Notes on Group Cohomology I: Central Extension and Projective Representation

Derek Zeng*

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Before you begin reading this note, a warning is in order:

> *The first section of this note is not meant to teach you anything new; it is meant to serve as a filter. If you cannot follow even this part, you should probably stop reading here.*

If you find even this section difficult to follow, it is likely that you are not yet ready for the material that follows.

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*derek6@illinois.edu

1 Introduction and some Common Sense in Algebra

1.1 Background

The problem began with my self-study of *The Quantum Theory of Fields*, by S. Weinberg [1]. In chapter 2 of that book, Weinberg used the group representation technique to (not mathematically rigorously) define what a quantum theory is and discuss the behavior of the unitary representation of Lie groups. To be more precise, a quantum theory on Hilbert space $\mathcal{H}$ has a phase redundancy that $\forall \psi, \phi \in \mathcal{H}$

$$\psi \sim \phi \iff \psi = \mathrm{e}^{\mathrm{i}\theta}\phi$$

which means ψ and ϕ has the same physics, i.e., the state space if quantum mechanics is actually $\mathbb{C}\mathrm{P}^n$ (this is still not a rigorous claim since $\mathcal{H}$ often appears to be infinite dimensional). Since the probability in quantum mechanics is defined to be the Hermite product $\langle -, - \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, the symmetry, or transformation operator, acts as the group element of the unitary group $\mathrm{U}(\mathcal{H})$. Thus, the study of the transformation of some group G is the study of the representation $\rho : G \to \mathrm{U}(\mathcal{H})$, but with a phase redundancy: $\forall g, h \in G$

$$\rho(g)\rho(h) = \mathrm{e}^{\mathrm{i}\theta(g,h)}\rho(gh)$$

It is easy to see that the phase shall not depend on the vector in $\mathcal{H}$, but the redundancy still causes some trouble.

1.2 Preliminary on Algebra

To solve the problem, we shall review the following common sense of algebra, especially on the representation theory and Lie groups.

Definition 1.1 (Linear Representation)**.** A *linear representation* of the group G on the vector space V is a group action

$$\begin{aligned} \tilde{\rho} : G \times V &\to V \\ (g, v) &\mapsto g \cdot v \end{aligned}$$

such that $\forall g \in G : \tilde{\rho}(g, -) : V \to V$ is linear, and satisfies the following properties:

1. $\forall v \in V : e \cdot v = v$, i.e. $\tilde{\rho}(e, -) = \mathrm{id}_V$
2. $\forall g, h \in G : \forall v \in V : g \cdot (h \cdot v) = (gh) \cdot v$

In other words, a linear representation of group G on representation space V is a group homomorphism

$$\begin{aligned} \rho : G &\to \mathrm{GL}(V) \\ g &\mapsto \big(\rho(g) := \tilde{\rho}(g, -) : V \to V\big) \end{aligned}$$

In general, a projective representation is "nearly a linear representation", which, informally, in Weinberg's book (as we already mentioned), just means $\forall g, h \in G$, the representation map behaves as

$$\rho(g)\rho(h) = \mathrm{e}^{\mathrm{i}\theta(g,h)}\rho(gh)$$

The term "projective" in this case means that, compared with a linear representation, the representation differs by a phase parameter $\omega(g,h) := \mathrm{e}^{\mathrm{i}\phi(g,h)}$. A more precise definition will be given in the following passages, based on the definition given in [2, 3].

To deal with the group that appears in physics, we also need to know the basic definition of Lie groups and Lie algebras. The following introduction is mainly based on [2, 4]. More detail can be found in [5].

Definition 1.2 (Lie Groups)**.** The group G is a *Lie group* if G (with some topology) is a smooth manifold such that the multiplication $m : G \times G \to G$ and inverse map $i : G \to G$ defineds by $\forall g \in G : i(g) = g^{-1}$ are both smooth map.

The smoothness of multiplication can really imply the smoothness of the inversion map. Since a Lie group is also a smooth manifold, it is natural to consider its tangent space, the Lie algebra.

Definition 1.3 (Lie Algebra)**.** The *Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ of Lie group G is its tangent space at identity e, i.e., $\mathfrak{g} = T_eG$. More explicitly, for any smooth curve $g : (-\epsilon, \epsilon) \to G$ with $g(0) = e$, $\epsilon > 0$, the element of $\mathfrak{g}$ is just

$$X := \frac{\mathrm{d}}{\mathrm{d}t}\varphi \circ g(t)\Big|_{t=0}$$

for some coordinate $\varphi : U \subseteq G \to \mathbb{R}^n$.

In this level, the Lie "algebra" is yet not an algebra but a vector space, since we still need a "multiplication" map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ to make it an $\mathbb{R}$(or, $\mathbb{C}$ for complex Lie groups)-algebra. In fact, this algebra structure is natural since the base manifold is not only a manifold but a group, with the group multiplication, we are supposed to naturally induce such map on tangent space. To apply this claim, we need the *exponential map*, which reconstructs a subset of the manifold from the tangent space.

The existence of exponential map follows this proposition:

Proposition 1.1 (Single-Parameter Subgroup)**.** *Let G be a Lie group, and $\mathfrak{g} = \mathrm{Lie}(G)$. Then $\forall X \in G : \exists \gamma_X : (\mathbb{R}, +) \to G$ a single parameter subgroup (smooth curve that is a subgroup), that defined by*

$$\gamma_X(t) := \exp(tX)\ \forall t \in \mathbb{R}$$

where $\exp : \mathfrak{g} \to G$ *is defined by* $\forall X \in \mathfrak{g} : \exp(X) = \gamma_X(1)$

The proof is simply recall that there are two natural diffeomorphisms L_g and R_g on any Lie group G, known as the Left/Right translation. Base on these maps, one can always construct a left/right invariant vector field, and using the existence & uniqueness theorem of the solution of ODE on its flow to construct $\gamma_X(\mathbb{R}) \leq G$.

Finally, we are able to define the multiplication on the Lie algebra, named the *Lie bracket*. Consider the adjoint map $\mathrm{Ad} : G \times G \to G$ defined by $\forall g, h \in G : \mathrm{Ad}_g(h) = ghg^{-1}$ the conjugate action. To induce a map on the Lie algebra, we linearize the map in the following way: let $g(t) = \exp(tX)$ and $h(s) = \exp(sY)$ for some $X, Y \in \mathfrak{g}$, then, the induced map onthe Lie algebra $\mathfrak{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is

$$[X,Y] := \mathfrak{ad}_X Y := \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Ad}_{g(t)}\left(h(s)\right)\Big|_{s=0}\right)\Big|_{t=0}$$

It is easy to check that it satisfies the following properties:

1. $[X,Y] = -[Y,X]$
2. $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear.
3. The *Jacobi identity*: $\forall X, Y, Z \in \mathfrak{g}$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Often, at least in matrix groups, $[X, Y]_{\mathfrak{g}} = XY - YX$ (follows the Lie bracket in differential geometry). A more formal statement is that

Theorem 1.4 (Ado)**.** Any finite-dimensional Lie algebra $\mathfrak{g}$ over a field K of characteristic zero has a faithful (injective) representation that

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

where V is a K-vector space.

Note that $\mathfrak{gl}(V) \leq \mathrm{End}(V)$ is a subalgebra of associative algebra $\mathrm{End}(V)$, with Lie bracket $[X, Y]_{\mathfrak{gl}(V)}$.

These are all preliminary knowledge of representations and Lie groups.

So what did Weinberg do?

2 Physicist's Approach

To deal with the projective unitary representation, Weinberg considered two problems [1]:

1. **The problem on local (geometrical) behavior**: How to reduce the phase near the identity? More precisely, how to obtain the representation of the Lie algebra without the phase? The trick is to use the simply connectedness of some small neighborhood near the identity, and transform the local phase to the problem of redefining the generator (the basis of the Lie algebra).

2. **The problem on global (topological) behavior**: Does the corrected Lie algebra representation extend to the entire group? The key observation is that if the group is simply connected, then we can always extend the representation of the Lie algebra via a homotopic path. So a nice way to deal with the Lorentz group $\mathrm{SO}^+(1, 3)$ is to consider its universal cover $\mathrm{SL}(2, \mathbb{C})$, and the unitary representation of $\mathrm{SO}^+(1, 3)$ has a correspondence to the unitary representation of $\mathrm{SL}(2, \mathbb{C})$.

Recall the physicists' definition of projective representation, the group homomorphism $\rho : G \to \mathrm{U}(\mathcal{H})$ such that $\forall g, h \in G$

$$\rho(g)\rho(h) = \mathrm{e}^{\mathrm{i}\theta(g,h)}\rho(gh)$$

However, to make this projective representation act on the Hilbert space $\mathcal{H}$ in a legal way, we need: $\forall g, h, k \in G$

1. $\theta(g, e) = \theta(e, g) = 0$, and

2. (2-cocycle) by associtivity of the group G, $\theta(g, h) + \theta(gh, k) = \theta(g, hk) + \theta(h, k)$

This is the key observation, and we are going to use this property to solve the general lifting problem of projective representation. The reason that it is sufficient to claim that for may connected Lie group, we can extend the property is a small neighborhood $e \in U \subseteq G$ containing the identity is because of the following proposition:

Proposition 2.1. *Let G be a connected topological group, $e \in U \subseteq G$ is open. Then*

$$\bigcup_{n \geq 1} U^n = G$$

where $U^n = \{g_1 g_2 \cdots g_n \mid g_i \in U \ \forall i\}$

Proof. Defined $\forall U \subseteq G : U^{-1} := \{g^{-1} \mid g \in U\}$. Let $V = U \cap U^{-1}$, then V is also an open neighborhood of e and $V = V^{-1}$. Let

$$H = \bigcup_{n \geq 1} V^n \subseteq \bigcup_{n \geq 1} U^n$$

We claim that H is a subgroup of G. To show that $H = G$, we shall prove that H is both open and closed.

To show that H is a closed subgroup, note that since H is open, $\forall g \in G : gH$ is open (left translation is a homeomorphism). Thus

$$H = G \setminus \Big(\bigcup_{g \notin H} gH \Big)$$

is closed. Since H is nonempty clopen in a connected topological group G, $H = G$. Thus,

$$G = H \subseteq \bigcup_{n \geq 1} U^n \subseteq G$$

i.e. $\bigcup_{n \geq 1} U^n = G$. □

To reduce the phase, we first naïvly take it as redefining the representation with a phase:

$$\rho(g) \mapsto \tilde{\rho}(g) := \mathrm{e}^{-\mathrm{i}\xi(g)} U(g)$$

and the goal is to find some $\xi(g)$ that after defined the phase, $\tilde{\theta}(g,h) = 0 \ \forall g, h \in N$. Then, the 2-cocycle condition requires that

$$\begin{aligned}\tilde{\rho}(g)\tilde{\rho}(h) &= \mathrm{e}^{\mathrm{i}\tilde{\theta}(g,h)}\tilde{\rho}(gh) = \mathrm{e}^{\mathrm{i}(\tilde{\theta}(g,h) - \xi(gh))}\rho(gh) \\ &= \mathrm{e}^{-\mathrm{i}(\xi(g)+\xi(h))}\rho(g)\rho(h) = \mathrm{e}^{\mathrm{i}(\theta(g,h)-\xi(g)-\xi(h))}\rho(gh)\end{aligned}$$

i.e. $\mathrm{e}^{\mathrm{i}(\tilde{\theta}(g,h)-\xi(gh))} = \mathrm{e}^{\mathrm{i}(\theta(g,h)-\xi(g)-\xi(h))}$, which means the phase always satisfies the equation

$$\tilde{\theta}(g,h) = \theta(g,h) + \xi(gh) - \xi(g) - \xi(h)$$

With this phase equation, we shall begin to study the local problem.

2.1 Local Problem

As we have mentioned, consider the small neighborhood $e \in \mathcal{N} \subseteq G$, and suppose

1. The Lie group G is connected.
2. The representation $\rho : G \to U(\mathcal{H})$ is strongly continuous, i.e.,

$$T(t)x \to x \text{ as } t \to 0$$

3. $\theta : G \times G \to \mathrm{U}(1)$ continuous, and differentiable at (e, e).

Then, it is always sufficient to choose a coordinate α^i in the infinitesimal neighborhood $\mathcal{N}$ and the group element can be written as

$$g(\alpha) = \exp\big(-\mathrm{i}\alpha^i X_i\big), \quad \forall \alpha^i \in \mathbb{R}$$

Since the group can be parametrized, we can write the group product in a local coordinate form

$$g(\alpha)g(\beta) = g(\gamma(\alpha,\beta)), \quad \gamma^k(\alpha,\beta) = \alpha^k + \beta^k + \frac{1}{2}{c_{ij}}^k \alpha^i \beta^j + O(\alpha^2\beta, \alpha\beta^2)$$

where ${c_{ij}}^k$ is the structureal constant of the group G. On the other hand, one can expand $\theta(\alpha,\beta)$ around $(0,0)$:

$$\theta(\alpha,\beta) = \frac{1}{2}\omega_{ij}\alpha^i\beta^j + O(\alpha^2\beta, \alpha\beta^2)$$

the normalization condition that $\theta(e,g)=\theta(g,e)=0$ implies that $\theta(0,\beta)=\theta(\alpha,0)=0$. Using the Baker-CampbellHausdorff (BCH) formula to expand LHS,

$$\begin{aligned}\exp\bigl(-\mathrm{i}\alpha^i X_i\bigr)\exp\bigl(-\mathrm{i}\beta^j X_j\bigr) &= \exp(-\mathrm{i}\theta(\alpha,\beta))\exp\bigl(-\mathrm{i}\gamma^k(\alpha,\beta)T_k\bigr)\\ &= \exp\left(-\mathrm{i}(\alpha^i+\beta^i)X_i - \frac{\mathrm{i}}{2}\alpha^i\beta^j[X_i,X_j] + O(\alpha^2\beta,\alpha\beta^2)\right)\\ &= \exp\left(-\mathrm{i}(\alpha^i+\beta^i)X_i - \frac{\mathrm{i}}{2}\omega_{ij}\alpha^i\beta^j - \frac{\mathrm{i}}{2}c_{ij}{}^k\alpha^i\beta^j X_k + O(\alpha^2\beta,\alpha\beta^2)\right)\end{aligned}$$

Hence, we show that

$$[X_i,X_j]=\mathrm{i}c_{ij}{}^kT_k+\mathrm{i}\omega_{ij}\mathbf{1}$$

The projective representation of the Lie group G results in a central extension of the Lie algebra. For Poincaré algebra, it can be written explicitly as

$$\begin{aligned}[J_{\mu\nu},J_{\rho\sigma}] &= \mathrm{i}(g_{\mu\rho}J_{\nu\sigma}-g_{\nu\rho}J_{\mu\sigma}-g_{\mu\sigma}J_{\nu\rho}+g_{\nu\sigma}J_{\mu\rho})+\mathrm{i}C_{\mu\nu,\rho\sigma}\mathbf{1}\\ [J_{\mu\nu},P_\rho] &= \mathrm{i}(g_{\nu\rho}P_\mu-g_{\mu\rho}P_\nu)+\mathrm{i}C_{\mu\nu,\rho}\mathbf{1}\\ [P_\mu,P_\nu] &= \mathrm{i}C_{\mu\nu}\mathbf{1}\end{aligned}$$

Notice that the commutator requires the coefficient to be antisymmetric. For $C_{\mu\nu}$, to ensure the Lorentz invariance, the only possibility is that $C_{\mu\nu}=0$. Thus, Stonevon Neumann theory ensures that we can always diagonalize $\{P_\mu\}$ in the same basis and be able to use Wigner's classification [6].

For the rest of the central charge, we can redefine the generator by

$$\begin{aligned}P_\mu &\mapsto P'_\mu = P_\mu + a_\mu\mathbf{1}\\ J_{\mu\nu} &\mapsto J'_{\mu\nu} = J_{\mu\nu}+b_{\mu\nu}\mathbf{1}\end{aligned}$$

Since $\mathbf{1}$ commutes with every other operator

- $[P'_\mu,P'_\nu]=0$
- $[J'_{\mu\nu},P'_\rho]=[J_{\mu\nu},P_\rho]+\mathrm{i}(C_{\mu\nu,\rho}+g_{\nu\rho}a_\mu-g_{\mu\rho}a_\nu)\mathbf{1}$
- $[J'_{\mu\nu},J'_{\rho\sigma}]=[J_{\mu\nu},J_{\rho\sigma}]+\mathrm{i}(C_{\mu\nu,\rho\sigma}+g_{\mu\rho}b_{\nu\sigma}-g_{\nu\rho}b_{\mu\sigma}-g_{\mu\sigma}b_{\nu\rho}+g_{\nu\sigma}b_{\mu\rho})\mathbf{1}$

Thus, by choosing a_μ and b_μ, one can always let the central term vanish. Actually, there is a theorem that any semisimple Lie algebra has only a trivial (i.e., can be set to vanish via redefining generators) central charge (Theorem 7.1 in [3]).

More formally, at the end of this section, we will see that for general Lie groups G, the problem can be transformed into the integrability of a dynamical system. We will see that the integrable condition of that system is exactly the 2-cocycle condition.

2.2 Topological Problem

For convenience, we shall claim that the Lorentz group $\mathrm{SO}^+(1,3)$ has a universal cover $\mathrm{SL}(2,\mathbb{C})$ via the construction that $\forall V^\mu\in\mathbb{R}^{1,3}$

$$v=V^\mu\sigma_\mu=\begin{pmatrix}V^0+V^3 & V^1-\mathrm{i}V^2\\ V^1+\mathrm{i}V^2 & V^0-V^3\end{pmatrix}$$

We can represent any Lorentz 4-vector as a Hermite matrix. The Lorentz transformation will become an adjoint transformation

$$\lambda v\lambda^\dagger=(\Lambda^\mu{}_\nu(\lambda)V^\nu)\sigma_\mu$$

Where $|\det\lambda| = 1$, by changing the phase (since the additional phase does not physically change the system), we can always ensure that $\det\lambda = 1$. The corresponding group given by this transformation is the special linear group

$$\mathrm{SL}(2,\mathbb{C}) := \{\lambda \in \mathrm{GL}(2,\mathbb{C}) | \det\lambda = 1\}$$

It is simply connected (thus, the universal cover of $\mathrm{SO}^+(1,3)$). Since for any simply connected Lie group (we will explain this later), the topological obstruction caused by the fundamental group is trivial, one can simply use the proposition above to claim that the Lie algebra representation $\pi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{u}(\mathcal{H})$ defined by

$$\rho(\exp X) = \exp(-\mathrm{i}\pi(X))$$

can really vanish the phase in a small neighborhood $I \in \mathcal{N} \subseteq \mathrm{im}(\rho : \mathrm{SL}(2,\mathbb{C}) \to U(\mathcal{H}))$, i.e., by BCH formula

$$\rho(\exp X)\rho(\exp Y) = \exp(-\mathrm{i}X * Y)$$

where $X * Y := \ln(\exp X \exp Y)$. By the fact that $\exp : V \subseteq \pi(\mathfrak{sl}(2,\mathbb{C})) \to \mathcal{N} \subseteq \mathrm{im}(\rho)$ is a diffeomorphism, without global obstruction, we can extend this result to the entire $\mathrm{SL}(2,\mathbb{C})$ and thus we shall claim that:

1. The Lie group representation can be **uniquely** obtained via the single parameter subgroup $\rho \circ g_a(t) := \exp(-\mathrm{i}t\pi(X_a))$ by

$$\rho(g) = \prod_{a=1}^{n} \exp(-\mathrm{i}t\pi(X_a))$$

2. There exists a global $\xi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{U}(1)$ that ensures the projective phase $\tilde{\theta}(g,h) \equiv 0$ (Since in this case, with the simply connectivity of the group, we can directly apply the Proposition 2.1 to globalize the result).

2.3 More General Discussion

To conclude this section, we note that in the preceding discussion, we have deliberately glossed over many of the topological details, especially those aspects that differ from Weinbergs treatment of this problem. The reason is that this topological viewpoint is precisely what we intend to develop in the following sections of the article. Here, we shall briefly present a general approach to this problem on Lie groupswhich is slightly different from the standard one based on algebraic topologyas an overview of the subsequent discussion.

Recall that the modified phase for any given phase function $\theta(g,h)$ is given by

$$\tilde{\theta}(g,h) = \theta(g,h) + \xi(gh) - \xi(g) - \xi(h)$$

And our goal is to show whether there exists a function $\xi : G \to U(1)$ that $\tilde{\theta}(g,h) = 0\ \forall g,h \in G$, i.e.

$$\xi(gh) = \xi(g) + \xi(h) - \theta(g,h)$$

To solve this function equation, in the case that θ and ξ are at least C^1, one can choose a basis $\langle X_a \mid a = 1,\dots,n\rangle = \mathfrak{g}$, we can write the single parameter subgroup $h_a(t) := \exp(tX_a)$ and

$$\xi(gh_a(t)) = \xi(g) + \xi(h_a(t)) - \theta(g,h_a(t))$$

The key observation is that this function equation shows an action of right left-invariant vector field $X_a^L(g) := (L_g)_{*,e}X_a \in T_gG$, which more explicitly, $\forall\xi \in C^\infty(G)$

$$(X_a^L\xi)(g) := \frac{\mathrm{d}}{\mathrm{d}t}\xi\big(g\exp(tX_a)\big)\Big|_{t=0}$$

Then since $\phi : G \times G \to \mathrm{U}(1)$ is already be given, then the differentiation of the equation is given by a dynamical system:

$$X_a^L\xi(g) = c_a - A_a(g)$$

where

1. The constant is given by $c_a := X_a^K \xi(e)$.
2. The function $A_a(g)$ is
$$A_a(g) := \frac{\mathrm{d}}{\mathrm{d}t}\theta(g, h_a(t))\Big|_{t=0}$$

More geometrically, we claim that there is a 1-form $\alpha \in \Omega^1(G)$ such that
$$\alpha(X_a^L)(g) = c_a + A_a(g)$$
and the system is just $\mathrm{d}\xi = \alpha$.

Remark. *In this step, actually, we can already seen some kind of relation between the topology obstruction (de Rham Cohomology) and the central extension via the existance of such* ξ*.*

With a local chart of G contains $e \in G$, $\varphi := (x^1, \ldots, x^n) : \mathcal{N} \to V \subseteq \mathbb{R}^n$, for any curve $\gamma : I \to G$, denote $\xi(t) = \xi \circ \gamma(t)$,
$$\dot{\xi}(t) = \sum_{\alpha=1}^{n} \dot{\gamma}^\alpha(t)\big(c_\alpha + A_\alpha \circ \gamma(t)\big)$$
To deal with the integrability more explicitly, the system can be also written as
$$\sum_{i=1}^{n} (X_a^L)^i(x) \frac{\partial \xi}{\partial x^i}(x) = c_a - A_a(x)$$
The initial condition of this system is given by the equation at identity: $\xi(e) = 0$.

To test the integrability of the system, we use the Frobenius theorem. We shall apply again X_b^L to both side of the equation and obtain the Lie bracket of the derivator:
$$X_a^L\big(X_b^L\xi(g)\big) = -X_b^L A_a(g), \quad X_b^L\big(X_a^L\xi(g)\big) = -X_a^L A_b(g)$$
i.e., the Lie bracket of these two generators is given by
$$\begin{aligned}[X_a^L, X_b^L]\xi(g) &= f_{ab}{}^c X_c^L \xi(g) = f_{ab}{}^c\big(c_c - A_c(g)\big) \\ &= X_a^L A_a(g) - X_b^L A_b(g)\end{aligned}$$
where $f_{ab}{}^c$ is the structural constant of $\mathfrak{g}$. Thus, the integrability is determined by
$$X_a^L A_b(g) - X_b^L A_a(g) - f_{ab}{}^c A_c(g) = f_{ab}{}^c c_c$$
It is not hard to notice the fact that this is just $\mathrm{d}\alpha = 0$ locally. However, recall that the phase function we need requires $\mathrm{d}\xi = \alpha$, which means the regularity condition is actually the following:

Theorem 2.1. The projective representation $\hat{\rho} : G \to \mathrm{PU}(\mathcal{H})$ can be lift to a representation $\rho : G \to \mathrm{U}(\mathcal{H})$ if and only if $[\alpha] = [0] \in H^1_{\mathrm{dR}}(G)$.

3 (Co)Homology Theory and Group Cohomology

After stating some fundamental concepts in algebra and the physicists' approach, we also need some basic knowledge about (co)homology to understand the most general approach. Further reading includes [7, 8] in Chinese, and [9] in English.

3.1 The (Co)Chain Complexes and (Co)Homology

In this section, we use chain complex and homology as every formal definition, the cochain complex and cohomology are just the dual, i.e., reverse the direction of all arrows.

Definition 3.1 (Chain Complex)**.** Given a collection of graded Abelian groups (or vector spaces, in some cases) and maps between them $(C_\bullet, \partial_\bullet)$ (Note that $C_\bullet$ can be 0), named chain groups. Consider the following chain:

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

If the graded map $\partial_\bullet$ satisfies that $\forall n \in Z : \partial_{n-1} \circ \partial_n \equiv 0$, i.e. $\mathrm{im}(\partial_n) \subseteq \ker(\partial_{n-1})$, then the chain is called a *chain complex* and $\partial_n : C_n \to C_{n-1}$ is called a *boundary operator*.

Similarly, we called the chain $(C^\bullet, \delta^\bullet)$ with an increasing grade

$$\cdots \xrightarrow{\delta^{n-1}} C^{n-1} \xrightarrow{\delta^n} C^n \xrightarrow{\delta^{n+1}} C^{n+1} \xrightarrow{\delta^{n+2}} \cdots$$

and satisfied $\forall n \in \mathbb{Z} : \delta^n \circ \delta^{n-1} \equiv 0$ a *cochain complex*. The (co)chain complex is invented to "approximate" the following concept, the exact sequence.

Definition 3.2 (Exact Sequence)**.** A mapping sequence of groups $(G_\bullet, f_\bullet)$ given by

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} G_n$$

is said to be an *exact sequence* if it satisfies the property that $\mathrm{im}(f_k) = \ker(f_{k+1})\ \forall k$.

The exact sequence is an extremely important concept; actually, the entire homological algebra is the study of exact sequences and (co)chain complexes. To show the obstruction of a (co)chain complex from being an exact sequence, we defined the (co)homology groups as follows:

Definition 3.3 ((Co)Homology)**.** Given the (co)chain complex $(C_\bullet, \partial_\bullet)$, for each order n, defined

1. The *(co)cycle* $Z_n(C) := \ker(\partial_n)$.
2. The *(co)boundary* $B_n(C) := \mathrm{im}(\partial_{n+1})$

Since the (co)chain complex is defined to have $\partial_n \circ \partial_{n+1} \equiv 0$, we always have $B_n \subseteq Z_n$. The *(co)homology group* is given by

$$H_n(C) := \frac{Z_n(C)}{B_n(C)} = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}$$

It is easy to know that the (co)homology of an exact sequence is trivial. The (co)homology group $H_k(C)$ is known to be the obstruction of the (co)chain $(C_\bullet, \partial_\bullet)$ to be exact. For cochain complexes, the cohomology uses a superscript instead of a subscript:

$$H^k(C) := \frac{\ker(\delta^k : C^k \to C^{k+1})}{\mathrm{im}(\delta^{k-1} : C^{k-1} \to C^k)}$$

Finally, we need to define a chain map, which is the morphism between chains:

Definition 3.4 (Chain Map)**.** Given two (co)chain complexes, $(A_\bullet, \delta_\bullet)$ and $(B_\bullet, \partial_\bullet)$. A *chain map with degree* k is a collection of map $\{\Psi^n : A^n \to B^{n+k}\}$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_{n+2}"] & A_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "\Psi_{n+1}"] & A_n \arrow[r, "\delta_n"] \arrow[d, "\Psi_n"] & A_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "\Psi_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial_{n+k+2}"] & B_{n+k+1} \arrow[r, "\partial_{n+k+1}"] & B_{n+k} \arrow[r, "\partial_{n+k}"] & B_{n+k-1} \arrow[r, "\partial_{n+k-1}"] & \cdots
\end{tikzcd}$$

3.2 The Group Cohomology

After the general discussion of the (co)chain complexes, in this subsection, we are going to discuss the specific cochain complex and cohomology we are going to use: the inhomogeneous cochains.

Throughout the section, let G be the group we study, and $(A, +)$ be an Abelian group on which G acts (mostly trivially) on it.

Definition 3.5 (Inhomogeneous Cochain)**.** The *inhomogeneous cochain complex* $(C^\bullet_{\text{inh}}, \delta^\bullet)$ is given by the set of Abelian groups

$$\forall n \in \mathbb{N} \setminus \{0\} : C^n_{\text{inh}}(G, A) := \operatorname{Map}(G^n, A)$$

and $C^0_{\text{inh}}(G, A) = A$. The graded map $\delta^n : C^n_{\text{inh}}(G, A) \to C^{n+1}_{\text{inh}}(G, A)$ is defined by

1. For $n = 0$, $\forall a \in A$
$$(\delta^0 a)(g_1) = g_1 a - a$$
2. For $n \geq 1$, $\forall f \in C^n_{\text{inh}}(G, A) : \forall g_1, \ldots, g_{n+1} \in G$
$$\begin{aligned}(\delta^n f)(g_1, \ldots, g_{n+1}) &= g_1 \cdot f(g_2, \ldots, g_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n)\end{aligned}$$

It is generally not an easy thing to show the cochain condition $\delta^{n+1} \circ \delta^n \equiv 0$ for the inhomogeneous cochain complex. In fact, directly proving via calculation will lead to a very disastrous stacking of symbols. The standard method is to find a cochain complex $(C^\bullet_{\text{hom}}, d^\bullet)$ with better symmetry, named the homogeneous cochain complex, and check that $d^{n+1} \circ d^n \equiv 0$, then we shall use the chain map (morphism between chains) to show the inhomogeneous cochain complex satisfies the cochain condition.

Definition 3.6 (Homogeneous Cochain Complex)**.** We say a map $F : G^n \to A$ is equivariant if

$$F(gg_1, \ldots, gg_{n+1}) = gF(g_1, \ldots, g_{n+1}) \ \forall g, g_1, \ldots, g_{n+1} \in G$$

We called the subgroup of all equivariant map in $C^\bullet(G, A)$ the *homogeneous cochain complex*, denote as $C^\bullet_{\text{hom}}(G, A)$, with the ordered map $d^n : C^n_{\text{hom}}(G, A) \to C^{n+1}_{\text{hom}}(G, A)$ given by $\forall f \in C^n_{\text{hom}}(G, A)$

$$d^n f(g_1, \ldots . g_{n+1}) = \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, \widehat{g_i}, \ldots, g_{n+1})$$

Where $\widehat{g_i}$ means we skip the i-th group element in the map and take the rest of the elements.

It is easy to show that $d^{n+1} \circ d^n \equiv 0$, since $\forall f \in C^n_{\text{hom}}(G, A)$

$$\begin{aligned}(d^{n+1} \circ d^n)f(g_0, \dots, f_{n+2}) &= \sum_{i=1}^{n} (-1)^i (d^n f)(g_1, \dots, \widehat{g_i}, \dots, g_{n+2}) \\ &= \sum_{i=1}^{n+1} (-1)^i \sum_{j=1}^{n} (-1)^j f(g_1, \dots, \widehat{g_i}, \dots, \widehat{g_j}, \dots, g_{n+2}) = 0, \quad i \neq j\end{aligned}$$

Note that when we remove g_j, the element we remove is the j-th element after the original i-th element has been removed. Since for any possible pair of indices $1 \leq p < q \leq n+2$, it will exactly appear twice with opposite sign.

To complete this discussion of this group cohomology, we claim that the homogeneous chain complex is isomorphic to the inhomogeneous complex.

Proposition 3.1. *The chain map $\Psi_n : C^n_{\text{hom}}(G, A) \to C^n_{\text{inh}}(G, A)$ given by*

$$\Psi_n f(g_1, \dots, g_n) := f(e, g_1, g_1 g_2, \dots, g_1 \cdots g_n)$$

is a chain isomorphism with degree 0. If $n = 0$, we shall take $\Psi_0 = \text{id}_A : A \to A$.

Proof. We shall prove the bijectivity by constructing an inverse of this chain map. Consider

$$\Phi_n : C^n_{\text{inh}}(G, A) \to C^n_{\text{hom}}(G, A)$$

defined by $\Phi_n f(g_0, \dots, g_n) := g_0 \cdot f(g_0^{-1} g_1, g_1^{-1} g_2, \dots, g_{n-1}^{-1} g_n)$. First, we show that $\Phi_n f \in C^n_{\text{hom}}(G, A)$. $\forall h \in G$

$$\begin{aligned}\Phi_n f(hg_1, \dots, hg_{n+1}) &= hg_1 \cdot f\big((hg_1)^{-1} hg_2, \dots, (hg_n)^{-1} hg_{n+1}\big) \\ &= hg_1 \cdot f(g_1^{-1} g_1, \dots, g_n^{-1} g_{n+1}) = h \cdot \Phi_n f(g_1, \dots, g_{n+1})\end{aligned}$$

This ensures the well-definedness of this map.

Then, we shall show the bijectivity by showing that $\Phi_n = (\Psi_n)^{-1}$, $\forall f \in C^n_{\text{inh}}(G, A)$

$$\begin{aligned}(\Phi_n \circ \Psi_n f)(g_1, \dots, g_n) &= (\Phi_n f)(e, g_1, g_1 g_2, \dots, g_1 \cdots g_n) \\ &= e \cdot f\big(g_1, g_1^{-1} g_1 g_2, \dots, (g_1 \cdots g_{n-1})^{-1} g_1 \cdots g_n\big) \\ &= f(g_1, \dots, g_n)\end{aligned}$$

And also, $\forall f \in C^n_{\text{hom}}(G, A)$

$$\begin{aligned}(\Psi_n \circ \Phi_n f)(g_1, \dots, g_{n+1}) &= g_1 \cdot (\Psi_n f)(g_1^{-1} g_2, \dots, g_n^{-1} g_{n+1}) \\ &= g_1 \cdot f(e, g_1^{-1} g_2, \dots, g_1^{-1} g_{n+1}) = f(g_1, g_2, \dots, g_{n+1})\end{aligned}$$

Thus, $\Phi_n \circ \Psi_n = \text{id}_{C^n_{\text{hom}}(G,A)}$ and $\Phi_n \circ \Psi_n = \text{id}_{C^n_{\text{inh}}(G,A)}$,

Finally, we need to show that it is a chain map, i.e. $\Psi_{n+1} \circ d^n = \delta^n \circ \Psi_n$. $\forall f \in C^n_{\text{hom}}(G, A)$

$$\begin{aligned}(\delta^n \Psi_n f)(g_1, \dots, g_{n+1}) = g_1 \cdot \Psi_n f(g_2, \dots, g_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i \Psi_n f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} \Psi_n f(g_1, \dots, g_{n+1})\end{aligned}$$

We can expand this in terms: (in the following description, $\widehat{h_i}$ means remove h_i.)

1. The first term is given by $g_1 \cdot \Psi_n f(g_2, \dots, g_{n+1}) = g_1 \cdot f(e, g_2, g_2 g_3, \dots, g_2 \cdots g_{n+1}) = f(g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_{n+1})$, we shall defined that $h_k = \prod_{i=1}^{k} g_i$ and $h_0 = e$, then $g_1 \cdot \Psi_n f(g_2, \dots, g_{n+1}) = f(h_0, \dots, h_n)$.

2. For the summation term,

$$\Psi_n f(g_0, \dots, g_i g_{i+1}, \dots, g_{n+1}) = \sum_{i=0}^{n} (-1)^i f(h_0, \dots, \widehat{h_i}, \dots, h_{n+1})$$

3. $(-1)^{n+1}\Psi_n f(g_0, \dots, g_n) = (-1)^{n+1} f(h_0, \dots, h_n)$

Thus, $(\delta^n \Psi_n f)(g_0, \dots, g_n) = \sum_{j=1}^{n} (-1)^j f(h_0, \dots, \widehat{h_j}, \dots, h_n)$. Then, we shall compute $(\Psi_{n+1} \circ d^n) f$, by the definition of homogeneous chain

$$d^n f(g_0, \dots, g_{n+1}) := \sum_{j=0}^{n+1} f(g_0, \dots, \widehat{g_j}, \dots, g_{n+1})$$

Then, simply apply the definition of the chain map:

$$\begin{aligned} \Psi_{n+1}(d^n f)(g_1, \dots, g_{n+1}) &= (d^n f)(e, g_1, g_1 g_2, \dots, g_1 \cdots g_{n+1}) \\ &= (d^n f)(h_0, \dots, h_{n+1}) = \sum_{j=0}^{n+1} f(h_0, \dots, \widehat{h_j}, \dots, h_{n+1}) \end{aligned}$$

Thus, Ψ_n is indeed a chain map.

In this way, we show that $C^n_{\text{inh}}(G, A) \cong C^n_{\text{hom}}(G, A)$. □

A corollary is that the inhomogeneous chain complex $(C^\bullet_{\text{inh}}(G, A), \delta^n)$ is a cochain complex since it is isomorphic to the homogeneous cochain complex. Since these two cochain complexes are isomorphic, the cohomology group should also be isomorphic:

$$H^n_{\text{hom}}(G, A) \cong H^n_{\text{inh}}(G, A)$$

In particular, we are interested in the 1-cocycle and 2-cocycle:

1. The 1-cocycle $\ker \delta^1$ is asking for the function $f : G \to A$ to satisfies $\forall g_1, g_2 \in G$

$$\delta^1(f)(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1) = 0$$

   And the 1-coboundary is given by $\text{im}\delta^0$ is just $f : G \to A$ such that

$$f(g) = g \cdot a - a$$

2. The 2-cocycle $\ker \delta^2$ is given by the function $f : G^2 \to A$ such that

$$\delta^2(f)(g_1, g_2, g_3) = g_1 \cdot f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = 0$$

   The 2-coboundary $\text{im}\delta^1$, i.e. $\omega : G^2 \to A$ such that $\exists f \in \text{Map}(G, A)$

$$\omega(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1)$$

4 Projective Representation

The goal of this section is to formulate the "phase ambiguity" of quantum mechanics as a precise *lifting problem.* Concretely, we start from a homomorphism $G \to \text{PU}(\mathcal{H})$ and ask when it admits a lift to an honest homomorphism $G \to \text{U}(\mathcal{H})$.

4.1 Projective Linear and Unitary Groups

Definition 4.1 (Projective General Linear Group)**.** Let V be a complex vector space. The *projective general linear group* is

$$\mathrm{PGL}(V) := \mathrm{GL}(V)/\mathbb{C}^\times$$

where $A \sim B$ iff $A = \lambda B$ for some $\lambda \in \mathbb{C}^\times$. Denote by $\pi : \mathrm{GL}(V) \twoheadrightarrow \mathrm{PGL}(V)$ the quotient map.

Definition 4.2 (Projective Unitary Group)**.** Let $\mathcal{H}$ be a complex Hilbert space. The *projective unitary group* is

$$\mathrm{PU}(\mathcal{H}) := \mathrm{U}(\mathcal{H})/\,\mathrm{U}(1)$$

where $U_1 \sim U_2$ iff $U_1 = \lambda U_2$ for some $\lambda \in \mathrm{U}(1)$. We denote again by $\pi : \mathrm{U}(\mathcal{H}) \twoheadrightarrow \mathrm{PU}(\mathcal{H})$ the quotient map.

4.2 Projective Representations and Multipliers

Definition 4.3 (Projective representation)**.** Let G be a group. A *projective representation* of G on a complex vector space V is a group homomorphism

$$\bar{\rho} : G \longrightarrow \mathrm{PGL}(V)$$

If $V = \mathcal{H}$ is a Hilbert space and $\bar{\rho} : G \to \mathrm{PU}(\mathcal{H})$, we call it a *projective unitary representation.*

Remark. *A projective unitary representation does* not *choose a unitary operator for each group element uniquely; it only chooses it up to multiplication by a phase in* $\mathrm{U}(1)$*. This is exactly the "global phase" redundancy in quantum mechanics.*

Definition 4.4 (Lift and multiplier)**.** Let $\bar{\rho} : G \to \mathrm{PU}(\mathcal{H})$ be a projective unitary representation. A (set-theoretic) *lift* of $\bar{\rho}$ is a map $\rho : G \to \mathrm{U}(\mathcal{H})$ such that $\pi \circ \rho = \bar{\rho}$ and $\rho(e) = \mathbf{1}_{\mathcal{H}}$.
Given a lift ρ, define its *multiplier* (or *factor system*) $\omega_\rho : G \times G \to \mathrm{U}(1)$ by

$$\rho(g)\rho(h) = \omega_\rho(g,h)\,\rho(gh), \qquad g,h \in G \tag{4.1}$$

Proposition 4.1 (Normalization and cocycle condition)**.** *Let ρ be a lift of $\bar{\rho}$ with multiplier ω_ρ. Then:*

1. *(Normalization) $\omega_\rho(e,g) = \omega_\rho(g,e) = 1$ for all $g \in G$.*
2. *(2-cocycle condition)*

$$\omega_\rho(g,h)\,\omega_\rho(gh,k) = \omega_\rho(h,k)\,\omega_\rho(g,hk), \qquad \forall g,h,k \in G. \tag{4.2}$$

Proof. Normalization follows by setting $g = e$ or $h = e$ in (4.1) and using $\rho(e) = \mathbf{1}$.

For the cocycle condition, use associativity in $\mathrm{U}(\mathcal{H})$:

$$(\rho(g)\rho(h))\rho(k) = \rho(g)(\rho(h)\rho(k)).$$

Insert (4.1) twice:

$$\omega_\rho(g,h)\rho(gh)\rho(k) = \omega_\rho(h,k)\rho(g)\rho(hk).$$

Apply (4.1) again to $\rho(gh)\rho(k)$ and $\rho(g)\rho(hk)$, and cancel $\rho(ghk)$ on both sides. This yields (4.2). ☐

Definition 4.5 (Gauge change of lifts and cohomology class)**.** Let ρ be a lift of $\bar{\rho}$. For any map $\beta : G \to \mathrm{U}(1)$ with $\beta(e) = 1$, define a new lift

$$\rho^{\beta}(g) := \beta(g)\rho(g)$$

We call this a *phase redefinition* (or *gauge change*) of the lift.

Proposition 4.2 (Coboundary transformation law)**.** *Let ρ be a lift with multiplier ω_ρ, and let ρ^β be as in Definition 4.5. Then the new multiplier is*

$$\omega_{\rho^\beta}(g, h) = \beta(g)\beta(h)\beta(gh)^{-1}\,\omega_\rho(g, h). \tag{4.3}$$

In particular, ω_{ρ^β} and ω_ρ differ by a 2-coboundary.

Proof. Compute:

$$\rho^\beta(g)\rho^\beta(h) = \beta(g)\rho(g)\,\beta(h)\rho(h) = \beta(g)\beta(h)\,\omega_\rho(g, h)\rho(gh) = \Big(\beta(g)\beta(h)\beta(gh)^{-1}\omega_\rho(g, h)\Big)\rho^\beta(gh).$$

Comparing with the defining equation of the multiplier gives (4.3). □

Definition 4.6 (Multiplier class)**.** Let $\bar{\rho} : G \to \mathrm{PU}(\mathcal{H})$ be a projective unitary representation. Choose any lift ρ. By Proposition 4.1, ω_ρ is a normalized 2-cocycle in $Z^2(G, \mathrm{U}(1))$ (with trivial G-action on $\mathrm{U}(1)$), and by Proposition 4.2, changing the lift changes ω_ρ by a coboundary. Hence, the cohomology class

$$[\omega_{\bar{\rho}}] \;:=\; [\omega_\rho] \;\in\; H^2(G, \mathrm{U}(1))$$

is well-defined; we call it the *multiplier class* of $\bar{\rho}$.

4.3 The Lifting Problem

Theorem 4.7 (Lifting criterion)**.** Let $\bar{\rho} : G \to \mathrm{PU}(\mathcal{H})$ be a projective unitary representation. The following are equivalent:

1. There exists a (unitary) representation $\tilde{\rho} : G \to \mathrm{U}(\mathcal{H})$ such that $\pi \circ \tilde{\rho} = \bar{\rho}$.
2. For some (equivalently, any) lift ρ, the multiplier ω_ρ is a 2-coboundary: there exists $\beta : G \to \mathrm{U}(1)$ with $\beta(e) = 1$ such that
$$\omega_\rho(g, h) = \beta(g)^{-1}\beta(h)^{-1}\beta(gh) \quad \text{for all } g, h \in G.$$
3. The multiplier class vanishes:
$$[\omega_{\bar{\rho}}] = 0 \in H^2(G, \mathrm{U}(1))$$

Moreover, if $[\omega_{\bar{\rho}}] = 0$ and β is as in (2), then $\tilde{\rho}(g) := \beta(g)^{-1}\rho(g)$ is a genuine unitary representation lifting $\bar{\rho}$.

Proof. (1)⇒(2): if $\tilde{\rho}$ is a genuine representation, then $\tilde{\rho}(g)\tilde{\rho}(h) = \tilde{\rho}(gh)$, so its multiplier is identically 1. Any other lift differs by a phase, hence has a multiplier of a coboundary.

(2)⇒(3): being a 2-coboundary means exactly that the cohomology class is 0.

(3)⇒(1): if $[\omega_\rho] = 0$, then $\omega_\rho = \delta\beta^{-1}$ for some β, i.e. $\omega_{\rho^\beta} \equiv 1$ by Proposition 4.2. Then ρ^β is a genuine representation lifting $\bar{\rho}$. □

Remark. *If one writes* $\omega_\rho(g,h) = \exp(i\theta(g,h))$ *and* $\beta(g) = \exp(i\xi(g))$*, then* (4.3) *becomes exactly Weinberg's "phase redefinition"*

$$\theta \longmapsto \theta + \xi(g) + \xi(h) - \xi(gh) \pmod{2\pi}.$$

Thus the physicist's condition "can we remove the phase by redefining $\rho(g)$*?" is precisely the question whether the* 2*-cocycle is cohomologically trivial.*

5 The Most General Approach: Central Extension via Group Cohomology

In Section 4 we associated to each projective unitary representation $\bar{\rho} : G \to \mathrm{PU}(\mathcal{H})$ a cohomology class $[\omega_{\bar{\rho}}] \in H^2(G, \mathrm{U}(1))$. In this section we explain why $H^2(G, A)$ is *the natural home* of the obstruction: it classifies group extensions, and projective representations are precisely linear representations of certain *central extensions.*

5.1 Extensions and central extensions

Throughout, let A be an abelian group (written additively) equipped with a left action of G by automorphisms, denoted $g \cdot a$.

Definition 5.1 (Group Extension)**.** An *extension* of G by A is a short exact sequence of groups

$$1 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ p\ } G \longrightarrow 1 \tag{5.1}$$

such that conjugation in E induces the given G-action on A: for $g \in G$ and $a \in A$,

$$g \cdot a \ := \ \iota^{-1}\big(s(g)\,\iota(a)\,s(g)^{-1}\big)$$

where $s : G \to E$ is any set-theoretic section of p (this is independent of s). Two extensions E and E' are *equivalent* if there exists a group isomorphism $\Phi : E \to E'$ commuting with ι and p.

Definition 5.2 (Central Extension)**.** An extension (5.1) is called *central* if $\iota(A) \subseteq Z(E)$. Equivalently, the induced G-action on A is trivial: $g \cdot a = a$ for all $g \in G$, $a \in A$.

5.2 Classification of Extensions

Proposition 5.1 (Factor set from a section)**.** *Let* $1 \to A \xrightarrow{\iota} E \xrightarrow{p} G \to 1$ *be an extension, and let* $s : G \to E$ *be a set-theoretic section with* $s(e) = e$*. Define* $\omega_s : G \times G \to A$ *by*

$$\iota(\omega_s(g,h)) := s(g)s(h)s(gh)^{-1}. \tag{5.2}$$

Then ω_s *is a normalized* 2*-cocycle in the inhomogeneous complex:*

1. $\omega_s(e,g) = \omega_s(g,e) = 0$ *for all* g*.*
2. $\delta^2\omega_s = 0$*, i.e.*

$$g \cdot \omega_s(h,k) - \omega_s(gh,k) + \omega_s(g,hk) - \omega_s(g,h) = 0. \tag{5.3}$$

Moreover, if $s'(g) = \iota(b(g))s(g)$ *for some* $b : G \to A$ *with* $b(e) = 0$*, then*

$$\omega_{s'} = \omega_s + \delta^1 b,$$

so the cohomology class $[\omega_s] \in H^2(G, A)$ *is independent of the choice of section.*

Proof. Normalization is immediate from $s(e) = e$.

For the cocycle condition, compare $(s(g)s(h))s(k)$ with $s(g)(s(h)s(k))$. Using (5.2) twice gives

$$s(g)s(h) = \iota(\omega_s(g, h))s(gh), \qquad s(h)s(k) = \iota(\omega_s(h, k))s(hk).$$

Then

$$\begin{aligned}(s(g)s(h))s(k) &= \iota(\omega_s(g, h))\, s(gh)s(k) = \iota(\omega_s(g, h))\, \iota(\omega_s(gh, k))\, s(ghk),\\ s(g)(s(h)s(k)) &= s(g)\, \iota(\omega_s(h, k))\, s(hk) = \iota(g \cdot \omega_s(h, k))\, s(g)s(hk)\\ &= \iota(g \cdot \omega_s(h, k))\, \iota(\omega_s(g, hk))\, s(ghk).\end{aligned}$$

Cancel $s(ghk)$ and apply ι^{-1} to obtain (5.3).

If $s'(g) = \iota(b(g))s(g)$, then a direct substitution into (5.2) yields $\omega_{s'} = \omega_s + \delta^1 b$. ☐

Proposition 5.2 (Twisted product extension)**.** *Let* $\omega \in Z^2(G, A)$ *be a normalized* 2*-cocycle. Define a multiplication on the set* $E_\omega := A \times G$ *by*

$$(a, g) \cdot (b, h) := \big(a + g \cdot b + \omega(g, h),\ gh\big). \tag{5.4}$$

Then E_ω *is a group with identity* $(0, e)$*. Moreover,*

$$1 \longrightarrow A \xrightarrow{\ i\ } E \xrightarrow{\ p\ } G \longrightarrow 1\ , \qquad \iota(a) = (a, e), \quad p(a, g) = g,$$

is an extension of G *by* A*, and its extension class in* $H^2(G, A)$ *is* $[\omega]$*.*

Proof. The only nontrivial point is associativity. Compute

$$\big((a, g) \cdot (b, h)\big) \cdot (c, k) = \Big(a + g \cdot b + \omega(g, h) + (gh) \cdot c + \omega(gh, k),\ ghk\Big),$$

and

$$(a, g) \cdot \big((b, h) \cdot (c, k)\big) = \Big(a + g \cdot (b + h \cdot c + \omega(h, k)) + \omega(g, hk),\ ghk\Big).$$

These coincide exactly because $\delta^2\omega = 0$. The rest is routine: identity and inverses can be written down explicitly, and exactness is clear. ☐

> **Theorem 5.3** (Extensions are Classified by H^2)**.** Equivalence classes of extensions of G by A (with the prescribed G-action on A) are in bijection with the cohomology group $H^2(G, A)$. Under this bijection, a 2-cocycle ω corresponds to the twisted product extension E_ω in Proposition 5.2.

Proof. Proposition 5.1 assigns to an extension a well-defined class in $H^2(G, A)$. Proposition 5.2 constructs an extension from a cocycle.

These constructions are inverse up to equivalence: if $\omega' = \omega + \delta^1 b$, then $(a, g) \mapsto (a + b(g), g)$ is an isomorphism $E_\omega \to E_{\omega'}$ commuting with ι and p. Conversely, equivalent extensions yield cohomologous factor sets. ☐

5.3 Application: projective unitary representations

Now specialize to the case $A = \mathrm{U}(1)$ with *trivial* G-action. Then extensions are central.

Proposition 5.3 (Central extension associated to a multiplier)**.** *Let $\bar{\rho} : G \to \mathrm{PU}(\mathcal{H})$ be a projective unitary representation and choose a lift ρ. Let $\omega = \omega_\rho \in Z^2(G, \mathrm{U}(1))$ be the multiplier. Define the group $\widetilde{G}_\omega := \mathrm{U}(1) \times G$ with multiplication*

$$(z, g) \cdot (w, h) := \big(zw\,\omega(g, h),\ gh\big) \tag{5.5}$$

Then $\widetilde{G}_\omega$ is a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widetilde{G}_\omega \xrightarrow{\ p\ } G \longrightarrow 1$$

and its extension class is exactly $[\omega_{\bar{\rho}}] \in H^2(G, \mathrm{U}(1))$.

Proof. Associativity of (5.5) is equivalent to the 2-cocycle condition (4.2) for ω. Centrality holds because $\mathrm{U}(1)$ is in the first factor. Changing the lift changes ω by a 2-coboundary, which yields an equivalent extension by Theorem 5.3. □

Theorem 5.4 (Projective representations are linear representations of a central extension)**.** Let $\bar{\rho} : G \to \mathrm{PU}(\mathcal{H})$ be a projective unitary representation with multiplier cocycle ω. Then:

1. (Linearization on the extension) The map

$$\widetilde{\rho} : \widetilde{G}_\omega \to \mathrm{U}(\mathcal{H}), \qquad \widetilde{\rho}(z, g) := z\rho(g)$$

   is a genuine unitary representation of the *central extension* $\widetilde{G}_\omega$.

2. (Obstruction = splitting) The projective representation $\bar{\rho}$ lifts to a genuine representation of G iff the central extension $\widetilde{G}_\omega$ splits (admits a group homomorphic section), iff $[\omega] = 0$ in $H^2(G, \mathrm{U}(1))$.

Proof. (1) Using (4.1):

$$\widetilde{\rho}(z, g)\widetilde{\rho}(w, h) = zw\,\rho(g)\rho(h) = zw\,\omega(g, h)\rho(gh) = \widetilde{\rho}(zw\omega(g, h), gh) = \widetilde{\rho}\big((z, g) \cdot (w, h)\big).$$

Thus $\widetilde{\rho}$ is a homomorphism.

(2) A lift $G \to \mathrm{U}(\mathcal{H})$ is exactly a homomorphic section of the pullback extension; by Theorem 4.7 this is equivalent to $[\omega] = 0$. □

In the second part of this note, we are going to study in detail the group cohomology on Lie groups ecpecially its relation with Lie algebra cohomology and a transgression map to the de Rham cohomology.

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