From 2D TQFT to Gromov-Witten Theory

Derek Zeng*

February 27, 2026

Abstract

In this explanatory article, we begin with a brief introduction of 2-dimensional topological quantum field theories (2D TQFTs) and then introduce the Gromov-Witten Theory as a Cohomological Field Theory (CohFT).

Contents

*derek6@illinois.edu

1 Introduction

Topological ideas entered quantum field theory in a particularly concrete way in the late 1980s, when Witten introduced *topological sigma models* and related constructions that produce invariants insensitive to metric deformations [Wit88]. In parallel, Atiyah (building on the functorial viewpoint advocated by Segal) distilled the essential structural axiomatics into what is now called a *topological quantum field theory* (TQFT): a symmetric monoidal functor from a bordism category to vector spaces [Ati88]. This framework elevated "cutting and gluing" from a heuristic principle to a precise algebraic constraint: the value of a theory on a composite cobordism must factor through the state space associated with the cut.

In two dimensions, the functorial axioms become remarkably rigid. If one restricts to oriented bordisms, the entire theory is determined by the single vector space

$$A := Z(S^1),$$

together with the linear maps assigned to a small collection of elementary surfaces (pair-of-pants, cap, cup). By the mid-1990s, it was understood (and is now standard folklore with many clean expositions) that *2D oriented TQFTs are classified by commutative Frobenius algebras* [Abr96; Koc04]. Concretely, the pair-of-pants cobordism induces a multiplication $\mu : A \otimes A \to A$, a disk induces a unit $\eta : \mathbb{k} \to A$, and a second disk (viewed oppositely) induces a trace $\epsilon : A \to \mathbb{k}$, whose associated bilinear form

$$\langle a, b \rangle := \epsilon(\mu(a, b))$$

is nondegenerate and satisfies the Frobenius compatibility. From this perspective, the slogan "2D TQFT $\Longleftrightarrow$ Frobenius algebra" is not an analogy but a theorem: any amplitude of a closed surface is computable purely from the algebraic data, via pants decompositions and invariance under the bordism relations [Abr96; Koc04].

However, the same decade also made it clear that many of the most interesting "topological" theories in geometry are not exhausted by this purely topological dependence on the underlying surface. In enumerative geometry and in the sigma-model picture, correlation functions typically vary with the complex structure of the source curve. This naturally forces one to work over the moduli space of curves. The foundational construction of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ [DM69] (see also the textbook treatment [HM98]) provides exactly the geometric arena in which cutting and gluing become boundary phenomena: degenerations to nodal curves define boundary strata, and the corresponding *gluing morphisms* (both separating and non-separating) encode factorization in cohomology rather than merely in numbers.

At roughly the same time, Gromov's theory of J-holomorphic curves catalyzed a modern definition of *Gromov–Witten (GW) invariants* as intersection numbers on moduli spaces of stable maps. On the algebraic side, Kontsevich introduced stable maps as a compactification amenable to intersection theory and localization, leading to effective computations of rational curve counts [Kon95]. On the symplectic side, Ruan–Tian formulated a rigorous theory of quantum cohomology and GW invariants in a differential-geometric setting [RT95]. A central technical obstacle is that moduli spaces of maps are seldom smooth of the expected dimension. The solution is to replace the fundamental class by a *virtual* fundamental class (or cycle), constructed via different but ultimately compatible analytic and algebro-geometric frameworks [BF97; FO99; LT98]. This virtual class is precisely what makes the gluing and deformation principles true at the level needed to build field-theoretic structures.

The conceptual synthesis of these ideas appears prominently in the work of Kontsevich–Manin, who introduced the notion of a *cohomological field theory* (CohFT) as a refinement of 2D TQFT in which correlation functions become cohomology classes on $\overline{\mathcal{M}}_{g,n}$ and satisfy functorial gluing axioms [KM94]. In this language, the state space is typically

$$V := H^*(X;)$$

for a target space X, equipped with the Poincaré pairing η and unit $\mathbf{1} \in V$. The field theory is a system of multilinear maps

$$\Omega_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$

that is S_n-equivariant and compatible with the boundary gluing morphisms by contraction with η^{-1}; a distinguished unit axiom controls compatibility with forgetting marked points [KM94; Man99]. Gromov–Witten theory furnishes a canonical example: using the stabilization map $st : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$ and evaluation maps ev_i, one pushes forward the virtual class with insertions to obtain $\Omega_{g,n}$, and the virtual gluing theorems imply the CohFT axioms [BM96; FP97; KM94]. In genus zero, the resulting structure recovers quantum cohomology: three-point invariants define a deformation of the cup product, and the gluing axioms along boundary divisors encode associativity [Dub96; Man99; RT95].

Beyond existence, the subsequent history emphasizes structure and classification. Dubrovin and Manin recast genus-zero GW theory in the language of *Frobenius manifolds*, linking it to integrable systems and providing a systematic organizational principle for quantum cohomology [Dub96; Man99]. Givental developed a powerful formalism expressing higher-genus potentials via quantization procedures on symplectic loop spaces, with far-reaching consequences for computation and comparison theorems (e.g. "quantum Riemann–Roch") [CG07; Giv01]. Teleman later proved a classification theorem for *semisimple* CohFTs, showing that (under semisimplicity) the higher-genus theory is determined by genus-zero data, in precise agreement with the predictions of Givental's framework [Tel12].

This paper follows this historical and conceptual progression. We begin with the functorial definition of 2D oriented TQFT and its classification by commutative Frobenius algebras [Abr96; Ati88; Koc04]. We then explain why moduli of curves and the Deligne-Mumford compactification are the natural geometric enhancement needed to capture families of theories [DM69; HM98]. Next, we introduce CohFTs in the sense of Kontsevich–Manin [KM94; Man99] and describe how Gromov–Witten theory produces a CohFT via virtual fundamental classes and stabilization maps [BF97; BM96; FP97; LT98]. Finally, we illustrate the genus-zero consequences in quantum cohomology and briefly indicate modern structural results (Givental–Teleman) that clarify how much of the higher-genus theory is controlled by genus zero [Giv01; Tel12].

2 TQFT as a Symmetric Monoidal Functor

Michael Atiyah, in the 1980s, had defined the topological quantum field theory (TQFT) in the following way as a functor [Ati88]:

$$\text{A TQFT is a symmetrical monoidal functor } Z : \mathsf{Bord}_{\langle n-1,n\rangle} \to \mathsf{Vect}_{\Bbbk}.$$

and we shall follows the notation in the lecture note *Bordism: Old and New* [Fre12], denote symmetric monoidal category of n-dimensional TQFT's as

$$\mathsf{TQFT}_n = \mathrm{Hom}^{\otimes}(\mathsf{Bord}_{\langle n-1,n\rangle}, \mathsf{Vect}_{\Bbbk}) \quad \big(:= \mathrm{Fun}^{\otimes}(\mathsf{Bord}_{\langle n-1,n\rangle}, \mathsf{Vect}_{\Bbbk})\big)$$

where $\mathsf{Vect}_{\Bbbk}$ is the category of vector spaces over the field $\Bbbk$.

2.1 Cobordisms and Categroy $\mathsf{Bord}^{\mathfrak{X}(n)}_{\langle n-1,n\rangle}$

To illustrate the definition of TQFT, recall that a cobordism is a generalization of homotopy, more precisely:

Definition 2.1 (n-Cobordism)**.** Let M_0, M_1 be $(n-1)$-dimensional closed manifold, a cobordism $(W, p, \theta_-, \theta_+)$ between M_0 and M_1 is given by a n-dimensional compact manifold W with boundary, and the partition $p : \partial W \to \{-, +\}$ on the boundary that gives embeddings

$$\theta_- : [0, +1) \times M_0 \to W$$
$$\theta_+ : (-1, 0] \times M_1 \to W$$

and also $\theta_\pm(0, M_i) = (\partial W)_\pm := p^{-1}(\pm)$ for $i = 0, 1$.

The standard way to think about the configuration above is to take $(\partial W)_\pm$ to be the disjoint union of certain boundary components, and $[0, 1) \times M_i \cong (-1, 0] \times (\partial W)_\pm$ can be identified as a collar neighborhood of the boundary component. We connect given $(n-1)$-dimensional manifolds with the (diffeomorphism class of) n-dimensional manifolds W with the boundary identification defined above. We shall also write $W : M_0 \to M_1$ refers to W being a cobordism from M_1 to M_2. The idea can be shown as the following diagram: In this

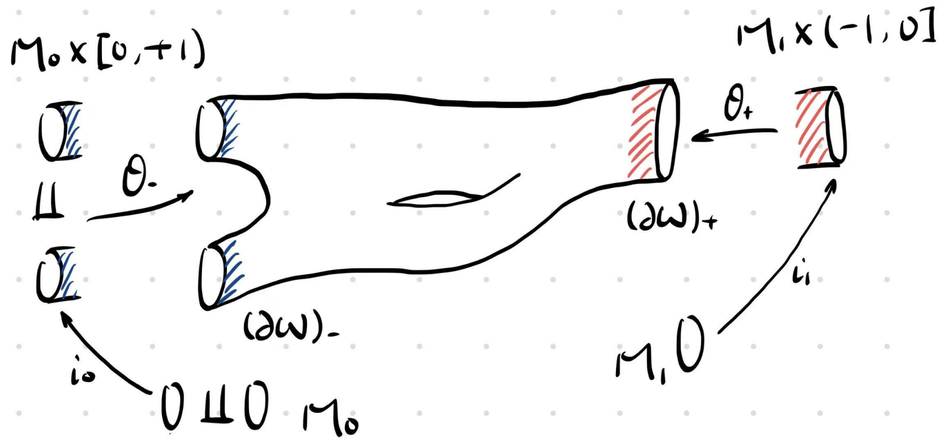

Figure 1: W as a cobordism from M_1 to M_2.

article, we shall only consider the orientable case where M_0, M_1, and W are all orientable manifolds and the boundary of W satisfies

$$\partial W = (\partial W)_- \sqcup (\partial W)_+ \cong \overline{M_0} \sqcup M_1$$

where $\overline{M_1}$ represents M_1 with orientation reversed, and we shall choose $\theta_\pm$ to be orientation compatible embedding (that will ensure the further definition of the gluing operation to generate a well-defined orientation).

Definition 2.2 (Dual Cobrodism)**.** Let $(W, p, \theta_-, \theta_+)$ be a cobordism from M_0 to M_1, the dual cobordism $(W^\vee, p^\vee, \theta_-^\vee, \theta_+^\vee)$ is a cobordism from M_1 to M_0 given by $p^\vee := 1 - p$ and

$$\begin{aligned} \theta_-^\vee(t, x) &:= \theta_+(-t, x) \quad \forall t \in [0, +1)\ \forall x \in M_1 \\ \theta_+^\vee(t, y) &:= \theta_-(-t, x) \quad \forall t \in (-1, 0]\ \forall y \in M_0 \end{aligned}$$

The dual cobordism W is just $W^\vee$ "turned around".

With the additional orientation information, the dual should include orientation reversing on the boundary manifold M_0 and M_1 that take the n-manifold $W^\vee$ with

$$\partial W = (\partial W)_- \sqcup (\partial W)_+ \cong \overline{M_1} \sqcup M_0$$

As the generalization of homotopy, we shall define the "gluing" operator between two cobordisms with the boundary identification that coincide with each other.

Definition 2.3 (Gluing of Cobordisms)**.** Suppose $W_0 : M_0 \to M_1$ and $W_1 : M_1 \to M_2$ are oriented cobordisms, the gluding operation gives a cobordism $W : M_0 \to M_2$ which topologically be

$$W := (W_0 \sqcup W_1)/\sim$$

with the equivalence relation given by a diffeomorphism $\varphi := \theta^1_- \circ (\theta^0_+)^{-1} : (\partial W_0)_+ \xrightarrow{\sim} (\partial W_1)_-$ and

$$\forall x \in (\partial W_0)_+ : \forall y \in (\partial W_1)_- : \big(x \sim y \iff y = \varphi(x)\big)$$

We denote the gluing cobordism $W = W_2 * W_1 : M_0 \to M_2$.

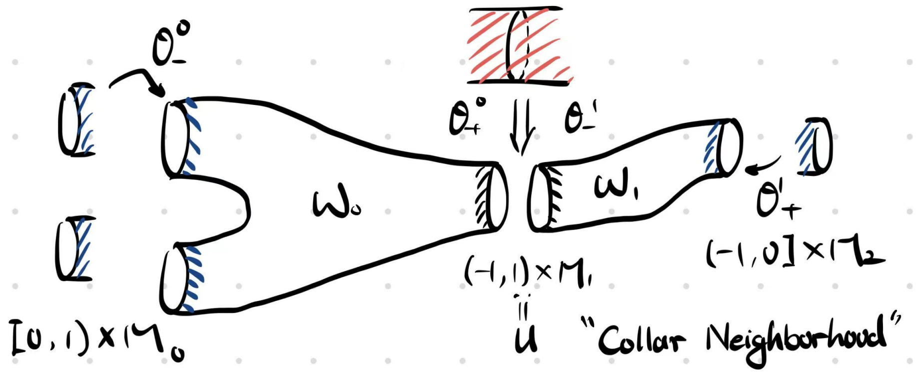

Figure 2: Gluing cobordisms.

The "cobordism" generated by the gluing obviously has a boundary

$$\partial W = (\partial W_0)_- \sqcup (\partial W_1)_+$$

However, we have to show that it admits a smooth structure that is consistent with the smooth structure on W_0 and W_1, i.e., by restricting the smooth structure onto the components from W_0 and W_1, we shall have the original smooth structure on the two cobordisms. The result (and actually, all previous definition of cobordisms) rely on the following theorem stating that the boundary inclusion can always be extended to an open neighborhood of the boundary on the manifold.

The detailed proof of the following theorem can be found in any standard smooth manifold textbook, for example, Theorem 9.25 in [Lee03]. We will only give a sketch of the proof, which skips most technical details of constructing a certain smooth map using the partition of unity.

Theorem 2.4 (Collar Neighborhood Theorem)**.** Let W be a smooth manifold with boundary, then there exists a smooth embedding (collar)

$$c : [0,1) \times \partial W \hookrightarrow W$$

i.e., the boundary ∂W always has a open neighborhood that diffeomorphic to $[0,1) \times \partial W$

Sketch of Proof. The proof is through three steps:

1. Construct a smooth vector field V that "points strictly inwards" at the boundary ∂W, and everywhere non-zero on the boundary, i.e., a vector field that everywhere is transverse to the boundary.

2. Consider the flow Φ_t of V that gives a local collar neighborhood on an open boundary chart.

3. Reparametrize the "time" of the flow to fit all local collar neighborhoods into $[0,1) \times \partial W$.

To construct the vector field, $\forall p \in \partial W$, we shall take the boundary chart (U_p, ϕ_p) with $\phi_p : U_p \to \mathbb{H}^n$ such that $\phi_p(p) = 0$, note that $\mathbb{H}^n = \mathbb{R}^{n-1} \times [0, +\infty)$, we shall let $\phi_p := (u, r)$ which $u(q) \in \mathbb{R}^{n-1}$ and $r(q) > 0$ for any $q \in U_p$. A simple choice of the inward vector field is just to consider $\partial/\partial r$ on $\mathbb{H}^n$ and use the coordinate map to pushforward the vector field onto the manifold:

$$V|_{U_p} := (\phi_p^{-1})_* \frac{\partial}{\partial r} \in \Gamma^\infty(TU_p)$$

and then by applying the partition of unity $\{\rho_i\}$ of an open cover $\{U_{p_i}\}$ of W that covers ∂W, and consider

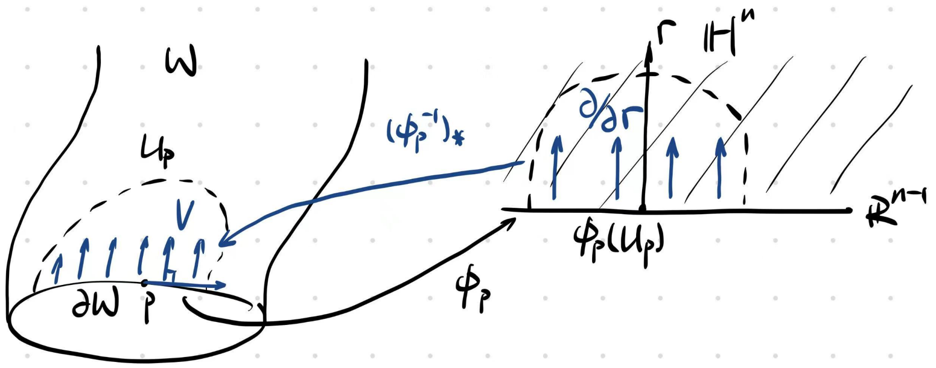

Figure 3: The inward pointing vector field.

all boundary charts in the open cover (i.e., take $p_i \in \partial W$)

$$V := \sum_i \rho_i V|_{U_{p_i}}$$

By the existence and uniqueness of the ODE solution, one shall claim that $\exists \epsilon(p) > 0$ such that the flow $\Phi_t(p)$ is well-defined on W when $t \in [0, \epsilon(p))$. Let $c_{\text{loc}}(t,p) := \Phi_t(p)$, the pushforward

$$(c_{\text{loc}})_{*,(0,p)} : \mathbb{R} \oplus T_p\partial W \xrightarrow{\sim} T_pW$$

is a linear isomorphism. By the inverse function theorem, there exists $V := [0, \delta(p)) \times O_p$ such that $c_{\text{loc}}|_V$ is diffeomorphism. Finally, one shall use the partition of unity to show the global existence of the C^∞-function $\tau : \partial W \to (0, +\infty)$ such that $\forall p \in \partial W$, $\Phi_t(p)$ exists for $[0, 2\tau(p))$. The glocal collar neighborhood is given by

$$c : [0,1) \times \partial W \to W, \quad c(s,p) := \Phi_{s\tau(p)}(p)$$

since $\Phi_t(p)$ is well-defined for $[0, 2\tau(p))$, $c(s,t)$ exists $\forall s \in [0,1)$. $\square$

With the collar neighborhood being well-defined, we shall prove the existence of a smooth structure on W:

Theorem 2.5. W can be equipped with a smooth structure and an orientation that coincides with the boundary identification as an orientable smooth cobordism.

Proof. We shall first construct the smooth structure, and then check the orientation.
Let the map $q : W_0 \sqcup W_1 \to W$ be the quotient map; the gluing happens on the set

$$S := q\big((\partial W_0)_+\big) = q\big((\partial W_1)_-\big) \subseteq W$$

which, after gluing, is supposed to be in the interior of the cobordism W. To work in a (open) neighborhood of S, defined $\Psi : (-\epsilon, \epsilon) \times M_1 \to W$ be

$$\Psi(t,x) := \begin{cases} q \circ \theta_+^0(t,x), & t \in (-\epsilon, 0] \\ q \circ \theta_-^1(t,x), & t \in [0, \epsilon) \end{cases}$$

since at $t = 0$, $q \circ \theta_+^0(0,x) = q \circ \theta_-^1(0,x)$, Ψ is well-defined. By the universal property of quotient, let $A := (-\epsilon, 0] \times M_1$ and $B := [0, \epsilon) \times M_1$, since both q and $\theta_\pm^i$ are continuous, $\Psi|_{C_i} = q \circ \theta_\pm^i$ is continuous. Thus, $\exists! : \Psi^\sqcup : A \sqcup B \to W$ such that the following left diagram commutes

$$\begin{tikzcd} A \arrow[r, hook, "i_A"] \arrow[dr, "\Psi|_A"'] & A \sqcup B \arrow[d, dashed, "\Psi^\sqcup"] & B \arrow[l, hook', "i_B"'] \arrow[dl, "\Psi|_B"] \\ & W & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \sqcup B \arrow[d, "q"'] \arrow[dr, "\Psi^\sqcup"] & \\ (-\epsilon, \epsilon) \times M_1 \arrow[r, dashed, "\Psi"'] & W \end{tikzcd}$$

Since $(-\epsilon, \epsilon) \times M_1 = (A \sqcup A)/\sim$ with $0 \in A \sim 0 \in B$, and

$$\Psi^\sqcup|_A(0,x) = \Psi|_A(0,x) = \Psi|_B(0,x) = \Psi^\sqcup|_B(0,x)$$

is a well-defined map with $\Psi \circ q = \Psi^\sqcup$, we know that Ψ is the unique continuous map such that the right diagram above commutes, which shows the continuity of Ψ. Then, by the fact that $\theta_\pm^i$ are embeddings, $A \cong C_0 := \theta_+^0(A)$ and $A \cong C_1 := \theta_-^1(B)$, the following diagram commutes:

$$\begin{tikzcd} A \sqcup B \arrow[r, shift left, "\cong"] \arrow[d, "\tilde{q}"] \arrow[dr] & C_0 \sqcup C_1 \arrow[l, shift left] \arrow[r, hook, "\iota"] \arrow[d, "q|_{C_0 \sqcup C_1}"] \arrow[dl] & W_0 \sqcup W_1 \arrow[d, "q"] \\ (-\epsilon, \epsilon) \times M_1 \arrow[r, dashed, shift left, "\cong"] & q(C_0 \sqcup C_1) \arrow[l, dashed, shift left] \arrow[r, hook, "\iota"] & W = (W_0 \sqcup W_1)/\sim \end{tikzcd}$$

the existence of continuous Ψ^{-1} is just by apply the universal property on the map $f : C_0 \sqcup C_1 \to (-\epsilon, \epsilon) \times M$ which $q : A \sqcup B \xrightarrow{\sim} C_0 \sqcup C_1 \xrightarrow{f} (-\epsilon, \epsilon) \times M$. Thus, Ψ is an homeomorphism onto its image $(C_0 \sqcup C_1)/\sim$. Let

$$U := q(C_0 \sqcup C_1) \subseteq W$$

The, by the homeomorphism, we consider three types of charts:

(A) For (V, κ) on W_0 such that $V \cap (\partial W_0)_+ = \emptyset$, then $\forall x \in V : x \sim x$ (i.e., $q|_V = \mathrm{id}|_V$ and thus, $q(V) \cong V$), we shall defined the corresponding chart $(q(V), \kappa \circ q^{-1}|_V)$.

(B) Similarly, for (V', κ') on W_1 such that $V' \cap (\partial W_1)_- = \emptyset$, take chart $(q(V'), \kappa' \circ (q|_{V'})^{-1})$

(C) Since $(-\epsilon, \epsilon) \times M_1$ is a smooth manifold, take any chart (O, α) on M_1 with $\alpha : O \to \mathbb{R}^{n-1}$, define the local chart (U_O, χ_O) as

$$U_O := \Psi((-\epsilon, \epsilon) \times O) \subseteq W, \quad \chi_O\big(\Psi(t,x)\big) := \big(t, \alpha(x)\big) \in \mathbb{R}^n$$

This is a well-defined chat since Ψ is a homeomorphism onto its image.

The smooth structure on W is simply given by the maximal atlas that is compatible with the following smooth atlas

$$\mathcal{A}_W := \{\text{Type (A) chart}\} \cup \{\text{Type (B) chart}\} \cup \{\text{Type (C) chart}\}$$

It is easy to check that transition maps are all smooth by the compatibility on W_i and M_1 and the smoothness of every map.
Since θ_-^0 and θ_-^1 are assumed to be orientation-compatible collars (with the fixed boundary orientation convention), the induced boundary identification $\varphi := \theta_-^1 \circ (\theta_+^0)^{-1} : (\partial W_0)_+ \to (\partial W_1)_+$ is automatically orientation-reversing. Hence, the orientations pushed forward from W_0 and W_1 agree on the overlap of the glued collar neighborhood, and define an orientation on W. Thus, we complete the proof of the proposition. □

A result of the previous gluing operation and dual cobordism is that:

Corollary. *Cobordism defines an equivalence relation. M and N are called bordant if there exists $W : M \to N$.*

Also, suppose $f : M \to N$ gives a diffeomorphism between closed manifolds M and N. Consider the cobordism $W := [0,1] \times N$ with the boundary identification be $f : M \to N \cong (\partial W)_-$ and id_N, we know that

Corollary. *Diffeomorphic manifolds are bordant.*

Since the cobordism, similar to homotopy, is a topological object, we are supposed to be able to consider the cobordism up to certain equivalence:

Definition 2.6 (Diffeomorphism of Cobordisms)**.** Cobordisms $W_0, W_1 : M_- \to M_+$ are said to be diffeomorphic if there exists a diffeomorphism $f : W_0 \to W_1$ of smooth manifolds that commutes with p_i and $\theta^i_\pm$, i.e., let $I_+ := [0,1)$ and $I_- := (-1,0]$the following diagrams commutes

$$\begin{tikzcd} W_0 \arrow[rr, "f"] \arrow[dr, "p_0"'] & & W_1 \arrow[dl, "p_1"] \\ & \{\pm\} & \end{tikzcd} \qquad \begin{tikzcd} W_0 \arrow[rr, "f"] & & W_1 \\ & I_\pm \times M_\pm \arrow[ul, "\theta^0_\pm"] \arrow[ur, "\theta^1_\pm"'] & \end{tikzcd}$$

With all results above, the class of n-cobordism gives the following category:

Definition 2.7 (Cobordism Category)**.** The n-dimensional cobordism category $\mathsf{Bord}_{\langle n-1,n\rangle}$ is a symmetrical monoidal category that consists of the following data:

1. The objects are closed $n-1$ manifolds.
2. The morphism $\mathsf{Bord}_{\langle n-1,n\rangle}(M_0, M_1) := \{$Diffeomorphism class of cobordisms $W : M_0 \to M_1\}$.
3. Composition of cobordisms $(W_1, W_2) \mapsto W_2 \circ W_1$ is given by gluing with the identity morphism $\mathrm{id}_M \in \mathsf{Bord}_{\langle n-1,n\rangle}(M, M)$ given by the cobordism $W := I \times M$.
4. The monoidal bifunctor is given by the disjoint union with the unit object given by the empty manifold $\emptyset^{n-1}$.

For any $A, B, C \in \mathrm{ob}\,\mathsf{Bord}_{\langle n-1,n\rangle}$ the disjoint union, as the monoidal product also been required to satisfies the following natural isomorphisms: $\alpha_{A,B,C} : (A \sqcup B) \sqcup C \cong A \sqcup (B \sqcup C)$ and $s_{A,B} : A \sqcup B \cong B \sqcup A$ such that the following diagram commutes: The inverse law (left) and the unit coherence (right)

$$\begin{tikzcd} & B \sqcup A \arrow[dr, "s_{B,A}"] & \\ A \sqcup B \arrow[ur, "s_{A,B}"] \arrow[rr, equal, "\mathrm{id}_{A \sqcup B}"] & & A \sqcup B \end{tikzcd} \qquad \begin{tikzcd} \emptyset^{n-1} \sqcup A \arrow[rr, "\cong"] \arrow[dr, "l_A"'] & & A \sqcup \emptyset^{n-1} \arrow[dl, "r_A"] \\ & A & \end{tikzcd}$$

And finally, the associative coherence

$$\begin{tikzcd} (A \sqcup B) \sqcup C \arrow[r, "s_{A,B} \sqcup \mathrm{id}_C"] \arrow[d, "\alpha_{A,B,C}"] & (B \sqcup A) \sqcup C \arrow[d, "\alpha_{B,A,C}"] \\ A \sqcup (B \sqcup C) \arrow[d, "s_{A,B \sqcup C}"] & B \sqcup (A \sqcup C) \arrow[d, "\mathrm{id}_B \sqcup s_{A,C}"] \\ (B \sqcup C) \sqcup A \arrow[r, "\alpha_{B,C,A}"] & B \sqcup (C \sqcup A) \end{tikzcd}$$

Often, people need to define additional structures on cobordisms (as a smooth manifold) to describe further topological properties; such a structure is the tangential structure. We have talked about oriented n-cobordisms $\mathsf{Bord}^{\mathrm{SO}(n)}_{\langle n-1,n\rangle}$ (which have an $\mathrm{SO}(n)$ structure as the frame that gives orientation), and this definition can give another perspective on understanding such a structure.

Recall that given a Hausdorff paracompact space, the classifying space BG of a principal G-bundle satisfies

$$[X, BG] \cong \{\text{Principal } G\text{-bundle } P \to X\}/\sim$$

where $[X, BG]$ is the homotopy classes of continuous maps $X \to BG$, and $\sim$ is the isomorphism between principal G-bundle given by $(P_1, p_1, X) \sim (P_2, p_2, X)$ if exists homeomorphism $\phi : P_1 \to P_2$ such that the following diagram commutes

$$\begin{tikzcd} P_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & P_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

On the classifying space, there exists a principal G-bundle

$$G \longrightarrow EG \xrightarrow{\gamma} BG$$

such that

1. EG is contractible.
2. G act freely on EG,
3. For any principal G-bundle $P \to X$, exists a classifying map $f : X \to BG$ that $P \cong f^*(EG)$.
4. The classifying map f above is unique up to homotopy.

And such a principal G-bundle is called a universal bundle.

A tangential space is given by the classifying space $B\mathrm{O}(n)$ with an additional lifting property:

Definition 2.8 (Tangential Strucutre)**.** A n-dimensional tangential structure is a topological space $\mathfrak{X}(n)$ with a fibration $\pi(n) : \mathfrak{X}(n) \to B\mathrm{O}(n)$ with the rank-n universal bundle $\gamma^n \to B\mathrm{O}(n)$ being pullback as

$$S(n) := \pi^*\gamma^n \to \mathfrak{X}(n)$$

Similarly, defined the $\mathfrak{X}(n)$ structure on manifold M with dimension $k \leq n$ as

$$\begin{tikzcd} \mathbb{R}^{n-k} \oplus TM \arrow[r] \arrow[d] & S(n) \arrow[d] \\ M \arrow[r, "f"] & \mathfrak{X}(n) \end{tikzcd}$$

which the map $f : M \to \mathfrak{X}(n)$ is the lifting of the forgetful map $\tau : M \to B\mathrm{O}(n)$, and the diagram pullback the universal bundle as

$$\mathbb{R}^{n-k} \oplus TM \cong \tau^* S(n) \cong (\pi \circ \tau)^* \gamma^n$$

We take the structure on M as $\mathbb{R}^{n-k} \oplus TM$ since M can maximally contains rank k (actually, $\dim M$) nontrivial structures.

Example (Orientation as a Tangential Structure)**.** *Consider the fibration induced by the natural inclusion map* $\mathrm{SO}(n) \hookrightarrow \mathrm{O}(n)$*,*

$$\pi(n) : E\mathrm{SO}(n) \to B\mathrm{O}(n)$$

For any given vector bundle $E \to M$ with rank n, the frame on E is given by $\mathrm{Fr}(E) \to M$ *as a principal $O(n)$-bundle. So, given an orientation on E is just given by the lifting of the classifying map*

$$\begin{tikzcd} & B\mathrm{SO}(n) \arrow[d, "\pi(n)"] \\ M \arrow[ur, "\tilde{c}_E"] \arrow[r, "c_E"'] & B\mathrm{O}(n) \end{tikzcd}$$

Furthermore, from an obstruction point of view, the group homomorphism $\det : \mathrm{O}(n) \to \mathbb{Z}_2 \cong \{\pm 1\}^{\times}$ *induce short exact sequence*

$$1 \longrightarrow \mathrm{SO}(n) \hookrightarrow \mathrm{O}(n) \xrightarrow{\det} \mathbb{Z}_2 \longrightarrow 1$$

and we shall first show the following proposition:

Proposition 2.9. For $w := B(\det) : B\mathrm{O}(n) \to B\mathbb{Z}_2$, the short exact sequence above induces the Serre fibration $B\mathrm{SO}(n) \simeq \mathrm{hofib}(w : B\mathrm{O}(n) \to \mathbb{RP}^\infty)$, which in other word, the homotopy fiber sequence (we shall use the known fact that $B\mathbb{Z}_2 \cong \mathbb{RP}^\infty$):

$$B\mathrm{SO}(n) \longrightarrow B\mathrm{O}(n) \xrightarrow{w} B\mathbb{Z}_2 \cong \mathbb{RP}^\infty$$

To prove the proposition, we shall first show the following lemma:

Lemma. *Any principal G-bundle $P \to X$ gives a homotopy fiber sequence*

$$P \longrightarrow X \longrightarrow BG$$

Proof. The proof is simply to consider the principal G-bundle $p : P \to X$ and the classifying map $f : X \to BG$. By the definition of a universal bundle

$$P \cong f^*(EG)$$

Since $EG \to BG$ is a fibration and EG is contractible, P is a homotopy fiber of P. Thus, the lemma was proved. □

To apply this lemma and prove the proposition, we need to find a principal $\mathbb{Z}_2$-bundle

$$p : B\mathrm{SO}(n) \to B\mathrm{O}(n)$$

such that the classifying map is $B(\det) : B\mathrm{O}(n) \to B\mathbb{Z}_2$.

Proof of Proposition 2.9. Take the contractible space $E\mathrm{O}(n)$ with free $\mathrm{O}(n)$-action as the total space of the universal bundle $E\mathrm{O}(n) \to B\mathrm{O}(n)$. Then we have $B\mathrm{O}(n) = E\mathrm{O}(n)/O(n)$. Restrict the $\mathrm{O}(n)$-action to the $\mathrm{SO}(n)$-action, the group action is still free and $E\mathrm{O}(n)$ remains to be contractible, i.e. the classifying space of $\mathrm{SO}(n)$-bundle is given by

$$B\mathrm{SO}(n) = E\mathrm{O}(n)/\mathrm{SO}(n)$$

Since $\mathrm{SO}(n) \lhd \mathrm{O}(n)$, and then quotient group

$$\mathrm{O}(n)/\mathrm{SO}(n) \cong \mathbb{Z}_2$$

has a natural free action on quotient space $E\mathrm{O}(n)/E\mathrm{SO}(n)$, the quotient space is given by

$$\big(E\mathrm{O}(n)/\mathrm{SO}(n)\big)/\big(\mathrm{O}(n)/\mathrm{SO}(n)\big) \cong E\mathrm{O}(n)/\mathrm{O}(n) = B\mathrm{O}(n)$$

Thus, $p : B\mathrm{SO}(n) = E\mathrm{O}(n)/\mathrm{SO}(n) \to B\mathrm{O}(n) = E\mathrm{O}(n)/\mathrm{O}(n)$ is a $\mathbb{Z}_2$-bundle.
Then, we shall show that the classifying map of this principal $\mathbb{Z}_2$-bundle is just $B(\det)$. Since $\det : \mathrm{O}(n) \to \mathbb{Z}_2$ is a group homomorphism, we shall write the associate $\mathbb{Z}_2$-bundle

$$E\mathrm{O}(n) \times_{\mathrm{O}(n)} \mathbb{Z}_2 \to B\mathrm{O}(n)$$

with the group action $\forall M \in \mathrm{O}(n)$ on $q \in \mathbb{Z}_2$ given by $M \cdot q = \det(M)q$ and the associate $\mathbb{Z}_2$-bundle has classifying map $B(\det)$ (The notation $E\mathrm{O}(n)\times_{\mathrm{O}(n)}\mathbb{Z}_2$ means we take $(E\mathrm{O}(n)\times\mathbb{Z}_2)/\mathrm{O}(n)$ by quotient the group action). The final goal remains to show that the associate $\mathbb{Z}_2$-bundle above is isomorphic to $E\mathrm{O}(n)/\mathrm{SO}(n) \to E\mathrm{O}(n)/\mathrm{O}(n)$. We can write down explicitly the isomorphism:

$$\Phi : E\mathrm{O}(n)/\mathrm{SO}(n) \to E\mathrm{O}(n) \times_{O(n)} \mathbb{Z}_2, \quad [x] \mapsto \Phi([x]) := [x, 1]$$

and we need to check that:

1. Φ is well-defined: suppose $x \sim Rx$ for $R \in \mathrm{SO}(n)$, then $[Rx] \mapsto [Rx, \det(R) \cdot 1] = [Rx, 1]$.
2. The continuity of Φ is due to the definition of quotient topology on $E\mathrm{O}(n)/\mathrm{SO}(n)$, and we can easily construct the inverse for $r \in \mathrm{O}(n)$ such that $\det(r) = -1$

   $$\Phi^{-1}([x, \epsilon]) := \begin{cases} [x], & \epsilon = 1 \\ [r \cdot x], & \epsilon = -1 \end{cases}$$

   and check the continuity.

Thus, the proposition been proved. ☐

With the Proposition 2.9, for any rank-n vector bundle $E \to M$, we have the classifying map $c_E : M \to B\mathrm{O}(n)$, which gives the lifting problem that determines the orientation as a tangential structure:

$$\begin{tikzcd} E \arrow[d] & & B\mathrm{SO}(n) \arrow[d, "\pi(n)"] & \\ M \arrow[rr, "c_E"] \arrow[urr, dashed, "\tilde{c}_E"] & & B\mathrm{O}(n) \arrow[r, "w"] & \mathbb{RP}^\infty \end{tikzcd}$$

on other hand, we have $H^1(\mathbb{RP}^\infty, \mathbb{Z}_2) \cong \mathbb{Z}_2$, by pullback the cohomology class u onto M, we can obtain $w_1(E) := (w \circ c_E)^* u \in H^1(M, \mathbb{Z}_2)$, namely the first Stiefel–Whitney class. Then

$$c_E \text{ can be lifted } \iff M \to \mathbb{RP}^\infty \text{ homotopically trivial } \iff w_1(E) = 0.$$

and thus, $E \to M$ orientiable if and only if Stiefel–Whitney class $w_1(E) = 0$ vanishes.

2.2 Topological Quantum Field Theory

With the cobordism and the category of cobordism, we shall now illustrate the definition of a TQFT of $\mathfrak{X}(n)$-manifold ($\mathfrak{X}(n)$ is a triangulation structure):

Definition 2.10 (Topological Quantum Field Theory)**.** A n-dimensional TQFT of $\mathfrak{X}(n)$-manifolds is a (strong) symmetrical monoidal functor (i.e., a functor between symmetrical monoidal categories that preserves the monoidal product)

$$Z : \mathsf{Bord}^{\mathfrak{X}(n)}_{\langle n-1, n\rangle} \to \mathsf{Vect}_{\Bbbk}$$

The cobordisms are allowed to exist with certain tangential structure, for example, orientations.

The definition of a orientiable TQFT functor $Z : \mathsf{Bord}^{\mathrm{SO}(n)}_{\langle n-1, n\rangle} \to \mathsf{Vect}_{\Bbbk}$, in an explicit way, is just:

1. On object level, each $n-1$ manifold M corresponds to a $\Bbbk$-vector space $Z(M)$ such that the monoidal unit is preserved, i.e., $Z(\emptyset^{n-1}) = \Bbbk$.

2. As morphisms, each diffeomorphism class of cobordisms $W : M_0 \to M_1$ is corresponding to a $\Bbbk$-linear map $Z(W) : Z(M_0) \to Z(M_1)$ such that if $W = I \times M : M \to M$, then $F(W) = \mathrm{id}_{F(M)}$.

3. Let $M := \coprod_{i=1}^n M_i$, then the corresponding vector space is given by

$$Z(M) := \bigotimes_{i=1}^n Z(M_i)$$

and for $W_i : M_i \to N_i$ be cobordisms, $W := \coprod_{i=1}^n W_i$

$$Z(W) := \bigotimes_{i=1}^n Z(W_i) : \bigotimes_{i=1}^n Z(M_i) \to \bigotimes_{i=1}^n Z(N_i)$$

and Z preserves composition.

4. The functor should preserves duality in $\mathsf{Bord}_{\langle n-1,n\rangle}^{\mathfrak{X}(n)}$, i.e., on object level, $Z(\overline{M}) \cong Z(M)^*$; and on morphism level,

$$\begin{aligned}
&W_0 : M_0 \to M_1, && Z(W_0) \in \mathrm{Hom}_{\mathsf{Vect}_\Bbbk}\big(Z(M_0), Z(M_1)\big) \cong Z(M_0)^* \otimes Z(M_1)\\
&W_1 : \emptyset^{n-1} \to \overline{M_0} \sqcup M_1, && Z(W_1) \in \mathrm{Hom}_{\mathsf{Vect}_\Bbbk}\big(\Bbbk, Z(\overline{M_0} \sqcup M_1)\big) \cong Z(\overline{M_0}) \otimes Z(M_1)\\
&W_2 : M_0 \sqcup \overline{M_1} \to \emptyset^{n-1}, && Z(W_2) \in \mathrm{Hom}_{\mathsf{Vect}_\Bbbk}\big(Z(M_0 \sqcup \overline{M_1}), \Bbbk\big) \cong Z(M_0) \otimes Z(\overline{M_1})
\end{aligned}$$

the three different morphisms in $\mathsf{Bord}_{\langle n-1,n\rangle}^{\mathfrak{X}(n)}$ can be viewed as somehow the same linear map. Which we shall illustrate in the following diagram

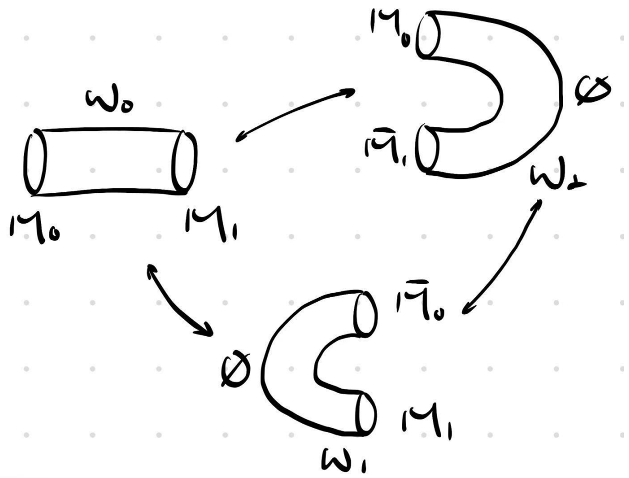

Figure 4: Involutivity of the cobordisms.

It is easy to check that the category of n-dimensional $\mathfrak{X}(n)$-TQFTs, denoted as $\mathsf{TQFT}_{\langle n-1,n\rangle}^{\mathfrak{X}(n)}$, is also an symmetrical monodial functor.

Before we really come into the 2D TQFT, there are some propositions that we need for the fitness of TQFT:

Theorem 2.11. Let $(Z : \mathsf{Bord}_{\langle n-1,n\rangle}^{\mathrm{SO}(n)} \to \mathsf{Vect}_\mathbb{C}) \in \mathsf{TQFT}_n^{\mathrm{SO}(n)}$ be a n-dimensional orientable TQFT, then $\forall M \in \mathsf{Bord}_{\langle n-1,n\rangle}$, the corresponding vector space $Z(M) \in \mathsf{Vect}_\mathbb{C}$ is finite dimensional.

Proof. Fix $M \in \mathsf{Bord}_{\langle n-1,n\rangle}$, $V := Z(M) \in \mathsf{Vect}_{\mathbb{C}}$, and $W := V^* = Z(\overline{M})$. Consider cobordisms given in the following (left) diagram of the "rainbow shape" that $c : \emptyset^{n-1} \to \overline{M} \sqcup M$ and $e : M \sqcup \overline{M} \to \emptyset^{n-1}$, then, consider the cbordism $(e \sqcup \mathrm{id}_M) * (\mathrm{id}_M \sqcup c)$ in the following (right) diagram with the "S" shape (We can call it the "zigzag identity"):

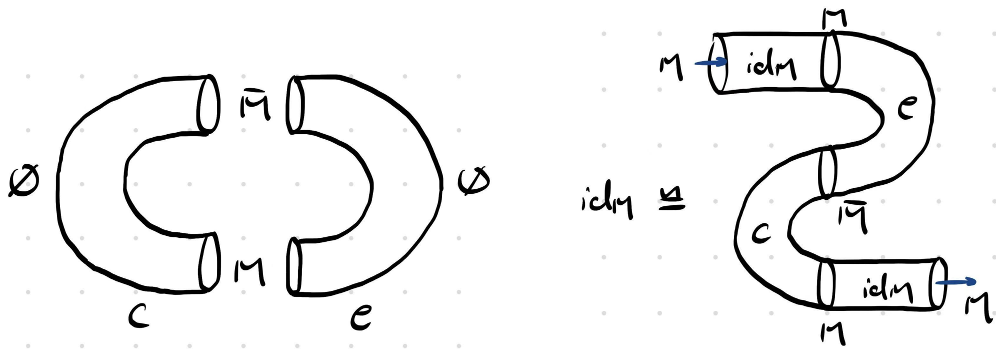

Figure 5: (Left) The "rainbow" shape cobordism. (Right) The "S" shape cobordism.

The "S" shape cobordism is diffeomorphic to the identity $M \times I$, and the TQFT functor on it gives the following diagram:

$$\begin{tikzcd}
M \sqcup \emptyset^{n-1} \cong M \arrow[r, "\mathrm{id}_M \sqcup c"] \arrow[rr, bend left, "\mathrm{id}_M"] \arrow[d, dashed] & M \sqcup \overline{M} \sqcup M \arrow[r, "e \sqcup \mathrm{id}_M"] \arrow[d, dashed] & \emptyset^{n-1} \sqcup M \cong M \arrow[d, dashed] \\
V \arrow[r, "\mathrm{id}_V \otimes \mathrm{coev}"] & V \otimes W \otimes V \arrow[r, "\mathrm{ev} \otimes \mathrm{id}_V"] & V \\
\xi \arrow[r, mapsto] & \xi \otimes \left(\sum_i w_i \otimes v_i\right) \arrow[r, mapsto] & \sum_i \mathrm{ev}(\xi \otimes w_i) v_i
\end{tikzcd}$$

Where $\mathrm{ev} := Z(e) : V \otimes W \to \mathbb{C}$ and $\mathrm{coev} := Z(c) : \mathbb{C} \to V \otimes W$, since

$$\mathrm{coev}(1) = \sum_{i=1}^{n} w_i \otimes v_i$$

are only allowed to be a finite sum, the commutative diagram gives $\forall \xi \in V$

$$\xi = Z\big((e \sqcup \mathrm{id}_M) * (\mathrm{id}_M \sqcup c)\big) = \sum_{i=1}^{n} \mathrm{ev}(\xi \otimes w_i) \otimes v_i$$

i.e., $V = \mathrm{Span}\{v_i \mid i = 1, \dots, n\}$ is finite dimensional. We shall also call the identity $(\mathrm{ev} \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \mathrm{coev}) = \mathrm{id}_V$ the "zigzag" identity due to the shape of the corresponding cobordism. $\square$

We shall end this section with the classification theorem of 1-dimensional oriented TQFT:

Theorem 2.12. Every finite-dimensional vector space $V \in \mathsf{Vect}_{\Bbbk}$ gives a unique (up to natural isomorphism) 1-dimensional oriented TQFT functor $Z \in \mathsf{TQFT}_1^{\mathrm{SO}(1)}$ with $Z(\mathrm{pt}_+) = V$, where pt_+ is a point with positive orientation.

Proof. (Existence) By classification of 1-dimensional manifolds, the compact 1-dimensional manifold can be either a closed interval $[0, 1]$ or a circle S^1. Given a n-dimensional $\Bbbk$-vector space V, we have the functor Z

1. On objects: $Z(\emptyset) = \Bbbk$, $Z(\mathrm{pt}_+) = V$, $Z(\mathrm{pt}_-) = V^*$ and for disjoint union, $Z(X \sqcup Y) = Z(X) \otimes Z(Y)$
2. On morphisms: Consider the 1-cobordisms "c" and "e" defined as in Figure 5. For "c", we defined the corresponding linear map

$$\mathrm{coev} := Z(c) : \Bbbk \to V^* \otimes V, \quad 1 \mapsto \mathrm{coev}(1) := \sum_i \epsilon_i \otimes e_i$$

where ϵ_i and e_i are (dual) basis for V^* and V; and for "e", the linear map to be the evaluation map of a linear functional in V^*

$$\mathrm{ev} := Z(e) : V \otimes V^* \to \Bbbk, \quad \phi \otimes v \mapsto \mathrm{ev}(\phi \circ v) := \phi(v)$$

Also, for the cobordism W that exchanges the order as shown in Figure **??**, we defined the linear map

$$Z(W) : V \otimes V^* \to V^* \otimes V, \quad v \otimes \phi \mapsto \phi \otimes v$$

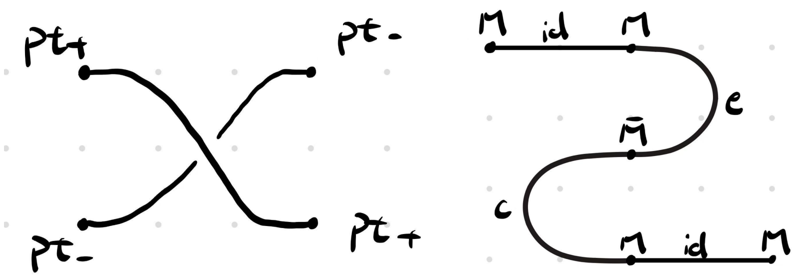

Figure 6: (Left) The "swap order" cobordism. (Right) The "S" shape cobordism.

And we shall check the consistency of these definitions of morphism given by the "S" shape cobordism in (Right) Figure 6, which, in $\mathsf{Vect}_{\Bbbk}$, is just the "zigzag" identity given by the following diagram:

$$\begin{tikzcd} V \arrow[r, "\mathrm{id}_V \otimes \mathrm{coev}"'] \arrow[rr, bend left, "\mathrm{id}_V"] & V \otimes V^* \otimes V \arrow[r, "\mathrm{ev} \otimes \mathrm{id}_V"'] & V \\ v \arrow[r, mapsto] & v \otimes \left(\sum_i \epsilon_i \otimes e_i\right) \arrow[r, mapsto] & \sum_i \epsilon_i(v) e_i \end{tikzcd}$$

or just $(\mathrm{ev} \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \mathrm{coev}) = \mathrm{id}_V$. Take the vector $v \in V$ such that the definition of the dual basis gives

$$v = \sum_{i=1}^n v_i e_i, \quad \epsilon_j(v) = \sum_{i=1}^n v_i \delta_{ij} = v_i$$

which proves the "zigzag" identity since

$$(\mathrm{ev} \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \mathrm{coev}) v = \sum_{i=1}^n \mathrm{ev}(v \otimes \epsilon_i) \otimes \mathrm{id}_V(e_i) = \sum_{i=1}^n \epsilon_i(v) e_i = v$$

This shows the existence of TQFT.
(Uniqueness) Let Z be any oriented TQFT with $V = Z(\mathrm{pt}_+)$ and $W = Z(\mathrm{pt}_-)$, we claim that the linear map $Z_e : V \otimes W \to \Bbbk$ corresponding to the cobordism e is a perfect pairing, which gives a linear isomorphism $\Phi : W \to V^*$ by $\Phi(w)v = Z_e(w \otimes v)\ \forall w \in W$. The linearity is by definition, and we shall write the inverse as

$$\Phi^{-1}(\alpha) := (\alpha \otimes \mathrm{id}_W)\big(Z_c(1)\big)\ \forall \alpha \in V^*$$

with $Z_c : \Bbbk \to V \otimes W$ be the linear map corresponding to c. We shall check that it is indeed the inverse:

$$[(\Phi \circ \Phi^{-1})\alpha]v := Z_e\big(\Phi^{-1}(\alpha) \otimes v\big) = Z_e\big((\alpha \otimes \mathrm{id}_W)\big(Z_c(1)\big) \otimes v\big), \quad Z_c(1) := \sum_i v_i \otimes w_i$$
$$= \sum_i \alpha(v_i) Z_e(w_i \otimes v) = \alpha\big((\mathrm{id}_V \otimes Z_e) \circ (Z_c \otimes \mathrm{id}_V)v\big) \quad \forall \alpha \in V^*$$

But by the "S" shape figure, we know that $v = (\mathrm{id}_V \otimes Z_e) \circ (Z_c \otimes \mathrm{id}_V)v$. Thus, $(\Phi \circ \Phi^{-1})\alpha(v) = \alpha(v)\ \forall v \in V$, i.e., $\Phi \circ \Phi^{-1} = \mathrm{id}_{V^*}$. Also, the other composition is given by

$$(\Phi^{-1} \circ \Phi)(w) = \big[\big(\Phi(w) \otimes \mathrm{id}_W\big) \circ Z_c\big](1) = (Z_e \otimes \mathrm{id}_W) \circ (\mathrm{id}_W \otimes Z_c)w = w$$

The final step is still due to the "S" shape cobordism. Thus, given $V = Z(\mathrm{pt}_+)$, we can automatically get $W = Z(\mathrm{pt}_-) \cong V^*$. By setting the functor preserves the monoidal product, i.e., for $X = \mathrm{pt}_\pm \sqcup \cdots \sqcup \mathrm{pt}_\pm$, given a natural isomorphism

$$\theta_X : Z(X) \to Z_V(X)$$

which take id_V on all positive points pt_+ and Φ on all negative points pt_-, we have the unique symmetry monoidal functor $Z \in \mathsf{TQFT}^{\mathrm{SO}(x)}_{\langle 0,1\rangle}$ up to isomorphisms. This completes the proof. □

3 Frobenius Algebras and 2D TQFT

In the last section, we have defined TQFT to be the analogy of path homotopy, with the "boundary points" being closed manifolds and "path class" being the diffeomorphism class of cobordisms. In the following passages, we shall denote the n-dimensional TQFT with tangential structure $\mathfrak{X}(n)$ and take values in symmetrical monoidal category C as

$$\mathsf{TQFT}^{\mathfrak{X}(n)}_n[C] := \mathrm{Fun}^{\otimes}\left(\mathsf{Bord}^{\mathfrak{X}(n)}_{\langle n-1,n\rangle}, C\right)$$

In this section, we shall revisit the famous result of [Abr96] that there is an equivalence of categories between oriented 2D TQFT and commutative Frobenius algebra:

$$\mathsf{TQFT}^{\mathrm{SO}(2)}_2[\mathsf{Vect}_\Bbbk] \cong \mathsf{cFrob}_\Bbbk$$

and for simplicity of notations, we denote $\Bbbk\mathsf{TQFT}^{\mathfrak{X}(n)}_n := \mathsf{TQFT}^{\mathfrak{X}(n)}_n[\mathsf{Vect}_k]$.

In the following passages, we shall first introduce the definition of a commutative Frobenius algebra. Without further explanation, by TQFT, we refer to oriented TQFT, i.e., objects in $\mathsf{TQFT}^{\mathrm{SO}(n)}_n$.

3.1 Frobenius Algebra

Fix a field $\Bbbk$; all the algebras we mentioned in this section will be assumed to be finite-dimensional and commutative. The (commutative) Frobenius algebra over $\Bbbk$ is defined as follows:

Definition 3.1 (Commutative Frobenius Algebra)**.** A commutative Frobenius algebra is an $\Bbbk$-algebra (A, m, Θ) such that

1. $m : A \otimes A \to A$ is an associative and commutative product with unit 1_A.
2. $\Theta : A \to \Bbbk$ is a $\Bbbk$-linear form such that $\ker \Theta$ contains no non-zero ideal.
3. A $\Bbbk$-linear form $\Theta : A \to \Bbbk$ (called "Frobenius trace") such that $\ker \Theta$ contains no non-zero ideal.

We shall name the second condition the "Frobenius structure".

Remark. *Here are some remarks on notations. We shall always write* $ab := m(a \otimes b)$ *be the multiplication of* $a, b \in A$*, and write* $\langle a, b\rangle$ *is the perfect pairing given by* $\langle a, b\rangle := \Theta \circ m(a \otimes b)\ \forall a, b \in A$*.*

To simplify our following discussion of the algebraic structure of 2D TQFT, we shall remark that there are several different forms of Frobenius structure:

Proposition 3.2. The following structures are equivalent as Frobenius structures on (A, m):

1. A $\Bbbk$-linear form $\Theta : A \to \Bbbk$ (called "Frobenius form") such that $\ker \Theta$ contains no non-zero ideal.
2. An associative and commutative non-degenerate pairing $\beta : A \otimes A \to \Bbbk$.
3. Coassociative and cocommutative comultiplication (coalgebra structure) $\Delta : A \to A \otimes A$ with a counit, such that $\Delta \circ m = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta)$ (we may call this "Frobenius relation").

For the equivalence of the first two conditions, it is clear that we can simply take the non-degenerate pairing to be $\beta := \Theta \circ m$, or, conversely, take $\Theta := \beta(1_A, a)$, where 1_A is the multiplication unit. The proof is mostly by computation, and we may assume the equivalence of these three definitions for a commutative Frobenius algebra without proof.

3.2 Algebraic Structure of 2D TQFT

We shall now discuss the algebraic structure of 2D TQFTs. Note that the objects in $\mathsf{Bord}_{\langle 1,2\rangle}$ are closed 1-manifolds, and we have the following classification theorem:

Theorem 3.3 (Classification of 1-Manifolds)**.** Let M be a connected 1-manifold, then M is either

1. Compact: S^1 (without boundary) or closed interval $[0, 1]$ (with boundary).
2. Noncompact: Open interval $(0, 1)$ (without boundary) or half open interval $(0, 1]$ (with boundary).

and any other 1-manifold would be the disjoint union of the case listed above.

Thus, the only object of the 2-bordism category is S^1. Thus,

Proposition 3.4. To defined $Z \in \Bbbk\mathsf{TQFT}_2$ on object level, we only need to specify $Z(S^1) := A \in \mathsf{Vect}_{\Bbbk}$.

With $A := Z(S^1)$, we shall claim that a TQFT gives a commutative Frobenius $\Bbbk$-algebra (A, m, Δ) where

$$m := Z(W_1 : S^1 \sqcup S^1 \to S^1), \quad \Delta := Z(W_1^\vee : S^1 \to S^1 \sqcup S^1)$$

We may intuitively give the correspondence between 2D TQFT and a Frobenius algebra as shown in the following figure: where 1_A and Θ are the unit/counit (trace) and the $\Delta : \Bbbk \to A \otimes A$ is the comultiplication

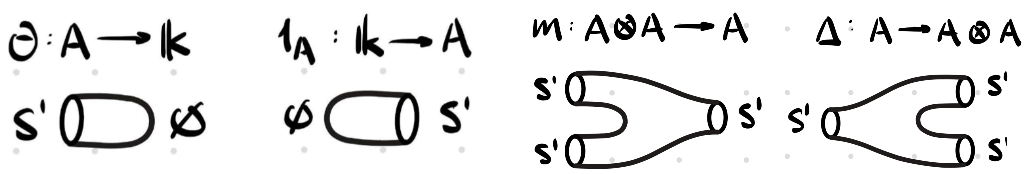

Figure 7: The correspondence between commutative Frobenius algebra and 2D TQFT.

given in the Frobenius structures in Proposition 3.2.

The Frobenius relation is motivated by the following cobordism: Note that we can also write the algebraic

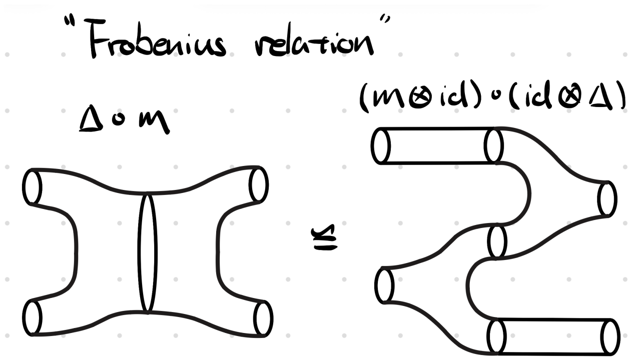

Figure 8: The cobordism corresponding to the "Frobenius relation".

structure above using indices: By the fact that we choose $\Bbbk$ to be a field, the commutative Frobenius $\Bbbk$-algebra we are considering is a finite-dimensional (by Theorem 2.11) $\Bbbk$-vector space. We may take basis $\{e_i \mid i = 1, \dots, n\}$, and then the multiplication is given by (we apply Einstein notation of summation)

$$m(e_i \otimes e_j) := c^k{}_{ij} e_k, \quad g_{ij} := \langle e_i, e_j \rangle := \Theta \circ m(e_i \otimes e_j) = c_{ij}{}^k \Theta(e_k)$$

and then, by the nondegeneracy of the pairing $\Theta \circ m$, matrix $[g_{ij}]$ is invertible, and we shall write the inverse as g^{ij} (thus, $g^{ij} g_{jk} = \delta^i{}_k$). Then, the multiplication unit is given by the $\Bbbk$-linear map $1_A : \Bbbk \to A$, which only depends on its value at $1 \in \Bbbk$, $1_A(1) = \eta = \eta^i e_i$ and the multiplication gives $c^k{}_{ij} \eta^i = \delta^k{}_j$.
Recall that the involutivity of the cobordisms gives an equivalence between the identity cobordism, the "c" cobordism, and the e cobordism. We shall make a further illustration on that: The perfect pairing $\langle -, - \rangle : A \otimes A \to \Bbbk$ gives a natural isomorphism (as aaninner product) $\varphi : A \to A^*$ between A and its dual space. Then, in finite-dimensional linear algebra, we have the isomorphism

$$A \otimes A \cong \mathrm{Hom}_{\mathsf{Vect}_{\Bbbk}}(A^*, A), \text{ given by } \sum_i a_i \otimes b_i \mapsto \left(\sum_i f(a_i) b_i \right)$$

which, in this case, we shall find the element $\Gamma \in A \otimes A$ (or, equivalently, the map $\Bbbk \ni 1 \mapsto \Gamma \in A \otimes A$) gives the corresponding element of $\varphi^{-1} : \langle a, - \rangle \to a$

$$\Gamma := g^{ij} e_i \otimes e_j, \quad a = \varphi^{-1}(\langle a, - \rangle) = (\langle a, - \rangle \otimes \mathrm{id})(\Gamma) = g^{ij} \langle a, e_i \rangle e_j = g^{ij} a_i e_j = a^j e_j$$

and thus, we have an explicit formula of the coproduct:

Proposition 3.5. The coproduct $\Delta : A \to A \otimes A$ is given by $\Delta := (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Gamma)$, which can be explicitly given by

$$\Delta(e_i) = g^{kl} c^j{}_{ik} e_j \otimes e_l$$

The proposition above can be shown from the following cobordism: As a summary of all the statements we

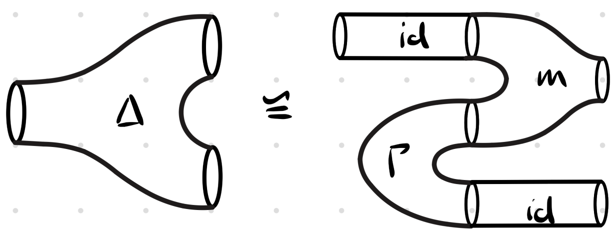

Figure 9: Expression of coproduct

have made above, we have the classification theorem of 2D-TQFT:

Theorem 3.6 (Classification of 2D Oriented TQFT)**.** Let $Z \in \Bbbk\mathsf{TQFT}_2^{\mathrm{SO}(2)}$ be a oriented 2D TQFT with $Z(S^1) = A$ and $A \in \mathsf{Vect}_\Bbbk$ be a vector space over field $\Bbbk$. Then, A admits the structure of a finite-dimensional commutative Frobenius algebra given by the 2D TQFT.
Conversely, every finite-dimensional commutative Frobenius algebra A, there is a unique (up to isomorphism) $Z_A \in \Bbbk\mathsf{TQFT}_2^{\mathrm{SO}(2)}$ such that $Z(S^1) = A$.

The previous argument in this section already showed that every 2D TQFT gives a commutative Frobenius algebra. In the following proof, we shall show the converse, that every commutative Frobenius algebra over $\Bbbk$ gives a unique 2D TQFT.

The idea of proof follows from the fact that any closed 1-manifold (as the boundary of cobordisms) is S^1 up to diffeomorphism, and any cobordism can be properly cut into the basic components of 2D TQFT that correspond to the algebraic structure in Frobenius algebra, as we have intuitively known from Fig. 7. An example of this canonical decomposition (without any cup/cap) can be shown as the following figure:

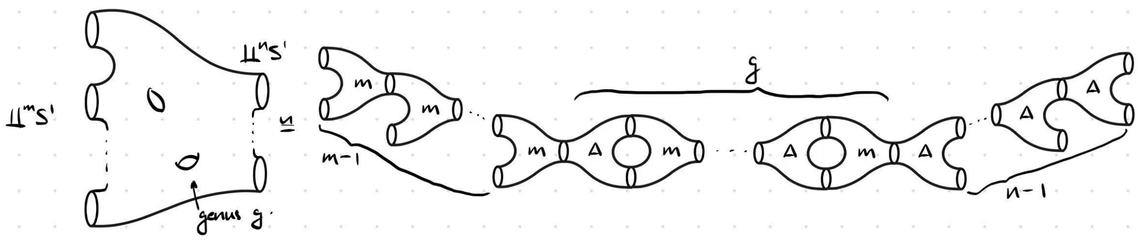

Figure 10: The decomposition of cobordism $W : \coprod^m S^1 \to \coprod^n S^1$ with out any "cup" or "cap"

Proof of the Classification Theorem 3.6. We consider the commutative unital associative $\Bbbk$-algebra $(A, m, 1_A)$ and Frobenius trace $\Theta : A \to \Bbbk$. Given basis $\{e_i \mid i = 1, \dots, n\}$ of A and $e^i = g^{ij} e_j$ as defined. The coproduct

$$\Delta(a) = ae_i \otimes e^i = a^i c_{ij}{}^k g^{jl} e_k \otimes e_l, \quad \forall a = a^i e_i \in A$$

and $(A, m, 1_A, \Delta, \Theta)$ satisfies the Frobenius relation

$$(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = \Delta \otimes m = (m \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id})$$

Since $A \in \mathsf{Vect}_\Bbbk$, we shall defined $Z_A \in \Bbbk\mathsf{TQFT}_2^{\mathrm{SO}(n)}$ such that $Z_A(S^1) = A$. Then, we shall consider the following "generating cobordisms" shown in Figure 7 using the fundamental structures on A as a Frobenius algebra:

1. Cup ($\emptyset \to S^1$): $Z_A(\mathrm{cup}) = 1_A : \Bbbk \to A$
2. Cap ($S^1 \to \emptyset$): $Z_A(\mathrm{cap}) = \Theta : A \to \Bbbk$
3. Pair-of-pants ($S^1 \sqcup S^1 \to S^1$): $Z_A(\mathrm{pants}) = m : A \otimes A \to A$
4. Copair-of-pants ($S^1 \to S^{\sqcup} S^1$): $Z_A(\mathrm{copants}) = \Delta : A \to A \otimes A$

And to ensure the functor is symmetrical, we need to make the conversion that each time a specific calculation is performed, a homeomorphism is selected to number/sort the boundary branches; changing the sorting only introduces commutative isomorphisms of tensor factors (given by symmetry).
Then, let $W : M_- \to M_+$ be any 2-cobordism, where

$$M_- \cong \coprod^m S^1, \qquad M_+ \cong \coprod^n S^1$$

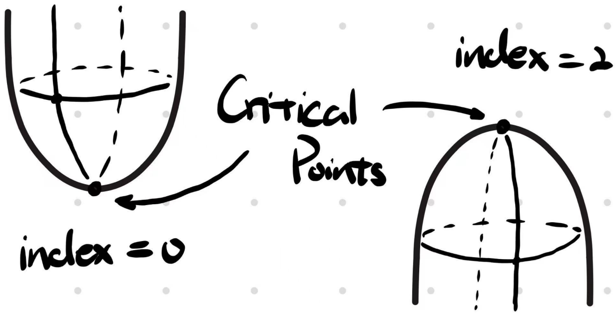

Figure 11: The corresponding cobordism segment of each interval between regular values.

We shall construct the corresponding linear map due to the TQFT by the following decomposition: Take a Morse function $f : W \to [0,1]$ such that $f^{-1}(0) = M_-$ and $f^{-1}(1) = M_+$, with all critical points in the interior of W. Suppose the regular values are

$$0 < c_1 < \cdots < c_r < 1$$

and regular values $t_i \in (c_i, c_{i+1})$ $\forall i \in \{0, \ldots, r+1\}$ such that $t_0 = 0$ and $t_{r+1} = 1$. Let the "time slice" be $W_i := f^{-1}([t_i, t_{i+1}])$ and each time slice contains at most 1 critical point with index either 0, 1, or 2.
For each regular value t_i, the preimage is given by

$$f^{-1}(t_i) \cong \coprod^{k_i} S^1$$

and thus, we shall choose an identification (up to permutation) $\phi_i : f^{-1}(t_i) \xrightarrow{\cong} \coprod^{k_i} S^1$. For any permutation $\sigma \in S_k$, there is canonical isomorphism

$$\tau_\sigma : A^{\otimes k} \to A^{\otimes k}, \quad \tau_\sigma(a_1 \otimes \cdots \otimes a_k) = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}$$

which is the symmetrical structure of $\mathsf{Vect}_{\Bbbk}$. We shall claim that each slice is a cobordism $W_i : f^{-1}(t_i) \to f^{-1}(t_{i+1})$, which is corresponding to a $\Bbbk$-linear map under the TQFT, given by

$$Z_i : A^{\otimes k_i} \to A^{\otimes k_{i+1}}$$

We deal with the Morse index of each critical point of the Morse function to get the linear map. The relation between Morse indexes and the decomposition law can be shown as the following figure

1. Index 0 (birth): This implies that along the direction of cobordism (from input to output), there is a local minimum, i.e., we shall insert a cup $1_A : \Bbbk \to A$. More precisely, suppose the newly generated circle has index p in the disjoint union, then the linear map corresponds to a cobordism segment is

$$Z_i := \mathrm{id}^{\otimes(p-1)} \otimes 1_A \otimes \mathrm{id}^{\otimes(k_1-p+1)} : A^{\otimes k_i} \to A^{\otimes(k_i+1)}$$

2. Index 2 (death): Similarly, index 2 gives a local maximum, which corresponds to a cap $\Theta : A \to \Bbbk$ or the "Frobenius trace" as la inear map. The linear map corresponding to the cobordism segment is given by (assume the trace kills the q-th component)

$$Z_i := \mathrm{id}^{\otimes(q-1)} \otimes \Theta \otimes \mathrm{id}^{\otimes(k_i-q)} : A^{\otimes k_i} \to A^{\otimes(k_i-1)}$$

3. Index 1 (merge/split): In this case, we have a saddle of the Morse function, which can be either multiplication $m : A \otimes A \to A$ or comultiplication $\Delta : A \to A \otimes A$. Thus, we may check the number of connected components of $f^{-1}(t_i)$ and $f^{-1}(t_{i+1})$.
If $|f^{-1}(t_i)| > |f^{-1}(t_{i+1})|$, then we have

$$Z_i := \mathrm{id}^{\otimes(q-1)} \otimes m \otimes \mathrm{id}^{\otimes(k_i-q-1)} : A^{\otimes k_i} \to A^{\otimes(k_i-1)}$$

and if $|f^{-1}(t_i)| < |f^{-1}(t_{i+1})|$, then we have

$$Z_i := \mathrm{id}^{\otimes(q-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes(k_i-q)} : A^{\otimes k_i} \to A^{\otimes(k_i+1)}$$

Finally, we shall composite all time slices as

$$Z_A(W) := Z_{r+1} \circ Z_r \circ \cdots \circ Z_1 : A^{\otimes m} \to A^{\otimes n}$$

which gives the corresponding $\Bbbk$-linear map for any cobordism.
It remains to show that the above definition of $Z_A(W)$ is independent of the choice of Morse data. Let $f_0, f_1 : W \to [0,1]$ be two relative Morse functions with $f_i^{-1}(0) = M_-$, $f_i^{-1}(1) = M_+$, and all critical points in the interior of W. By relative Cerf theory, there exists a generic homotopy $\{f_s\}_{s\in[0,1]}$ from f_0 to f_1 such that the induced handle decomposition changes only at finitely many parameters $s = s^*$, and at each such parameter one encounters only the following local Cerf moves:

1. Birth–death: creation/annihilation of a cancelable pair of critical points of indices $(0,1)$ or $(1,2)$;
2. Critical value crossing: two critical values interchange, equivalently swapping the order of two adjacent elementary cobordisms.

On the algebraic side, Move (1) corresponds to inserting/removing one of the composites

$$m \circ (1_A \otimes \mathrm{id}), \qquad m \circ (\mathrm{id} \otimes 1_A), \qquad (\Theta \otimes \mathrm{id}) \circ \Delta, \qquad (\mathrm{id} \otimes \Theta) \circ \Delta$$

which equal id by the unit/counit axioms. Move (2) corresponds to changing parentheses and/or permuting adjacent tensor factors; invariance under such changes follows from associativity/coassociativity/commutativity together with the Frobenius relation

$$\Delta \circ m = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = (\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id})$$

Therefore, $Z_A(W)$ is unchanged under each Cerf move, hence is independent of the chosen Morse function, regular values, and boundary orderings (which only contribute canonical symmetry isomorphisms in $\mathsf{Vect}_\Bbbk$). Consequently, $Z_A(W)$ depends only on the diffeomorphism class of W relative to the boundary, so the assignment is well-defined. Compatibility with gluing and disjoint union is immediate from the construction, hence Z_A defines a symmetric monoidal functor $Z_A \in \Bbbk\mathsf{TQFT}_2^{\mathrm{SO}(2)}$. □

Moreover, this one-to-one correspondence gives rise to an equivalence of categories:

Theorem 3.7. The functor $F : \Bbbk\mathsf{TQFT}_2^{\mathrm{SO}(2)} \to \mathsf{cFrob}_\Bbbk$ gives a equivalence of category.

Recall that equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ which exists another functor $G : \mathcal{D} \to \mathcal{C}$ such that there exists natural isomorphisms $\alpha_x : x \to GFx$ and $\beta_y : y \to FGy$ for any $x \in \mathrm{ob}\,\mathcal{C}$ and any $y \in \mathrm{ob}\,\mathcal{D}$. To prove Theorem 3.7, we just need to show the relation above.

Sketch Proof of Theorem 3.7. Consider the functor $F : \Bbbk\mathsf{TQFT}_2^{\mathrm{SO}(2)} \to \mathsf{cFrob}_\Bbbk$ as the correspondence given in Theorem 3.6. For natural transformation $\theta : Z \Rightarrow Z'$ as morphism in $\Bbbk\mathsf{TQFT}_2^{\mathrm{SO}(2)}$, let the corresponding morphism in $\mathsf{cFrob}_\Bbbk$ be

$$F(\theta) := \theta_{S^1} : Z(S^1) \to Z'(S^1)$$

Then, given $G : \mathsf{cFrob}_\Bbbk \to \Bbbk\mathsf{TQFT}_2^{\mathrm{SO}(2)}$ as follows: On object level, the construction of 2D TQFT corresponding to a finite-dimensional commutative Frobenius algebra is given in the previous proof of Theorem 3.6. For any morphism in $\mathsf{cFrob}_\Bbbk$, $f : A \to B$, consider the natural transformation that induced by $f : A \to B$ which send S^1 to itself and $Z(S^1) = A \xrightarrow{f} B = Z'(S^1)$. More precisely, the cobordism $W : M_0 \to M_1$ gives the linear map

$$G(A)(W) : A^{\otimes|M_0|} \to B^{\otimes|M_0|}$$

Where $|M_i|$ is the number of connected components in M_i. Then, for $f \in \mathrm{Hom}_{\mathsf{cFrob}_{\Bbbk}}(A, B)$

$$G(f)_M := f^{\otimes|M|} : A^{\otimes|M|} \to B^{\otimes|M|}$$

By the functornality of symmetrical monoidal product, G is clearly a functor.
For the natural isomorphism, an observation is that

$$FG(A) = G(A)(S^1) = A$$

which allows us to take the natural isomorphism be just $\beta_A := \mathrm{id}_A$. Then, consider $\alpha_Z : GF(Z) \to Z$ given by

1. On object $M = \coprod^n S^1$, take the monoidal structure isomorphism:
$$\alpha_{Z,M} : Z(M) \to Z(S^1)^{\otimes n} = A^{\otimes m} = GF(Z)(M)$$
2. For any cobordism $W : M_0 \to M_1$, the naturality: $\alpha_{Z,M_1} \circ Z(W) = Z(W) \circ \alpha_{Z,M_0}$. We shall check the condition for just the four generating cobordism (cup/cap/pants/copants), but $GF(Z)$ by definition is the same as Z on those cobordisms, thus, α is a natural isomorphism.

Thus, there is equivalence of categories $\Bbbk\mathsf{TQFT}_2^{\mathrm{SO}(2)} \cong \mathsf{cFrob}_{\Bbbk}$. □

Note that $\mathsf{TQFT}_n^{\mathfrak{X}(n)}[C]$ is a groupoid (category where every morphism is an isomorphism). We shall end our discussion of the details of TQFT by pointing out the central problem of TQFT.

Central Problem: Consider the n-dimensional TQFT with tangential structure $\mathfrak{X}(n)$ and codomain category C. The goal of studying TQFT is to "compute" the groupoid $\mathsf{TQFT}_n^{\mathfrak{X}(n)}[C]$. More precisely, we want to find the equivalence of groupoids between $\mathsf{TQFT}_n^{\mathfrak{X}(n)}[C]$ and some other groupoid that we are more familiar with. Theorem 2.12 and 3.6 show that we have a nice answer for $n = 1$ and the orientable cobordism case when $n = 2$.

At the end of this section, we may give a remark on the motivation of TQFT and an overview of further steps, which explain how to use the idea of TQFT to give more detailed information about complex structures on manifolds and counting curves on algebraic varieties.

3.3 Limitations of 2D TQFT: Why the Atiyah-Segal axioms are not enough?

Before moving on to moduli spaces of (stable) curves, it is helpful to clarify what the Atiyah–Segal notion of a 2-dimensional TQFT does extremely well, and what it cannot do. The axioms are optimized for cut-and-paste topology, but the geometric problems in symplectic/algebraic geometry force us to remember extra structure (complex structure, marked points, degenerations) that bordisms forget.

A guiding principle in topology is that meaningful invariants should be (i) functorial and (ii) computable by decomposition. Euler's polyhedron formula is the classical prototype:

$$\chi = V - E + F.$$

In modern language, χ is a homotopy invariant expressible via homology:

$$\chi(X) := \sum_{k \geq 0} (-1)^k \dim H_k(X, \mathbb{Q}),$$

and the basic "decompose-and-reconstruct" mechanism is excision encoded by Mayer–Vietoris.

Theorem (Mayer–Vietoris sequence)**.** Let $X = U \cup V$ with $U, V \subset X$ open (or suitably nice subspaces). Then there is a natural long exact sequence in homology

$$\cdots \longrightarrow H_k(U \cap V) \xrightarrow{(i_*, -j_*)} H_k(U) \oplus H_k(V) \xrightarrow{+} H_k(X) \xrightarrow{\partial} H_{k-1}(U \cap V) \longrightarrow \cdots$$

Philosophically: global invariants are constrained—and often determined—by how local pieces glue.

Atiyah–Segal TQFT as "cut-and-paste linear algebra". A d-dimensional TQFT in the Atiyah–Segal sense is a symmetric monoidal functor

$$Z : \mathsf{Bord}_d^{\mathrm{or}} \to \mathsf{Vect}_{\Bbbk}$$

sending a closed oriented $(d-1)$-manifold to a vector space ("states"), and a d-dimensional cobordism to a linear map ("time evolution"), such that:

- disjoint union corresponds to tensor product;
- gluing cobordisms along matching boundaries corresponds to the composition of linear maps;
- orientation reversal corresponds (morally) to dualization.

In $d = 2$, objects are disjoint unions of circles and morphisms are compact oriented surfaces. The axioms make surface-cutting into algebra: decompose into pairs of pants/caps/cylinders, translate to operations, then compose.

Concretely, for a 2D TQFT one sets $A = Z(S^1)$, and the values of Z on basic cobordisms encode

$$m : A \otimes A \to A \quad \text{(pair-of-pants)}, \quad 1_A : \Bbbk \to A \quad \text{(cap)}, \quad \Theta : A \to \Bbbk \quad \text{(cup)},$$

with nondegenerate pairing $\langle a, b\rangle = \Theta(m(a, b))$. Equivalently, a 2D TQFT is a (commutative) Frobenius algebra package: once $(A, m, 1_A, \Theta)$ is fixed, $Z(\Sigma)$ is computed from any pants decomposition.

So what is missing? From the point of view of symplectic/algebraic geometry, the Atiyah–Segal framework is too coarse in three related ways.

(1) No dependence on complex structure, hence no room for moduli. A bordism is defined up to diffeomorphism, so $Z(\Sigma)$ depends only on genus. But GW theory needs marked Riemann surfaces varying in families; the natural objects live over $\overline{\mathcal{M}}_{g,n}$, not as a single number for a topological type.

(2) No place to store cohomological data ("gravitational descendants"). GW theory inserts cohomology classes and characteristic classes (e.g. ψ-classes) and naturally produces cohomology classes on $\overline{\mathcal{M}}_{g,n}$ (numbers arise only after integration). A 2D TQFT outputs linear maps/scalars, not classes in $H^*(\overline{\mathcal{M}}_{g,n})$.

(3) The relevant "gluing" is degeneration to nodal curves, not cutting along circles. The key identities in GW theory come from nodal degenerations forming boundary strata of $\overline{\mathcal{M}}_{g,n}$, with gluing morphisms such as

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}, \quad \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}.$$

This is sewing along the boundary of moduli, not along geometric boundary circles of a bordism. This is exactly why the next step must be the geometry/topology of $\overline{\mathcal{M}}_{g,n}$ and its compactification.

4 Moduli of Stable Curves and Deligne–Mumford Compactification

The purpose of this section is to introduce the necessary information about the moduli space of stable curves and its compactification, since this will be the working space of the CohFT. In many cases (especially in symplectic geometry), we often study the moduli space $\mathcal{M}_{g,n}$ of J-holomorphic map from genus-g curves to a target space X (often some symplectic manifold or algebraic variety)

$$u : (C; p_1 \ldots, p_n) \to X$$

Even if the target space X is compact, the moduli space of curves is often not compact.

The noncompactness of the moduli space is due to the "nodes" that may be generated from the "limit" on a series of algebraic curves (pinching or bubbling). For example, consider curve C, we shall take the limit of a simple closed curve γ on C such that the geodesic length of the curve converges to zero in the limit, as shown in Fig. 12.

We give an example here to show the first taste of the idea on the node by a pinching process:

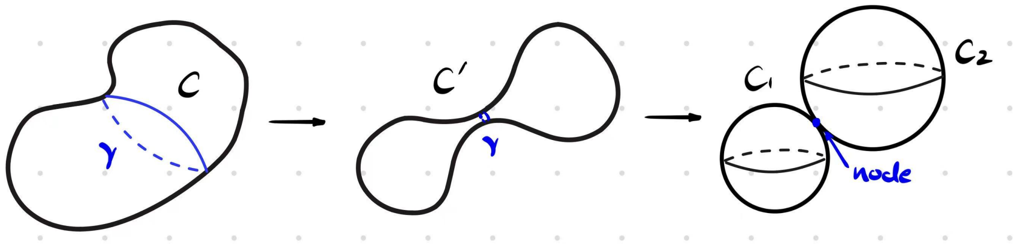

Figure 12: The nodal curve, where the node is locally isomorphic to $zw = 0$ in $\mathbb{C}^2$.

Example (Node of two disks)**.** *Take $D_z := \{z \in \mathbb{C} \mid |z| < \epsilon\}$ and $D_w := \{w \in \mathbb{C} \mid |w| < \epsilon\}$, consider a small parameter $q \in \mathbb{C}$ with $0 \neq |q| < \epsilon^2 \ll 1$. The equation $zw = q$ defines the curve*

$$C_q := \{(z, w) \in D_z \times D_w \mid zw = q\}$$

on $D_z \times D_w \subseteq \mathbb{C}^2$.

Claim. $zw = q$ is a smooth curve if $q \neq 0$.

The proof of the proposition is easy since this is the level set of the holomorphic function $F(z, w) = zw - q$. Then, defined the anulus

$$A_q = \{z \in \mathbb{C} \mid |q|/\epsilon < |z| < \epsilon\}$$

A clear observation is the following proposition:

Claim. For $q \neq 0$, $\phi : A_q \to C_q$, $\phi(z) = (z, q/z)$ is biholomorphism.

Then, we shall define the gluing operation. Let $D_z^* := D_z \setminus \{0\}$ and $D_w^* := D_w \setminus \{0\}$, consider the small annulus $U_z(q) := \{z \in \mathbb{C} \mid |q|/\epsilon < |z| < \epsilon\} \subseteq D_z^*$ and similar for $U_w(q)$. The gluing is given by the pushout of the diagram

$$D_z^* \xleftarrow{\;i_z\;} U_z(q) \cong U_w(q) \xrightarrow{\;i_w\;} D_w^*$$

where $U_z(q) \cong U_w(q)$ is given by $g(z) = q/z$. g is indeed a biholomorphism since multiplication are holomorphic and $g^{-1}(w) = q/w$. We shall denote the gluing as $X_q := D_z^* \sqcup_{U(q)} D_w^*$. It is easy to check that X_q is a smooth curve.

For now, what we construct is nothing but a connected sum $D^2 \# D^2$. However, if we consider the limit of this construction (since the construction depends on q) as $q \to 0$,

$$C := C_0 = \{(z, w) \in D_z \times D_w \mid zw = 0\} = (D_z \times \{0\}) \cup (\{0\} \times D_w)$$

The nodal curve C_0 is smooth everywhere except $(0, 0)$, and the conformal modulus of the annulus

$$\operatorname{mod}(A_q) = \frac{1}{2\pi} \ln \frac{\epsilon^2}{|q|} \longrightarrow +\infty$$

which shows that the length of the annulus "neck" $\to \infty$ as $q \to 0$, this is exactly a node from pinching.

In more algebraic geometrical language, the analytic local ring $\mathcal{O}_{C,p}^{\mathrm{an}}$ (analytic/holomorphic function germs) near $p \in C$ has the completion:

$$\widehat{\mathcal{O}}_{C,p} = \mathbb{C}[[z, w]]/(zw)$$

which means this "pinching a neck" limit makes the curve locally isomorphic to $zw = 0$ analytically near p.

The key idea for solving this problem goes back to Deligne and Mumford [DM69], who introduced the notion of stable curves and the Deligne–Mumford compactification. In what follows, the author recasts their method from a more geometric viewpoint, which the author finds helpful for clarifying the objects involved. To study this method, we first need to understand what a node is and the properties of curves with nodes. We still use Fig.12 as an example. In such a limit process, the remaining parts C_1 and C_2 independently are both well-defined smooth curves, and thus, a limit can be viewed as the one-point union of two curves, in this case, topologically being $S^2 \vee S^2$. By the homogeneity of the manifolds, we shall replace the elements of the moduli space from an algebraic curve C_g with genus g to a curve $(C_g; p_1, \ldots, p_n)$ with genus g and n marked points. And we may pick random points on both C_1 and C_2 and glue them together to generate the limit elements in the moduli space $\overline{\mathcal{M}}_{g,n}$ and thus, make the compactification of the moduli space. An example of such a gluing operation is shown in Fig. 13.

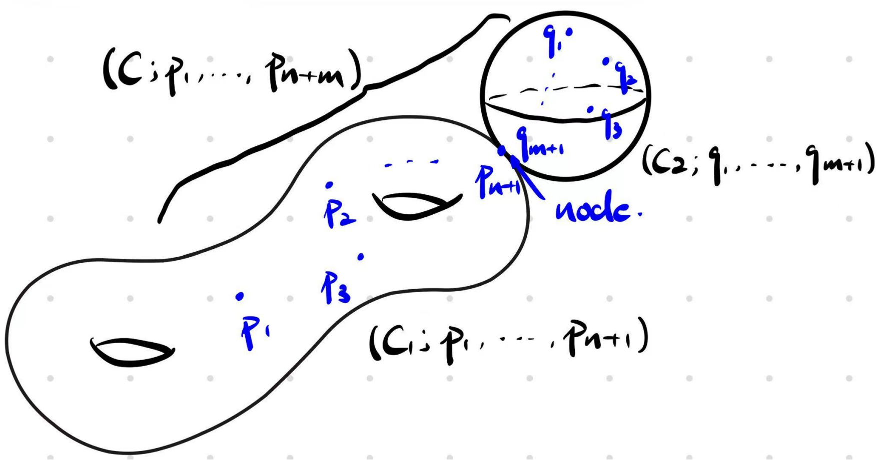

Figure 13: The gluing of curves with marked points $(C_1; p_1, \ldots, p_{m+1})$ and $(C_2; q_1, \ldots, q_{m+1})$ and obtain $(C; p_1, \ldots, p_{m+n})$ with a node p_{m+n+1}.

4.1 Prestable curves and stability

To formalize the idea we have talked about above, we need to clarify the definition of "a curve with n marked points" as the elements of the moduli space. More precisely, we need a stabilization condition on the moduli space.

Fix integers g, n in the stable range $2g - 2 + n > 0$. A smooth n-pointed curve of genus g means a compact Riemann surface C of genus g with n ordered, distinct marked points $(p_1, \ldots, p_n) \in C^n$. Isomorphisms are biholomorphisms preserving the labels.

Definition 4.1 (Nodal Curve). A (complex) curve C is nodal if it is a compact connected 1-dimensional complex analytic space whose only singularities are ordinary double points (nodes), i.e., every singular point admits local coordinates in which C is analytically isomorphic to $\{zw = 0\} \subset \mathbb{C}^2$.

Definition 4.2 (Prestable Curve). A prestable n-pointed curve is a nodal curve C together with n pairwise distinct marked smooth points $p_1, \ldots, p_n \in C$.

However, recall that we defined the marked points on curves to define the gluing operation shown in Fig. 13. When counting special points, we should not only consider the marked points but also take the node as a special point.

Note that each node is locally $zw = 0$, then we shall consider the following normalization: Note that locally biholomorphic to $zw = 0$ means that we can take two branches of the small neighborhood $(z, 0)$ and $(0, w)$, which intersect transversely at $(0, 0)$. We shall denote the set of nodes as N, consider a compact (may not be connected) Riemann surface $\widetilde{C}$:

$$\widetilde{C} := \coprod_v \widetilde{C}_v$$

and consider surjection $\nu : \widetilde{C} \to C$ such that:

1. $\nu|_{\widetilde{C}\setminus\nu^{-1}(N)} : \widetilde{C} \setminus \nu^{-1}(N) \to C \setminus N$ is biholomorphism.
2. For any node $p \in N$, there are exactly two preimage $\nu^{-1}(p) = \{p_+, p_-\}$.
3. For small neighborhood U_+ and U_- of p_+ and p_-, ν locally behave as
$$U_+ \sqcup U_- \subseteq D_z \sqcup D_w \to C, \quad z \mapsto (z, 0),\ w \mapsto (0, w)$$
which sends both center $0 \mapsto (0, 0)$.

Also, since the normalization map is biholomorphic, for all marked points $p_i \notin N$, $\exists! \tilde{p}_i \in C$ that $\nu(\tilde{p}_i) = p_i$. Thus, we shall define the set of special points to be

$$S := \nu^{-1}(N) \cup \{\tilde{p}_1, \dots, \tilde{p}_n\} \subseteq \tilde{C}$$

and for each connected component of $\tilde{C}$, consider $S_v := S \cap \tilde{C}_v$ and $n_v = |S_v|$. Recall that the Euler characteristic of a closed curve with genus g is given by

$$\chi(\Sigma_g) = 2 - 2g$$

Then, since after we remove all special points, C_v is then a well-defined smooth surface that is holomorphic to $\tilde{C}_v \setminus S_v$, we shall compute its Euler characteristic:

Proposition 4.3. Let C_v be a irreducible component of C with genus g (well-defined by the genus of $\tilde{C}_v = \nu^{-1}(C_v)$), then let C_v^o be C_v with all nodes and marked points been removed

$$\chi(C_v^o) = \chi(\tilde{C}_v \setminus S_v) = 2 - 2g_v - n_v$$

Proof. The first equal sign $\chi(C_v^o) = \chi(\tilde{C}_v \setminus S_v)$ is obvious since $\nu : \tilde{C}_v^o \to C_v$ is biholomoirphism. Then, homotopically, we may consider

$$C_v^o \cong \tilde{C}_v \setminus S_v \simeq C_v \setminus \bigcup_{x \in S_v} \mathrm{int}(D_x)$$

And then Euler's characteristic is just

$$\chi(C_v^o) = 2 - 2g - \#\text{ of boundary components} = 2 - 2g - n_v$$

which completes the proof. ∎

Following the description of the special points on curves, we consider the following stability condition:

Definition 4.4 (Stability)**.** A prestable n-pointed curve $(C; p_1, \dots, p_n)$ is stable if for every component $\widetilde{C}_v$ one has $2g_v - 2 + n_v > 0$. Equivalently, $\chi(C_v^o) < 0$.

The idea of the definition of the stable curve above is to reduce the automorphism of the curve to 0-dimensional (discrete): The idea is to require special points to be the fixed points of automorphisms. For each smooth component $(\tilde{C}_v, S_v)$ of the nodal curve, we define the automorphism to be

$$\mathrm{Aut}(\tilde{C}_v, S_v) := \{\text{Biholomorphism } h : \tilde{C}_v \to \tilde{C}_v \text{ that } \forall p_i \in S_v : h(p_i) = p_i\}$$

and the automorphism is just the map $h : (C, S) \to (C, S)$ that is an automorphism when we restrict it to any smooth component.

Theorem 4.5. Let C be a nodal curve with genus g and set of special points S ($n = |S|$), then (C, S) stable implies $\mathrm{Aut}(C, S)$ finite.

Proof. Note that the stability condition $\chi(\tilde{C}_v) < 0$ ($\tilde{C}_v := \tilde{C}\backslash S$) means $\tilde{C}_v$ is hyperbolic. By the uniformization theorem, there exists a unique complete Riemannian metric with constant curvature -1 on a given conformal structure of X, denoted as $\rho_{\tilde{C}_v}$, and $(\tilde{C}_v, \rho_{\tilde{C}_v})$ has finite area. More precisely, by the Gauss-Bonnet Theorem,

$$\mathrm{Area}(\tilde{C}_v, \rho_{\tilde{C}_v}) = -2\pi\chi(\tilde{C}_v) = 2\pi(2g - 2 + n) < \infty$$

Then, $\forall \varphi \in \mathrm{Aut}(C, S)$, the restriction on $\tilde{C}_v$ gives biholomorphism $\varphi|_{\tilde{C}_v} : \tilde{C}_v \to \tilde{C}_v$. Since $\rho_{\tilde{C}_v}$ is compatible with the complex structure, the pullback $\varphi^*\rho_{\tilde{C}_v}$ is also a complete metric with curvature -1. Thus, by uniqueness, $\rho_{\tilde{C}_v} = \varphi^*\rho_{\tilde{C}_v}$, i.e., we have the incusion

$$\mathrm{Aut}(\tilde{C}_v, S_v) \hookrightarrow \mathrm{Isom}^+(\tilde{C}_v, \rho_{\tilde{C}_v})$$

and thus, it is enough to claim that $|\mathrm{Aut}(C_v, S_V)| < \infty$ by the classical fact that the isometric group of a complete hyperbolic surface with finite area is finite. The proof of this fact is in Appendix A. Since the automorphism of (C, S) is just combining the automorphism on each stable component with the permutation of the components

$$\mathrm{Aut}(C, S) = \left(\prod_{v=1}^{k} \mathrm{Aut}(C_v, S_v)\right) \rtimes S_k$$

where k is the number of irreducible components on the curve. The finiteness of automorphism of each irreducible component $\mathrm{Aut}(C_v, S_v)$ implies the finitness of the automorphism of the entire stable curve. Thus, we proved $|\mathrm{Aut}(C, S)| < \infty$. □

4.2 Deligne-Mumford Stack and Gluing Operation

We begin with the formal definition of Deligne-Mumford (DM) stack:

Definition 4.6 (Deligne-Mumford Stack)**.** The Deligne-Mumford stack (or orbifold) $\overline{\mathcal{M}}_{g,n}$ is the space of genus g stable curve with n marked points up to biholomorphism.

Note that by Theorem 4.5 there is an alternative definition that defined

$$\overline{\mathcal{M}}_{g,n} := \{(C_g, S) \mid |S| = n, |\mathrm{Aut}(C, S)| < \infty\}$$

which replace the condition on number of marked points to just requiring the automorphism to be finite.

There is an obvious forgetful morphism $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ given by dropping the last marked point: $p(C; p_1, \ldots, p_{n+1}) = (C; p_1, \ldots, p_n)$, where C is a nodal curve and p_i are marked points. However, dropping a marked point may cause the curve component to be unstable when the component has $g = 0, 1$. We shall discuss the stability case by case on each irreducible component:

1. Genus 0 ($\widehat{\mathbb{C}}$ component): The stable component C_v need at least 3 special points.
2. Genus 1 (complex torus component): The stable component C_v needs at least 1 special point.
3. Genus ≥ 2: Automatically stable.

The method that deals with this problem is to contract the unstable components after remove the marked points, an intuitive example is shown in the Fig. 14, where under the forgetful map makes the $\widehat{\mathbb{C}}$ component

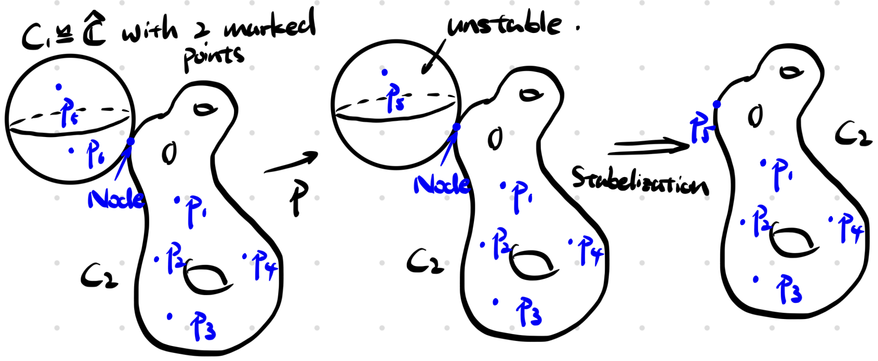

Figure 14: The forgetful map and stabilization

remains only two marked points, which cause unstability. Thus, under the stabilization, we shall forget this component and the other part remains sable.

Another natural morphism is given by the gluing of stable curves, which is already intuitivly gived in Fig 13, which is similar to the structure of TQFT:

Definition 4.7 (Gluings). The gluing operation on the DM stack is to glue two nodes and form a node on the curve, which can be formally classified by the property of generated nodes:

1. Non-separating node/Self gluing: $q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$, which glue two marked points on a single curve.
2. Separating node: $r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$ where $g = g_1 + g_2$ and $n = n_1 + n_2$.

The idea of gluing is shown in Fig. 15

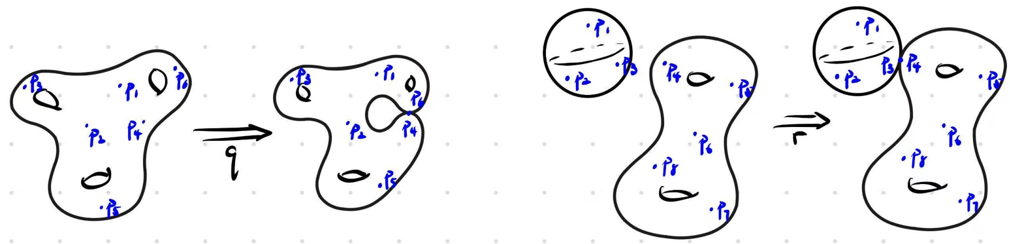

Figure 15: The gluing map r and q.

Since we are counting special points, the preimage of a node under normalization maps is a pair of special points; we can argue that the gluing operation does not break the stability condition. However, it remains an important problem: What is the topology on the moduli spaces? Without any topology, there is no way to talk about any of the following steps and the compactification of the DM stack. To give DM stack a topology naturally, we need to introduce the Teichmüller space: Let C be a closed orientable smooth surface, then we define the *space of complex structure* as

$$\mathcal{J}(C) := \{J \in \Gamma^\infty(\mathrm{End}(TC)) \mid J^2 = -\mathrm{id}\}$$

An observation is that the (almost) complex structure[1] natually induces an orientation. Thus, we shall consider

[1]This definition gives an almost complex structure in general. However, any almost complex structure gives rise to a complex structure in dimension 2 (the Nijenhuis tensor is constant zero, and thus, every almost complex structure is integrable).

the space of the complex structure that is compatible with the given orientation:

$$\mathcal{J}^+(C) := \{J \in \mathcal{J}(C) \mid J \text{ compatible with the given oridentation on } C\}$$

Note that $\mathcal{J}(C)$ is a contractable space, a proof is in Appendix B. We may give $\mathcal{J}^+(C) \subseteq \Gamma^\infty\big(\mathrm{End}(TC)\big)$ as a set of C^∞-tensor field on C and equip with the C^∞-topology.

Consider $\mathrm{Diff}^+(C) \trianglelefteq \mathrm{Diff}(C)$ consists of all orientation-preserving diffeomorphisms, and $\mathrm{Diff}^0(C) \trianglelefteq \mathrm{Diff}^+(C)$ be the diffeomorphisms that are isotopic to the identity on C. All diffeomorphism groups above act naturally on $\mathcal{J}^+(C)$. Then, the Teichmüller space is given by:

Definition 4.8 (Teichmüller Space)**.** The Teichmüller space contains the class of conformal structures under the identity components of the diffeomorphism group:

$$\mathcal{T}(C) := \mathcal{J}^+(C)/\mathrm{Diff}^0(C)$$

Instead, the moduli space of complex structures on surface C with genus g is given by

$$\mathcal{M}_g := \mathcal{J}(C)/\mathrm{Diff}(C) = \mathcal{J}^+(C)/\mathrm{Diff}^+(C)$$

To connect these two concepts, we shall define the mapping class group as follows:

Definition 4.9 (Mapping Class Group)**.** The mapping class group on C is given by

$$\mathrm{MCG}(C) := \mathrm{Diff}^+(C)/\mathrm{Diff}^0(C)$$

and thus, the moduli space of genus g curve (Riemann surface) is just

$$\mathcal{M}_g = \mathcal{T}(C)/\mathrm{MCG}(C)$$

which contains the quotient topology on from $\mathcal{T}(C)$. More detail about the C^k-topology and the inherent topology on the moduli space of curve can be find in Appendix C. Finally, as we have stated in 4.6, we shall take the moduli space of genus g curve with n marked points

$$\mathcal{M}_{g,n} := \mathcal{J}(C)/\mathrm{Diff}(C,S) = \mathcal{J}^+(C)/\mathrm{Diff}^+(C,S)$$

where the DM stack $\overline{\mathcal{M}}_{g,n}$ is given by the space $\mathcal{M}_{g,n}$ with all nodal cases been added in it.

We may also list some useful properties of Deligne-Mumford stacks without proof: Recall that an orbifold is a generalization of a manifold, which allows a finite group action on each chart (Thus, instead of locally Euclidean, locally modeled as quotients of $U \subseteq \mathbb{R}$).

1. $\overline{\mathcal{M}}_{g,n}$ is a closed nonsingular, irreducible orbifold of (complex) dimension $3g-3+n$.
2. The "boundary"[2] of the Deligne-Mumford compactification given by the curves with at least one node:

$$\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$$

   This can be illustrate as the "points" that been added into $\mathcal{M}_{g,n}$, which are obtained by the limit that broke smoothness.
3. $\mathrm{im}(r)$ and $\mathrm{im}(q)$ are both in the boundary $\partial\overline{\mathcal{M}}_{g,n}$ (since gluing generates nodal curves).

Here are some simple examples of DM stack:

[2]Since the previous statement already states that $\overline{\mathcal{M}}_{g,n}$ is a closed nonsingular orbifold, the boundary here is NOT in the sense of orbifold with boundary, instead, by "boundary", we mean the limit case of nodal curves.

Example. *The moduli space of genus 0 curve with 3 marked points* $\overline{\mathcal{M}}_{0,3} = \{\mathrm{pt}\}$ *is just a point, since three given fixed points can fix a Mobiüs transformation on Riemann sphere.*

Recall that at the beginning of this section, we mentioned that we are considering the map from a genus g nodal curve with marked points to some target space X (often a symplectic manifold or projective algebraic manifold) given by

$$f : (C, S) \to X$$

The automorphism of (C, S) induces the automorphism of stable maps, given by the following diagram

$$\begin{tikzcd}
(C,S) \arrow[r, "f"] \arrow[d, "h"'] & X \\
(C,S) \arrow[ur, "f"'] &
\end{tikzcd}$$

where $h \in \mathrm{Aut}(C, S)$. This allows us to say such a map is stable if and only if the prestable curve (C, S) is stable.

Definition 4.10 (Moduli Space of Stable Maps)**.** The moduli space we consider is given by

$$\overline{\mathcal{M}}_{g,n}(X) := \{f : (C, S) \to X \mid f \text{ stable}\}$$

where (C, S) is a genus g curve with n special points, i.e., $|S| = n$. Furthermore, we can choose a characteristic homology class $\beta \in H_2(X)$ and restrict the map as

$$\overline{\mathcal{M}}_{g,n}(X, \beta) = \{f : (C, S) \to X \mid f_*[C] = \beta,\ f \text{ stable}\}$$

5 Definition of Cohomological Field Theory

6 Gromov-Witten Invariants

7 Example: Quantum Cohomology as Genus-Zero Part

A Automorphisms of Complete Hyperbolic Surface

In this appendix, we may prove the statement that the automorphism group of a complete hyperbolic surface is finite, which we used in the proof of Theorem 4.5.

Theorem A.1. Suppose Σ is a compact Riemann surface with $\chi(\Sigma) < 0$ or the genus $g \geq 2$. Then the automorphism $|\,\mathrm{Aut}(\Sigma)| < \infty$.

Proof. □

B Contractibility of $\mathcal{J}^+(S)$

C C^∞ Topology and Gromov Topology on Moduli Space

References

[Abr96] Lowell Abrams. "Two-Dimensional Topological Quantum Field Theories and Frobenius Algebras". In: *Journal of Knot Theory and Its Ramifications* 5.5 (1996), pp. 569–587.

[Ati88] Michael F. Atiyah. "Topological Quantum Field Theories". In: *Publications Mathématiques de l'IHÉS* 68 (1988), pp. 175–186.

[BF97] Kai Behrend and Barbara Fantechi. "The Intrinsic Normal Cone". In: *Inventiones Mathematicae* 128 (1997), pp. 45–88.

[BM96] Kai Behrend and Yuri Manin. "Stacks of Stable Maps and Gromov–Witten Invariants". In: *Duke Mathematical Journal* 85 (1996).

[CG07] Tom Coates and Alexander Givental. "Quantum Riemann–Roch, Lefschetz and Serre". In: *Annals of Mathematics* 165 (2007), pp. 15–53.

[DM69] Pierre Deligne and David Mumford. "The Irreducibility of the Space of Curves of Given Genus". In: *Publications Mathématiques de l'IHÉS* 36 (1969), pp. 75–109.

[Dub96] Boris Dubrovin. "Geometry of 2D Topological Field Theories". In: *Integrable Systems and Quantum Groups*. Vol. 1620. Lecture Notes in Mathematics. Springer, 1996.

[Fre12] Daniel S. Freed. *Bordism: Old and New*. 2012. eprint: https://people.math.harvard.edu/~dafr/M392C-2012/index.html.

[FO99] Kenji Fukaya and Kaoru Ono. "Arnold Conjecture and Gromov–Witten Invariant". In: *Topology* 38 (1999).

[FP97] William Fulton and Rahul Pandharipande. "Notes on Stable Maps and Quantum Cohomology". In: *Algebraic Geometry—Santa Cruz 1995*. Vol. 62. Proceedings of Symposia in Pure Mathematics. American Mathematical Society, 1997.

[Giv01] Alexander Givental. "Gromov–Witten Invariants and Quantization of Quadratic Hamiltonians". In: *Moscow Mathematical Journal* 1 (2001).

[HM98] Joe Harris and Ian Morrison. *Moduli of Curves*. Vol. 187. Graduate Texts in Mathematics. Springer, 1998.

[Koc04] Joachim Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories*. Vol. 59. London Mathematical Society Student Texts. Cambridge University Press, 2004.

[Kon95] Maxim Kontsevich. "Enumeration of Rational Curves via Torus Actions". In: *The Moduli Space of Curves*. Ed. by Robbert Dijkgraaf, Carel Faber, and Gerard van der Geer. Vol. 129. Progress in Mathematics. Birkhäuser, 1995.

[KM94] Maxim Kontsevich and Yuri Manin. "Gromov–Witten Classes, Quantum Cohomology, and Enumerative Geometry". In: *Communications in Mathematical Physics* 164 (1994), pp. 525–562.

[Lee03] J.M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer, 2003. ISBN: 9780387954486. URL: https://link.springer.com/book/10.1007/978-1-4419-9982-5.

[LT98] Jun Li and Gang Tian. "Virtual Moduli Cycles and Gromov–Witten Invariants of Algebraic Varieties". In: *Journal of the American Mathematical Society* 11 (1998), pp. 119–174.

[Man99] Yuri Manin. *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*. Vol. 47. American Mathematical Society Colloquium Publications. American Mathematical Society, 1999.

[RT95] Yongbin Ruan and Gang Tian. "A Mathematical Theory of Quantum Cohomology". In: *Journal of Differential Geometry* 42 (1995), pp. 259–367.

[Tel12] Constantin Teleman. “The Structure of 2D Semi-Simple Field Theories”. In: *Inventiones Mathematicae* 188 (2012).

[Wit88] Edward Witten. “Topological Sigma Models”. In: *Communications in Mathematical Physics* 118 (1988), pp. 411–449.