University of Illinois at Urbana-Champaign

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Amid the Cloud-Sea

Notes on Geometry and Topology

从微分流形开始的异世界魔法笔记

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Introduction

Historical Survey and Scope

Modern geometry and topology can be read as a long commentary on two early gestures. The first is Euler's decision, in the problem of the bridges of Königsberg, to forget distances and angles and look only at how things are connected [15]. The second is Gauss's insistence that curvature is an intrinsic feature of a surface, invisible to any attempt to embed it in a higher-dimensional Euclidean space [18]. Euler's later work on polyhedra [16] distilled topology into the integer

$$\chi = V - E + F,$$

while Gauss's *Theorema Egregium* revealed that χ and curvature are two faces of the same phenomenon. From these two moves—forgetting length, and internalizing curvature—the modern subject begins.

Riemann's 1854 habilitation lecture [40] lifted Gauss's ideas from surfaces to abstract manifolds. By allowing a smoothly varying inner product g on each tangent space, he introduced the notion of a Riemannian manifold and suggested that geometry is not a rigid Euclidean background, but rather a dynamical field. This point of view lies behind both general relativity and the basic objects studied in these notes: smooth manifolds equipped with metrics, forms, and connections.

The first recognizably modern topology appears in Poincaré's *Analysis Situs* [36]. There, the fundamental group, homology groups, and early versions of cohomology are introduced, together with the idea that spaces should be probed by maps into and out of them. Hurewicz clarified the bridge between homotopy and homology [25], while Hopf's study of maps between spheres revealed the power of algebraic invariants such as linking numbers and Hopf fibrations [23]. What began as a combinatorial shadow of geometry became a subject in its own right: algebraic topology.

A decisive analytical turn came with de Rham's reinterpretation of cohomology in terms of differential forms [39]. By integrating closed forms over cycles and proving the de Rham theorem, he showed that topological invariants can be computed by solving differential equations. Leray's work on sheaves and spectral sequences [28] provided a conceptual framework for cohomology and for the Čech–de Rham complexes that appear later in these notes. From this point on, geometry, topology, and analysis cease to be separate disciplines: they share the same language.

The technical apparatus that makes smooth manifolds workable was assembled in the mid-twentieth century. Whitney constructed the general theory of differentiable manifolds and embedded them into Euclidean space [48, 49], showing that, from a distance, all smooth manifolds are submanifolds of some $\mathbb{R}^N$. Sard's theorem [41] explains why generic smooth maps have well-behaved critical sets, and Thom's transversality theory [46] turns genericity into a tool: intersection numbers and the Euler class can be defined and computed in a stable way. Ehresmann's work on fiber bundles and connections [12] supplies the geometric background for tubular neighborhoods, foliations, and many of the constructions in Part I of these notes.

Riemannian geometry, already implicit in Riemann's lecture, acquired a precise analytic form in Levi-Civita's theory of parallel transport and covariant differentiation [29]. Hopf and Rinow related metric completeness to geodesic completeness [24], while Myers and the comparison theorems of Rauch and others [34, 38] connected curvature bounds to global geometric and topological information. Geodesics, Jacobi fields, and comparison inequalities—treated at length in these notes—are the probes with which one tests the large-scale shape of a manifold.

Lie groups and homogeneous spaces form a second, algebraic laboratory for geometry. Cartan's method of moving frames and his analysis of symmetric spaces [6, 7] describe curvature, torsion, and symmetry in terms of differential forms on principal bundles. This language feeds directly into the treatment of Lie groups, homogeneous spaces, and symmetric spaces that appears in the later chapters on Riemannian geometry and on connections on principal bundles.

The synthesis of analysis, topology, and algebra reaches a particularly transparent form in Hodge theory and in Chern–Weil theory. Hodge's work on harmonic forms on compact Kähler manifolds [22] produces canonical representatives of cohomology classes. Chern's intrinsic proof of the Gauss–Bonnet theorem [8] and his joint work with Weil on characteristic classes [9] show that curvature forms can be integrated to yield topological invariants. Pontryagin's characteristic classes [37] and Hirzebruch's signature theorem [21] deepen this correspondence between curvature and topology and provide a primary motivation for the treatment of characteristic classes and the Gauss–Bonnet–Chern theorem in these notes.

In parallel, algebraic topology was axiomatized by Eilenberg and Mac Lane [13] and systematized by Eilenberg and Steenrod [14]. Their functorial language and axioms for homology and cohomology still underlie the constructions in modern topology, from the fundamental group and covering spaces to generalized cohomology theories. The algebraic-topological interlude in these notes follows this tradition: topology is organized by its invariants and by the natural transformations between them.

At the interface of analysis and topology lies Morse theory. Morse's original work on the relation between critical points of smooth functions and the topology of level sets [33] was recast by Milnor into the now-standard language of handle decompositions and cell attachments [32]. In these notes, Morse functions appear both as tools for understanding the topology of manifolds and as a conceptual bridge to infinite-dimensional variants such as Floer homology.

The arrival of gauge theory transformed four-dimensional topology. Yang and Mills introduced non-abelian gauge fields into physics [51], and Atiyah and Bott analyzed the Yang–Mills equations on Riemann surfaces via Morse theory on infinite-dimensional spaces of connections [3]. Donaldson used moduli spaces of anti-self-dual connections to extract new invariants of smooth four-manifolds, revealing unexpected rigidity in dimension four [11]. The Seiberg–Witten equations [43] provided a more flexible gauge-theoretic framework that remains central to the study of smooth four-manifolds; the gauge-theoretic material in these notes is written with this story in the background.

Symplectic geometry grew out of Hamiltonian mechanics and early work of Poincaré on dynamical systems [35]. In the twentieth century, Arnold reinterpreted classical mechanics in symplectic terms and tied it to topological invariants [1]. Gromov's introduction of pseudo-holomorphic curves [20] and Floer's infinite-dimensional Morse theory for Lagrangian intersections [17] turned symplectic rigidity into a rich homological theory. Kontsevich's deformation quantization theorem [27] makes precise the passage from Poisson geometry to noncommutative algebras and informs the discussion of quantization in the symplectic chapter.

Complex geometry weaves into this picture through Kodaira's work on compact complex and Kähler manifolds [26] and Calabi's conjectures on Ricci-flat Kähler metrics [5], resolved by Yau's solution of the complex Monge–Ampère equation and the Calabi–Yau theorem [52]. These developments silently underlie many of the examples in the sections on complex and Kähler geometry.

Metric geometry and noncommutative geometry push the notion of "space" still further. Gromov's metric viewpoint on groups and manifolds [19] provides tools for studying collapse, limits, and rigidity beyond the smooth category. Connes' noncommutative geometry [10] replaces spaces with operator algebras and recovers topological invariants via cyclic cohomology and index theory, offering a natural home for examples arising in representation theory and quantum physics. The brief discussion of noncommutative spaces in these notes should be read against this backdrop.

Finally, topology returns to physics in the guise of topological quantum field theory. Atiyah formulated axioms for TQFT as functors from bordism categories to tensor categories [2], while Witten constructed concrete models whose partition functions and correlation functions reproduce classical invariants such as Donaldson and

Jones polynomials [50]. Segal's definition of conformal field theory [42] and the higher-categorical perspectives of Baez–Dolan and Lurie [4, 30] articulate a functorial viewpoint that interacts naturally with higher categories and homotopy-theoretic ideas. The final chapter of these notes offers only a sketch of this landscape, but it points toward the categorical and quantum horizons of contemporary geometry and topology.

These notes are organized with this historical arc in mind. Part I develops smooth manifolds, Lie groups, and the analytic tools (partitions of unity, Sard's theorem, Whitney embedding, tubular neighborhoods, transversality) needed to do calculus on manifolds. Part II turns to fundamental groups, covering spaces, and Riemannian geometry, emphasizing geodesics and curvature. Part III introduces fiber bundles, connections, and characteristic classes, then moves toward gauge theory, Morse theory, metric geometry, algebraic topology, noncommutative geometry, complex and symplectic geometry, and finally TQFT. The aim is not to reproduce the original works cited above, but to offer a coherent path through the landscape they opened: from Euler's combinatorial shadows and Gauss's curved surfaces to the analytic, categorical, and quantum structures that shape modern geometry and topology.

Because the author's mathematical training and research experience are still developing, these notes are not intended to serve as a textbook, nor do they claim completeness or optimal exposition. Rather, they are meant to present the author's current understanding, derivations, and intuitions in a form that is open to scrutiny—serving as a stepping stone for communication, discussion, and subsequent refinement.

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Part I

Smooth Manifold and Fundamental Structure

Chapter 1

Smooth Manifolds and Submanifolds

1.1 Topological Manifold and Smooth Structure

The goal of Chapter 1 is to define the central object of modern geometry, the smooth manifold. To define a smooth manifold, we first need to study a more basic case called a topological manifold, which is a special type of topological space that locally resembles Euclidean space.

The key to defining a topological manifold is the following property: locally Euclidean.

Definition 1.1 (Locally Euclidean)**.** A topological space X is said to be locally Euclidean iff $\forall x \in X$, there is some open neighborhood $x \in U \subseteq X$, such that we can find a map $\varphi : U \to V \subseteq \mathbb{R}^n$ that is a homeomorphism.

Definition 1.2 (Topological Manifolds)**.** A topological manifold is an n-dimensional topological space M iff M is second countable, Hausdorff, and locally Euclidean to some fixed $\mathbb{R}^n$. A chart is a pair $(U_\alpha, \varphi_\alpha)$ that $U_\alpha \subseteq M$ is a open subset and $\varphi_\alpha : U_\alpha \to \phi(U_\alpha) \subset \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. The set of charts that can cover the entire manifold M is called an atlas.

From my previous experiences, it seems that it is often more important to have a taste of what is not a topological manifold, so here are some examples:

Example 1.1. *If we consider the set $S = (\mathbb{R} \setminus \{0\}) \cup \{A, B\}$, with the topology defined by the following laws:*

- *In $\mathbb{R} \setminus \{0\}$, take the subset topology inherit from $\mathbb{R}$.*
- *The open set that contains A (or B), for some $c, d \in \mathbb{R}_{\geq 0}$ consider the set $I_A(c,d) = (-c, 0) \cup (0, d) \cup \{A\}$ and the basis is given by $\mathcal{B} = \{I_A(c,d) : c, d \in \mathbb{R}_{\geq 0}\} \cup \{I_B(c,d) : c, d \in \mathbb{R}_{\geq 0}\} \cup \{(a,b) \subseteq \mathbb{R} : 0 \notin (a,b)\}$*

This set is locally Euclidean, second countable, but not Hausdorff.

Proof. It is quite obvious that the set is locally Euclidean, since if we consider $I \subseteq S$ such that $A, B \notin I$, then $I \subseteq \mathbb{R}$. For some open set containing A or B, we just take $(I \setminus \{A, B\}) \subseteq \mathbb{R}$ and take $A, B \mapsto 0$. The continuity of this map is obvious. In addition, the second countability of the topological space S is also easy to prove. However, consider any open neighborhood of A and B, $I_A(a,b)$ and $I_B(c,d)$, then there will always intersect. Let $\alpha := \max\{a, c\}$ and $\beta := \min\{b, d\}$, then $I_A(a,b) \cap I_B(c,d) = (\alpha, 0) \cup (0, \beta)$ always nonempty. Thus, points A and B are not disjoint, and the space S is not Hausdorff. $\square$

Example 1.2. *As a counterexample of a topological space that is locally Euclidean, Hausdorff, but not second-countable, consider the uncountable index set I and the topological space*

$$X = \coprod_{i \in I} S_i := \{(x, i) \mid x \in S^1, i \in I\}$$

with the coproduct topology. This space is not second-countable. Recall that the coproduct has a universal property that the following diagram commutes with $f : Y \to X_1 \sqcup X_2$ is unique:

$$\begin{tikzcd} & Y & \\ X_1 \arrow[ur, "f_1"] \arrow[r, "i_1"'] & X_1 \sqcup X_2 \arrow[u, dashed, "\exists! f"] & X_2 \arrow[ul, "f_2"'] \arrow[l, "i_2"] \end{tikzcd}$$

and the coproduct topology is defined to be the coarsest topology such that i_1, i_2 are continuous and f is continuous if f_1, f_2 are continuous. Then, $U \in X$ is open if $i_k^{-1}(U)$ is open in S^1.

Proof. Suppose X has a topological basis $\exists \mathcal{B}$. For any $k \in I$, S_k^1 denotes the i-th 1-sphere in the coproduct. Then, since $S_k^1 \subseteq X$ such that $p_k \in B_k \subseteq S_k^1$. Notice that $\forall k \neq l : S_k^1 \cap S_l^1 = \emptyset$, thus, for any S_i^1, we can take B_i and each two of the B_i with different indices are disjoint. Then we can take the map $I \to \mathcal{B}$ sends each i to B_i, which means $|B| \geq |I|$ is uncountable. Thus X is not second countable. □

The topological manifold only allows the continuity (C^0) to be defined, which is not enough for calculus. To define calculus on a smooth manifold, which is a generalization of smooth curves and surfaces in Euclidean space, we need to define a smooth structure.

Definition 1.3 (C^k-Structure)**.** A C^k-structure on a manifold is an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ for some indexes set I, and satisfies the following properties:

- $\{U_\alpha \mid \alpha \in I\}$ covers M, $\bigcup_\alpha U_\alpha = M$.
- For any $\alpha, \beta \in I$, $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.
- The collection $\mathcal{A}$ is maximal: $\forall \alpha \in I$, if charts (U, φ) and $(U_\alpha, \varphi_\alpha)$ are either $U \cap U_\alpha = \emptyset$, or $\varphi \circ \varphi_\alpha$ is a diffeomorphism, then $(U, \varphi) \in \mathcal{A}$, i.e., $\mathcal{A}$ does not been properly contained in any other C^k-atlas.

Remark. *An alternative way is that we defined the equivalence relation between C^k-atlas $\mathcal{A}_1$, $\mathcal{A}_2$ such that*

$$\mathcal{A}_1 \sim \mathcal{A}_2 \iff \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \text{ is still a } C^k\text{-atlas.}$$

The smooth structure can be represented by the equivalence class of C^k-atlas $[\mathcal{A}]$.

A smooth manifold $(M, \mathcal{O}_M, \mathcal{A})$ is a topological manifold $(M, \mathcal{O}_M)$ with a smooth structure $\mathcal{A}$ on it. However, a potential problem in constructing the smooth structure requires maximal, which is extremely hard to prove. To deal with this, we have the following proposition.

Proposition 1.1. *Every C^k-atlas is contained in a unique maximal C^k-atlas.*

Proof. Consider the topological manifold $(M, \mathcal{O}_M)$ and a smooth atlas $\mathcal{A}_1 = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. Then, we have to apply the following lemma

Lemma. *Given smoothly compatible charts (U_1, φ_1) and (U_2, φ_2), if a chart (V, ψ) is compatible with one of these two charts, then it is also compatible with the other one.*

Without loss of generality, we can assume the open sets U_1, U_2, and V are not disjoint. The proof of the lemma is simply to consider that the compatible condition implies that $\varphi_2 \circ \varphi_1^{-1}$ and $\varphi_1 \circ \varphi_2^{-1}$ are both smooth. Without loss of generality, consider (V, ψ) smoothly compatible with (U_1, φ_1), then check the transition maps

$$\varphi_2 \circ \psi^{-1} = \varphi_2 \circ (\varphi_1^{-1} \circ \varphi_1) \circ \psi^{-1} = (\varphi_2 \circ \varphi_1^{-1}) \circ (\varphi_1 \circ \psi^{-1})$$

Since both $\varphi_2 \circ \varphi_1^{-1}$, $\varphi_1 \circ \psi^{-1}$ are smooth, $\varphi_2 \circ \psi^{-1}$ is smooth. Same argument also apply for $\psi \circ \varphi_2^{-1}$. Thus, the lemma was proved.
Then, to prove the proposition, take $(U_\alpha, \varphi_\alpha) \in \mathcal{A}_1$, we can defined the atlas $\mathcal{A}$ such that

$$(V, \psi) \in \mathcal{A} \iff \exists (U_\alpha, \varphi_\alpha) \in \mathcal{A}_1 : (U_\alpha, \varphi_\alpha) \text{ smoothly compatible with } (V, \psi)$$

We claim that $\mathcal{A}$ is unique and is a maximum atlas. □

A corollary is that the equivalence relation of the atlas that has been used to define the C^k-structure can also be written as

$$\mathcal{A}_1 \sim \mathcal{A}_2 \iff \mathcal{A}_1, \mathcal{A}_2 \text{ is contained in the same maximum atlas } \mathcal{A}$$

and also, the proposition also shows that to prove the topological manifold M has a C^k-structure, it is enough to find a single smooth atlas $\mathcal{A}$ on M.

As examples of topological and smooth manifolds, consider the following sets:

Example 1.3. *As a topological manifold, $\mathbb{R}^n$ can be constructed by a single chart $(\mathbb{R}^n, \mathrm{id})$, and the smoothness is trivial. However, as a critical counterexample to the uniqueness of the smooth structure, we can take another chart $(\mathbb{R}^n, \varphi)$, where $\varphi(u) = u^3$ is indeed a diffeomorphism. The two smooth structures above are not compatible.*

Proof. Consider the map $\varphi \circ \mathrm{id} = \varphi$ is smooth. However, the inverse map $(\varphi \circ \mathrm{id})^{-1} = \mathrm{id}^{-1} \circ \varphi^{-1} = \varphi^{-1}$, which is defined by $\varphi^{-1}(v) = v^{1/3}$ is not globally smooth since $(\varphi^{-1})'(v) = v^{-2/3}/3$, which is undefined at the origin. □

Example 1.4 (Unite Circle S^1). *A classical example of a smooth manifold is the unit circle S^1, take*

$$S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \cong \{\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in [0, 2\pi]\}$$

As a topological subset of $\mathbb{R}^2$, consider the following chats:

$$U_1 = S^1 \setminus \{(1, 0)\} \cong \{\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in (0, 2\pi)\}$$
$$U_2 = S^1 \setminus \{(-1, 0)\} \cong \{\mathrm{e}^{\mathrm{i}\eta} \mid \eta \in (\pi, 3\pi)\}$$

Then, a defined homeomorphism

$$\varphi_1 : U_1 \to (0, 2\pi), \quad \varphi_2 : U_2 \to (\pi, 3\pi)$$

where φ_1 and φ_2 are homeomorphism. The transformation map is given by

$$\varphi_2 \circ \varphi_1^{-1} : (0, \pi) \cup (\pi, 2\pi) \to (\pi, 2\pi) \cup (2\pi, 3\pi)$$

which is defined by

$$\varphi_2 \circ \varphi_1^{-1}(\theta) = \begin{cases} \theta + 2\pi, & \theta \in (0, \pi) \\ \theta, & \theta \in (\pi, 2\pi) \end{cases}$$

is smooth, and also we can also check $\varphi_1 \circ \varphi_2^{-1}$ in the same way. Thus, $\mathcal{A} = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is a smooth structure, and S^1 is a smooth manifold.

Another important example of a smooth manifold is the product manifold, which is given by the following proposition:

Proposition 1.2 (Product Manifold)**.** *M and N are smooth manifolds with dimension m and n $\iff$ $M \times N$ is smooth manifold, and* $\dim M \times N = m + n$*.*

Proof. The Hausdorff and second countable properties are preserved in the Cartesian product, and the Hausdorff and second countable properties on a superset can be directly extended to a subset; the proof is in Appendix A.
($\Rightarrow$) If M and N are orientable C^∞-manifolds with dimension m and n, then we can take the orientable smooth structures $\mathcal{A}_M = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ and $\mathcal{A}_N = \{(V_\beta, \psi_\beta) \mid \beta \in J\}$. Consider the inherent smooth structure $\mathcal{A}_{M\times N} = \{(U \times V, \varphi) \mid U \in \mathcal{A}_M, V \in \mathcal{A}_N\}$ and the coordinate map is given by

$$\forall (p, q) \in M \times N : \varphi(p, q) = (\phi^1(p), \dots, \phi^m(p), \psi^1(q), \dots, \psi^n(q))$$

Since $\phi : M \to \mathbb{R}^m$ and $\psi : N \to \mathbb{R}^n$ are homomorphisms onto their image, the coordinate map given above on $M \times N$ is a homomorphism onto its image $\varphi(U \times V) = \phi(U) \times \psi(V) \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Then we have to check the smoothness of the manifold, and the transition map is given by

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x) = \big(\phi_\alpha \circ \phi_\beta^{-1}(x^1, \dots, x^m); \psi_\alpha \circ \psi_\beta^{-1}(x^{m+1}, \dots, x^{m+n})\big)$$

By the smoothness of ϕ and ψ, the transition map $\varphi_\alpha \circ \varphi_\beta^{-1}$ is smooth. Thus, $M \times N$ is a smooth manifold with dimension $m + n$.
($\Leftarrow$) In the similar way, $\varphi_\alpha \circ \varphi_\beta^{-1}(x) = (y^1(x), \dots, y^{m+n}(x)) \in C^\infty(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$ indicates that each of its component is a smooth function on $\mathbb{R}^{m+n}$, which leads to the smooth structure on M and N separately. □

The smooth manifold is the generalization of a well-behaved subset of $\mathbb{R}^n$, so it is natural to talk about the smoothness of the map.

Definition 1.4 (C^k-Map)**.** For some smooth manifold M and N of dimension m and n, a continuous map $f : M \to N$ is said to be C^k iff the local coordinate representation of chart (U, φ) on M and (V, ψ) on N

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is C^k as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.

We shall check that the smoothness is well-defined, i.e., independent of the choice of charts in the given smooth structure

Proposition 1.3 (Well-Definedness of Smooth Map)**.** *Given $f \in C^\infty(M, N)$, given smooth atlas $\mathcal{A}$ and $\mathcal{B}$ on M and N. If $\mathcal{A}_1$ and $\mathcal{B}_1$ compatible with $\mathcal{A}$ and $\mathcal{B}$, then f is smooth also under the smooth atlas $\mathcal{A}_1$, $\mathcal{B}_1$.*

The proof of this proposition is to simply notice that $\forall (U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{A}_1$ such that $U \cap V \neq \emptyset$, the compatibility implies that both $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are diffeomorphism. A similar argument also applies to $\mathcal{B}$.

Proposition 1.4 (Composition of C^k Maps)**.** *The composition of finite C^k maps $f_1, \dots, f_n$ is still a C^k-map.*

Proof. Because continuity and differentiability are defined chart-wise and composition is a local operation, it suffices to consider maps between open subsets of Euclidean spaces. Let

$$f : V \subset \mathbb{R}^n \to \mathbb{R}^m, \qquad g : U \subset \mathbb{R}^p \to V$$

1. If f and g are continuous, then for any open set $O \subset \mathbb{R}^m$,

   $$(f \circ g)^{-1}(O) = g^{-1}\big(f^{-1}(O)\big),$$

   which is open because $f^{-1}(O)$ is open and g is continuous. Thus $f \circ g$ is continuous, i.e., lies in C^0.

2. Fix $k \geq 1$ and assume that if $f, g \in C^{k-1}$, then $f \circ g \in C^{k-1}$.

3. Since $f, g \in C^k$, they are C^1. For each $x \in U$,

$$D(f \circ g)(x) = Df\big(g(x)\big) \circ Dg(x)$$

The map $x \mapsto Df(g(x))$ is the composition of g (class C^k) with Df (class C^{k-1}), hence is C^{k-1} by the induction hypothesis. Likewise $x \mapsto Dg(x)$ is C^{k-1}. Since both Dg and Df are C^{k-1}, by the hypothesis, $D(f \circ g)(x) = D_{g(x)}f(g(x)) \circ D_x g(x)$ is a C^{k-1} map.

Therefore $f \circ g \in C^k$. Together with the base case, induction on k completes the proof. □

Definition 1.5 (Diffeomorphism)**.** A map $f : M \to N$ is said to be a diffeomorphism between smooth manifold M and N iff

1. f is bijective
2. f and f^{-1} are both smooth

Consider smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ on a n-dimensional manifold M, the coordinate transformation is given by $\varphi_\beta \circ \varphi_\alpha^{-1}(x) = (y^1, \ldots, y^m)$, where $x = (x^1, \ldots, x^n) \in \varphi_\alpha(U_\alpha \cap U_\beta)$, the Jacobi matrix is given by

$$J(\varphi_\beta \circ \varphi_\alpha^{-1}) = \left(\frac{\partial y^i}{\partial x^j}\right)_{1 \leq i,j \leq n}$$

And the Jacobian is

$$\det J(\varphi_\beta \circ \varphi_\alpha^{-1})(p) = \frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)}(\varphi_\alpha(p))$$

With the Jacobian, the orientability of a smooth manifold can be defined as follows.

Definition 1.6 (Orientability of Manifolds)**.** Let M be a smooth manifold with (at least C^1) cover $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$, M is orientable iff

$$\forall \alpha, \beta \in I : \big(U_\alpha \cap U_\beta \neq \emptyset \implies J(\varphi_\alpha \circ \varphi_\beta^{-1}) > 0\big)$$

And $\mathcal{A}$ is said to be an orientable cover. If orientable covers do not exist, then M is not orientable.

An obvious fact about the orientation is the following proposition:

Proposition 1.5. *Smooth manifolds M and N orientable $\iff$ smooth manifold $M \times N$ orientable.*

Proof. We defined the smooth structure on $M \times N$ to be induced by the smooth structure on M and N; the smooth structure and transition map are given by the following equations:

$$\begin{aligned}
&\mathcal{A}_{M\times N} = \{(U_\alpha \times V_\beta, \varphi_{(\alpha,\beta)} = (\phi_\alpha, \psi_\beta)) \mid (\alpha, \beta) \in I \times J\} \\
&\varphi_{(\alpha_1,\beta_1)} \circ \varphi_{(\alpha_2,\beta_2)}^{-1}(x) = \big(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1}(x^1, \ldots, x^m), \psi_{\beta_1} \circ \psi_{\beta_2}^{-1}(x^{m+1}, \ldots, x^{m+n})\big)
\end{aligned}$$

And the Jacobi matrix is defined by

$$J\big(\varphi_{(\alpha_1,\beta_1)} \circ \varphi_{(\alpha_2,\beta_2)}^{-1}\big)(x) = \begin{pmatrix} \partial_j(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})^i(x) & \\ & \partial_j(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})^i(x) \end{pmatrix}$$

By the orientability of M and N, we can always choose the open cover to make the following Jacobian positive

$$\det J(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})(x) = \det \left(\frac{\partial(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})^i}{\partial x^j}(x) \right)_{m\times m} > 0$$
$$\det J(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})(x) = \det \left(\frac{\partial(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})^i}{\partial x^j}(x) \right)_{n\times n} > 0$$

Thus, the Jacobi matrix of the corresponding chart transition map on $M \times N$

$$\det J\big(\varphi_{(\alpha_1,\beta_1)} \circ \varphi^{-1}_{(\alpha_2,\beta_2)}\big)(x) = \det J(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})(x) \cdot \det J(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})(x) > 0$$

Thus, there exists an orientable atlas on $M \times N$; $M \times N$ is orientable. □

Example 1.5 (Real Projective Space $\mathbb{RP}^n$)**.** *The real projective space is given by $\mathbb{RP}^n := (\mathbb{R}^{n+1} - \{0\})/\sim$, where the equivalence relation is given by*

$$\forall x, y \in (\mathbb{R}^{n+1} \setminus \{0\}) : \exists \lambda \in \mathbb{R} \setminus \{0\} : (x \sim y \iff x = \lambda y)$$

Consider the quotient map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$, the topology on $\mathbb{RP}^n$ is induced by the quotient map, i.e.

$$V \subseteq \mathbb{RP}^n \text{ open} \iff \pi^{-1}(V) \text{ is open in } \mathbb{R}^{n+1}$$

It is obvious that the quotient topology ensures that π is continuous and surjective, and thus $\mathbb{RP}^n$ is second countable and Hausdorff. Real projective space can also be written as a quotient space of $S^n/\sim$, where $\forall x, y \in S^n$,

$$x \sim y \iff x = -y$$

The chart on $\mathbb{RP}^n$ is given by

$$U_k = \{[x] \in \mathbb{RP}^n \mid x = (x^1, \dots, x^{n+1}), x^k \neq 0\}, \quad \text{where } [x^1 : \dots : x^k] \sim \left[\frac{x^1}{x^{n+1}} : \dots : \frac{x^{k-1}}{x^k} : 1 : \frac{x^{k+1}}{x^k} : \dots : \frac{x^{n+1}}{x^k}\right]$$

The overlapping region of two charts with $k \neq l$ is just

$$U_k \cap U_l = \{[x] = [x^1 : \dots : x^{n+1}] \mid x^k, x^l \neq 0\}$$

By definition $\mathbb{RP}^n = \bigcup_{k=1}^{n+1} U_k$, the coordinate map is given by $\varphi_k : U_k \to \mathbb{R}^n$

$$\varphi_k([x]) = (\eta^1, \eta^2, \dots, \eta^n)$$

where $\eta^i = x^i/x^k$ if $i < k$ and $\eta^i = x^{i+1}/x^k$ if $i \geq k$. To compute the transition map, denote ${}_j\xi^i = x^i/x^j$, then

$$\varphi_l \circ \varphi_k^{-1}({}_k\xi^1, \dots, {}_k\xi^{k-1}, {}_k\xi^{k+1}, \dots, {}_k\xi^{n+1}) = ({}_l\xi^1, \dots, {}_l\xi^{l-1}, {}_l\xi^{l+1}, \dots, {}_l\xi^{n+1})$$

Since the coordinate has the relation

$$\begin{aligned} {}_l\xi^h &= x^h/x^l = \left(\frac{x^h}{x^k}\right) / \left(\frac{x^l}{x^k}\right) = {}_k\xi^h / {}_k\xi^l, \quad h \neq l, k \\ {}_l\xi^k &= x^k/x^l = ({}_k\xi^l)^{-1} \end{aligned}$$

By the fact that x^l and x^k are nonzero, then it is obvious that $\varphi_l \circ \varphi_k^{-1} : \varphi_k(U_k \cap U_l) \to \varphi_l(U_l \cap U_k)$ is a smooth map. Since this statement is general, U_k and U_l are smoothly compatible $\forall k, l$. Thus, $\mathbb{RP}^n$ is a smooth manifold.

The generalization of the projective space is called the Grassmannian, which is also a smooth manifold.

Example 1.6 (Grassmannian)**.** *Given an finite dimensional vector space V such that $\dim V = n$, $n \geq k$, then the Grassmannian on V is given by*

$$\mathrm{Gr}_k(V) := \{W \subseteq V \textit{ linear subspace} \mid \dim W = k\}$$

Remark. *The Grassmannian can reduce to projective space $\mathrm{Gr}_1(\mathbb{R}^{n+1}) \cong \mathrm{Gr}_n(\mathbb{R}^{n+1}) \cong \mathbb{R}\mathrm{P}^n$.*

Theorem 1.1. Grassmannian $\mathrm{Gr}_k(V)$ wirh V being n-dimensional $\mathbb{R}$-vector space is a $(n-k)k$ dimensional C^∞-manifold.

Proof. WOLG, take $V = \mathbb{R}^n$. It is an obvious fact that the Grassmannian is a quotient manifold from the set of k-frames in V, denoted as

$$\mathrm{Fr}_k(\mathbb{R}^n) := \{(v_1, \dots, v_k) \in \mathbb{R}^{nk} \mid (v_1, \dots, v_k) \text{ linear independent}\} \cong \{F \in M_{n\times k}(\mathbb{R}) \mid \operatorname{rank} F = k\}$$

Grassmannian has the topology as a quotient topology from $\mathrm{Fr}_k(\mathbb{R}^n)$ with the equivland class $\forall F_1, F_2 \in \mathrm{Fr}_k(\mathbb{R}^n)$

$$F_1 \sim F_2 \iff \exists M \in \mathrm{GL}_k(\mathbb{R}) : F_1 = F_2 M$$

i.e., $\mathrm{Gr}_k(\mathbb{R}^n) \cong \mathrm{Fr}_k(\mathbb{R}^n)/\,\mathrm{GL}_k(\mathbb{R}^n)$. Thus, the topology on the Grassmannian is defined by the quotient map $\pi : \mathrm{Fr}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$

$$U \subseteq \mathrm{Gr}_k(\mathbb{R}^n) \text{ is open} \iff \pi^{-1}(U) \text{ is open.}$$

Open sets in the k-frame are given by $U_I = \{F \in \mathrm{Fr}_k(\mathbb{R}^n) \mid \det F_I \neq 0\}$. Where $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ such that $i_a \neq i_b\ \forall a \neq b$, and $F_I \in M_k(\mathbb{R})$ is defined by $(F_I)_{ab} = F_{i_a b}$, also, since U_I is invariant under $\mathrm{GL}_k(\mathbb{R})$-right action $\rho : \mathrm{Fr}_k(\mathbb{R}^n) \times \mathrm{GL}_k(\mathbb{R}) \to \mathrm{Fr}_k(\mathbb{R}^n)$, $\rho(A, M) = AM$. Thus, U_I can be viewed as the open set in $\mathrm{Gr}_k(\mathbb{R}^n)$.
Take the local coordinate of each chart U_I be

$$\begin{aligned} \phi_I : U_I &\to M_{(n-k)\times k}(\mathbb{R}) \cong \mathbb{R}^{(n-k)k} \\ F_I &\to F_{I^c} \end{aligned}$$

As an example, consider $n = 4$, $k = 2$. Take $I = (2, 3)$ (i.e., $I^c = (1, 4)$), then the frame and the coordinate are given by

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \\ f_{41} & f_{42} \end{pmatrix} \sim A = FM = \begin{pmatrix} * & * \\ 1 & 0 \\ 0 & 1 \\ * & * \end{pmatrix}, \qquad \begin{aligned} &F_I = \begin{pmatrix} f_{21} & f_{22} \\ f_{31} & f_{32} \end{pmatrix},\ M = (F_I)^{-1} \\ &\phi_I(F) = F_{I^c} = \begin{pmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{pmatrix} \end{aligned}$$

Where $\phi_I(F) = F_{I^c}$ up to the fixed identification $M_{(n-k)\times k}(\mathbb{R}) \cong \mathbb{R}^{(n-k)k}$. For $M \in M_{(n-k)\times k}(\mathbb{R})$, let $A^{(J)}(M) \in M_{n\times k}(\mathbb{R})$ be the matrix whose J-rows equal I_k and whose J^c-rows equal M (in order). Set

$$B_{I\leftarrow J}(M) := \big(A^{(J)}(M)\big)_I, \qquad C_{I\leftarrow J}(M) := \big(A^{(J)}(M)\big)_{I^c}$$

Then, on the intersection domain

$$U_I \cap U_J = \{\, M \in M_{(n-k)\times k}(\mathbb{R}) \mid \det B_{I\leftarrow J}(M) \neq 0 \,\}$$

The transition map is then given by

$$\phi_I \circ \phi_J^{-1}(M) = C_{I\leftarrow J}(M)\big(B_{I\leftarrow J}(M)\big)^{-1}$$

Since the transition map is linear, the Grassmannian is a C^∞-map. ☐

For the next proposition about orientability, we need to introduce a topological operation first.

Definition 1.7 (The Connected Sum)**.** Let M_1 and M_2 be connected n-dimensional smooth manifolds, take points $p_1 \in M_1$ and $p_2 \in M_2$. Take local coordinate systems on M_1 and M_2, denotes as (U_1, φ_1) and (U_2, φ_2) such that $\varphi_1(p_1) = \varphi_2(p_2) = 0 \in \mathbb{R}^n$, and

$$\varphi_1(U_1) = \varphi_2(U_2) = B_2(0) = \Big\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n}(x^i)^2 < 4\Big\}$$

Denote the set

$$A(1/2, 2) = B_2(0) - \overline{B_{1/2}(0)} = \Big\{x \in \mathbb{R}^n \mid \frac{1}{4} < \sum_{i=1}^{n}(x^i)^2 < 4\Big\}$$

and its preimage are $V_1 = \varphi_1^{-1}(A(1/2, 2))$, $V_2 = \varphi_2^{-1}(A(1/2, 2))$. Consider the map

$$\phi : A(1/2, 2) \to A(1/2, 2), \phi(x) = x\Big(\sum_{i=1}^{n}(x^i)^2\Big)^{-1}$$

The following lemma is significant:

Lemma. *ϕ is a diffeomorphism.*

Proof. By the smoothness and bijectivity of $f(x) = x$ and $f(x) = 1/|x|$ when $x > 0$, the lemma is obvious. □

By the lemma above, by the smoothness of the manifolds M_1 and M_2, the map $\varphi_2^{-1} \circ \phi \circ \varphi_1 : V_1 \to V_2$ is also a diffeomorphism. Consider the quotient space:

$$M_1 \# M_2 = \big(M_1 - \varphi_1^{-1}(\overline{B_{1/2}(0)})\big) \sqcup \big(M_2 - \varphi_2^{-1}(\overline{B_{1/2}(0)})\big) / \sim$$

Where the quotient is being defined based on the map $\varphi_2^{-1} \circ \phi \circ \varphi_1$

$$\forall x \in V_1 : \forall y \in V_2 : \big(x \sim y \iff y = \varphi_2^{-1} \circ \phi \circ \varphi_1(x)\big)$$

The smooth manifold $M_1 \# M_2$ is called the connected sum of M_1 and M_2.

Proposition 1.6. *The connected sum of two orientable n-dimensional C^∞-manifolds is still orientable.*

Proof. Consider connected smooth manifolds M_1 and M_2 in the definition of connected sum above, suppose M_1 and M_2 are both orientable. By the given definition of the connected sum of manifolds, the only place that needs to be examined is the open set that includes the quotient part. For arbitrary open sets V_1 and V_2 that contain the quotient part of the manifold, the chart transition map is given by

$$\varphi_2 \circ (\varphi_2^{-1} \circ \phi \circ \varphi_1) \circ \varphi_1^{-1} = \phi$$

which we already know is a diffeomorphism, since M_2 is given to be orientable, it is safe to change a coordinate map by composition with a reflection transformation

$$\hat{\varphi}_2 = \mathcal{P} \circ \varphi_2(x), \quad \mathcal{P}x = (-x^1, \dots, x^n)$$

Where the chart transition map can be expanded in the new chart as $\hat{\varphi}_2 \circ (\varphi_2^{-1} \circ \phi \circ \varphi_1) \circ \varphi_1^{-1}$, the Jacobian of the transition map is then given by

$$\det J(\hat{\varphi}_2 \circ \varphi_2^{-1} \circ \phi \circ \varphi_1 \circ \varphi_1^{-1}) = \det J(\mathcal{P} \circ \phi) = -\det J(\phi)$$

Take $r^2 = \|x\|^2 = \sum_i (x_i)^2$, then $\phi(x) = x/r^2$, then we can calculate the partial derivative

$$J_{ij}(\phi) = \frac{\partial \phi^i}{\partial x^j} = \delta^i{}_j r^{-2} + x^i \frac{\partial r^{-2}}{\partial x^j} = \delta^i{}_j r^{-2} - 2x^i x_j r^{-4}$$

which, in matrix notation, is

$$J(\phi) = (r^{-2}I - 2r^{-4}xx^T)$$

And the Jacobian is just the determinant of this matrix

$$\det J(\phi) = (r^{-2})^n \det\bigl(I - 2r^{-2}xx^T\bigr)$$

To compute the determinant, we will need a lemma

Lemma (Sylvester Identity of Determinants)**.** *If $A \in M_{m\times n}(\mathbb{R})$, and $B \in M_{n\times m}(\mathbb{R})$, then*

$$\det(I + AB) = \det(I + BA)$$

Consider the following block matrix in $(m + k) \times (m + k)$,

$$M = \begin{pmatrix} I_m & A \\ B & I_k \end{pmatrix}$$

By elementary transformation, we can get

$$\det M = \det \begin{pmatrix} I_m - AB & O \\ B & I_k \end{pmatrix} = \det \begin{pmatrix} I_m & A \\ O & I_k - BA \end{pmatrix}$$

Which proves the lemma $\det(I_m - AB) = \det(I_k - BA)$

With the lemma, $\det J(\phi) = r^{-2n}(1 - 2r^{-2}\|x\|^2) = -r^{-2n} < 0$. Then, $\det(\mathcal{P} \circ \phi) = r^{-2n} > 0$. This means $M_1\#M_2$ is orientable. □

The orientation of smooth manifolds can also be defined by differential forms. In Chapter 2, we will prove the equivalence of the two definitions. In the last part of this section, we will introduce the connectivity of manifolds and their relation with orientation.

Proposition 1.7. *If M is a connected topological manifold, then M is path-connected.*

Proof. Note that M is connected, which indicates that there do not exist any open sets $U, V \in \mathcal{O}_M$ like $M = U \sqcup V$. Locally, take $p \in M$ and the chart (U, φ) contains p. By the definition of a chart,

$$\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$$

By the definition of open sets, we can take $B_r(\varphi(p)) \subseteq \varphi(U)$. Let $V = \varphi^{-1}\bigl(B_r(\varphi(p))\bigr)$. By the given coordinate map, V is open and path-connected as an inherent property of Euclidean space. Thus, M is locally path-connected.

Thus, for some $p \in M$, it is sufficient to take a branch connected to a path not empty $C_p \subseteq M$. By path-connectivity, $\forall q \in C_p : \exists V \in \mathcal{O}_M$ such that $q \in V \subseteq C_p$, makes C_p open.

Lemma. *Let X be a locally path-connected topological space, and the path-connected branch containing $x \in X$ is given by*

$$C_x = \bigl\{y \in X \mid \exists\gamma \in C^0\bigl([0, 1], X\bigr), \gamma(0) = x, \gamma(1) = y\bigr\}$$

Then, $\forall x, y \in X$ either $C_x = C_y$ or $C_x \cup C_y = \emptyset$ and $\bigcup_{x\in X} C_x = X$, i.e., the locally path-connected branch constructs an equivalence class.

To prove the lemma, we need to check reflexivity, symmetry, and transitivity to prove that the relation above is an equivalence relation. First, define the relation as

$$\forall x, y \in X : \Bigl(x \sim y \iff \exists\gamma \in C^0\bigl([0, 1], X\bigr) : \gamma(0) = x, \gamma(1) = y\Bigr)$$

1. (Reflexivity) $x \sim x$ by the constant map $\forall a \in [0, 1] : \gamma(a) = x$.

2. (Symmetry) If $x \sim y$, i.e., $\gamma : [0,1] \to X$ connected x and y, then $\forall t \in [0,1] : \bar{\gamma}(t) = \gamma(1-t)$ ensures that $y \sim x$.
3. (Transitivity) Suppose for $x, y, z \in X$, $x \sim y$ by path γ_1 and $y \sim z$ by path γ_2, then consider

$$\gamma(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le 1/2 \\ \gamma_2(2t-1), & 1/2 \le t \le 1 \end{cases}$$

thus, $(x \sim y) \wedge (y \sim z) \implies (x \sim z)$.

Thus, $\sim$ is an equivalence relation, and the lemma has been proved.

By the lemma, if $C_p \neq M$, then $\exists p \neq q \in M : C_p \sqcup C_q = M$, which disobeys the connectivity of M. Thus, $C_p = M$, M is a path-connected set. $\square$

Definition 1.8 (Orientation)**.** Let M be an orientable smooth manifold, and $\mathcal{D}$ be an orientable cover. If every chart that is compatible (in the sense of orientation) with $(U, \varphi) \in \mathcal{D}$ is in $\mathcal{D}$, then $\mathcal{D}$ is said to be an orientation.

Proposition 1.8. *Connected orientable smooth manifolds always have exactly two orientations.*

Proof. Let M be the connected orientable smooth manifold, take an orientable atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta) : \beta \in J\}$. We take $(U_\alpha, \phi_\alpha) \in \mathcal{A}$ and $(V_\beta, \psi_\beta) \in \mathcal{B}$, such that $U_\alpha \cap V_\beta \neq \emptyset$, and $\forall p \in U_\alpha \cap V_\beta$:

$$f(p) = \frac{J(\phi_\alpha \circ \psi_\beta^{-1})}{|J(\phi_\alpha \circ \psi_\beta^{-1})|}(p)$$

Since A and B are both orientable atlas, f can be either 1 or -1, which indicates $\mathcal{B} = \mathcal{A}$ or $\mathcal{B} = \mathcal{A}^-$ $\square$

Example 1.7 (Orientability of Real Projective Space)**.** *By the previous example 1.5 of projective spaces, the transition map of $\mathbb{R}\mathrm{P}^n$ is given by*

$$\varphi_l \circ \varphi_k^{-1}({}_k\xi^1, \dots, {}_k\xi^{k-1}, {}_k\xi^{k+1}, \dots, {}_k\xi^{n+1}) = ({}_l\xi^1, \dots, {}_l\xi^{l-1}, {}_l\xi^{l+1}, \dots, {}_l\xi^{n+1})$$

where ${}_j\xi^i = x^i/x^j$. Thus, for $h \neq k, l$

$$\frac{\partial_l \xi^h}{\partial_k \xi^\beta} = \begin{cases} \frac{1}{{}_k\xi^l}, & \beta = h \\ -\frac{{}_k\xi^h}{({}_k\xi^l)^2}, & \beta = l \\ 0, & \textit{otherwise} \end{cases}$$

And for $h = k$

$$\frac{\partial_l \xi^h}{\partial_k \xi^\beta} = \begin{cases} -\frac{1}{({}_k\xi^l)^2}, & \beta = l \\ 0, & \textit{otherwise} \end{cases}$$

With proper coordinate transformation (exchange the order of coordinates) that moves $\beta = l$ to the last column and $h = k$ to the last row, the upper-triangular form of the Jacobi matrix is given by

$$J(\varphi_l \circ \varphi_k^{-1}) = \begin{pmatrix} \frac{1}{{}_k\xi^l} I_{n-1} & \left[\frac{-{}_k\xi^h}{({}_k\xi^l)^2}\right]_{h\neq k,l} \\ 0 & -\frac{1}{({}_k\xi^l)^2} \end{pmatrix}$$

Thus, the determinant is given by

$$\det J(\varphi_l \circ \varphi_k^{-1}) = ({}_k\xi^l)^{-n-1}$$

Notice that if $n+1$ (i.e. n is odd) is even, then $({}_k\xi^l)^{-n-1}$ is always positive, and when n is even, the $({}_k\xi^l)^{-n-1}$ does not have a certain sign. Thus, $\mathbb{R}\mathrm{P}^n$ orientable is n is odd.

Moreover, we can define the category of C^p-manifolds:

Definition 1.9 (Category of C^p-Manifolds)**.** The category of C^p-manifolds, Diff^p, is the category given by

1. $\mathrm{Obj}(\mathsf{Diff}^k) = \text{All } C^k\text{-manifolds}$
2. $\forall M, N \in \mathsf{Diff}^k : \mathrm{Hom}^k_{\mathsf{Diff}}(M, N) = C^k(M, N)$
3. The composition of morphisms is given by the composition of maps.

1.2 Submanifolds

The study of submanifolds is naturally introduced from the concept of a subset and the rank of a map.

Example 1.8 (Invertible Linear Map)**.** *Consider the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, from undergraduate linear algebra, A is invertible iff $\det A \neq 0$; we also say that the linear map is "full rank". From the perspective of smooth manifolds, we can extend the concept of full rank to the map on manifolds.*
Any linear map on $\mathbb{R}^n$ can be written in matrix form, which can also be viewed as the elements in $\mathbb{R}^{n^2}$, on which we can induce a norm. For $A : \mathbb{R}^n \to \mathbb{R}^n$, the L^2-norm on $\mathbb{R}$ can be defined by

$$\begin{aligned} \|\cdot\| :&\mathbb{R}^n \to [0, \infty) \\ x \mapsto& \|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \end{aligned}$$

Then, we can induce the operator norm in the following way:

Definition 1.10 ((Linear) Operator Norm)**.** With given vector norm $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$, the linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ has norm:

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

Then we have the following proposition

Proposition 1.9. *The linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ satisfies that $\|B\| < 1$, then $I_n - B$ is invertible.*

Proof. To prove the bijectivity, in this case, we only need to check the kernel of the linear map $I_n - B$. Consider the equation $(I_n - B)x = 0$, then $x = Bx$. Thus, the 2-norm is given by

$$\|x\| = \|Bx\| \leq \|B\|\|x\|$$

Which indicates $(1 - \|B\|)\|x\| \leq 0$. Known that $1 - \|B\| > 0$, then

$$(I_n - B)x = 0 \iff x = 0$$

Which means $\ker(I_n - B) = \{0\}$. Since $B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$, B is bijective. □

A significant observation from the example above is that the identity matrix is still nondegenerate after a small perturbation. Thus, we have the following generalization.

Definition 1.11 (Rank of C^k-Maps)**.** Let $f : M \to N$ to be a C^k-map $(k \geq 1)$ between smooth manifolds, let $p \in M$ and $q = f(p) \in N$. Taking local maps (U, φ) on M and (V, ψ) on N, the rank of map f at point p is given by

$$\mathrm{rank}_p f := \mathrm{rank}\, J(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}$$

Proposition 1.10. *The definition of rank does not depend on the choice of coordinates.*

Proof. Consider the chart transition map, $\Theta := \tilde{\varphi} \circ \varphi^{-1}$ and $\Phi : \tilde{\psi} \circ \psi^{-1}$, by definition, $\Theta : \varphi(\tilde{U} \cap U) \to \tilde{\varphi}(\tilde{U} \cap U)$ and $\Phi : \psi(\tilde{V} \cap V) \to \tilde{\psi}(\tilde{V} \cap V)$ are C^k-diffeomorphisms ($k \geq 1$). Thus, consider $U, \tilde{U} \subseteq M$, $V, \tilde{V} \subseteq N$ and $\dim M = m$, $\dim N = n$.

$$P := J(\Theta) = \Big(\frac{\partial(\tilde{\varphi} \circ \varphi)^i}{\partial x^j}\Big)_{1 \leq i,j \leq m}, \quad Q := J(\Phi) = \Big(\frac{\partial(\tilde{\psi} \circ \psi)^i}{\partial y^j}\Big)_{1 \leq i,j \leq n}$$

are invertible matrices. Then, consider the local coordinate representation: $\tilde{f} = \psi \circ f \circ \varphi$ and $\tilde{\psi} \circ f \circ \tilde{\varphi} = \Phi \circ \tilde{f} \circ \Theta^{-1}$, we can compute the rank of the Jacobi matrix:

$$\operatorname{rank} J(\Phi \circ \tilde{f} \circ \Theta^{-1})_{\varphi(p)} = \operatorname{rank}(PJ(\tilde{f})Q)_{\varphi(p)} = \operatorname{rank} J(\psi \circ f \circ \varphi)_{\varphi(p)} = \operatorname{rank}_p f$$

Thus, the rank of a function at a point $\operatorname{rank}_p f$ is well defined under coordinate transformations. □

An important way to define a submanifold is to consider the image (preimage) of a smooth map, formally via the inverse function theorem (IFT). The IFT on smooth manifolds is directly induced by the IFT on Euclidean spaces.

Theorem 1.2 (Inverse Function Theorem)**.** Let $f : M^n \to N^n$ be the C^k-map between C^∞-manifold M and N, if $\operatorname{rank}_p f = n$, then there exists an open neighborhood $U \subseteq M$ that contains p and $V \subseteq N$ contains $q = f(p)$, such that $f|_U : U \to V$ is a C^k-diffeomorphism.

Proof. As a local constraint to differentiable functions on C^∞-manifolds, it is sufficient to take $M = N = \mathbb{R}^n$, and take $p = 0 \in \mathbb{R}^m$. Since $\operatorname{rank}_0 F = n$, it is sufficient to consider the Jacobi matrix of the map $J(f)_0 = I_n$ by the composition of a linear map, which will not change the result. Thus, at the origin $p = 0$, the map F is a perturbation of the identity map. Let

$$\forall x \in \mathbb{R}^n : \ g = f(x) - x$$

Then, $J(g)_0 = J(f)_0 - I_n = 0$, then $\exists \epsilon > 0$,

$$\forall x \in \overline{B_\epsilon(0)} : \ \|J(g)_x\| \leq \frac{1}{2}$$

And by the mean value theorem

$$\forall x_1, x_2 \in \overline{B_\epsilon(0)} : \ \|g(x_1) - g(x_2)\| \leq \|J(g)_\xi\| \|x_1 - x_2\| \leq \frac{1}{2}\|x_1 - x_2\|$$

Which indicates that g is 1/2-Lipschitz continuous, $\forall \xi \in B_\epsilon(0)$, solving the equation $y = f(x)$ is just finding the fixed point of $g_y(x) = x - (f(x) - y)$. The function is also 1/2-Lipschitz, since $g_y(x_1) - g_y(x_2) = g(x_1) - g(x_2)$. By the Banach fixed-point theorem, $\exists x_y \in B_\epsilon(0) : g_y(x_y) = x_y + y - f(x_y) = x_y$, i.e., $f(x_y) = y$ has a unique solution $x_y = h(y)$ such that

$$f(h(y)) = y \iff y - g(h(y)) = h(y)$$

Given $V = B_{\epsilon/2}(0)$ and $U = f^{-1}(V) \cap B_\epsilon(0)$, to prove that $h : V \to U$ is a C^k-diffeomorphism, we only need to check the continuity and C^k of the map.

- h is continuous. Since $\forall y_1, y_2 \in B_\epsilon(0)$

$$\begin{aligned}\|h(y_1) - h(y_2)\| &\leq \|y_1 - y_2\| - \|g(h(y_1)) - g(h(y_2))\| \\ &\leq \|y_1 - y_2\| - \frac{1}{2}\|h(y_1) - h(y_2)\|\end{aligned}$$

  Thus, $\|h(y_1) - h(y_2)\| \leq \frac{2}{3}\|y_1 - y_2\|$ is Lipschitz. Thus, h is continuous.

- h is C^k. Firx a base point $y_0 \in V$, given any $y \in V$,

$$\begin{aligned} h(y) - h(y_0) &= (y - y_0) + \big(g(h(y)) - g(h(y_0))\big) \\ &= (y - y_0) + J(g)_{h(y_0)}\big(h(y) - h(y_0)\big) + O(\|h(y) - h(y_0)\|) \\ \Longrightarrow y - y_0 &= \big[I_n - J(g)_{h(y_0)}\big](h(y) - h(y_0)) + O(\|h(y) - h(y_0)\|) \end{aligned}$$

  Which implies that the Jacobi matrix exists and $Jh(y_0) = I_n - J(g)_{h(y_0)} = (Jf(h(y)_0))^{-1}$. Repeat this procedure, by the C^k-differentiability of f, h is a C^k-map.

Thus, we proved that f is locally a C^k-diffeomorphism. □

Based on the IFT, it is sufficient to define the following behavior of maps, which provides us with a nice hint on how to study the submanifold.

Definition 1.12 (Immersion, embedding, and submersion)**.** Let $f : M^m \to N^n$ be a C^k-map between differentiable manifolds. If $\operatorname{rank}_p f = m \ \forall p \in M$, f is said to be a C^k-immersion. If $f : M \to f(M)$ is a C^k-immersion and also a homeomorphism ($f(M) \subseteq N$ has the subset topology of N), then f is a C^k-embedding. If $\operatorname{rank}_p f = n \ \forall p \in M$, then f is a C^k-submersion.

Here are some examples:

Example 1.9 (An immersion that is not embedding (not injective))**.** *Consider the map* $f : \mathbb{R} \to \mathbb{R}^2$ *such that*

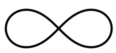
$$f(\theta) = (\cos\theta, \sin\theta)$$

It is easy to check the smoothness and $\operatorname{rank} f = 1$*. However, since* $f(\theta) = f(\theta + 2n\pi)$*,* f *is not injective, and thus, not an embedding.*

Example 1.10 (An injective immersion that is not embedding)**.** *Consider the map* $f : \mathbb{R} \to \mathbb{R}^2$ *defined by*

$$f(t) = \left(\frac{t^3 + t}{t^4 + 1}, \frac{t^3 - t}{t^4 + 1}\right) \quad \forall t \in \mathbb{R}$$

f *is injective and* $\operatorname{rank} f = 1$*. However, it is not a submersion since the image of the map is the lemniscate shown in the following figure:*



Which is clearly not homeomorphic to $\mathbb{R}$ *and thus, not diffeomrorphic to* $\mathbb{R}$

Example 1.11 (Embedding of torus)**.** *The torus is given by* $T^2 = S^1 \times S^1$*. Consider the map* $f : T^2 \to \mathbb{R}^3$ *defined by taking* $R > r > 0$ *and*

$$f(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\phi}) = \big((R - r\cos\phi)\cos\theta,\ (R - r\cos\phi)\sin\theta,\ r\sin\theta\big), \quad \forall \theta, \phi \in [0, 2\pi]$$

is a smooth embedding of T^2 *into* $\mathbb{R}^3$*.*

Theorem 1.3 (Local Form of Immersion)**.** Let $f : M^m \to N^n$ be an immersion between differentiable manifolds, $f(p) = q$. Then, exist charts (U, φ) on M contains p and (V, ψ) on N contains q such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ given by

$$\psi \circ f \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0)$$

Proof. Since the result is local, we can consider only the case that $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ $(m < n)$. Then, let the component form of the immersion be given by $f(x) = (f_1(x), \dots, f_n(x))$ $\forall x \in \mathbb{R}^m$, consider the Jacobi matrix of the immersion (which is a $n \times m$ matrix with rank m)

$$J(f)_x = \frac{\partial(f_1, \dots, f_n)}{\partial(x^1, \dots, x^m)}(x)$$

By Gauss reduction, we can take the $m \times m$ component nondegenerate near $x = 0$

$$\left|\frac{\partial(f_1, \dots, f_m)}{\partial(x^1, \dots, x^m)}\right|(x) \neq 0$$

And the map defined by $g(x^1, \dots, x^n) = (f_1, \dots, f_m, f_{m+1} - x^{m+1}, \dots, f_n - x^n)$ has Jacobi matrix

$$J(g) = \begin{pmatrix} \left[\frac{\partial f_i}{\partial x^j}\right]_{m\times m} & 0 \\ * & I_{n-m} \end{pmatrix}$$

It is clear that this Jacobi matrix is nondegenerate near $x = 0$. By IFT (Theorem 1.2), $\exists U', V$ charts such that $g|_{U'} : U \to V$, then $\exists \psi : V \to U'$ be the inverse of $g|_{U'} \circ \psi = \mathrm{id}_V$. Let

$$U = \{(x^1, \dots, x^m) \in \mathbb{R}^m \mid (x^1, \dots, x^m, 0, \dots, 0) \in U'\}$$

Then, $f|_U : U \to V$ has the local coordinate representation with the local coordinate (U, φ)

$$\psi \circ f(x^1, \dots, x^m) = \psi(f_1(x), \dots, f_n(x)) = (x^1, \dots, x^m, 0, \dots, 0)$$

which proves the theorem. □

Corollary. *Any immersion is locally embedded.*

The corollary is directly from the local form. Since the results show that immersions can be expressed (using proper coordinates) in local level sets of coordinate functions.

Theorem 1.4 (Local Form of Submersion)**.** Let $f : M^m \to N^n$ be a submersion between differentiable manifolds, $f(p) = q$. Then, exist charts (U, φ) on M contains p and (V, ψ) on N contains q such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ given by

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n)$$

Proof. It is still sufficient to take $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ $(m > n)$, and $f(x) = (f_1(x), \dots, f_n(x))$. Since f is a submersion, rank $f(x) = n$. Thus,

$$J(f)_x = \frac{\partial(f_1, \dots, f_n)}{\partial(x^1, \dots, x^m)}(x) \text{ is a } n \times m \text{ matrix with rank } n$$

That means, using Gauss reduction, the full rank block Jacobi matrix can be written as

$$\left|\frac{\partial(f_1, \dots, f_n)}{\partial(x^1, \dots, x^n)}\right| \neq 0$$

Then, take the following construction: consider the function defined by $g(x^1, \dots, x^m) = (f_1, \dots, f_n, x^{n+1}, \dots, x^m)$

$$J(g) = \begin{pmatrix} \left[\frac{\partial f_i}{\partial x^j}\right]_{n\times n} & * \\ 0 & I_{m-n} \end{pmatrix}$$

Thus, by IFT (Theorem 1.2), the function g is a local diffeomorphism. Let U' be the chart that $g|_{U'} : U' \to V$ invertible, $\psi(f_1, \dots, f_n, x^{n+1}, x^m) = (x^1, \dots, x^m)$ be the local inverse. Then, take $\forall x \in V$ and ψ as the local coordinate map on V

$$\psi \circ f(x^1, \dots, x^m) = (x^1, \dots, x^n)$$

Which proves the theorem. □

Similarly, we have the following corollary

Corollary. *Submersion on manifolds without boundary must be an open map (and thus, a quotient map).*

Definition 1.13 (Immersion/Regular (Embedded) Submanifold)**.** Consider M^m, N^n be smooth manifolds. If $i : M \hookrightarrow N$ is an injective immersion, then M (with the induced topology by i) is the immersion submanifold of N; if i is an embedding, then M is the regular submanifold (or embedded submanifold) of N.

In the folloing example, let M^m, N^n be smooth manifolds, $(U, \varphi = (x^1, \dots, x^m))$ and $(V, \psi = (y^1, \dots, y^n))$ be coodrdinate maps.

Example 1.12 (Graphs are regular submanifolds)**.** *Consider the smooth map $f : M^m \to N^n$, the graph of f is defined by*

$$\Gamma_f := \{(m, n) \in M \times N \mid n = f(m)\}$$

The Graph of f is an m-dimensional regular submanifold of $M \times N$. Let the local coordinate representation of f be $g = \psi \circ f \circ \varphi^{-1}$, the Jacobi matrix of the inclusion map $\Gamma_f \overset{i}{\hookrightarrow} M \times N$ in local coordinate should be a $(m+n) \times m$ dimensional matrix

$$J(i) = \begin{pmatrix} I_m \\ J(g) \end{pmatrix}$$

Where $J(g)$ is the Jacobi matrix of g in dimension $n \times m$. It is obvious that $\operatorname{rank} i = m$, which means the inclusion map is an immersion. To prove it is an embedding, it is sufficient to prove that the projection map $p_M : M \times N \to M$ that sends $(p, q) \mapsto p$ is the continuous inverse from Γ_f to M, which means that i is an embedding.

Remark. *The immersion submanifolds in N do not require a subset topology from N. A counterexample is that the Example 1.10 is an immersion submanifold of $\mathbb{R}^2$, but is not a manifold if it contains a subset topology from $\mathbb{R}^2$.*

Example 1.13 (Immersion Submanifold that Dense in T^2)**.** *Consider the inclusion map $i : \mathbb{R} \to T^2 = S^1 \times S^1$ defines by*

$$i(t) = (e^{2\pi i t}, e^{2\pi \alpha i t})$$

for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The image $i(\mathbb{R})$ is dense in T^2 and thus, the inclusion map is not proper when the image is in the subset topology induced from T^2. That means the image cannot be a regular manifold in T^2.

If $\pi : M \to N$ is some continuous map, a smooth local section $\sigma : U \to M$ of π is a right inverse of π in some open neighborhood U of $\pi(p)$ in N, i.e., $\pi \circ \sigma = \mathrm{id}|_U$. Based on the local section, the submersion also has the following property:

Theorem 1.5 (Local Section Theorem)**.** Suppose M and N are smooth manifolds and $\pi : M \to N$ is a smooth map. Then it is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.

Proof. First, suppose $\pi : M \to N$ is a submersion, then, by the local coordinate representation of the submersion (Theorem 1.4), there exists a local coordinate (U, φ) such that

$$\psi \circ \pi \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^n)$$

Then, we can find the local inverse $\sigma : V \subseteq N \to M$ with the local coordinate representation

$$\sigma(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$$

Since the choice of $p \in M$ is arbitrary, and we can always find a coordinate φ that maps p to $(0, \dots, 0)$ and $\varphi(U) = B_\epsilon(0)$, every point of M is in the image of a smooth local section of π.

Conversely, suppose $\forall p \in N : V \subseteq N$ is an open neighborhood of p, there exists $\sigma : V \subseteq N \to M$ such that $\pi \circ \sigma = \mathrm{id}|_V$. Then the Jacobi matrix of the map is

$$\begin{aligned} J(\psi \circ \pi \circ \sigma \circ \psi^{-1}) &= J(\psi \circ \pi \circ \varphi^{-1}) J(\varphi \circ \sigma \circ \psi^{-1}) \\ &= I_n, \quad \text{which is in rank } n \end{aligned}$$

Then, $J(\psi \circ \pi \circ \varphi^{-1})$ is in rank n, and thus, $\forall p \in M : \operatorname{rank}_p \pi = n$, which means π is a immersion. ☐

We will see an important example of submersion in this type, called the fiber bundle.

Theorem 1.6 (The Chateceristic of Regular Submanifold)**.** Consider smooth manifolds M^m and N^n. M is the regular submanifold of $N \iff M$ is the topological subspace of N and $\forall p \in M$, exists a local chart U contains p with coordinate map $\varphi = (x^1, \ldots, x^n)$ such that

$$M \cap U = \{q \in U \mid x^i(q) = 0 \ \forall i = m+1, \ldots, n\}$$

Proof. ($\Rightarrow$) Suppose M is the regular submanifold of N. Then, the inclusion map $M \overset{i}{\hookrightarrow} N$ is an immersion. By the local coordinate form of immersion (Theorem 1.3), we can find the local coordinate (U, φ) and (V, ψ) on M and N

$$\begin{aligned} \psi \circ i \circ \varphi^{-1} &: \varphi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n \\ \varphi(p) &= (x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0) \end{aligned}$$

Then it is natural to defined the chart $(\tilde{U}, \tilde{\varphi})$ on N that

$$\tilde{U} := \{q \in V \mid (x^1(q), \ldots, x^m(q)) \in \varphi(U)\}$$

Where $\psi(q) = (x^1(q), \ldots, x^n(q))$. Then we can check the coordinate representation on $M \cap \tilde{U}$. Let $q = i(p)$ for some $p \in U$. Thus

$$\psi(q) = \psi \circ i(p) = (x^1(q), \ldots, x^m(q), 0, \ldots, 0)$$

And thus $x^{m+1}(q) = \cdots = x^n(q) = 0$. Then, if we take $q \in \tilde{U}$ and $x^{m+1}(q) = \cdots = x^n(q) = 0$. Denote

$$u = (u^1, \ldots, u^m) := (x^1(q), \ldots, x^m(q)) \in \varphi(U)$$

Then $p = \varphi^{-1}(u^1, \ldots, u^m) \in U$, and thus

$$\psi(q) = \psi \circ i \circ \varphi^{-1}(u^1, \ldots, u^m) = (u^1, \ldots, u^m, 0, \ldots, 0)$$

Thus, $q \in M$, and $M \cap \tilde{U} = \{q \in \tilde{U} \mid x^i(q) = 0, m+1 \leq i \leq n\}$.
($\Leftarrow$) It is easy to show that the inclusion map satisfies the assumption in the theorem is an embedding. ☐

With the characteristics of a regular submanifold, we can prove the following theorem.

Theorem 1.7 (Constant Rank Theorem)**.** Let M^m and N^n be smooth manifolds. $f : M^m \to N^n$ is C^∞-map, if $\operatorname{rank}_p f = k \ \forall p \in M$. Then, $\forall p, q \in M, N$, there is a pair of locoal coordinates (U, φ) and (V, ψ) such that f has local coordinate form

$$\psi \circ f \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

Proof. Since the claim is local, we can let $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ and f be smooth map with constant rank k, written as

$$f(x^1, \ldots, x^m) = (f_1(x), \ldots, f_n(x))$$

And similar to the previous proofs, we can assume the matrix

$$M_f = \left[\frac{\partial f_i}{\partial x}\right]_{1\leq i,j\leq k}$$

has ranl k, i.e. M_f is the largest invertible block in $J(f)$. The IFT inspires us to define the function

$$\varphi(x^1,\ldots,x^m) = (f_1,\ldots,f_k(x),x^{k+1},\ldots,x^m)$$

The Jacobi matrix of φ is given by

$$J(\varphi) = \begin{pmatrix} M_f & * \\ 0 & I_{m-k} \end{pmatrix}$$

which is nondegenerate at the origin, and implies that we can use IFT (Theorem 1.2) in an open neighborhood. Consider open neighborhood $U,V \subseteq \mathbb{R}^m$ of $0 \in \mathbb{R}^m$ such that $\varphi|_U : U \to V$ has local inverse $\varphi^{-1} : V \to U$. Then, take φ as a local coordinate map and

$$F(x) = f\circ\varphi^{-1}(x^1,\ldots,x^m) = (x^1,\ldots,x^k,F^{k+1},\ldots,F^n)$$

Since $\operatorname{rank} f \equiv k$,

$$\frac{\partial F^i}{\partial x^j} = 0, \quad k+1 \leq i \leq n,\ k+1\leq j\leq m$$

Which implies the function $F^i = F^i(x^1,\ldots,x^k)$. Then, defined $x = (x^1,\ldots,x^k)$, consider the coordinate transformation such that

$$\psi(x^1,\ldots,x^n) = \big(x^1,\ldots,x^k,x^{k+1}-F^{k+1}(x),\ldots,x^n-F^n(x)\big)$$

With this coordinate map

$$\begin{aligned}\psi\circ f\circ\varphi^{-1}(x^1,\ldots,x^m) &= \big(x^1,\ldots,x^k,(F^{k+1}-F^{k+1})(x),\ldots,(F^n-F^n)(x)\big)\\ &= \big(x^1,\ldots,x^k,0,\ldots,0\big)\end{aligned}$$

In this way, we proved the theorem. □

An application of the constant rank theorem (Theorem 1.7) can provide us with the construction of a class of regular submanifolds:

Theorem 1.8 (Level Set Theorem)**.** Let $f : M^m \to N^n$ be a smooth map with constant rank $\operatorname{rank} f = l$, then $\forall q \in N$

$$f^{-1}(q) := \{p\in M \mid f(p) = q\}$$

is either empty, or a $m-l$ dimensional submanifold of M.

Proof. Let $S = f^{-1}(q)$ for some $q \in N$. Suppose S is nonempty, by the constant rank theorem (Theroem 1.7), $\forall p \in S$, there exists chart (U,φ) contains p and chart (V,ψ) on N contains q such that the local coordinate representation of f take the form

$$\psi\circ f\circ\varphi^{-1}(x^1,\ldots,x^m) = (x^1,\ldots,x^l,0,\ldots,0)$$

Without the loss of generality, we can let $\psi(q) = 0 \in \mathbb{R}^n$, then

$$\psi\circ f\circ\varphi^{-1}(x^1,\ldots,x^m) = (x^1,\ldots,x^l,0,\ldots,0)$$

which always be 0 $\forall p \in S$. In this way, we shall claim that there $\forall p \in S$, there exists chart (U,φ) such that for coordinate map $\varphi = (x^1,\ldots,x^m)$

$$S\cap U = \{p\in U \mid x^1(p) = \cdots = x^l(p) = 0\}$$

By the characteristic of the regular submanifold (Theorem 1.6), S is a regular submanifold. □

1.3 Tangent and Cotangent Spaces with the Induced Linear Maps

1.3.1 Tangent Space and Pushforward

As a locally Euclidean topological space, the open coordinate charts are homeomorphic to open sets in $\mathbb{R}^n$. Recall that in multivariable calculus, the key property is that every (differentiable) map $f : \mathbb{R}^m \to \mathbb{R}^n$ is locally linearized at a small neighborhood p as a linear transformation, or "matrix". More precisely, a map $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be differentable at $a \in \mathbb{R}^m$ if

$$\exists! L \in \mathrm{Hom}_{\mathsf{Vect}}(\mathbb{R}^m, \mathbb{R}^n) : \lim_{h \to 0} \frac{\|f(a+h) - f(a) - Lh\|}{\|h\|} = 0$$

Which means the global differentiability is just saying $f(x+h) - f(x) = Lh + \alpha(x,h)$, where $\forall x \in \mathbb{R}^n : \alpha(x,h) = o(h)$. To construct a similar (locally) linear structure on C^∞-manifolds, we need the tangent space associated to each point on the manifold. In the following passage, $C^\infty(M)$ is the global smooth function on M and C_p^∞ is the function germ.

Definition 1.14 (C^∞-Function Germ at p)**.** The function germ is defined by $C_p^\infty = C^\infty(M)/\sim_p$, where the equivlance relation is given by

$$\forall f, g \in C^\infty(M) : f \sim_p g \iff \exists U_p \subseteq M : f|_{U_p} = g|_{U_p}$$

Where U_p is an open neighborhood of p.

It is easy to show that C_p^∞ is an $\mathbb{R}$-module.

Definition 1.15 (Tangent Vector)**.** Let M be a smooth manifold, fix a point $p \in M$. A tangent vector at point p is a $\mathbb{R}$-linear map $X_p : C_p^\infty \to \mathbb{R}$ such that $\forall f, g \in C_p^\infty$

$$X_p(f \cdot g) = f(p)X_p(g) + X_p(f)g(p)$$

The space of all vector spaces at p is called the tangent space at p, denoted as T_pM.

Remark. *A more abstract way to define a derivation is that, given a ring R, let A be an R-algebra and B be an A-bimodule. A derivation is a R-linear map $D : A \to B$ satisfies the Leibniz rule: $\forall a, b \in A$*

$$D(ab) = aD(b) + D(a)b$$

The purpose of this definition is to define the differential of a smooth function on a manifold. In Euclidean space

$$\boxed{\text{Vectors } v_p = (v^1, \dots, v^n)_p} \xleftrightarrow{\text{One to one correspondence}} \boxed{\text{derivations } D_p = v^1\partial_1|_p + \cdots + v^n\partial_n|_p}$$

each vector v has a corresponding directional derivativor $v \cdot \nabla$. The derivativor (algebraic) definition of a tangent vector is simply to use the correspondence relation above, which uses the linearity and Leibniz law to define the tangent vector.

From the definition, the following properties are easy to check:

Proposition 1.11 (Properties of Tangent Vectors)**.** *Let M be a smooth manifold, $p \in M$. $\forall v_p \in T_pM$, the following statements are true*

1. $\forall f \in C^\infty(M) : f = \text{const} \implies v_p(f) = 0$
2. $\forall f, g \in C^\infty(M) : f(p) = g(p) = 0 \implies v_p(f \cdot g) = 0$

The proof is simply to apply the linearity and Leibniz law.

Other than the algebraic definition, there is a more geometrical way to define a tangent vector using smooth curves on a manifold $\gamma : I \to M$, where $I = (-\epsilon, \epsilon)$ for some $\epsilon > 0$.

Definition 1.16 (Tangent Vector as Velocity of Curves)**.** Let M be a smooth manifolds, $\gamma : I \to M$, a tangent vector along γ is defined by

$$X(f) := \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0} \in T_pM$$

Remark. *The equivalent form of this definition is that given a chart (U, φ) on M contains p, $\varphi = (x^1, \dots, x^m)$, $T_pM = \{\text{Curves cross } p \in M\}/\sim$ where the equivalence relation is defined by $\forall \gamma_1, \gamma_2 : I \to \mathbb{R}$.*

$$\gamma_1 \sim \gamma_2 \iff \frac{\mathrm{d}}{\mathrm{d}t}\big(x^i \circ \gamma_1(t) - x^i \circ \gamma_2(t)\big)\Big|_{t=0} = 0$$

The definition using curves is significant since this is the only way that one can define the tangent space and, more importantly, the chart-induced basis on a C^1-manifold.

It is easy to check that X is linear and Leibniz. We also need to show that the tangent vector in the curve definition is well defined on function germs, i.e.

Proposition 1.12 (Well-Definedness of Tangent Vectors)**.** *Let M be a smooth manifold, $p \in M$. $f, g \in C^\infty(M)$ where U_p is an open neighborhood of p. Then $\forall X_p \in T_pM$*

$$f|_{U_p} = g|_{U_p} \implies X_p(f) = X_p(g)$$

Proof. By the definition of tangent vector using curves, take $I_\epsilon := (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and the curve $\gamma : I_\epsilon \to M$ such that $\gamma(0) = p$

$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0}$$

Since $f|_{U_p} = g|_{U_p}$, it is sufficient to take $0 < \epsilon' < \epsilon$ such that $\gamma(I_{\epsilon'}) \subseteq U_p$ and thus

$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} g \circ \gamma(t)\Big|_{t=0} = X_p(g)$$

This proves the well-definedness. □

This interpretation of tangent vectors also provides us with a canonical choice of the basis in the tangent space.

Theorem 1.9 (Chart-Induced Basis)**.** Let M be a smooth manifold and $p \in M$. Consider the chart (U, φ) contains p, where $\varphi : U \to \mathbb{R}^m$ has coordinatte forms $\varphi = (x^1, \dots, x^m)$, then the chart induced basis is given by the curve $\gamma(t) := \varphi^{-1}(\varphi(p) + te^i)$, where e^i is the i-th basis of $\mathbb{R}^n$

$$\frac{\partial}{\partial x^i}\Big|_p f := \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0}, \quad i = 1, \dots, m$$

In simpler words, $\varphi(p) = 0 \in \mathbb{R}^m$

$$\frac{\partial}{\partial x^i}\Big|_p f := \frac{\partial f \circ \varphi^{-1}}{\partial x^i}\big(\varphi(p)\big)$$

It is easy to check that the two definitions given above are equivalent.

Proof. With the local chart (U, φ) that $\varphi = (x^1, \dots, x^m)$ on smooth manifold M, it is easy to check that

$$\frac{\partial}{\partial x^j}\Big|_p x^i = \delta^i{}_j$$

Thus, the set $\{\partial/\partial x^i|_p \mid i = 1, \dots, m\}$ is a linear independent set.
(Proof of generation with the algebraic definition). It follows the well-definedness proposition above that this construction satisfies the Leibniz rule. Then, $\forall f \in C^\infty(M)$, take $p, q \in U \subseteq M$ which $\varphi = (x^1, \dots, x^m) : U \to \varphi(U) \subseteq \mathbb{R}^m$ be the coordinate map, $x := \varphi(q)$ and $a := \varphi(p)$

$$\begin{aligned} f(q) = f \circ \varphi^{-1}(x) &= f \circ \varphi^{-1}(a) + \int_0^1 \mathrm{d}t \Big[\frac{\mathrm{d}}{\mathrm{d}t} f \circ \varphi^{-1}\big(a - t(x-a)\big)\Big] \\ &= f \circ \varphi^{-1}(a) + \sum_{i=1}^m (x^i - a^i) g_i(x), \quad g_i(x) = \int_0^1 \mathrm{d}t \frac{\partial f \circ \varphi^{-1}}{\partial x^i}(a - t(x-a)) \end{aligned}$$

By the definition,

$$g(a) = \frac{\partial f \circ \varphi^{-1}}{\partial x^i}\big(\varphi(p)\big) = \frac{\partial}{\partial x^i}\Big|_p f$$

Then, given a tangent vector $X_p \in T_pM$

$$\begin{aligned} X_p f &= X_p\Big(f \circ \varphi^{-1}(a) + \sum_{i=1}^m (x^i - a^i) g_i(x)\Big) \\ &= \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}\Big|_p f, \quad \text{where } a^i = X_p(x^i) \end{aligned}$$

Which proves the theorem.
(Proof of generation using the velocity of curves). For any curve $\gamma : I \to M$ with $\gamma(0) = p$, the velocity is given by

$$\begin{aligned} \dot\gamma(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\tilde f \circ \tilde\gamma(t)\Big|_{t=0}, \quad \text{where } \tilde f = f \circ \varphi^{-1} \text{ and } \tilde\gamma = \varphi \circ \gamma \\ &= \sum_{i=1}^m a_i \frac{\partial f \circ \varphi^{-1}}{\partial x^i}\big(\varphi(p)\big) = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}\Big|_p f, \quad \text{where } a^i = \frac{\mathrm{d}x^i \circ \gamma}{\mathrm{d}t}\Big|_{t=0} \end{aligned}$$

Which proves the theorem. □

The proof above also shows that the two definitions of tangent spaces are equivalent, and $\mathcal{D}_pM \cong T_pM = \mathrm{Span}\{\partial_i|_p \mid i = 1, \dots, n\}$ (without ambiguity, we write $\partial_i|_p := \partial/\partial x^i|_p$ when there is only one coordinate), and for any differentiable n-dimensional manifolds, the tangent space is n-dimensional vector space. Another thing that needs to be checked is the behavior of tangent vectors under a change of charts:

Proposition 1.13. *Given charts $\big(U, \varphi = (x^1, \dots, x^m)\big)$ and $\big(V, \psi = (y^1, \dots, y^m)\big)$ be charts on smooth manifold M contains p. Then $\forall X_p \in T_pM$*

$$X_p = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^m b^i \frac{\partial}{\partial y^i}\Big|_p$$

where the coefficient $\mathbf{a} = (a^1, \dots, a^n)^T$ and $\mathbf{b} = (b^1, \dots, b^n)^T$ satisfies $\mathbf{a} = J(\varphi \circ \psi^{-1})\mathbf{b}$, or, in component

$$a^i = \sum_{j=1}^m \frac{\partial (x^i \circ \psi^{-1})}{\partial y^j}\big(\psi(p)\big) b^j$$

Proof. By the definition of a chart-induced basis, one can write

$$X_p = \sum_{i=1}^{m}(X_p x^i)\frac{\partial}{\partial x^i}|_p$$

Then, let $f_j = (\varphi \circ \psi^{-1})_j$

$$\frac{\partial}{\partial y^i}\Big|_p = \sum_{j=1}^{m}\left(\frac{\partial}{\partial y^i}\Big|_p x^j\right)\frac{\partial}{\partial x^j}\Big|_p = \sum_{j=1}^{m}\frac{\partial f_j}{\partial y^i}(\psi(p))\frac{\partial}{\partial x^j}\Big|_p$$

Where $\partial f_j/\partial y^i$ is the (i,j)-th entry of the Jacobi matrix $J(\varphi \circ \psi^{-1})$. Thus, the proposition was proved. □

As the final topic of the "space" level illustration of linearization, here is a quick overview of differential topology that shows how a certain decomposition of the tangent space gives a submanifold.

Theorem 1.10. Let M be a smooth manifold, given $p \in M$ and $H_p \subseteq T_pM$ be a linear subspace. Then exists a regular submanifold $S_p \subseteq T_pM$ that $p \in S_p$ and

$$T_pS_p = H_p \subseteq T_pM$$

The key to the proof is to use the level set theorem (Theorem 1.8).

Proof. Suppose $\dim M = m$, $\dim H_p = n$, one shall define the submersion F that for some chart $(U, \psi : U \to \mathbb{R}^m)$ contains p with $\psi(p) = 0 \in \mathbb{R}^m$, and consider the linear subspace

$$W = \psi_{*,p}(H_p) \subseteq \mathbb{R}^m$$

We shall give a linear map (for example, a projection) $L : \mathbb{R}^m \to \mathbb{R}^{m-n}$ such that $\ker L = W$. Then, there exists a natural submersion

$$F := L \circ \psi : U \to \mathbb{R}^{m-n}$$

(it is clearly a submersion since ψ is a difeomorphism and linear map L always have constant rank.) such that the pushforward of F is given by

$$F_{*,p} = L \circ \psi_{*,p} : T_pM \to \mathbb{R}^{m-n}$$

The kernel of the linear map above is given by

$$\ker F_{*,p} = (\psi_{*,p})^{-1}(\ker L) = (\psi_{*,p})^{-1}(W) = H_p$$

Using the level set theorem (Theorem 1.8), since $F : M \to \mathbb{R}^{m-n}$ has constant rank, $S_p = F^{-1}(0) \subseteq M$ is an regular submanfold. Since $F(p) = L \circ \psi(p) = L(0) = 0$, $p \in S_p$. By the constant rank theorem (Theorem 1.7), one shall pick local coordinates such that F hasa local coordinate representation

$$\tilde{F}(u^1, \dots, u^m) = (u^1, \dots, u^{m-n})$$

and thus $\tilde{F}^{-1}(0) \cong \{0\}^{m-n} \times \mathbb{R}^n$ with $F^{-1}(0) \cap V \cong \tilde{F}^{-1}(0) \cap \varphi(V)$ for some chart (V, φ) in M. Defined

$$\tilde{S}_p := \tilde{F}^{-1}(0) \cap \varphi(V) = \{(u^1, \dots, u^m) \in \varphi(V) \mid u^1 = \dots = u^{m-n} = 0\} \cong S_p$$

Then $T_{\varphi(p)}\tilde{S}_p \cong T_pS_p$ with explicitly coordinate representation that

$$T_{\varphi(p)}\tilde{S}_p = \{(v^1, \dots, v^m) \in \mathbb{R}^m \mid v^1 = \dots = v^{m-n} = 0\}$$

On the other hand, since $\tilde{F}_{*,p}(v^1, \dots, v^m) = (v^1, \dots, v^{m-n})$

$$\ker F_{*,p} \cong \ker \tilde{F}_{*,\varphi(p)} = \{(v^1, \dots, v^m) \in \mathbb{R}^m \mid v^1 = \dots = v^{m-n} = 0\} = T_{\varphi(p)}\tilde{S}_p \cong T_pS_p$$

Which shows that $T_pS_p \cong \ker F_{*,p}$. □

After stating the space level linearization, there must be some mapping level linearization locally, which, in more precise terms, is a local induced map on tangent spaces from a differentiable map between manifolds.

Definition 1.17 (Pushforwards / Tangent Maps)**.** Given smooth manifolds M^m and N^n and a differentiable map $f : M \to N$. Then, the pushforward corresponding to f is defined by $\forall X_p \in T_pM : \forall g \in C^\infty_{f(p)}$

$$f_{*,p}X_p(g) = X_p(g \circ f)$$

In this way, we obtain a map $f_{*,p} : T_pM \to T_{f(p)}N$.

Proposition 1.14. *For smooth map $f : M \to N$, its pushforwards $f_{*,p}$ has following properties:*

1. *$f_{*,p}$ is a linear map.*
2. $(\mathrm{id}_M)_{*,p} = \mathrm{id}_{T_pM}$
3. *With the curve definition, $\forall g \in C^\infty_p : \forall h \in C^\infty_{f(p)}$*

$$X_p(g) = \frac{\mathrm{d}}{\mathrm{d}t} g \circ \gamma(t)\Big|_{t=0} \iff f_{*,p}X_p(g) = \frac{\mathrm{d}}{\mathrm{d}t} g \circ f \circ \gamma(t)\Big|_{t=0}$$

The proof is simple; we shall only prove the linearity.

Proof. Suppose $X_p = a_1X_1 + a_2X_2$, directly by the definition $\forall g \in C^\infty_{f(p)}$

$$\begin{aligned} f_{*,p}X_p(g) &= X_p(g \circ f) = a_1X_1(g \circ f) + a_2X_2(g \circ f) \\ &= a_1(f_{*,p}X_1)(g) + a_2(f_{*,p}X_2)(g) \end{aligned}$$

This shows the linearity. □

As we have defined above, the tangent vector can be written as the velocity of smooth curves. Now, with the pushforward of smooth maps, we can say that for smooth curves $\gamma : I \to M$ such that $\gamma(t_0) = p$, then the velocity vector $\gamma'(t_0) \in T_pM$

$$\gamma'(t_0) := \gamma_*\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0}\Big), \quad \text{where } \mathrm{d}/\mathrm{d}t|_{t_0} \text{ is just the unit vector on } T_{t_0}\mathbb{R}.$$

Similar to the differential in Euclidean spaces, the chain rule also applies to the pushforward on smooth manifolds

Theorem 1.11 (Chain Rule)**.** Let M, N, and P be smooth manifolds. If $f : M \to N$ and $g : N \to P$ be smooth maps between manifolds, $\forall p \in M$

$$(g \circ f)_{*,p} = g_{*,f(p)} \circ f_{*,p}$$

Proof. Given arbitary $X_p \in T_pM$, and $\forall h \in C^\infty_{g \circ f(p)}$. Then

$$(g \circ f)_{*,p}X_p(h) = X_p(h \circ g \circ f)$$

and

$$(g_{*,f(p)} \circ f_{*,p})X_p(h) = f_{*,p}X_p(h \circ g) = X_p(h \circ g \circ f)$$

which completes the proof. □

In the following passages, we will find that under the chart-induced basis, the pushforward can be written in matrix form as a Jacobi matrix, and in this case, the chain rule directly follows from the chain rule of the Jacobi matrix on Euclidean space. Furthermore, a fancy way of describing the chain rule and the induced local linear map on (co)tangent space is known as functorial, which means the

Recall that the pushforward is a generalization of differentiation in Euclidean space, which means the pushforward is expected to have the form of a Jacobi matrix in local coordinates:

Theorem 1.12 (Local Coordinate Expression of Pushforwards)**.** Consider chart $(U, \varphi = (x^1, \dots, x^m))$ and $(V, \psi = (y^1, \dots, y^n))$ on M and N where $f : M \to N$ has local coordinate representation

$$\psi \circ f \circ \varphi^{-1} = \big(f^1(x), \dots, f^n(x)\big)$$

With the chart-induced basis, $\forall X_p \in T_pM$, $X_p = \sum_{i=1}^m a^i \partial/\partial x^i|_p$

$$f_{*,p} X_p = \sum_{j=1}^n b^j \frac{\partial}{\partial y^j}\Big|_{f(p)}, \quad b^j = \sum_{i=1}^m a^i \frac{\partial f^j}{\partial x^i}\big(\varphi(p)\big)$$

The proof is direct by computation.

Proof. By the definition of a chart-induced basis

$$f_{*,p} \frac{\partial}{\partial x^i}\Big|_p = \sum_{j=1}^n \Big(f_{*,p} \frac{\partial}{\partial x^i}\Big|_p\Big)(f^i) \frac{\partial}{\partial y^j}\Big|_{f(p)}$$

To compute the coefficient, it is sufficient to show the transformation law on the basis of:

$$\begin{aligned} f_{*,p} \frac{\partial}{\partial x^i}\Big|_p (f^j) &= \frac{\partial}{\partial x^i}\Big|_p (y^j \circ f) = \frac{\partial}{\partial x^i}(y^j \circ f \circ \varphi^{-1})\big(\varphi(p)\big) \\ &= \frac{\partial}{\partial x^i}(\psi \circ f \circ \varphi^{-1})_j\big(\varphi(p)\big) = \frac{\partial f^j}{\partial x^i}\big(\varphi(p)\big) \end{aligned}$$

Thus, the pushforward of a general vector $X_p \in T_pM$ has coordinate representation given by

$$\begin{aligned} f_{*,p} X_p &= \sum_{i=1}^m a^i f_{*,p} \frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^m a^i \sum_{j=1}^m \frac{\partial f^j}{\partial x^i}\big(\varphi(p)\big) \frac{\partial}{\partial y^j}\Big|_{f(p)} \\ &= \sum_{j=1}^m \Big(\sum_{i=1}^m a^i \frac{\partial f^j}{\partial x^i}\big(\varphi(p)\big)\Big) \frac{\partial}{\partial y^j}\Big|_{f(p)} = \sum_{j=1}^m b^j \frac{\partial}{\partial y^j}\Big|_{f(p)} \end{aligned}$$

Where $(a^1, \dots, a^m)^T = J(\psi \circ f \circ \varphi^{-1})(b^1, \dots, b^m)^T$ gives the transformation on coefficients. $\square$

The proposition above also shows that the matrix form we obtain from the chart induced basis $\{\partial/\partial x^i \mid i = 1, \dots m\}$ and $\{\partial/\partial y^j \mid j = 1, \dots n\}$ is the Jacobi matrix of the map f in local coordinate:

$$[f_{*,p}] = J(\psi \circ f \circ \varphi^{-1})\big(\varphi(p)\big)$$

With the local coordinate representation of pushforward, it is obvious that the rank of the smooth map $f : M \to N$ at $p \in M$ can be represented by the rank of pushforward $f_{*,p} : T_pM \to T_{f(p)}N$ as a linear map. Thus, we have the following lemma of differentiable maps with constant rank:

Proposition 1.15. *Let M and N be smooth manifolds. The smooth map $f : M \to N$ is said to be an immersion if $\forall p \in M : f_{*,p} : T_pM \to T_{f(p)}N$ is injective, and a submmersion if $\forall p \in M : f_{*,p} : T_pM \to T_{f(p)}N$ is surjective.*

The proof is simply by definition.

After this section, we are going to slightly abuse a notation. In the following passages, without ambiguity, for any $f \in C^\infty(M)$, we are going to use the notation under the chart-induced basis

$$\frac{\partial f}{\partial x^i}\Big|_p := \frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial(f \circ \varphi^{-1})}{\partial x^j}(\varphi(p))$$

1.3.2 Cotangent Space and Pullback

Definition 1.18 (Cotangent Space)**.** The cotangent space is defined to be the dual space of the tangent space, i.e., let M be a differentiable manifold, $\forall p \in M : T_p^*M = (T_pM)^*$

There is a class of cotangent vectors induced from the smooth function on the manifold M,

Definition 1.19 (Differential of $\mathbb{R}$-Valued Functions)**.** Let $f \in C^\infty(M)$, then $\forall p \in M : \exists \mathrm{d}f_p \in T_p^*M$ defined by

$$\mathrm{d}f_p(X_p) := X_p(f)$$

In the previous passage, we have defined the pushforward of a general smooth map between smooth manifolds. We shall show that the above definition of the differential of a real-valued function coincides with the previous definition.

Proposition 1.16. *Let $f \in C^\infty(M)$, then $\forall p \in M : \forall X_p \in T_pM$*

$$f_{*,p}X_p = \mathrm{d}f_p(X_p)\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{f(p)}$$

where $\mathrm{d}/\mathrm{d}t|_{f(p)}$ is the unit vector on $T_{f(p)}\mathbb{R}$.

Proof. Since $T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is a 1-dimensional vector space, one can write the tangent vector as

$$f_{*.p}X_p = a\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{f(p)}$$

To evaluate the coefficient a, recall the definition of the chart-induced basis of the tangent space, the coordinate map of $\mathbb{R}$ canonically is given by the identity $\varphi(t) = t$, and thus, the coefficient is given by

$$a = f_{*,p}X_p(t) = X_p(t \circ f) = X_p(f) = \mathrm{d}f_p(X_p)$$

Thus, the proof is complete. $\square$

Recall the definition of dual space in linear algebra, the cotangent space (on a real manifold) is just $T_p^*M := \mathrm{Hom}_{\mathsf{Vect}}(T_pM, \mathbb{R})$, the collection of all linear maps from the tangent space at $p \in M$ to real numbers. Then, similar to vectors, there should be a chart-induced basis of cotangent spaces:

Theorem 1.13 (Local-Coordinate Expression of Cotangent Vectors)**.** Let M be a smooth manifold and (U, φ) be a chart on M, where $\varphi = (x^1, \dots, x^m)$. The cotangent space has the dual basis $T_p^*M = \langle (\mathrm{d}x^1)_p, \dots, (\mathrm{d}x^m)_p \rangle$ such that

$$(\mathrm{d}x^i)_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \delta^i{}_j$$

Proof. The key point is to show that the cotangent vector $\mathrm{d}x^i$ is a dual basis, and it automatically generates the cotangent space and is linearly independent.
By the previous definition of the differential of a real-valued function,

$$(\mathrm{d}x^i)_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \frac{\partial}{\partial x^j}\Big|_p x^i = \frac{\partial(x^i \circ \psi^{-1})}{\partial x^j}\big(\psi(p)\big) = \delta^i{}_j$$

Thus, $\{\mathrm{d}x^i\}_{i=1}^m$ is a dual basis. $\square$

Thus, $\forall \omega_p \in T_pM$, the cotangent vector can be written in the chart-induced basis

$$\omega_p = \sum_{i=1}^m a_i \mathrm{d}x^i|_p$$

Where the coefficient is given by $a_i = \omega_p(\partial/\partial x^i|_p)$ since $\mathrm{d}x^i|_p(\partial/\partial x^j|_p) = \delta^i{}_j$.

Also, recall that in real analysis, with some $U \subseteq \mathbb{R}^m$, $\forall f \in C^\infty(U) : \mathrm{d}f = \sum_{i=1}^n (\partial f/\partial x^i)\mathrm{d}x^i$. An analogous local expression holds for smooth manifolds. However, while this formula is globally valid in $\mathbb{R}^m$, its globalization to an arbitrary smooth manifold requires the machinery of the tangent and cotangent bundles. Suppose

$$\mathrm{d}f_p = \sum_{i=1}^m a_i(\mathrm{d}x^i)_p$$

By the expression of the coefficient we get above,

$$\mathrm{d}f_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \sum_{i=1}^m a_i(\mathrm{d}x^i)_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \sum_{i=1}^m a_i\delta^i{}_j = a_j$$

and by the definition of $\mathrm{d}f_p(\partial/\partial x^j|_p) = \partial f/\partial x^j|_p$ (this is slightly a notation abuse since the derivation is in the sense of a tangent vector acting on a smooth function). Thus, $a_i = \partial f/\partial x^i|_p$, and

$$\mathrm{d}f_p = \sum_{i=1}^m \frac{\partial f}{\partial x^i}\Big|_p(\mathrm{d}x^i)_p$$

Similar to the pushforward induced from the smooth map, the smooth map also induces a linear map between the cotangent spaces.

Definition 1.20 (Codifferential/Pullback of Smooth Maps)**.** Consider M and N be differential manifolds, $f \in C^\infty(M,N)$. $\forall p \in M$, the pullback induced by f is the dual of pushforward

$$f_p^* := (f_{*,p})^\vee : T^*_{f(p)}N \to T^*_pM$$

which $\forall \omega_{f(p)} \in T^*_{f(p)}N$ and $X_{f(p)} \in T_pM$, the following construction has is the explicit illustration of the term "dual", denotes as $(f_{*,p})^\vee$

$$f_p^*\omega_{f(p)}(X_p) := \big((f_{*,p})^\vee\omega_{f(p)}\big)(X_p) = \omega_{f(p)}(f_{*,p}X_p)$$

Remark. *One can also pullback the function on manifolds via $\forall g \in C^\infty(N) : f^*g = g \circ f \in C^\infty(M)$. This map is linear and distributive on the multiplication of smooth functions. As the dual of pullback, the pushforward of $X_p \in T_pM$ can then be written as*

$$\forall g \in C^\infty(N) : f_{*,p}X_p(g) = X_p(f_p^*g)$$

The generalization (globalization) will be useful in the following discussion of tangent and cotangent vector fields.

Similar to pushforwards, the pullback has the following properties:

Proposition 1.17. *For smooth map $f : M \to N$, its pullback f_p^* has the following properties:*

1. *f_p^* is a linear map.*
2. *$(\mathrm{id}_M)_p^* = \mathrm{id}_{T_p^* M}$*

The proof simply follows the properties of pushforwards.

As the dual of pushforward/differential of the smooth map, the pullback/codifferential should also have a local coordinate representation:

Theorem 1.14 (Local Coordinate Expression of Pullbacks)**.** Consider chart $(U, \varphi = (x^1, \ldots, x^m))$ and $(V, \psi = (y^1, \ldots, y^n))$ on M and N where $f : M \to N$ has local coordinate representation

$$\psi \circ f \circ \varphi^{-1} = \big(f^1(x), \ldots, f^n(x)\big)$$

With the chart-induced basis, $\forall \omega_{f(p)} \in T_{f(p)}^* N$, $\omega_{f(p)} = \sum_{i=1}^n a_i (\mathrm{d}y^i)_{f(p)}$

$$f_p^* \omega_{f(p)} = \sum_{j=1}^m b_j (\mathrm{d}x^j)_p, \quad b_j = \sum_{i=1}^n a_i \frac{\partial f^i}{\partial x^j}\big(\varphi(p)\big)$$

Proof. We can prove directly by definition, and it is sufficient to prove the transformation rule on the chart-induced basis. Consider the chart-induced basis $\{(\mathrm{d}y^i)_{f(p)} \mid i = 1, \ldots, n\}$, $\forall X_p \in T_p M$, by the definition of pullback

$$f_p^* (\mathrm{d}y^i)_{f(p)}(X_p) = (\mathrm{d}y^i)_{f(p)}(f_{*,p} X_p)$$

Given that $X_p = \partial/\partial x^i|_p$ the base vector of $T_p M$,

$$f_{*,p} \frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^n \frac{\partial f^j}{\partial x^i}\big(\varphi(p)\big) \frac{\partial}{\partial y^j}\Big|_{f(p)}$$

Then, it is sufficient to apply the cotangent vector on the basis

$$\begin{aligned}
f_p^* (\mathrm{d}y^i)_{f(p)} \Big(\frac{\partial}{\partial x^j}\Big|_p\Big) &= (\mathrm{d}y^i)_{f(p)} \Big(f_{*,p} \frac{\partial}{\partial x^j}\Big|_p\Big) \\
&= \sum_{k=1}^n \frac{\partial f^k}{\partial x^j}\big(\varphi(p)\big) (\mathrm{d}y^i)_{f(p)} \Big(\frac{\partial}{\partial y^k}\Big|_{f(p)}\Big) = \frac{\partial f^i}{\partial x^j}\big(\varphi(p)\big) \\
\Longrightarrow f_p^* (\mathrm{d}y^i)_{f(p)} &= \sum_{j=1}^m f_p^* (\mathrm{d}y^i)_{f(p)} \Big(\frac{\partial}{\partial x^j}\Big|_p\Big) (\mathrm{d}x^j)_p = \sum_{j=1}^m \frac{\partial f^i}{\partial x^j}\big(\varphi(p)\big) (\mathrm{d}x^j)_p
\end{aligned}$$

Thus, for a general cotangent vector $\omega_{f(p)} = \sum_{i=1}^n a_i (\mathrm{d}y^i)_{f(p)} \in T_{f(p)}^* N$, the pullback is given by

$$\begin{aligned}
f_p^* \omega_{f(p)} &= \sum_{i=1}^n a_i f_p^* (\mathrm{d}y^i)_{f(p)} = \sum_{i=1}^n a_i \sum_{j=1}^m \frac{\partial f^i}{\partial x^j}\big(\varphi(p)\big) (\mathrm{d}x^j)_p \\
&= \sum_{j=1}^m \Big(\sum_{i=1}^n a_i \frac{\partial f^i}{\partial x^j}\big(\varphi(p)\big)\Big) (\mathrm{d}x^j)_p = \sum_{j=1}^m b_j (\mathrm{d}x^j)_p
\end{aligned}$$

Thus, the theorem was proved. □

We have observed that in the formalism of tangent and cotangent spaces, not only has the base space been changed between manifolds and vector spaces, but each (smooth) map is also being mapped to a corresponding induced map. Such a phenomenon can be formalized in the language of category theory and is known as the functorial. In the following section, our goal is to to show that the relation of tangent and cotangent spaces is an nice example of an widely existed phenomena called the dual of category. The use of categorical language is not particularly beneficial here, but more as a concise example for reader to get familiar with the concepts in the category theory.

A Functor Approach of Tangent and Cotangent Spaces*

In this section, we admit the Axiom of Choice and fix a Grothendieck universe $\mathscr{U}$; throughout, all categories are $\mathscr{U}$-categories. We assume the reader is familiar with the basic notions of categories and functors; background relevant to this section is collected in Appendix E. For further details on category theory, as well as rigorous set-theoretic foundations that avoid size issues, see [31, 53]. The purpose of this section is to familiarize the reader with categorical language; it is not strictly necessary for the topics discussed here. A more algebraic (categorical) treatment of differential geometry requires the use of sheaves; see [47].

The most classical way of describing the tangent and cotangent spaces as functors is to use the category of C^k-manifolds Diff_k and the category of vector bundles VBdl. However, at this point, we have not yet introduce the vector bundle as the nontrivial globalization of point-wise attached vector spaces (like tangent spaces). Thus, in this section, the category we are considering is the category of C^k-manifolds with a marked base point, $\mathsf{Mani}^k_\bullet$; here is the precise definition:

Definition 1.21 (Manifold with a Base Point)**.** The category $\mathsf{Mani}^k_\bullet$ consists of the following data as a category:

1. $\mathrm{Obj}(\mathsf{Mani}^k_\bullet)$ = the pair (M, p) such that M is a C^k-manifolds and $p \in M$ is the marked point.
2. The morphism is given by $\forall (M, p), (N, q) \in \mathrm{Obj}(\mathsf{Mani}^k_\bullet)$, the morphism between them is given by
$$\mathrm{Hom}_{\mathsf{Mani}^k_\bullet}\big((M, p), (N, q)\big) := \{f \in C^k(M, N) \mid f(p) = q\}$$
and the composition of the morphism is the composition of maps (as in Set).

As we have defined in the previous sections, in this category, there are two most direct (contra/co)variant functors, valued only in the "tangent/cotangent space":

Definition 1.22 (Tangent Functor)**.** Then tangent function $T : \mathsf{Mani}^k_\bullet \to \mathsf{Vect}_{\mathbb{R}}$ is a covariant functor that defined by
$$(M, p) \mapsto T_pM, \quad \big(f : (M, p) \to (N, q)\big) \mapsto (f_{*,p} : T_pM \to T_qN)$$

The functorial is given by the properties of pushforwards that
$$(\mathrm{id}_M)_{*,p} = \mathrm{id}_{T_pM}, \quad (f \circ g)_{*,f(p)} = f_{*,p} \circ g_{*,p}$$

Which, in our new language,
$$T(\mathrm{id}_{(M,p)}) = \mathrm{id}_{T_pM}, \quad T(f \circ g) = T(f) \circ T(g)$$

This shows the covariant functorial.

To provide a dual description of the category, we need to define the opposite category.

*This part lies somewhat out of the main line of development of the chapter, and may be omitted in a first reading

Definition 1.23 (Opposite (Dual) Category)**.** Given a category $\mathcal{C}$, the dual of $\mathcal{C}$ is named the opposite category, denoted as $\mathcal{C}^\vee$, such that

1. $\mathrm{Obj}(\mathcal{C}) = \mathrm{Obj}(\mathcal{C}^\vee)$.
2. $\forall X, Y \in \mathrm{Obj}(\mathcal{C}^\vee) : \big(f \in \mathrm{Hom}_{\mathcal{C}^\vee}(X,Y) \iff f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)\big)$.
3. The composition $f \circ g$ in $\mathcal{C}^\vee$ is just $g \circ f$ in $\mathcal{C}$.

The same trick also applies to cotangent cases, which we just replace the category $\mathsf{Mani}^k_\bullet$ by the opposite category (reverse all morphisms), $(\mathsf{Mani}^k_\bullet)^\vee$.

Definition 1.24 (Cotangent Functor)**.** The cotangent functor is the dual of the tangent functor. Explicitly, $T^* = (T)^\vee : (\mathsf{Mani}^k_\bullet)^\vee \to \mathsf{Vect}_{\mathbb{R}}$ is a covariant functor on the opposite category, defined by

$$(M,p) \mapsto T_p^* M, \quad \big(f : (M,p) \to (N,q)\big) \mapsto (f_p^* : T_q N \to T_p M)$$

Remark. *Note that for categories $\mathcal{C}, \mathcal{D}$, the contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is the same as the covariant functor on the opposite category $F : \mathcal{C}^\vee \to \mathcal{D}$, since the opposite category is just the category with the same objects but with reversed direction on all morphisms.*

In the same way, the cotangent functor is indeed the contravariant functorial on $\mathsf{Mani}^k_\bullet$

$$T^*(\mathrm{id}_{(M,p)}) = \mathrm{id}_{T_p^* M}, \quad T^*(f \circ g) = T^*(g) \circ T^*(g)$$

The final result of this section is to consider the behavior of the tangent functor T under (Cartesian) product (Note that we are not allowed to take infinite product on manifold). Consider the product of differentiable manifolds as

$$\forall (M,p), (N,q) \in \mathsf{Mani}^k_\bullet : (M,p) \times (N,q) := \big(M \times N, (p,q)\big)$$

It is sufficient to show that this satisfies the universal property of the product, such that the following diagram commutes

$$\begin{tikzcd}
 & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[d, "f"] \arrow[dr, "\pi_2"] & \\
X_1 \arrow[r, "f_1"] & Y & X_2 \arrow[l, "f_2"']
\end{tikzcd}$$

Then, by the functoriality of the tangent/cotangent functor, we know that $T_{(p,q)}(M \times N) = T_p M \times T_q N$ and $T^*_{(p,q)}(M \times N) = T_p^* M \oplus T_q^* N$. As finite-dimensional vector spaces, the product $\times$ and coproduct $\oplus$ are equivalent (called biproduct), we can also write

$$T_{(p,q)}(M \times N) = T_p M \oplus T_q N$$
$$T^*_{(p,q)}(M \times N) = T_p^* M \oplus T_q^* N$$

A more algebraic geometrical taste of the smooth manifold is in Appendix F, which introduces the C^∞-ring and C^∞-Scheme.

1.4 The Essential Fact of Topological Groups and Lie Groups

The final topic of this chapter concerns the action of transformation groups on manifolds. In general topology, it is important to understand under what conditions a group action (or the corresponding quotient space) preserves Hausdorffness. In geometry, our focus shifts to smooth actions of Lie groups on fiber bundles, as these play a central role in the study of topological invariants known as characteristic classes, which will be introduced in Section 7.7 in Chapter 7 about homogeneous spaces and Section 12.1 in Chapter 12 about the Lie groups and Lie algebras prerequisite of mathematical gauge theory.

In particular, we are interested in topological groups that also possess a smooth manifold structure—these are the Lie groups. Lie groups constitute fundamental examples of smooth manifolds, and they provide the mathematical framework for describing most geometric symmetries.

1.4.1 Topological Groups

Definition 1.25 (Topological Groups/Lie Groups)**.** The group G is a topological group if G (with some topology) is a topological space such that the multiplication $m : G \times G \to G$ and inverse map $i : G \to G$ defineds by $\forall g \in G : i(g) = g^{-1}$ are both continuous map.
Furthermore, if G is a C^r-manifolds and μ, i are C^r-maps, then G is a C^r-Lie group.

Without other examination of the differentiability of the Lie group, then the group is a C^∞-Lie group.

Remark. *Note that the product and any embedded submanifold of Lie groups are reasonably expected to be Lie groups, but not every smooth manifold can be given a Lie group structure.*

We are going to just list some basic properties of topological groups without proof, since most of the proofs use nothing but basic group theory and general topology. More detailed study of topological group can be find in [44].

Topological groups have remarkable separation properties:

Proposition 1.18. *For a topological group G, the following properties are equivalent:*

1. *G is T_0.*
2. *G is T_1.*
3. *G is Hausdorff.*

Moreover, every topological group is T_3.

Also, the multiplication on a topological group gives a homeomorphism (automorphism) of the group G, more precisely:

Proposition 1.19. *Let G be a topological group, the the following maps are all homeomorphisms:*

1. *The left and right translation $\forall g \in G : L_g, R_g : G \to G$ defines by*
$$\forall h \in G : L_g(h) = gh, \quad R_g(h) = hg$$
2. *The inverse map $x \mapsto x^{-1}$.*
3. *The conjugate map $\forall g, h \in G : c_g(h) = ghg^{-1}$.*

When we consider subgroups,

Proposition 1.20. *Let G be a topological group, then*

1. *If $H \leq G$ is a subgroup, the so is $\overline{H}$*
2. *If $H \trianglelefteq G$, then so is $\overline{H}$*

And also,

Proposition 1.21. *The center $Z(G)$ of a Hausdorff topological group G is a closed normal subgroup.*

Proposition 1.22. *The connected component contains e of a topological group G is a normal subgroup.*

We are going to use the following two propositions in the discussion of Lie groups:

Proposition 1.23. *Let G be a connected topological group, $e \in U \subseteq G$ is open. Then*

$$\bigcup_{n \geq 1} U^n = G$$

where $U^n = \{g_1 g_2 \cdots g_n \mid g_i \in U \ \forall i\}$

Proof. Defined $\forall U \subseteq G : U^{-1} := \{g^{-1} \mid g \in U\}$. Let $V = U \cap U^{-1}$, then V is also an open neighborhood of e and $V = V^{-1}$. Let

$$H = \bigcup_{n \geq 1} V^n \subseteq \bigcup_{n \geq 1} U^n$$

We claim that H is a subgroup of G. To show that $H = G$, we shall prove that H is both open and closed.

To show that H is a closed subgroup, note that since H is open, $\forall g \in G : gH$ is open (left translation is an homeomorphism.). Thus

$$H = G \setminus \Big(\bigcup_{g \notin H} gH \Big)$$

is closed. Since H is nonempty clopen in a connected topological group G, $H = G$. Thus,

$$H \subseteq \bigcup_{n \geq 1} U^n \subseteq G$$

i.e. $\bigcup_{n \geq 1} U^n = G$. □

Proposition 1.24. *Let H and G be topological groups, $H \subseteq G$. If H is locally closed (i.e., $\forall h \in H : \exists U \subseteq G$ open and contains h such that $U \cap H$ is closed in U), then $H \subseteq G$ is a closed subset.*

To prove the proposition, we need the following lemma:

Lemma. *If $H \leq G$ is a Lie subgroup, then $K = \overline{H}$ is also a subgroup of G.*

Proof. Since inverse map i and multiplication m are both continuous,

$$i(K) \subseteq \overline{i(H)} = K, \quad m(K \times K) \subseteq \overline{m(H \times H)} = K$$

Thus, K is a subgroup of G. □

Now we can prove the proposition.

Proof of Proposition 1.24. Let $H \leq G$ be locally closed, and $K = \overline{H}$. Let the open set that associate to the element $h \in H$ be $U_h \subseteq G$ such that $U_h \cap H$ closed in U_h. Let $V_h = K \cap U_h$, then V_h is open in K (with subset topology). Also $H \cap V_h = H \cap U_h$, which close in U_h (since it is closed in U_h). Suppose $y \in V_h \setminus H$, then $\exists O \subseteq K$ open that

$$\emptyset \neq O \subseteq V_h \setminus H$$

Which contradict to $K = \overline{H}$, i.e. $V_h \subseteq H$. Thus,

$$H = \bigcup_{h \in H} V_h$$

is open in K. Then, $\forall k \in K$, the left coset kH is open (since left translation is homeomorphism), and

$$K = \coprod_{k \in K/H} H$$

Thus,

$$K \setminus H = \coprod_{kH \neq H} kH$$

is open, i.e. H is closed in K. Since $K = \overline{H}$ is closed in G, $H \leq G$ is a closed subgroup. □

Finally, we defined the topological group action:

Definition 1.26 (Topological Group Action)**.** Given the topological group G and topological space X, G act on X continuously if there exists a continuous map

$$\begin{aligned} \rho : G \times X &\to X \\ (g, x) &\mapsto g \cdot x \in M \end{aligned}$$

satisfies the following conditions:

1. $\forall x \in X : e \cdot x = x$
2. $\forall g_1, g_2 \in G : g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

The action of topological groups on spaces does not always preserve the Hausdorffness of the topological space (under the quotient).

Definition 1.27 (Proper Action)**.** A topological group action $\rho : G \times X \to X$ on a topological space X is said to be proper if

$$\tilde{\rho} : G \times X \to X \times X \tag{1.4.1}$$

$$(g, x) \mapsto \big(\rho(g, x), x\big) \tag{1.4.2}$$

is a proper map.

Then, we shall state a fact about the action os topological groups.

Theorem 1.15. Consider a topological group G and a space X that are both locally compact and Hausdorff groups. If $G \times X \to X$ is a proper action, then X/G is Hausdorff.

1.4.2 Lie Groups

Similar to topological groups, the most fundamental structure on a Lie group is the multiplication:

Proposition 1.25 (Left/Right Translation)**.** *Let G be a Lie group, then we can define the following diffeomorphisms:*

1. *The left/right translation $\forall g \in G : L_g, R_g : G \to G$ defined by $\forall h \in G$*

$$L_g(h) = gh, \quad R_g(h) = hg$$

2. *The conjugate map $\forall g \in G : c_g : G \to G$ defined by*

$$\forall h \in G : c_g(h) = ghg^{-1}$$

Also, the pushforward corresponding to the multiplication, $m_{*,(g,h)} : T_{(g,h)}(G \times G) \cong T_gG \times T_hG \to T_{gh}G$ *satisfies the relation*

$$m_{*,(g,h)}(X_g, Y_h) = (R_h)_{*,g}X_g + (L_g)_{*,h}Y_h$$

Proof. To show $L_g : G \to G$ is a diffeomorphism, notice that L_g has inverse $L_g^{-1} = L_{g^{-1}}$, thus, L_g is bijective. Since

$$L_g : G \xrightarrow{h \mapsto (g,h)} G \times G \xrightarrow{m} G$$

is the composition of smooth map, $\forall g \in G : L_g$ smooth. In the same way, we shall prove that R_g is smooth, and $c_g = L_g \circ R_{g^{-1}}$ is smooth diffeomorphism.
To show the property of the multiplication, we apply the linearity of the pushforward, since $T_{(g,h)}(G \times G) = T_gG \times T_hG$

$$m_{*,(g,h)}(X_g, Y_h) = m_{*,(g,h)}(X_g, 0) + m_{*,(g,h)}(0, Y_h)$$

Which we claim that $m_{*,(g,h)}(X_g, 0) = (R_h)_{*,g}X_g$ and $m_{*,(g,h)}(0, Y_h) = (L_g)_{*,h}Y_h$. □

Proposition 1.26. *Let G be a topological group as well as a C^∞-manifold, then the smoothness of multiplication implies the smoothness of the inverse.*

Proof. The key idea of the proof is to construct the inverse map i by the composition of diffeomorphisms. Consider the map

$$\begin{aligned} f : G \times G &\to G \times G \\ (g,h) &\mapsto (g, gh) \end{aligned}$$

This is clearly a bijection. Since the pushforward of f in any neighborhood of arbitrary $(g,h) \in G \times G$ is given by proposition 1.25:

$$\begin{aligned} f_{*,(g,h)} : T_gG \oplus T_hG &\to T_gG \oplus T_{gh}G \\ (X_g, Y_h) &\mapsto \big(X_g, (R_h)_{*,g}X_g + (L_g)_{*,h}Y_h\big) \end{aligned}$$

Since both L_g and R_h are diffeomorphism, $f_{*,(g,h)}$ non-degenerate. By IFT (Theorem 1.2), $\forall (g,h) \in G \times G$, there is a neighborhood in which f is a diffeomorphism. By the bijectivity, f is a global diffeomorphism. Then, the inverse of f

$$\begin{aligned} f^{-1} : G \times G &\to G \times G \\ (g,h) &\mapsto (g, g^{-1}h) \end{aligned}$$

is a smooth map. Then we shall write

$$\begin{aligned} i : G \hookrightarrow G \times G &\xrightarrow{f^{-1}} G \times G \xrightarrow{\pi_2} G \\ g \longmapsto (g, e) &\longmapsto (g, g^{-1}) \longmapsto g^{-1} \end{aligned}$$

is a diffeomorphism. □

Note that the same proposition does NOT apply in general topological groups.

Example 1.14 (Galilean Group)**.** *The Galilean group is the spacetime symmetry in classical physical systems, which includes the spacetime translation, rotation, and Galilean boost:*

$$\mathbf{x} \mapsto R\mathbf{x} + \mathbf{v}t + \mathbf{a}, \quad t \mapsto t + s$$

Such that $R \in \mathrm{SO}\,3$, $\mathbf{v}, \mathbf{a} \in \mathbb{R}^3$, and $s \in \mathbb{R}$. Thus, the Galilean group has a semidirect product structure

$$\mathrm{Ga} = (\mathrm{SO}(3) \ltimes \mathbb{R}^3) \ltimes \mathbb{R}^4$$

with multiplication given by $(R_2, \mathbf{v}_2, \mathbf{a}_2, s_2) \cdot (R_1, \mathbf{v}_1, \mathbf{a}_1, s_1) = (R_2R_1, \mathbf{v}_2 + R_2\mathbf{v}_1, \mathbf{a}_2 + R_2\mathbf{a}_1, s_2 + s_1)$.

Another extremely important example of Lie groups is the Heisenberg group.

Example 1.15 (Heisenberg Group)**.** *The Heisenberg group (in a narrow sense) is defined by the matrix group defined by the following* 3×3 *matrices:*

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad \forall x, y, z \in \mathbb{R}$$

Obviously, as a smooth manifold, $H \cong \mathbb{R}^3$*, and* $H \leq \mathrm{GL}(3, \mathbb{R})$*. Thus,* H *is a Lie group.*

Another fact is that:

$$S^n \text{ is a Lie group} \iff n = 1, 3.$$

Then, we defined the map between groups:

Definition 1.28 (Lie Group Homomorphism)**.** The homomorphism between Lie groups is a smooth group homomorphism $\varphi : G \to H$, i.e. $\forall g_q, g_2 \in G$

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$$

Example 1.16 (Determinant)**.** *The determinant* $\det : \mathrm{GL}(n, \mathbb{F}) \to \mathbb{F}^{\times}$ *is a Lie group homomorphism, where* $\mathbb{F}^{\times} = (\mathbb{F} \setminus \{0\}, \times)$ *is the multiplication group over field* $\mathbb{F}$*.*

The most remarkable property about Lie group homomorphisms is:

Proposition 1.27. *Any Lie group homomorphism has constant rank.*

Proof. Let G, H be Lie groups, and $\varphi : G \to H$ be a homomorphism. Then, consider the left translation $L_g : G \to G$

$$\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$$

So if we compute the pushforward of the homomorphism, one can obtain

$$\varphi_{*,g} \circ (L_g)_{*,e} = (L_{\varphi(g)})_{*,\tilde{e}} \circ \varphi_{*,e}$$

where $\tilde{e} \in H$ is the identity in H. Since $(L_g)_{*,e}$ and $(L_{\varphi(g)})_{*,\tilde{e}}$ both are linear isomorphisms (by Proposition 1.25), the homomorphism has constant rank since $\operatorname{rank} \varphi_{*,g} = \operatorname{rank} \varphi_{*,e}$. □

Corollary. *Let* $\varphi : G \to H$ *be a Lie group homomorphism. Let* $h \in H$*, if* $\varphi^{-1}(h) \neq \emptyset$*, then it is a* $\dim G - \dim \ker \varphi$ *dimensional regular submanifold of* G*. In particular,* $\ker \varphi$ *is a Lie subgroup.*

The proof of the corollary uses the fact that Lie group homomorphisms always have constant rank, and simply to apply Theorem 1.8. Another corollary is that:

Corollary. *Injective Lie group homomorphisms must be immersions, surjective Lie group homomorphisms must be submersions, and Lie group isomorphisms must be diffeomorphisms.*

The proof needs Sard's theorem (Theorem 2.6) to show that constant rank surjection is submersion.

Definition 1.29 (Lie Subgroup)**.** Let H, G be Lie groups, $i : H \hookrightarrow G$ be Lie group homomorphism, if i is injective, then H is Lie subgroup of G. Moreover, if i is a embedding, then H is a closed Lie subgroup of G

It is easy to check that:

Proposition 1.28. *Closed Lie subgroup is a closed set.*

Proof. By Theorem 1.6, for all $h \in H$, one can find a chart $(U, \varphi = (x^1, \ldots, x^n))$ such that

$$H \cap U = \{q \in U \mid x^j(q) = 0 \ \forall j > \dim H\}$$

Thus, we know that H is locally closed. By Proposition 1.24, $H \subset G$ is a closed set. □

Next, we are going to see that the pushforward of the Lie group homomorphism can strongly determaine the map itself, this also provides us a hint that one can actually study a Lie group via study its tangent space, which we will introduce in Subsection 7.7.1

Proposition 1.29. *Let $\varphi : G \to H$ be Lie group homomorphism and H be connected Lie groups, if $\varphi_{*,e}$ surjective, then φ is surjective.*

Proof. By the Proposition 1.27, since φ has constant rank, φ is a submersion, and thus, is an open map. In particular, $\exists V \subseteq G$ contains e such that $\varphi(V) \subseteq H$ open. Since H is connected, by Proposition 1.23

$$H = \bigcup_{n\in\mathbb{N}} \varphi(V)^n \subseteq \varphi(G)$$

i.e. φ is surjective. □

In the following part, we are going to study the further properties that appears on Lie group homomorphisms if its pushforward is an linear isomorphism. To do that, we will need to know about Lie algebra.

1.4.3 Lie Algebra and Covering

This subsection is aimed to answer a question: How much does an isomorphism between tangent spaces (or Lie algebra) determines the property of the homomorphism between Lie groups.

Definition 1.30 (Lie Algebra)**.** An Lie algebra over a $\mathbb{R}$-vector space[a] V is a pair $(V, [-,-])$ where the Lie bracket is given by

$$\begin{aligned}[-,-] : V \times V &\to V \\ (v, w) &\mapsto [v, w]\end{aligned}$$

such that $\forall u, v, w \in V$ and $\forall a, b \in \mathbb{R}$,

1. (Antisymmetry) $[u, v] = -[v, u]$
2. (Bilinear) $[au + bv, w] = a[u, w] + b[v, w]$
3. (Jacobi Identity) $\big[u, [v, w]\big] + \big[v, [w, u]\big] + \big[w, [u, v]\big] = 0$

[a]The Lie algebra can be defined in general field, we only consider $\mathbb{R}$ here.

Remark. *The Jacobi identity can be obtain by antisymmetry, we use it as a part of definition to emphasize its importance.*

We are going to show that the structure of finite dimensional Lie algebra is naturally exists on the tangent space from multiplication.

Definition 1.31 (Lie Algebra Associate to Lie Group)**.** The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ associate to a Lie group G is the Lie algebra $(T_eG, [-,-])$ with the Lie bracket inherent from the group structure.

More precisely, the Lie bracket is the local linearization of adjoint action. Consider the adjoint map $\mathrm{Ad} : G \times G \to G$ defined by $\forall g, h \in G : \mathrm{Ad}_g(h) = ghg^{-1}$ the conjugate action. To induce a map on the Lie algebra, we linearize the map in the following way: let $g(t) = \exp(tX)$ and $h(s) = \exp(sY)$ for some $X, Y \in \mathfrak{g}$, then, the induced map on Lie algebra $\mathfrak{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is

$$[X, Y] := \mathfrak{ad}_X Y := \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Ad}_{g(t)}\left(h(s)\right)\Big|_{s=0}\right)\Big|_{t=0}$$

It is easy to check that it satisfies the following properties:

1. $[X, Y] = -[Y, X]$.
2. $[-, -] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear.
3. The *Jacobi identity*: $\forall X, Y, Z \in \mathfrak{g}$, $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$.

Often, at least in matrix groups, $[X, Y]_{\mathfrak{g}} = XY - YX$ (follows the Lie bracket in differential geometry, which will be introduced in Chapter 3). A more formal statement is that

Theorem 1.16 (Ado)**.** Any finite-dimensional Lie algebra $\mathfrak{g}$ over a field K of characteristic zero has a faithful (injective) representation that

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

where V is a K-vector space.

Note that $\mathfrak{gl}(V) \leq \mathrm{End}(V)$ is a subalgebra of associative algebra $\mathrm{End}(V)$, with Lie bracket $[X, Y]_{\mathfrak{gl}(V)}$. With the basic knowledge of Lie algebra, we also need to introduce the covering sp;ce, modetailsail will be introduced in Chapter 6. Continuous surjection between topological spaces $p : Y \to X$ is a covering map if $\forall x \in X : \exists V_x \subseteq X$ open neighborhood of X such that

$$p^{-1}(V_x) = \coprod_{\alpha} U_\alpha$$

where U_α are disjoint open sets in Y and $p|_{U_\alpha} : U_\alpha \cong V_x\ \forall \alpha$ (i.e., exists a discrete set F, $p^{-1}(V_x) \cong V_x \times F$). If Y is simply connected, then Y is the universal covering of X.

Example 1.17 (Universal Cover of S^1)**.** *The Lie group homomorphism* $\exp : \mathbb{R} \to S^1$ *defined by* $\exp(t) = \mathrm{e}^{2\pi \mathrm{i} t}$ *shows that* $\mathbb{R}$ *is the universal cover of* S^1.

We can finally prove the proposition:

Proposition 1.30. *Let $\varphi : G \to H$ be a homomorphism between connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then φ is a covering map if and only if $\varphi_{*,e} : \mathfrak{g} \to \mathfrak{h}$ is a linear isomorphism.*

Proof. ($\Rightarrow$) If $\varphi : G \to H$ is a covering map as well as a Lie group homomorphism, then φ is surjective, and thus, and submersion (second corollary of Proposition 1.27). By the definition of a covering map, for any open neighborhood V of e, $\varphi^{-1}(V) \cong V \times F$, such that F is discrete. Thus, $\ker \varphi = \varphi^{-1}(e) \cong F$ is a discrete subgroup of G, i.e.

$$\operatorname{rank} \varphi = \dim G - \dim \ker \varphi = \dim G$$

Which means $\varphi_{*,e}$ is also injective and thus, a linear isomorphism.
($\Leftarrow$) Since $\varphi_{*,e}$ is a linear isomorphism, Proposition 1.27, the constant rank map (Lie group homomorphism) $\varphi : G \to H$ is both immersion and submersion, i.e.,

$$\operatorname{rank} \varphi = \dim G = \dim H$$

Since φ is both an immersion and a submersion with constant rank, using Sard's theorem (Theorem 2.6), one can show that it is surjective, and also an open map. Since

$$\dim \ker \varphi = \dim T_e G - \operatorname{rank} \varphi_{*,e} = 0$$

and $\ker\varphi$ is an regular submanifold, by Theorem 1.6, we can find open neighborhood $W \subseteq G$ of the identity e, such that $W \cap \ker\varphi = \{e\}$ and $\varphi|_W : W \to \varphi(W)$ is an diffeomorphism. Then, one shall take $V \subseteq W$ contains e such that $V^{-1}V \subseteq W$. Let $U = \varphi(V)$, clearly, $g_1 \cdot V \cap g_2 \cdot V = \emptyset$ for any $g_1 \neq g_2 \in \ker\varphi$. Since $\forall g \in \ker\varphi$, $\varphi(g \cdot V) = U$ is diffeomorphism, then,

$$\varphi^{-1}(U) = \bigcup_{g\in\ker\varphi} g \cdot V$$

Which proves the statement. □

The statement above is really showing that the isomorphism of Lie algebra somehow provided us with the global information of the topology on the Lie group. In the further discussion about Lie group in Section 7.7, we will see that the reason for this phenomenon is the group structure.

1.4.4 Basic Facts about the Action of Lie Groups

In this section, we denote the smooth action of G on differentiable manifolds M as

$$\rho_M : G \times M \to M$$

We need to clarify certain terms we used about group action:

1. The orbit of point $p \in M$ is
$$G \cdot p := \{g \cdot p \in M \mid \forall g \in G\}$$
2. The stabilizer of $p \in M$ is
$$G_p = \{g \in G \mid g \cdot p = p\}$$
3. The action of G on M is a transitive action if $\forall p, q \in M : \exists g \in G$
$$p = g \cdot q$$
i.e., there is only one orbit for the group action.
4. The action of G on M is effective (or faithful) if $\forall p \in M : g \cdot p = p$ implies $g = e$.
5. The action is free if $\forall g \neq e : \forall p \in M : g \cdot p \neq p$, i.e., the stabilizer $G_p = \{e\}\ \forall p \in M$.

Proposition 1.31. *Suppose $\rho_M : G \times M \to M$ is an smooth action of Lie group G, then*

$$\rho_g^M := \rho_M(g, -) : M \to M$$

is a diffeomorphism.

Proof. Since ρ_g^M is smooth with smooth inverse $\rho_{g^{-1}}^M$. □

Definition 1.32 (Equivariant)**.** $f \in C^\infty(M, N)$ is said to be equivariant under group action ρ_M, ρ_N of G on M and N if $\forall g \in G$, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M(g,-)"'] & N \arrow[d, "\rho_N(g,-)"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Proposition 1.32. *If $f \in C^\infty(M, N)$ is equivariant under smooth action ρ_M, ρ_N of Lie group G, then f has constant rank on each orbit $G \cdot p$ of the group action.*

Proof. By the Proposition 1.32, we already know that the map $\rho_g^M : M \to M$ and $\rho_g^N : N \to N$ $\forall g \in G$ are both diffeomorphism, thus $\forall p \in M$ and $\forall q \in N$

$$(\rho_g^M)_{*,p} : T_pM \to T_{g\cdot p}M, \quad (\rho_g^N)_{*,q} : T_qN \to T_{g\cdot q}N$$

are both linear isomorphisms. Thus, consider the equivariant smooth map f, we shall consider the pushforward given by the commutative diagram, which gives $\forall g \in G$, $\forall p \in M$, and $q = f(p) \in N$:

$$(\rho_g^N)_{*,q} \circ f_{*,p} = f_{*,g\cdot p} \circ (\rho_g^M)_{*,p}$$

Since composition with linear isomorphism does not change the rank, $\forall \tilde{p} \in G \cdot p$

$$\operatorname{rank}_{\tilde{p}} f = \operatorname{rank}_p f$$

i.e. f has constant rank on each orbit of group action. □

If the group action is transitive, since one can always find $g \in G$ such that $\forall p_1, p_2 \in M : \rho_M(g, p_1) = p_2$, f has constant rank.

Next, we study the property of orbit map $\rho^{(p)} : G \to M$, $\rho^{(p)}(g) = g \cdot p \in M$, which gives $G \cdot p$ a unique smooth structure such that $\rho^{(p)} : G \to G \cdot p$ be a surjective submersion. More precisely, consider the stabilizer G_p of p and its Lie subalgebra $\mathfrak{g}_p = T_eG_p \subseteq T_eG$, $\forall X \in \mathfrak{g}_p$

$$\rho_{*,e}^{(p)}(X) = \frac{\mathrm{d}}{\mathrm{d}t}\rho\big(\exp(tX), p\big)\Big|_{t=0} = 0$$

since $\rho\big(\exp(tX), p\big) = p$ is a constant path. Thus, $\dim G \cdot p = \dim G - \dim G_p$. The following theorem shows how group action orbits on smooth manifold M (and the Lie group itself) be a submanifold of M.

Theorem 1.17. Let G be a Lie group, $\rho : G \times M \to M$ defines a smooth action of G on a smooth manifold M, and $g \cdot p := \rho(g, p)$ $\forall p \in M$. Then any orbits $G \cdot p$ for $p \in M$ are immersion submanifolds on M. Furthermore, if the action ρ is freely and properly on M, then $G \cdot p$ are embedded submanifolds.

Proof. We shall first show that the inclusion map $i : G \cdot p \hookrightarrow M$ has constant rank. Notice that the following diagram commutes by the definition of group action:

$$\begin{tikzcd} G \cdot p \arrow[r, hook, "i"] \arrow[d, "\rho(g,-)"'] & M \arrow[d, "\rho(g,-)"] \\ G \cdot p \arrow[r, hook, "i"] & M \end{tikzcd}$$

i.e., i is equivariant under the action. By Proposition 1.32, i has constant rank on $G \cdot p$. To explicitly compute the rank of the inclusion map, note that one can defined the submersion $\alpha : G \to G \cdot p$, and $i \circ \alpha = \rho^{(p)}$. The tangent map is then given by

$$\rho_{*,g}^{(p)} = i_{*,g\cdot p} \circ \alpha_{*,g} : T_gG \to T_{g\cdot p}M, \quad \operatorname{rank} \rho_{*,e}^{(p)} = \dim G - \dim G_p = \dim G \cdot p$$

Since α is a submersion, $i = \mathrm{id}_{G\cdot p} : G \cdot p \to G \cdot p \subseteq M$ is bijective onto its image,

$$\operatorname{rank}(i_{*,g\cdot p} \circ \alpha_{*,g}) = \operatorname{rank} \alpha_{*,g} = \dim G \cdot p$$

Which shows that $i : G \cdot p \hookrightarrow M$ is an immersion.
Furthermore, if the group action $\rho : G \times M \to M$ is free and proper. By definition of free action, since the stabilizer $G_p = \{e\}$ $\forall p \in G$, $\mathfrak{g}_p = \{0\}$ and if $\rho^{(p)}(g_1) = \rho^{g_2}(p)$, then

$$\rho(g_2^{-1}g_1, p) = p \implies g_1 = g_2$$

which shows $\rho^{(p)} : G \to M$ is an injective immersion with image $G \cdot p$, and thus, $i : G \cdot p \hookrightarrow M$ is an injective submersion. Also, since the action is proper, i.e.

$$\tilde{\rho} : G \times M \to M \times M, \quad \tilde{\rho}(g, p) := (g \cdot p, p)$$

is a proper map. Then, for any compact set $C \subseteq M$, consider $C \times \{p\} \subseteq M \times M$

$$\tilde{\rho}(C \times \{p\}) = \{(g, p) \in G \times M \mid g \cdot p \in C\} = (\rho^{(p)})^{-1}(C) \times \{p\} \subseteq G \times M$$

is a compact set. Thus, $\rho^{(p)}$ is also a proper map so that (i is also a proper map). Since manifolds are Hausdorff, proper maps are closed. The closeness together with the fact that i is an injective smooth immersion shows that i is an embedding. □

Then, we can introduce the basic theorem about the action of Lie groups on smooth manifolds.

We shall use the following proposition:

Proposition 1.33 (Universdal Property of Quotient)**.** *Let $\pi : M \to N$ be a smooth surjective submersion (quotient map), then consider the following diagram:*

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "f \circ \pi"'] & N \arrow[d, "f"] \\ & P \end{tikzcd}$$

Then the smoothness of $f \circ \pi$ implies the smoothness of f.

Proof. We shall use Theorem 1.5 to prove this proposition. Suppose π is a quotient map (and thus, immersion) and $f \circ \pi$ is a smooth map, $\forall q \in N : \exists U \subseteq N$ be an open neighborhood of q. There is a smooth local section

$$\sigma : V \to U \subseteq M, \quad \pi \circ \sigma = \mathrm{id}_V$$

Then $f|_U = (f \circ \pi) \circ \sigma : U \to P$ is smooth since it is the composition of smooth maps. Since the point $q \in N$ is arbitrary, f is smooth on N. □

Theorem 1.18. Let G be a Lie group. If the smooth action of G acts on a smooth manifold M freely and properly, then M/G is a topological manifold and admits a unique smooth structure such that the quotient map is a submersion.

Proof. (Uniqueness) Suppose there are two smooth structures $\mathcal{A}_1$, $\mathcal{A}_2$ on M. We shall just prove that $\mathrm{id} : M/G \to M/G$ is a diffeomorphism. To prove diffeomorphism, we only need to prove the smoothness. But the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & (M/G, \mathcal{A}_1) \arrow[d, "\mathrm{id}"] \\ & (M/G, \mathcal{A}_2) \end{tikzcd}$$

Since π_1 and π_2 are both smooth quotient maps (i.e., surjective submersions), by the Proposition 1.33 id is smooth (so it is a diffeomorphism).

(Existence) The proof inculuse two parts: First, we shall prove that M/G is a differentiable manifold; Second, the quotient map induces a smooth structure. To prove M/G is a topological manifold, the Hausdorffness is preserved by Theorem 1.15. Consider the map

$$\begin{aligned} \Theta : G \times M &\to M \times M \\ (g, x) &\mapsto (g \cdot x, x) \end{aligned}$$

This map is clearly smooth. We shall construct a countable basis of M to prove second countable, take $\{U_i\}_{i\in I}$ to be a countable basis of M, then $\{\pi(U_i)\}_{i\in I}$ is countable, and since π is open, it is still a basis. Finally, we shall prove that it is locally Euclidean. Let M be an n-dimensional smooth manifold. By Theorem 1.17, orbits of free and proper action of Lie groups are embedded submanifolds in M. Thus, one shall define $\mathcal{O}_p := G \cdot p$ with $T_p\mathcal{O}_p \subseteq T_pM$ such that $\dim T_p\mathcal{O}_p = \dim G =: k$. Then, one shall pick the complement H_p of $T_p\mathcal{O}_p$ such that

$$T_pM = T_p\mathcal{O}_p \oplus H_p, \quad \dim H_p = \dim T_pM - k = n - k$$

Using Theorem 1.10, we shall find an embedded submanifold S_p such that $T_pS_p \cong H_p$. We shall conclude that in some open neighborhood of p, S_p always gives a direct sum decomposition $T_pM = T_p\mathcal{O}_p \oplus T_pS_p$.
With the restriction of group action on S_p

$$\rho|_{G\times S_p} : G \times S_p \to M$$

For any $(\xi, v) \in \mathfrak{g} \times T_pS_p$, the pushforward $\rho_{*,(e,p)} : T_p\mathcal{O}_p \times T_pS_p \to T_pM$ is given by

$$\rho_{*,(e,p)}(\xi, v) = X_\xi(p) + v \in T_pM$$

where $X_\xi(p) := \dot{\rho}\big(\exp(t\xi), p\big)|_{t=0}$, and $\mathrm{Span}\{X_\xi(p) \mid \xi \in \mathfrak{g}\} = T_p\mathcal{O}_p$. Thus, $\mathrm{im}(\rho_{*,(e,p)}) = T_p\mathcal{O}_p \oplus T_pS_p$, i.e., $\rho_{*,(e,p)}$ is linear isomorphism. By inverse function theorem (Theorem 1.2), exists neighborhood $e \in U_G \subseteq G$, $p \in U_S \subseteq S_p$, and $p \in U \subseteq M$ such that

$$\rho|_{U_G\times U_S} : U_G \times U_S \xrightarrow{\sim} U$$

is a local diffeomorphism. Then, we need a lemma

Lemma. *We shall find U_G and U_S such that $\forall g \in G \setminus \{e\}$*

$$g \cdot U_S \cap U_S = \emptyset$$

Which means every orbit $G \cdot p$ $\forall p \in U_S$ can only intersect U_S once.

To prove the lemma, we already know that $\rho|_{U_G\times U_S} : U_G \times U_S \xrightarrow{\sim} U$ is diffeomorphism. Then, $\forall s_1, s_2 \in U_S$, $g \cdot s_2 = s_1$, then

$$s_1 = \rho|_{U_G\times U_S}(e, s_1) = \rho|_{U_G\times U_S}(g, s_2)$$

by the bijectivity, $(e, s_1) = (g, s_2)$, and thus, $g = e$, $s_1 = s_2$. That completes the proof of the lemma.
By the lemma, one shall find U_S such that $g \cdot U_S \cap U_S = \emptyset$ $\forall g \neq e$. Let

$$W_p := G \cdot U_S = \bigcup_{g\in G} g \cdot U_S$$

Hence, we have $\forall q \in W_p$, $\exists! g \in G : \exists! s \in U_S : q = g \cdot s$. Thus, the restriction of group action $\rho|_{G\times U_S} : G \times U_S \xrightarrow{\sim} W_p$ is also a diffeomrophism, i.e., we have construct the local open set $W_p \subseteq M$ contains p that $W_p \cong G \times U_S$. We exam the quotient map $\pi : M \to M/G$, we shall restrict the map to U_S, and

$$\pi|_{U_S} : U_S \xrightarrow{\sim} \pi(U_S)$$

is a homeomorphism. To see this, we know that

1. Continuity and openness is from the definition of quotient topology.
2. Bijective since $\forall s \in U_S$, $G \cdot s$ intersect with U_S exactly once when the group element is e.
3. The inverse map $(\pi|_{U_S})^{-1} : \pi(U_S) \to U_S$ is also continuous by the universal property of the quotient topology.

Thus, on the given neighborhood $U_S^p \subseteq S_p$ contains $p \in M$ with a local chart $\kappa_p : U_S^p \xrightarrow{\sim} V_p \subseteq \mathbb{R}^{n-k}$, defined the chart on $\pi(U_S^p)$ as

$$\varphi_p := \kappa_p \circ (\pi|_{U_S^p})^{-1} : \pi(U_S^p) \xrightarrow{\sim} V_p \subseteq \mathbb{R}^{n-1}$$

which gives an atlas $\{\pi(U_S^p), \varphi_p\}_{p\in M}$ on M/G. In conclusion, M/G is an topological manifold. To show the smoothness, consider $U_{pq} := \pi(U_S^p) \cap \pi(U_S^q) \neq \emptyset$, to prove the transition map

$$\varphi_q \circ \varphi_p^{-1} = \kappa_q \circ (\pi|_{U_S^q})^{-1} \circ (\pi|_{U_S^p}) \circ \kappa_p : \varphi_p(U_{pq}) \to \varphi_q(U_{pq})$$

is smooth. This statement is equivalent to $H := \sigma_q \circ \sigma_p^{-1}$ is smooth. We shall defined the following maps:

$$\Psi_p := \rho|_{G\times U_S^p} : G \times U_S^p \xrightarrow{\sim} W_p, \quad \Psi_q := \rho|_{G\times U_S^q} : G \times U_S^q \xrightarrow{\sim} W_q$$

Where W_p and W_q are open sets in M defined previously. Now we consider the following map on $W_{pq} := W_p \cap W_q \neq \emptyset$

$$\Psi_q^{-1} \circ \Psi_p : \Psi_p^{-1}(W_{pq}) \subset (G \times U_S^p) \xrightarrow{\sim} \Psi_q^{-1}(W_{pq}) \subset (G \times U_S^q)$$

It is also an diffeomorphism, thus, one shall write $\Psi_q^{-1} \circ \Psi_p = (G_{pq}(g,s), H_{pq}(g,s))$ where G_{pq} and H_{pq} are both smooth maps. In particular, with $g = e$, $H_{pq}(e,s) \in U_S^q$ and $\pi(H_{pq}(e,s)) = \pi(s)$. Thus, H_{pq} is actually the map

$$H_{pq} = (\pi|_{U_S^1})^{-1} \circ \pi|_{U_S^p} : (\pi|_{U_S^p})^{-1}(U_{pq}) \to U_S^q$$

since the projection π is diffeomorphism on U_S, H_{pq} is the composition of smooth maps, and thus is a smooth map. Then, then transition map

$$\varphi_q \circ \varphi_p^{-1} = \kappa_q \circ (\pi|_{U_S^q})^{-1} \circ (\pi|_{U_S^p}) \circ \kappa_p = \kappa_q \circ H_{pq} \circ \kappa_p$$

is a smooth map. Thus, M/G admits a smooth structure.
Finally, it remains to show that the quotient map $\pi : M \to M/G$ is a submersion. Since we have diffeomorphism $\Psi_p : G \times U_S^p \xrightarrow{\sim} W_p$, we can restrict the quotient map to local open set:

$$\pi|_{W_p} : W_p \to \pi(W_p) \cong U_S$$

Here, we claim that $\pi(W_p) \cong \pi(U_S) \cong U_S$ since by the definition that $W_p = G \cdot U_S$ and the choice of U_S, every orbit of $p \in U_S$ only intersect with U_S at a single point p. We shall apply the diffeomorphism Ψ_p to W_p here and thus, the quotient map is corresponding to the projection map

$$\tilde{\pi} = (\pi|_{U_S})^{-1} \circ \pi \circ \Psi_p = \mathrm{proj}_2 : U_G \times U_S \to U_S$$

Thus, π is a submersion since proj_2 is a submersion. $\square$

Chapter 2

Fundamental Facts related to Functions on Manifolds

After we have studied the main object of modern geometry: manifolds, we are going to study more details about one of the most essential tools we can use to study manifolds (scalar-valued) functions on manifolds.

2.1 Partition of Unity

In the first section, we are going to study the tool that ensures that a smooth function defined locally can be globalized to a smooth function on the entire manifold.

First of all, we need a smooth function here:

Proposition 2.1. *The function $\phi : \mathbb{R} \to \mathbb{R}$ defines by*

$$\phi(x) = \begin{cases} \mathrm{e}^{-1/x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

is smooth.

The proof is simply undergraduate analysis. The function above leads to the following theorem:

Theorem 2.1 (Existence of Bump Function)**.** The following two statements holds:

1. There exists $h \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $h(x) = 0\ \forall |x| \geq 1$, $h(x) \in (0, 1]\ \forall x \in (-1, 1)$, and $h(x) = 1\ \forall x \in [-1/2, 1/2]$.
2. There exists $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f(x) = 0\ \forall \|x\| \geq 1$, $h(x) \in (0, 1]\ \forall \|x\| < 1$, and $h(x) = 1\ \forall \|x\| \leq 1/2$.

Proof. The proof is based on the smooth function we take in Proposition 2.1.

1. Consider the following function

$$\tilde{h}(x) = \frac{\phi(x)}{\phi(x) + \phi(1 - x)},\ \forall x \in \mathbb{R}$$

it is obvious that $\tilde{h}$ is smooth and

$$\tilde{h}(x) = 0 \text{ if } x \leq 0;\ \tilde{h}(x) > 0 \text{ if } x \in (0, 1);\ \tilde{h}(x) = 1 \text{ if } x \geq 1.$$

Then, take $h_1(x) = \tilde{h}(2x+2)$, and

$$h(x) = h_1(|x|) = \begin{cases} h_1(x), & x \leq 0 \\ h_1(-x), & x > 0 \end{cases}$$

is the function we need in the first proposition.

2. The function we need in the second proposition is the smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = h(\|x\|)$$

where $\|\cdot\|$ is just the standard norm in $\mathbb{R}^n$.

Thus, the theorem was proved. □

Recall that the support of a function is defined by:

Definition 2.1 (Support of a Function)**.** The support of a function $f : X \to \mathbb{R}$ is the closure of the preimage of $\mathbb{R} \setminus \{0\}$, i.e.,

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$$

The goal of the partition of unity is to find a collection of functions $\{\rho_\alpha : X \to [0,1]\}$ such that $\forall x \in X$, $\sum_{\alpha \in A} \rho_\alpha(x) = 1$. Or, more formally

Definition 2.2 (Partition of Unity)**.** Suppose M be a differentiable manifold with open cover $\{U_\alpha\}_{\alpha \in A}$. A partition of unity on M associate to the cover $\{U_\alpha\}_{\alpha \in A}$ is defined to be a collection of smooth functions $\{\rho_i : M \to \mathbb{R}\}_{i \in I}$ where I is at most countable, such that:

1. $\rho_i(x) \in [0,1]$ $\forall x \in M$.
2. $\forall i \in I : \exists \alpha(i) \in A : \operatorname{supp}(g_i) \subseteq U_{\alpha(i)}$
3. $\{\operatorname{supp}(\rho_i)\}_{i \in I}$ is a locally finite cover of M.
4. $\sum_i \rho_i(x) \equiv 1$ $\forall x \in M$.

However, this summation is pathological if A is an infinite set, i.e., $\forall x \in X$, we need there to exist $U \subseteq X$ a neighborhood of x such that $\rho_{\alpha'}|_U \neq 0$ for only finitely many $\alpha' \in A$.

The formal solution to the problem is require the paracompactness.

Definition 2.3 (Paracompactness)**.** X is paracompact if every open cover has a locally finite refinement.

Then, we shal show that the partition of unity always exists on differentiable manifolds.

Proposition 2.2 (Exhaustion)**.** *For any differential manifold, there exists a series of open sets $\{G_i\}_{i \geq 1}$ such that $\overline{G_i}$ are compact and*

$$\overline{G_i} \subseteq G_{i+1} \ \forall i \geq 1, \quad \bigcup_{i \geq 1} G_i = M$$

Proof. The statement is trivial if M is compact. If not, since manifolds are all locally compact, for some $p \in M$, take an open neighborhood $V_p \subseteq M$ of p such that $\overline{V_p}$ is compact, and such a collection gives a cover of M. By Lindelöf lemma, there exists a countable collection of points $\{p_i\}_{i \geq 1}$ such that $\{V_{p_i}\}_{i \geq 1} \subset \{V_p\}_{p \in M}$

covers the manifold M. Based on this fact, we can define the exhaustion recursively: Given $G_1 := V_{p_1}$, suppose $G_1, \dots, G_n$ have been defined. Then, let

$$G := \bigcup_{i=1}^{n} G_i \subseteq M$$

Since $\{V_{p_i}\}_{i\geq 1}$ covers M, there exists index set I such that

$$\bigcup_{i=1}^{n} \overline{G_i} \subseteq \bigcup_{i\in I} V_{p_i} =: G_{n+1}$$

Then, $\{G_i\}_{i\geq 1}$ is the exhaustion we need. □

An application of the exhaustion is to prove the paracompactness.

Proposition 2.3. *Every topological manifold is paracompact.*

Proof. Take $\{G_i\}_{i\geq 1}$ be an exhaustion on manifold M, $\forall i \in \mathbb{N}\setminus\{0\}$ we can defined compact sets $V_i := \overline{G_{i+1}} \setminus G_i$ and open sets $W_i := G_{i+2} \setminus \overline{G_{i-1}}$, $V_i \subseteq W_i$. Consider the open cover $\{U_\alpha\}_{\alpha\in A}$, by the compactness of V_i, we shall find a subcover $\{U_{ij}\}_{j\in I}$ for some finite index set I such that covers V_i. We shall take

$$K_{ij} := U_{ij} \cap W_i$$

which also covers V_i and be a countable refinement of $\{U_{ij}\}$. Consider $p \in M$ with open neighborhood U_p such that $\overline{U_p}$ compact. Then finite many K_{ij} can cover $\overline{U}$, with the maximum index $i = s$, i.e., $K_{ij} \subseteq G_{i+1} \subseteq G_{s+1}$ for any possible i, j. Thus, $\overline{U_p} \subseteq G_{s+1}$. If $i \geq s+2$, since $W_{s+2} := G_{s+2} \setminus \overline{G_{s+1}}$ is disjoint with G_{s+1} and $V_{ij} = U_{ij} \cap W_i$, $V_{ij} \cap G_{s+2} = \emptyset$. Thus, only finitely map U_p intersect with G_{s+2}, which proves the theorem. □

Then, we shall prove the existence of a partition of unity:

> **Theorem 2.2** (Existence of Partition of Unity)**.** For any open cover $\{U_\alpha\}_{\alpha\in A}$ of a smooth manifold M, there exists a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in A}$.

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M.
By Proposition 2.3, M is paracompact, hence every open cover admits a locally finite open refinement. Thus, there exists an index set P and a locally finite family of open sets

$$\{V_p\}_{p\in P}, \quad V_p \subset U_{\alpha(p)} \text{ for some } \alpha(p) \in A$$

such that $\{V_p\}_{p\in P}$ still covers M.
Since M is a smooth manifold, we may (by shrinking each V_p if necessary) assume that for every $p \in P$ there is a smooth chart

$$(V_p, \varphi_p), \quad \varphi_p : V_p \xrightarrow{\sim} B_1(0) \subset \mathbb{R}^n$$

where $n = \dim M$ and $B_1(0)$ denotes the open unit ball. Shrinking each V_p preserves both the refinement property $V_p \subset U_{\alpha(p)}$ and local finiteness.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be the bump function given by Theorem 2.1, so that

$$\operatorname{supp} f \subset \{x \in \mathbb{R}^n : \|x\| \leq 1\} = \overline{B_1(0)}, \quad f(x) \in (0, 1] \text{ for } \|x\| < 1$$

In particular, $f > 0$ on $B_1(0)$. For each $p \in P$, define a smooth function $f_p : M \to \mathbb{R}$ by

$$f_p(x) := \begin{cases} f\big(\varphi_p(x)\big), & x \in V_p, \\ 0, & x \in M \setminus V_p \end{cases}$$

Then $f_p \in C^\infty(M)$, and

$$\operatorname{supp} f_p \subset \varphi_p^{-1}\big(\operatorname{supp} f\big) \subset \varphi_p^{-1}\big(\overline{B_1(0)}\big) \subset V_p \subset U_{\alpha(p)}$$

Moreover, since $f > 0$ on $B_1(0)$ and $\varphi_p(V_p) = B_1(0)$, we have

$$f_p(x) > 0 \quad \text{for all } x \in V_p$$

The family $\{V_p\}_{p\in P}$ is locally finite, hence the family of supports $\{\operatorname{supp} f_p\}_{p\in P}$ is also locally finite. Define

$$\psi(x) := \sum_{p\in P} f_p(x), \qquad x \in M$$

Because for each $x \in M$ only finitely many p satisfy $x \in V_p$ (local finiteness), the above sum is finite at every point x, so ψ is well-defined and smooth. Furthermore, since $\{V_p\}$ covers M, for every $x \in M$ there exists $p_0 \in P$ with $x \in V_{p_0}$, hence $\psi(x) \geq f_{p_0}(x) > 0$. Thus $\psi(x) > 0$ for all $x \in M$.
Now define, for each $p \in P$,

$$\rho_p(x) := \frac{f_p(x)}{\psi(x)}, \quad x \in M$$

Since f_p and ψ are smooth and ψ has no zeros, each ρ_p is smooth. We have $0 \leq \rho_p(x) \leq 1$ for all x, and

$$\operatorname{supp} \rho_p \subset \operatorname{supp} f_p \subset U_{\alpha(p)}$$

The local finiteness of $\{V_p\}$ implies that for each $x \in M$ only finitely many $\rho_p(x)$ are nonzero, and

$$\sum_{p\in P} \rho_p(x) = \frac{1}{\psi(x)} \sum_{p\in P} f_p(x) = \frac{\psi(x)}{\psi(x)} = 1, \qquad \forall x \in M$$

Therefore, $\{\rho_p\}_{p\in P}$ is a partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$. This completes the proof of the theorem. ☐

2.2 Some Application of the Partition of Unity

In this section, we will see how powerful the partition of unity is in dealing with problems related to smooth functions on manifolds. We will discuss some of the most important applications of the partition of unity, including a weaker version of the Whitney embedding theorem, the Whitney approximation theorem, the smooth extension of a smooth function on a subset of the manifold, and related topics.

Theorem 2.3 ((Weak) Whitney Embedding Theorem)**.** Every compact differentiable manifold M can be embedded into an Euclidean space $\mathbb{R}^N$.

Proof. With the compactness of M, consider $\dim M = n$, one shall take finite local charts $\{(U_i, \varphi_i) \mid i = 1, \dots, k\}$ such that $\varphi_i(U_i) = B_2(0)$ and $\{V_i = \varphi_i^{-1}(B_{1/2}(0)) \mid i = 1, \dots, k\}$. Consider the bump function of the partition of unity $\{\rho_i : M \to [0, 1]\}$ associate to this covering, such that

$$\rho_i|_{V_i} \equiv 1, \quad \operatorname{supp}(\rho_i) \subseteq \varphi_i^{-1}(B_1(0))$$

Defined the map $F : M \to \mathbb{R}^{kn} \times \mathbb{R}^k$ as

$$F(x) = \big(\rho_1(x)\varphi_1(x), \dots, \rho_k(x)\varphi_k(x), \rho_1(x), \dots, \rho_k(x)\big)$$

which by zero extension using ρ_i, we treat $\rho_i\varphi_i$ as the smooth function on M. We know the following facts:

- F is injective: Since if $F(x) = F(y)$, $\rho_i(x) = \rho_j(y)$ $\forall 1 \leq i \leq k$. Let $x \in V_i$, then $\rho_i(x) = 1$ and $\rho_i(y) = \rho_i(x) = 1$ and thus $\varphi_i(x) = \varphi_i(y)$. Since φ_i is a local diffeomorphism, $x = y$.

- F has nondegenerate Jacobi matrix: Since $\rho_i(x) \equiv 1\ \forall x \in V_i$, by definition F is embedded on V_i.

Thus, we get the injective immersion from M into $\mathbb{R}^N$, $N = k(n+1)$, since M compact, it is an embedding. Hence, we complete the proof of the theorem. □

Actually, H. Whitney [48, 49] proved that any n-dimensional smooth manifold can be embedded into $\mathbb{R}^{2n}$. The stronger version of this theorem will be include in the following section since the proof demands the use of Sard's theorem.

Theorem 2.4 (Smooth Extension of Functions)**.** The smooth real-valued function on a differentiable manifold M can be extended in the following ways:

1. Let C be a closed set on differentiable manifold M, $U \subseteq M$ is a open neighborhood of C. Then there exists $f : M \to \mathbb{R}$ such that $f|_C \equiv 1$ and $f|_{X \setminus U} \equiv 0$.
2. Let A be the subset of M, $f \in C^\infty(A, \mathbb{R})$. If $\forall x \in B : \exists U_x \subseteq M$ be a open neighborhood of x and smooth function $f_x : U_x \to \mathbb{R}$ such that $f|_{B \cap U_x} = f_x|_{B \cap U_x}$, then $\exists V \subseteq M$ open neighborhood of A with smooth function $\tilde{f} : V \to \mathbb{R}$ such that $\tilde{f}$ is the extension of f, i.e., $\tilde{f}|_B = f$.

Proof. (1) One shall consider the open cover $\{C, M \setminus C\}$ and the partition of unity $\{\phi, \psi\}$ associate to this covering. Let supp$\phi \subseteq C$, and supp$\psi \subseteq M \setminus C$. Since $\psi|_A = 0$, $\phi|_A + \psi_A \equiv 1$, we know that $\phi|_A \equiv 1$. Then ϕ is the function we want.
(2) Consider the open set defined by

$$V = \bigcup_{x \in B} U_x$$

Let $\{\rho_i\}$ be the partition of unity associated to the open cover $\{U_x \mid x \in B\}$. By second countability, one shall take a countable covering labeled with $x_i \in B$, and take supp$(\rho_i) \subseteq U_{x_i}$. The function we need is

$$\tilde{f} := \sum_i \rho_i(x) f_{x_i}(x) \quad \forall x \in V$$

where $\rho_i f_{x_i}$ can be viewed as the extension of f_{x_i} on V, which equals to zero for any $x \in V \setminus U_{x_i}$. □

Furthermore, the extension theorem can also be generalized to any smooth map between manifolds.

Corollary. *For a smooth map $f \in C^\infty(M, S)$ where $M \subseteq N$ is a closed regular submanifold. Then $\exists \tilde{f} \in C^\infty(N, S)$ such that $\tilde{f}|_M = f$.*

Proof. This corollary follows from the characterization of regular submanifolds (Theorem 1.6). The regular submanifold is locally closed. Thus, from the second proposition in Theorem 2.4, the smooth map can be extended to an open neighborhood, and by the closeness together with the first proposition in the theorem, one can find a global extension. □

If we remove the closeness, the local coordinate is still applicable, but the function can no longer have a global extension.

Theorem 2.5 (Smooth Approximation of Continuous Functions)**.** Let $f : M \to \mathbb{R}^k$ be continuous map, then $\forall \epsilon \in C(M, \mathbb{R}_{>0})$, there exists a smooth map $g : M \to \mathbb{R}^k$ such that

$$\|f(x) - g(x)\| \leq \epsilon(x) \quad \forall x \in M$$

Proof. Since both f and ϵ are both continuous, for any $x \in M$ one can find neighborhood $U_x \subseteq M$ such that $\forall y \in U_x$, the following constraint of continuous function holds:

$$\epsilon(y) \geq \frac{1}{2}\epsilon(x), \quad \|f(y) - f(x)\| \leq \frac{1}{2}\epsilon(x)$$

Consider $\{U_x \mid x \in M\}$, and let $\{\rho_i \in C^\infty(M, [0,1]) \mid x_i \in M\}$ be the partition of unity associated to the open cover. For any i, let $\mathrm{supp}(\rho_i) \subseteq U_{x_i}$ with some $x_i \in M$. Let the smooth function be defined by

$$g(x) := \sum_i \rho_i(x) f(x_i), \quad \forall x \in M$$

The smoothness is ensured by the smoothness of the partition of unity, since $f(x_i)$ is just a constant. We shall check the approximation condition:

$$\begin{aligned}
\|g(x) - f(x)\| &\leq \sum_i g_i(x)\|f(x_i) - f(x)\| \\
&= \sum_{g_i(x)\neq 0} g_i(x)\|f(x_i) - f(x)\| \leq \frac{1}{2}\sum_i g_i(x)\epsilon(x) \\
&\leq \sum_{g_i(x)\neq 0} g_i(x)\epsilon(x) = \epsilon(x)
\end{aligned}$$

Thus, the theorem was proved. □

Two maps $f, g : M \to N$ are homotopy if $\exists H \in C(I \times M, N)$ with $I := [0,1]$, such that $\forall x \in M$

$$H(0, x) = f(x), \quad H(1, x) = g(x)$$

The homotopy is an equivalent relation on $C(M, N)$, denoted as $H : f \sim g$.

Proposition 2.4. *The following claim holds for the homotopy of maps on manifolds:*

1. $\forall f \in C(M) : \exists g \in C^\infty(M, N)$ *such that* $f \sim g$.
2. $\forall f_0, f_1 \in C^\infty(M, N)$ *that* $f_0 \sim f_1$*, then* $\exists F \in C^\infty(I \times M, N)$ *such that* $F : f_0 \sim f_1$.

Proof. (1) We shall first consider the continuous homotopy

$$H(s, x) = (1 - s)f(x) + sg(x)$$

with $g(x)$ be the smooth function in Theorem 2.5. Thus, the first proposition got proved.
(2) To prove the second statement, we shall first consider continuous homotopy

$$H : I \times M \to N, \quad H : f_0 \sim f_1$$

The key idea is to use the Theorem 2.5 claim the existence of the function. However, we need to preserve the value on the boundary of the interval to ensure the smooth map still being a homotopy, so we need some modify on the condition. Let $\varphi : I \to I$ such that for some $0 < \delta < 1/2$, the function satisfies

$$\varphi|_{[0,\delta]} \equiv 0, \quad \varphi|_{[1-\delta,1]} \equiv 1$$

Let the new function $H_1(t, x) := H(\varphi(t), x)$ $\forall x \in M$, by the definition of homotopy,

$$H_1(t, x) = f_0(x) \ \forall t \in [0, \delta]; \quad H(t, x) = f_1(x) \ \forall t \in [1 - \delta, 1]$$

Then, by Theorem 2.5, we shall take $G \in C^\infty([\delta, 1 - \delta] \times M, N)$ such that

$$\left\| H|_{[\delta,1-\delta]\times M} - G \right\| \leq \epsilon(x)$$

and by Theorem 2.2, we shall take bump function such that

$$\beta(t) \equiv 0,\ t \in [0, \delta] \cup [1-\delta, 0]; \quad \beta(t) \equiv 1,\ t \in [\delta, 1-\delta]$$

Let the smooth homotopy be

$$\tilde{H}(t,x) = (1-\beta(t))H_1(t,x) + \beta(t)G(t,x)$$

which proves the theorem. □

Proposition 2.5 (Existence of Smooth Proper Maps)**.** *Smooth, proper maps always exist on differentiable manifolds.*

Proof. Let M be a smooth manifold with open cover $\{U_i\}$ such that $\overline{U_i}$ compact. Let $\{\rho_i\}$ be the partition of unity associated with the open cover. We shall construct the smooth function $\rho : M \to \mathbb{R}$

$$\rho(x) := \sum_k k\rho_k(x)$$

The smoothness is preserved by the definition of partition of unity. If for $x \in M$, $\rho_i(x) = 0\ \forall i < k$

$$\rho(x) = \sum_{i \geq k} i\rho_i(x) \geq k \cdot \sum_{i \geq k} \rho_i(x) = k$$

In other words,

$$\rho^{-1}[0,k] \subseteq \bigcup_{i=1}^{k} \operatorname{supp}(\rho_i) \subseteq \bigcup_{i=1}^{k} \overline{U_i}$$

Since closed subsets of compact sets are compact, $\rho^{-1}[0,k]$ is compact. □

2.3 Critical Points and Sard's Theorem

In Section 1.2, we introduced the rank of a smooth map as a basic measure of its regularity. In the previous chapter, we focused on the strongest situation—maps of constant rank—and showed that constant rank hypotheses lead naturally to the existence of submanifolds. In differential topology, however, one cannot generally expect such uniform regularity. Instead, one often perturbs a given map slightly to achieve the desired generic properties (and the perturbation preserves the generic property we have).

In this section, we prove Sard's theorem [41], due to A. Sard (1942), which asserts that the set of critical values of a smooth map has measure zero. Equivalently, the image of the set of non-regular (critical) points is a null set in the target.

Definition 2.4 (Regular/Critical Points)**.** For C^∞-map $f : M \to N$, $p \in M$ is a regular point if $f_{*,p}$ is surjective. The point $q \in N$ is said to be a regular value if $f^{-1}(q)$ contains only regular points. Otherwise, p is a critical point and $f(p)$ is a critical value.

The first observation that we can made is that the "number of preimages" $|f^{-1}(q)|$ of a regular value is locally constant.

Proposition 2.6. *Let $f : M \to N$ be a C^∞-function with regular value $q \in N$ and M compact. Then, there exists an neighborhood $V \subseteq N$ contains q that $\forall q' \in V$*

$$|f^{-1}(q')| = |f^{-1}(q)|$$

Proof. The proof is simply by inverse function theory (Theroem 1.2).By the definition of regular value, $\forall p_i \in f^{-1}(q)$, there exists a open neighborhood U_i such that $f|_{U_i} : U_i \to V_i \subseteq N$ is a diffeomorphism. Then, we need the following lemma:

Lemma. *For M compact and $\forall f \in C^\infty(M, N)$, if $q \in N$ is a regular value, then*

$$f^{-1}(q) = \{p_1, \ldots, p_k\}$$

is finite subset in M.

The proof of the lemma is simply follows the fact that since $q \in N$ is regular value, $\dim f^{-1}(q) = 0$, i.e., $f^{-1}(q)$ is discrete and closed (since singleton $\{q\}$ is a close set in N). By the compactness of M, since discrete subset $f^{-1}(q)$ is closed, $f^{-1}(q)$ is compact, and thus, is finite set.
By the Hausdorffness of manifolds, one shall always let $U_i, \ldots, U_k$ disjoint and diffeomorphic to $V_1, \ldots, V_k$. Then, the open set

$$V = \left(\bigcap_{i=1}^k V_i\right) \setminus f\left(M \setminus \bigcup_{i=1}^k U_i\right)$$

Every point $q \in V$ has number of preimage $|f^{-1}(q)| = k$. □

After this small, useful observation, we need to first introduce the concept of measure-zero sets: Recall that in analysis, we defined the open box in $\mathbb{R}^n$ as

$$B := \prod_{i=1}^n (a_i, b_i)$$

with its volumn given by

$$\mathrm{vol}(B) = \prod_{i=1}^n (b_i - a_i)$$

Definition 2.5 (Measure-Zero Sets)**.** A set $S \subseteq \mathbb{R}^n$ is said to be measure-zero if $\forall \epsilon > 0$, there exists mostly countable open box $\{B_i\}$ that

$$S \subseteq \bigcup_i B_i \quad \text{and} \quad \sum_i \mathrm{vol}(B_i) < \epsilon$$

We also denote this property as $\mu(S) = 0$.

In more serious real analysis text, such set is named as the subset in $\mathbb{R}^n$ with Lebesgue measure zero. The following facts are obvious for measure-zero sets:

Proposition 2.7. *The following facts holds for measure zero sets:*

1. *Subsets of measure-zero sets are measure-zero.*
2. *Countable union of measure-zero sets still been measure-zero.*
3. *Nonempty open sets in $\mathbb{R}^n$ are not measure-zero.*
4. *If $m < n$, the $\mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$ is measure zero in $\mathbb{R}^n$.*

The proof is simply and can be find in may classical text of analysis, for example [45]. Recall from the undergraduate mathematical analysis that, a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz if $\forall p, q \in \mathbb{R}^n$

$$\|f(p) - f(q)\| \leq K\|p - q\| \text{ for some } K > 0$$

where the norm is the ordinary Euclidean norm on $\mathbb{R}^n$. If the condition only satisfied in certain subspace S, we shall say that the map is Lipschitz on the subspace S. We shall claim that the Lipschitz is enough for the map to preserve measure-zero, and differentiability also implies the same result.

Proposition 2.8. *The following proposition related to $f : \mathbb{R}^n \to \mathbb{R}^n$ holds:*

1. *Lipschtiz map preserves measure-zero.*
2. *C^1-map preserves measure-zero.*

Proof. For the Lipschitz case, suppose f has Lipschitz constant K, then the box with volume V has volume upper bound $(K\sqrt{n})^n V$ (since the distance between two points x, y in box Q with side length a has estimation $\|x - y\| \leq \sqrt{n}a$). Then consider $E \subseteq \mathbb{R}^n$ with measure-zero:

$$E \subseteq \bigcup_i Q_i, \quad \sum_i \mathrm{vol}(Q_i) \leq \epsilon$$

Then, each Q_i has image contains in a larger box $f(Q_i) \subseteq Q_i'$ with $\mathrm{vol}(Q_i') \leq (K\sqrt{n})^n \mathrm{vol}(Q_i)$. Thus, $f(E) \subseteq \bigcup_i Q_i'$, and

$$\sum_i \mathrm{vol}(Q_i') \leq (K\sqrt{n})^n \sum_i \mathrm{vol}(Q_i) \leq (K\sqrt{n})^n \epsilon \to 0$$

It is also important to note that the claim above only require locally Lipshitz. For the C^1-function case, oen shall consider the Jacobian of the map Jf and $\forall x, y \in \mathbb{R}^n$

$$f(x) - f(y) = \int_0^1 Jf\big(y + t(x - y)\big)(x - y)\mathrm{d}t$$

With the operator norm, let $\gamma(t) := y + t(x - y)$ and $r = \|x - y\|$

$$\|f(x) - f(y)\| \leq \int_0^1 \big\|Jf\big(\gamma(t)\big)\big\|_{\mathrm{op}} \dot{\gamma}(t)\mathrm{d}t \leq M\|x - y\|, \quad M = \sup_{z \in \overline{B_r(x)}} \big(Jf(z)\big)$$

Thus, C^1-function are locally Lipschitz. Since $\mathbb{R}^n$ is second countable, we shall conclude that the arbitrariness (on the choice of $\overline{B_r(x)}$) in the claim above proves the proposition. □

Theorem 2.6 (Sard's Theorem)**.** For any $f \in C^\infty(M, N)$, the set of critical points of f has measure-zero in N.

2.4 Whitney Embedding Theorem and Whitney Approximation Theorem

2.5 Tubular Neighborhood Theorem

2.6 Transversality

2.7 Intersection Numbers and the Euler Characteristic

Chapter 3

Tensors and Calculus on Manifold

The main purpose of the study of geometry is to understand the global invariant properties through out the study of local structure. For this purpose, one should be concern about the induced global phenomena from the fundamental structures we have defined in the first chapter. The best example of this procedure is the fiber bundle:

Definition 3.1 (Fiber Bundles)**.** A fiber bundle (E, B, π, F) consists the following data:

1. Topological spaces E (total space) and B (base space).
2. Continuous surjection $\pi : E \to B$ (projection).
3. Topological space F (fiber).

We shall assume that the base space B is connected and satisfies local trivialization condition, i.e., for any open subset $U \subseteq B$, the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi_U", "\sim"'] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_U"] \\ U & \end{tikzcd}$$

If E and B are both smooth mmanifolds then to state the smoothness of the fiber bundle, π is required to be a smooth map, and thus, a submersion. The following fact describes the topology on fiber bundles:

Proposition 3.1. *Consider fiber bundle (E, B, π, F), the following statement holds:*

1. *$\pi : E \to B$ is an open map, i.e., B has the quotient topology induced by π*
2. *The topology on E can be generate by basis*

$$\mathcal{B} := \bigcup_U \{\phi_\alpha^{-1}(W \times V) \mid W \subseteq U \ \textit{open},\ V \subseteq F \ \textit{open}\}$$

The proof is simple.

3.1 Tangent Bundles and Vector Fields

We shall begin with the definition of the tangent bundle, the global structure induced by the tangent space.

Definition 3.2 (Tangent Bundle). The tangent bundle on smooth manifold M^m is the fiber bundle (TM, M, π, F) which

1. The total space is $TM = \coprod_{p \in M} T_pM \cong \{(p, v_p) \mid \forall p \in M : v_p \in T_pM\}$ and the base space is M. The fiber $F \cong \mathbb{R}^m$ is some m-dimensional $\mathbb{R}$-vector space.

2. The projection is given by $\pi(p, v_p) = p$.

The tangent bundle TM is a $2m$-dimensional differentiable manifold.

3.2 Integration Curve and Flow of Vector Fields

3.3 Integrability Theorem and Foliation

3.4 Vector Bundles

3.5 Tensors

3.6 Differential Forms and Exterior Algebra

3.7 Manifolds with Boundary

3.8 Stokes Formula

Chapter 4

de Rham Theory and Poincaré Lemma

4.1 Poincaré Lemma and Degree of Maps

Theorem 4.1 (Poincaré Lemma)**.** Let M be a differentiable manifold. Every closed form on simply connected region on M is exact.

4.2 The de Rham Cohomology

One of the most profound impact of the Poincaré lemma (Theorem 4.1) [**Derek:** To be complete.]

The de Rham cohomology can be ectend to relative cohomology on any manifold with boundary. First, we need to defined the relative k-forms on smooth manifold M.

Definition 4.1 (Relative k-Forms)**.** Let M be a smooth manifold with boundary ∂M (can be empty). Then the relative k-forms is defined by

$$\Omega^k(M, \partial M) := \{\omega \in \Omega^k(M) \mid \omega|_{\partial M} = 0\}$$

It is easy to prove that the the exterior derivative restrict on the k-forms relative to the boundary sends every relative k-forms to relative $k+1$-forms. Formally,

Proposition 4.1. $\mathrm{d}\big(\Omega^k(M, \partial M)\big) \subseteq \Omega^{k+1}(M, \partial M)$.

Proof. [**Derek:** To be complete.] □

4.3 Homotopy Invariance and Mayer-Vietoris Sequence

4.4 The de Rham Theorem

4.5 Poincaré Duality

4.6 Introduction to Sheaves and Cěch-de Rham Complex

Chapter 5

Distribution and Foliation on Manifolds

Part II

A First Step to Geometry and Topology of Manifolds

Chapter 6

Fundamental Groups and Covering Spaces of Manifolds

6.1 Homotopy and Fundamental Groups

6.2 Van Kampen Theorem

6.3 Covering Spaces and their Classification

6.4 Deck Transformations and Group Actions

6.5 Computation of Some Classical Cases

6.6 Homotopy and Fibrations

Chapter 7

First Step to Geometry on Manifolds

7.1 Riemannian Manifolds as Metric Spaces

7.2 Connections

7.3 Geodisics and Jacobi Fields

7.4 Curvature

7.5 Comparison Theory

7.6 Geometry of Submanifolds

7.7 Homogeneous Space

7.7.1 Geometrical Structure of Lie Groups

7.7.2 Homogeneous Space

7.7.3 Symmetric Space

7.8 Hodge Theory and Harmonic Forms

Chapter 8

Geodesics and Jacobi Fields

8.1 Geodesics and the Variation of Arclength & Energy

8.2 Jacobi Fields

8.3 Conjugate Points and Distance Minimizing Geodesics

8.4 The Rauch Comparison Theorems and Other Jacobi Field Estimates

8.5 Geometric Applications of Jacobi Field Estimates

8.6 Approximate Fundamental Solutions and Representation Formula

Chapter 9

Additional Topics on the Variation Problems

This chapter lies somewhat out of the book's main line of development, and may be omitted in a first reading

Part III

More Advanced Topics on Geometry and Topology

Chapter 10

Geometry on Smooth Fiber Bundles and Chern-Weil Theory

10.1 Fiber Bundles and Principal Bundles

10.2 Connection and Curvature on Principal Bundles

10.3 Chern-Weil Theorem

10.4 Characteristic Classes

10.5 Bott Vanishing Theorem for Foliations

10.6 Bott and Duistermaat-Heckman Formulas

10.7 Gauss-Bonnet-Chern Theorem

Chapter 11

Introduction to Morse Theory

Chapter 12

Introduction to Gauge Theory

12.1 More about Lie Groups: Representations and Group Extensions

12.2 Spinors and Dirac Operator

12.3 Linear Elliptic Operators on Manifolds

12.4 Fredholm Maps

12.5 Gauge Theory with Finite Structure Group

12.6 The Seiberg–Witten Gauge Theory

This chapter lies somewhat out of the book's main line of development, and may be omitted in a first reading

Chapter 13

A Concise Introduction to Metric Geometry

Chapter 14

More Algebraic Topology

This chapter aimed to rewrite all algebraic topology in previous chapters (Fundamental groups, de Rham Cohomology, etc.) in a more modern, more categorical way, as well as introduce the algebraic topology in a more systematic and comprehensive way. This chapter requires reader to have the knowledge of some basic category theory. An introductory level category theory is in the Appendix E.

This chapter lies somewhat out of the book's main line of development, and may be omitted in a first reading

Chapter 15

Geometry and Topology Reflected by Noncommutative Algebras

Chapter 16

Introduction to Complex Geometry

Chapter 17

Introduction to Symplectic Geometry and Quantization

Chapter 18

A Brief Overview of TQFT

18.1 Atiyah's Axiomatic Topological Field Theory

18.2 2D TQFT and Foubinius Algebra

18.3 Higher Categories

18.4 Homotopy Sigma Models (of Finite Type)

18.5 Segral's Conformal Field Theory

18.6 A Survey on Further Topics

Part IV

Appendix

Appendix A

Review on General Topology

The study of modern geometry focuses on connecting the (local) geometrical quantities (e.g., curvature, length, volume, etc.) and the (global) topological properties (genus, Euler's characteristics, fundamental groups, etc.). Often, this connection is given by an integral or some other global operation. Furthermore, the study of geometry focuses on a concept called "topological manifolds", which is a topological space with some "good properties". Thus, it is useful to have a brief review of the concepts in general topology.

Definition A.1 (Topological Space, Open and Closed Sets)**.** A topological space is a tuple $(\mathcal{T}, \mathcal{O})$, where $\mathcal{T}$ is a nonempty (ZFC) set and the additional structure is the topology $\mathcal{O} \subseteq \mathcal{P}(\mathcal{T})$. The element of topology $\mathcal{O}$ is defined to be an open subset of $\mathcal{T}$, which satisfy:

- $\mathcal{T}, \emptyset \in \mathcal{O}$
- $\forall U, V \in \mathcal{O} : U \cap V \in \mathcal{O}$
- $\forall \{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{O} : \bigcup_{\alpha \in I} U_\alpha \in \mathcal{O}$

If $A \subseteq \mathcal{T}$ is open, then $\mathcal{T} \backslash A$ is defined to be a closed set. Without the discussion of topology, we can denote the topological space as X.

Proposition A.1. *For any topological space $(X, \mathcal{O}_X)$, X itself and $\emptyset$ are both open and closed.*

Proof. The proof is simple. Notice that the definition of topology requires $X, \emptyset \in \mathcal{O}_X$, i.e., both X and $\emptyset$ are open. Also, $X \backslash X = \emptyset$ is closed, and $X \backslash \emptyset$ is closed. $\square$

Remark. *For a topological space $(X, \mathcal{O}_X)$,*

- *The definition of topological spaces indicates that the infinite intersection of open sets may not be open, which is reasonable in Euclidean space since the infinite intersection of $\{(-1/n, 1/n) : n \in \mathbb{N}\}$ it s a single point $\{0\}$ that is closed.*
- *Not all subsets of X can be classified as "open" or "closed", i.e., there can be a set that is neither open nor closed.*
- *For some topological space (not a connected space, actually), there can be a subset other than X and $\emptyset$ that is both open and closed.*

It is quite obvious for readers familiar with real analysis that this is a generalization of open sets in $\mathbb{R}^n$. We use the universal properties of open sets in $\mathbb{R}^n$ as the definition of open sets in more general topological spaces. Here are some examples:

Example A.1. *Consider sets* $S = \{1, 2, 3, 4, 5, 6\}$*,*

- $\big\{\{1\}, \{2\}, \{6\}, \{1, 2\}, \{1, 6\}, \{2, 6\}, \{1, 2, 6\}\big\}$ *is a topology on* S*.*
- $\big\{\{1\}, \{2\}, \{5\}, \{1, 2, 5\}\big\}$ *is not a topology on* S*.*

Example A.2. *For any nonempty (ZFC) set* S*, two trivial topologies can be given*

- $\mathcal{O}_{chaotic} = \{\emptyset, S\}$ *is the chaotic topology on* S*.*
- $\mathcal{O}_{descrete} = \mathcal{P}(S)$ *is the discrete topology on* A*.*

For finite sets $|M| \geq 1$, the topology that can be established is given by the following table:

Cardinality $\lvert M\rvert < 1$	Numbers of Topology
1	1
2	4
3	29
4	355
5	6942
6	209527
$\vdots$	$\vdots$

The most "basic" form of open sets is the open neighborhood:

Definition A.2 (Neighborhood)**.** For some topological space X and the point $x \in X$, $V \subseteq X$ is a neighborhood of x if $\exists U \in \mathcal{O}_X : x \in U \subseteq V$. If V is open (closed), then it is an open (closed) neighborhood of x.

Remark. *As a remark, it is important to mention that the neighborhood does not necessarily have to be "small". As an example, since any neighborhood* U *satisfies* $x \in U \subset X$*, then* X *is both an open and a closed neighborhood of any point* $x \in X$*.*

Proposition A.2. *Let* X *be a topological space, and* $x \in X$ *is a point. For some* $V \subseteq X$*, the following statements are equivlant:*

1. *V is the open neighborhood of x.*
2. *V is open, and $x \in V$.*

Proof. ($1 \Rightarrow 2$) By the definition of open neighborhood, we know the following information: V is open (the requirement of "open" neighborhood) and $\exists U \in \mathcal{O}_X : x \in U \subseteq V \implies x \in V$, which proves 2
($2 \Rightarrow 1$) If V is open and $x \in V$, then just take $U = V$ and $x \in V \subseteq V$, which makes V being an open neighborhood of x. □

With the concept of neighborhood being introduced, we can easily distinguish whether a set is open or not.

Theorem A.1 (Determination of Open Sets)**.** Let X be a topological space, $U \subseteq X$ is nonempty subset of X, the following proposition are equivalent:

1. U is open.
2. $\forall x \in U$, $\exists V \subseteq U$ such that V is a open neighborhood of x.

Proof. ($1 \Rightarrow 2$) For some open set U, take arbitrary $x \in U$, then U itself is a open neighborhood of x.
($2 \Rightarrow 1$) By the given condition, $\forall x \in U$ take the corresponding open neighborhood $V_x \subseteq U$

$$\bigcup_{x \in U} V_x = U$$

Since $\forall x \in U : V_x$ are open, then, by the axiom of topological space, any union of open sets is open. Thus, U is open. $\square$

To describe the topology structure on any set (often uncountable infinite sets, like surfaces in $\mathbb{R}^n$), it is useful to discuss some generating sets of the topology, called a (topological) basis. We often choose some of the most "representative" open sets in the topology to form a topological basis.

Definition A.3 (Topological Basis)**.** Let $(X, \mathcal{O}_X)$ be a topological space. For some $\mathcal{B} \in \mathcal{O}_X$, $\mathcal{B}$ is said to be a (topological) basis of the topological space iff

$$\forall U \in \mathcal{O}_X : \forall x \in U : \exists B \in \mathcal{B} : x \in B \subseteq U$$

The definition of basis has an equivalent statement:

Proposition A.3 (Equivalent Description of Topological Basis)**.** *For some set contains open sets $\mathcal{B} \in \mathcal{O}_X$ in the topological space $(X, \mathcal{O}_X)$, the following statements are equivalent:*

1. *$\mathcal{B}$ is the basis of X*
2. *$\forall U \in \mathcal{O}_X : \exists \mathcal{B}' \subseteq \mathcal{B} : U = \bigcup_{B \in \mathcal{B}'} B$*

Proof. ($1 \Rightarrow 2$) For any open set U, for any $x \in U$. By the definition of basis, we can always find $x \in B_x \in \mathcal{B}$ such that $B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. WE can just take $\mathcal{B}' = \{B_x\}_{x \in U} \subseteq \mathcal{B}$, which proves the second statement.
($2 \Rightarrow 1$) By the given condition that $\forall U \in \mathcal{O}_X : U = \bigcup_{B \in \mathcal{B}'} B$, where $\mathcal{B}' \subseteq \mathcal{B}$. Thus, there must be some set $B \in \mathcal{B}'$ that $x \in B \in \mathcal{B}$, which is the definition of basis. $\square$

Example A.3. *The natural topology on $\mathbb{R}^n$ is generated by the following basis:*

$$\mathcal{B} = \{B(x, r) | x \in \mathbb{Q}^n, r \in \mathbb{Q}_{\geq 0}\}, \quad B(x, r) = \{y \in \mathbb{R}^n | x \in \mathbb{R}^n, d(x, y) < r\}$$

which is the topology we used for most metric spaces.

Definition A.4 (Second Countable, A_2)**.** A topological space is said to be second countable iff it can be generated by a countable basis.

By the given example of a topological basis, an obvious fact is that $\mathbb{R}^n$ is second countable. A further result is that any metric space is second countable. The topological basis of a topological space has the following properties:

Proposition A.4 (Properties of Basis)**.** *Let $\mathcal{B} \subset \mathcal{O}_X$ is a basis of topological space $(X, \mathcal{O}_X)$, then $\mathcal{B}$ has the following properties:*

1. *$\forall x \in X : \exists B \in \mathcal{B} : x \in B$*
2. *$\forall B_1, B_2 \in \mathcal{B} : \forall x \in B_1 \cap B_2 : \exists B \in \mathcal{B} : x \in B \subseteq B_1 \cap B_2$*

With the properties above, an important technique is to use the basis defined topology on a set. [**Derek:** To be finished.]

After all of the previous definition and proposition of topological spaces, here is a better explanation of the motivation for the definition of topology.

Topology is the minimum structure to define the continuity of the map.

As a generalization of the open set and continuity of functions in Euclidean space $\mathbb{R}^n$. We can have the following definition:

Definition A.5 (Continuity of functions). Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, the map $f : X \to Y$ is continuous if

$$\forall V \in \mathcal{O}_Y : f^{-1}(V) \in \mathcal{O}_X$$

A necessary step to check the well-definedness of continuity is to consider the $\mathbb{R}^n$ and the natural topology of it.

Proposition A.5 (Continuity on $\mathbb{R}^n$ and Topological Continuity). *We take the topological space $X = \mathbb{R}^n$ and the standard topology in Euclidean space. The continuity of real functions in analysis and topological continuity are equivalent.*

Appendix B

Linear Algebra

Appendix C

Real and Functional Analysis

Appendix D

Sobolev Space and Linear PDEs

Appendix E

A Brief Introduction to Category

Appendix F

A Functorial Way on Defining the Smooth Manifolds: C^∞-Schemes

In this appendix, we are going to illustrate a famous quote:

> The smoothness of a manifold is equivalent to the given (legal) collection of C^∞-functions.

Thus, another aspect of defining a smooth structure is to start with the "smooth functions".

F.1 C^∞-Ring

Definition F.1 (C^∞-Ring)**.** A C^∞-ring consists the pair (A, Ψ) such that A is some nonempty set and $\Psi := \{\Psi_f \mid \forall n \in N : f \in C^\infty(\mathbb{R}^n)\}$ which $\forall n \in \mathbb{N} : \forall f \in C^\infty(\mathbb{R}^n)$

$$\Psi_f : A^n \to A$$

The pair satisfy the following axioms:

1. (Projection Axiom): Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the i-th coordinate projection $(1 \leq i \leq n)$, then

$$\forall (a_1, \dots, a_n) \in A^n : \Psi_{\pi_i}(a_1, \dots, a_n) = a_i$$

2. (Composition Axiom): Take smooth map $f = (f_1, \dots, f_m) \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$. $\forall g \in C^\infty(\mathbb{R}^m)$, the composition of g and x is defines by

$$\forall x \in \mathbb{R}^n : h(x) = g \circ f(x) := g\big(f_1(x), \dots, f_m(x)\big)$$

Then, $\forall a = (a_1, \dots, a_n) \in A^n$

$$\Psi_h(a) = \Psi_g\big(\Psi_{f_1}(a), \dots, \Psi_{f_m}(a)\big)$$

3. (Unitality Axiom) As an additional convention, we defined $A^0 = \mathbb{R}^0 = \{\emptyset\}$, and any constant $c \in \mathbb{R}$ can be viewed as $c : \mathbb{R}^0 \to \mathbb{R}$ which has image $\{c\}$. By this axiom, we can define the map

$$\iota : \mathbb{R} \to A, \quad \forall c \in \mathbb{R} : \iota(c) = \Psi_c(\emptyset) \in A$$

The following facts can be directly observed from the axiom:

1. For any permutation of coordinates $\sigma : \mathbb{R}^n \to \mathbb{R}^n$, $\forall f \in C^\infty(\mathbb{R})$ the map $\Psi_{f\circ\sigma}$ is uniquely determined by the map Ψ_f and the projection map.
2. Cinsider the identity map $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, $\mathrm{id}_A = \Psi_{\mathrm{id}_R}$.

As the most fundamental result (that makes A really be the collection of some smooth function) of the axioms, A is a commutative, associative, unital $\mathbb{R}$-algebra. In more detail,

Proposition F.1. *The C^∞-ring provides a commutative, associative, unital $\mathbb{R}$-algebra defines by*

1. *The addition $\Psi_+ : A^2 \to A$ is induced by the addition $+ : \mathbb{R}^2 \to \mathbb{R}$.*
2. *The multiplication $\Psi_\times : A^2 \to A$ is also induced by the multiplication on $\mathbb{R}$.*
3. *The addition identity is given by $0_A = \Psi_0 = \iota(0) \in A$ and the multiplication identity is given by $1_A = \Psi_1 = \iota(1) \in A$. The addition and muliplication identity are unique.*
4. *The addition inverse is given by $\Psi_- : A \to A$ whcih induced by the map $\forall x \in \mathbb{R} : i(x) = -x$.*
5. *The scalar product is naturally induced by the multiplication on $\mathbb{R}$. Let $\forall \lambda, x \in \mathbb{R} : m_\lambda(x) = \lambda x$, then $\lambda \cdot a := \Psi_{m_\lambda}(a)$ and it is distributive on addition.*

And one can check that the addition and multiplication are both commutative and associative.

Proof. Firstly, we check the commutativity of Ψ_+. We shall use the fact that consider the order reversing map $\forall (x, y) \in \mathbb{R}^2 : \tau(x, y) = (y, x)$ by the commutativity of addition in $\mathbb{R}$ and the composition axiom, $\forall a_1, a_2 \in A$

$$\begin{aligned} a_1 + a_2 &:= \Psi_+(a_1, a_2) = \Psi_{+\circ\tau}(a_1, a_2) = \Psi_+\big(\Psi_{\pi_1\circ\tau}(a_1, a_2), \Psi_{\pi_2\circ\tau}(a_1, a_2)\big) \\ &= \Psi_+\big(\Psi_{\pi_2}(a_1, a_2), \Psi_{\pi_1}(a_1, a_2)\big) = \Psi_+(a_2, a_1) =: a_2 + a_1 \end{aligned}$$

The same trick also applies to multiplication.
Secondly, we shall prove the associativity of addition (and the same trick also works for multiplication). Consider arbitrary $a_1, a_2, a_3 \in A$

$$\Psi_+(a_1, \Psi_+(a_2, a_3)) = \Psi_{+\circ(\mathrm{id}_{\mathbb{R}},\times)}(a_1, a_2, a_3) = \Psi_{+\circ(\times,\mathrm{id}_{\mathbb{R}})}(a_1, a_2, a_3) = \Psi_+(\Psi_+(a_1, a_2), a_3)$$

Furthermore, using the associative property of addition and multiplication, any finite sum/product in A can be defined.
Thirdly, we shall check the distributive property of scalar multiplication. $\forall \lambda \in \mathbb{R} : \forall a_1, a_2 \in A$

$$\lambda \cdot (a_1 + a_2) := \Psi_{m_\lambda}\big(\Psi_+(a_1, a_2)\big) = \Phi_{m_\lambda\circ+}(a_1, a_2) = \Psi_+\big(\Psi_{m_\lambda}(a_1), \Psi_{m_\lambda}(a_1)\big) =: \lambda \cdot a_1 + \lambda \cdot a_2$$

This shows the distributive property of scalar multiplication. Using the same trick, one can also show that $\Psi_{m_1} = \mathrm{id}_A$.
Finally, we shall show that the existence of additional identities and inverses, the existence of the multiplication identity, and the uniqueness of identities are easy to show, and we will leave them as exercises. $\forall a \in A :$

$$\begin{aligned} 0_A + a &:= \Psi_+(\Psi_0, a) = \Psi_{+(0,\mathrm{id}_{\mathbb{R}})}(a) = \Psi_{\mathrm{id}_R}(a) = a \\ 1_A \cdot a &:= \Psi_\times(\Psi_1, a) = \Psi_{\times(1,\mathrm{id}_{\mathbb{R}})}(a) = a \end{aligned}$$

Thus, Ψ_0 and Ψ_1 are the addition and multiplication identities. □

The axioms have an equivalent form as a functor. Let **Euc** be the category of all $\mathbb{R}^n$ with $n \in \mathbb{N}$ and morphisms be smooth maps, the product in the category is given by the Cartesian product.

Definition F.2 (C^∞-Ring as Functor)**.** A C^∞-ring is a functor

$$A : \mathbf{Euc} \to \mathsf{Set}$$

That preserves the finite product, i.e., $\forall n \in \mathbb{N} : A(\mathbb{R}^n) \cong A(\mathbb{R})^n$. Then, $\forall f \in \mathrm{Hom}_{\mathbf{Euc}}(\mathbb{R}^n, \mathbb{R}^m)$, there is a corresponding morphism

$$\Psi_f := A(f) : A(\mathbb{R}^n) \to A(\mathbb{R}^m)$$

The category of C^∞-rings is demoted as $\mathbf{C^\infty Ring}$, $\forall A, B \in \mathbf{C^\infty Ring}$, the morphism is given by the map between sets that preserves the arithmetic, i.e., $\forall \varphi \in \mathrm{Hom}_{\mathbf{C^\infty Ring}}(A, B) : \varphi : A \to B$ such that

$$\varphi\big(\Psi_f^A(a_1, \ldots, a_n)\big) = \Psi_f^B\big(\varphi(a_1), \ldots, \varphi(a_n)\big)$$

Definition F.3. The ring of C^∞-functions on a manifold M is defined to be the functor

$$A_M : \mathbf{Euc} \to \mathsf{Set}, \quad A_M(\mathbb{R}^n) := \mathbb{C}^\infty(M, \mathbb{R}^n)$$

It is easy to check that A is in $\mathbf{C^\infty Ring}$.

In particular, there is a remarkable morphism (A natural transformation on A) called $\mathbb{R}$-point:

Definition F.4 (R-Point)**.** $\forall A \in \mathbf{C^\infty Ring}$, the R-point of A is a morphism (a natural transformation in functor category)

$$\xi : A \Longrightarrow \mathbb{R}$$

Where $\mathbb{R} \in \mathbf{C^\infty Ring}$ is the functor given by $\mathbb{R}(\mathbb{R}^n) = \mathbb{R}^n$, $\mathbb{R}(f) = f$. On object level, the natural transformation gives a map $\xi_\mathbb{R} : A(\mathbb{R}) \to \mathbb{R}$. By the definition of natural transformation, the following diagram commutes

$$\begin{tikzcd}
A(\mathbb{R}^n) \cong A(\mathbb{R})^n \arrow[r, "A(f)"] \arrow[d, "\xi_{\mathbb{R}^n} = (\xi_\mathbb{R})^{\times n}"'] & A(\mathbb{R}) \arrow[d, "\xi_\mathbb{R}"] \\
\mathbb{R}^n \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

The definition above naturally induced the evaluation map $\mathrm{ev}_\xi := \xi_\mathbb{R} : A(\mathbb{R}) \to \mathbb{R}$, which satisfies $\forall a_1, \ldots, a_n \in A : \forall f \in C^\infty(\mathbb{R}^n, \mathbb{R})$

$$\mathrm{ev}_\xi \Psi_f(a_1, \ldots, a_n) = \Psi_f(\mathrm{ev}_\xi(a_1), \ldots, \mathrm{ev}_\xi(a_n))$$

It can be shown that the abstract "$\mathbb{R}$-points" is equivalent to the "geometrical points" on the manifold.

[**Derek:** To be complete]

F.2 C^∞-Scheme

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