Homotopy Proof of Fundamental Theorem of Algebra

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The fundamental theorem of algebra states that:

Any non-constant polynomial $p \in \mathbb{C}[x]$ has at least one root.

We shall prove the theorem by homotopy.

Notation and Setup. Let the nonconstant complex polynomial be:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad n \geq 1$$

and $S^1_r := \{z \in \mathbb{C} \mid |z| = r\}$ be the circle with radius $r \in \mathbb{R}_{\geq 0}$.

We also need two fundamental facts on topology:

1. $\pi_1(\mathbb{C}) = \{\mathrm{id}\}$
2. $\pi_1(\mathbb{C} \setminus \{0\}) \cong \pi_1(S^1) = \mathbb{Z}$

Proof: We shall prove the theorem by contradiction. Suppose $p(z)$ has no zero, i.e. $p : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. Then p naturally induces a homomorphism of the fundamental group, since $\pi_1(\mathbb{C}) = \{1\}$

$$\begin{aligned} p_* : \{\mathrm{id}\} &\to \pi_1(\mathbb{C} \setminus \{0\}) \\ \mathrm{id}_{\mathbb{C}} &\mapsto \mathrm{id}_{\mathbb{C} \setminus \{0\}} \end{aligned}$$

To compute the pushforward of the fundamental group explicitly, rewrite the polynomial as

$$p(z) = a_n z^n + q(z)$$

and the degree (as a polynomial) of q is less than or equal to $n - 1$. With $z \in S^1_R$ for some R, there is estimation

$$|q(z)| \leq \sum_{k=0}^{n-1} |a_k| R^k$$

Thus, on S^1_R

$$\frac{|q(z)|}{|a_n z^n|} = \frac{|q(z)|}{|a_n|R} \leq \left|\frac{a_{n-1}}{a_n}\right| \frac{1}{R^n} + \cdots + \left|\frac{a_1}{a_n}\right| \frac{1}{R^{n-1}} + \left|\frac{a_0}{a_n}\right| \frac{1}{R^n}$$

As $R \to \infty$, the RHS has limit goes to zero, so $\exists R_0 \in \mathbb{R}_{\geq 0}$ such that $\forall r > R_0$, $|z| = r$

$$\frac{|q(z)|}{|a_n z^n|} \leq \frac{1}{2} \implies |q(z)| < |a_n z^n| \ \forall z \in S^1_r$$

With the inequality above, consider the polynomial restricted to S^1_r:

$$\begin{aligned} p|_{S^1_r} : S^1_r &\to \mathbb{C} \setminus \{0\} \\ z &\mapsto a_n z^n + q(z) \end{aligned}$$

and the polynomial with only the highest order term:

$$\begin{aligned} g : S^1_R &\to \mathbb{C} \setminus \{0\} \\ z &\mapsto a_n z^n \end{aligned}$$

One shall define the following homotopy betweem $p|_{S^1_r}(z)$ and $g(z) = a_n z^n$:

$$\begin{aligned} H : [0,1] \times S^1_r &\to \mathbb{C} \\ (t, z) &\mapsto a_n z^n + tq(z) \end{aligned}$$

For any $(t, z) \in [0,1] \times S^1_r$, there is $|tq(z)| \leq |q(z)| < |a_n z^n|$, and thus

$$|H(t,z)| = |a_n z^n + tq(z)| \geq |a_n z^n| - |tq(z)| > 0$$

Thus, $\forall (t,z) \in [0,1] \times S^1_r$, $|H(t,z)| > 0$, i.e., $H : [0,1] \times S^1_r \to \mathbb{C} \setminus \{0\}$. Also,

$$H(0,z) = a_n z^n, \quad H(1,z) = p(z)$$

Which means $H : g \simeq p|_{S^1_r}$, or, if we consider the inclution map $j : S^1_r \hookrightarrow \mathbb{C} \setminus \{0\}$, we have $H : p \circ j \simeq g$. By the homotopy, we know that the induced pushforward on the fundamental group is given by

$$(p \circ j)_* = g_* : \pi_1(S^1) \to \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$$

To compute the homomorphism induced by g, we shall consider the following homotopy equivalence:

$$h_1 : S^1_r \to S^1, \quad h_1(z) = \frac{z}{r}$$
$$h_2 : \mathbb{C}\setminus\{0\} \to S^1, \quad h_2(z) = \frac{z}{|z|}$$

And the composition $\tilde{g} = h_2 \circ g \circ h_1^{-1} : S^1 \to S^1$, which $\forall z \in S^1$

$$h_2 \circ g \circ h_1^{-1}(z) = \frac{a_n}{|a_n|} \mathrm{e}^{2\pi \mathrm{i} n\theta} = \mathrm{e}^{\mathrm{i}(\alpha + 2\pi n\theta)}$$

where $\alpha := \deg a_n$ is some real number and $g_* = \tilde{g}_*$. Thus, $\tilde{g}$ is given by $\tilde{g}(z) = \mathrm{e}^{\mathrm{i}\alpha} z^n$. Recall that the universal cover of S^1 is $\mathbb{R}$, with covering map $\exp : \mathbb{R} \to S^1$, $t \mapsto \mathrm{e}^{2\pi \mathrm{i}t}$ and the homotopy lifting problem shows that for continuous map $\tilde{g} : S^1 \to S^1$, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\hat{g}"] \arrow[d, "\exp"'] & \mathbb{R} \arrow[d, "\exp"] \\
S^1 \arrow[r, "\tilde{g}"] & S^1
\end{tikzcd}$$

From the definition of $\tilde{g}$, we shall find the lifting on convering space $\hat{g} : \mathbb{R} \to \mathbb{R}$ given by $\forall t \in \mathbb{R}$

$$\hat{g}(t) := \frac{\alpha}{2\pi} + nt$$

We shall redefine the polynomial such that $\alpha = 0$, i.e., $\hat{g}(t) = nt$. Thus, any non-trivial loop $\gamma : [0,1] \to S^1$ under the group homomorphism will transform to

$$\tilde{g}_*([\gamma]) = [\tilde{g} \circ \gamma], \quad \tilde{g}(1) = 1$$

And consider the homomorphism $\phi : \pi(S^1) \to \mathbb{Z}$ such that for $\gamma(t) = \mathrm{e}^{2\pi k \mathrm{i} t}$, $\phi : [\gamma] \mapsto k$, for some lifted path $\tilde{\gamma} : I \to \mathbb{R}$,

$$\begin{aligned}
\phi \circ \tilde{g}_*(p_*[\gamma]) &= \phi([\tilde{g} \circ \exp \circ \tilde{\gamma}]) = \phi([\exp \circ \hat{g} \circ \tilde{\gamma}]) \\
&= n \in \mathbb{Z} \quad \text{since } \exp(\hat{g} \circ \tilde{\gamma}) = \mathrm{e}^{2\pi n i t}
\end{aligned}$$

This is exactly the degree of the polynomial. However, in another way, consider $S_r^1 \overset{j}{\hookrightarrow} \mathbb{C}$, we have the following diagram commutes:

$$\begin{tikzcd}
S_r^1 \arrow[r, hook, "j"] \arrow[d, dashed, "\pi_1"'] & \mathbb{C} \arrow[r, "p"] \arrow[d, dashed, "\pi_1"'] & \mathbb{C} \setminus \{0\} \arrow[d, dashed, "\pi_1"'] \\
\pi_1(S^1) \cong \mathbb{Z} \arrow[r, "j_*"] \arrow[rr, bend right, "(p\circ j)_* = p_* \circ j_*"'] & \pi_1(\mathbb{C}) \cong \{1\} \arrow[r, "p_*"] & \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}
\end{tikzcd}$$

Thus, the composition

$$(p \circ j)_* = p_* \circ j_* = g_* = 0$$

which means $g_*([\gamma]) = n[\gamma] = 0$ and thus, $n = 0$, contradicts with the setup that p is nonconstant. $\square$