. Then, the transition map

$$\varphi_q \circ \varphi_p^{-1} = \kappa_q \circ (\pi|_{U_S^q})^{-1} \circ (\pi|_{U_S^p}) \circ \kappa_p = \kappa_q \circ H_{pq} \circ \kappa_p$$

is a smooth map. Thus, M/G admits a smooth structure.
Finally, it remains to show that the quotient map $\pi : M \to M/G$ is a submersion. Since we have diffeomorphism $\Psi_p : G \times U_S^p \xrightarrow{\sim} W_p$, we can restrict the quotient map to a local open set:

$$\pi|_{W_p} : W_p \to \pi(W_p) \cong U_S$$

Here, we claim that $\pi(W_p) \cong \pi(U_S) \cong U_S$ since by the definition that $W_p = G \cdot U_S$ and the choice of U_S, every orbit of $p \in U_S$ only intersects with U_S at a single point p. We shall apply the diffeomorphism Ψ_p to W_p here, and thus, the quotient map corresponds to the projection map

$$\tilde{\pi} = (\pi|_{U_S})^{-1} \circ \pi \circ \Psi_p = \mathrm{proj}_2 : U_G \times U_S \to U_S$$

Thus, π is a submersion since proj_2 is a submersion. $\square$

1.5 Manifolds with Boundary

This section will discuss a particular case of the manifold, where the manifold M, as a topological space, has a boundary. However, the concept "boundary" in a general topological sense cannot be defined intrinsically (since it is a relative concept):

Definition 1.36 (Topological Boundary)**.** Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$, then the boundary of A is given by

$$\partial A = \overline{A} \setminus \mathrm{int}(A)$$

Such definition means that the boundary could change if we change the choice of X. Here is an example:

Example 1.18. *Let $A = (0,1]$, when take $X = \mathbb{R}$, $\partial_{\mathbb{R}} A = \{1\}$. However, if we choose $X = A$, then $\partial_A A = \emptyset$.*

To define what a manifold with boundary is, we need either to embed the manifold M into some other topological space, or find a clever way to defined the boundary with the additional structure on M. To do that, first, we let

$$\mathbb{H}^n := \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$$

be the upper half of the Euclidean space with (topological) boundary $\partial\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x^n = 0\} \cong \mathbb{R}^{n-1}$.

Definition 1.37 (Manifold with Boundary)**.** Let M be a Hausdorff, second countable topological space, if M is locally $\mathbb{H}^n$, i.e., exists an open cover $\{U_\alpha\}$ of M with corresponding homeomorphism

$$\varphi_\alpha : U_\alpha \to \varphi(U_\alpha) \subseteq \mathbb{H}^n$$

such that $(U_\alpha, \varphi_\alpha)$, $(U_\alpha, \varphi_\alpha)$ are C^k-compatible $\forall \alpha, \beta$, then M is a C^k-manifold with boundary. The boundary of M is defined to be

$$\partial M := \{p \in M \mid \exists \alpha : \varphi_\alpha(p) \in \partial\mathbb{H}^n\}$$

We often call the manifold we defined in the first section a manifold without boundary, and a compact (connected) manifold without boundary is called a "closed manifold".

The definition of C^k-structure, (co)tangent spaces (bundles), orientations, and any other structures we mentioned before are the same as the previous definition. In particular, a manifold with boundary has the several properties. We shall first prove the following lemma:

Lemma. *Let U, V be open sets in $\mathbb{H}^n$, and $F : U \to V$ be diffeomorphism. Then*

$$F(U \cap \partial\mathbb{H}^n) = V \cap \partial\mathbb{H}^n$$

Proof. First we shall prove $F(U \cap \partial\mathbb{H}^n) \subseteq V \cap \partial\mathbb{H}^n$. Take $x \in U \cap \partial\mathbb{H}^n$, $y := F(x) \in F(U \cap \partial\mathbb{H}^n)$, since F is a diffeomorphism, if $y \in \mathrm{int}(\mathbb{H}^n) : \exists\epsilon > 0 : B_\epsilon(y) \subseteq \mathrm{int}(\mathbb{H}^n)$, and since the open ball has no intersection with the boundary $\partial\mathbb{H}^n$, it is the ordinary open set in $\mathbb{R}^n$. Take $W := F^{-1}\big(B_\epsilon(y)\big) \subseteq U \cap \partial\mathbb{H}$, since F^{-1} is a smooth map under the topology of $\mathbb{R}^n$, and $F_{y,*}$ invertible (since F is diffeomorphism), we shall find $W \cong B_\epsilon(y)$ be also open set in $\mathbb{R}^n$ and thus $W \subseteq \mathrm{int}(\mathbb{H}^n)$, contradict with the set up that $x \in \partial\mathbb{H}^n$.
The same logic also applies for F^{-1} since F is a diffeomorphism, and it is easy to show that

$$V \cap \partial\mathbb{H}^n \subseteq F(U \cap \partial\mathbb{H}^n) \iff F^{-1}(V \cap \partial\mathbb{H}^n) \subseteq U \cap \partial\mathbb{H}^n$$

In this way, we can prove the lemma. □

Theorem 1.24. Let M be a differentiable manifold with boundary ∂M. Then

1. ∂M is well-defined, i.e., if $\exists\alpha$ such that $\varphi_\alpha(p) \in \partial\mathbb{H}^n$, then $\forall\beta$ such that $p \in U_\beta$, $\varphi_\beta(p) \in \partial\mathbb{H}^n$.
2. ∂M is an $(n-1)$-dimensional differentiable manifold without boundary.

Proof. (1) Consider the boundary charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β). By the lemma we proved above, for the given diffeomorphism

$$\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

Thus, for $U_{\alpha\beta} = U_\alpha \cap U_\beta$,

$$\varphi_{\alpha\beta}\big(\varphi_\beta(U_{\alpha\beta}) \cap \partial\mathbb{H}^n\big) = \varphi_\alpha(U_{\alpha\beta}) \cap \partial\mathbb{H}^n$$

i.e. $\forall p \in U_{\alpha\beta}$ such that $\varphi_\beta(p) \in \partial\mathbb{H}^n$, $\varphi_\alpha(p) = \varphi_{\alpha\beta} \circ \varphi_\beta(p) \in \partial\mathbb{H}^n$, which proves the first statement.
(2) For every $p \in \partial M$, we shall find boundary chart U with $\varphi : U \to W \subseteq \mathbb{H}^n$ such that $\varphi(p) \in \partial\mathbb{H}^n$. Let

$$U_\partial := U \cap \partial M$$

and since $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1}$, we can consider map

$$\tilde{\varphi} : U_\partial \to \mathbb{R}^{n-1}, \quad \varphi(p) = (x^1, \dots, x^{n-1})$$

Since $\varphi : U \to W$ os homeomorphism,

$$U_\partial \cong \varphi(U_\partial) = W \cap \partial\mathbb{H}^n$$

and $\tilde{\varphi} = \pi \circ \varphi|_{U_\partial}$ where $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ is the natural projection, $(U_{\alpha,\partial}, \tilde{\varphi}_\alpha)$ gives a chart of topological $(n-1)$-manifold. Then, consider the induced transition map

$$\tilde{\varphi}_{\alpha\beta} := \tilde{\varphi}_\alpha \circ \tilde{\varphi}_\beta^{-1} : \tilde{\varphi}_\beta(U_{\alpha,\partial} \cap U_{\beta,\partial}) \to \tilde{\varphi}_\alpha(U_{\alpha,\partial} \cap U_{\beta,\partial})$$

Since $\tilde{\varphi}_\alpha = \pi \circ \varphi_\alpha|_{U_{\alpha,\partial}}$ where both π and $\varphi_\alpha|_{U_{\alpha,\partial}}$ are homeomorphisms, the transition map can be written as

$$\tilde{\varphi}_{\alpha\beta} := \tilde{\varphi}_\alpha \circ \tilde{\varphi}_\beta^{-1} = \pi \circ (\varphi_{\alpha\beta})|_{U_{\alpha,\partial}} \circ \pi^{-1}$$

By the fact that π and $\varphi_{\alpha\beta}$ are diffeomorphisms, the transition map $\tilde{\varphi}_{\alpha\beta}$ is also a diffeomorphism. Thus, ∂M is an differentiable $(n-1)$-manifold. Since $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1}$ has the subset topology from the inclusion $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$, every boundary chart gives a homeomorphism to open set in $\mathbb{R}^{n-1}$. Thus, ∂M is a manifold without boundary, $\partial(\partial M) = \emptyset$. This complete the proof of the proposition. □

An important example of a manifold with boundary is the following:

Example 1.19 (Möbus Strip)**.** *Consider space $X := [a, b] \times [0, 1]$, defined the following equivalance relation on X:*

$$(x, s) \sim (y, t) \iff x = a, y = b \text{ or } x = b, y = a; \text{ and } s + t = 1$$

The Möbus strip is given by $M := X/\sim$, and $\partial M = S^1$.

Another important example is the following:

Example 1.20 (Closure of Open Sets in Manifolds)**.** *Let $W \subseteq M$ be some open set in n-manifold, and as open set, the boundary $\partial W := \overline{W} \setminus \operatorname{int}(W)$ is a $(n-1)$-dimensional embedded submanifold of M. Then the closure $\overline{W}$ is a submanifold with boundary in M.*

We shall defined the interior of manifolds with boundary as $\operatorname{int}(M) := M \backslash \partial M$, and the interior is a manifold without boundary. The fundamental structures like tangent and cotangent spaces (bundles) are all well defined on manifold with boundary.

Note that the constant rank theorem and the Sard's theorem also applies for manifolds with boudnary:

Theorem 1.25 (Constant Rank Theorem for Manifolds with Boundary)**.** Let M^m be manifold with boundary and N^n be some manifold, $f : M \to N$ be a C^∞-map with $\operatorname{rank}_p f = k \ \forall p \in M$. Then if $\ker f_{*,p} \nsubseteq T_p\partial M \ \forall p \in \partial M$, then we shall find charts (U, φ) on M and (V, ψ) on N such that

$$\psi \circ f \circ \varphi = (x^1, \dots, x^k, 0, \dots, 0)$$

The proof is almost the same with Theorem 1.7, but just take care of the case with the boundary chart.

Theorem 1.26 (Sard's Theorem for Manifolds with Boudnary)**.** Let M be some manifold with boundary and N be some manifold without boundary, $f : M \to N$ be C^∞-map and $\partial f := f|_{\partial M}$. Then the union of critical values of f and ∂f has measure-zero in N.

The proof is also simple, just apply Sard's theorem twice on $\operatorname{int}(M)$ and ∂M respectively.

Also, let M be manifold with boundary and N be manifold without boundary, $S \subseteq N$ be embedded submanifold, if both f and ∂f (as defined above) are transitive to S, i.e.

$$f_{*,p}(T_pM) + T_{f(p)}S = T_{f(p)}N$$
$$(\partial f)_{*,p}(T_p\partial M) + T_{f(p)}S = T_{f(p)}N \quad (\text{if } p \in \partial M)$$

then we say that f transitive to S, denote as $f \pitchfork S$. Similar to the case in manifold with boundary (Theorem 1.14):

Theorem 1.27. If $f : M \to N$ is a smooth map from manifold with boundary to manifold without boundary, $S \subseteq N$ be a regular manifold. If $f \pitchfork S$, then $f^{-1}(S)$ be manifold with boundary, and

$$\partial f^{-1}(S) = f^{-1}(S) \cap \partial M$$

and $\operatorname{codim} f^{-1}(S) = \operatorname{codim} S$.

Finally, we shall prove some differential topological result of manifolds with boundary:

Definition 1.38 (Retraction)**.** Let X be a topological space with subspace $A \subseteq X$, a retraction is a continuous map $r : X \to A$ such that $r|_A = \mathrm{id}_A$.

as an application of Theorem 1.27, we have the following theorem:

Theorem 1.28 (No-Retraction Theorem)**.** Let M be compact manifold with boundary, then there is no smooth retraction $f : M \to \partial M$.

Proof. Let $q \in \partial M$ be a regular value of f, since $f|_{\partial M}$ is the identity map, q is also the regular value of ∂f (i.e., transverse to every embedded manifold that cross q). Thus, by Theorem 1.27, $f^{-1}(q)$ is a compact 1-dimensional embedded submanifold of M with

$$\partial f^{-1}(q) = f^{-1}(q) \cap \partial M = \{q\}$$

However, compact 1-dimensional manifold with boundary must be (homeomorphic/diffeomorphic to) a disjoint union of close intervals, i.e., boundary contains even point, which leads to contradiction. □

By this theorem, we can prove the famous Brouwer fixed point theorem. Sometime the smoothness condition in theorem above can be replaced by continuity, we shall first consider the retraction on unit close disk in $\mathbb{R}^n$ as an application of Theorem 1.28 as an example:

Proposition 1.35. *There is no continuous map $f : D^n \to S^{n-1}$ such that $f|_{\partial D^n} = \mathrm{id}_{S^{n-1}}$.*

Proof. This proposition can be proved by contradiction. Suppose exists such $f : D^n \to S^{n-1}$, then consider $\tilde{f} : D^n \to S^{n-1}$ defined by

$$\tilde{f}(x) := \begin{cases} f(2x), & \|x\| \leq 1/2 \\ x/\|x\|, & \|x\| \geq 1/2 \end{cases}$$

Note that $\tilde{f}$ is continuous and smooth on $(1/2, 1]$. A classical result of Stone-Weierstrass Theorem ensures that $\exists \tilde{g} \in C^\infty(D^n, S^{n-1})$ such that

1. The uniform approximation error is given by

$$\left\|\tilde{g}(x) - \tilde{f}(x)\right\| \leq \frac{1}{4}, \quad \forall x \in D^2$$

2. Let $S := \{x \in D^2 \mid 2/3 \leq \|x\| \leq 1\}$ be a closed subset of D^2, $\tilde{g}|_S \equiv \tilde{f}|_S$.

Let $g = r \circ \tilde{g}$ be the composition of $\tilde{g}$ with canonical projection $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ be $r(x) := x/\|x\|$ ($\tilde{g}(D^2) \subseteq \mathbb{R}^n \setminus \{0\}$ is due to the approximation condition since $\tilde{f}(D^2) \subseteq S^1$). Then, g is the composition of smooth maps, and thus, is smooth. Thus, we construct a smooth retraction $g : D^n \to \partial D^n \cong S^{n-1}$, which contradict to Theorem 1.28. □

The Brouwer fixed point theorem is a simple corollary of this proposition:

Theorem 1.29 (Brouwer Fixed Point Theorem)**.** For any continuous map $f : D^2 \to D^2$, $\exists x_0 \in D^2$ such that $f(x_0) = x_0$.

Proof. We still use proof by contradiction. Suppose $f : D^2 \to D^2$ continuous without fixed point. Then, defined the map $g : D^n \to S^{n-1}$ given by

$$g(x) = f(x) + t(x - f(x))$$

where $t = t(x)$ is the unique positive solution of the algebraic equation $y = f(x) + t(x - f(x)) \in S^1$, i.e., in more explicit equation:

$$t^2 + 2\langle f, u\rangle + (\|f\|^2 - 1) = 0, \text{ with } u := \frac{x - f(x)}{\|x - f(x)\|}$$

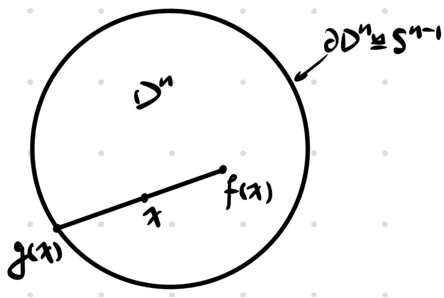


Figure of $x \mapsto g(x)$ in D^2

The configuration can be shown as the following figure. By simply solve the equation, the coefficient depends on x continuously. We claim that $g(x)$ gives a continuous retraction from D^n to S^{n-1}. The continuity is given by the continuity of $f(x)$ and $t(x)$, and when $x \in S^{n-1}$ (i.e., $\|x\| = 1$), then $t = 1$ and

$$g(x) = f(x) + x - f(x) = x$$

Thus, the result contradict with the Proposition 1.35. $\square$

Chapter 2

Fundamental Facts related to Functions on Manifolds

After we have studied the main object of modern geometry: manifolds, we are going to study more details about one of the most essential tools we can use to study manifolds (scalar-valued) functions on manifolds.

2.1 Partition of Unity

In the first section, we are going to study the tool that ensures that a smooth function defined locally can be globalized to a smooth function on the entire manifold.

First of all, we need a smooth function here:

Proposition 2.1. *The function $\phi : \mathbb{R} \to \mathbb{R}$ defines by*

$$\phi(x) = \begin{cases} \mathrm{e}^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

is smooth.

The proof is simply undergraduate analysis. The function above leads to the following theorem:

Theorem 2.1 (Existence of Bump Function)**.** The following two statements hold:

1. There exists $h \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $h(x) = 0\ \forall |x| \geq 1$, $h(x) \in (0,1]\ \forall x \in (-1,1)$, and $h(x) = 1\ \forall x \in [-1/2, 1/2]$.
2. There exists $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f(x) = 0\ \forall \|x\| \geq 1$, $h(x) \in (0,1]\ \forall \|x\| < 1$, and $h(x) = 1\ \forall \|x\| \leq 1/2$.

Proof. The proof is based on the smooth function we take in Proposition 2.1.

1. Consider the following function

   $$\tilde{h}(x) = \frac{\phi(x)}{\phi(x) + \phi(1-x)},\ \forall x \in \mathbb{R}$$

   it is obvious that $\tilde{h}$ is smooth and

   $$\tilde{h}(x) = 0 \text{ if } x \leq 0;\ \tilde{h}(x) > 0 \text{ if } x \in (0,1);\ \tilde{h}(x) = 1 \text{ if } x \geq 1.$$

Then, take $h_1(x) = \tilde{h}(2x+2)$, and

$$h(x) = h_1(|x|) = \begin{cases} h_1(x), & x \le 0 \\ h_1(-x), & x > 0 \end{cases}$$

is the function we need in the first proposition.

2. The function we need in the second proposition is the smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = h(\|x\|)$$

where $\|\cdot\|$ is just the standard norm in $\mathbb{R}^n$.

Thus, the theorem was proved. □

Recall that the support of a function is defined by:

Definition 2.1 (Support of a Function)**.** The support of a function $f : X \to \mathbb{R}$ is the closure of the preimage of $\mathbb{R} \setminus \{0\}$, i.e.,

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$$

The goal of the partition of unity is to find a collection of functions $\{\rho_\alpha : X \to [0,1]\}$ such that $\forall x \in X$, $\sum_{\alpha \in A} \rho_\alpha(x) = 1$. Or, more formally

Definition 2.2 (Partition of Unity)**.** Suppose M be a differentiable manifold with open cover $\{U_\alpha\}_{\alpha \in A}$. A partition of unity on M associate to the cover $\{U_\alpha\}_{\alpha \in A}$ is defined to be a collection of smooth functions $\{\rho_i : M \to \mathbb{R}\}_{i \in I}$ where I is at most countable, such that:

1. $\rho_i(x) \in [0,1]$ $\forall x \in M$.
2. $\forall i \in I : \exists \alpha(i) \in A : \operatorname{supp}(g_i) \subseteq U_{\alpha(i)}$
3. $\{\operatorname{supp}(\rho_i)\}_{i \in I}$ is a locally finite cover of M.
4. $\sum_i \rho_i(x) \equiv 1$ $\forall x \in M$.

However, this summation is pathological if A is an infinite set, i.e., $\forall x \in X$, we need there to exist $U \subseteq X$ a neighborhood of x such that $\rho_{\alpha'}|_U \neq 0$ for only finitely many $\alpha' \in A$.

The formal solution to the problem requires the paracompactness.

Definition 2.3 (Paracompactness)**.** X is paracompact if every open cover has a locally finite refinement.

Then, we shall show that the partition of unity always exists on differentiable manifolds.

Proposition 2.2 (Exhaustion)**.** *For any differential manifold, there exists a series of compact sets* $\{K_i\}_{i \ge 1}$ *such that*

$$K_i \subseteq \operatorname{int}(K_{i+1}) \ \forall i \ge 1, \quad \bigcup_{i \ge 1} K_i = M$$

Proof. The statement is trivial if M is compact. If not, since manifolds are all locally compact, for some $p \in M$, take an open neighborhood $V_p \subseteq M$ of p such that $\overline{V_p}$ is compact, and such a collection gives a cover of M. By Lindelöf lemma, there exists a countable collection of points $\{p_i\}_{i \ge 1}$ such that $\{V_{p_i}\}_{i \ge 1} \subset \{V_p\}_{p \in M}$

covers the manifold M. Based on this fact, we can define the exhaustion recursively: Given $K_1 := \overline{V_{p_1}}$, suppose $K_1, \dots, K_n$ have been defined. Then, let

$$K := \bigcup_{i=1}^{n} K_i \subseteq M$$

Since $\{V_{p_i}\}_{i\geq 1}$ covers M, there exists index set I such that

$$\bigcup_{i=1}^{n} K_i \subseteq \bigcup_{i\in I} V_{p_i} =: \operatorname{int}(K_{n+1}), \text{ i.e. } K_{i+1} := \overline{\bigcup_{i\in I} V_{p_i}}$$

Then, $\{K_i\}_{i\geq 1}$ is the exhaustion we need. □

An application of the exhaustion is to prove the paracompactness.

Proposition 2.3. *Every topological manifold is paracompact.*

Proof. Take $\{K_i\}_{i\geq 1}$ be an exhaustion on manifold M, $\forall i \in \mathbb{N} \setminus \{0\}$ we can defined compact sets $V_i := K_{i+1} \setminus \operatorname{int}(K_i)$ and open sets $W_i := \operatorname{int}(K_{i+2}) \setminus K_{i-1}$, $V_i \subseteq W_i$. Consider the open cover $\{U_\alpha\}_{\alpha\in A}$, by the compactness of V_i, we shall find a subcover $\{U_{ij}\}_{j\in I}$ for some finite index set I such that covers V_i. We shall take

$$K_{ij} := U_{ij} \cap W_i$$

which also covers V_i and be a countable refinement of $\{U_{ij}\}$. Consider $p \in M$ with open neighborhood U_p such that $\overline{U_p}$ compact. Then finite many K_{ij} can cover $\overline{U}$, with the maximum index $i = s$, i.e., $K_{ij} \subsetneq K_{i+1} \subsetneq K_{s+1}$ for any possible i, j. Thus, $\overline{U_p} \subseteq K_{s+1}$. If $i \geq s+2$, since $W_{s+2} := \operatorname{int}(K_{s+2}) \setminus K_{s+1}$ is disjoint with K_{s+1} and $V_{ij} = U_{ij} \cap W_i$, $V_{ij} \cap K_{s+2} = \emptyset$. Thus, only finitely many U_p intersect with K_{s+2}, which proves the theorem. □

Then, we shall prove the existence of a partition of unity:

> **Theorem 2.2** (Existence of Partition of Unity)**.** For any open cover $\{U_\alpha\}_{\alpha\in A}$ of a smooth manifold M, there exists a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in A}$.

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M.
By Proposition 2.3, M is paracompact, hence every open cover admits a locally finite open refinement. Thus, there exists an index set P and a locally finite family of open sets

$$\{V_p\}_{p\in P}, \quad V_p \subset U_{\alpha(p)} \text{ for some } \alpha(p) \in A$$

such that $\{V_p\}_{p\in P}$ still covers M.
Since M is a smooth manifold, we may (by shrinking each V_p if necessary) assume that for every $p \in P$ there is a smooth chart

$$(V_p, \varphi_p), \quad \varphi_p : V_p \xrightarrow{\sim} B_1(0) \subset \mathbb{R}^n$$

where $n = \dim M$ and $B_1(0)$ denotes the open unit ball. Shrinking each V_p preserves both the refinement property $V_p \subset U_{\alpha(p)}$ and local finiteness.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be the bump function given by Theorem 2.1, so that

$$\operatorname{supp} f \subset \{x \in \mathbb{R}^n : \|x\| \leq 1\} = \overline{B_1(0)}, \quad f(x) \in (0, 1] \text{ for } \|x\| < 1$$

In particular, $f > 0$ on $B_1(0)$. For each $p \in P$, define a smooth function $f_p : M \to \mathbb{R}$ by

$$f_p(x) := \begin{cases} f\big(\varphi_p(x)\big), & x \in V_p, \\ 0, & x \in M \setminus V_p \end{cases}$$

Then $f_p \in C^\infty(M)$, and

$$\operatorname{supp} f_p \subset \varphi_p^{-1}\big(\operatorname{supp} f\big) \subset \varphi_p^{-1}\big(\overline{B_1(0)}\big) \subset V_p \subset U_{\alpha(p)}$$

Moreover, since $f > 0$ on $B_1(0)$ and $\varphi_p(V_p) = B_1(0)$, we have

$$f_p(x) > 0 \quad \text{for all } x \in V_p$$

The family $\{V_p\}_{p\in P}$ is locally finite, hence the family of supports $\{\operatorname{supp} f_p\}_{p\in P}$ is also locally finite. Define

$$\psi(x) := \sum_{p\in P} f_p(x), \qquad x \in M$$

Because for each $x \in M$ only finitely many p satisfy $x \in V_p$ (local finiteness), the above sum is finite at every point x, so ψ is well-defined and smooth. Furthermore, since $\{V_p\}$ covers M, for every $x \in M$ there exists $p_0 \in P$ with $x \in V_{p_0}$, hence $\psi(x) \geq f_{p_0}(x) > 0$. Thus $\psi(x) > 0$ for all $x \in M$.
Now define, for each $p \in P$,

$$\rho_p(x) := \frac{f_p(x)}{\psi(x)}, \quad x \in M$$

Since f_p and ψ are smooth and ψ has no zeros, each ρ_p is smooth. We have $0 \leq \rho_p(x) \leq 1$ for all x, and

$$\operatorname{supp} \rho_p \subset \operatorname{supp} f_p \subset U_{\alpha(p)}$$

The local finiteness of $\{V_p\}$ implies that for each $x \in M$ only finitely many $\rho_p(x)$ are nonzero, and

$$\sum_{p\in P} \rho_p(x) = \frac{1}{\psi(x)} \sum_{p\in P} f_p(x) = \frac{\psi(x)}{\psi(x)} = 1, \qquad \forall x \in M$$

Therefore, $\{\rho_p\}_{p\in P}$ is a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$. This completes the proof of the theorem. ☐

2.2 Some Applications of the Partition of Unity

In this section, we will see how powerful the partition of unity is in dealing with problems related to smooth functions on manifolds. We will discuss some of the most important applications of the partition of unity, including a weaker version of the Whitney embedding theorem, the Whitney approximation theorem, the smooth extension of a smooth function on a subset of the manifold, and related topics.

Theorem 2.3 (Weak Whitney Embedding Theorem)**.** Every compact differentiable manifold M can be embedded into an Euclidean space $\mathbb{R}^N$.

Proof. With the compactness of M, consider $\dim M = n$, one shall take finite local charts $\{(U_i, \varphi_i) \mid i = 1, \dots, k\}$ such that $\varphi_i(U_i) = B_2(0)$ and $\{V_i = \varphi_i^{-1}(B_{1/2}(0)) \mid i = 1, \dots, k\}$. Consider the bump function of the partition of unity $\{\rho_i : M \to [0, 1]\}$ associate to this covering, such that

$$\rho_i|_{V_i} \equiv 1, \quad \operatorname{supp}(\rho_i) \subseteq \varphi_i^{-1}(B_1(0))$$

Defined the map $F : M \to \mathbb{R}^{kn} \times \mathbb{R}^k$ as

$$F(x) = \big(\rho_1(x)\varphi_1(x), \dots, \rho_k(x)\varphi_k(x), \rho_1(x), \dots, \rho_k(x)\big)$$

which by zero extension using ρ_i, we treat $\rho_i\varphi_i$ as the smooth function on M. We know the following facts:

- F is injective: Since if $F(x) = F(y)$, $\rho_i(x) = \rho_j(y)$ $\forall 1 \leq i \leq k$. Let $x \in V_i$, then $\rho_i(x) = 1$ and $\rho_i(y) = \rho_i(x) = 1$ and thus $\varphi_i(x) = \varphi_i(y)$. Since φ_i is a local diffeomorphism, $x = y$.

- F has nondegenerate Jacobi matrix: Since $\rho_i(x) \equiv 1 \ \forall x \in V_i$, by definition F is embedded on V_i.

Thus, we get the injective immersion from M into $\mathbb{R}^N$, $N = k(n+1)$, since M compact, it is an embedding. Hence, we complete the proof of the theorem. □

Actually, H. Whitney [49, 50] proved that any n-dimensional smooth manifold can be embedded into $\mathbb{R}^{2n}$. The stronger version of this theorem will be include in the following section since the proof demands the use of Sard's theorem.

Theorem 2.4 (Smooth Extension of Functions)**.** The smooth real-valued function on a differentiable manifold M can be extended in the following ways:

1. Let C be a closed set on differentiable manifold M, $U \subseteq M$ is a open neighborhood of C. Then there exists $f : M \to \mathbb{R}$ such that $f|_C \equiv 1$ and $f|_{X \setminus U} \equiv 0$.
2. Let A be the subset of M, $f \in C^\infty(A, \mathbb{R})$. If $\forall x \in B : \exists U_x \subseteq M$ be a open neighborhood of x and smooth function $f_x : U_x \to \mathbb{R}$ such that $f|_{B \cap U_x} = f_x|_{B \cap U_x}$, then $\exists V \subseteq M$ open neighborhood of A with smooth function $\tilde{f} : V \to \mathbb{R}$ such that $\tilde{f}$ is the extension of f, i.e., $\tilde{f}|_B = f$.

Proof. (1) One shall consider the open cover $\{C, M \setminus C\}$ and the partition of unity $\{\phi, \psi\}$ associate to this covering. Let $\operatorname{supp} \phi \subseteq C$, and $\operatorname{supp} \psi \subseteq M \setminus C$. Since $\psi|_A = 0$, $\phi|_A + \psi_A \equiv 1$, we know that $\phi|_A \equiv 1$. Then ϕ is the function we want.
(2) Consider the open set defined by

$$V = \bigcup_{x \in B} U_x$$

Let $\{\rho_i\}$ be the partition of unity associated to the open cover $\{U_x \mid x \in B\}$. By second countability, one shall take a countable covering labeled with $x_i \in B$, and take $\operatorname{supp}(\rho_i) \subseteq U_{x_i}$. The function we need is

$$\tilde{f} := \sum_i \rho_i(x) f_{x_i}(x) \quad \forall x \in V$$

where $\rho_i f_{x_i}$ can be viewed as the extension of f_{x_i} on V, which equals to zero for any $x \in V \setminus U_{x_i}$. □

Furthermore, the extension theorem can also be generalized to any smooth map between manifolds.

Corollary. *For a smooth map $f \in C^\infty(M, S)$ where $M \subseteq N$ is a closed regular submanifold. Then $\exists \tilde{f} \in C^\infty(N, S)$ such that $\tilde{f}|_M = f$.*

Proof. This corollary follows from the characterization of regular submanifolds (Theorem 1.6). The regular submanifold is locally closed. Thus, from the second proposition in Theorem 2.4, the smooth map can be extended to an open neighborhood, and by the closeness together with the first proposition in the theorem, one can find a global extension. □

If we remove the closeness, the local coordinate is still applicable, but the function can no longer have a global extension.

Theorem 2.5 (Smooth Approximation of Continuous Functions)**.** Let $f : M \to \mathbb{R}^k$ be continuous map, then $\forall \epsilon \in C(M, \mathbb{R}_{>0})$, there exists a smooth map $g : M \to \mathbb{R}^k$ such that

$$\|f(x) - g(x)\| \le \epsilon(x) \quad \forall x \in M$$

Proof. Since both f and ϵ are both continuous, for any $x \in M$ one can find neighborhood $U_x \subseteq M$ such that $\forall y \in U_x$, the following constraint of continuous function holds:

$$\epsilon(y) \geq \frac{1}{2}\epsilon(x), \quad \|f(y) - f(x)\| \leq \frac{1}{2}\epsilon(x)$$

Consider $\{U_x \mid x \in M\}$, and let $\{\rho_i \in C^\infty(M, [0,1]) \mid x_i \in M\}$ be the partition of unity associated to the open cover. For any i, let $\mathrm{supp}(\rho_i) \subseteq U_{x_i}$ with some $x_i \in M$. Let the smooth function be defined by

$$g(x) := \sum_i \rho_i(x) f(x_i), \quad \forall x \in M$$

The smoothness is ensured by the smoothness of the partition of unity, since $f(x_i)$ is just a constant. We shall check the approximation condition:

$$\begin{aligned}\|g(x) - f(x)\| &\leq \sum_i g_i(x)\|f(x_i) - f(x)\| \\ &= \sum_{g_i(x)\neq 0} g_i(x)\|f(x_i) - f(x)\| \leq \frac{1}{2}\sum_i g_i(x)\epsilon(x) \\ &\leq \sum_{g_i(x)\neq 0} g_i(x)\epsilon(x) = \epsilon(x)\end{aligned}$$

Thus, the theorem was proved. □

We can also show that any continuous homotopy between smooth maps of manifolds is homotopic to a smooth homotopy.

Proposition 2.4. *The following claim holds for the homotopy of maps on manifolds:*

1. $\forall f \in C(M) : \exists g \in C^\infty(M, N)$ *such that* $f \sim g$.
2. $\forall f_0, f_1 \in C^\infty(M, N)$ *that* $f_0 \sim f_1$, *then* $\exists F \in C^\infty(I \times M, N)$ *such that* $F : f_0 \sim f_1$.

Proof. (1) We shall first consider the continuous homotopy

$$H(s, x) = (1 - s)f(x) + sg(x)$$

with $g(x)$ be the smooth function in Theorem 2.5. Thus, the first proposition was proved.
(2) To prove the second statement, we shall first consider a continuous homotopy

$$H : I \times M \to N, \quad H : f_0 \sim f_1$$

The key idea is to use Theorem 2.5 claim the existence of the function. However, we need to preserve the value on the boundary of the interval to ensure the smooth map still being a homotopy, so we need some modify on the condition. Let $\varphi : I \to I$ such that for some $0 < \delta < 1/2$, the function satisfies

$$\varphi|_{[0,\delta]} \equiv 0, \quad \varphi|_{[1-\delta,1]} \equiv 1$$

Let the new function $H_1(t, x) := H(\varphi(t), x)\ \forall x \in M$, by the definition of homotopy,

$$H_1(t, x) = f_0(x)\ \forall t \in [0, \delta]; \quad H(t, x) = f_1(x)\ \forall t \in [1 - \delta, 1]$$

Then, by Theorem 2.5, we shall take $G \in C^\infty([\delta, 1 - \delta] \times M, N)$ such that

$$\left\|H|_{[\delta,1-\delta]\times M} - G\right\| \leq \epsilon(x)$$

and by Theorem 2.2, we shall take bump function such that

$$\beta(t) \equiv 0,\ t \in [0, \delta] \cup [1 - \delta, 0]; \quad \beta(t) \equiv 1,\ t \in [\delta, 1 - \delta]$$

Let the smooth homotopy be

$$\tilde{H}(t, x) = (1 - \beta(t))H_1(t, x) + \beta(t)G(t, x)$$

which proves the theorem. □

Proposition 2.5 (Existence of Smooth Proper Maps)**.** *Smooth, proper maps always exist on differentiable manifolds.*

Proof. Let M be a smooth manifold with open cover $\{U_i\}$ such that $\overline{U_i}$ compact. Let $\{\rho_i\}$ be the partition of unity associated with the open cover. We shall construct the smooth function $\rho : M \to \mathbb{R}$

$$\rho(x) := \sum_k k\rho_k(x)$$

The smoothness is preserved by the definition of partition of unity. If for $x \in M$, $\rho_i(x) = 0\ \forall i < k$

$$\rho(x) = \sum_{i \geq k} i\rho_i(x) \geq k \cdot \sum_{i \geq k} \rho_i(x) = k$$

In other words,

$$\rho^{-1}[0, k] \subseteq \bigcup_{i=1}^{k} \operatorname{supp}(\rho_i) \subseteq \bigcup_{i=1}^{k} \overline{U_i}$$

Since closed subsets of compact sets are compact, $\rho^{-1}[0, k]$ is compact. □

2.3 Critical Points and Sard's Theorem

In Section 1.2, we introduced the rank of a smooth map as a basic measure of its regularity. In the previous chapter, we focused on the strongest situation—maps of constant rank—and showed that constant rank hypotheses lead naturally to the existence of submanifolds. In differential topology, however, one cannot generally expect such uniform regularity. Instead, one often perturbs a given map slightly to achieve the desired generic properties (and the perturbation preserves the generic property we have).

In this section, we prove Sard's theorem [41], due to A. Sard (1942), which asserts that the set of critical values of a smooth map has measure zero. Equivalently, the image of the set of non-regular (critical) points is a measure-zero set in the target. First, recall that in Definition **??**, for C^∞-map $f : M \to N$, $p \in M$ is a regular point if $f_{*,p}$ is surjective, and the corresponding image $q \in N$ is said to be a regular value of f if $f^{-1}(q)$ contains only regular points. Otherwise, p is a critical point and $f(p)$ is a critical value.

The first observation that we can make is that the "number of preimages" $|f^{-1}(q)|$ of a regular value is locally constant.

Proposition 2.6. *Let $f : M \to N$ be a C^∞-function with regular value $q \in N$ and M compact. Then, there exists an neighborhood $V \subseteq N$ contains q that $\forall q' \in V$*

$$|f^{-1}(q')| = |f^{-1}(q)|$$

Proof. The proof is simply by inverse function theory (Theorem 1.2).By the definition of regular value, $\forall p_i \in f^{-1}(q)$, there exists a open neighborhood U_i such that $f|_{U_i} : U_i \to V_i \subseteq N$ is a diffeomorphism. Then, we need the following lemma:

Lemma. *For M compact and $\forall f \in C^\infty(M, N)$, if $q \in N$ is a regular value, then*

$$f^{-1}(q) = \{p_1, \ldots, p_k\}$$

is a finite subset in M.

The proof of the lemma simply follows the fact that since $q \in N$ is a regular value, $\dim f^{-1}(q) = 0$, i.e., $f^{-1}(q)$ is discrete and closed (since the singleton $\{q\}$ is a closed set in N). By the compactness of M, since discrete subset $f^{-1}(q)$ is closed, $f^{-1}(q)$ is compact, and thus, is finite set.

By the Hausdorffness of manifolds, one shall always let $U_i, \dots, U_k$ be disjoint and diffeomorphic to $V_1, \dots, V_k$. Then, the open set

$$V = \left(\bigcap_{i=1}^{k} V_i\right) \setminus f\left(M \setminus \bigcup_{i=1}^{k} U_i\right)$$

Every point $q \in V$ has number of preimage $|f^{-1}(q)| = k$. □

After this small, useful observation, we need to first introduce the concept of measure-zero sets: Recall that in analysis, we defined the open box in $\mathbb{R}^n$ as

$$B := \prod_{i=1}^{n}(a_i, b_i)$$

with its volume given by

$$\mathrm{vol}(B) = \prod_{i=1}^{n}(b_i - a_i)$$

Definition 2.4 (Measure-Zero Sets)**.** A set $S \subseteq \mathbb{R}^n$ is said to be measure-zero if $\forall \epsilon > 0$, there exists mostly countable open box $\{B_i\}$ that

$$S \subseteq \bigcup_i B_i \quad \text{and} \quad \sum_i \mathrm{vol}(B_i) < \epsilon$$

We also denote this property as $\mu(S) = 0$.

In a more serious real analysis text, such a set is named as the subset in $\mathbb{R}^n$ with Lebesgue measure zero. The following facts are obvious for measure-zero sets:

Proposition 2.7. *The following facts hold for measure zero sets:*

1. *Subsets of measure-zero sets are measure-zero.*
2. *Countable union of measure-zero sets is till been measure-zero.*
3. *Nonempty open sets in $\mathbb{R}^n$ are not measure-zero.*
4. *If $m < n$, the $\mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$ is measure zero in $\mathbb{R}^n$.*

The proof is simple and can be found in many classical texts of analysis, for example [**SteinShakarchi2005**]. Recall from the undergraduate mathematical analysis that, a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz if $\forall p, q \in \mathbb{R}^n$

$$\|f(p) - f(q)\| \leq K\|p - q\| \text{ for some } K > 0$$

where the norm is the ordinary Euclidean norm on $\mathbb{R}^n$. If the condition is only satisfied in a certain subspace S, we shall say that the map is Lipschitz on the subspace S. We shall claim that the Lipschitz condition is enough for the map to preserve measure-zero, and differentiability also implies the same result.

Proposition 2.8. *The following proposition related to $f : \mathbb{R}^n \to \mathbb{R}^n$ holds:*

1. *Lipschitz map preserves measure-zero.*
2. *C^1-map preserves measure-zero.*

Proof. For the Lipschitz case, suppose f has Lipschitz constant K, then the box with volume V has volume upper bound $(K\sqrt{n})^n V$ (since the distance between two points x, y in box Q with side length a has estimation $\|x-y\| \le \sqrt{n}a$). Then consider $E \subseteq \mathbb{R}^n$ with measure-zero:

$$E \subseteq \bigcup_i Q_i, \quad \sum_i \operatorname{vol}(Q_i) \le \epsilon$$

Then, each Q_i has image contains in a larger box $f(Q_i) \subseteq Q_i'$ with $\operatorname{vol}(Q_i') \le (K\sqrt{n})^n \operatorname{vol}(Q_i)$. Thus, $f(E) \subseteq \bigcup_i Q_i'$, and

$$\sum_i \operatorname{vol}(Q_i') \le (K\sqrt{n})^n \sum_i \operatorname{vol}(Q_i) \le (K\sqrt{n})^n \epsilon \to 0$$

It is also important to note that the claim above only requires locally Lipschitz. For the C^1-function case, oen shall consider the Jacobian of the map Jf and $\forall x, y \in \mathbb{R}^n$

$$f(x) - f(y) = \int_0^1 Jf\big(y + t(x-y)\big)(x-y)\mathrm{d}t$$

With the operator norm, let $\gamma(t) := y + t(x-y)$ and $r = \|x-y\|$

$$\|f(x) - f(y)\| \le \int_0^1 \big\|Jf\big(\gamma(t)\big)\big\|_{\mathrm{op}} \dot{\gamma}(t)\mathrm{d}t \le M\|x-y\|, \quad M = \sup_{z \in \overline{B_r(x)}} \big(Jf(z)\big)$$

Thus, C^1-functions are locally Lipschitz. Since $\mathbb{R}^n$ is second countable, we shall conclude that the arbitrariness (on the choice of $\overline{B_r(x)}$) in the claim above proves the proposition. $\square$

With the measure-zero set on Euclidean space, it is natural to pullback the measure-zero set onto differentiable manifolds:

Definition 2.5 (Measure-Zero Sets on Manfiolds)**.** Let $S \subseteq M$ be the subset of a differentiable manifold. S has measure-zero if exists an atlas $\mathcal{A} := \{(U_\alpha, \varphi_\alpha) \mid \alpha \in \Gamma\}$ on M, such that $\forall \alpha \in \Gamma : \varphi(C \cap U_\alpha) \subseteq \mathbb{R}^n$ has measure-zero.

From Proposition 2.8, since the transition map is smooth, the definition above does not depend on the local coordinates, and the properties in 2.7 also hold for measure-zero sets on differentiable manifolds.

We finally reach the main theorem of the section:

Theorem 2.6 (Sard's Theorem)**.** For any $f \in C^\infty(M, N)$, the set of critical points of f has measure-zero in N.

We shall show some applications before we prove Sard's theorem. Recall that we have shown that transversality is invariant under small perturbation in Theorem 1.15. Here, we shall now prove Theorem 1.15, which shows that transversality is generic: Recall that the theorem statement is given by

Theorem (Transversality Theorem)**.** Let $f : M \times P \to N$ be a smooth map with $f_p := f(-, p)$ for $p \in P$, and $f \pitchfork Z$ for some regular submanifold $Z \subseteq N$. Then, $f_p \pitchfork Z$ almost everywhere on N.

Proof. The key is to consider the preimage $W := f^{-1}(Z)$, by Theorem 1.14,

- W is an embedded submanifold.
- $\operatorname{codim}_{M \times P} W = \operatorname{codim}_N Z$.

Without loss of generality, we can consider only the case that $Z = g^{-1}(0)$ such that $g : N \to \mathbb{R}^k$ is a submersion, since by the second countability, the set of points B such that the transversality fails can always be covered by countably many local charts, and by Theorem 1.6, we can choose such charts (U, φ) such that $Z \cap U = g^{-1}(0)$ for some g. For any $(x, p) \in W$ and $z = f(x, p)$, the tangent space is given by

$$T_{(x,p)}W := \{(u, v) \in T_xM \oplus T_pP \mid f_{*,(x,p)}(u, v) \in T_zZ\}$$

Let $\mathrm{proj}_2 : M \times P \to P$ and $\pi := \mathrm{proj}\,|_W : W \to P$. Then, fix $p \in P$, the preimage

$$\pi^{-1}(p) = W \cap (M \times \{p\}) = \{(x, p) \in M \times P \mid f(x, p) \in Z\}$$

and its pushforward is given by $\pi_{*,p} : T_{(x,p)}W \to T_pP$

$$\pi_{*,(x,p)}(u, v) = v$$

Then we have the following lemma:

Lemma. *Fix $p \in P$, with $f \pitchfork Z$*

$$f_p \pitchfork Z \iff p \text{ is a regular value of } \pi$$

To prove this lemma, fix $(x, p) \in W$, let $z = f(x, p) \in Z$, we shall defined the decomposition of $f_{*,(x,p)}$

$$A := f_{*,(x,p)}|_{T_xM\times\{0\}} : T_xM \to T_zN, \quad B := f_{*,(x,p)}|_{\{0\}\times T_pP} : T_pP \to T_zN$$

it is obvious that $(f_p)_{*,x} = A$. Thus,

$$f_{*,(x,p)}(u, v) = Au + Bv$$

Then we shall compute the By the definition of transversality, $f \pitchfork Z$ means

$$f_{*,(x,p)}(T_xM \oplus T_pP) + T_zZ = T_zN$$

For simplicity, we shall introduce the quotient space T_zN/T_zZ with quotient map q. Also, by the given result of $T_{(x,p)}W$, $\forall (u, v) \in T_{(x,p)}W$

$$q \circ A(u) = -q \circ B(v)$$

($\Rightarrow$) If $f_p \pitchfork Z$, i.e. $A(T_xM) + T_zZ = T_zN$. Then we know that $q \circ A(T_xM) = T_zN/T_zZ$ is surjective, which mean for any given $v \in T_pP$, there always exists $u \in T_xM$ such that $q \circ A(u) = -q \circ B(v)$, or $(u, v) \in T_{(x,p)}W$. Thus, p is a regular point of π.
($\Leftarrow$) If p is a regular point of π, then $\forall v \in T_pP$, one shall find $u \in T_xM$ such that $q \circ B(v) = -q \circ A(u)$, which means

$$\mathrm{im}(q \circ B) \subseteq \mathrm{im}(q \circ A)$$

Which implies $\forall v \in T_pP : \forall u \in T_xM$, we have $Bv - Au \in T_zZ$. Thus,

$$\begin{aligned} f_{*,(x,p)}(T_xM \oplus T_pP) + T_zZ &= A(T_xM) + B(T_pP) + T_zZ \\ &= A(T_xM) + T_zZ = T_zN \end{aligned}$$

Thus, $f_p \pitchfork Z$, which completes the proof of the lemma.
With the lemma, we can simply apply Sard's theorem, which proves the theorem. □

The final goal of this section is to prove Sard's Theorem (Theorem 2.6). The following proof can be found in J. Milnor's book, *Topology from Differentiable Viewpoints* [**Milnor1965**], and originally given by Pontryagin [37]. The proof has been slightly modified since [37] gives a more general proof on C^k functions, with $k > \max\{\dim M - \dim N, 0\}$, and also since we do not assume readers to have knowledge about real analysis.

Proof of Sard's Theorem 2.6. We shall assume in the following proof that $m \geq n$, since if $m < n$, by Proposition 2.7, the image of $\mathbb{R}^m$ has measure-zero.
This theorem directly follows from the same theorem in Euclidean space, so we shall take $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. We shall use induction on m. The theorem holds obviously when $m = 0$, and we shall assume the theorem holds for $m-1$. Let $C := \mathrm{Crit}(f)$ be the set of critical points of f, and

$$C_s := \{x \in \mathbb{R}^m \mid \text{All } k\text{-th partial derivative of } f \text{ vanishes, } 1 \leq k \leq s\}$$

Obviously $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$, and we have

$$f(C) = f(C \setminus C_1) \cup f(C_1 \setminus C_2) \cup \cdots \cup f(C_{k-1} \setminus C_k) \cup f(C_k)$$

Thus, we shall prove the theorem based on the properties in Proposition 2.7, via the following steps:

1. $f(C \setminus C_1)$ has measure-zero.
2. $f(C_i \setminus C_{i+1})$ is measure-zero $\forall i$.
3. $f(C_k)$ is measure-zero for k sufficiently large.

Step 1: For any $x \in C \setminus C_1$, we shall find open neighborhood U_x such that $f(U_x \cap C)$ has measure-zero. We shall take $m \geq n > 1$ since if $n = 1$,

$$x \in C \iff \frac{\partial f}{\partial x^i}(x) = 0 \ \forall i \iff x \in C_1$$

Thus $C \setminus C_1 = \emptyset$, which must be measure-zero. When $n > 1$, without loos of generality, since $x \notin C_1$, we shall take $\partial f_1 / \partial x^1 \neq 0$ and

$$h : U \to \mathbb{R}^m, \quad h(x) = (f_1(x), x^2, \ldots, x^m)$$

Then the Jacobi matrix is given by

$$h_{*,x} = \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & * \\ 0 & I_{m-1} \end{pmatrix}$$

is nondegenerate. By inverse function theorem (Theorem 1.2), we shall find $U_x \cong V$ such that $h : U \xrightarrow{\sim} V$ gives the diffeomorphism. Let $g := f \circ h^{-1}$, then the crical value of the map is $f(U_x \cap C)$. By definition, g is given by

$$g(t, x^2, \ldots, x^m) = \big(t, g_2(x), \ldots, g_n(x)\big)$$

Thus, $\forall t \in \mathbb{R}$, the function induce a smooth map $g^t : \{t\} \times \mathbb{R}^{m-1} \cap V \to \{t\} \times \mathbb{R}^{n-1}$, and g has Jacobi matrix

$$g_{*,x} = \begin{pmatrix} 1 & 0 \\ * & \left(\frac{\partial g_i}{\partial x^j}\right)_{i,j \geq 2} \end{pmatrix}$$

Which shows that $x \in \{t\} \times \mathbb{R}^{m-1} \cap V$ is a critical point of g if and only if it is a critical point of g^t. However, based on the induction assumption, we know that Sard's theorem holds for all g^t. Consider the following lemma asserted by Fubini's theorem (see any standard real analysis textbook, e.g., [**SteinShakarchi2005**]):

Lemma. *Let $A \subseteq \mathbb{R}^m$ be a measurable set, then A is (m-dimensional) measure-zero if $\forall t \in \mathbb{R}$, the subset $A \cap \{t\} \times \mathbb{R}^{m-1}$ is (($m-1$)-dimensional) measure-zero.*

Directly follows from the lemma, we know that $f(U_x \cap C)$ is measure-zero. By the second countability of manifold, apply the Proposition 2.7, $f(C \setminus C_1)$ is a measure-zero set.
Step 2: For any $x \in C_i \setminus C_{i+1}$, take an index $\alpha = (\alpha_1, \ldots, \alpha_i)$ such that

- The partial derivative
$$w(x) := \partial^\alpha f|_x = \frac{\partial^i f}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_i}}(x) = 0$$

- Without loss of generality, $\partial w/\partial x^1 \neq 0$.

Then, similar to step 1, by the inverse function theorem, the function $h : U \to \mathbb{R}^m$ defined by $h(x) = (w(x), x^1, \ldots, x^m)$ diffeomorphically maps some open neighborhood U_x to V. Since $h(C_i \cap U_x) \subseteq \{0\} \times \mathbb{R}^{m-1}$, again, we shall consider $g := f \circ h^{-1}$ with the restriction

$$\tilde{g} := g|_{\{0\}\times\mathbb{R}^{n-1}} : \{0\} \times \mathbb{R}^{n-1} \cap V \to \mathbb{R}^n$$

and all critical points of g with "C_i-type" were included in the domain of $\tilde{g}$. By induction assumption, $\mathrm{Crit}(\tilde{g})$ has measure-zero in $\mathbb{R}^n$, but

$$\tilde{g} \circ h(C_i \cap U_x) = g(C_i \cap U_x) \text{ has measuire-zero.}$$

which completes the proof of step 2.
Step 3: Still, by the second countability, it is sufficient to prove that for (closed) box Q with edge δ, $f(C_k \cap Q)$ has measure-zero when k is sufficiently large. Since Q is compact (closed and bounded), by the definition of C_k, for any pair of (x, h) such that $x \in Q$ and $x + h \in Q$, Taylor's theorem gives

$$f(x + h) = f(x) + R(x, h)$$

with the remainder $\|R(x, h)\| < a\|h\|^{k+1}$ for some constant a that only depends on f and the choice of Q. We shall divide Q into l^m cubes with edge δ/l. Let Q_1 be the small cube with $x \in C_k \cap Q_1$, then, any other points in Q_1 can be written as $x + h$ with $\|h\| \leq \delta\sqrt{m}/l$. Then, from the Taylor's theorem approximation we did above, it follows that $f(Q_1)$ is in a cube with edge b/l^{k+1} and centered at $f(x)$, where

$$b = 2a(\delta\sqrt{m})^{k+1}$$

which implies $f(C_k \cap Q_1)$ is contains in the union of at most l^m shall cubes and with volume

$$V \leq l^m \left(\frac{b}{l^{k+1}}\right)^n = b^n l^{m-(k+1)n}$$

with $k > m/n - 1$, $V \to 0$ as $l \to \infty$, i.e., $f(C_k \cap Q)$ has measure-zero, which completes the proof of step 3. $\square$

At the end of this section, we shall show that the result of Sard's theorem requires a strong differentiability condition. Here is a counterexample (constructed by E. Grinberg) that f is not smooth enough:

Example 2.1. *Let $C \subseteq [0, 1]$ be the cantor set. We shall first construct a C^1 function $f : \mathbb{R} \to \mathbb{R}$ that contains the cantor set C. Note that the cantor set is construct based on*

$$C_0 := [0, 1] \text{ with } C_n := \frac{1}{3}C_{n-1} + \frac{2}{3}C_{n-1}, \quad C := \bigcap_{i=0}^{\infty} C_i$$

An obvious fact is that (since Cantor set is the intersection of closed set, and thus, closed)

$$[0, 1] \setminus C = \bigcup_{k=0}^{\infty} (a_k, b_k)$$

Then, we shall take the following C^0-bump function

$$\eta_k(t) := \begin{cases} \frac{6}{(a_k-b_k)^3}(t - a_k)(b_k - t), & t \in [a_k, b_k] \\ 0, & \text{otherwise} \end{cases}$$

Since $\eta_k(a_k) = \eta_k(b_k) = 0$, η_k continuous and

$$\int_{a_k}^{b_k} \eta_k(t)\mathrm{d}t = 1$$

take $\tilde{n}(k) = n$ if

$$(a_k, b_k) \subseteq [0,1] \setminus \Big(\bigcap_{i=0}^{n} C_i\Big)$$

then we shall take $\alpha_k = 2^{-n(k)}$ defined the function on $\mathbb{R}$

$$h(t) := \begin{cases} \alpha_k \eta_k(t), & t \in [a_k, b_k] \\ 0, & otherwise \end{cases}$$

we shall get a C^1-function by integrate $h(t)$:

$$f(t) = \int_0^x h(t)\mathrm{d}t, \quad f'(x) = h(x)$$

since $\forall x \in C$, $x \notin (a_k, b_k)$ $\forall k$ and $h(x) = 0$ for all such x, i.e., $C \subseteq \mathrm{Crit}(f)$. We shall use without a proof the fact that $C + C = [0,2]$, then, we shall defined $g : \mathbb{R}^2 \to \mathbb{R}$ such that $g(x,y) = f(x) + f(y)$.

Lemma. *The critical points of g contains $C \times C$.*

Proof. For any $(x, y) \in C \times C$, the Jacobi matrix (same as gradient) of g is given by

$$J(g)_{(x,y)} = (f'(x), f'(y)) = (h(x), h(y)) = 0$$

Thus, $C \times C \subseteq \mathrm{Crit}(g)$. □

Since $g(C \times C) = f(C) + f(C)$. We shall show that $f(C) + f(C) = [0,2]$: Since $\mathrm{supp}(\eta_k) \cap \mathrm{supp}(\eta_l) = \emptyset$,

$$\int_0^1 \eta_x(t)\mathrm{d}t = \sum_{k\geq 1} \alpha_k \int_{a_k}^{b_k} \eta_k(t)\mathrm{d}t = \sum_{k\geq 1} \alpha_k = 1$$

By the continuity of f, since $f(0) = 0$, $f(1) = 1$, and f is defined on $[0,1]$. We shall conclude that $f([0,1]) = [0,1]$. Thus

$$g(C \times C) = f(C) + f(C) = [0,2] \supseteq (0,2)$$

contains non-empty open sets, i.e., $g(C)$ is not measure-zero.

2.4 Whitney Embedding Theorem

In Section 2.2, we have proved a weak version of the Whitney embedding theorem (Theorem 2.3), which shows that every differentiable manifold can be embedded in $\mathbb{R}^N$ for some positive integer N. In this section, we shall show an important application of Sard's theorem, which proves the stronger version of Theorem 2.3, that gives the dimension information N of the Euclidean space in the embedding theorem. More precisely, we shall prove a stronger version of the Whitney embedding theorem:

Theorem 2.7 (Whitney Embedding Theorem)**.** Any m-dimensional smooth manifold can be embedded into $\mathbb{R}^{2m+1}$ and immersed into $\mathbb{R}^{2m}$.

In Theorem 2.3, we have already proved actually gives a stronger result:

Theorem 2.8. Any differentiable manifold M can be embedded into $\mathbb{R}^N$ for some $N > 2m+1$ if M can be covered by finitely many charts.

The proof is exactly the same as 2.3.

However, with the Sard's theorem, we can prove the following theorem that gives a lower bound of the dimension of Euclidean space:

Theorem 2.9. If differentiable manifold M^m can be immersed into $\mathbb{R}^N$ injectively with $N > 2m+1$, then M can be injectively immerse into $\mathbb{R}^{N-1}$.

Proof. Suppose $\Phi : M \to \mathbb{R}^N$ is an injective immersion with $N > 2m+1$. To construct the injective immersion from M to $\mathbb{R}^{N-1}$, consider the set of all $N-1$ dimensional subspaces $\mathrm{Gr}_{N-1}(\mathbb{R}^N) \cong \mathbb{RP}^{N-1}$ (Consider the collection of linear complement of all subsets $V \in \mathrm{Gr}_{N-1}(\mathbb{R}^N)$). $\forall [v] \in \mathbb{RP}^{N-1}$

$$P_{[v]} := \{u \in \mathbb{R}^N \mid u \cdot v = 0\} = [v]^{\perp} \cong \mathbb{R}^{N-1}$$

and consider the projection $\pi_{[v]} : \mathbb{R}^N \to P_{[v]}$, $\Phi_{[v]} := \pi_{[v]} \circ \Phi : M \to \mathbb{R}^{K-1}$. Then, with Sard's theorem (Theorem 2.6), we shall claim that

Lemma. *The set* $\{[v] \in \mathbb{RP}^{N-1} \mid \Phi_{[v]}$ *is not an injective immersion*$\}$ *has measure-zero in* $\mathbb{RP}^{N-1}$.

To prove the lemma, for any $[v] \in \mathbb{RP}^{N-1}$, there are two possibilities:
First of all, we shall consider the $[v]$ such that $\Phi_{[v]}$ is not injective. We shall find $p_1 \neq p_2 \in M$ such that $\Phi_{[v]}(p_1) = \Phi_{[v]}(p_2)$, in other word

$$[\Phi(p_1) - \Phi(p_2)] = [v]$$

and note that $\Phi(p_1) - \Phi(p_2) \neq 0$ (since Φ is a injective immersion). Thus, $[v]$ is in the image of the C^∞-map

$$\alpha : (M \times M) \setminus \Delta_M \to \mathbb{RP}^{N-1}, \quad \alpha(p_1, p_2) := [\Phi(p_1) - \Phi(p_2)]$$

Where $\Delta_M := \{(p,p) \mid p \in M\}$. Since $\dim((M \times M) \setminus \Delta_M) \leq 2m < N-1$, by Sard's theorem, $\mathrm{im}\,\alpha$ is measure-zero in $\mathbb{RP}^{N-1}$, and thus, the set of $[v]$ such that $\Phi_{[v]}$ is not injective is measure-zero.
Secondly, consider the set of $[v]$ such that $\Phi_{[v]}$ is not a immersion, i.e., $\exists p \in M : \exists 0 \neq X_p \in T_pM$ such that

$$0 = (\Phi_{[v]})_{*,p} X_p = (\pi_{[v]})_{*,\Phi(p)} \circ \Phi_{*,p}(X_p)$$

Since $\pi_{[v]}$ is a linear projection, $(\pi_{[v]})_{*,\Phi(p)} = \pi_{[v]}$ and thus $[\Phi_{*,p}(X_p)] = [v]$. In other words, $[v]$ is in the image of C^∞-map

$$\beta : TM \setminus \{0\} \to \mathbb{RP}^{N-1}, \quad \beta(p, X_p) := [\Phi_{*,p} X_p]$$

Where the manifold $TM := \{(p, X_p) \mid p \in M : X_p \in T_pM\}$ is called the tangent bundle, it is easy to check that this is a $2m$-dimensional C^∞-manifold. Since $2m < N-1$, we shall apply Sard's theorem again to state that the image of β is also measure-zero in $\mathbb{RP}^{N-1}$.
Thus, the lemma was proved, and the map $\Phi_{[v]} = \pi_{[v]} \circ \Phi$ is the injective immersion from M to $\mathbb{R}^{N-1}$. □

Notice the fact that when the manifold been immersed/embedded into $\mathbb{R}^N$, we shall take the canonical norm on $\mathbb{R}^N$ as the norm on T_pM, and thus

$$\Phi_{*,p}(X_p) = 0 \iff \Phi_{*,p}\Big(\frac{X_p}{\|X_p\|}\Big) = 0$$

Thus, with a similar technique, if we do not require the immersion to be injective, we shall prove the following result:

Theorem 2.10. If m-dimensional smooth manifold M can be embedded into $\mathbb{R}^{2m+1}$, then it can be immerse into $\mathbb{R}^{2m}$.

Proof. Suppose $\Phi : M \to \mathbb{R}^{2m+1}$ is an embedding, then we shall use a similar method in the last proof. We shall similarly consider $\Phi_{[v]} := \pi_{[v]} \circ \Phi$ with the chosen $X_p \in T_pM$ such that $\|X_p\| = 1$. The smooth map β in the previous proof can be changed to

$$\tilde{\beta} : SM \setminus \{0\} \to \mathbb{RP}^{2m}, \quad \tilde{\beta}(p, X_p) := [\Phi_{*,p}(X_p)]$$

Where $SM := \{(p, X_p) \in TM \mid \|X_p\| = 1\}$ is a $2m - 1$ dimensional smooth manifold called the unit sphere bundle on M. The Sard's theorem gives that $\Phi_{[v]}$ is the immersion from M to $\mathbb{R}^{2m}$. $\square$

2.4.1 Compact Manifolds

We shall now prove the theorem for compact manifolds:

Theorem 2.11 (Whitney Embedding Theorem for Compact Manifolds)**.** m-dimensional compact differentiable manifold M can always immersed into $\mathbb{R}^{2m}$ and embedded into $\mathbb{R}^{2m+1}$.

Proof. Let M be an m-dimensional compact smooth manifold. Thus, any smooth map $f : M \to \mathbb{R}^{2m+1}$ is proper. By Theorem 2.9, exists injective immersion $f : M \to \mathbb{R}^{2m+1}$. By the compactness, f is proper, the inverse $f^{-1} : f(M) \to M$ (existence by injectiveness) is continuous. Thus, f gives an homeomorphism between M and $f(M)$. Thus $f : M \to \mathbb{R}^{2m+1}$ is an embedding. By Theorem 2.10, M can be immersed into $\mathbb{R}^{2m}$. $\square$

2.4.2 Noncompact Manifolds

To generalize the result from compact manifolds to general cases, it will be sufficient to show the existence of a proper injective immersion. We shall first prove a generalization of Theorem 2.3:

Theorem 2.12. Any differentiable manifold M can be embedded into Euclidean space $\mathbb{R}^N$ for sufficiently large N.

The idea of proving this theorem is to use exhaustion to find relatively "good" slices on M, and embed them into some Euclidean space.

Proof. We shall use the exhaustion function on M to define the slice on the manifold. By Proposition 2.2, we shall find a compact exhausion $\{K_i\}$ such that $K_i \forall i$ is compact and

$$\bigcup_i K_i = M, \quad K_i \subseteq \operatorname{int}(K_{i+1}) \ \forall i$$

Then, we need the following function, named a smooth exhaustion function:

Lemma (Smooth Exhaustion Function)**.** *There exists a smooth function $f : M \to \mathbb{R}$ such that $f^{-1}\big((-\infty, 0]\big)$ compact.*

It is not hard to see that the partition of unity associated with the covering $\{\operatorname{int}(K_i)\}$. More precisely, we take smooth functions $\phi_i : M \to [0, 1]$ such that

$$\phi_i|_{K_i} \equiv 1, \qquad \operatorname{supp}(\phi_i) \subseteq \operatorname{int}(K_{i+1})$$

and the exhaustion function we need is given by

$$f = g(x) - 1, \quad g(p) = \sum_i \phi_i(p)$$

since if $p \in K_N$, then $\forall n \le N$, $\phi_n \equiv 1$. Thus

$$x \in K_N \implies g(x) \ge N$$

which means $g^{-1}([0, N]) \subseteq K_N$. By the continuity of g and compactness of K_N, $g^{-1}([0, N])$ is compact. Thus, $f^{-1}\big((-\infty, 0]\big) = \{x \mid g(x) \le 1\} = g^{-1}\big((-\infty, 1]\big) = g^{-1}([0, 1])$ is compact.
With the exhaustion function f, we shall define the compact set

$$M_i := f^{-1}([i, i+1])$$

and its open cover $\{U_1, \dots, U_k\}$. Let

$$N_i = (U_1 \cup \cdots \cup U_k) \cap f^{-1}\big((i - 0.1, i + 1.1)\big)$$

Then, $N_i \subseteq M$ is a (open) submanifold of M, and $M_i \subseteq N_i$. Also, for $|i - j| \ge 2$, $N_i \cap N_j = \emptyset$. By the given construction, N_i can be covered by finitely many coordinate charts, which means we can still apply the proof of Theorem 2.3 on N_i even if it may not be compact. Since $\dim N_i = \dim M = m$, by Theorem 2.9, N_i can be injectively immersed into $\mathbb{R}^{2m+1}$ by some map φ_i.
Take a smooth bump function ρ_i such that $\rho_i \equiv 1$ on some open neighborhood of M_i and $\operatorname{supp}(\rho_i) \subseteq N_i$. Take $\Phi : M \to \mathbb{R}^{4m+3}$ defined by

$$\Phi(p) := \Big(\sum_{i \text{ odd}} \rho_i(p)\varphi_i(p), \sum_{i \text{ even}} \rho_i(p)\varphi_i(p), f(p)\Big)$$

The smoothness is obvious; we shall check that it is an injective immersion:

1. Φ is an injection: Suppose $\Phi(p_1) = \Phi(p_2)$, then for some $i \in \mathbb{N}$, $f(p_1) = f(p_2) \in [i, i+1]$. Thus, $p_1, p_2 \in M_i \subseteq N_i$ with $\varphi_i(p_1) = \varphi_i(p_2)$. By the injectivity of φ_i, $p_1 = p_2$.
2. Φ is an immersion: Let $p \in M_i$, without loss of generality, let i be an odd number. Then, for any $0 \ne X_p \in T_pM$
$$\Phi_{*,p}X_p = \big((\varphi_i)_{*,p}(p), *, *\big)$$
Since φ_i is an immersion, $\Phi_{*,p}X_p \ne 0$ is also an immersion.

Thus, we complete the proof of the Theorem. □

Then we shall show that there always be a proper injective immersion, which makes the poof of Theorem 2.11also applies for Theorem 2.7.

Theorem 2.13. If an m-dimensional differentiable manifold M can be injectively immersed into ^{N}, with $N > 2m$, then it can be properly injectively immersed into $\mathbb{R}^N$.

Proof. Let $\Phi : M \to \mathbb{R}^N$ be injective immersion, and consider the diffeomorphism

$$\mathbb{R}^N \to B^N := \{x \in \mathbb{R}^n \mid |x| < 1\}, \quad x \mapsto \frac{x}{1 + |x|^2}$$

We shall assume $\forall p \in M$, $|\Phi(p)| < 1$ by compose Φ with the diffeomorphism above. For any positive exhaustion function $f : M \to [0, +\infty)$, let

$$\widetilde{\Phi} := (\Phi, f) : M \to \mathbb{R}^{N+1}\text{fi} \quad p \mapsto \big(\Phi(p), f(p)\big)$$

Then $\widetilde{\Phi}$ is also an injective immersion and $N + 1 > 2m + 1$. We shall also consider the same function $\pi_{[v]} : \mathbb{R}^{N+1} \to P_{[v]} \cong \mathbb{R}^N$ with the proof of Theorem 2.9, we defined another injective immersion

$$\Psi := \pi_{[v]} \circ \widetilde{\Phi} : M \to \mathbb{R}^N$$

Without loss of generality, we can choose coordinates such that $[v] \neq [0 : \cdots : 0 : 1]$.
We claim that the immersion Ψ is proper. Without loss of generality, let v be unit vector and $v = (v', v^{N+1})$. The condition that $v \neq [0 : \cdots : 1]$ means that $|v^{N+1}| < 1$, and $\pi_{[v]}(x) = x - (x \cdot v)v$. Thus, we shall write Ψ explicitly as:

$$\begin{aligned}\Psi(p) &= \big(\Phi(p), f(p)\big) - \big[\Phi(p) \cdot v' + f(p)v^{N+1}\big](v'.v^{N+1}) \\ &= \big(*, f(p)\big[1 - (v^{N+1})^2\big] - (\Phi(p) \cdot v')v^{N+1}\big)\end{aligned}$$

For any compact set $C \subseteq P_{[v]} \cong \mathbb{R}^N$. We shall find $A > 0$ such that

$$C \subseteq \{x \mid |x^{K+1}| < A\}$$

Since $\Phi(p) \leq 1$, $|v^{K+1}| \leq 1$ and $|v'| \leq 1$, we can know that $\forall p \in \Psi^{-1}(C)$

$$\begin{aligned}&\left|f(p)\big[1 - (v^{N+1})^2\big] - (\Phi(p) \cdot v')v^{N+1}\right| < A \\ \Longrightarrow\ &\left|f(p)\big[1 - (v^{N+1})^2\big]\right| \leq A + \left|(\Phi(p) \cdot v')v^{N+1}\right| \leq A + 1 \\ \Longrightarrow\ &\Psi^{-1}(C) \subseteq f^{-1}\left(\left[-\frac{A+1}{1-(v^{N+1})^2}, \frac{A+1}{1-(v^{N+1})^2}\right]\right)\end{aligned}$$

But by the continuity of Ψ, $\Psi^{-1}(C)$ is closed, and by the properness of f we know that $\Psi^{-1}(C)$ is compact. Thus, we have shown that Ψ is a proper map. □

Finally, we shall prove the main theorem of this section. We first recall the theorem:

> **Theorem** (Whitney Embedding Theorem)**.** Any m-dimensional smooth manifold can be embedded into $\mathbb{R}^{2m+1}$ and immersed into $\mathbb{R}^{2m}$.

The proof is rather simple with so much work been done previously:

Proof of Whitney Embedding Theorem 2.7. Let M be an m-dimensional differentiable manifold. By Theorem 2.12 and 2.9, M can be injectively immersed into $\mathbb{R}^{2m+1}$. Then, we claim that we can find a proper injective immersion by applying 2.13. By the same logic in the proof of the Whitney embedding theorem for compact manifolds (Theorem 2.11), the proper injective immersion is an embedding, which means M can be embedded into $\mathbb{R}^{2m+1}$. Finally, directly by Theorem 2.10, we know that M can be immersed into $\mathbb{R}^{2m}$. □

In this way, we have shown that every differentiable manifold can be viewed as a certain subset in $\mathbb{R}$. Actually, using a completely different technique (named Whitney trick), Whitney had proven the following theorem in 1944 [**Whitney1944**]:

> **Theorem 2.14** (Strong Whitney Embedding theorem)**.** Any m-dimensional ($m \geq 2$) smooth manifold M can be embedded into $\mathbb{R}^{2m}$ and immersed into $\mathbb{R}^{2m-1}$.

The proof of this theorem is far beyond our note, and the Whitney technique has been developed to the h-cobordism theory by Smale and successfully proved the Poincaré conjecture in ≥ 5 dimension [45, 46]. As further reading, J. Milnor wrote a more well-written text on h-cobordism theory [**Milnor1965HCobordism**].

In the late 20th century, people found more results related to the embedding problem. The result has been summarized as the following problem [5]:

Open Problem. *Any m-dimensional closed (compact without boundary) manifold can be embedded into $\mathbb{R}^{2m-\alpha(m)+1}$ and immersed into $\mathbb{R}^{2m-\alpha(m)}$, where $\alpha(m)$ is the number of 1's in the binary expansion of n.*

In the paper [5], Brown successfully proved that the statement above holds up to cobordisms. For the more general case, the embedding part of this problem is still open, and the immersion part was proved by Cohen in 1982 [**Cohen1982**, **Cohen1985**]. For manifolds with boundary (or noncompact), we probably need more conditions in the statement to add more constraints on the boundary (or properness of the embedding).

2.5 Morse Function Preliminary

Another significant application of Sard's theorem is that it provides the existence of Morse functions, which are roughly smooth functions on M with non-degenerate critical points. The Morse function is an extremely important technique for studying the topology of differentiable manifolds. A more detailed introduction to Morse theory will be in Chapter 12. We shall define the Hessian of the function first: Recall that a critical point of a C^∞-function $f : M \to \mathbb{R}$ is the point $p \in M$ such that $f_{*,p} \equiv 0$.

Definition 2.6 (Non-degenerate Critical Points)**.** A critical point $p \in M$ of $f \in C^\infty(M)$ is said to be nondegenerate if and only if the Hessian

$$\mathrm{Hess}_p(f) := \Big(\frac{\partial^2 f}{\partial x^i \partial x^j}\Big|_p\Big)_{1\le i,j\le n}$$

is non-singular as a matrix.

It can be checked directly that this non-degeneracy does not depend on the coordinates. We shall also define the Hessian intrinsically:

Definition 2.7 (Hessian)**.** The Hessian of f at critical point $p \in M$ is a symmetric bilinear form given by $\forall v \in T_pM$ with smooth curve $\gamma : I \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$

$$\mathrm{Hess}_p(f)(v, v) = Q_p(v) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}(f \circ \gamma)(t)\Big|_{t=0}$$

Using the polarization identity, we shall write

$$\mathrm{Hess}_p(f)(v, w) = \frac{1}{2}\big(Q_p(v + w) - Q_p(v) - Q_p(w)\big)$$

In the following remark, for simplicity of notation, we shall use the summation convention that a pair of upper and lower indices indicates summation on the indices from 1 to n.

Remark. *The definition here only applies to the case that $p \in M$ is a critical point of f. Since if not, consider the local coordinate expression of the Hessian given by taking local coordinates $\varphi = (x^1, \ldots, x^n)$ and $\psi = (y^1, \ldots, y^n)$ with the chart induced basis $e_i := \partial/\partial x^i|_p$ and $\tilde e_i := \partial/\partial y^i|_p$, then the coordinate transformation on Hessian is given by*

$$\begin{aligned}\mathrm{Hess}_p(f)(e_i, e_j) &= \frac{\partial^2 f}{\partial x^i \partial x^j}\Big|_p = \frac{\partial}{\partial x^i}\Big(\frac{\partial y^s}{\partial x^j}\frac{\partial f}{\partial y^s}\Big)\Big|_p = \frac{\partial y^r}{\partial x^i}\frac{\partial}{\partial y^r}\Big(\frac{\partial y^s}{\partial x^j}\frac{\partial f}{\partial y^s}\Big)\Big|_p \\ &= \frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j}\frac{\partial^2 f}{\partial y^r \partial y^s}\Big|_p + \frac{\partial^2 y^s}{\partial x^i \partial x^j}\frac{\partial f}{\partial y^s}\Big|_p = \frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j}\mathrm{Hess}_p(f)(\tilde e_i, \tilde e_j) + \frac{\partial^2 y^s}{\partial x^i \partial x^j}\frac{\partial f}{\partial y^s}\Big|_p\end{aligned}$$

which does not transform as usual with the Jacobi matrix if $\partial f/\partial y^s|_p \neq 0$. Thus, the calculation above indicates that our definition of Hessian requires that $f_{*,p} \equiv 0$.

With the Hessian and non-degeneracy of the critical points being defined, we shall now talk about the Morse function:

Definition 2.8 (Morse Function)**.** A Morse function $f \in C^\infty(M)$ is a smooth function such that all critical points are non-degenerate.

The following theorem (Morse Lemma) defines the index of the Hessian as a bilinear form. The index, or Morse index of a smooth function f at the non-degenerate critical point p, is the maximal dimension of a subspace $V \subseteq T_pM$ such that $f_{*,p}|_V$ is negative definite.

Theorem 2.15 (Morse Lemma)**.** If $p \in M$ is a non-degenerate critical point of $f \in C^\infty(M)$, then exists a chart (U, φ) that contains p such that

$$f \circ \varphi^{-1}(y) = f(p) - (y^1)^2 - \cdots - (y^r)^2 + (y^{r+1})^2 + \cdots + (y^n)^2$$

where $0 \leq r \leq n$ is called the Morse index of f at p.

Proof. Without loss of generality, we can take $M = \mathbb{R}^n$ with $p = 0$. Similar with the proof of Theorem 1.9, since

$$f(x) - f(0) = \int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\big(f(tx)\big) = \int_0^1 \mathrm{d}t \sum_{i=1}^n \frac{\partial f}{\partial x^i}(tx) x^i$$

Thus, f can be expressed based on a collection of smooth functions g_i

$$f(x) = f(0) + \sum_{i=1}^n x^i g_i(x), \quad g_i(x) := \int_0^1 \mathrm{d}t \frac{\partial f}{\partial x^i}(tx)$$

since g_i are also smooth functions with $g_i(0) = \partial f / \partial x^i(0) = 0$, so we can apply the expansion again on g_i, which gives

$$f = f(0) + \sum_{i,j=1}^n x^i x^j g_{ij}(x)$$

where $g_{ij} = g_{ji}$ are still smooth functions. Also,

$$g_{ij}(0) = \frac{1}{2} \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{1}{2} \mathrm{Hess}_0(f)(e_i, e_j)$$

is non-degenerate. Then, we shall apply the following lemma:

Lemma (Hirsch)**.** *Let $Q(x)$ be a symmetrical n-dimensional real matrix that smoothly defines near $0 \in \mathbb{R}^n$. If $Q(0)$ non-degenerate, then exists non-degenerate smooth n-dimensional matrix $P(x)$ defined near 0 such that*

$$P(x)Q(x)P(x)^T = \begin{pmatrix} -I_r & \\ & I_{n-r} \end{pmatrix}$$

where r is the number of negative eigenvalues of Q.

The lemma can be proved by induction. When $n = 1$, the non-degeneracy is just $Q(x) \neq 0$, and we shall just let $P(x) = |Q(x)|^{-1/2}$. Assume the lemma hold for $n - 1$ dimensional matrix, consider N dimensional matrix Q, we know that since $Q(x)$ $\forall x$ is symmetrical, it always been diagnosable, i.e. exists $U \in O(n)$ such that

$$\Lambda = UQ(0)U^T, \quad \Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

Then, we shall take $S := \mathrm{diag}(|\lambda_1|^{-1/2}, \ldots, |\lambda_n|^{-1/2})$, and $P_0 := SU$. Then

$$PQP^T = S\Lambda S^T = \mathrm{diag}(\mathrm{sgn}\,\lambda_1, \ldots, \mathrm{sgn}\,\lambda_n)$$

Finally, we shall find a permutation R such that the first r entry are all -1, let $P = RP_0$, then

$$PQP^T = \begin{pmatrix} I_r & \\ & I_{n-r} \end{pmatrix}$$

Then, we shall let $Q'(x) := PQP$, and since the (n, n) entry of $Q'(0)$ is 1, we shall let $q_{nn}(x)$ be the (n, n) entry of $Q'(x)$. By the continuity, $q_{nn}(x) > 0$ near 0. Thus, we shall let $P_1(x) := \mathrm{diag}(1, \ldots, 1, |q_{nn}(x)|^{-1/2})$ such that the (n, n) entry of $Q''(x) := P_1(x)Q'(x)P_1(x)^T$ is 1, more precisely

$$Q''(x) = \begin{pmatrix} A(x) & b(x) \\ b(x)^T & 1 \end{pmatrix}$$

where $b(x)$ is a $(n-1)\times 1$ matrix (column vector) and $A(x)$ is a $(n-1)$-dimensional matrix. Consider the matrix

$$P_2(x) := \begin{pmatrix} I_{n-1} & b(x) \\ 0 & 1 \end{pmatrix}$$

Then, by simple computation

$$P_2(x)Q''(x)P_2(x)^T = \begin{pmatrix} A(x) - b(x)b(x)^T & 0 \\ 0 & 1 \end{pmatrix}$$

Finally, apply the induction assumption to $A(x) - b(x)b(x)^T$, the lemma got proved.
With the lemma, take non-degenerate $P(x)$ defines near 0 such that

$$P(x)\big(g_{ij}(x)\big)P(x)^T = \begin{pmatrix} -I_r & \\ & I_{n-r} \end{pmatrix}$$

With this matrix, we shall define another local coordinate

$$\varphi(x) = (y^1, \dots, y^n) := (x^1, \dots, x^n)P^{-1}(x)$$

and thus,

$$\begin{aligned} f\circ\varphi^{-1}(y) &= f(p) + x^T(g_{ij}(x))x = f(p) + y^T P(x)^T(g_{ij}(x))P(x)y \\ &= f(p) - (y^1)^2 - \cdots - (y^r)^2 + (y^{r+1})^2 + \cdots + (y^n)^2 \end{aligned}$$

This completes the proof of the Morse lemma. $\square$

An important corollary of the Morse lemma is that

Corollary. *The set of non-degenerate critical points of smooth functions is discrete. In particular, smooth functions have only finitely many non-degenerate critical points on compact manifolds.*

The proof of this corollary is simply notice that non-degenerate critical points are isolated since $f_{*,p}$ has a coordinate representation $f_{*,p} = 2(-y^1, \dots, -y^r, y^{r+1}, \dots, y^n)$ and the critical point $f_{*,p} = 0$ is equivalence to $y^i = 0\ \forall i$, which is the unique point $p \in \mathrm{Crit}(f)\cap U$. Finally, we shall prove the existence of the Morse function.

Theorem 2.16 (Existence of the Morse Function)**.** Let (U, φ) be a local coordinate chart on M, $K \subseteq U$ be a compact set. Then

1. If all critical points of smooth function $f : U \to \mathbb{R}$ on K are non-degenerate, then $\exists\delta > 0$ such that if smooth function $g : U \to \mathbb{R}$ satisfies

$$\big\|J(f\circ\varphi^{-1}) - J(g\circ\varphi^{-1})\big\| < \delta, \quad \big\|\mathrm{Hess}(f\circ\varphi^{-1}) - \mathrm{Hess}(g\circ\varphi^{-1})\big\| < \delta$$

   Then its critical points on K are all non-degenerate.

2. Let $f : U \to \mathbb{R}$ be smooth function, then $\forall\epsilon > 0$, exists smooth function $l : U \to \mathbb{R}$ such that

$$\mathrm{supp}(l) \subseteq U, \quad \big\|J(l\circ\varphi^{-1})\big\| < \epsilon \text{ and } \big\|\mathrm{Hess}(l\circ\varphi^{-1})\big\| < \epsilon$$

   and all critical points of $f + l$ on K are non-degenerate.

3. If M is a compact differentiable manifold with $f : M \to \mathbb{R}$ be a smooth function. Then $\forall\epsilon > 0$, there is a smooth Morse function $g : M \to \mathbb{R}$ such that

$$|g(x) - f(x)| < \epsilon\ \forall x \in M$$

Proof. (1). To prove the first statement, we shall consider the following function defined on U

$$\left\|J(f\circ\varphi^{-1})\right\| + \left\|\mathrm{Hess}(f\circ\varphi^{-1})\right\|$$

Since all critical points of f on K are non-degenerate, the function above has no zeros on K. By the compactness of K, this means the function above has a positive minimum on K. Thus, we shall always find $\delta > 0$ such that if

$$\left\|J(f\circ\varphi^{-1}) - J(g\circ\varphi^{-1})\right\| < \delta, \quad \left\|\mathrm{Hess}(f\circ\varphi^{-1}) - \mathrm{Hess}(g\circ\varphi^{-1})\right\| < \delta$$

then $\left\|J(f\circ\varphi^{-1})\right\| + \left\|\mathrm{Hess}(f\circ\varphi^{-1})\right\|$ is also positive, i.e., every critical points of g on K are non-degenerate.
(2). Since the statement is local, we shall take $U = \mathbb{R}^n$ with $\varphi = \mathrm{id}$. We shall take an open set V such that $K \subseteq V \subseteq U$ and $\overline{V} \subseteq U$ is compact. Let ϕ be the bump function on U such that

$$\phi|_{\overline{V}} \equiv 1, \quad \mathrm{supp}(\phi) \subseteq U$$

For $a \in \mathbb{R}^n$, consider the function given by $l_a : \mathbb{R}^n \to \mathbb{R}$

$$l_a(x) := a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

and $f_a : U \to \mathbb{R}$ defined by $f_a(x) := f(x) + \phi(x)l_a(x)$. We need to show that for almost every $a \in \mathbb{R}^n$, the critical points of f_a on K are non-degenerate. Consider $F_a : U \to \mathbb{R}^n$

$$F_a(x) := \Big(\frac{\partial f_a}{\partial x^1}(x), \dots, \frac{\partial f_a}{\partial x^n}\Big)$$

be the Jacobian of f_a. Thus, if x is the critical points of f_a, if and only if $F_a(x) = 0$, i.e., $J(f)(x) + a = 0$. Also

$$F_{*,x} = \mathrm{Hess}_x(f)$$

which also means that $x \in F^{-1}(a)$ is non-degenerate as a critical point if and only if $F_{*,x}$ surjective. Thus, we have the statement that

$$\text{All critical points of } f_a \text{ on } K \text{ are non-degenerate} \iff a \text{ is a regular value of } F_{(-)}(x)$$

By Sard's theorem (Theorem 2.6), for almost every a, all critical points of f_a on K are non-degenerate. Since we can find such a in an arbitrary interval, the second statement got proved.
(3) Since M is compact, we shall take a finite coordinate chart $\{(U_i, \varphi_i)\}_{i=1}^N$ with compact set

$$K_i \subseteq U_i, \quad \bigcup_{i=1}^N K_i = M$$

Let $f : M \to \mathbb{R}$ be a smooth function. By (2), we shall find $\tilde{l}_i$ and $l_i(x) := \phi_i(x)\tilde{l}_i(x)$ where ϕ_i is the bump function on U_i such that

$$\phi_i|_{K_i} \equiv 1, \quad \mathrm{supp}(\phi_i) \subseteq U$$

We shall defined $f_0 = f$, $f_i = f_{i-1} + l_i$ and let all critical points of f_i on K_i is non-degenerate and keeps the non-degeneracy of critical points in $K_1 \cup \cdots \cup K_{i-1}$ unchanged (this can be done due to the statement (1), since in the following part of the proof we will see that the contribution of the function l_i can be controlled in a small range). Since in (2) we already show that the number a such that $f + l_a$ is a Morse function on U is generic. Since the norm of l_a is given by

$$\|\phi(x)l_a\|_\infty := \sup_{x\in U}|\phi(x)l_a(x)| \leq \|a\| \cdot \|x\|$$

since $\|\phi\|_\infty = 1$. The generic on a gives the generic on the norm $\|l\|$. Thus, we can take

$$\|l_i\|_\infty := \sup_{x\in M}|l_i(x)| < \epsilon/2^i$$

Thus
$$\|f_N - f\|_\infty \leq \sum_{i=0}^{N} \|l_i\|_\infty < \sum_{i=0}^{N} \frac{\epsilon}{2^i} < \epsilon$$
Take $g = f_N$ to be the Morse function such that
$$|g(x) - f(x)| \leq \|f_N - f\|_\infty < \epsilon$$
Thus, the theorem was proved. □

2.6 The Degree of Maps Modulo 2 and Homotopy

In the previous section of Sard's theorem, we have proved that the number of preimages is locally constant (2.6). In this section, we shall study further about the number of preimages, which has a more general name called the degree of a map, we will show that given smooth map $f : M \to N$ with M be a closed manifold and N be a connected manifold (we may also assume it to be a closed manifold), the degree $|f^{-1}(q)|$ modulo 2 does not depends on the choice of regular value q. To show this, we first show that the degree modulo 2 is homotopy invariant. We shall write $H : f \sim g$ if $H : [0,1] \times M \to N$ gives a homotopy between f and g, and without any specific explanation, $I := [0,1]$.

Theorem 2.17 (Homotopy Invariance of Degree Modulo 2)**.** Suppose $f, g : M \to N$ are both smooth maps and $f \sim g$ homotopic smoothly, then, for $q \in N$ be regular value of both f and g
$$|f^{-1}(q)| \equiv |g^{-1}(q)| \pmod 2$$

Proof. Let $H : I \times M \to N$ gives the homotopy and $H_t := H(t, -)$. First, consider the case that q is also the regular value of H. Then the preimage $H^{-1}(q)$ is a compact 1-manifold, (in general) with boundary equals to
$$\partial H^{-1}(q) = H^{-1}(q) \cap \partial(M \times I) = (f^{-1}(q) \times \{0\}) \cup (f^{-1}(q) \times \{1\})$$
and the number of boundary points is given by
$$|\partial H^{-1}(q)| = |f^{-1}(q)| + |g^{-1}(q)|$$
Since the compact 1-manifold always have even number of boundary points, thus, $|f^{-1}(q)| + |g^{-1}(q)|$ is even, and therefore
$$|f^{-1}(q)| \equiv |g^{-1}(q)| \pmod 2$$
Then, suppose q is not a regular value of H. Then recall from the Proposition 2.6 that the cardinality of preimages $|f^{-1}(q')|$ and $|g^{-1}(q')|$ are locally constant (if q' is not critical value). Thus, there is a neighborhood $U_1 \subseteq N$ of q' such that
$$|f^{-1}(q)| = |f^{-1}(q')| \quad \forall q \in U$$
and same statement applies for g with the neighborhood $U_2 \subseteq N$. Take regular value $x \in V_1 \cap V_2$ of H, then
$$|f^{-1}(q)| = |f^{-1}(x)| = |g^{-1}(x)| = |g^{-1}(q)|$$
which completes the proof of the theorem. □

We also need the following result, which shows the homogeneity of a connected smooth manifold:

Theorem 2.18 (Homogeneity Lemma)**.** For a connected manifold M and arbitrary two interior points $p, q \in M$, there exists a diffeomorphism $f : M \to M$ that is smoothly isotopic to id_M and $f(p) = q$.

Proof. We shall first work on $\mathbb{R}^m$, let $B := B_1(0) \subseteq \mathbb{R}^m$ be the unit open ball. We shall take the bump function $\rho \in C^\infty(\mathbb{R}^m)$ such that $\rho(x) \in [0,1]\ \forall x \in \mathbb{R}^n$

$$\rho(x) = 1 \text{ when } \|x\| \leq \frac{1}{2}, \quad \rho(x) = 0 \text{ when } \|x\| \geq 1$$

and let $F_t(x) := x + t\rho(x)a\ \forall x \in \mathbb{R}^m$, then

- $F_0 = \mathrm{id}_{\mathbb{R}^m}$ since $F_0(x) = x + 0 \cdot \rho(x)a = x\ \forall x$.
- $F_t|_{\mathbb{R}^n \setminus B} = \mathrm{id}_{\mathbb{R}^n \setminus B}\ \forall t$ since $\rho|_{\mathbb{R}^n \subseteq B} = 0$.
- $F_1(0) = a$ since $\rho(0) = 1$ and when $t = 1$, $F_1(0) = \rho(0)a = a$.

Then, we claim that F_t is an smooth isotropy, i.e., $\forall x \in \mathbb{R}^m : F_{(-)(x)}$ smoothly depends on T and $\forall t \in [0,1]$, $F_t : \mathbb{R}^m \to \mathbb{R}^m$ is a diffeomorphism. $\square$

Chapter 3

Calculus on Manifold

The main purpose of the study of geometry is to understand the global invariant properties throughout the study of local structure. For this purpose, one should be concern about the induced global phenomena from the fundamental structures we have defined in the first chapter. The best example of this procedure is the fiber bundle:

Definition 3.1 (Fiber Bundles)**.** A fiber bundle (E, B, π, F) consists the following data:

1. Topological spaces E (total space) and B (base space).
2. Continuous surjection $\pi : E \to B$ (projection).
3. Topological space F (fiber).

We shall assume that the base space B is connected and satisfies the local trivialization condition, i.e., for any point $p \in B$, there exists an open subset $U \subseteq B$, such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi_U", "\sim"'] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_U"] \\ U & \end{tikzcd}$$

3.1 Tangent Bundles and Vector Fields

We shall begin with the definition of the tangent bundle, the global structure induced by the tangent space.

Definition 3.2 (Tangent Bundle)**.** The tangent bundle on smooth manifold M^m is the fiber bundle (TM, M, π, F) which

1. The total space is $TM = \coprod_{p\in M} T_pM \cong \{(p, v_p) \mid \forall p \in M : v_p \in T_pM\}$ and the base space is M. The fiber $F \cong \mathbb{R}^m$ is some m-dimensional $\mathbb{R}$-vector space.
2. The projection is given by $\pi(p, v_p) = p$.

The tangent bundle TM is a $2m$-dimensional differentiable manifold.

3.2 Normal Bundles and Tubular Neighborhood Theorem

Theorem 3.1 (Tubular Neighborhood Theorem)**.**

Now we can prove the homotopy transversality theorem (Theorem 1.17):

Theorem (Homotopy Transversality Theorem)**.** Let $f \in C^\infty(M,N)$, and $Y \subseteq N$ be a regular submanifold. Then exists $g : M \to N$ such that $f \sim g$ and $g \pitchfork Y$. Also, if $X \subseteq M$ is a (topologically) closed regular submanifold, and $\forall p \in f^{-1}(Y) \cap X$, $f \pitchfork Y$, then we shall choose g such that $g|_X = f|_X$.

Proof. □

Corollary. *Let M be a connected m-dimensional differentiable manifold, $S \subseteq N$ be a regular submanifold such that* $\operatorname{codim} S \geq 2$*. Then $M \setminus S$ is connected.*

Proof. Consider any $x, y \in M \setminus S$ with path $\gamma : [0,1] \to M$ copnnected x and y. We can apply the (weak) Whitney approximation theorem (Theorem 2.5). We shall take a smooth path $\tilde{\gamma}_1 : I \to M$ such that $\gamma \sim \tilde{\gamma}_1$ rel$\{0,1\}$, and by Theorem 1.17, we can also take smooth path $\gamma \sim \tilde{\gamma}_2$ rel$\{0,1\}$ such that $\tilde{\gamma}_2 \pitchfork S$. We shall prove that the only case in which transversality is possible is that $\operatorname{im}(\tilde{\gamma}_2) \cap S = \emptyset$. By Theorem 1.14, if $\operatorname{im}(\gamma) \cap S$ nonempty

$$I \supseteq \operatorname{codim} \gamma^{-1}(S) = \operatorname{codim} S \geq 2$$

Thus, $\operatorname{codim} \gamma^{-1}(S) \geq 2 > \dim I = 1$, which is impossible. In this way, we show that for any two points that can be connected by a continuous path in M, there is a smooth path in $M \setminus S$ that connects them, i.e., $M \setminus S$ is connected. □

Also, we can go one step further: Recall that for a connected manifold M, the fundamental group with the base point $x_0 \in M$ is given by

$$\pi_1(M, x_0) := \{\gamma : I \to M \mid \gamma(0) = \gamma(1) = x_0\}$$

Since M is connected (and path-connected, by Proposition 1.7), the path between any two arbitrary points induces the isomorphism between the fundamental group with different base points. Thus, for any connected manifold, we can skip the base point in the fundamental group and denote it as $\pi_1(M)$.

Corollary. *Let M be a connected m-dimensional differentiable manifold, $S \subseteq N$ be a regular submanifold such that* $\operatorname{codim} S \geq 3$*. Then $\pi_1(M \setminus S) = \pi_1(M)$.*

Proof. The proof is similar to the previous corollary. We shall show that the inclusion map

$$i : M \setminus S \hookrightarrow M, \quad i(x) = x$$

induce the group isomorphism $i_* : \pi_1(M \setminus S) \xrightarrow{\sim} \pi_1(M)$.
(Surjectivity) We shall show that for any loop class $[\gamma] \in \pi_1(M)$, by Theorem 1.17, we shall find a representative loop $\alpha \in [\gamma]$ in the homotopy class such that $\alpha \pitchfork S$. By the previous corollary, since $\operatorname{codim} S \geq 2$, we can let $\operatorname{im}(\alpha) \cap S = \emptyset$, which means $\operatorname{im}(\alpha) \subseteq M \setminus S$, and thus, we have found $[\alpha] \in \pi_1(M \setminus S)$.
(Injectivity) $\forall [\beta] \in \pi_1(M \setminus S)$, suppose $i_*([\beta]) = 1$ (i.e., $[\beta] \in \ker(i_*)$). We shall introduce the following lemma that extends the contractable loop to a contractable region in the manifold M with the loop as the boundary:

Lemma. *The loop $\beta : S^1 \to M$ is contractable if and only if it can be extended to a map*

$$F : D^2 \to M, \quad F|_{\partial D^2} = \operatorname{im}(\beta)$$

Where $D^2 := \{r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \mid r \in [0,1]\}$ is the unit disk.

The proof is simply consider the map $F : (D^2, S^1) \to (M, \beta(S^1))$ defined by

$$F(r\mathrm{e}^{\mathrm{i}\theta}) = H(\mathrm{e}^{\mathrm{i}\theta}, 1 - r)$$

where $H : S^1 \times I \to M$ is the smooth homotopy that $H(-, 0) = \beta$ and $H(-, 1) = p$.
With the lemma, we can do exactly the same thing with the previous proof that applies Theorem 1.17 and let $F \sim F'$ such that $\tilde{F} \pitchfork S$, and we do the codimension calculation,

$$\operatorname{codim} \tilde{F}^{-1}(S) = \operatorname{codim} S \geq 3 \geq \dim D^2 = 2$$

Which means the only possibility is that $\tilde{F}(D^2) \cap S = \emptyset$. Thus, $F'(D^2) \subseteq M \setminus S$ is contractable in $M \setminus S$, i.e., $[\beta] = 1 \in \pi_1(M \setminus S)$, which shows $\ker(i_*) = \{1\}$ and thus i_* is injective.
In this way, we show that $i_* : \pi_1(M \setminus S) \to \pi_1(M)$ is an isomorphism. □

[**Derek:** To be complete.]

3.3 Vector Field and the Euler Characteristics

3.4 Distribution and Integrability Theorem

3.5 Tensors

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4.6.2 Homogeneous Space

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This chapter lies somewhat out of the book's main line of development, and may be omitted in a first reading

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Part V

Appendix

Appendix A

Review on General Topology

The study of modern geometry focuses on connecting the (local) geometrical quantities (e.g., curvature, length, volume, etc.) and the (global) topological properties (genus, Euler's characteristics, fundamental groups, etc.). Often, this connection is given by an integral or some other global operation. Furthermore, the study of geometry focuses on a concept called "topological manifolds", which is a topological space with some "good properties". Thus, it is useful to have a brief review of the concepts in general topology.

Definition A.1 (Topological Space, Open and Closed Sets)**.** A topological space is a tuple $(\mathcal{T}, \mathcal{O})$, where $\mathcal{T}$ is a nonempty (ZFC) set and the additional structure is the topology $\mathcal{O} \subseteq \mathcal{P}(\mathcal{T})$. The element of topology $\mathcal{O}$ is defined to be an open subset of $\mathcal{T}$, which satisfy:

- $\mathcal{T}, \emptyset \in \mathcal{O}$
- $\forall U, V \in \mathcal{O} : U \cap V \in \mathcal{O}$
- $\forall \{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{O} : \bigcup_{\alpha \in I} U_\alpha \in \mathcal{O}$

If $A \subseteq \mathcal{T}$ is open, then $\mathcal{T} \backslash A$ is defined to be a closed set. Without the discussion of topology, we can denote the topological space as X.

Proposition A.1. *For any topological space $(X, \mathcal{O}_X)$, X itself and $\emptyset$ are both open and closed.*

Proof. The proof is simple. Notice that the definition of topology requires $X, \emptyset \in \mathcal{O}_X$, i.e., both X and $\emptyset$ are open. Also, $X \backslash X = \emptyset$ is closed, and $X \backslash \emptyset$ is closed. $\square$

Remark. *For a topological space $(X, \mathcal{O}_X)$,*

- *The definition of topological spaces indicates that the infinite intersection of open sets may not be open, which is reasonable in Euclidean space since the infinite intersection of $\{(-1/n, 1/n) : n \in \mathbb{N}\}$ it s a single point $\{0\}$ that is closed.*
- *Not all subsets of X can be classified as "open" or "closed", i.e., there can be a set that is neither open nor closed.*
- *For some topological space (not a connected space, actually), there can be a subset other than X and $\emptyset$ that is both open and closed.*

It is quite obvious for readers familiar with real analysis that this is a generalization of open sets in $\mathbb{R}^n$. We use the universal properties of open sets in $\mathbb{R}^n$ as the definition of open sets in more general topological spaces. Here are some examples:

Example A.1. *Consider sets* $S = \{1, 2, 3, 4, 5, 6\}$,

- $\big\{\{1\}, \{2\}, \{6\}, \{1, 2\}, \{1, 6\}, \{2, 6\}, \{1, 2, 6\}\big\}$ *is a topology on* S.
- $\big\{\{1\}, \{2\}, \{5\}, \{1, 2, 5\}\big\}$ *is not a topology on* S.

Example A.2. *For any nonempty (ZFC) set* S*, two trivial topologies can be given*

- $\mathcal{O}_{chaotic} = \{\emptyset, S\}$ *is the chaotic topology on* S.
- $\mathcal{O}_{descrete} = \mathcal{P}(S)$ *is the discrete topology on* A.

For finite sets $|M| \geq 1$, the topology that can be established is given by the following table:

Cardinality $\lvert M\rvert < 1$	Numbers of Topology
1	1
2	4
3	29
4	355
5	6942
6	209527
$\vdots$	$\vdots$

The most "basic" form of open sets is the open neighborhood:

Definition A.2 (Neighborhood)**.** For some topological space X and the point $x \in X$, $V \subseteq X$ is a neighborhood of x if $\exists U \in \mathcal{O}_X : x \in U \subseteq V$. If V is open (closed), then it is an open (closed) neighborhood of x.

Remark. *As a remark, it is important to mention that the neighborhood does not necessarily have to be "small". As an example, since any neighborhood* U *satisfies* $x \in U \subset X$*, then* X *is both an open and a closed neighborhood of any point* $x \in X$.

Proposition A.2. *Let* X *be a topological space, and* $x \in X$ *is a point. For some* $V \subseteq X$*, the following statements are equivlant:*

1. V *is the open neighborhood of* x.
2. V *is open, and* $x \in V$.

Proof. ($1 \Rightarrow 2$) By the definition of open neighborhood, we know the following information: V is open (the requirement of "open" neighborhood) and $\exists U \in \mathcal{O}_X : x \in U \subseteq V \implies x \in V$, which proves 2
($2 \Rightarrow 1$) If V is open and $x \in V$, then just take $U = V$ and $x \in V \subseteq V$, which makes V being an open neighborhood of x. $\square$

With the concept of neighborhood being introduced, we can easily distinguish whether a set is open or not.

Theorem A.1 (Determination of Open Sets)**.** Let X be a topological space, $U \subseteq X$ is nonempty subset of X, the following proposition are equivalent:

1. U is open.
2. $\forall x \in U$, $\exists V \subseteq U$ such that V is a open neighborhood of x.

Proof. ($1 \Rightarrow 2$) For some open set U, take arbitrary $x \in U$, then U itself is a open neighborhood of x.
($2 \Rightarrow 1$) By the given condition, $\forall x \in U$ take the corresponding open neighborhood $V_x \subseteq U$

$$\bigcup_{x \in U} V_x = U$$

Since $\forall x \in U : V_x$ are open, then, by the axiom of topological space, any union of open sets is open. Thus, U is open. □

To describe the topology structure on any set (often uncountable infinite sets, like surfaces in $\mathbb{R}^n$), it is useful to discuss some generating sets of the topology, called a (topological) basis. We often choose some of the most "representative" open sets in the topology to form a topological basis.

Definition A.3 (Topological Basis)**.** Let $(X, \mathcal{O}_X)$ be a topological space. For some $\mathcal{B} \in \mathcal{O}_X$, $\mathcal{B}$ is said to be a (topological) basis of the topological space iff

$$\forall U \in \mathcal{O}_X : \forall x \in U : \exists B \in \mathcal{B} : x \in B \subseteq U$$

The definition of basis has an equivalent statement:

Proposition A.3 (Equivalent Description of Topological Basis)**.** *For some set contains open sets $\mathcal{B} \in \mathcal{O}_X$ in the topological space $(X, \mathcal{O}_X)$, the following statements are equivalent:*

1. *$\mathcal{B}$ is the basis of X*
2. $\forall U \in \mathcal{O}_X : \exists \mathcal{B}' \subseteq \mathcal{B} : U = \bigcup_{B \in \mathcal{B}'} B$

Proof. ($1 \Rightarrow 2$) For any open set U, for any $x \in U$. By the definition of basis, we can always find $x \in B_x \in \mathcal{B}$ such that $B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. WE can just take $\mathcal{B}' = \{B_x\}_{x \in U} \subseteq \mathcal{B}$, which proves the second statement.
($2 \Rightarrow 1$) By the given condition that $\forall U \in \mathcal{O}_X : U = \bigcup_{B \in \mathcal{B}'} B$, where $\mathcal{B}' \subseteq \mathcal{B}$. Thus, there must be some set $B \in \mathcal{B}'$ that $x \in B \in \mathcal{B}$, which is the definition of basis. □

Example A.3. *The natural topology on $\mathbb{R}^n$ is generated by the following basis:*

$$\mathcal{B} = \{B(x, r) | x \in \mathbb{Q}^n, r \in \mathbb{Q}_{\geq 0}\}, \quad B(x, r) = \{y \in \mathbb{R}^n | x \in \mathbb{R}^n, d(x, y) < r\}$$

which is the topology we used for most metric spaces.

Definition A.4 (Second Countable, A_2)**.** A topological space is said to be second countable iff it can be generated by a countable basis.

By the given example of a topological basis, an obvious fact is that $\mathbb{R}^n$ is second countable. A further result is that any metric space is second countable. The topological basis of a topological space has the following properties:

Proposition A.4 (Properties of Basis)**.** *Let $\mathcal{B} \subset \mathcal{O}_X$ is a basis of topological space $(X, \mathcal{O}_X)$, then $\mathcal{B}$ has the following properties:*

1. $\forall x \in X : \exists B \in \mathcal{B} : x \in B$
2. $\forall B_1, B_2 \in \mathcal{B} : \forall x \in B_1 \cap B_2 : \exists B \in \mathcal{B} : x \in B \subseteq B_1 \cap B_2$

With the properties above, an important technique is to use the basis defined topology on a set. [**Derek:** To be finished.]

After all of the previous definitions and propositions of topological spaces, here is a better explanation of the motivation for the definition of topology.

Topology is the minimum structure to define the continuity of the map.

As a generalization of the open set and continuity of functions in Euclidean space $\mathbb{R}^n$. We can have the following definition:

Definition A.5 (Continuity of functions)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, the map $f : X \to Y$ is continuous if

$$\forall V \in \mathcal{O}_Y : f^{-1}(V) \in \mathcal{O}_X$$

A necessary step to check the well-definedness of continuity is to consider the $\mathbb{R}^n$ and the natural topology of it.

Proposition A.5 (Continuity on $\mathbb{R}^n$ and Topological Continuity)**.** *We take the topological space $X = \mathbb{R}^n$ and the standard topology in Euclidean space. The continuity of real functions in analysis and topological continuity are equivalent.*

Appendix B

Linear Algebra

Appendix C

Real and Functional Analysis

Appendix D

Elliptic Regularities

Appendix E

A Brief Introduction to Categorical Languages

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