Notes on MATH 535 Fall 2025

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Lecture 1: Topology on Metric Spaces

1. Metric spaces

Notation. *Throughout,* $\mathbb{R}_+ := [0, \infty)$*. If* X *is a set, we denote by* $\mathcal{P}(X)$ *its power set.*

Definition 1 (Euclidean distance)**.** For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, the *Euclidean distance* is

$$d_2(x, y) := \Big(\sum_{i=1}^{n} (x_i - y_i)^2\Big)^{1/2}.$$

Definition 2 (Metric)**.** Let X be a set. A *metric* (or *distance*) on X is a map $d : X \times X \to \mathbb{R}_+$ such that, for all $x, y, z \in X$,

(M1) (*separation*) $d(x, y) = 0 \iff x = y$;

(M2) (*symmetry*) $d(x, y) = d(y, x)$;

(M3) (*triangle inequality*) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a *metric space.*

Remark. *Condition* (M1) implies $d(x, y) \geq 0$ for all x, y since $d(x, y) \in \mathbb{R}_+$ by definition.

Example.
1. *On* $\mathbb{R}^n$*, the map* d_2 *of Definition 1 is a metric.*
2. *(*Discrete metric*) On any set* X*, the map*

$$d_{\text{disc}}(x, y) := \begin{cases} 0, & x = y, \\ 1, & x \neq y \end{cases}$$

is a metric.

3. *For* $p \in [1, \infty)$*, on* $\mathbb{R}^n$ *the formula*

$$d_p(x, y) := \Big(\sum_{i=1}^{n} |x_i - y_i|^p\Big)^{1/p}$$

defines a metric; for $p = \infty$ *one sets* $d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|$*.*

2. Balls and open sets

Definition 3 (Balls)**.** Let (X, d) be a metric space, $x \in X$, and $r \in \mathbb{R}_+$.

1. The *open ball* of center x and radius r is

$$B_d(x, r) := \{y \in X : \ d(x, y) < r\}.$$

2. The *closed ball* of center x and radius r is

$$\overline{B}_d(x, r) := \{y \in X : \ d(x, y) \leq r\}.$$

When the metric is clear from the context, we write $B(x, r)$ and $\overline{B}(x, r)$.

Lemma (Balls are stable under shrinking at interior points)**.** *Let (X,d) be a metric space, $x \in X$, and $r > 0$. For every $y \in B(x,r)$ there exists $\rho > 0$ such that*

$$B(y,\rho) \subseteq B(x,r).$$

Proof. Fix $y \in B(x,r)$, so $d(x,y) < r$. Set $\rho := r - d(x,y) > 0$. If $z \in B(y,\rho)$, then $d(y,z) < \rho$, hence by the triangle inequality,

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \rho = r,$$

so $z \in B(x,r)$. □

Definition 4 (Open subset of a metric space)**.** Let (X,d) be a metric space. A subset $U \subseteq X$ is said to be *open (for d)* if for every $x \in U$ there exists $r > 0$ such that

$$B(x,r) \subseteq U.$$

Proposition 1. *In a metric space (X,d), the sets $\varnothing$ and X are open; moreover, every open ball $B(x,r)$ is open.*

Proof. The cases $\varnothing$ and X are immediate. Let $U = B(x,r)$ and let $y \in U$. By Lemma , there exists $\rho > 0$ such that $B(y,\rho) \subseteq B(x,r) = U$, hence U is open. □

Proposition 2 (Open sets form a topology)**.** *Let (X,d) be a metric space and let $\mathcal{T}_d$ be the collection of all open subsets of X (in the sense of Definition 4). Then:*

1. *$\varnothing \in \mathcal{T}_d$ and $X \in \mathcal{T}_d$;*
2. *if $U, V \in \mathcal{T}_d$, then $U \cap V \in \mathcal{T}_d$;*
3. *if $(U_\alpha)_{\alpha \in A}$ is a family in $\mathcal{T}_d$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_d$.*

Proof. (1) is Proposition 1.

(2) Let $x \in U \cap V$. Since U is open, there exists $r > 0$ with $B(x,r) \subseteq U$; since V is open, there exists $s > 0$ with $B(x,s) \subseteq V$. Put $t := \min\{r,s\} > 0$. Then $B(x,t) \subseteq B(x,r) \cap B(x,s) \subseteq U \cap V$, so $U \cap V$ is open.

(3) Let $x \in \bigcup_{\alpha \in A} U_\alpha$. Choose $\alpha_0 \in A$ with $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $r > 0$ with $B(x,r) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in A} U_\alpha$. Hence the union is open. □

3. Abstract topological spaces

Definition 5 (Topology; topological space)**.** Let X be a set. A *topology* on X is a subset $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

(T1) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;

(T2) for every family $(U_\alpha)_{\alpha \in A}$ in $\mathcal{T}$, one has $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$;

(T3) for all $U, V \in \mathcal{T}$, one has $U \cap V \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called *open sets*. The pair $(X, \mathcal{T})$ is called a *topological space*.

Remark. *By induction using* (T3), a finite intersection of open sets is open. Conversely, (T3) follows from closure under finite intersections.

Definition 6 (Topology induced by a metric)**.** Let (X, d) be a metric space and let $\mathcal{T}_d$ be the collection of d-open subsets of X (Definition 4). The topology $\mathcal{T}_d$ is called the *topology induced by d*. The corresponding topological space $(X, \mathcal{T}_d)$ is called the *topological space associated to the metric space* (X, d).

Proposition 3. *For every metric space (X, d), $\mathcal{T}_d$ is a topology on X.*

Proof. This is exactly Proposition 2. □

4. Bases and neighborhoods

Definition 7 (Basis of a topology)**.** Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* of $\mathcal{T}$ if:

(B1) for every $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$;

(B2) for all $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

Proposition 4 (Open balls form a basis)**.** *Let (X, d) be a metric space and let*

$$\mathcal{B}_d := \{B(x, r) : x \in X,\ r > 0\}.$$

Then $\mathcal{B}_d$ is a basis of the topology $\mathcal{T}_d$.

Proof. (B1) holds since for each $x \in X$ one has $x \in B(x, 1)$.

For (B2), let $x \in B(x_1, r_1) \cap B(x_2, r_2)$. Then $x \in B(x_1, r_1)$, so by Lemma there exists $\rho_1 > 0$ with $B(x, \rho_1) \subseteq B(x_1, r_1)$. Similarly, there exists $\rho_2 > 0$ with $B(x, \rho_2) \subseteq B(x_2, r_2)$. Put $\rho := \min\{\rho_1, \rho_2\} > 0$. Then $B(x, \rho) \subseteq B(x_1, r_1) \cap B(x_2, r_2)$. □

Definition 8 (Neighborhood)**.** Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A subset $V \subseteq X$ is a *neighborhood* of x if there exists an open set $U \in \mathcal{T}$ such that $x \in U \subseteq V$.

Remark. *In a metric space (X, d), a set V is a neighborhood of x if and only if there exists $r > 0$ such that $B(x, r) \subseteq V$.*

5. Closed sets, interior, and closure

Definition 9 (Closed set)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $F \subseteq X$ is *closed* if $X \setminus F$ is open.

Proposition 5. *Let $(X, \mathcal{T})$ be a topological space. Then:*

1. *$\varnothing$ and X are closed;*
2. *an arbitrary intersection of closed sets is closed;*
3. *a finite union of closed sets is closed.*

Proof. This follows from Definition 9 and the stability properties (T2)–(T3) by taking complements. □

Definition 10 (Interior)**.** Let $(X,\mathcal{T})$ be a topological space and $A \subseteq X$. The *interior* of A, denoted $\operatorname{int}(A)$, is the union of all open subsets contained in A:

$$\operatorname{int}(A) := \bigcup\{U \in \mathcal{T} :\ U \subseteq A\}.$$

Definition 11 (Closure)**.** Let $(X,\mathcal{T})$ be a topological space and $A \subseteq X$. The *closure* of A, denoted $\overline{A}$, is the intersection of all closed subsets containing A:

$$\overline{A} := \bigcap\{F \subseteq X :\ F \text{ is closed and } A \subseteq F\}.$$

Proposition 6. *Let $(X,\mathcal{T})$ be a topological space and $A \subseteq X$.*

1. $\operatorname{int}(A)$ *is open and* $\operatorname{int}(A) \subseteq A$*; moreover, it is the largest open subset contained in A.*
2. $\overline{A}$ *is closed and $A \subseteq \overline{A}$; moreover, it is the smallest closed subset containing A.*

Proof. Both statements are immediate from the defining union/intersection and the stability properties of open/closed sets. □

Proposition 7 (Neighborhood characterization of closure)**.** *Let $(X,\mathcal{T})$ be a topological space and $A \subseteq X$. For $x \in X$, the following are equivalent:*

1. *$x \in \overline{A}$;*
2. *for every neighborhood V of x, one has $V \cap A \neq \varnothing$;*
3. *for every open set $U \in \mathcal{T}$ with $x \in U$, one has $U \cap A \neq \varnothing$.*

Proof. (2) $\Leftrightarrow$ (3) is immediate from Definition 8.

(1) $\Rightarrow$ (3): if $x \in \overline{A}$ and U is open with $x \in U$, then $X \setminus U$ is closed and does not contain x; hence it cannot contain $\overline{A}$. Thus $\overline{A} \not\subseteq X \setminus U$, so $A \not\subseteq X \setminus U$, i.e. $A \cap U \neq \varnothing$.

(3) $\Rightarrow$ (1): let F be any closed set with $A \subseteq F$. If $x \notin F$, then $U := X \setminus F$ is open, contains x, and satisfies $U \cap A = \varnothing$, contradicting (3). Hence $x \in F$ for every such F, i.e. $x \in \overline{A}$. □

6. Subspaces

Definition 12 (Induced (subspace) topology)**.** Let $(X,\mathcal{T})$ be a topological space and let $Y \subseteq X$. The *topology induced on Y* is

$$\mathcal{T}|_Y := \{U \cap Y :\ U \in \mathcal{T}\} \subseteq \mathcal{P}(Y).$$

The topological space $(Y, \mathcal{T}|_Y)$ is called a *subspace* of $(X,\mathcal{T})$.

Proposition 8. *Let (X,d) be a metric space and $Y \subseteq X$. Let d_Y be the restriction of d to $Y \times Y$. Then the topology induced by d_Y on Y coincides with the subspace topology $\mathcal{T}_d|_Y$.*

Proof. Let $U \subseteq Y$. By definition, U is open in (Y, d_Y) iff for every $y \in U$ there exists $r > 0$ such that

$$\{z \in Y :\ d(y,z) < r\} \subseteq U,$$

i.e. $(B_d(y,r) \cap Y) \subseteq U$. This is equivalent to U being open in the induced topology $\mathcal{T}_d|_Y$. □

7. Continuous maps (metric and topological formulations)

Definition 13 (Continuity between metric spaces)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is *continuous at* $x_0 \in X$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X) \quad d_X(x, x_0) < \delta \;\Rightarrow\; d_Y\big(f(x), f(x_0)\big) < \varepsilon.$$

It is *continuous* if it is continuous at every point of X.

Definition 14 (Continuity between topological spaces)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is *continuous* if for every $V \in \mathcal{T}_Y$ one has $f^{-1}(V) \in \mathcal{T}_X$.

Proposition 9. *Let (X, d_X) and (Y, d_Y) be metric spaces, and endow X, Y with the induced topologies $\mathcal{T}_{d_X}, \mathcal{T}_{d_Y}$. A map $f : X \to Y$ is continuous in the metric sense (Definition 13) if and only if it is continuous in the topological sense (Definition 14).*

Proof. Assume first that f is continuous in the metric sense. Let $V \in \mathcal{T}_{d_Y}$ and let $x_0 \in f^{-1}(V)$. Then $f(x_0) \in V$, hence there exists $\varepsilon > 0$ such that $B_{d_Y}(f(x_0), \varepsilon) \subseteq V$. By continuity at x_0, there exists $\delta > 0$ such that

$$d_X(x, x_0) < \delta \;\Rightarrow\; d_Y\big(f(x), f(x_0)\big) < \varepsilon,$$

i.e. $f\big(B_{d_X}(x_0, \delta)\big) \subseteq B_{d_Y}(f(x_0), \varepsilon) \subseteq V$, so $B_{d_X}(x_0, \delta) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is open.

Conversely, assume f is continuous in the topological sense. Fix $x_0 \in X$ and $\varepsilon > 0$. The ball $B_{d_Y}(f(x_0), \varepsilon)$ is open in Y, hence its preimage is open in X and contains x_0. Therefore there exists $\delta > 0$ such that $B_{d_X}(x_0, \delta) \subseteq f^{-1}\big(B_{d_Y}(f(x_0), \varepsilon)\big)$, which is exactly the ε–δ condition at x_0. $\square$

Definition 15 (Homeomorphism)**.** A bijection $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a *homeomorphism* if both f and f^{-1} are continuous. In this case X and Y are said to be *homeomorphic*.

Lecture 2: Topology

1. Topological spaces

Notation. *Let X be a set. We write $\mathcal{P}(X)$ for the power set of X. If $A \subseteq X$, we denote its complement in X by $A^c := X \setminus A$. We also fix $\mathbb{N} := \{1, 2, 3, \dots\}$.*

Definition 16 (Topology; topological space)**.** Let X be a set. A *topology* on X is a subset $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

(T1) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;

(T2) for every family $(U_\alpha)_{\alpha \in I}$ in $\mathcal{T}$, one has $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$;

(T3) for all $U, V \in \mathcal{T}$, one has $U \cap V \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*; the elements of $\mathcal{T}$ are called *open subsets* (or *open sets*).

Remark. *By induction using* (T3), any finite intersection of open sets is open.

Definition 17 (Closed subset)**.** Let $(X, \mathcal{T})$ be a topological space. A subset $F \subseteq X$ is *closed* if $F^c \in \mathcal{T}$.

Definition 18 (Clopen subset)**.** A subset $A \subseteq X$ is *clopen* if it is both open and closed.

Remark. *A subset of a topological space may be open, closed, clopen, or neither open nor closed.*

Proposition 10 (Stability properties of closed sets)**.** *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{F}$ be the family of closed subsets of X. Then:*

1. *$\varnothing \in \mathcal{F}$ and $X \in \mathcal{F}$;*
2. *for every family $(F_\alpha)_{\alpha \in I}$ in $\mathcal{F}$, one has $\bigcap_{\alpha \in I} F_\alpha \in \mathcal{F}$;*
3. *for all $F_1, F_2 \in \mathcal{F}$, one has $F_1 \cup F_2 \in \mathcal{F}$ (hence any finite union of closed sets is closed).*

Proof. (1) Since $\varnothing^c = X \in \mathcal{T}$ and $X^c = \varnothing \in \mathcal{T}$, both $\varnothing$ and X are closed.
(2) If each F_α is closed, then each F_α^c is open, hence $\bigcup_{\alpha \in I} F_\alpha^c$ is open by (T2). But

$$\Big(\bigcap_{\alpha \in I} F_\alpha\Big)^c = \bigcup_{\alpha \in I} F_\alpha^c,$$

so $\bigcap_{\alpha \in I} F_\alpha$ is closed.
(3) If F_1, F_2 are closed, then F_1^c, F_2^c are open, hence $F_1^c \cap F_2^c$ is open by (T3). But

$$(F_1 \cup F_2)^c = F_1^c \cap F_2^c,$$

so $F_1 \cup F_2$ is closed. $\square$

2. Topologies induced by metrics

Definition 19 (Metric topology)**.** Let (X, d) be a metric space. A subset $U \subseteq X$ is called *d-open* if for every $x \in U$ there exists $r > 0$ such that

$$\{y \in X : \; d(x, y) < r\} \subseteq U.$$

The family $\mathcal{T}_d$ of all d-open subsets is a topology on X, called the *topology induced by d*.

Example (Standard topology on $\mathbb{R}^n$)**.** *On $\mathbb{R}^n$ equipped with the Euclidean metric d_2, the induced topology $\mathcal{T}_{d_2}$ is called the* standard topology *(or* usual topology*) on $\mathbb{R}^n$.*

3. Extremal topologies

Definition 20 (Indiscrete and discrete topologies)**.** Let X be a nonempty set.

1. The *indiscrete topology* on X is $\mathcal{T}_{\min} := \{\varnothing, X\}$.
2. The *discrete topology* on X is $\mathcal{T}_{\max} := \mathcal{P}(X)$.

Proposition 11. *For any topology $\mathcal{T}$ on X one has $\mathcal{T}_{\min} \subseteq \mathcal{T} \subseteq \mathcal{T}_{\max}$.*

Proof. By (T1), $\varnothing, X \in \mathcal{T}$, hence $\mathcal{T}_{\min} \subseteq \mathcal{T}$. Trivially $\mathcal{T} \subseteq \mathcal{P}(X) = \mathcal{T}_{\max}$. □

Lemma (Discrete topology as a metric topology)**.** *Let X be a set and let $d_{\text{disc}} : X \times X \to [0, \infty)$ be defined by*

$$d_{\text{disc}}(x, y) := \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Then the topology induced by d_{disc} is the discrete topology $\mathcal{T}_{\max}$.

Proof. For $x \in X$ one has

$$B(x, 1) := \{y \in X : \; d_{\text{disc}}(x, y) < 1\} = \{x\}.$$

Thus every singleton $\{x\}$ is open. Since every subset of X is a union of singletons, every subset is open, i.e. $\mathcal{T}_{d_{\text{disc}}} = \mathcal{P}(X)$. □

4. Separation: Hausdorff spaces

Definition 21 (Hausdorff (separated) space)**.** A topological space $(X, \mathcal{T})$ is *Hausdorff* (or *separated*) if for any two distinct points $x, y \in X$ there exist $U, V \in \mathcal{T}$ such that

$$x \in U, \quad y \in V, \quad U \cap V = \varnothing.$$

Proposition 12. *Every metric space is Hausdorff for its induced topology.*

Proof. Let (X, d) be a metric space and let $x, y \in X$ with $x \neq y$. Set $r := d(x, y) > 0$ and put $\rho := r/2$. Assume $B(x, \rho) \cap B(y, \rho) \neq \varnothing$, and take z in the intersection. Then

$$d(x, z) < \rho, \qquad d(y, z) < \rho.$$

By the triangle inequality,
$$d(x, y) \leq d(x, z) + d(z, y) < \rho + \rho = r,$$
a contradiction. Hence $B(x, \rho) \cap B(y, \rho) = \varnothing$. □

5. A non-metrizable topology: the cofinite topology

Definition 22 (Cofinite topology)**.** Let X be a set. The *cofinite topology* on X is the family
$$\mathcal{T}_{\text{cof}} := \{\varnothing\} \ \cup \ \{U \subseteq X : \ U^c \text{ is finite}\}.$$

Proposition 13. *For every set X, the family $\mathcal{T}_{\text{cof}}$ is a topology on X.*

Proof. (T1): $\varnothing \in \mathcal{T}_{\text{cof}}$ by definition, and $X \in \mathcal{T}_{\text{cof}}$ since $X^c = \varnothing$ is finite.

(T2): Let $(U_\alpha)_{\alpha \in I}$ be a family in $\mathcal{T}_{\text{cof}}$. If some $U_\alpha = \varnothing$, it does not affect the union, so we may assume $U_\alpha \neq \varnothing$ for all α. Then each U_α^c is finite, and
$$\Big(\bigcup_{\alpha \in I} U_\alpha\Big)^c = \bigcap_{\alpha \in I} U_\alpha^c.$$
An intersection of finite sets is finite; hence the complement of the union is finite, so $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_{\text{cof}}$.

(T3): If $U, V \in \mathcal{T}_{\text{cof}}$ and $U = \varnothing$ or $V = \varnothing$, then $U \cap V = \varnothing \in \mathcal{T}_{\text{cof}}$. Otherwise U^c and V^c are finite and
$$(U \cap V)^c = U^c \cup V^c,$$
a finite union of finite sets, hence finite. Therefore $U \cap V \in \mathcal{T}_{\text{cof}}$. □

Proposition 14. *Assume X is infinite and equip it with the cofinite topology $\mathcal{T}_{\text{cof}}$.*

1. *For every $x \in X$, the singleton $\{x\}$ is closed.*
2. *Any two nonempty open sets intersect.*
3. *In particular, $(X, \mathcal{T}_{\text{cof}})$ is not Hausdorff.*

Proof. (1) One has $X \setminus \{x\}$ cofinite, hence open; therefore $\{x\}$ is closed.

(2) Let U, V be nonempty open. Then U^c and V^c are finite, hence $(U \cap V)^c = U^c \cup V^c$ is finite. Since X is infinite, a set with finite complement cannot be empty; hence $U \cap V \neq \varnothing$.

(3) If $x \neq y$, any open neighborhoods $U \ni x$ and $V \ni y$ are nonempty, hence $U \cap V \neq \varnothing$ by (2). Thus x and y cannot be separated by disjoint open neighborhoods, so the space is not Hausdorff. □

Corollary. *If X is infinite, the cofinite topology on X is not induced by any metric.*

Proof. Every metric topology is Hausdorff by Proposition 12, whereas $(X, \mathcal{T}_{\text{cof}})$ is not Hausdorff by Proposition 14. □

Remark. *If X is finite, the cofinite topology coincides with the discrete topology (every subset has finite complement), hence is metrizable.*

6. Topology defined by closed sets

Theorem 1 (Closed-set axioms)**.** Let X be a set and let $\mathcal{F} \subseteq \mathcal{P}(X)$. Define $\mathcal{T} := \{F^c : F \in \mathcal{F}\}$. Then the following are equivalent:

1. $\mathcal{T}$ is a topology on X.
2. $\mathcal{F}$ satisfies:

   (C1) $\varnothing \in \mathcal{F}$ and $X \in \mathcal{F}$;

   (C2) for every family $(F_\alpha)_{\alpha\in I}$ in $\mathcal{F}$, one has $\bigcap_{\alpha\in I} F_\alpha \in \mathcal{F}$;

   (C3) for all $F_1, F_2 \in \mathcal{F}$, one has $F_1 \cup F_2 \in \mathcal{F}$ (hence any finite union of sets in $\mathcal{F}$ belongs to $\mathcal{F}$).

In this case, $\mathcal{F}$ is exactly the family of closed subsets of $(X, \mathcal{T})$.

Proof. Assume (1). Let $\mathcal{F}$ be the family of complements of elements of $\mathcal{T}$. Then (C1) follows from (T1). For (C2), if $F_\alpha = X \setminus U_\alpha$ with $U_\alpha \in \mathcal{T}$, then

$$\bigcap_{\alpha\in I} F_\alpha = X \setminus \bigcup_{\alpha\in I} U_\alpha \in \mathcal{F}$$

by (T2). For (C3), if $F_i = X \setminus U_i$ with $U_i \in \mathcal{T}$, then

$$F_1 \cup F_2 = X \setminus (U_1 \cap U_2) \in \mathcal{F}$$

by (T3).

Conversely, assume (2) and define $\mathcal{T} = \{F^c : F \in \mathcal{F}\}$. Then (T1) follows from (C1). For (T2), let $U_\alpha = F_\alpha^c$ with $F_\alpha \in \mathcal{F}$. Then

$$\bigcup_{\alpha\in I} U_\alpha = \Big(\bigcap_{\alpha\in I} F_\alpha\Big)^c \in \mathcal{T}$$

by (C2). For (T3), if $U_i = F_i^c$ with $F_i \in \mathcal{F}$, then

$$U_1 \cap U_2 = (F_1 \cup F_2)^c \in \mathcal{T}$$

by (C3). The last assertion is by construction. □

7. Two standard counterexamples

Example. *In* $(\mathbb{R}, \mathcal{T}_{\text{standard}})$*:*

1. *The intersection of countably many open sets need not be open:*

$$\bigcap_{n\in\mathbb{N}} \Big(-\frac{1}{n}, \frac{1}{n}\Big) = \{0\},$$

   which is not open in the standard topology.

2. *The union of countably many closed sets need not be closed:*

$$\bigcup_{n\in\mathbb{N}} \Big[\frac{1}{n}, 2 - \frac{1}{n}\Big] = (0, 2),$$

   which is not closed in the standard topology.

Lecture 3: Continuous Maps

1. Preimages

Notation. *Let $f : X \to Y$ be a map and $A \subseteq Y$. We write*

$$f^{-1}(A) := \{x \in X : \ f(x) \in A\} \subseteq X$$

for the preimage *of A under f.*

Lemma (Elementary properties of preimages)**.** *Let $f : X \to Y$ be a map and let $(A_\alpha)_{\alpha \in I}$ be a family of subsets of Y. Then:*

1. $f^{-1}(\varnothing) = \varnothing$ *and* $f^{-1}(Y) = X$*;*
2. $f^{-1}\big(\bigcup_{\alpha \in I} A_\alpha\big) = \bigcup_{\alpha \in I} f^{-1}(A_\alpha)$*;*
3. $f^{-1}\big(\bigcap_{\alpha \in I} A_\alpha\big) = \bigcap_{\alpha \in I} f^{-1}(A_\alpha)$*;*
4. $f^{-1}(A^c) = \big(f^{-1}(A)\big)^c$*.*

Proof. All statements follow directly from the definition of $f^{-1}(A)$ by unwinding membership. $\square$

2. Continuity in topological spaces

Definition 23 (Continuous map)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is *continuous* if

$$(\forall V \in \mathcal{T}_Y) \qquad f^{-1}(V) \in \mathcal{T}_X.$$

Definition 24 (Continuity at a point via neighborhoods)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, $f : X \to Y$, and $x_0 \in X$. We say that f is *continuous at x_0* if for every neighborhood W of $f(x_0)$ in Y there exists a neighborhood U of x_0 in X such that

$$f(U) \subseteq W.$$

Proposition 15. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. The following are equivalent:*

1. *f is continuous (Definition 23);*
2. *for every $x_0 \in X$, f is continuous at x_0 in the sense of Definition 24;*
3. *for every $x_0 \in X$ and every open set $V \in \mathcal{T}_Y$ with $f(x_0) \in V$, there exists $U \in \mathcal{T}_X$ such that*

$$x_0 \in U \quad \text{and} \quad f(U) \subseteq V.$$

Proof. (1) $\Rightarrow$ (3): Let $x_0 \in X$ and $V \in \mathcal{T}_Y$ with $f(x_0) \in V$. By continuity, $U := f^{-1}(V) \in \mathcal{T}_X$ and $x_0 \in U$; moreover $f(U) \subseteq V$ by definition of preimage.

(3) $\Rightarrow$ (2): Let W be a neighborhood of $f(x_0)$. By definition, there exists an open set $V \in \mathcal{T}_Y$ such that $f(x_0) \in V \subseteq W$. By (3) there exists $U \in \mathcal{T}_X$ with $x_0 \in U$ and $f(U) \subseteq V \subseteq W$. Thus f is continuous at x_0.

(2) $\Rightarrow$ (1): Let $V \in \mathcal{T}_Y$. We show that $f^{-1}(V) \in \mathcal{T}_X$. If $f^{-1}(V) = \varnothing$ there is nothing to prove. Otherwise let $x_0 \in f^{-1}(V)$, i.e. $f(x_0) \in V$. The set V is a neighborhood of $f(x_0)$, hence by (2) there exists a neighborhood U of x_0 such that $f(U) \subseteq V$, i.e. $U \subseteq f^{-1}(V)$. Therefore $f^{-1}(V)$ is a neighborhood of each of its points, hence open. $\square$

3. Continuity in metric spaces

Definition 25 (Continuity between metric spaces)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

1. We say that f is *continuous at* $x_0 \in X$ if
$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in X) \qquad d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon.$$
2. We say that f is *continuous* if it is continuous at every point of X.

Lemma (Ball inclusion reformulation)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces, let $f : X \to Y$, and fix $x_0 \in X$. Then f is continuous at x_0 (Definition 25) if and only if*
$$(\forall \varepsilon > 0)(\exists \delta > 0) \qquad B_{d_X}(x_0, \delta) \subseteq f^{-1}\big(B_{d_Y}(f(x_0), \varepsilon)\big),$$
equivalently,
$$(\forall \varepsilon > 0)(\exists \delta > 0) \qquad f\big(B_{d_X}(x_0, \delta)\big) \subseteq B_{d_Y}(f(x_0), \varepsilon).$$

Proof. Unwind the definitions:
$$x \in B_{d_X}(x_0, \delta) \iff d_X(x, x_0) < \delta \quad \text{and} \quad f(x) \in B_{d_Y}(f(x_0), \varepsilon) \iff d_Y\big(f(x), f(x_0)\big) < \varepsilon.$$
Thus the ε–δ implication is exactly the stated inclusion of sets. □

Theorem 2 (Compatibility of metric and topological continuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, and endow X and Y with the induced topologies $\mathcal{T}_{d_X}$ and $\mathcal{T}_{d_Y}$. For a map $f : X \to Y$, the following are equivalent:

1. f is continuous in the metric sense (Definition 25);
2. $f : (X, \mathcal{T}_{d_X}) \to (Y, \mathcal{T}_{d_Y})$ is continuous in the topological sense (Definition 23).

Proof. (1) $\Rightarrow$ (2): Let $V \in \mathcal{T}_{d_Y}$ and let $x_0 \in f^{-1}(V)$. Then $f(x_0) \in V$, hence there exists $\varepsilon > 0$ such that $B_{d_Y}(f(x_0), \varepsilon) \subseteq V$. By metric continuity at x_0 and Lemma , there exists $\delta > 0$ such that
$$B_{d_X}(x_0, \delta) \subseteq f^{-1}\big(B_{d_Y}(f(x_0), \varepsilon)\big) \subseteq f^{-1}(V).$$
Thus $f^{-1}(V)$ is open in X.

(2) $\Rightarrow$ (1): Fix $x_0 \in X$ and $\varepsilon > 0$. The ball $B_{d_Y}(f(x_0), \varepsilon)$ is open in $(Y, \mathcal{T}_{d_Y})$, hence $f^{-1}\big(B_{d_Y}(f(x_0), \varepsilon)\big)$ is open in $(X, \mathcal{T}_{d_X})$ by (2), and it contains x_0. Therefore there exists $\delta > 0$ such that
$$B_{d_X}(x_0, \delta) \subseteq f^{-1}\big(B_{d_Y}(f(x_0), \varepsilon)\big),$$
which is exactly the ε–δ condition by Lemma . □

4. Functorial properties

Proposition 16. *Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, $(Z, \mathcal{T}_Z)$ be topological spaces.*

1. *The identity map $\mathrm{id}_X : X \to X$ is continuous.*
2. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

3. *If $f : X \to Y$ is continuous and $Y' \subseteq Y$ is equipped with the subspace topology, then the corestriction*

$$f : X \to Y', \qquad x \mapsto f(x),$$

is continuous provided $f(X) \subseteq Y'$.

Proof. (1) For every open $U \subseteq X$, $\mathrm{id}_X^{-1}(U) = U$ is open.
(2) For every open $W \subseteq Z$,

$$(g \circ f)^{-1}(W) = f^{-1}\big(g^{-1}(W)\big)$$

is open since $g^{-1}(W)$ is open in Y and then f^{-1} of it is open in X.
(3) Let $V \subseteq Y'$ be open in the subspace topology, so $V = Y' \cap U$ for some open $U \subseteq Y$. Then

$$f^{-1}(V) = f^{-1}(Y' \cap U) = f^{-1}(Y') \cap f^{-1}(U) = X \cap f^{-1}(U) = f^{-1}(U),$$

which is open in X. □

5. Comparing topologies: coarser and finer

Definition 26 (Coarser and finer topologies)**.** Let X be a set and let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_1$ is *coarser* (or *weaker*) than $\mathcal{T}_2$, and that $\mathcal{T}_2$ is *finer* (or *stronger*) than $\mathcal{T}_1$.

Proposition 17 (Order characterization via identity maps)**.** *Let X be a set and let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X. Then the following are equivalent:*

1. $\mathcal{T}_1 \subseteq \mathcal{T}_2$*;*
2. *the identity map $\mathrm{id}_X : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous.*

Proof. By definition, id_X is continuous if and only if for every $U \in \mathcal{T}_1$ one has $\mathrm{id}_X^{-1}(U) = U \in \mathcal{T}_2$, i.e. $\mathcal{T}_1 \subseteq \mathcal{T}_2$. □

Remark. *Fix a set-map $f : X \to Y$ and a topology on Y. If the topology on X is made* finer*, continuity of f is preserved. Dually, if the topology on Y is made* coarser*, continuity of f is preserved.*

6. Subspace topology and its universal property

Definition 27 (Subspace topology)**.** Let $(X, \mathcal{T}_X)$ be a topological space and let $Y \subseteq X$. The *subspace topology* on Y is

$$\mathcal{T}_X|_Y := \{Y \cap U : \ U \in \mathcal{T}_X\} \subseteq \mathcal{P}(Y).$$

The corresponding space $(Y, \mathcal{T}_X|_Y)$ is called a *subspace* of $(X, \mathcal{T}_X)$.

Lemma (Subspace topology is the coarsest making the inclusion continuous)**.** *Let $(X, \mathcal{T}_X)$ be a topological space and $Y \subseteq X$. Let $i : Y \hookrightarrow X$ be the inclusion map, $i(y) = y$, and let $\mathcal{T}_Y := \mathcal{T}_X|_Y$. Then:*

1. *$i : (Y, \mathcal{T}_Y) \to (X, \mathcal{T}_X)$ is continuous;*
2. *if $\mathcal{S}$ is any topology on Y such that $i : (Y, \mathcal{S}) \to (X, \mathcal{T}_X)$ is continuous, then $\mathcal{T}_Y \subseteq \mathcal{S}$.*

In particular, $\mathcal{T}_Y$ is the coarsest topology on Y for which the inclusion i is continuous.

Proof. First, $\mathcal{T}_Y$ is a topology (verification as in Lecture 2).

(1) Let $U \in \mathcal{T}_X$ be open. Then
$$i^{-1}(U) = U \cap Y \in \mathcal{T}_Y$$
by definition of $\mathcal{T}_Y$. Hence i is continuous.

(2) Let $\mathcal{S}$ be a topology on Y such that $i : (Y, \mathcal{S}) \to (X, \mathcal{T}_X)$ is continuous. For every $U \in \mathcal{T}_X$, continuity gives $i^{-1}(U) \in \mathcal{S}$. But $i^{-1}(U) = U \cap Y$, so every element of $\mathcal{T}_Y$ belongs to $\mathcal{S}$. Hence $\mathcal{T}_Y \subseteq \mathcal{S}$. $\square$

Proposition 18 (Universal property of the subspace topology)**.** *Let $(X, \mathcal{T}_X)$ be a topological space, let $Y \subseteq X$ with the subspace topology, and let $i : Y \hookrightarrow X$ be the inclusion. For any topological space Z and any map $g : Z \to Y$, the following are equivalent:*

1. *$g : Z \to Y$ is continuous;*
2. *$i \circ g : Z \to X$ is continuous.*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 16(2), since i is continuous by Lemma (1).

(2) $\Rightarrow$ (1): Let $V \subseteq Y$ be open in the subspace topology, so $V = Y \cap U$ for some $U \in \mathcal{T}_X$. Then
$$g^{-1}(V) = g^{-1}(Y \cap U) = g^{-1}\big(i^{-1}(U)\big) = (i \circ g)^{-1}(U),$$
which is open in Z because $i \circ g$ is continuous. $\square$

Lecture 4: Topological Bases

1. Bases of a topology

Definition 28 (Basis)**.** Let X be a set and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a family of subsets of X. We say that $\mathcal{B}$ is a *basis* (or *base*) on X if the following conditions hold:

(B1) (*covering*) $\bigcup_{B\in\mathcal{B}} B = X$;

(B2) (*refinement*) for all $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subseteq B_1 \cap B_2.$$

Theorem 3 (Topology generated by a basis)**.** Let X be a set and let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a basis in the sense of Definition 28. Define

$$\mathcal{T}(\mathcal{B}) := \Big\{ U \subseteq X : \exists \mathcal{A} \subseteq \mathcal{B} \text{ such that } U = \bigcup_{B\in\mathcal{A}} B \Big\}.$$

Then:

1. $\mathcal{T}(\mathcal{B})$ is a topology on X;
2. $\mathcal{B} \subseteq \mathcal{T}(\mathcal{B})$ and $\mathcal{B}$ is a basis of the topological space $(X, \mathcal{T}(\mathcal{B}))$, i.e. every $U \in \mathcal{T}(\mathcal{B})$ is a union of members of $\mathcal{B}$;
3. $\mathcal{T}(\mathcal{B})$ is the *coarsest* topology on X containing $\mathcal{B}$ (equivalently, the intersection of all topologies on X that contain $\mathcal{B}$).

Proof. (1) We verify the axioms of a topology.

(T1). By definition, $\varnothing = \bigcup \varnothing \in \mathcal{T}(\mathcal{B})$. Moreover, by (B1),

$$X = \bigcup_{B\in\mathcal{B}} B \in \mathcal{T}(\mathcal{B}).$$

(T2). Let $(U_\alpha)_{\alpha\in I}$ be a family in $\mathcal{T}(\mathcal{B})$. For each α choose $\mathcal{A}_\alpha \subseteq \mathcal{B}$ such that $U_\alpha = \bigcup_{B\in\mathcal{A}_\alpha} B$. Then

$$\bigcup_{\alpha\in I} U_\alpha = \bigcup_{\alpha\in I} \bigcup_{B\in\mathcal{A}_\alpha} B = \bigcup_{B\in\bigcup_{\alpha\in I}\mathcal{A}_\alpha} B \in \mathcal{T}(\mathcal{B}).$$

(T3). Let $U, V \in \mathcal{T}(\mathcal{B})$. Choose $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ such that $U = \bigcup_{B\in\mathcal{A}} B$ and $V = \bigcup_{C\in\mathcal{C}} C$. We show that $U \cap V \in \mathcal{T}(\mathcal{B})$ by proving that it is a union of members of $\mathcal{B}$.

Let $x \in U \cap V$. Then there exist $B \in \mathcal{A}$ and $C \in \mathcal{C}$ such that $x \in B$ and $x \in C$, hence $x \in B \cap C$. By (B2), there exists $D \in \mathcal{B}$ such that $x \in D \subseteq B \cap C \subseteq U \cap V$. Therefore each $x \in U \cap V$ belongs to some element of $\mathcal{B}$ contained in $U \cap V$, so

$$U \cap V = \bigcup \{D \in \mathcal{B} : D \subseteq U \cap V\} \in \mathcal{T}(\mathcal{B}).$$

This proves (T3), hence $\mathcal{T}(\mathcal{B})$ is a topology.

(2) For each $B \in \mathcal{B}$, one has $B = \bigcup\{B\} \in \mathcal{T}(\mathcal{B})$, so $\mathcal{B} \subseteq \mathcal{T}(\mathcal{B})$. By definition of $\mathcal{T}(\mathcal{B})$, every open set is a union of elements of $\mathcal{B}$, so $\mathcal{B}$ is a basis of $(X, \mathcal{T}(\mathcal{B}))$.

(3) Let $\mathcal{T}$ be any topology on X with $\mathcal{B} \subseteq \mathcal{T}$. Since $\mathcal{T}$ is stable under arbitrary unions, every union of elements of $\mathcal{B}$ belongs to $\mathcal{T}$, hence $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{T}$. Thus $\mathcal{T}(\mathcal{B})$ is contained in every topology containing $\mathcal{B}$, i.e. it is the coarsest such topology. □

Proposition 19 (Bases inside a given topology). *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \subseteq \mathcal{T}$. Then the following are equivalent:*

1. *$\mathcal{B}$ is a basis of $\mathcal{T}$ in the sense that every $U \in \mathcal{T}$ is a union of members of $\mathcal{B}$;*
2. *$\mathcal{B}$ satisfies* (B1)–(B2) of Definition 28 and $\mathcal{T} = \mathcal{T}(\mathcal{B})$.

Proof. (1) ⇒ (2): Since $X \in \mathcal{T}$ and X is a union of members of $\mathcal{B}$, we obtain (B1). Let $B_1, B_2 \in \mathcal{B}$. Then $B_1 \cap B_2 \in \mathcal{T}$, hence $B_1 \cap B_2$ is a union of members of $\mathcal{B}$. For $x \in B_1 \cap B_2$, choose $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$, giving (B2). Finally, since $\mathcal{B} \subseteq \mathcal{T}$, every union of members of $\mathcal{B}$ is in $\mathcal{T}$, so $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{T}$. Conversely every $U \in \mathcal{T}$ is a union of members of $\mathcal{B}$, hence $U \in \mathcal{T}(\mathcal{B})$; thus $\mathcal{T} \subseteq \mathcal{T}(\mathcal{B})$.

(2) ⇒ (1) is immediate from $\mathcal{T} = \mathcal{T}(\mathcal{B})$. □

Example (Metric spaces). *Let (X, d) be a metric space. The family of all open balls*

$$\mathcal{B}_d := \{B_d(x, r) : x \in X,\ r > 0\}$$

is a basis of the metric topology. In particular, for $\mathbb{R}^n$ with the Euclidean metric, the standard topology admits $\mathcal{B}_{d_2}$ as a basis.

Example (A countable basis for $\mathbb{R}^n$). *In $\mathbb{R}^n$ with the standard topology, the family*

$$\mathcal{B}_{\mathbb{Q}} := \Big\{B_{d_2}(q, r) :\ q \in \mathbb{Q}^n,\ r \in \mathbb{Q},\ r > 0\Big\}$$

is a countable basis. Indeed, $\mathbb{Q}^n \times (\mathbb{Q} \cap (0, \infty))$ is countable, hence so is $\mathcal{B}_{\mathbb{Q}}$, and every Euclidean open ball contains some ball in $\mathcal{B}_{\mathbb{Q}}$.

2. Continuity tests using a basis

Lemma (Composition of continuous maps). *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $g : (Y, \mathcal{T}_Y) \to (Z, \mathcal{T}_Z)$ be continuous maps. Then $g \circ f : (X, \mathcal{T}_X) \to (Z, \mathcal{T}_Z)$ is continuous.*

Proof. Let $W \in \mathcal{T}_Z$. Since g is continuous, $g^{-1}(W) \in \mathcal{T}_Y$. Since f is continuous, $f^{-1}(g^{-1}(W)) \in \mathcal{T}_X$. But $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$, hence $g \circ f$ is continuous. □

Proposition 20 (Continuity can be tested on a basis). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f : X \to Y$ be a map. Assume $\mathcal{B}$ is a basis of $\mathcal{T}_Y$. Then the following are equivalent:*

1. *f is continuous;*
2. *for every $B \in \mathcal{B}$, the set $f^{-1}(B)$ is open in X (i.e. $f^{-1}(B) \in \mathcal{T}_X$).*

Proof. (1) ⇒ (2) is immediate from the definition of continuity.

(2) ⇒ (1): Let $U \in \mathcal{T}_Y$. Since $\mathcal{B}$ is a basis, there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{A}} B$. Then

$$f^{-1}(U) = f^{-1}\Big(\bigcup_{B \in \mathcal{A}} B\Big) = \bigcup_{B \in \mathcal{A}} f^{-1}(B),$$

which is open in X as an arbitrary union of open sets. □

3. Subbases

Definition 29 (Finite intersections)**.** Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X)$. We denote by $\mathrm{FI}(\mathcal{S})$ the family of all finite intersections of elements of $\mathcal{S}$, with the convention that the intersection of an empty family is X:

$$\mathrm{FI}(\mathcal{S}) := \left\{ \bigcap_{i=1}^{n} S_i : \ n \in \mathbb{N}, \ S_i \in \mathcal{S} \right\} \ \cup \ \{X\}.$$

Definition 30 (Subbasis)**.** Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{S} \subseteq \mathcal{T}$ is a *subbasis* of $\mathcal{T}$ if the family $\mathrm{FI}(\mathcal{S})$ (Definition 29) is a basis of $\mathcal{T}$. Equivalently, every open set of $\mathcal{T}$ is a union of finite intersections of elements of $\mathcal{S}$.

Theorem 4 (Topology generated by a subbasis)**.** Let X be a set and let $\mathcal{S} \subseteq \mathcal{P}(X)$. Define

$$\mathcal{T}(\mathcal{S}) := \left\{ U \subseteq X : \ U = \bigcup_{B \in \mathcal{A}} B \text{ for some } \mathcal{A} \subseteq \mathrm{FI}(\mathcal{S}) \right\}.$$

Then:

1. $\mathcal{T}(\mathcal{S})$ is a topology on X;
2. $\mathcal{S} \subseteq \mathcal{T}(\mathcal{S})$;
3. $\mathrm{FI}(\mathcal{S})$ is a basis of $(X, \mathcal{T}(\mathcal{S}))$ (hence $\mathcal{S}$ is a subbasis of $\mathcal{T}(\mathcal{S})$);
4. $\mathcal{T}(\mathcal{S})$ is the coarsest topology on X containing $\mathcal{S}$.

Proof. Apply Theorem 3 to the family $\mathcal{B} := \mathrm{FI}(\mathcal{S})$. Indeed, $\mathcal{B}$ covers X since $X \in \mathcal{B}$ by definition, and $\mathcal{B}$ is stable under finite intersections. Thus $\mathcal{T}(\mathcal{S}) = \mathcal{T}(\mathcal{B})$ is a topology and $\mathcal{B}$ is a basis of it. Since $\mathcal{S} \subseteq \mathcal{B}$, we also have $\mathcal{S} \subseteq \mathcal{T}(\mathcal{S})$. The minimality is the minimality statement of Theorem 3. □

Proposition 21 (Continuity can be tested on a subbasis)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \to Y$ be a map. Assume $\mathcal{S}$ is a subbasis of $\mathcal{T}_Y$. Then f is continuous if and only if $f^{-1}(S) \in \mathcal{T}_X$ for all $S \in \mathcal{S}$.*

Proof. The "only if" direction is trivial. For the converse, assume $f^{-1}(S)$ is open for all $S \in \mathcal{S}$. Then for any finite intersection $B = \bigcap_{i=1}^{n} S_i$ with $S_i \in \mathcal{S}$, one has

$$f^{-1}(B) = \bigcap_{i=1}^{n} f^{-1}(S_i),$$

hence $f^{-1}(B)$ is open in X. Thus, by Proposition 20 applied to the basis $\mathrm{FI}(\mathcal{S})$ of $\mathcal{T}_Y$, the map f is continuous. □

4. Second countability

Definition 31 (Second countable)**.** A topological space $(X, \mathcal{T})$ is *second countable* if $\mathcal{T}$ admits a countable basis, i.e. there exists a basis $\mathcal{B} \subseteq \mathcal{T}$ such that $\mathcal{B}$ is countable.

Example. *For each $n \geq 1$, the space $\mathbb{R}^n$ with the standard topology is second countable (Example).*

Lecture 5–6: Homeomorphisms, Product and Coproduct Topologies

1. Homeomorphisms

Definition 32 (Homeomorphism)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is a *homeomorphism* if

1. f is bijective;
2. f is continuous;
3. the inverse map $f^{-1} : Y \to X$ is continuous.

In this case, X and Y are said to be *homeomorphic*.

Proposition 22. *Let $f : X \to Y$ be a homeomorphism.*

1. *For every $U \subseteq X$, U is open in X if and only if $f(U)$ is open in Y.*
2. *For every $F \subseteq X$, F is closed in X if and only if $f(F)$ is closed in Y.*

Proof. (1) If U is open in X, then $f(U) = (f^{-1})^{-1}(U)$ is open in Y since f^{-1} is continuous. Conversely, if $f(U)$ is open in Y, then $U = f^{-1}(f(U))$ is open in X since f is continuous.

(2) Apply (1) to complements. □

Proposition 23 (Characterizations of homeomorphisms)**.** *Let $f : X \to Y$ be a bijection between topological spaces. The following are equivalent:*

1. *f is a homeomorphism;*
2. *f is continuous and* open, *i.e. it maps open sets of X to open sets of Y;*
3. *f is continuous and* closed, *i.e. it maps closed sets of X to closed sets of Y.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow from Proposition 22.

(2) $\Rightarrow$ (1): assume f is continuous and open. To show that f^{-1} is continuous, let $U \subseteq X$ be open. Then $(f^{-1})^{-1}(U) = f(U)$ is open in Y by openness of f, hence f^{-1} is continuous.

(3) $\Rightarrow$ (1) is analogous, using complements or directly closed sets. □

Lemma. *Homeomorphisms are stable under composition and inversion:*

1. $\mathrm{id}_X : X \to X$ *is a homeomorphism for every topological space X;*
2. *if $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms, then $g \circ f : X \to Z$ is a homeomorphism;*
3. *if $f : X \to Y$ is a homeomorphism, then $f^{-1} : Y \to X$ is a homeomorphism.*

Proof. (1) is immediate. (2) and (3) follow from the corresponding facts for bijections and for continuous maps. □

Corollary. *The relation "X is homeomorphic to Y" on topological spaces is an equivalence relation.*

Proof. Reflexivity is Lemma (1), symmetry is (3), and transitivity is (2). □

Remark (A continuous bijection need not be a homeomorphism)**.** *Continuity of f does not imply continuity of f^{-1}, even when f is bijective.*

Let $I := [0, 2\pi)$ with the subspace topology induced from $\mathbb{R}$, and let $S^1 \subseteq \mathbb{R}^2$ carry the subspace topology. Define

$$f : I \to S^1, \qquad f(\theta) := (\cos\theta, \sin\theta).$$

Then f is continuous and bijective. However f^{-1} is not continuous at $(1, 0)$. Indeed, the set $[0, \pi/2) \subseteq I$ is open in the subspace topology (since $[0, \pi/2) = (-1, \pi/2) \cap I$), but

$$f\big([0, \pi/2)\big) = \{(\cos\theta, \sin\theta) : 0 \le \theta < \pi/2\}$$

is an arc in S^1 which is not open in S^1 (every neighborhood of $(1, 0)$ in S^1 contains points with angle close to 2π as well). By Proposition 23, f cannot be a homeomorphism.

2. Initial and final topologies

Definition 33 (Initial topology)**.** Let X be a set, let $(Y_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces, and let $f_i : X \to Y_i$ be maps. The *initial topology* on X induced by the family $(f_i)_{i\in I}$ is the *coarsest* topology $\mathcal{T}$ on X such that each $f_i : (X, \mathcal{T}) \to (Y_i, \mathcal{T}_i)$ is continuous.

Proposition 24 (Construction of the initial topology)**.** *In the situation of Definition 33, let*

$$\mathcal{S} := \{f_i^{-1}(U) : \ i \in I, \ U \in \mathcal{T}_i\} \subseteq \mathcal{P}(X).$$

Then $\mathcal{S}$ is a subbasis of the initial topology on X; equivalently, the initial topology is the topology generated by $\mathcal{S}$.

Proof. Let $\mathcal{T}(\mathcal{S})$ be the topology generated by the subbasis $\mathcal{S}$. For each i, and each open $U \in \mathcal{T}_i$, one has $f_i^{-1}(U) \in \mathcal{S} \subseteq \mathcal{T}(\mathcal{S})$; hence each f_i is continuous for $\mathcal{T}(\mathcal{S})$.

If $\mathcal{T}'$ is any topology on X making all f_i continuous, then $f_i^{-1}(U) \in \mathcal{T}'$ for all i and all $U \in \mathcal{T}_i$, hence $\mathcal{S} \subseteq \mathcal{T}'$. Since $\mathcal{T}'$ is a topology, it contains the topology generated by $\mathcal{S}$, i.e. $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{T}'$. Thus $\mathcal{T}(\mathcal{S})$ is coarsest among topologies making all f_i continuous, i.e. it is the initial topology. $\square$

Proposition 25 (Universal property of the initial topology)**.** *Let $\mathcal{T}$ be the initial topology on X induced by $(f_i : X \to Y_i)_{i\in I}$. For every topological space Z and every map $g : Z \to X$, the following are equivalent:*

1. *$g : (Z, \mathcal{T}_Z) \to (X, \mathcal{T})$ is continuous;*
2. *for every $i \in I$, the composite $f_i \circ g : Z \to Y_i$ is continuous.*

Proof. (1) $\Rightarrow$ (2): each f_i is continuous by definition of the initial topology, hence $f_i \circ g$ is continuous as a composite of continuous maps.

(2) $\Rightarrow$ (1): by Proposition 24, it suffices to check continuity of g on the subbasis $\mathcal{S} = \{f_i^{-1}(U)\}$. Let $S = f_i^{-1}(U) \in \mathcal{S}$ with $U \in \mathcal{T}_i$. Then

$$g^{-1}(S) = g^{-1}\big(f_i^{-1}(U)\big) = (f_i \circ g)^{-1}(U),$$

which is open in Z because $f_i \circ g$ is continuous. Hence g is continuous. $\square$

Definition 34 (Final topology)**.** Let Y be a set, let $(X_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces, and let $u_i : X_i \to Y$ be maps. The *final topology* on Y induced by the family $(u_i)_{i\in I}$ is the *finest* topology $\mathcal{T}$ on Y such that each $u_i : (X_i, \mathcal{T}_i) \to (Y, \mathcal{T})$ is continuous.

Proposition 26 (Construction and universal property of the final topology)**.** *In the situation of Definition 34, set*

$$\mathcal{T} := \{U \subseteq Y : \ \forall i \in I, \ u_i^{-1}(U) \in \mathcal{T}_i\}.$$

Then:

1. *$\mathcal{T}$ is a topology on Y and is the final topology induced by $(u_i)_{i\in I}$;*
2. *for every topological space Z and every map $h : Y \to Z$, the following are equivalent:*
   (a) *$h : (Y, \mathcal{T}) \to Z$ is continuous;*
   (b) *for every $i \in I$, the composite $h \circ u_i : X_i \to Z$ is continuous.*

Proof. (1) We verify the topology axioms.

(T1). For every i, $u_i^{-1}(\varnothing) = \varnothing$ and $u_i^{-1}(Y) = X_i$ are open in X_i, hence $\varnothing, Y \in \mathcal{T}$.

(T2). Let $(U_\alpha)_{\alpha\in A} \subseteq \mathcal{T}$. Then for each i,

$$u_i^{-1}\Big(\bigcup_{\alpha\in A} U_\alpha\Big) = \bigcup_{\alpha\in A} u_i^{-1}(U_\alpha)$$

is open in X_i as a union of open sets. Hence $\bigcup_{\alpha\in A} U_\alpha \in \mathcal{T}$.

(T3). If $U, V \in \mathcal{T}$, then for each i,

$$u_i^{-1}(U \cap V) = u_i^{-1}(U) \cap u_i^{-1}(V)$$

is open in X_i as an intersection of open sets. Hence $U \cap V \in \mathcal{T}$.

Thus $\mathcal{T}$ is a topology, and by construction each u_i is continuous. If $\mathcal{T}'$ is any topology on Y such that all u_i are continuous, then every $U \in \mathcal{T}'$ satisfies $u_i^{-1}(U) \in \mathcal{T}_i$ for all i, hence $U \in \mathcal{T}$. Thus $\mathcal{T}'$ is contained in $\mathcal{T}$, so $\mathcal{T}$ is the finest such topology, i.e. the final topology.

(2) $(a) \Rightarrow (b)$ is stability of continuity under composition. For $(b) \Rightarrow (a)$, let $W \subseteq Z$ be open. Then for every i,

$$u_i^{-1}\big(h^{-1}(W)\big) = (h \circ u_i)^{-1}(W)$$

is open in X_i by continuity of $h \circ u_i$. Hence $h^{-1}(W) \in \mathcal{T}$ by definition of $\mathcal{T}$, so h is continuous. □

3. Products in Top

3.1. Product of sets

Definition 35 (Product of a family of sets)**.** Let $(X_\alpha)_{\alpha\in A}$ be a family of sets. Define

$$\prod_{\alpha\in A} X_\alpha := \{x : A \to \bigcup_{\alpha\in A} X_\alpha : \ x(\alpha) \in X_\alpha \text{ for all } \alpha \in A\}.$$

For each $\beta \in A$, the *projection* $p_\beta : \prod_{\alpha\in A} X_\alpha \to X_\beta$ is defined by $p_\beta(x) := x(\beta)$.

Proposition 27 (Universal property in **Set**)**.** *Let $(X_\alpha)_{\alpha\in A}$ be a family of sets and let p_β be the projections. For every set Z and every family of maps $(f_\beta : Z \to X_\beta)_{\beta\in A}$, there exists a unique map $f : Z \to \prod_{\alpha\in A} X_\alpha$ such that $p_\beta \circ f = f_\beta$ for all $\beta \in A$.*

Proof. Define $f(z) \in \prod_{\alpha\in A} X_\alpha$ by $f(z)(\beta) := f_\beta(z)$ for each $\beta \in A$. Then $p_\beta(f(z)) = f(z)(\beta) = f_\beta(z)$, so $p_\beta \circ f = f_\beta$. Uniqueness follows because any f satisfying $p_\beta \circ f = f_\beta$ must satisfy $f(z)(\beta) = f_\beta(z)$ for all β, hence must equal the above map. □

3.2. Binary products and the product topology

Notation. *For two sets X, Y, write $X \times Y := \prod_{\alpha \in \{1,2\}} X_\alpha$ with $X_1 = X$ and $X_2 = Y$. We denote the projections by*

$$p_X : X \times Y \to X, \qquad p_Y : X \times Y \to Y.$$

Definition 36 (Product topology: binary case)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The *product topology* on the set $X \times Y$ is the initial topology induced by the projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$. We denote it by $\mathcal{T}_{X \times Y}$.

Proposition 28 (A basis for the binary product topology)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The family*

$$\mathcal{B} := \{U \times V : \ U \in \mathcal{T}_X, \ V \in \mathcal{T}_Y\}$$

is a basis of the product topology on $X \times Y$.

Proof. By Proposition 24, a subbasis for the product topology is

$$\mathcal{S} = \{p_X^{-1}(U) : U \in \mathcal{T}_X\} \ \cup \ \{p_Y^{-1}(V) : V \in \mathcal{T}_Y\}.$$

Since $p_X^{-1}(U) = U \times Y$ and $p_Y^{-1}(V) = X \times V$, finite intersections of subbasic elements are exactly the sets

$$(U \times Y) \cap (X \times V) = U \times V, \qquad U \in \mathcal{T}_X, \ V \in \mathcal{T}_Y.$$

Hence $\mathcal{B}$ is the family of finite intersections of elements of $\mathcal{S}$, so $\mathcal{B}$ is a basis of the topology generated by $\mathcal{S}$. ☐

Proposition 29 (Universal property of the product in **Top**: binary case)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and equip $X \times Y$ with the product topology.*

1. *The projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous.*
2. *For every topological space Z and every pair of continuous maps $f_X : Z \to X$ and $f_Y : Z \to Y$, the unique map*
$$f : Z \to X \times Y, \qquad f(z) := \big(f_X(z), f_Y(z)\big),$$
is continuous.
3. *For every topological space Z and every map $g : Z \to X \times Y$, the following are equivalent:*
   (a) *g is continuous;*
   (b) *$p_X \circ g : Z \to X$ and $p_Y \circ g : Z \to Y$ are continuous.*

Proof. (1) holds by definition of the product topology as an initial topology.

(2) It suffices to test continuity on the basis $\mathcal{B}$ of Proposition 28. Let $U \times V \in \mathcal{B}$. Then

$$f^{-1}(U \times V) = \{z \in Z : \ f_X(z) \in U \text{ and } f_Y(z) \in V\} = f_X^{-1}(U) \cap f_Y^{-1}(V),$$

which is open in Z since f_X and f_Y are continuous.

(3) This is the universal property of the initial topology (Proposition 25) applied to the family of maps $\{p_X, p_Y\}$. ☐

Remark (Diagrammatic form)**.** *The universal property of Proposition 29(2) can be written as:*

$$\begin{tikzcd} & Z \arrow[dl, "f_X"'] \arrow[d, dashed, "f"] \arrow[dr, "f_Y"] & \\ X & X \times Y \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

where f is uniquely determined by $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$.

3.3. Arbitrary products

Definition 37 (Product topology: arbitrary family)**.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ be a family of topological spaces. Equip the set $\prod_{\alpha\in A} X_\alpha$ with the initial topology induced by the projections $p_\beta : \prod_{\alpha\in A} X_\alpha \to X_\beta$ $(\beta \in A)$. This topology is called the *product topology* and is denoted $\mathcal{T}_\prod$.

Proposition 30 (Subbasis and basis for the product topology)**.** *Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ be a family of topological spaces, and let $X := \prod_{\alpha\in A} X_\alpha$ with the product topology.*

1. *A subbasis of $\mathcal{T}_\prod$ is*
$$\mathcal{S} := \{p_\alpha^{-1}(U) : \ \alpha \in A, \ U \in \mathcal{T}_\alpha\}.$$
2. *A basis of $\mathcal{T}_\prod$ is the family of* cylinder sets
$$\mathcal{B} := \left\{\bigcap_{i=1}^n p_{\alpha_i}^{-1}(U_i) : \ n \in \mathbb{N}, \ \alpha_1, \dots, \alpha_n \in A, \ U_i \in \mathcal{T}_{\alpha_i}\right\} \ \cup \ \{X\}.$$
Equivalently, $\mathcal{B}$ consists of sets of the form $\prod_{\alpha\in A} V_\alpha$ where $V_\alpha = X_\alpha$ for all but finitely many α, and V_α is open in X_α for the remaining finitely many indices.

Proof. (1) is Proposition 24 applied to the family $(p_\alpha)_{\alpha\in A}$.

(2) Finite intersections of elements of $\mathcal{S}$ have the displayed form, and by the general theory of subbases they form a basis of the generated topology. The equivalence with $\prod_\alpha V_\alpha$ is obtained by setting $V_{\alpha_i} := U_i$ and $V_\alpha := X_\alpha$ for $\alpha \notin \{\alpha_1, \dots, \alpha_n\}$. ☐

Theorem 5 (Universal property of products in **Top**)**.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ be a family of topological spaces and let $X = \prod_{\alpha\in A} X_\alpha$ with the product topology.

1. Each projection $p_\beta : X \to X_\beta$ is continuous.
2. For every topological space Z and every family of continuous maps $(f_\alpha : Z \to X_\alpha)_{\alpha\in A}$, the unique map
$$f : Z \to X, \qquad f(z) := (f_\alpha(z))_{\alpha\in A},$$
is continuous.
3. For every topological space Z and every map $g : Z \to X$, the following are equivalent:
   (a) g is continuous;
   (b) for every $\alpha \in A$, the map $p_\alpha \circ g : Z \to X_\alpha$ is continuous.

Proof. (1) holds by construction (initial topology).

(2) It suffices to test continuity on the basis $\mathcal{B}$ of Proposition 30. Let $B = \bigcap_{i=1}^n p_{\alpha_i}^{-1}(U_i) \in \mathcal{B}$. Then
$$f^{-1}(B) = \bigcap_{i=1}^n f^{-1}\big(p_{\alpha_i}^{-1}(U_i)\big) = \bigcap_{i=1}^n (p_{\alpha_i} \circ f)^{-1}(U_i) = \bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_i),$$
which is open in Z since each f_{α_i} is continuous.

(3) is the universal property of the initial topology (Proposition 25) applied to the family $(p_\alpha)_{\alpha\in A}$. ☐

Remark (On finite vs. infinite products)**.** *If A is infinite, the product topology is* not *generated by all "boxes" $\prod_{\alpha\in A} U_\alpha$ with each U_α open: it is generated by those boxes for which $U_\alpha = X_\alpha$ for all but finitely many α.*

4. Coproducts in Top

4.1. Coproduct of sets

Definition 38 (Disjoint union of a family of sets)**.** Let $(X_\alpha)_{\alpha\in A}$ be a family of sets. Define the *disjoint union*

$$\coprod_{\alpha\in A} X_\alpha := \bigcup_{\alpha\in A} (X_\alpha \times \{\alpha\}).$$

For each $\alpha \in A$, define the canonical injection

$$i_\alpha : X_\alpha \to \coprod_{\beta\in A} X_\beta, \qquad i_\alpha(x) := (x, \alpha).$$

Proposition 31 (Universal property in **Set**)**.** *Let $(X_\alpha)_{\alpha\in A}$ be a family of sets and let i_α be the canonical injections. For every set Z and every family of maps $(h_\alpha : X_\alpha \to Z)_{\alpha\in A}$, there exists a unique map $h : \coprod_{\alpha\in A} X_\alpha \to Z$ such that $h \circ i_\alpha = h_\alpha$ for all $\alpha \in A$.*

Proof. Define $h(x, \alpha) := h_\alpha(x)$ for $(x, \alpha) \in X_\alpha \times \{\alpha\}$. This is well-defined because the union is disjoint. The identities $h \circ i_\alpha = h_\alpha$ are immediate. Uniqueness is clear since every element of the disjoint union lies in the image of some i_α. □

4.2. Coproduct topology (disjoint union topology)

Definition 39 (Coproduct topology)**.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ be a family of topological spaces and let $X := \coprod_{\alpha\in A} X_\alpha$ with canonical injections i_α. The *coproduct topology* on X is the final topology induced by the family $(i_\alpha)_{\alpha\in A}$. We denote it by $\mathcal{T}_\coprod$.

Proposition 32 (Characterization of open sets in the coproduct topology)**.** *Let $X = \coprod_{\alpha\in A} X_\alpha$ with the coproduct topology. A subset $U \subseteq X$ is open if and only if, for every $\alpha \in A$, the subset*

$$i_\alpha^{-1}(U) = \{x \in X_\alpha : \ i_\alpha(x) \in U\}$$

is open in X_α. Equivalently, U is open if and only if $U \cap i_\alpha(X_\alpha)$ is open in the subspace $i_\alpha(X_\alpha) \subseteq X$ for every α.

Proof. This is the explicit description of the final topology from Proposition 26(1). The equivalence with $U \cap i_\alpha(X_\alpha)$ follows because $i_\alpha : X_\alpha \to i_\alpha(X_\alpha)$ is a bijection and $i_\alpha^{-1}(U) = i_\alpha^{-1}\big(U \cap i_\alpha(X_\alpha)\big)$. □

Proposition 33. *Let $X = \coprod_{\alpha\in A} X_\alpha$ with the coproduct topology.*

1. *Each injection $i_\alpha : X_\alpha \to X$ is continuous.*
2. *Each subset $i_\alpha(X_\alpha) \subseteq X$ is open and closed.*

Proof. (1) is true by definition of the final topology. (2) Let $U_\alpha := i_\alpha(X_\alpha)$. For $\beta \in A$,

$$i_\beta^{-1}(U_\alpha) = \begin{cases} X_\beta, & \beta = \alpha, \\ \varnothing, & \beta \neq \alpha, \end{cases}$$

which is open in X_β for all β. Hence U_α is open by Proposition 32. Its complement is $\bigcup_{\beta\neq\alpha} U_\beta$, hence open, so U_α is also closed. □

Theorem 6 (Universal property of coproducts in **Top**)**.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ be a family of topological spaces and let $X = \coprod_{\alpha\in A} X_\alpha$ with the coproduct topology. For every topological space Z and every family of maps $(h_\alpha : X_\alpha \to Z)_{\alpha\in A}$, there exists a unique map $h : X \to Z$ such that $h \circ i_\alpha = h_\alpha$ for all $\alpha \in A$, and the following are equivalent:

1. h is continuous;
2. for every $\alpha \in A$, the map h_α is continuous.

Proof. Existence and uniqueness of h as a *set map* are given by Proposition 31.

For continuity, apply the universal property of the final topology (Proposition 26(2)) to the family $(i_\alpha)_{\alpha\in A}$: the map h is continuous if and only if each composite $h \circ i_\alpha = h_\alpha$ is continuous. □

Remark (Diagrammatic form)**.** *The universal property of Theorem 6 can be written as:*

$$\begin{tikzcd} X_\alpha \arrow[r, "i_\alpha"] \arrow[dr, "h_\alpha"'] & \coprod_{\beta\in A} X_\beta \arrow[d, dashed, "h"] \\ & Z \end{tikzcd} \qquad (\alpha \in A),$$

meaning that h is uniquely determined by $h \circ i_\alpha = h_\alpha$ for all α.

Lecture 7: Open and Closed Maps; Quotient Topology

1. Open and closed maps

Definition 40 (Open and closed maps)**.** Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map of topological spaces.

1. f is called *open* if for every open set $U \in \mathcal{T}_X$, the image $f(U)$ is open in Y.
2. f is called *closed* if for every closed set $F \subseteq X$, the image $f(F)$ is closed in Y.

Remark. *The properties "open" and "closed" are logically independent of continuity. In particular, an open map need not be continuous and a continuous map need not be open. If f is a homeomorphism, then f is both open and closed.*

Lemma (Testing openness on a basis)**.** *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map, and let $\mathcal{B} \subseteq \mathcal{T}_X$ be a basis of $\mathcal{T}_X$. Assume that $f(B) \in \mathcal{T}_Y$ for every $B \in \mathcal{B}$. Then f is an open map.*

Proof. Let $U \in \mathcal{T}_X$ be open. Since $\mathcal{B}$ is a basis, there exists a family $(B_i)_{i \in I}$ in $\mathcal{B}$ such that $U = \bigcup_{i \in I} B_i$. Then

$$f(U) = f\Big(\bigcup_{i \in I} B_i\Big) = \bigcup_{i \in I} f(B_i),$$

and the right-hand side is open in Y as a union of open sets. Hence $f(U)$ is open for every open U, i.e. f is open. $\square$

2. Projections from products are open

Theorem 7 (Projections are open)**.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ be a family of topological spaces, and let

$$X := \prod_{\alpha \in A} X_\alpha$$

equipped with the product topology. For every $\beta \in A$, the projection

$$p_\beta : X \to X_\beta, \qquad p_\beta\big((x_\alpha)_{\alpha \in A}\big) := x_\beta,$$

is an open map.

Proof. Let $\mathcal{B}$ be the standard basis of the product topology (cf. Proposition 30), i.e. the family of *cylinder sets*

$$B = \bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(U_i), \qquad n \in \mathbb{N},\ \alpha_1, \dots, \alpha_n \in A,\ U_i \in \mathcal{T}_{\alpha_i}.$$

By Lemma , it suffices to show that $p_\beta(B)$ is open in X_β for every such B.

Fix B as above. Let $J := \{i \in \{1, \dots, n\} : \alpha_i = \beta\}$ and set

$$U_\beta := \bigcap_{i \in J} U_i,$$

with the convention that $U_\beta = X_\beta$ if $J = \varnothing$. We claim that either $p_\beta(B) = \varnothing$ or $p_\beta(B) = U_\beta$; in particular $p_\beta(B)$ is open.

Indeed, always $p_\beta(B) \subseteq U_\beta$ because membership in B forces the β-coordinate to lie in each U_i with $\alpha_i = \beta$. If $B = \varnothing$, then $p_\beta(B) = \varnothing$ and we are done. Assume now that $B \neq \varnothing$ and choose $x^0 = (x^0_\alpha)_{\alpha\in A} \in B$. For any $y \in U_\beta$, define $x = (x_\alpha)_{\alpha\in A} \in X$ by

$$x_\alpha := \begin{cases} y, & \alpha = \beta, \\ x^0_\alpha, & \alpha \neq \beta. \end{cases}$$

Since the defining conditions of B only constrain finitely many coordinates, and the only constraint on the β-coordinate is $x_\beta \in U_\beta$, we have $x \in B$. Therefore $y = p_\beta(x) \in p_\beta(B)$, showing $U_\beta \subseteq p_\beta(B)$. Hence $p_\beta(B) = U_\beta$, as claimed.

Thus $p_\beta(B)$ is open for every basic open B, and Lemma implies that p_β is open. □

Remark (Projections need not be closed)**.** *In general, projections from products are* not *closed maps.*

For example, let $p : \mathbb{R}^2 \to \mathbb{R}$ be the projection $p(x, y) = x$, and consider

$$F := \{(x, y) \in \mathbb{R}^2 : \; xy = 1\}.$$

The set F is closed in $\mathbb{R}^2$ since it is the preimage of $\{1\}$ under the continuous map $(x, y) \mapsto xy$. However,

$$p(F) = \mathbb{R} \setminus \{0\},$$

which is not closed in $\mathbb{R}$.

3. Quotient sets and factorization

Definition 41 (Quotient set)**.** Let X be a set and let $\sim$ be an equivalence relation on X. The set of equivalence classes is denoted $X/\sim$, and the canonical surjection (quotient map of sets) is

$$q : X \to X/\sim, \qquad q(x) := [x].$$

Proposition 34 (Set-theoretic universal property)**.** *Let X be a set with an equivalence relation $\sim$, and let $q : X \to X/\sim$ be the canonical map. Let Z be a set and $f : X \to Z$ a map such that*

$$x \sim x' \implies f(x) = f(x').$$

Then there exists a unique map $\bar{f} : X/\sim \to Z$ such that $f = \bar{f} \circ q$.

Proof. Define $\bar{f}([x]) := f(x)$. This is well-defined because f is constant on each equivalence class. Then $(\bar{f} \circ q)(x) = \bar{f}([x]) = f(x)$, so $f = \bar{f} \circ q$. Uniqueness follows from surjectivity of q: if $\bar{f}_1 \circ q = \bar{f}_2 \circ q$, then $\bar{f}_1 = \bar{f}_2$. □

4. Quotient topology

Definition 42 (Quotient topology)**.** Let $(X, \mathcal{T}_X)$ be a topological space and let $\sim$ be an equivalence relation on X with canonical surjection $q : X \to X/\sim$. The *quotient topology* on $X/\sim$ is the topology

$$\mathcal{T}_{\text{quot}} := \{U \subseteq X/\sim : \; q^{-1}(U) \in \mathcal{T}_X\}.$$

The space $(X/\sim, \mathcal{T}_{\text{quot}})$ is called the *quotient* of X by $\sim$.

Proposition 35. *$\mathcal{T}_{\text{quot}}$ is a topology on $X/\sim$, and the canonical map*

$$q : (X, \mathcal{T}_X) \to (X/\sim, \mathcal{T}_{\text{quot}})$$

is continuous.

Proof. We check the topology axioms using the elementary properties of preimages.
(T1). One has $q^{-1}(\varnothing) = \varnothing \in \mathcal{T}_X$ and $q^{-1}(X/\sim) = X \in \mathcal{T}_X$, hence $\varnothing, X/\sim \in \mathcal{T}_{\text{quot}}$.
(T2). Let $(U_\alpha)_{\alpha\in I} \subseteq \mathcal{T}_{\text{quot}}$. Then each $q^{-1}(U_\alpha)$ is open in X, and

$$q^{-1}\Big(\bigcup_{\alpha\in I} U_\alpha\Big) = \bigcup_{\alpha\in I} q^{-1}(U_\alpha)$$

is open in X. Hence $\bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}_{\text{quot}}$.
(T3). If $U, V \in \mathcal{T}_{\text{quot}}$, then $q^{-1}(U)$ and $q^{-1}(V)$ are open in X, and

$$q^{-1}(U \cap V) = q^{-1}(U) \cap q^{-1}(V)$$

is open in X. Hence $U \cap V \in \mathcal{T}_{\text{quot}}$.
Thus $\mathcal{T}_{\text{quot}}$ is a topology. By definition of $\mathcal{T}_{\text{quot}}$, the map q is continuous. □

Theorem 8 (Universal property of the quotient topology)**.** Let $(X, \mathcal{T}_X)$ be a topological space with an equivalence relation $\sim$, and let $q : X \to X/\sim$ be the canonical map equipped with the quotient topology. Let $(Z, \mathcal{T}_Z)$ be a topological space and let $f : X \to Z$ be a map constant on $\sim$-equivalence classes. Let $\bar{f} : X/\sim \to Z$ be the induced map (Proposition 34), so that $f = \bar{f} \circ q$. Then:

1. $\bar{f}$ is continuous if and only if f is continuous.

2. Equivalently, for every topological space Z and every map $\bar{f} : X/\sim \to Z$, the map $\bar{f}$ is continuous if and only if $\bar{f} \circ q$ is continuous.

Proof. If $\bar{f}$ is continuous, then $f = \bar{f} \circ q$ is continuous as a composite of continuous maps.
Conversely, assume f is continuous. Let $W \in \mathcal{T}_Z$ be open. Then

$$q^{-1}\big(\bar{f}^{-1}(W)\big) = (\bar{f} \circ q)^{-1}(W) = f^{-1}(W) \in \mathcal{T}_X.$$

Hence $\bar{f}^{-1}(W) \in \mathcal{T}_{\text{quot}}$ by definition of the quotient topology, so $\bar{f}$ is continuous. The reformulation in (2) is the same statement with $f = \bar{f} \circ q$. □

5. Quotient maps and fibers

Definition 43 (Quotient map)**.** Let $q : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map of topological spaces. We say that q is a *quotient map* if:

1. q is surjective;

2. a subset $U \subseteq Y$ is open if and only if $q^{-1}(U)$ is open in X.

Equivalently, $\mathcal{T}_Y$ is the final topology on Y induced by q.

Remark. *If q is a quotient map, then q is continuous (the "only if" direction is precisely continuity). Conversely, if q is surjective and $\mathcal{T}_Y = \{U \subseteq Y : \ q^{-1}(U) \in \mathcal{T}_X\}$, then q is a quotient map by definition.*

Proposition 36. *Let* $q : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ *be a surjective continuous map.*

1. *If* q *is open, then* q *is a quotient map.*
2. *If* q *is closed, then* q *is a quotient map.*

Proof. We prove (1); the proof of (2) is analogous by passing to complements.

Assume q is surjective, continuous, and open. Let $U \subseteq Y$ such that $q^{-1}(U)$ is open in X. Then $U = q(q^{-1}(U))$ by surjectivity, and the right-hand side is open in Y since q is open. Hence U is open. Thus U is open in Y if and only if $q^{-1}(U)$ is open in X, i.e. q is a quotient map. □

Definition 44 (Fiber (level set))**.** Let $q : X \to Y$ be a map and let $y \in Y$. The *fiber* (or *level set*) of q over y is

$$q^{-1}(\{y\}) = \{x \in X : \ q(x) = y\}.$$

Lecture 8: Limit Points and Sequences

1. Neighborhoods, adherence, and closure

Definition 45 (Neighborhood)**.** Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. A subset $N \subseteq X$ is called a *neighborhood* of x if there exists an open set $U \in \mathcal{T}$ such that

$$x \in U \subseteq N.$$

Notation. *For $x \in X$, we denote by $\mathcal{N}(x)$ the family of neighborhoods of x.*

Lemma. *Let $(X, \mathcal{T})$ be a topological space and $x \in X$.*

1. *If $N \in \mathcal{N}(x)$ and $N \subseteq N'$, then $N' \in \mathcal{N}(x)$.*
2. *If $N_1, N_2 \in \mathcal{N}(x)$, then $N_1 \cap N_2 \in \mathcal{N}(x)$.*
3. *Every open set $U \in \mathcal{T}$ with $x \in U$ is a neighborhood of x.*

Proof. (1) Choose open U with $x \in U \subseteq N \subseteq N'$, then N' is a neighborhood.

(2) Choose open U_i with $x \in U_i \subseteq N_i$ $(i = 1, 2)$. Then $U_1 \cap U_2$ is open, contains x, and satisfies $U_1 \cap U_2 \subseteq N_1 \cap N_2$.

(3) Take U itself. □

Definition 46 (Adherent point; closure)**.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.

1. A point $x \in X$ is called an *adherent point* of A if

$$(\forall N \in \mathcal{N}(x)) \qquad N \cap A \neq \varnothing.$$

2. The *closure* of A, denoted $\overline{A}$, is the set of adherent points of A.

Proposition 37. *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.*

1. *$\overline{A}$ is closed and satisfies $A \subseteq \overline{A}$.*
2. *If $F \subseteq X$ is closed and $A \subseteq F$, then $\overline{A} \subseteq F$.*

In particular, $\overline{A}$ is the smallest closed subset of X containing A; it exists and is unique.

Proof. Let $\mathcal{F}$ be the family of closed subsets of X containing A, and set $C := \bigcap_{F \in \mathcal{F}} F$. Then C is closed and contains A, and is contained in every closed F containing A.

We claim $C = \overline{A}$.

(i) $C \subseteq \overline{A}$. Let $x \in C$ and let $N \in \mathcal{N}(x)$. Choose an open U with $x \in U \subseteq N$. If $U \cap A = \varnothing$, then $X \setminus U$ is closed, contains A, and does not contain x, contradicting $x \in C$. Hence $U \cap A \neq \varnothing$, and therefore $N \cap A \neq \varnothing$. Thus $x \in \overline{A}$.

(ii) $\overline{A} \subseteq C$. Let $x \notin C$. Then $x \notin F$ for some closed $F \in \mathcal{F}$. Thus $U := X \setminus F$ is open, contains x, and satisfies $U \cap A = \varnothing$ (since $A \subseteq F$). Hence there exists a neighborhood of x disjoint from A, so $x \notin \overline{A}$.

Thus $\overline{A} = C = \bigcap_{F \supseteq A,\ F \text{ closed}} F$, proving (1) and (2). Uniqueness is immediate from minimality. □

Remark. *A subset $A \subseteq X$ is closed if and only if $\overline{A} = A$.*

2. Limit points and the derived set

Definition 47 (Limit point; derived set)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. A point $x \in X$ is called a *limit point* (or *accumulation point*) of A if

$$(\forall N \in \mathcal{N}(x)) \qquad N \cap (A \setminus \{x\}) \neq \varnothing.$$

The set of limit points of A is denoted by A' and is called the *derived set* of A.

Example. *In* $(\mathbb{R}, \mathcal{T}_{\text{standard}})$*, let* $A = [0,1) \cup \{2\}$*. Then* $A' = [0,1]$*; in particular,* $2 \notin A'$ *and* $\overline{A} = [0,1] \cup \{2\}$*.*

Proposition 38. *For every topological space* X *and every* $A \subseteq X$ *one has*

$$\overline{A} = A \cup A'.$$

Proof. Let $x \in X$.

If $x \in A$, then every neighborhood of x meets A (at least in x), hence $x \in \overline{A}$. If $x \in A'$, then every neighborhood of x meets $A \setminus \{x\}$, hence also meets A, so $x \in \overline{A}$. Thus $A \cup A' \subseteq \overline{A}$.

Conversely, assume $x \in \overline{A}$. If $x \notin A$, then $A \setminus \{x\} = A$, so the condition $(\forall N \in \mathcal{N}(x))$ $N \cap A \neq \varnothing$ implies $(\forall N \in \mathcal{N}(x))$ $N \cap (A \setminus \{x\}) \neq \varnothing$, hence $x \in A'$. Therefore $\overline{A} \subseteq A \cup A'$. $\square$

Definition 48 (T_1-space)**.** A topological space X is a T_1*-space* if every singleton $\{x\}$ is closed (equivalently, $X \setminus \{x\}$ is open for every $x \in X$).

Proposition 39. *If* X *is a* T_1*-space, then for every* $A \subseteq X$ *the derived set* A' *is closed.*

Proof. Let $x \notin A'$. Then there exists an open set U with $x \in U$ and

$$U \cap (A \setminus \{x\}) = \varnothing.$$

Let $y \in U$. We show $y \notin A'$.

If $y = x$, this is true by choice of U. If $y \neq x$, then since X is T_1, the set $U \setminus \{x\} = U \cap (X \setminus \{x\})$ is open and contains y; moreover

$$(U \setminus \{x\}) \cap A = \varnothing$$

because $U \cap A \subseteq \{x\}$. Hence y has a neighborhood disjoint from A, so in particular it has a neighborhood disjoint from $A \setminus \{y\}$; thus $y \notin A'$. Therefore $U \subseteq X \setminus A'$, showing that $X \setminus A'$ is open and A' is closed. $\square$

Remark. *The* T_1 *hypothesis is necessary: in the indiscrete topology* $\mathcal{T} = \{\varnothing, X\}$ *on a set* X *with* $\#X \geq 2$*, if* $A = \{a\}$*, then* $A' = X \setminus \{a\}$*, which is not closed.*

3. Sequences and convergence

Definition 49 (Sequence)**.** Let X be a set. A *sequence* in X is a map $x : \mathbb{N} \to X$, written $n \mapsto x_n$.

Definition 50 (Convergence of a sequence)**.** Let $(X, \mathcal{T})$ be a topological space and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. We say that (x_n) *converges* to $x \in X$ (or that x is a *limit* of the sequence) if

$$(\forall N \in \mathcal{N}(x))(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \qquad x_n \in N.$$

We write $x_n \to x$.

Lemma (Testing convergence on a neighborhood basis)**.** *Let $(X, \mathcal{T})$ be a topological space, let $x \in X$, and let $\mathcal{B}_x \subseteq \mathcal{N}(x)$ be a neighborhood basis at x (in the sense of Definition 51 below). Then $x_n \to x$ if and only if*

$$(\forall B \in \mathcal{B}_x)(\exists n_0)(\forall n \geq n_0) \qquad x_n \in B.$$

Proof. The "only if" direction is immediate since $\mathcal{B}_x \subseteq \mathcal{N}(x)$. Conversely, let $N \in \mathcal{N}(x)$. Choose $B \in \mathcal{B}_x$ with $B \subseteq N$. If the sequence is eventually in B, then it is eventually in N. □

Proposition 40 (A sequential limit is an adherent point)**.** *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. If (a_n) is a sequence in A and $a_n \to x$, then $x \in \overline{A}$.*

Proof. Let N be a neighborhood of x. Since $a_n \to x$, there exists n_0 such that $a_{n_0} \in N$. But $a_{n_0} \in A$, hence $N \cap A \neq \varnothing$. By Definition 46, $x \in \overline{A}$. □

Proposition 41 (Uniqueness of sequential limits in Hausdorff spaces)**.** *Let X be a Hausdorff space. If a sequence (x_n) converges to both x and y, then $x = y$.*

Proof. Assume $x \neq y$. Since X is Hausdorff, there exist disjoint open sets $U \ni x$ and $V \ni y$. Since $x_n \to x$, the sequence is eventually in U; since $x_n \to y$, it is eventually in V. For large n, we would then have $x_n \in U \cap V = \varnothing$, a contradiction. □

Example (Non-uniqueness in a non-Hausdorff space)**.** *Let $X = \{a, b, c\}$ and*

$$\mathcal{T} := \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}.$$

Then $(X, \mathcal{T})$ is a topological space which is not Hausdorff. The constant sequence $x_n = b$ converges to a, to b, and to c, since every open neighborhood of a (resp. c) contains b.

4. Neighborhood bases and first countability

Definition 51 (Neighborhood basis at a point)**.** Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A family $\mathcal{B}_x \subseteq \mathcal{N}(x)$ is called a *neighborhood basis* at x if

$$(\forall N \in \mathcal{N}(x))(\exists B \in \mathcal{B}_x) \qquad B \subseteq N.$$

Definition 52 (First countable)**.** A topological space X is *first countable* if for every $x \in X$ there exists a countable neighborhood basis at x.

Proposition 42. *Every metric space is first countable.*

Proof. Let (X, d) be a metric space and $x \in X$. The family

$$\mathcal{B}_x := \{B_d(x, 1/n) : n \in \mathbb{N}\}$$

is a countable neighborhood basis at x. Indeed, if N is any neighborhood of x, then there exists $r > 0$ with $B_d(x, r) \subseteq N$. Choose $n \in \mathbb{N}$ with $1/n < r$; then $B_d(x, 1/n) \subseteq B_d(x, r) \subseteq N$. □

Theorem 9 (First countability implies closure is detected by sequences)**.** Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$, and let $x \in \overline{A}$. Assume X is first countable at x. Then there exists a sequence (a_n) in A such that $a_n \to x$.

Proof. Let $(B_n)_{n\in\mathbb{N}}$ be a countable neighborhood basis at x. Define

$$N_n := \bigcap_{k=1}^{n} B_k \qquad (n \in \mathbb{N}).$$

Then each N_n is a neighborhood of x, and the family (N_n) is decreasing: $N_1 \supseteq N_2 \supseteq \cdots$. Since $x \in \overline{A}$, every neighborhood of x meets A; in particular $N_n \cap A \neq \varnothing$ for all n. Choose $a_n \in N_n \cap A$.

We claim $a_n \to x$. Let N be any neighborhood of x. Since (B_n) is a neighborhood basis, choose $B_m \subseteq N$. Then for all $n \geq m$ one has $N_n \subseteq B_m \subseteq N$, hence $a_n \in N$. This is exactly $a_n \to x$. ☐

Corollary. *In a metric space (X, d), for every $A \subseteq X$ and every $x \in X$,*

$$x \in \overline{A} \quad\Longleftrightarrow\quad \exists (a_n) \subseteq A \text{ such that } a_n \to x.$$

Proof. ($\Leftarrow$) is Proposition 40. ($\Rightarrow$) follows from Theorem 9 since metric spaces are first countable (Proposition 42). ☐

Remark. *The converse of Theorem 9 is* false*: there exist spaces in which every closure point is the limit of a sequence, yet the space is not first countable. This leads to the notions of* Fréchet–Urysohn spaces *and* sequential spaces*, which are strictly weaker than first countability.*

Example (Box topology: closure not detected by sequences)**.** *Let $X := \mathbb{R}^{\mathbb{N}}$ endowed with the* box topology*, i.e. the topology generated by the basis*

$$\mathcal{B} := \Big\{ \prod_{n\in\mathbb{N}} U_n : \ U_n \subseteq \mathbb{R} \text{ open for all } n \Big\}.$$

Let $0 := (0, 0, \dots) \in X$ and let

$$A := (0, \infty)^{\mathbb{N}} \subseteq X.$$

Then $0 \in \overline{A}$, but there is no sequence $(a^m)_{m\in\mathbb{N}}$ in A such that $a^m \to 0$.

Proof. Step 1: $0 \in \overline{A}$. Let W be a neighborhood of 0. Then W contains some basic neighborhood

$$B = \prod_{n\in\mathbb{N}} (-\varepsilon_n, \varepsilon_n)$$

with $\varepsilon_n > 0$ for all n. The point $x = (\varepsilon_n/2)_{n\in\mathbb{N}}$ belongs to A and to B, hence $W \cap A \neq \varnothing$. Thus every neighborhood of 0 meets A, so $0 \in \overline{A}$.

Step 2: no sequence in A converges to 0. Let $(a^m)_{m\in\mathbb{N}}$ be any sequence in A, where $a^m = (a^m_n)_{n\in\mathbb{N}}$ and $a^m_n > 0$ for all m, n. Define

$$U := \prod_{n\in\mathbb{N}} \Big(-\frac{a^n_n}{2}, \frac{a^n_n}{2} \Big).$$

Then U is an open neighborhood of 0 in the box topology. For each $m \in \mathbb{N}$, the m-th coordinate of a^m equals a^m_m, which does not belong to $\big(-a^m_m/2,\, a^m_m/2\big)$. Hence $a^m \notin U$ for all m, so the sequence is not eventually in U and therefore cannot converge to 0. ☐

Lecture 9: Interior and Boundary

1. Interior

Notation. *Let $(X, \mathcal{T})$ be a topological space. For $A \subseteq X$ we write $A^c := X \setminus A$ and denote the closure of A by $\overline{A}$.*

Definition 53 (Interior)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The *interior* of A, denoted $\operatorname{int}(A)$ (or A°), is defined by

$$\operatorname{int}(A) := \bigcup\{U \in \mathcal{T} : \ U \subseteq A\}.$$

Equivalently, $\operatorname{int}(A)$ is the largest open subset of X contained in A.

Proposition 43. *For every $A \subseteq X$:*

1. $\operatorname{int}(A)$ *is open and* $\operatorname{int}(A) \subseteq A$*;*
2. *if U is open and $U \subseteq A$, then $U \subseteq \operatorname{int}(A)$.*

In particular, the interior exists and is uniquely determined by these properties.

Proof. By definition, $\operatorname{int}(A)$ is a union of open sets; hence it is open. Every set appearing in the union is contained in A, hence $\operatorname{int}(A) \subseteq A$. If U is open with $U \subseteq A$, then U occurs among the sets being unioned, so $U \subseteq \operatorname{int}(A)$. Uniqueness follows from the "largest" characterization: if V is another open subset of A containing all open subsets of A, then $V \subseteq \operatorname{int}(A)$ and $\operatorname{int}(A) \subseteq V$, hence $V = \operatorname{int}(A)$. □

Proposition 44 (Complement identities)**.** *For every $A \subseteq X$ one has*

$$X \setminus \operatorname{int}(A) = \overline{A^c}, \qquad X \setminus \overline{A} = \operatorname{int}(A^c).$$

Equivalently,

$$\operatorname{int}(A) = X \setminus \overline{A^c}, \qquad \overline{A} = X \setminus \operatorname{int}(A^c).$$

Proof. We prove the first identity; the second follows by replacing A with A^c.

Let $x \in X$. Then $x \notin \operatorname{int}(A)$ if and only if there is no open set U with $x \in U \subseteq A$, i.e. if and only if every neighborhood of x meets A^c. By the neighborhood characterization of closure, this is equivalent to $x \in \overline{A^c}$. Thus $X \setminus \operatorname{int}(A) = \overline{A^c}$. □

Proposition 45 (Monotonicity and idempotence)**.** *For all $A, B \subseteq X$:*

1. $A \subseteq B \implies \operatorname{int}(A) \subseteq \operatorname{int}(B)$ *and* $\overline{A} \subseteq \overline{B}$*;*
2. $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$ *and* $\overline{\overline{A}} = \overline{A}$*.*

Proof. (1) If U is open and $U \subseteq A \subseteq B$, then $U \subseteq B$, hence the union defining $\operatorname{int}(A)$ is taken over a subfamily of that defining $\operatorname{int}(B)$; thus $\operatorname{int}(A) \subseteq \operatorname{int}(B)$. For closures, use the characterization $\overline{A} = \bigcap\{F : F \text{ closed and } A \subseteq F\}$: if $A \subseteq B$ then every closed set containing B contains A, hence the intersection defining $\overline{B}$ is taken over a subfamily, so $\overline{A} \subseteq \overline{B}$.

(2) Since $\operatorname{int}(A)$ is open and contained in A, it is also contained in $\operatorname{int}(A)$, so it is the largest open subset of itself; hence $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$. Similarly, $\overline{A}$ is closed and contains A, hence it is the smallest closed set containing itself, so $\overline{\overline{A}} = \overline{A}$. □

Proposition 46 (Finite unions and intersections)**.** *For all $A, B \subseteq X$:*

$$\overline{A \cup B} = \overline{A} \cup \overline{B}, \qquad \operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B).$$

Moreover,

$$\operatorname{int}(A) \cup \operatorname{int}(B) \subseteq \operatorname{int}(A \cup B), \qquad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

Proof. Closure of a union. The set $\overline{A} \cup \overline{B}$ is closed and contains $A \cup B$, hence $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ by minimality of closure. Conversely, $A \subseteq A \cup B$ implies $\overline{A} \subseteq \overline{A \cup B}$, and similarly $\overline{B} \subseteq \overline{A \cup B}$. Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$, proving equality.

Interior of an intersection. Since $\operatorname{int}(A) \subseteq A$ and $\operatorname{int}(B) \subseteq B$, one has $\operatorname{int}(A) \cap \operatorname{int}(B) \subseteq A \cap B$, and the left-hand side is open; hence $\operatorname{int}(A) \cap \operatorname{int}(B) \subseteq \operatorname{int}(A \cap B)$ by maximality of interior. Conversely, if $U \subseteq A \cap B$ is open then $U \subseteq A$ and $U \subseteq B$, hence $U \subseteq \operatorname{int}(A)$ and $U \subseteq \operatorname{int}(B)$. Taking the union over all such U yields $\operatorname{int}(A \cap B) \subseteq \operatorname{int}(A) \cap \operatorname{int}(B)$.

Inclusions. The inclusion $\operatorname{int}(A) \cup \operatorname{int}(B) \subseteq \operatorname{int}(A \cup B)$ follows from monotonicity: $\operatorname{int}(A) \subseteq \operatorname{int}(A \cup B)$ and $\operatorname{int}(B) \subseteq \operatorname{int}(A \cup B)$. Similarly, $A \cap B \subseteq A$ and $A \cap B \subseteq B$ give $\overline{A \cap B} \subseteq \overline{A}$ and $\overline{A \cap B} \subseteq \overline{B}$, hence $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. ☐

Remark (A standard counterexample)**.** *In $(\mathbb{R}, \mathcal{T}_{\text{standard}})$, let $A = [0,1]$ and $B = [1,2]$. Then*

$$\operatorname{int}(A) \cup \operatorname{int}(B) = (0,1) \cup (1,2) = (0,2) \setminus \{1\}, \qquad \operatorname{int}(A \cup B) = \operatorname{int}([0,2]) = (0,2),$$

so $\operatorname{int}(A) \cup \operatorname{int}(B) \neq \operatorname{int}(A \cup B)$.

2. Boundary

Definition 54 (Boundary)**.** Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The *boundary* of A is

$$\partial A := \overline{A} \cap \overline{A^c}.$$

Example. *In $(\mathbb{R}, \mathcal{T}_{\text{standard}})$ one has $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$, hence*

$$\partial\mathbb{Q} = \overline{\mathbb{Q}} \cap \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}.$$

Proposition 47 (Alternative descriptions)**.** *For every $A \subseteq X$,*

$$\partial A = \overline{A} \setminus \operatorname{int}(A) = X \setminus \big(\operatorname{int}(A) \cup \operatorname{int}(A^c)\big), \qquad \partial A = \partial(A^c).$$

In particular, ∂A is closed.

Proof. Using Proposition 44, one has $\overline{A^c} = X \setminus \operatorname{int}(A)$, hence

$$\partial A = \overline{A} \cap \overline{A^c} = \overline{A} \cap (X \setminus \operatorname{int}(A)) = \overline{A} \setminus \operatorname{int}(A).$$

Again by Proposition 44, $X \setminus \overline{A} = \operatorname{int}(A^c)$, so

$$\overline{A} \setminus \operatorname{int}(A) = X \setminus \big((X \setminus \overline{A}) \cup \operatorname{int}(A)\big) = X \setminus \big(\operatorname{int}(A^c) \cup \operatorname{int}(A)\big).$$

The identity $\partial A = \partial(A^c)$ is immediate from symmetry of the definition. Finally, ∂A is an intersection of closed sets, hence closed. ☐

Proposition 48 (Decompositions)**.** *For every $A \subseteq X$:*

1. *$\overline{A} = A \cup \partial A$;*

2. $\operatorname{int}(A) = A \setminus \partial A$*;*

3. one has the disjoint decomposition

$$X = \operatorname{int}(A) \ \sqcup \ \partial A \ \sqcup \ \operatorname{int}(A^c).$$

Proof. (1) The inclusion $A \cup \partial A \subseteq \overline{A}$ is clear since $A \subseteq \overline{A}$ and $\partial A \subseteq \overline{A}$. Conversely, let $x \in \overline{A}$. If $x \in A$ we are done. If $x \notin A$, then $x \in A^c$, hence every neighborhood of x meets A^c (at least in x). Since $x \in \overline{A}$, every neighborhood of x also meets A. Thus $x \in \overline{A} \cap \overline{A^c} = \partial A$. Hence $\overline{A} \subseteq A \cup \partial A$.

(2) If $x \in \operatorname{int}(A)$, then some neighborhood of x is contained in A, hence disjoint from A^c; therefore $x \notin \overline{A^c}$ and in particular $x \notin \partial A$. Thus $\operatorname{int}(A) \subseteq A \setminus \partial A$. Conversely, if $x \in A \setminus \partial A$, then $x \notin \overline{A^c}$, hence there exists a neighborhood N of x with $N \cap A^c = \varnothing$, i.e. $N \subseteq A$. Therefore $x \in \operatorname{int}(A)$.

(3) By Proposition 47, the three sets are pairwise disjoint and their union is X. □

Corollary. *For $A \subseteq X$, one has $\partial A = \varnothing$ if and only if A is open and closed.*

Proof. If $\partial A = \varnothing$, then by Proposition 48(1) and (2),

$$\overline{A} = A \cup \partial A = A, \qquad \operatorname{int}(A) = A \setminus \partial A = A,$$

so A is closed and open. Conversely, if A is open and closed, then $\overline{A} = A$ and $\overline{A^c} = A^c$, hence $\partial A = A \cap A^c = \varnothing$. □

Lecture 10: Limits of Nets

1. Preorders and directed sets

Definition 55 (Preordered set)**.** A *preorder* on a set Λ is a binary relation $\leq$ on Λ such that, for all $\lambda, \mu, \nu \in \Lambda$,

(P1) (*reflexivity*) $\lambda \leq \lambda$;

(P2) (*transitivity*) $\lambda \leq \mu$ and $\mu \leq \nu$ imply $\lambda \leq \nu$.

A set equipped with a preorder is called a *preordered set.*

Remark. *A preorder need not be antisymmetric: it may happen that $\lambda \leq \mu$ and $\mu \leq \lambda$ with $\lambda \neq \mu$. The relation*

$$\lambda \sim \mu \quad :\Longleftrightarrow \quad (\lambda \leq \mu \text{ and } \mu \leq \lambda)$$

is an equivalence relation on Λ, and the induced relation on $\Lambda/\sim$ is a partial order.

Definition 56 (Directed set)**.** A *directed set* is a nonempty preordered set $(\Lambda, \leq)$ such that for all $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda_3 \in \Lambda$ with

$$\lambda_1 \leq \lambda_3 \quad \text{and} \quad \lambda_2 \leq \lambda_3.$$

Such an element λ_3 is called a *common upper bound* of λ_1 and λ_2.

Example. 1. *$(\mathbb{N}, \leq)$ is directed.*

2. *Let $(X, \mathcal{T})$ be a topological space and $x \in X$. The set $\mathcal{N}(x)$ of neighborhoods of x, ordered by* reverse inclusion*,*

$$N_1 \leq N_2 \quad :\Longleftrightarrow \quad N_1 \supseteq N_2,$$

is a directed set: given $N_1, N_2 \in \mathcal{N}(x)$, the intersection $N_1 \cap N_2$ is a neighborhood of x and satisfies $N_1 \leq N_1 \cap N_2$ and $N_2 \leq N_1 \cap N_2$.

2. Nets and convergence

Definition 57 (Net)**.** Let X be a set and let $(\Lambda, \leq)$ be a directed set. A *net* in X indexed by Λ is a map

$$x : \Lambda \to X, \qquad \lambda \mapsto x_\lambda.$$

We write $(x_\lambda)_{\lambda \in \Lambda}$, or simply (x_λ) when Λ is understood.

Remark. *A sequence $(x_n)_{n \in \mathbb{N}}$ is exactly a net indexed by the directed set $(\mathbb{N}, \leq)$. Thus nets extend sequences by allowing more flexible index sets.*

Definition 58 (Eventually; frequently)**.** Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in a set X, and let $E \subseteq X$.

1. The net is *eventually in E* if there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in E$ for all $\lambda \geq \lambda_0$.
2. The net is *frequently in E* if for every $\lambda_0 \in \Lambda$ there exists $\lambda \geq \lambda_0$ such that $x_\lambda \in E$.

Definition 59 (Convergence of a net)**.** Let $(X, \mathcal{T})$ be a topological space and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X. We say that (x_λ) *converges* to $x \in X$ if for every neighborhood N of x the net is eventually in N, i.e.

$$(\forall N \in \mathcal{N}(x))(\exists \lambda_0 \in \Lambda)(\forall \lambda \geq \lambda_0) \qquad x_\lambda \in N.$$

We write $x_\lambda \to x$.

Remark. *Let $\mathcal{B}_x$ be a neighborhood basis at x. Then $x_\lambda \to x$ if and only if for every $B \in \mathcal{B}_x$ the net is eventually in B.*

3. Closure characterized by nets

Proposition 49. *Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$, and let $y \in X$. Then*

$$y \in \overline{A} \quad \Longleftrightarrow \quad \exists \text{ a net } (a_\lambda)_{\lambda\in\Lambda} \text{ in } A \text{ such that } a_\lambda \to y.$$

Proof. ($\Leftarrow$) Assume there exists a net (a_λ) in A such that $a_\lambda \to y$. Let N be a neighborhood of y. Since $a_\lambda \to y$, the net is eventually in N, hence there exists λ_0 such that $a_{\lambda_0} \in N$. But $a_{\lambda_0} \in A$, so $N \cap A \neq \varnothing$. Since N was arbitrary, $y \in \overline{A}$.

($\Rightarrow$) Assume $y \in \overline{A}$, i.e. every neighborhood of y meets A. Let $\Lambda := \mathcal{N}(y)$ be the set of neighborhoods of y, ordered by reverse inclusion:

$$N_1 \leq N_2 \quad :\Longleftrightarrow \quad N_1 \supseteq N_2.$$

By Example (2), $(\Lambda, \leq)$ is a directed set. For each $N \in \Lambda$, choose $a_N \in N \cap A$ (possible since $N \cap A \neq \varnothing$). This defines a net $(a_N)_{N\in\Lambda}$ in A.

We show $a_N \to y$. Let W be a neighborhood of y, i.e. $W \in \Lambda$. Take $\lambda_0 := W$. If $N \geq \lambda_0$, then $\lambda_0 \leq N$, i.e. $W \supseteq N$. Hence $a_N \in N \subseteq W$ for all $N \geq W$, proving that the net is eventually in W. Thus $a_N \to y$. ☐

Remark. *Proposition 49 holds in* every *topological space. In contrast, the sequential statement "$y \in \overline{A}$ implies there exists a sequence in A converging to y" may fail unless additional hypotheses (e.g. first countability) are imposed.*

4. Continuity characterized by nets

Proposition 50 (Continuity and preservation of net limits)**.** *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map of topological spaces. Then the following are equivalent:*

1. *f is continuous;*
2. *for every directed set Λ and every net $(x_\lambda)_{\lambda\in\Lambda}$ in X, one has*

$$x_\lambda \to x \quad \Longrightarrow \quad f(x_\lambda) \to f(x).$$

Proof. (1) $\Rightarrow$ (2): Let $x_\lambda \to x$ in X and let V be a neighborhood of $f(x)$ in Y. By continuity, $f^{-1}(V)$ is a neighborhood of x in X. Hence there exists λ_0 such that $\lambda \geq \lambda_0$ implies $x_\lambda \in f^{-1}(V)$, i.e. $f(x_\lambda) \in V$. Thus $f(x_\lambda) \to f(x)$.

(2) $\Rightarrow$ (1): We prove the contrapositive. Assume f is not continuous. Then there exists an open set $V \subseteq Y$ such that

$$K := f^{-1}(V)$$

is not open in X. Choose $x \in K \setminus \mathrm{int}(K)$. Then $f(x) \in V$, and V is a neighborhood of $f(x)$. Since $x \notin \mathrm{int}(K)$, every neighborhood N of x meets $X \setminus K$, hence $N \cap (X \setminus K) \neq \varnothing$.

Let $\Lambda := \mathcal{N}(x)$ ordered by reverse inclusion, as in Proposition 49. For each $N \in \Lambda$, choose $x_N \in N \setminus K$. Then $(x_N)_{N\in\Lambda}$ is a net in $X \setminus K$. Exactly as in Proposition 49, one has $x_N \to x$.

However, for all N one has $x_N \notin K = f^{-1}(V)$, hence $f(x_N) \notin V$. Thus the net $(f(x_N))$ is *never* in the neighborhood V of $f(x)$, so it cannot converge to $f(x)$. This contradicts (2). Therefore f must be continuous. □

5. Hausdorff spaces and uniqueness of limits

Proposition 51. *A topological space X is Hausdorff if and only if every net in X has at most one limit.*

Proof. ($\Rightarrow$) Assume X is Hausdorff and let (x_λ) be a net with $x_\lambda \to a$ and $x_\lambda \to b$. If $a \neq b$, choose disjoint open neighborhoods $U \ni a$ and $V \ni b$. Since $x_\lambda \to a$, there exists λ_1 such that $\lambda \geq \lambda_1$ implies $x_\lambda \in U$. Since $x_\lambda \to b$, there exists λ_2 such that $\lambda \geq \lambda_2$ implies $x_\lambda \in V$. Choose $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$ (directedness). Then $x_{\lambda_3} \in U \cap V = \varnothing$, a contradiction. Hence $a = b$.

($\Leftarrow$) Assume X is not Hausdorff. Then there exist distinct $a, b \in X$ such that for every neighborhoods $U \in \mathcal{N}(a)$ and $V \in \mathcal{N}(b)$, one has $U \cap V \neq \varnothing$. Let

$$\Lambda := \mathcal{N}(a) \times \mathcal{N}(b)$$

and define a preorder on Λ by

$$(U_1, V_1) \leq (U_2, V_2) \quad :\Longleftrightarrow \quad U_1 \supseteq U_2 \text{ and } V_1 \supseteq V_2.$$

Then $(\Lambda, \leq)$ is directed, since

$$(U_1, V_1) \leq (U_1 \cap U_2,\ V_1 \cap V_2), \qquad (U_2, V_2) \leq (U_1 \cap U_2,\ V_1 \cap V_2).$$

For each $(U, V) \in \Lambda$, pick a point

$$x_{(U,V)} \in U \cap V,$$

which is possible by the non-Hausdorff assumption. This defines a net $(x_{(U,V)})_{(U,V)\in\Lambda}$ in X.

We show $x_{(U,V)} \to a$. Let $U_0 \in \mathcal{N}(a)$ be a neighborhood of a and fix any $V_0 \in \mathcal{N}(b)$. Set $\lambda_0 := (U_0, V_0) \in \Lambda$. If $\lambda = (U, V) \geq \lambda_0$, then $\lambda_0 \leq \lambda$, hence $U_0 \supseteq U$, so $x_\lambda \in U \subseteq U_0$. Thus the net is eventually in every neighborhood of a, hence converges to a. Similarly, it converges to b. Therefore this net has two distinct limits, contradicting uniqueness of net limits. □

Corollary. *If X is Hausdorff, then limits of sequences are unique.*

Proof. A sequence is a net (Remark), so uniqueness for nets implies uniqueness for sequences. □

Lecture 11–12: Compactness

1. Covers and compactness

Definition 60 (Cover; open cover; subcover)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.

1. A *cover* of A is a family $(U_\alpha)_{\alpha \in I}$ of subsets of X such that
$$A \subseteq \bigcup_{\alpha \in I} U_\alpha.$$
2. A cover $(U_\alpha)_{\alpha \in I}$ of A is an *open cover* if $U_\alpha \in \mathcal{T}$ for all $\alpha \in I$.
3. A *subcover* of a cover $(U_\alpha)_{\alpha \in I}$ is a subfamily $(U_\alpha)_{\alpha \in J}$ with $J \subseteq I$ which still covers A.
4. A *finite subcover* is a subcover indexed by a finite subset $J \subseteq I$.

Definition 61 (Quasi-compactness; compactness)**.** A topological space $(X, \mathcal{T})$ is called *quasi-compact* if every open cover of X admits a finite subcover.
A subset $K \subseteq X$ is called *quasi-compact* if it is quasi-compact for the subspace topology.

Terminology. Many texts (especially in basic courses) use the word *compact* for "quasi-compact" as above. Bourbaki distinguishes: *compact = quasi-compact + Hausdorff.* Unless stated otherwise, we follow the course convention and write **compact** for the open-cover property.

Remark. *The empty space is compact (every open cover has the empty finite subcover).*

2. Compact subsets and continuous images

Proposition 52 (Compactness in a subspace)**.** *Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$ be endowed with the subspace topology. Then the following are equivalent:*

1. *Y is compact (as a topological space);*
2. *for every family $(U_\alpha)_{\alpha \in I}$ of open subsets of X such that $Y \subseteq \bigcup_{\alpha \in I} U_\alpha$, there exist $\alpha_1, \dots, \alpha_n \in I$ such that*
$$Y \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Proof. (1) ⇒ (2): Let $(U_\alpha)_{\alpha \in I}$ be open in X and cover Y. Then $(Y \cap U_\alpha)_{\alpha \in I}$ is an open cover of Y in the subspace topology, hence admits a finite subcover:
$$Y = (Y \cap U_{\alpha_1}) \cup \cdots \cup (Y \cap U_{\alpha_n}) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

(2) ⇒ (1): Let $(V_\beta)_{\beta \in J}$ be an open cover of Y in the subspace topology. Write $V_\beta = Y \cap U_\beta$ with U_β open in X. Then
$$Y \subseteq \bigcup_{\beta \in J} U_\beta.$$
By (2), there exist $\beta_1, \dots, \beta_n$ with $Y \subseteq U_{\beta_1} \cup \cdots \cup U_{\beta_n}$, hence
$$Y = (Y \cap U_{\beta_1}) \cup \cdots \cup (Y \cap U_{\beta_n}) = V_{\beta_1} \cup \cdots \cup V_{\beta_n}.$$
Thus Y is compact. □

Proposition 53 (Continuous image of a compact space)**.** *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be continuous. If X is compact, then $f(X) \subseteq Y$ is compact (in the subspace topology).*

Proof. Let $(V_\alpha)_{\alpha\in I}$ be an open cover of $f(X)$ in Y. Then $(f^{-1}(V_\alpha))_{\alpha\in I}$ is an open cover of X. By compactness of X, there exist $\alpha_1, \dots, \alpha_n$ such that

$$X = f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}).$$

Applying f gives

$$f(X) \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n},$$

so $(V_{\alpha_i})_{1\le i\le n}$ is a finite subcover of $f(X)$. □

Corollary. *If $q : X \to Y$ is continuous and surjective and X is compact, then Y is compact.*

Proof. Apply Proposition 53 to $f = q$ and note that $q(X) = Y$. □

Proposition 54 (Closed subsets of compact spaces)**.** *Let X be compact and let $C \subseteq X$ be closed. Then C is compact (in the subspace topology).*

Proof. Let $(V_\alpha)_{\alpha\in I}$ be an open cover of C in X. Then $(V_\alpha)_{\alpha\in I} \cup \{X\setminus C\}$ is an open cover of X. By compactness, there exist $\alpha_1, \dots, \alpha_n$ such that

$$X = (X \setminus C) \cup V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

Intersecting with C yields

$$C = (C \cap V_{\alpha_1}) \cup \cdots \cup (C \cap V_{\alpha_n}) \subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n},$$

so $(V_{\alpha_i})_{1\le i\le n}$ is a finite subcover of C. □

Remark. *The converse of Proposition 54 is false in general: a compact subset of a non-Hausdorff space need not be closed. For example, in the indiscrete topology on a set X with $\#X \ge 2$, every subset is compact, but the only closed subsets are $\varnothing$ and X.*

Proposition 55 (Compact subsets of Hausdorff spaces are closed)**.** *Let X be Hausdorff and let $K \subseteq X$ be compact. Then K is closed in X.*

Proof. Let $x \in X \setminus K$. For each $k \in K$, since X is Hausdorff, there exist open sets $U_k \ni x$ and $V_k \ni k$ such that $U_k \cap V_k = \varnothing$. Then $(V_k)_{k\in K}$ is an open cover of K, hence admits a finite subcover

$$K \subseteq V_{k_1} \cup \cdots \cup V_{k_n}.$$

Set $U := U_{k_1} \cap \cdots \cap U_{k_n}$, which is open and contains x. We claim $U \cap K = \varnothing$: if $y \in U \cap K$, then $y \in V_{k_i}$ for some i, but also $y \in U \subseteq U_{k_i}$, contradicting $U_{k_i} \cap V_{k_i} = \varnothing$. Thus U is an open neighborhood of x disjoint from K, so $x \notin \overline{K}$. Hence $\overline{K} \subseteq K$, i.e. K is closed. □

Corollary (Continuous bijection compact $\to$ Hausdorff is a homeomorphism)**.** *Let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. Let $C \subseteq X$ be closed. By Proposition 54, C is compact, hence $f(C)$ is compact by Proposition 53. Since Y is Hausdorff, $f(C)$ is closed by Proposition 55. Thus f is a closed map. A bijective continuous closed map has continuous inverse, so f is a homeomorphism. □

3. Compact intervals and the circle

Lemma (Nested intervals)**.** *Let $(I_n)_{n\in\mathbb{N}}$ be a decreasing sequence of nonempty closed intervals in $\mathbb{R}$,*

$$I_n = [a_n, b_n], \qquad I_{n+1} \subseteq I_n,$$

and assume $b_n - a_n \to 0$. Then $\bigcap_{n\in\mathbb{N}} I_n$ consists of exactly one point.

Proof. Since $I_{n+1} \subseteq I_n$, the sequence (a_n) is increasing and bounded above, and (b_n) is decreasing and bounded below. Let

$$c := \sup_{n\in\mathbb{N}} a_n.$$

Then $a_n \le c$ for all n. Also $c \le b_n$ for all n: indeed, since $a_m \le b_n$ for all $m \ge n$ (because $a_m \in I_m \subseteq I_n$), taking sup over m gives $c \le b_n$. Hence $c \in [a_n, b_n] = I_n$ for all n, so $c \in \bigcap_n I_n$.

If $c' \in \bigcap_n I_n$, then $a_n \le c' \le b_n$ for all n, so $0 \le c' - c \le b_n - a_n$ for all n. Letting $n \to \infty$ yields $c' = c$. Thus the intersection is $\{c\}$. □

Theorem 10 (Borel–Lebesgue; compactness of $[0,1]$)**.** The interval $[0,1] \subseteq \mathbb{R}$ is compact (for the standard topology).

Proof. Assume by contradiction that $[0,1]$ is not compact. Then there exists an open cover $(U_\alpha)_{\alpha\in I}$ of $[0,1]$ with no finite subcover.

Let $I_0 := [0,1]$. Since $I_0 = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$, at least one of these two closed subintervals cannot be covered by finitely many U_α's; otherwise, the union would also admit a finite subcover. Choose such an interval and denote it I_1. Iterating, we obtain a decreasing sequence of closed intervals

$$I_n = [a_n, b_n] \qquad (n \in \mathbb{N})$$

such that

$$I_{n+1} \subseteq I_n, \qquad b_n - a_n = 2^{-n},$$

and such that I_n cannot be covered by finitely many sets from $(U_\alpha)_{\alpha\in I}$.

By Lemma , $\bigcap_{n\in\mathbb{N}} I_n = \{c\}$ for some $c \in [0,1]$. Since (U_α) covers $[0,1]$, there exists $\beta \in I$ with $c \in U_\beta$. Because U_β is open, there exists $\varepsilon > 0$ such that $(c-\varepsilon, c+\varepsilon) \subseteq U_\beta$. Since $I_n \to \{c\}$ (in the sense that $b_n - a_n \to 0$ and $c \in I_n$), there exists N such that

$$I_N \subseteq (c-\varepsilon, c+\varepsilon) \subseteq U_\beta.$$

Thus I_N is covered by the single open set U_β, contradicting the construction of I_N. Hence $[0,1]$ is compact. □

Corollary. *Every closed interval $[a,b] \subseteq \mathbb{R}$ is compact.*

Proof. If $a = b$ this is trivial. If $a < b$, the affine map

$$\varphi : [0,1] \to [a,b], \qquad \varphi(t) = a + (b-a)t,$$

is a homeomorphism. Since $[0,1]$ is compact by Theorem 10, its homeomorphic image $[a,b]$ is compact. □

Example (The circle as a compact quotient)**.** *Let $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map. Since $[0,1]$ is compact and $q([0,1]) = \mathbb{R}/\mathbb{Z}$, Corollary shows that $\mathbb{R}/\mathbb{Z}$ is compact.*

Define

$$\Phi : \mathbb{R}/\mathbb{Z} \to S^1 \subseteq \mathbb{C}, \qquad \Phi([t]) = \exp(2\pi i t).$$

The map is well-defined because $\exp(2\pi i(t+n)) = \exp(2\pi i t)$ for $n \in \mathbb{Z}$. It is continuous because $\Phi \circ q$ is the continuous map $t \mapsto \exp(2\pi i t)$ on $\mathbb{R}$. It is bijective, and since $\mathbb{R}/\mathbb{Z}$ is compact and S^1 is Hausdorff, Corollary implies that Φ is a homeomorphism. In particular, S^1 is compact.

4. The tube lemma and finite products

Lemma (Tube lemma)**.** *Let X and Y be topological spaces and assume that Y is compact. Let $x_0 \in X$ and let $U \subseteq X \times Y$ be an open set such that*

$$\{x_0\} \times Y \subseteq U.$$

Then there exists an open neighborhood V of x_0 in X such that

$$V \times Y \subseteq U.$$

Proof. For each $y \in Y$, the point (x_0, y) belongs to the open set U, hence there exist open neighborhoods $V_y \subseteq X$ of x_0 and $W_y \subseteq Y$ of y such that

$$V_y \times W_y \subseteq U.$$

Then $(W_y)_{y \in Y}$ is an open cover of Y, hence admits a finite subcover $W_{y_1}, \ldots, W_{y_n}$. Set

$$V := V_{y_1} \cap \cdots \cap V_{y_n},$$

which is open and contains x_0. We claim $V \times Y \subseteq U$. Let $(x, y) \in V \times Y$. Choose i with $y \in W_{y_i}$. Then $x \in V \subseteq V_{y_i}$, hence $(x, y) \in V_{y_i} \times W_{y_i} \subseteq U$. □

Corollary (Finite products of compact spaces)**.** *If X and Y are compact, then $X \times Y$ is compact (for the product topology). Consequently, any finite product of compact spaces is compact.*

Proof. Let $(U_\alpha)_{\alpha \in I}$ be an open cover of $X \times Y$. Fix $x \in X$. Then $\{x\} \times Y$ is compact (homeomorphic to Y), hence is covered by finitely many $U_{\alpha_1}, \ldots, U_{\alpha_m}$. Let

$$U_x := U_{\alpha_1} \cup \cdots \cup U_{\alpha_m},$$

an open set with $\{x\} \times Y \subseteq U_x$. By Lemma , there exists an open neighborhood $V_x \subseteq X$ of x such that

$$V_x \times Y \subseteq U_x \subseteq \bigcup_{\alpha \in I} U_\alpha.$$

The family $(V_x)_{x \in X}$ is an open cover of X, hence admits a finite subcover $V_{x_1}, \ldots, V_{x_n}$. Then

$$X \times Y = \bigcup_{j=1}^{n} (V_{x_j} \times Y) \subseteq \bigcup_{j=1}^{n} U_{x_j}.$$

Each U_{x_j} is a finite union of U_α's, hence $\bigcup_{j=1}^{n} U_{x_j}$ is a finite union of U_α's. Therefore $(U_\alpha)_{\alpha \in I}$ has a finite subcover, and $X \times Y$ is compact.

The statement for finite products follows by induction. □

5. Compactness in Euclidean space: Heine–Borel

Definition 62 (Bounded subset of $\mathbb{R}^n$)**.** A subset $A \subseteq \mathbb{R}^n$ is *bounded* if there exists $M > 0$ such that

$$A \subseteq [-M, M]^n.$$

Lemma. *For every $M > 0$ and every $n \geq 1$, the cube $[-M, M]^n \subseteq \mathbb{R}^n$ is compact.*

Proof. The interval $[-M, M]$ is compact by Corollary . By Corollary , the n-fold product $[-M, M]^n$ is compact. □

Theorem 11 (Heine–Borel)**.** In $\mathbb{R}^n$ with the standard topology, a subset $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. ($\Rightarrow$) Assume K is compact. Since $\mathbb{R}^n$ is Hausdorff, K is closed by Proposition 55. To see that K is bounded, consider the open cover

$$\mathbb{R}^n = \bigcup_{m \in \mathbb{N}} B(0, m),$$

where $B(0, m)$ is the Euclidean open ball of radius m. Then $(B(0, m))_{m \in \mathbb{N}}$ is an open cover of K; by compactness, there exist $m_1, \ldots, m_r$ such that

$$K \subseteq B(0, m_1) \cup \cdots \cup B(0, m_r) = B(0, M), \quad \text{where } M := \max\{m_1, \ldots, m_r\}.$$

Thus K is bounded.

($\Leftarrow$) Assume K is closed and bounded. Choose $M > 0$ such that $K \subseteq [-M, M]^n$. By Lemma , the cube $[-M, M]^n$ is compact. Since K is closed in $\mathbb{R}^n$, it is also closed in the subspace $[-M, M]^n$, hence compact by Proposition 54. □

6. Extreme values on compact spaces

Proposition 56 (Extreme value theorem)**.** *Let X be a compact topological space and let $f : X \to \mathbb{R}$ be continuous. Then f attains a maximum and a minimum on X, i.e. there exist $x_{\min}, x_{\max} \in X$ such that*

$$f(x_{\min}) = \min f(X), \qquad f(x_{\max}) = \max f(X).$$

Proof. By Proposition 53, the set $f(X) \subseteq \mathbb{R}$ is compact. By Theorem 11 in dimension 1, $f(X)$ is closed and bounded in $\mathbb{R}$. Hence $\alpha := \inf f(X)$ and $\beta := \sup f(X)$ exist and are real. Since $f(X)$ is closed, it contains its infimum and supremum, so $\alpha, \beta \in f(X)$. Choose $x_{\min}, x_{\max} \in X$ such that $f(x_{\min}) = \alpha$ and $f(x_{\max}) = \beta$. □

Remark. *The conclusion fails without compactness: for example, $f(x) = x$ on $(0, 1)$ has neither maximum nor minimum.*

Lecture 13: Compactness, Cluster Points, and Nets

1. Finite intersection property

Definition 63 (Finite intersection property)**.** Let X be a set and let $(C_\alpha)_{\alpha\in A}$ be a family of subsets of X. We say that $(C_\alpha)_{\alpha\in A}$ has the *finite intersection property* (abbreviated: *FIP*) if for every finite subset $F \subseteq A$ one has

$$\bigcap_{\alpha\in F} C_\alpha \neq \varnothing.$$

Proposition 57 (Compactness via closed sets)**.** *Let $(X, \mathcal{T})$ be a topological space. The following are equivalent:*

1. *X is compact.*
2. *For every family $(C_\alpha)_{\alpha\in A}$ of closed subsets of X having the finite intersection property, one has*

$$\bigcap_{\alpha\in A} C_\alpha \neq \varnothing.$$

Proof. (1) $\Rightarrow$ (2): Assume X is compact and let $(C_\alpha)_{\alpha\in A}$ be a family of closed sets with FIP. Assume for contradiction that $\bigcap_{\alpha\in A} C_\alpha = \varnothing$. Then

$$X = \bigcup_{\alpha\in A} (X \setminus C_\alpha)$$

is an open cover of X. By compactness, there exist $\alpha_1, \dots, \alpha_n$ such that

$$X = \bigcup_{i=1}^{n} (X \setminus C_{\alpha_i}).$$

Taking complements yields

$$\bigcap_{i=1}^{n} C_{\alpha_i} = X \setminus \bigcup_{i=1}^{n} (X \setminus C_{\alpha_i}) = \varnothing,$$

contradicting the FIP. Hence $\bigcap_{\alpha\in A} C_\alpha \neq \varnothing$.

(2) $\Rightarrow$ (1): Assume (2). Let $(U_i)_{i\in I}$ be an open cover of X. If it admits no finite subcover, then for every finite $F \subseteq I$ one has $\bigcup_{i\in F} U_i \neq X$, hence

$$\bigcap_{i\in F} (X \setminus U_i) \neq \varnothing.$$

Set $C_i := X \setminus U_i$; then each C_i is closed, and the family $(C_i)_{i\in I}$ has the FIP. By (2), $\bigcap_{i\in I} C_i \neq \varnothing$, i.e.

$$X \setminus \bigcup_{i\in I} U_i \neq \varnothing,$$

contradicting that (U_i) covers X. Therefore every open cover has a finite subcover, so X is compact. $\square$

2. Cluster points and subnets

Definition 64 (Cluster point of a net)**.** Let $(X, \mathcal{T})$ be a topological space and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X. A point $p \in X$ is called a *cluster point* (or *accumulation point*) of the net if for every neighborhood W of p and every $\lambda_0 \in \Lambda$ there exists $\lambda \in \Lambda$ such that $\lambda \geq \lambda_0$ and $x_\lambda \in W$. Equivalently: the net is *frequently* in every neighborhood of p.

Example. *In $X = \mathbb{R}$ with the standard topology, the sequence $x_n := (-1)^n$ has two cluster points, namely* 1 *and* -1.

Definition 65 (Subnet)**.** Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in a set X. A *subnet* of (x_λ) consists of a directed set $(M, \leq)$ and a map $\varphi : M \to \Lambda$ such that:

(S1) (*monotonicity*) $\mu_1 \leq \mu_2 \implies \varphi(\mu_1) \leq \varphi(\mu_2)$;

(S2) (*cofinality*) for every $\lambda_0 \in \Lambda$ there exists $\mu_0 \in M$ such that $\lambda_0 \leq \varphi(\mu_0)$.

The associated net $(x_{\varphi(\mu)})_{\mu\in M}$ is then called a subnet of (x_λ), and is often written $(x_{\lambda_\mu})_{\mu\in M}$ where $\lambda_\mu := \varphi(\mu)$.

Proposition 58 (Cluster points and convergent subnets)**.** *Let $(X, \mathcal{T})$ be a topological space and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X. A point $p \in X$ is a cluster point of (x_λ) if and only if there exists a subnet of (x_λ) converging to p.*

Proof. ($\Rightarrow$) Assume p is a cluster point. Let $\mathcal{N}(p)$ be the set of neighborhoods of p and consider the set

$$M := \{(\lambda, W) \in \Lambda \times \mathcal{N}(p) : \ x_\lambda \in W\}.$$

We define a preorder on M by

$$(\lambda_1, W_1) \leq (\lambda_2, W_2) \quad :\Longleftrightarrow \quad \lambda_1 \leq \lambda_2 \text{ and } W_1 \supseteq W_2.$$

We claim $(M, \leq)$ is directed. Let $(\lambda_1, W_1), (\lambda_2, W_2) \in M$. Choose $\lambda_0 \in \Lambda$ with $\lambda_0 \geq \lambda_1, \lambda_2$ (directedness of Λ), and set $W := W_1 \cap W_2 \in \mathcal{N}(p)$. Since p is a cluster point, there exists $\lambda \geq \lambda_0$ with $x_\lambda \in W$. Then $(\lambda, W) \in M$ and

$$(\lambda_1, W_1) \leq (\lambda, W), \qquad (\lambda_2, W_2) \leq (\lambda, W),$$

so M is directed.

Define $\varphi : M \to \Lambda$ by $\varphi(\lambda, W) := \lambda$. Then (S1) holds by definition of the order on M. For (S2), given $\lambda_0 \in \Lambda$, the element $(\lambda_0, X) \in M$ (since X is a neighborhood of p and $x_{\lambda_0} \in X$), and $\lambda_0 \leq \varphi(\lambda_0, X)$, so cofinality holds.

Thus $(x_{\varphi(\mu)})_{\mu\in M}$ is a subnet of (x_λ). We show it converges to p. Let $W \in \mathcal{N}(p)$. Since p is a cluster point, choose $\lambda_W \in \Lambda$ with $x_{\lambda_W} \in W$; then $(\lambda_W, W) \in M$. Let $\mu_0 := (\lambda_W, W) \in M$. If $\mu = (\lambda, U) \in M$ satisfies $\mu \geq \mu_0$, then $\mu_0 \leq \mu$, hence $W \supseteq U$ and therefore

$$x_{\varphi(\mu)} = x_\lambda \in U \subseteq W.$$

Thus the subnet is eventually in W, so it converges to p.

($\Leftarrow$) Conversely, if a subnet $(x_{\varphi(\mu)})_{\mu\in M}$ converges to p, then for any neighborhood W of p and any $\lambda_0 \in \Lambda$, choose μ_1 such that $\mu \geq \mu_1 \Rightarrow x_{\varphi(\mu)} \in W$ (convergence), and choose μ_0 with $\lambda_0 \leq \varphi(\mu_0)$ (cofinality). Since M is directed, pick $\mu \geq \mu_0, \mu_1$. Then $\varphi(\mu) \geq \lambda_0$ and $x_{\varphi(\mu)} \in W$, proving that p is a cluster point of the original net. $\square$

3. Tails and the finite intersection property

Definition 66 (Tail of a net)**.** Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in a set X and let $\lambda_0 \in \Lambda$. The λ_0-*tail* of the net is the subset

$$\Gamma_{\lambda_0} := \{x_\lambda : \ \lambda \geq \lambda_0\} \subseteq X.$$

Lemma. *Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in a set X. Then the family $(\Gamma_\lambda)_{\lambda\in\Lambda}$ of tails has the finite intersection property.*

Proof. Let $\lambda_1,\dots,\lambda_n \in \Lambda$. Since Λ is directed, there exists $\lambda_* \in \Lambda$ with $\lambda_i \le \lambda_*$ for all i. Then $x_{\lambda_*} \in \Gamma_{\lambda_i}$ for all i, hence

$$\bigcap_{i=1}^{n} \Gamma_{\lambda_i} \neq \varnothing.$$

□

Lemma. *If $(E_\alpha)_{\alpha\in A}$ is a family of subsets of a topological space $(X,\mathcal{T})$ having the FIP, then the family $(\overline{E_\alpha})_{\alpha\in A}$ also has the FIP.*

Proof. Let $F \subseteq A$ be finite. If $\bigcap_{\alpha\in F}\overline{E_\alpha} = \varnothing$, then the complements $X\setminus\overline{E_\alpha}$ are open and cover X, hence cover $\bigcap_{\alpha\in F} E_\alpha$. But $E_\alpha \subseteq \overline{E_\alpha}$ implies

$$\bigcap_{\alpha\in F} E_\alpha \subseteq \bigcap_{\alpha\in F}\overline{E_\alpha} = \varnothing,$$

contradicting the FIP of (E_α). Therefore $\bigcap_{\alpha\in F}\overline{E_\alpha} \neq \varnothing$. □

4. Compactness and cluster points of nets

Theorem 12 (Compactness and cluster points of nets)**.** Let $(X,\mathcal{T})$ be a topological space. The following are equivalent:

1. X is compact.
2. Every net in X has a cluster point.
3. Every net in X admits a convergent subnet.

Proof. (2) $\Leftrightarrow$ (3) is Proposition 58: a net has a cluster point p if and only if it has a subnet converging to p.

(1) $\Rightarrow$ (2): Assume X is compact and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X. For each $\lambda\in\Lambda$ let Γ_λ be the λ-tail (Definition 66). By Lemma , the family $(\Gamma_\lambda)_{\lambda\in\Lambda}$ has the FIP; hence by Lemma , so does $(\overline{\Gamma_\lambda})_{\lambda\in\Lambda}$. Each $\overline{\Gamma_\lambda}$ is closed, so compactness plus Proposition 57 yield

$$\bigcap_{\lambda\in\Lambda}\overline{\Gamma_\lambda} \neq \varnothing.$$

Choose p in this intersection. We claim that p is a cluster point of (x_λ).

Let W be a neighborhood of p and let $\lambda_0\in\Lambda$. Since $p\in\overline{\Gamma_{\lambda_0}}$, the neighborhood W meets Γ_{λ_0}, i.e.

$$W \cap \Gamma_{\lambda_0} \neq \varnothing.$$

Thus there exists $\lambda \ge \lambda_0$ with $x_\lambda \in W$, which is exactly the definition of p being a cluster point.

(2) $\Rightarrow$ (1): Assume every net in X has a cluster point. We prove compactness using Proposition 57(2). Let $(C_\alpha)_{\alpha\in A}$ be a family of closed subsets of X with the FIP. Let $\mathcal{G}$ be the family of finite intersections of members of (C_α):

$$\mathcal{G} := \Big\{ \bigcap_{\alpha\in F} C_\alpha : \ F \subseteq A \text{ finite} \Big\}.$$

Each $G\in\mathcal{G}$ is nonempty by the FIP. Order $\mathcal{G}$ by reverse inclusion:

$$G_1 \le G_2 \quad :\Longleftrightarrow \quad G_1 \supseteq G_2.$$

Then $(\mathcal{G}, \leq)$ is a directed set, since $G_1 \cap G_2 \in \mathcal{G}$ and satisfies $G_1 \leq G_1 \cap G_2$ and $G_2 \leq G_1 \cap G_2$.

Choose for each $G \in \mathcal{G}$ a point $x_G \in G$. This defines a net $(x_G)_{G \in \mathcal{G}}$ in X. By assumption, the net has a cluster point $p \in X$.

We claim $p \in C_\alpha$ for every $\alpha \in A$. Fix $\alpha \in A$ and set $G_\alpha := C_\alpha \in \mathcal{G}$. Let W be any neighborhood of p. Since p is a cluster point, taking $\lambda_0 := G_\alpha$ in Definition 64 yields some $G \in \mathcal{G}$ with $G \geq G_\alpha$ (i.e. $G \subseteq G_\alpha = C_\alpha$) and $x_G \in W$. But $x_G \in G \subseteq C_\alpha$, hence $W \cap C_\alpha \neq \varnothing$ for every neighborhood W of p. Therefore $p \in \overline{C_\alpha} = C_\alpha$ since C_α is closed.

Thus $p \in \bigcap_{\alpha \in A} C_\alpha$, showing that the intersection is nonempty. By Proposition 57, X is compact. □

Corollary. *A topological space X is compact if and only if every net in X has a convergent subnet.*

Proof. This is the equivalence (1) $\Leftrightarrow$ (3) in Theorem 12. □

Lecture 14: Compactness in Metric Spaces

1. Diameter

Definition 67 (Diameter)**.** Let (X, d) be a metric space and let $Y \subseteq X$. The *diameter* of Y is the extended real number

$$\operatorname{diam}(Y) := \sup\{d(y_1, y_2) : \ y_1, y_2 \in Y\} \in [0, +\infty].$$

By convention, $\operatorname{diam}(\varnothing) := 0$.

Remark. *If $Y \subseteq B(x, r)$ for some $x \in X$ and $r > 0$, then* $\operatorname{diam}(Y) \leq 2r$*.*

2. Lebesgue numbers

Notation. *In a metric space (X, d), the open ball of radius $r > 0$ centered at $x \in X$ is*

$$B(x, r) := \{y \in X : \ d(x, y) < r\}.$$

Definition 68 (Lebesgue number)**.** Let (X, d) be a metric space and let $(U_\alpha)_{\alpha \in A}$ be an open cover of X. A *Lebesgue number* for the cover is a real number $\delta > 0$ such that

$$(\forall x \in X) \ (\exists \alpha \in A) \qquad B(x, \delta) \subseteq U_\alpha.$$

Proposition 59 (Lebesgue number lemma)**.** *Let (X, d) be a compact metric space and let $(U_\alpha)_{\alpha \in A}$ be an open cover of X. Then the cover admits a Lebesgue number.*

Proof. For each $x \in X$, choose $\alpha(x) \in A$ with $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is open, there exists $\varepsilon(x) > 0$ such that

$$B(x, 2\varepsilon(x)) \subseteq U_{\alpha(x)}.$$

Then the family $\{B(x, \varepsilon(x))\}_{x \in X}$ is an open cover of X. By compactness, there exist $x_1, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} B(x_i, \varepsilon(x_i)).$$

Set

$$\delta := \min_{1 \leq i \leq n} \varepsilon(x_i) > 0.$$

Let $x \in X$. Choose i such that $x \in B(x_i, \varepsilon(x_i))$, i.e. $d(x, x_i) < \varepsilon(x_i)$. We claim that $B(x, \delta) \subseteq U_{\alpha(x_i)}$. Indeed, if $y \in B(x, \delta)$, then

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \delta + \varepsilon(x_i) \leq 2\varepsilon(x_i),$$

so $y \in B(x_i, 2\varepsilon(x_i)) \subseteq U_{\alpha(x_i)}$. Thus $B(x, \delta) \subseteq U_{\alpha(x_i)}$. $\square$

Corollary (Diameter form)**.** *Let (X, d) be compact and let $(U_\alpha)_{\alpha \in A}$ be an open cover. Then there exists $\delta > 0$ such that for every nonempty subset $Y \subseteq X$,*

$$\operatorname{diam}(Y) < \delta \implies (\exists \alpha \in A) \ \ Y \subseteq U_\alpha.$$

Proof. Let $\delta > 0$ be a Lebesgue number. Fix nonempty $Y \subseteq X$ with $\operatorname{diam}(Y) < \delta$ and choose $y_0 \in Y$. There exists α such that $B(y_0, \delta) \subseteq U_\alpha$. For any $y \in Y$, $d(y, y_0) \leq \operatorname{diam}(Y) < \delta$, hence $y \in B(y_0, \delta) \subseteq U_\alpha$. $\square$

3. Cauchy sequences, completeness, total boundedness

Definition 69 (Cauchy sequence)**.** A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) is *Cauchy* if

$$(\forall\varepsilon>0)(\exists N\in\mathbb{N})(\forall n,m\geq N)\qquad d(x_n,x_m)<\varepsilon.$$

Definition 70 (Completeness)**.** A metric space (X,d) is *complete* if every Cauchy sequence converges in X.

Definition 71 (Total boundedness)**.** A metric space (X,d) is *totally bounded* if for every $\varepsilon>0$ there exist points $x_1,\dots,x_n\in X$ such that

$$X\subseteq\bigcup_{i=1}^{n}B(x_i,\varepsilon).$$

Equivalently, for every $\varepsilon>0$ there exists a finite ε-*net* in X.

Remark. *Total boundedness implies boundedness, but the converse is false in general (e.g. the closed unit ball of an infinite-dimensional normed space is bounded but not totally bounded).*

4. Total boundedness and Cauchy subsequences

Proposition 60. *Let (X,d) be a metric space. The following are equivalent:*

1. *X is totally bounded.*
2. *Every sequence in X admits a Cauchy subsequence.*

Proof. (1) $\Rightarrow$ (2): Let (x_n) be a sequence in X. Since X is totally bounded, it can be covered by finitely many balls of radius 1; hence one such ball contains infinitely many terms of (x_n). Choose a subsequence $(x_{n_k^{(1)}})$ contained in a ball of radius 1. Repeat with radius $1/2$: cover X by finitely many balls of radius $1/2$ and extract a further subsequence $(x_{n_k^{(2)}})$ contained in a ball of radius $1/2$. Continuing inductively, we obtain nested subsequences

$$(x_{n_k^{(1)}})\supseteq(x_{n_k^{(2)}})\supseteq\cdots$$

such that $(x_{n_k^{(m)}})$ is contained in some ball of radius 2^{-m} for each m. Define the diagonal subsequence $y_m:=x_{n_m^{(m)}}$. Then for $p,q\geq m$ one has $y_p,y_q\in B(c_m,2^{-m})$ for a suitable center c_m, hence

$$d(y_p,y_q)\leq 2\cdot 2^{-m}=2^{1-m}.$$

Given $\varepsilon>0$, choose m with $2^{1-m}<\varepsilon$. Then $p,q\geq m$ implies $d(y_p,y_q)<\varepsilon$. Thus (y_m) is Cauchy.

(2) $\Rightarrow$ (1): Assume X is not totally bounded. Then there exists $\varepsilon>0$ such that X cannot be covered by finitely many balls of radius ε. Construct a sequence (x_n) inductively as follows: choose $x_1\in X$, and having chosen $x_1,\dots,x_n$, choose

$$x_{n+1}\in X\setminus\bigcup_{i=1}^{n}B(x_i,\varepsilon),$$

which is nonempty by the choice of ε. Then for $m\neq n$ one has $d(x_m,x_n)\geq\varepsilon$. Hence (x_n) admits no Cauchy subsequence, contradicting (2). Therefore X is totally bounded. $\square$

5. Sequential compactness and compactness in metric spaces

Definition 72 (Sequential compactness)**.** A metric space (X, d) is *sequentially compact* if every sequence in X admits a convergent subsequence.

Lemma (Sequential compactness $\Rightarrow$ existence of Lebesgue numbers)**.** *Let (X, d) be a sequentially compact metric space and let $(U_\alpha)_{\alpha\in A}$ be an open cover of X. Then the cover admits a Lebesgue number (Definition 68).*

Proof. Assume for contradiction that no Lebesgue number exists. Then for each $n \in \mathbb{N}$ (with $\delta_n := 1/n$) there exists $x_n \in X$ such that

$$(\forall \alpha \in A) \qquad B(x_n, 1/n) \not\subseteq U_\alpha.$$

By sequential compactness, there exists a subsequence (x_{n_k}) converging to some $x \in X$.

Choose $\alpha_0 \in A$ with $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U_{\alpha_0}$. Choose k large enough that $d(x_{n_k}, x) < \varepsilon/2$ and $1/n_k < \varepsilon/2$. Then for any $y \in B(x_{n_k}, 1/n_k)$ one has

$$d(y, x) \leq d(y, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + \frac{\varepsilon}{2} < \varepsilon,$$

hence $y \in B(x, \varepsilon) \subseteq U_{\alpha_0}$. Therefore $B(x_{n_k}, 1/n_k) \subseteq U_{\alpha_0}$, contradicting the choice of x_{n_k}. $\square$

Lemma. *Let (X, d) be a totally bounded metric space and let $(U_\alpha)_{\alpha\in A}$ be an open cover of X having a Lebesgue number $\delta > 0$. Then the cover admits a finite subcover.*

Proof. Since X is totally bounded, there exist $x_1, \ldots, x_n \in X$ such that

$$X \subseteq \bigcup_{i=1}^{n} B(x_i, \delta/2).$$

By the Lebesgue number property, for each i there exists $\alpha_i \in A$ such that

$$B(x_i, \delta) \subseteq U_{\alpha_i}.$$

In particular, $B(x_i, \delta/2) \subseteq U_{\alpha_i}$, hence

$$X \subseteq \bigcup_{i=1}^{n} B(x_i, \delta/2) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Thus $(U_{\alpha_i})_{1\leq i\leq n}$ is a finite subcover. $\square$

Proposition 61. *Every sequentially compact metric space is compact.*

Proof. Let $(U_\alpha)_{\alpha\in A}$ be an open cover of X. By Proposition 60 and sequential compactness, X is totally bounded. By Lemma , the cover admits a Lebesgue number. Thus Lemma yields a finite subcover. $\square$

6. Main characterization theorem

Theorem 13 (Compactness in metric spaces)**.** Let (X, d) be a metric space. The following are equivalent:

1. X is compact.
2. X is sequentially compact (Definition 72).
3. X is complete and totally bounded (Definitions 70 and 71).

Proof. (1) ⇒ (2): Let (x_n) be a sequence in X. Assume for contradiction that it has no convergent subsequence. We claim that for every $x \in X$ there exists $r_x > 0$ such that $B(x, r_x)$ contains only finitely many terms of (x_n). Indeed, if this failed for some x, then for every $k \in \mathbb{N}$ the ball $B(x, 1/k)$ would contain infinitely many terms, and one could construct a subsequence (x_{n_k}) with $x_{n_k} \in B(x, 1/k)$; then $x_{n_k} \to x$, contradicting the assumption.

Thus $\{B(x, r_x)\}_{x \in X}$ is an open cover of X. By compactness, extract a finite subcover $B(x_1, r_{x_1}), \ldots, B(x_m, r_{x_m})$. Each $B(x_i, r_{x_i})$ contains only finitely many terms of the sequence, hence their union contains only finitely many terms, contradicting that (x_n) is infinite. Therefore (x_n) has a convergent subsequence.

(2) ⇒ (3): *Completeness.* Let (x_n) be a Cauchy sequence. By sequential compactness, it has a convergent subsequence $(x_{n_k}) \to x$. We claim $x_n \to x$. Let $\varepsilon > 0$. Choose N such that $m, n \geq N$ implies $d(x_n, x_m) < \varepsilon/2$. Choose k such that $n_k \geq N$ and $d(x_{n_k}, x) < \varepsilon/2$. Then for any $n \geq N$,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence (x_n) converges to x, so X is complete.

Total boundedness. If X were not totally bounded, then by the construction in the proof of Proposition 60 there would exist $\varepsilon > 0$ and a sequence (x_n) in X such that $d(x_m, x_n) \geq \varepsilon$ for $m \neq n$. Such a sequence has no Cauchy subsequence, hence no convergent subsequence, contradicting sequential compactness. Thus X is totally bounded.

(3) ⇒ (2): Let (x_n) be any sequence in X. Since X is totally bounded, Proposition 60 yields a Cauchy subsequence. Since X is complete, that Cauchy subsequence converges in X. Hence X is sequentially compact.

(2) ⇒ (1): This is Proposition 61. □

Corollary. *Every compact metric space is complete and totally bounded.*

Proof. (1) ⇒ (3) in Theorem 13. □

7. (Optional) Heine–Cantor uniform continuity theorem

Theorem 14 (Heine–Cantor)**.** Let (X, d_X) be a compact metric space, let (Y, d_Y) be a metric space, and let $f : X \to Y$ be continuous. Then f is uniformly continuous: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon.$$

Proof. Fix $\varepsilon > 0$. For each $x \in X$, set

$$U_x := f^{-1}\big(B(f(x), \varepsilon/2)\big),$$

which is open and contains x by continuity of f. Then $(U_x)_{x \in X}$ is an open cover of X. By the Lebesgue number lemma (Proposition 59), there exists $\delta > 0$ such that for every $x \in X$ there exists $x_0 \in X$ with

$$B(x, \delta) \subseteq U_{x_0}.$$

Now let $x, x' \in X$ with $d_X(x, x') < \delta$, and choose x_0 such that $B(x, \delta) \subseteq U_{x_0}$. Then $x, x' \in U_{x_0}$, hence

$$d_Y(f(x), f(x_0)) < \varepsilon/2, \qquad d_Y(f(x'), f(x_0)) < \varepsilon/2.$$

By the triangle inequality,

$$d_Y(f(x), f(x')) \leq d_Y(f(x), f(x_0)) + d_Y(f(x_0), f(x')) < \varepsilon.$$

Thus f is uniformly continuous. □

Lecture 15–16: Compactness of Product Spaces (Tychonoff Theorem)

1. Filters and filter bases

Definition 73 (Filter)**.** Let X be a set. A *filter* on X is a nonempty family $\mathcal{F} \subseteq \mathcal{P}(X)$ such that

(F1) $\varnothing \notin \mathcal{F}$;

(F2) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;

(F3) if $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

Elements of $\mathcal{F}$ are called *members* of the filter.

Remark. *If $\mathcal{F}$ is a filter, then $X \in \mathcal{F}$ (take any $A \in \mathcal{F}$ and use $A \subseteq X$ in* (F3)*).*

Example. *1. If $X \neq \varnothing$, then $\{X\}$ is a filter (the* trivial filter*).*

2. If $(X, \mathcal{T})$ is a topological space and $x \in X$, the family $\mathcal{N}(x)$ of neighborhoods of x is a filter, called the neighborhood filter *at x (Lemma).*

Lemma (Filters have the finite intersection property)**.** *Let $\mathcal{F}$ be a filter on X. Then every finite intersection of members of $\mathcal{F}$ is nonempty. Equivalently, $\mathcal{F}$ has the finite intersection property.*

Proof. If $A_1, \dots, A_n \in \mathcal{F}$, then $\bigcap_{i=1}^n A_i \in \mathcal{F}$ by repeated use of (F2). Since $\varnothing \notin \mathcal{F}$, the intersection cannot be empty. □

Definition 74 (Filter base)**.** Let X be a set. A family $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *filter base* on X if

(B1) $\mathcal{B} \neq \varnothing$ and $\varnothing \notin \mathcal{B}$;

(B2) for all $B_1, B_2 \in \mathcal{B}$ there exists $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Definition 75 (Filter generated by a base)**.** Let $\mathcal{B}$ be a filter base on X. The *filter generated by $\mathcal{B}$* is

$$\langle \mathcal{B} \rangle := \{F \subseteq X : \exists B \in \mathcal{B} \text{ with } B \subseteq F\}.$$

Proposition 62. *If $\mathcal{B}$ is a filter base on X, then $\langle \mathcal{B} \rangle$ is a filter on X and $\mathcal{B}$ is a base of $\langle \mathcal{B} \rangle$ in the sense that*

$$(\forall F \in \langle \mathcal{B} \rangle)\ (\exists B \in \mathcal{B}) \qquad B \subseteq F.$$

Moreover, $\langle \mathcal{B} \rangle$ is the smallest filter on X containing $\mathcal{B}$.

Proof. We check the axioms.

(F1). If $\varnothing \in \langle \mathcal{B} \rangle$, then there exists $B \in \mathcal{B}$ with $B \subseteq \varnothing$, hence $B = \varnothing$, contradicting $\varnothing \notin \mathcal{B}$.

(F2). Let $F, G \in \langle \mathcal{B} \rangle$. Choose $B_1, B_2 \in \mathcal{B}$ with $B_1 \subseteq F$ and $B_2 \subseteq G$. By (B2), there exists $B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2 \subseteq F \cap G$. Hence $F \cap G \in \langle \mathcal{B} \rangle$.

(F3). If $F \in \langle \mathcal{B} \rangle$ and $F \subseteq H \subseteq X$, choose $B \in \mathcal{B}$ with $B \subseteq F$; then $B \subseteq H$, so $H \in \langle \mathcal{B} \rangle$.

Thus $\langle \mathcal{B} \rangle$ is a filter and the displayed base property is built into its definition. If $\mathcal{G}$ is any filter containing $\mathcal{B}$ and $F \in \langle \mathcal{B} \rangle$, choose $B \in \mathcal{B}$ with $B \subseteq F$. Then $B \in \mathcal{G}$ and $F \in \mathcal{G}$ by upward closure; hence $\langle \mathcal{B} \rangle \subseteq \mathcal{G}$. Therefore $\langle \mathcal{B} \rangle$ is the smallest filter containing $\mathcal{B}$. □

Corollary (From FIP families to filters). *Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a nonempty family with the finite intersection property. Let $\mathcal{B}$ be the family of all finite intersections of members of $\mathcal{S}$:*

$$\mathcal{B} := \Big\{ \bigcap_{i=1}^{n} S_i : \ n \in \mathbb{N}, \ S_i \in \mathcal{S} \Big\}.$$

Then $\mathcal{B}$ is a filter base and there exists a filter $\mathcal{F} = \langle \mathcal{B} \rangle$ such that $\mathcal{S} \subseteq \mathcal{F}$.

Proof. Since $\mathcal{S}$ has the FIP, no finite intersection of elements of $\mathcal{S}$ is empty; hence $\varnothing \notin \mathcal{B}$ and $\mathcal{B} \neq \varnothing$. Moreover, the intersection of two finite intersections is again a finite intersection, so (B2) holds. Thus $\mathcal{B}$ is a filter base and Proposition 62 applies. Finally, each $S \in \mathcal{S}$ is itself a finite intersection, hence belongs to $\mathcal{B} \subseteq \langle \mathcal{B} \rangle$. ☐

Definition 76 (Filter generated by a family)**.** Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a family with the FIP. The *filter generated by* $\mathcal{S}$ is the smallest filter containing $\mathcal{S}$; equivalently, it is $\langle \mathcal{B} \rangle$ where $\mathcal{B}$ is the family of finite intersections of sets in $\mathcal{S}$ (Corollary).

2. Image filters and convergence

Definition 77 (Pushforward (image) of a filter)**.** Let $f : X \to Y$ be a map and let $\mathcal{F}$ be a filter on X. The *pushforward filter* (or *image filter*) $f_*\mathcal{F}$ on Y is

$$f_*\mathcal{F} := \{ B \subseteq Y : \ f^{-1}(B) \in \mathcal{F} \}.$$

Lemma. *If $\mathcal{F}$ is a filter on X, then $f_*\mathcal{F}$ is a filter on Y.*

Proof. Nonemptiness: $Y \in f_*\mathcal{F}$ since $f^{-1}(Y) = X \in \mathcal{F}$. Also $\varnothing \notin f_*\mathcal{F}$ since $f^{-1}(\varnothing) = \varnothing \notin \mathcal{F}$. If $B_1, B_2 \in f_*\mathcal{F}$, then $f^{-1}(B_1), f^{-1}(B_2) \in \mathcal{F}$, hence

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \in \mathcal{F},$$

so $B_1 \cap B_2 \in f_*\mathcal{F}$. Finally, if $B \in f_*\mathcal{F}$ and $B \subseteq C$, then $f^{-1}(B) \subseteq f^{-1}(C)$ and $f^{-1}(C) \in \mathcal{F}$ by upward closure. ☐

Definition 78 (Convergence of a filter)**.** Let $(X, \mathcal{T})$ be a topological space, let $\mathcal{F}$ be a filter on X, and let $x \in X$. We say that $\mathcal{F}$ *converges* to x, and write $\mathcal{F} \to x$, if

$$\mathcal{N}(x) \subseteq \mathcal{F},$$

equivalently, if every neighborhood of x belongs to $\mathcal{F}$.

Lemma (Continuity and convergence of filters)**.** *Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be continuous. If $\mathcal{F}$ is a filter on X and $\mathcal{F} \to x$, then $f_*\mathcal{F} \to f(x)$.*

Proof. Let V be a neighborhood of $f(x)$ in Y. Then $f^{-1}(V)$ is a neighborhood of x in X by continuity. Since $\mathcal{F} \to x$, we have $f^{-1}(V) \in \mathcal{F}$, hence $V \in f_*\mathcal{F}$ by definition. Thus $\mathcal{N}(f(x)) \subseteq f_*\mathcal{F}$. ☐

3. Ultrafilters and Zorn's lemma

Definition 79 (Ultrafilter)**.** An *ultrafilter* on a set X is a filter $\mathcal{U}$ maximal for inclusion among filters on X: if $\mathcal{F}$ is a filter on X and $\mathcal{U} \subseteq \mathcal{F}$, then $\mathcal{F} = \mathcal{U}$.

Definition 80 (Chain; upper bound)**.** Let $(S, \leq)$ be a partially ordered set. A subset $A \subseteq S$ is a *chain* if it is totally ordered by $\leq$. An element $u \in S$ is an *upper bound* of A if $a \leq u$ for all $a \in A$.

Theorem 15 (Zorn's lemma)**.** Let $(S, \leq)$ be a partially ordered set. Assume that every chain in S has an upper bound in S. Then S has a maximal element.

Theorem 16 (Ultrafilter lemma)**.** Let X be a set and let $\mathcal{F}_0$ be a filter on X. Then there exists an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F}_0 \subseteq \mathcal{U}$.

Proof. Let
$$S := \{\mathcal{F} : \ \mathcal{F} \text{ is a filter on } X \text{ and } \mathcal{F}_0 \subseteq \mathcal{F}\},$$
ordered by inclusion. Let $\mathscr{C} \subseteq S$ be a chain and set $\mathcal{F} := \bigcup_{\mathcal{G} \in \mathscr{C}} \mathcal{G}$. We claim that $\mathcal{F} \in S$, i.e. $\mathcal{F}$ is a filter containing $\mathcal{F}_0$.

Nonemptiness and $\mathcal{F}_0 \subseteq \mathcal{F}$ are clear since $\mathcal{F}_0 \in \mathscr{C}$ or at least $\mathcal{F}_0 \subseteq \mathcal{G}$ for all $\mathcal{G} \in \mathscr{C}$. Moreover $\varnothing \notin \mathcal{F}$ because $\varnothing$ belongs to none of the filters in $\mathscr{C}$.

If $A, B \in \mathcal{F}$, choose $\mathcal{G}_1, \mathcal{G}_2 \in \mathscr{C}$ with $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$. Since $\mathscr{C}$ is a chain, either $\mathcal{G}_1 \subseteq \mathcal{G}_2$ or $\mathcal{G}_2 \subseteq \mathcal{G}_1$; in either case, both A and B lie in one of them, say $\mathcal{G}$. Then $A \cap B \in \mathcal{G} \subseteq \mathcal{F}$. Upward closure is immediate: if $A \in \mathcal{F}$ and $A \subseteq C$, choose $\mathcal{G} \in \mathscr{C}$ with $A \in \mathcal{G}$, then $C \in \mathcal{G} \subseteq \mathcal{F}$.

Thus $\mathcal{F} \in S$ and is an upper bound of $\mathscr{C}$. By Zorn's lemma (Theorem 15), S has a maximal element $\mathcal{U}$, which is an ultrafilter containing $\mathcal{F}_0$. □

Proposition 63 (Characterization of ultrafilters)**.** *Let $\mathcal{U}$ be a filter on X. The following are equivalent:*

1. *$\mathcal{U}$ is an ultrafilter;*
2. *for every subset $A \subseteq X$, one has*
$$A \in \mathcal{U} \quad \text{or} \quad X \setminus A \in \mathcal{U}.$$

Proof. (1) ⇒ (2): Let $A \subseteq X$ and assume $A \notin \mathcal{U}$. Consider $\mathcal{U} \cup \{A\}$. If this family had the finite intersection property, then the filter it generates would strictly contain $\mathcal{U}$, contradicting maximality. Hence $\mathcal{U} \cup \{A\}$ fails the FIP, so there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that
$$A \cap U_1 \cap \cdots \cap U_n = \varnothing.$$
Set $U := U_1 \cap \cdots \cap U_n \in \mathcal{U}$. Then $U \subseteq X \setminus A$, hence $X \setminus A \in \mathcal{U}$ by upward closure.

(2) ⇒ (1): Let $\mathcal{F}$ be a filter with $\mathcal{U} \subseteq \mathcal{F}$. Take any $A \in \mathcal{F}$. If $A \notin \mathcal{U}$, then $X \setminus A \in \mathcal{U} \subseteq \mathcal{F}$, hence
$$\varnothing = A \cap (X \setminus A) \in \mathcal{F},$$
contradicting the definition of a filter. Therefore $A \in \mathcal{U}$. Thus $\mathcal{F} \subseteq \mathcal{U}$, so $\mathcal{F} = \mathcal{U}$. □

Proposition 64 (Finite unions)**.** *Let $\mathcal{U}$ be an ultrafilter on X and let $X = Y_1 \cup \cdots \cup Y_n$. Then $Y_k \in \mathcal{U}$ for some $k \in \{1, \ldots, n\}$.*

Proof. If $Y_k \notin \mathcal{U}$ for all k, then by Proposition 63 one has $X \setminus Y_k \in \mathcal{U}$ for all k. Hence

$$\bigcap_{k=1}^{n} (X \setminus Y_k) = X \setminus \bigcup_{k=1}^{n} Y_k = \varnothing$$

belongs to $\mathcal{U}$, a contradiction. $\square$

Lemma (Image of an ultrafilter)**.** *Let $f : X \to Y$ be a map and let $\mathcal{U}$ be an ultrafilter on X. Then $f_*\mathcal{U}$ is an ultrafilter on Y.*

Proof. By Lemma , $f_*\mathcal{U}$ is a filter. Let $A \subseteq Y$. Then $f^{-1}(A) \subseteq X$. Since $\mathcal{U}$ is an ultrafilter, either $f^{-1}(A) \in \mathcal{U}$ or $X \setminus f^{-1}(A) \in \mathcal{U}$. But $X \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$. Therefore either $A \in f_*\mathcal{U}$ or $Y \setminus A \in f_*\mathcal{U}$. By Proposition 63, $f_*\mathcal{U}$ is an ultrafilter. $\square$

4. Compactness via ultrafilters

Theorem 17 (Ultrafilter characterization of compactness)**.** Let $(X, \mathcal{T})$ be a topological space. Then X is compact if and only if every ultrafilter on X converges (to at least one point of X).

Proof. ($\Rightarrow$) Assume X is compact and let $\mathcal{U}$ be an ultrafilter on X. Suppose $\mathcal{U}$ does *not* converge. Then for each $x \in X$ there exists a neighborhood $W_x \in \mathcal{N}(x)$ with $W_x \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, Proposition 63 implies $X \setminus W_x \in \mathcal{U}$ for all x.

The family $(W_x)_{x \in X}$ is an open cover of X. By compactness, there exist $x_1, \dots, x_n \in X$ such that $X = W_{x_1} \cup \cdots \cup W_{x_n}$. Then

$$\bigcap_{i=1}^{n} (X \setminus W_{x_i}) = X \setminus \bigcup_{i=1}^{n} W_{x_i} = \varnothing$$

belongs to $\mathcal{U}$ (finite intersection property), contradicting $\varnothing \notin \mathcal{U}$. Thus $\mathcal{U}$ must converge.

($\Leftarrow$) Assume every ultrafilter on X converges. We prove compactness using the closed-set/FIP criterion (Proposition 57). Let $(C_\alpha)_{\alpha \in A}$ be a family of closed sets with the finite intersection property. Let $\mathcal{F}_0$ be the filter generated by the finite intersections of the C_α's (Definition 76); then $C_\alpha \in \mathcal{F}_0$ for all α. By the ultrafilter lemma (Theorem 16), choose an ultrafilter $\mathcal{U}$ with $\mathcal{F}_0 \subseteq \mathcal{U}$. By assumption, $\mathcal{U} \to x$ for some $x \in X$.

Fix $\alpha \in A$. Since $C_\alpha \in \mathcal{U}$, if $x \notin C_\alpha$ then $X \setminus C_\alpha$ is an open neighborhood of x, hence $X \setminus C_\alpha \in \mathcal{U}$ (because $\mathcal{U} \to x$), and then

$$\varnothing = C_\alpha \cap (X \setminus C_\alpha) \in \mathcal{U},$$

a contradiction. Therefore $x \in C_\alpha$ for all α, and $\bigcap_{\alpha \in A} C_\alpha \neq \varnothing$. By Proposition 57, X is compact. $\square$

5. Convergence of filters in product spaces

Notation. *Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ be a family of topological spaces and let*

$$X := \prod_{\alpha \in A} X_\alpha$$

with the product topology. For each $\alpha \in A$, denote the projection by $p_\alpha : X \to X_\alpha$.

Lemma (Filter convergence is coordinatewise in products)**.** *Let $\mathcal{F}$ be a filter on $X = \prod_{\alpha \in A} X_\alpha$ and let $x = (x_\alpha)_{\alpha \in A} \in X$. Then*

$$\mathcal{F} \to x \quad \Longleftrightarrow \quad (\forall \alpha \in A) \;\; p_{\alpha *}\mathcal{F} \to x_\alpha \text{ in } X_\alpha.$$

Proof. (⇒) Since each p_α is continuous, Lemma yields $p_{\alpha *}\mathcal{F} \to p_\alpha(x) = x_\alpha$.

(⇐) Assume $p_{\alpha *}\mathcal{F} \to x_\alpha$ for all α. Let W be a neighborhood of x in X. By the definition of the product topology, there exists a basic neighborhood V of x with $V \subseteq W$, of the form

$$V = \bigcap_{i=1}^{n} p_{\alpha_i}^{-1}(V_i),$$

where $\alpha_1, \dots, \alpha_n \in A$ and each V_i is an open neighborhood of x_{α_i} in X_{α_i}. Since $p_{\alpha_i *}\mathcal{F} \to x_{\alpha_i}$, we have $V_i \in p_{\alpha_i *}\mathcal{F}$, hence $p_{\alpha_i}^{-1}(V_i) \in \mathcal{F}$ for each i. By closure under finite intersections, $V \in \mathcal{F}$. Finally, since $V \subseteq W$ and $\mathcal{F}$ is upward closed, $W \in \mathcal{F}$. Thus every neighborhood of x lies in $\mathcal{F}$, i.e. $\mathcal{F} \to x$. □

6. Tychonoff theorem

Theorem 18 (Tychonoff)**.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ be a family of compact topological spaces. Then the product space

$$X := \prod_{\alpha \in A} X_\alpha$$

is compact for the product topology.

Proof. By Theorem 17, it suffices to show that every ultrafilter on X converges.

Let $\mathcal{U}$ be an ultrafilter on X. For each $\alpha \in A$, the pushforward $p_{\alpha *}\mathcal{U}$ is an ultrafilter on X_α (Lemma). Since X_α is compact, Theorem 17 implies that $p_{\alpha *}\mathcal{U}$ converges to at least one point of X_α. Choose $x_\alpha \in X_\alpha$ such that

$$p_{\alpha *}\mathcal{U} \to x_\alpha.$$

Let $x := (x_\alpha)_{\alpha \in A} \in X$. By Lemma , the coordinatewise convergence of the projection filters implies $\mathcal{U} \to x$ in X. Thus every ultrafilter on X converges, and therefore X is compact. □

Corollary. *If each X_α is compact and Hausdorff, then $\prod_{\alpha \in A} X_\alpha$ is compact and Hausdorff.*

Proof. Compactness is Theorem 18. Hausdorffness follows from the fact that products of Hausdorff spaces are Hausdorff (proved earlier using the product topology). □

Remark. *The proof uses the ultrafilter extension theorem (Theorem 16), which depends on Zorn's lemma (hence on a form of the axiom of choice). Accordingly, Tychonoff's theorem for arbitrary products is not provable in ZF alone.*

Lecture 17–18: Connectedness and Path Connectedness

1. Connected spaces

Definition 81 (Separation)**.** Let $(X, \mathcal{T})$ be a topological space. A *separation* of X is a pair (U, V) of open subsets of X such that

$$U \neq \varnothing, \quad V \neq \varnothing, \quad U \cap V = \varnothing, \quad U \cup V = X.$$

Definition 82 (Connected space; connected subset)**.** A topological space X is *connected* if it admits no separation.
A subset $A \subseteq X$ is *connected* if it is connected for the subspace topology.

Definition 83 (Clopen)**.** A subset $C \subseteq X$ is *clopen* if it is both open and closed.

Proposition 65. *For a topological space X, the following are equivalent:*

1. *X is connected;*
2. *the only clopen subsets of X are $\varnothing$ and X.*

Proof. (1) $\Rightarrow$ (2): Let $C \subseteq X$ be clopen and assume $C \neq \varnothing$ and $C \neq X$. Then C and $X \setminus C$ are disjoint nonempty open sets whose union is X, i.e. a separation, contradicting connectedness.

(2) $\Rightarrow$ (1): If $X = U \cup V$ is a separation, then U is clopen (its complement V is open) and $U \neq \varnothing, X$, contradicting (2). ☐

Example (The rationals are disconnected)**.** *Let $\mathbb{Q}$ carry the subspace topology induced by $\mathbb{R}$. Choose $a \in \mathbb{R} \setminus \mathbb{Q}$ (e.g. $a = \sqrt{2}$) and set*

$$U := \mathbb{Q} \cap (-\infty, a), \qquad V := \mathbb{Q} \cap (a, \infty).$$

Then U and V are disjoint nonempty open subsets of $\mathbb{Q}$ and $U \cup V = \mathbb{Q}$, so $\mathbb{Q}$ is not connected.

Proposition 66 (Continuous images preserve connectedness)**.** *Let $f : X \to Y$ be continuous. If X is connected, then $f(X)$ is connected (for the subspace topology on Y).*

Proof. Suppose $f(X) = U \cup V$ is a separation in $f(X)$, with U, V open in $f(X)$, disjoint and nonempty. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint nonempty open subsets of X and their union is X, giving a separation of X. ☐

Proposition 67 (Characterization via discrete targets)**.** *For a topological space X, the following are equivalent:*

1. *X is connected;*
2. *for every discrete topological space D, every continuous map $f : X \to D$ is constant;*
3. *every continuous map $f : X \to \{0, 1\}$ (with the discrete topology) is constant.*

Proof. (1) $\Rightarrow$ (2): Let D be discrete and $f : X \to D$ continuous. If f is not constant, pick $d_1 \neq d_2$ in $f(X)$. Then $f^{-1}(\{d_1\})$ and $X \setminus f^{-1}(\{d_1\})$ are disjoint nonempty open subsets of X (since $\{d_1\}$ is open in D), contradicting connectedness.

(2) $\Rightarrow$ (3) is immediate.

(3) $\Rightarrow$ (1): If X is not connected, choose a separation $X = U \cup V$. Define $f : X \to \{0,1\}$ by $f|_U \equiv 0$ and $f|_V \equiv 1$. Then f is continuous (preimages of $\{0\}$ and $\{1\}$ are U and V) and nonconstant, contradicting (3). ☐

Theorem 19 (Connectedness of $[0,1]$)**.** The interval $[0,1] \subseteq \mathbb{R}$ is connected (with the subspace topology).

Proof. Assume $[0,1] = U \cup V$ with U, V disjoint open subsets of $[0,1]$. Without loss of generality, $0 \in U$. Define
$$S := \{x \in [0,1] : \ [0,x] \subseteq U\}.$$
Since U is open in $[0,1]$ and contains 0, there exists $\varepsilon > 0$ such that $[0,\varepsilon) \subseteq U$; hence $\varepsilon/2 \in S$. Thus $S \neq \varnothing$. Let $c := \sup S \in [0,1]$.

Claim 1: $c \in U$. If $c \notin U$, then $c \in V$. Since V is open in $[0,1]$, there exists $\delta > 0$ such that
$$(c-\delta, c+\delta) \cap [0,1] \subseteq V.$$
Choose $x \in S$ with $c - \delta/2 < x \leq c$ (possible by definition of sup). Then $x \in U$ because $[0,x] \subseteq U$. But also $x \in (c-\delta, c+\delta) \cap [0,1] \subseteq V$, contradicting $U \cap V = \varnothing$. Hence $c \in U$.

Claim 2: $c = 1$. If $c < 1$, since U is open in $[0,1]$ and $c \in U$, there exists $\eta > 0$ such that
$$(c-\eta, c+\eta) \cap [0,1] \subseteq U.$$
Pick $c' < \min\{1, c+\eta/2\}$ with $c < c'$. We show $c' \in S$: let $t \in [0,c']$. If $t \leq c$, choose $x \in S$ with $t \leq x \leq c$; then $t \in [0,x] \subseteq U$. If $c < t \leq c'$, then $t \in (c-\eta, c+\eta) \cap [0,1] \subseteq U$. Thus $[0,c'] \subseteq U$, so $c' \in S$, contradicting $c = \sup S$. Therefore $c = 1$.

Hence $[0,1] \subseteq U$, so $V = \varnothing$. Thus no separation exists and $[0,1]$ is connected. ☐

Corollary. *$\mathbb{R}$ is connected (with the standard topology).*

Proof. For $n \in \mathbb{N}$, the interval $[-n,n]$ is homeomorphic to $[0,1]$, hence connected by Theorem 19. Moreover $[-n,n] \cap [-(n+1), n+1] \neq \varnothing$, and
$$\mathbb{R} = \bigcup_{n \in \mathbb{N}} [-n,n].$$
By Proposition 68 below, $\mathbb{R}$ is connected. ☐

Proposition 68 (Unions of connected sets with overlapping)**.** *Let $(X, \mathcal{T})$ be a topological space and let $(C_i)_{i \in I}$ be a family of connected subsets of X. Assume that $C_i \cap C_j \neq \varnothing$ for all $i, j \in I$. Then $C := \bigcup_{i \in I} C_i$ is connected.*

Proof. Assume for contradiction that $C = U \cup V$ is a separation in the subspace C, with U, V open in C, disjoint and nonempty. For each i, the sets $U \cap C_i$ and $V \cap C_i$ are open in C_i, disjoint, and their union is C_i. Since C_i is connected, one of them is empty; hence for each i we have either $C_i \subseteq U$ or $C_i \subseteq V$. If $C_i \subseteq U$ and $C_j \subseteq V$, then $C_i \cap C_j \subseteq U \cap V = \varnothing$, contradicting the hypothesis. Thus all C_i lie in the same side, so $C \subseteq U$ or $C \subseteq V$, contradicting that both U and V are nonempty. ☐

Proposition 69 (Intermediate set between a connected set and its closure)**.** *Let A be a connected subset of a topological space X, and let E satisfy*
$$A \subseteq E \subseteq \overline{A}.$$
Then E is connected. In particular, $\overline{A}$ is connected.

Proof. Assume E is not connected. Then there exist nonempty disjoint subsets $F, G \subseteq E$, closed in E, such that $E = F \cup G$. Write $F = E \cap F_0$ and $G = E \cap G_0$ with F_0, G_0 closed in X. Then

$$A = (A \cap F) \cup (A \cap G)$$

is a decomposition of A into two disjoint closed subsets of A; since A is connected, one is empty. Without loss of generality, $A \cap G = \varnothing$, hence $A \subseteq F_0$. Since F_0 is closed, $\overline{A} \subseteq F_0$, and therefore $E \subseteq \overline{A} \subseteq F_0$. Thus $F = E \cap F_0 = E$ and $G = \varnothing$, contradiction. $\square$

Theorem 20 (Intermediate value property)**.** Let X be connected and let $f : X \to \mathbb{R}$ be continuous. Fix $a, b \in X$ with $f(a) < f(b)$, and let $c \in (f(a), f(b))$. Then there exists $x \in X$ such that $f(x) = c$.

Proof. Assume for contradiction that $c \notin f(X)$. Set $U := (-\infty, c)$ and $V := (c, \infty)$. Then $f(X) \subseteq U \cup V$, and $f(a) \in U$, $f(b) \in V$, so both $f(X) \cap U$ and $f(X) \cap V$ are nonempty. Moreover, $f(X) \cap U$ and $f(X) \cap V$ are disjoint and open in the subspace $f(X)$. Thus they form a separation of $f(X)$, contradicting Proposition 66 since X is connected. $\square$

2. Connected components

Definition 84 (Connected component)**.** Let X be a topological space. Define a relation $\sim$ on X by

$$x \sim y \quad :\Longleftrightarrow \quad \text{there exists a connected subset } A \subseteq X \text{ with } x, y \in A.$$

The equivalence classes of $\sim$ are called the *connected components* of X.

Proposition 70. *The relation $\sim$ of Definition 84 is an equivalence relation.*

Proof. Reflexivity and symmetry are immediate. For transitivity, suppose $x \sim y$ and $y \sim z$. Then there exist connected subsets A, B with $x, y \in A$ and $y, z \in B$. Since $A \cap B \supseteq \{y\} \neq \varnothing$, Proposition 68 implies $A \cup B$ is connected and contains x and z. Thus $x \sim z$. $\square$

Proposition 71 (Basic properties of components)**.** *Let X be a topological space.*

1. *Each connected component is connected.*
2. *Each connected component is maximal among connected subsets (for inclusion).*
3. *Each connected component is closed in X.*
4. *If $Y \subseteq X$ is connected, then Y is contained in exactly one connected component of X.*

Proof. Fix $x \in X$ and let $C(x)$ denote the component of x.

(1) By definition, $C(x)$ is the union of all connected subsets containing x. Any two such subsets intersect (at least at x), hence their union is connected by Proposition 68. Therefore $C(x)$ is connected.

(2) If D is connected and $C(x) \subseteq D$, then for any $y \in D$ we have $x \sim y$ (take $A := D$), hence $y \in C(x)$, so $D \subseteq C(x)$. Thus $D = C(x)$.

(3) The closure $\overline{C(x)}$ is connected by Proposition 69 and contains $C(x)$. By maximality (2), $\overline{C(x)} = C(x)$, hence $C(x)$ is closed.

(4) If Y is connected and $x \in Y$, then $Y \subseteq C(x)$ by (2). Uniqueness follows because components form a partition of X. $\square$

Remark. *Connected components need not be open.*

3. Path connectedness

Definition 85 (Path; path connected)**.** Let X be a topological space.

1. A *path* in X is a continuous map $\gamma : [0,1] \to X$.
2. For $x, y \in X$, a path γ is said to *join* x to y if $\gamma(0) = x$ and $\gamma(1) = y$.
3. X is *path connected* if for all $x, y \in X$ there exists a path joining x to y.

Proposition 72 (Path connected $\Rightarrow$ connected)**.** *Every path connected space is connected.*

Proof. Let X be path connected and choose $x_0 \in X$. For each $x \in X$, let $\gamma_x : [0,1] \to X$ be a path joining x_0 to x. Since $[0,1]$ is connected (Theorem 19), the image $\gamma_x([0,1])$ is connected (Proposition 66) and contains x_0. Hence the union $\bigcup_{x \in X} \gamma_x([0,1]) = X$ is connected by Proposition 68. □

Definition 86 (Path components)**.** Define a relation $\sim_p$ on X by

$$x \sim_p y \quad :\Longleftrightarrow \quad \text{there exists a path joining } x \text{ to } y.$$

The equivalence classes are called the *path components* of X.

Proposition 73. *Path components are path connected, hence connected. In particular, every path component is contained in the connected component of each of its points.*

Proof. Path connectedness is tautological from the definition. Connectedness follows from Proposition 72. Thus a path component is a connected subset containing a given point, hence it lies in the connected component of that point by maximality of connected components. □

Definition 87 (Locally path connected)**.** A topological space X is *locally path connected* if for every $x \in X$ and every neighborhood U of x, there exists a *path connected open* set V such that

$$x \in V \subseteq U.$$

Proposition 74 (Path components are open in locally path connected spaces)**.** *If X is locally path connected, then every path component of X is open.*

Proof. Let P be the path component of a point $x \in X$ and let $y \in P$. Choose a path γ joining x to y. By local path connectedness at y, there exists a path connected open neighborhood V of y. For any $z \in V$, since V is path connected there exists a path joining y to z; concatenating with γ yields a path joining x to z. Hence $z \in P$ and $V \subseteq P$. Therefore P is a neighborhood of each of its points, hence open. □

Theorem 21 (Connected components vs. path components in the locally path connected case)**.** Let X be locally path connected. Then connected components of X coincide with path components. Consequently, in a locally path connected space one has

$$X \text{ connected} \quad \Longleftrightarrow \quad X \text{ path connected.}$$

Proof. Let C be a connected component and let P be a path component contained in C (existence: pick $x \in C$ and take P the path component of x; then $P \subseteq C$ by Proposition 73). By Proposition 74, P is open in X, hence open in the subspace C. The complement $C \setminus P$ is a union of path components, hence open in X and therefore open in C. Thus P is clopen in C. Since C is connected, Proposition 65 implies $P = C$. Hence each connected component is a path component, and the two partitions coincide.

The equivalence X connected $\Leftrightarrow$ X path connected now follows: X is connected iff it has exactly one connected component, iff it has exactly one path component, i.e. is path connected. $\square$

Remark. *Path connectedness does* not *imply local path connectedness in general. Thus the implication "path connected $\Rightarrow$ connected" always holds, but the converse "connected $\Rightarrow$ path connected" requires extra hypotheses (e.g. local path connectedness as in Theorem 21).*

4. A standard example: connected but not path connected

Definition 88 (Topologist's sine curve)**.** Let

$$S := \{(x, \sin(1/x)) \in \mathbb{R}^2 : \ 0 < x \leq 1\}, \qquad T := \overline{S} \subseteq \mathbb{R}^2.$$

Then

$$T = S \ \cup \ \big(\{0\} \times [-1, 1]\big).$$

Theorem 22. The set T of Definition 88 is connected but not path connected.

Proof. Connectedness. The map $f : (0, 1] \to \mathbb{R}^2$, $f(x) = (x, \sin(1/x))$, is continuous, hence $S = f((0, 1])$ is connected because $(0, 1]$ is connected. By Proposition 69, the closure $T = \overline{S}$ is connected.

Failure of path connectedness. Assume for contradiction that there exists a path $\gamma : [0, 1] \to T$ with $\gamma(0) = (0, 0)$ and $\gamma(1) \in S$. Write $\gamma(t) = (x(t), y(t))$ with continuous $x, y : [0, 1] \to \mathbb{R}$. Since $\gamma(1) \in S$, we have $x(1) > 0$. Set

$$t_0 := \sup\{t \in [0, 1] : x(t) = 0\}.$$

Then $t_0 < 1$ and $x(t_0) = 0$. Moreover, by definition of t_0, one has $x(t) > 0$ for all $t \in (t_0, 1]$, hence $\gamma(t) \in S$ for all $t \in (t_0, 1]$. Therefore, for $t \in (t_0, 1]$,

$$y(t) = \sin\big(1/x(t)\big).$$

Since x is continuous and $x(t_0) = 0$, we have $x(t) \to 0$ as $t \downarrow t_0$.

For $n \in \mathbb{N}$, set

$$a_n := \frac{1}{\frac{\pi}{2} + 2\pi n}, \qquad b_n := \frac{1}{\frac{3\pi}{2} + 2\pi n}.$$

Then $a_n, b_n \downarrow 0$, and

$$\sin(1/a_n) = 1, \qquad \sin(1/b_n) = -1.$$

Fix $\varepsilon > 0$ with $t_0 + \varepsilon \leq 1$ and set $t_1 := t_0 + \varepsilon$. Then $x(t_1) > 0$. Choose n so large that $a_n < b_n < x(t_1)$. By the intermediate value theorem applied to the continuous function x on $[t_0, t_1]$, there exist $s_n, r_n \in (t_0, t_1)$ such that $x(s_n) = a_n, \qquad x(r_n) = b_n$. Hence

$$y(s_n) = \sin(1/a_n) = 1, \qquad y(r_n) = \sin(1/b_n) = -1.$$

Since $\varepsilon > 0$ was arbitrary, this shows that in every right-neighborhood of t_0 the function y takes both values 1 and -1. Therefore $y(t)$ has no limit as $t \downarrow t_0$, contradicting continuity of y at t_0. This contradiction shows that no such path exists, hence T is not path connected. $\square$

Remark. *The space T is not locally path connected at any point of $\{0\} \times [-1, 1]$. This is consistent with Theorem 21.*

Lecture 19: Separation Axioms

The *separation axioms* stratify topological spaces according to the extent to which points and closed sets can be distinguished by open sets. They form a basic toolkit for recognizing when a topology behaves "as if it came from a metric".

Throughout, $(X, \mathcal{T})$ denotes a topological space, and $A^c := X \setminus A$.

1. The axioms T_0, T_1, T_2

Definition 89 (T_0 (Kolmogorov))**.** The space X is called T_0 if for any two distinct points $x \neq y$ there exists an open set $U \in \mathcal{T}$ such that

$$\big(x \in U,\ y \notin U\big) \quad \text{or} \quad \big(y \in U,\ x \notin U\big).$$

Definition 90 (T_1 (Fréchet))**.** The space X is called T_1 if for any two distinct points $x \neq y$ there exist open sets $U_x, U_y \in \mathcal{T}$ such that

$$x \in U_x,\ y \notin U_x, \qquad y \in U_y,\ x \notin U_y.$$

Definition 91 (T_2 (Hausdorff))**.** The space X is called T_2 (or *Hausdorff*) if for any two distinct points $x \neq y$ there exist disjoint open sets $U, V \in \mathcal{T}$ such that

$$x \in U, \quad y \in V, \quad U \cap V = \varnothing.$$

Proposition 75 (Implications)**.** *One has*

$$T_2 \implies T_1 \implies T_0.$$

Proof. $T_2 \Rightarrow T_1$: if $x \neq y$, choose disjoint open $U \ni x$ and $V \ni y$. Then $x \notin V$ and $y \notin U$.

$T_1 \Rightarrow T_0$: if $x \neq y$, the set U_x in Definition 90 contains x but not y. □

Proposition 76 (T_1 and closed singletons)**.** *A space X is T_1 if and only if every singleton $\{x\}$ is closed.*

Proof. ($\Rightarrow$) Fix $x \in X$. For each $y \in X \setminus \{x\}$, by the T_1 property there exists an open neighborhood U_y of y such that $x \notin U_y$. Hence $U_y \subseteq X \setminus \{x\}$. Therefore

$$X \setminus \{x\} = \bigcup_{y \neq x} U_y$$

is open, so $\{x\}$ is closed.

($\Leftarrow$) If all singletons are closed, then $X \setminus \{x\}$ and $X \setminus \{y\}$ are open. For $x \neq y$, set $U_x := X \setminus \{y\}$ and $U_y := X \setminus \{x\}$. These satisfy Definition 90. □

Example (Cofinite topology)**.** *Let X be an infinite set. The* cofinite topology *on X is*

$$\mathcal{T}_{\text{cof}} := \{\varnothing\} \ \cup\ \{U \subseteq X :\ X \setminus U \text{ is finite}\}.$$

Then $(X, \mathcal{T}_{\text{cof}})$ is T_1 but not Hausdorff.

Proof. T_1: for $x \in X$, the complement $X \setminus \{x\}$ is cofinite, hence open. Thus $\{x\}$ is closed by Proposition 76.
Not T_2: if U, V are nonempty open sets, then $X \setminus U$ and $X \setminus V$ are finite, hence

$$U \cap V = X \setminus \big((X \setminus U) \cup (X \setminus V)\big)$$

is the complement of a finite set, hence nonempty. Therefore no two nonempty open sets are disjoint. □

2. Regularity and normality

Definition 92 (Regular space)**.** The space X is called *regular* if:

(R0) X is T_1;

(R1) for every $x \in X$ and every closed set $C \subseteq X$ with $x \notin C$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that

$$x \in U, \qquad C \subseteq V, \qquad U \cap V = \varnothing.$$

In many conventions this is denoted T_3.

Definition 93 (Normal space)**.** The space X is called *normal* if:

(N0) X is T_1;

(N1) for any two disjoint closed sets $A, B \subseteq X$ there exist disjoint open sets $U, V \in \mathcal{T}$ such that

$$A \subseteq U, \qquad B \subseteq V, \qquad U \cap V = \varnothing.$$

In many conventions this is denoted T_4.

Remark. *If X is normal, then X is regular: indeed, for $x \notin C$ with C closed, the set $\{x\}$ is closed because X is T_1, so* (N1) separates the disjoint closed sets $\{x\}$ and C. Thus one has (at least under the T_1 convention)

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0.$$

3. A useful characterization of regularity

Proposition 77. *Assume X is Hausdorff. Then the following are equivalent:*

1. *X is regular.*
2. *For every $x \in X$ and every open neighborhood N of x, there exists an open set U such that*

$$x \in U, \qquad \overline{U} \subseteq N.$$

3. *For every $x \in X$ and every neighborhood N of x, there exists a* closed neighborhood *M of x such that*

$$M \subseteq N.$$

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and let N be an open neighborhood of x. Set $C := X \setminus N$, which is closed and does not contain x. By regularity, choose disjoint open sets U, V with $x \in U$ and $C \subseteq V$. Then $U \subseteq V^c$ and V^c is closed, hence

$$\overline{U} \subseteq V^c \subseteq C^c = N.$$

(2) $\Rightarrow$ (3): Let N be a neighborhood of x. Choose an open set N_0 with $x \in N_0 \subseteq N$. By (2), there exists open U with $x \in U$ and $\overline{U} \subseteq N_0 \subseteq N$. Set $M := \overline{U}$. Then M is closed, contains x, and satisfies $M \subseteq N$; moreover M is a neighborhood of x because U is open, $x \in U \subseteq M$.

(3) $\Rightarrow$ (1): Assume (3). Let $x \in X$ and let $C \subseteq X$ be closed with $x \notin C$. Then $N := X \setminus C$ is an open neighborhood of x. By (3), choose a closed neighborhood M of x with $M \subseteq N$. Let $U := \mathrm{int}(M)$ (open) and $V := X \setminus M$ (open). Then $x \in U$ because M is a neighborhood of x. Also $C \subseteq X \setminus N \subseteq X \setminus M = V$, and

$$U \cap V \subseteq M \cap (X \setminus M) = \varnothing.$$

Thus U, V separate x and C, so X is regular. □

4. A Hausdorff space that is not regular (the K-topology)

Example (The K-topology)**.** *Let*

$$K := \left\{ \frac{1}{n} : \ n \in \mathbb{N}, \ n \geq 1 \right\} \subseteq \mathbb{R}.$$

Let $\mathcal{T}_K$ be the smallest topology on $\mathbb{R}$ containing the standard topology $\mathcal{T}_{\mathrm{st}}$ and the set $\mathbb{R} \setminus K$. Equivalently, a basis for $\mathcal{T}_K$ is

$$\mathcal{B} := \{(a, b) : a < b\} \ \cup \ \{(a, b) \setminus K : \ a < b\}.$$

Then $(\mathbb{R}, \mathcal{T}_K)$ is Hausdorff but not regular.

Proof. Hausdorff. Since $\mathcal{T}_K$ contains the standard topology and the standard topology is Hausdorff, any two distinct points can be separated by disjoint standard open intervals, which are in particular $\mathcal{T}_K$-open.

Failure of regularity. The set K is closed because its complement $\mathbb{R} \setminus K$ is open by construction, and $0 \notin K$. We claim there do not exist disjoint open sets $U, V \in \mathcal{T}_K$ such that $0 \in U$ and $K \subseteq V$.

Indeed, let U be an open set with $0 \in U$. By the basis description, there exists $\varepsilon > 0$ such that either

$$(-\varepsilon, \varepsilon) \subseteq U \quad \text{or} \quad (-\varepsilon, \varepsilon) \setminus K \subseteq U.$$

In either case we have

$$(0, \varepsilon) \setminus K \subseteq U. \tag{$*$}$$

Now let V be an open set with $K \subseteq V$. For each $n \geq 1$, since $1/n \in V$ and $(a, b) \in \mathcal{B}$, there exists an open interval $I_n = (a_n, b_n)$ such that

$$\frac{1}{n} \in I_n \subseteq V.$$

Choose n large enough so that $1/n < \varepsilon$. Then $I_n \subseteq (0, \varepsilon)$ after shrinking if necessary. Since I_n is an interval, it contains points not in the countable set K, so pick $t \in I_n \setminus K$. Then $t \in (0, \varepsilon) \setminus K \subseteq U$ by $(*)$, and also $t \in I_n \subseteq V$. Hence $U \cap V \neq \varnothing$, as claimed.

Therefore 0 and the closed set K cannot be separated by disjoint open sets, so $(\mathbb{R}, \mathcal{T}_K)$ is not regular. □

5. Metric spaces are normal

Notation. *Let (X, d) be a metric space and let $A \subseteq X$ be nonempty. Define the* distance to A *by*

$$d(x, A) := \inf_{a \in A} d(x, a) \in [0, +\infty).$$

(For $A = \varnothing$ one may set $d(x, \varnothing) := +\infty$, but this will not be needed below.)

Lemma. *For any nonempty $A \subseteq X$, the map $x \mapsto d(x, A)$ is 1-Lipschitz:*

$$|d(x, A) - d(y, A)| \leq d(x, y) \qquad (\forall x, y \in X).$$

In particular, $x \mapsto d(x, A)$ is continuous.

Proof. Fix $x, y \in X$. For every $a \in A$, the triangle inequality gives

$$d(x, a) \leq d(x, y) + d(y, a).$$

Taking the infimum over $a \in A$ yields $d(x, A) \leq d(x, y) + d(y, A)$, i.e. $d(x, A) - d(y, A) \leq d(x, y)$. Exchanging x and y gives $d(y, A) - d(x, A) \leq d(x, y)$, hence the desired bound. □

Proposition 78 (Metric spaces are normal)**.** *Every metric space is normal.*

Proof. Let (X, d) be a metric space and let $A, B \subseteq X$ be disjoint closed sets. For $x \in X$ define the continuous functions

$$u(x) := d(x, A), \qquad v(x) := d(x, B).$$

Since A and B are closed and disjoint, one has $u(x) = 0$ if and only if $x \in A$, and similarly $v(x) = 0$ if and only if $x \in B$. In particular, if $x \in A$ then $v(x) > 0$, and if $x \in B$ then $u(x) > 0$.

Define

$$U := \{x \in X : \ u(x) < v(x)\}, \qquad V := \{x \in X : \ v(x) < u(x)\}.$$

Since $u - v$ and $v - u$ are continuous, both U and V are open. Moreover $A \subseteq U$ (because on A one has $u = 0 < v$) and $B \subseteq V$ (because on B one has $v = 0 < u$). Finally, $U \cap V = \varnothing$ because $u(x) < v(x)$ and $v(x) < u(x)$ cannot hold simultaneously. Thus U and V are disjoint open neighborhoods of A and B, so X is normal. □

Corollary. *If a topological space $(X, \mathcal{T})$ is not normal, then it is not metrizable (i.e. there is no metric inducing $\mathcal{T}$).*

Proof. If $\mathcal{T}$ were induced by a metric, then $(X, \mathcal{T})$ would be normal by Proposition 78, contradiction. □

6. Metrizability

Definition 94 (Metrizable space)**.** A topological space $(X, \mathcal{T})$ is called *metrizable* if there exists a metric

$$d : X \times X \to [0, +\infty)$$

such that the topology $\mathcal{T}_d$ induced by d coincides with $\mathcal{T}$.

Lecture 20: Urysohn's Lemma

1. Compact Hausdorff spaces: regularity and normality

Proposition 79. *Every compact Hausdorff space is regular.*

Proof. Let X be compact and Hausdorff, let $x \in X$, and let $C \subseteq X$ be closed with $x \notin C$. For each $c \in C$, by the Hausdorff property there exist open sets $U_c \ni c$ and $V_c \ni x$ such that $U_c \cap V_c = \varnothing$. Then $(U_c)_{c \in C}$ is an open cover of C. Since C is closed in a compact space, it is compact; hence there exist $c_1, \dots, c_n \in C$ such that

$$C \subseteq U_{c_1} \cup \dots \cup U_{c_n}.$$

Set

$$U := U_{c_1} \cup \dots \cup U_{c_n}, \qquad V := V_{c_1} \cap \dots \cap V_{c_n}.$$

Then U and V are open, $C \subseteq U$, $x \in V$, and $U \cap V = \varnothing$. Thus X is regular. □

Proposition 80. *Every compact Hausdorff space is normal.*

Proof. Let X be compact and Hausdorff, and let $A, B \subseteq X$ be disjoint closed sets. Fix $a \in A$. For each $b \in B$, choose disjoint open neighborhoods $U_{a,b} \ni a$ and $V_{a,b} \ni b$ (Hausdorff). Then $(V_{a,b})_{b \in B}$ is an open cover of B, hence admits a finite subcover

$$B \subseteq V_{a,b_1} \cup \dots \cup V_{a,b_m}.$$

Define

$$U_a := U_{a,b_1} \cap \dots \cap U_{a,b_m}, \qquad V_a := V_{a,b_1} \cup \dots \cup V_{a,b_m}.$$

Then U_a is open and contains a, V_a is open and contains B, and $U_a \cap V_a = \varnothing$.

Now $(U_a)_{a \in A}$ is an open cover of A. Since A is compact, there exist $a_1, \dots, a_n \in A$ with

$$A \subseteq U_{a_1} \cup \dots \cup U_{a_n}.$$

Set

$$U := U_{a_1} \cup \dots \cup U_{a_n}, \qquad V := V_{a_1} \cap \dots \cap V_{a_n}.$$

Then U, V are open, $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$ (because $V \subseteq V_{a_i}$ for each i, hence $U_{a_i} \cap V = \varnothing$, and taking unions yields $U \cap V = \varnothing$). Thus X is normal. □

2. Complete regularity and Tychonoff spaces

Definition 95 (Completely regular)**.** A topological space X is *completely regular* if

(CR0) X is T_1;

(CR1) for every point $x \in X$ and every closed set $C \subseteq X$ with $x \notin C$, there exists a continuous map

$$f : X \to [0,1]$$

such that $f(x) = 0$ and $f|_C \equiv 1$.

Equivalently, points and disjoint closed sets can be separated by continuous functions into $[0,1]$.

Definition 96 (Tychonoff space)**.** A *Tychonoff space* is a completely regular T_0 space. Equivalently, a *Tychonoff space* is a completely regular Hausdorff space.

Proposition 81. *Every Tychonoff space is regular (hence T_3 in the T_1 convention).*

Proof. Let X be Tychonoff. Let $x \in X$ and let $C \subseteq X$ be closed with $x \notin C$. Choose $f : X \to [0,1]$ continuous with $f(x) = 0$ and $f|_C \equiv 1$. Set

$$U := f^{-1}\big([0, \tfrac{1}{2})\big), \qquad V := f^{-1}\big((\tfrac{1}{2}, 1]\big).$$

Then U and V are open, $x \in U$, $C \subseteq V$, and $U \cap V = \varnothing$. Thus X is regular. □

3. Urysohn's lemma

Theorem 23 (Urysohn)**.** Let X be a normal space, and let $A, B \subseteq X$ be disjoint closed sets. Then there exists a continuous function $f : X \to [0,1]$ such that

$$f|_A \equiv 0, \qquad f|_B \equiv 1.$$

The proof relies on a standard shrinking lemma.

Lemma (Shrinking lemma)**.** *Let X be a normal space, let $A \subseteq X$ be closed, and let $U \subseteq X$ be open with $A \subseteq U$. Then there exists an open set $V \subseteq X$ such that*

$$A \subseteq V \subseteq \overline{V} \subseteq U.$$

Proof. Since A and $X \setminus U$ are disjoint closed subsets of X, normality yields disjoint open sets V, W such that

$$A \subseteq V, \qquad X \setminus U \subseteq W, \qquad V \cap W = \varnothing.$$

Then $\overline{V} \cap W = \varnothing$ (because W is open), hence $\overline{V} \subseteq X \setminus W$. But $X \setminus W$ is closed and contains U (since $X \setminus U \subseteq W$), therefore $X \setminus W \subseteq U$ is not correct; rather

$$X \setminus U \subseteq W \implies X \setminus W \subseteq U.$$

Hence $\overline{V} \subseteq X \setminus W \subseteq U$. □

Proof of Theorem 23. Let

$$\mathbb{Q}_{[0,1]} := \mathbb{Q} \cap [0,1].$$

We shall construct open sets $(U_r)_{r \in \mathbb{Q}_{[0,1]}}$ such that:

(U0) $U_0 \supseteq A$ and $U_1 = X \setminus B$;

(U1) if $r, s \in \mathbb{Q}_{[0,1]}$ and $r < s$, then

$$\overline{U_r} \subseteq U_s.$$

Construction of (U_r). Fix an enumeration of $\mathbb{Q}_{[0,1]}$,

$$\mathbb{Q}_{[0,1]} = \{q_0, q_1, q_2, \dots\},$$

with $q_0 = 0$ and $q_1 = 1$. Set $U_{q_1} := X \setminus B$, which is open and contains A because $A \cap B = \varnothing$. By Lemma applied to $A \subseteq U_{q_1}$, choose an open set U_{q_0} such that

$$A \subseteq U_{q_0} \subseteq \overline{U_{q_0}} \subseteq U_{q_1}.$$

Assume by induction that for some $n \geq 1$ we have defined $U_{q_0}, \dots, U_{q_n}$ such that

$$q_i < q_j \implies \overline{U_{q_i}} \subseteq U_{q_j} \qquad (0 \leq i, j \leq n).$$

Consider $q_{n+1} \in \mathbb{Q}_{[0,1]}$. If $q_{n+1} \in \{q_0, \dots, q_n\}$, there is nothing to do. Otherwise, define

$$r := \max\{q_i : \ 0 \leq i \leq n, \ q_i < q_{n+1}\}, \qquad s := \min\{q_i : \ 0 \leq i \leq n, \ q_{n+1} < q_i\}.$$

Then $r < q_{n+1} < s$ and, by the induction hypothesis,

$$\overline{U_r} \subseteq U_s.$$

Apply Lemma to the closed set $\overline{U_r}$ and the open set U_s to obtain an open set $U_{q_{n+1}}$ such that

$$\overline{U_r} \subseteq U_{q_{n+1}} \subseteq \overline{U_{q_{n+1}}} \subseteq U_s.$$

This preserves the inductive condition. Hence the family $(U_r)_{r \in \mathbb{Q}_{[0,1]}}$ exists and satisfies (U0), (U1).

Definition of f. Define $f : X \to [0,1]$ by

$$f(x) := \begin{cases} 1, & x \in B, \\ \inf\{r \in \mathbb{Q}_{[0,1]} : \ x \in U_r\}, & x \in X \setminus B. \end{cases}$$

This is well-defined: if $x \in X \setminus B = U_1$, then $1 \in \{r \in \mathbb{Q}_{[0,1]} : \ x \in U_r\}$, so the infimum is taken over a nonempty subset of $[0,1]$.

Separation of A and B. If $x \in A$, then $x \in U_0$, hence $f(x) \leq 0$, so $f(x) = 0$. If $x \in B$, then by definition $f(x) = 1$. Thus $f|_A \equiv 0$ and $f|_B \equiv 1$.

Continuity. Let

$$\Delta := \{[0,a) : \ 0 < a \leq 1\} \ \cup \ \{(b,1] : \ 0 \leq b < 1\}.$$

The family Δ is a subbasis of the usual topology on $[0,1]$. It suffices to show $f^{-1}(D)$ is open in X for each $D \in \Delta$.

(1) Let $0 < a \leq 1$. We claim

$$f^{-1}([0,a)) = \bigcup_{\substack{r \in \mathbb{Q}_{[0,1]} \\ r < a}} U_r.$$

Indeed, if $x \in \bigcup_{r<a} U_r$, then for some $r < a$ we have $x \in U_r$, hence $f(x) \leq r < a$. Conversely, if $f(x) < a$, choose $r \in \mathbb{Q}_{[0,1]}$ with $f(x) < r < a$. By definition of $f(x)$ as an infimum, $x \in U_r$. Hence x belongs to the union. Therefore $f^{-1}([0,a))$ is open.

(2) Let $0 \leq b < 1$. We claim

$$f^{-1}((b,1]) = \bigcup_{\substack{r \in \mathbb{Q}_{[0,1]} \\ r > b}} \big(X \setminus \overline{U_r}\big).$$

First, let $x \in f^{-1}((b,1])$. If $x \in B$, then $x \notin \overline{U_r}$ for every $r < 1$ because $U_r \subseteq U_1 = X \setminus B$, so x lies in the union. If $x \notin B$, then $f(x) \in (b,1)$. Choose $r \in \mathbb{Q}_{[0,1]}$ with $b < r < f(x)$. Then $x \notin U_r$ (otherwise $f(x) \leq r$), and moreover $x \notin \overline{U_r}$: if $x \in \overline{U_r}$, pick $s \in \mathbb{Q}_{[0,1]}$ with $r < s < f(x)$; then $\overline{U_r} \subseteq U_s$ by (U1), so $x \in U_s$, forcing $f(x) \leq s < f(x)$, contradiction. Hence $x \in X \setminus \overline{U_r}$, so x lies in the union.

Conversely, if $x \in X \setminus \overline{U_r}$ for some $r > b$, then $x \notin U_r$ and, since $q < r$ implies $U_q \subseteq \overline{U_q} \subseteq U_r$ by (U1), we have $x \notin U_q$ for all $q < r$. Hence $\inf\{q : \ x \in U_q\} \geq r > b$, i.e. $f(x) > b$ (or $f(x) = 1$ if $x \in B$). Thus the claimed equality holds, and $f^{-1}((b,1])$ is open.

Since preimages of a subbasis are open, f is continuous. $\square$

4. Embeddings

Definition 97 (Embedding)**.** Let $f : X \to Y$ be a map of topological spaces. We say that f is an *embedding* if the induced map

$$f : X \longrightarrow f(X)$$

is a homeomorphism, where $f(X) \subseteq Y$ is given the subspace topology. Equivalently: f is injective, continuous, and a set $U \subseteq X$ is open if and only if $U = f^{-1}(V)$ for some open $V \subseteq Y$.

Lecture 21: Urysohn's Metrization Theorem

1. Function spaces and separating families

Definition 98 (Continuous function space)**.** Let X, Y be topological spaces. We denote by

$$C(X,Y) := \{f : X \to Y : \ f \text{ is continuous}\}$$

the set of continuous maps from X to Y.

Definition 99 (Separating points from closed sets)**.** Let X be a topological space and let $A \subseteq C(X,[0,1])$. We say that A *separates points from closed sets* if for every closed set $C \subseteq X$ and every $x \in X \setminus C$ there exists $f \in A$ such that

$$f(x) = 0, \qquad f|_C \equiv 1.$$

Definition 100 (Evaluation map)**.** Let X be a set and let $A \subseteq [0,1]^X$ be a family of functions $X \to [0,1]$. Define the *evaluation map*

$$e_A : X \to [0,1]^A, \qquad e_A(x) := \big(f(x)\big)_{f \in A},$$

where $[0,1]^A$ is endowed with the product topology.

Lemma (Evaluation is an embedding under separation)**.** *Let X be a T_1 space and let $A \subseteq C(X,[0,1])$ separate points from closed sets (Definition 99). Then the evaluation map $e_A : X \to [0,1]^A$ is an embedding.*

Proof. For $f \in A$, let $p_f : [0,1]^A \to [0,1]$ denote the canonical projection. By construction,

$$p_f \circ e_A = f \qquad (\forall f \in A),$$

hence e_A is continuous because all p_f are continuous and each f is continuous.

Injectivity. Let $x \neq y$ in X. Since X is T_1, the singleton $\{y\}$ is closed. By the separation hypothesis, there exists $f \in A$ such that $f(x) = 0$ and $f(y) = 1$. Hence $p_f(e_A(x)) \neq p_f(e_A(y))$, so $e_A(x) \neq e_A(y)$.

Topological embedding. It suffices to show that e_A is a homeomorphism of X onto $e_A(X)$ with the subspace topology, i.e. that the inverse

$$e_A^{-1} : e_A(X) \to X$$

is continuous. For this it is enough to show: for every closed $C \subseteq X$, the set $e_A(C)$ is closed in $e_A(X)$.

Let $C \subseteq X$ be closed and let $z \in e_A(X) \setminus e_A(C)$. Write $z = e_A(x)$ with $x \in X \setminus C$. Choose $f \in A$ with $f(x) = 0$ and $f|_C \equiv 1$. Set

$$O := p_f^{-1}\big([0, \tfrac{1}{2})\big) \subseteq [0,1]^A,$$

which is open. Then $z = e_A(x) \in O$ and $O \cap e_A(C) = \varnothing$ (because $p_f(e_A(c)) = f(c) = 1$ for $c \in C$). Hence z does not lie in the closure of $e_A(C)$ inside $e_A(X)$. Therefore $e_A(C)$ is closed in $e_A(X)$, as required. □

2. The Hilbert cube is metrizable

Lemma. *The product space $[0,1]^{\mathbb{N}}$ (with the product topology) is metrizable. More precisely, the formula*

$$d\big((x_n),(y_n)\big) := \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$$

defines a metric on $[0,1]^{\mathbb{N}}$ *inducing the product topology.*

Proof. *(i) d is a metric.* The series converges absolutely because $|x_n - y_n| \le 1$ and $\sum 2^{-n} = 1$. Nonnegativity and symmetry are clear. If $d((x_n),(y_n)) = 0$, then each term $2^{-n}|x_n - y_n| = 0$, hence $x_n = y_n$ for all n. For the triangle inequality, for each n we have $|x_n - z_n| \le |x_n - y_n| + |y_n - z_n|$; multiplying by 2^{-n} and summing over n yields

$$d(x,z) \le d(x,y) + d(y,z).$$

(ii) The metric topology equals the product topology. Let $\mathcal{T}_d$ be the topology induced by d, and $\mathcal{T}_{\mathrm{prod}}$ the product topology.

First, each coordinate projection $p_j : [0,1]^{\mathbb{N}} \to [0,1]$, $p_j((x_n)) = x_j$, is 2^j-Lipschitz:

$$|p_j(x) - p_j(y)| = |x_j - y_j| \le 2^j \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n| = 2^j d(x,y).$$

Hence p_j is continuous for $\mathcal{T}_d$. Since $\mathcal{T}_{\mathrm{prod}}$ is the initial topology for the family $(p_j)_{j\in\mathbb{N}}$, it follows that

$$\mathcal{T}_{\mathrm{prod}} \subseteq \mathcal{T}_d.$$

Conversely, fix $x = (x_n) \in [0,1]^{\mathbb{N}}$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{n>N} 2^{-n} < \varepsilon/2$ (e.g. take N with $2^{-N} < \varepsilon/2$). Define a basic product neighborhood

$$U := \Big(\prod_{n=1}^{N} (x_n - \varepsilon/2,\ x_n + \varepsilon/2) \cap [0,1]\Big) \times \prod_{n>N} [0,1].$$

Then $U \in \mathcal{T}_{\mathrm{prod}}$ and $x \in U$. If $y \in U$, then $|x_n - y_n| < \varepsilon/2$ for $1 \le n \le N$, and $|x_n - y_n| \le 1$ for $n > N$; hence

$$d(x,y) \le \sum_{n=1}^{N} 2^{-n} \cdot \frac{\varepsilon}{2} + \sum_{n>N} 2^{-n} \cdot 1 \le \frac{\varepsilon}{2} \sum_{n=1}^{N} 2^{-n} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $U \subseteq B_d(x,\varepsilon)$, so every d-ball contains a product-neighborhood of its center. Hence $\mathcal{T}_d \subseteq \mathcal{T}_{\mathrm{prod}}$.

Therefore $\mathcal{T}_d = \mathcal{T}_{\mathrm{prod}}$. □

Remark. *If A is countable, then $[0,1]^A$ is homeomorphic to $[0,1]^{\mathbb{N}}$ (choose a bijection $\mathbb{N} \to A$). Hence $[0,1]^A$ is metrizable for every countable A.*

3. A countable separating family in the second countable case

Lemma. *Every second countable space is Lindelöf: every open cover has a countable subcover.*

Proof. Let $\mathcal{B}$ be a countable basis and let $\mathcal{U}$ be an open cover of X. For each $B \in \mathcal{B}$ with $B \neq \varnothing$, choose (if possible) some $U_B \in \mathcal{U}$ such that $B \subseteq U_B$. Then the subfamily $\{U_B\}$ is countable and covers X because for each $x \in X$ one can choose $U \in \mathcal{U}$ with $x \in U$ and then some $B \in \mathcal{B}$ with $x \in B \subseteq U$. □

Lemma. *Let X be second countable and completely regular (in the sense of Lecture 20: for every closed C and $x \notin C$ there exists $f : X \to [0,1]$ with $f(x) = 0$ and $f|_C \equiv 1$). Then there exists a* countable *subset $A \subseteq C(X,[0,1])$ which separates points from closed sets (Definition 99).*

Proof. Fix once and for all a continuous map $\phi : [0,1] \to [0,1]$ such that

$$\phi(t) = 0 \text{ for } 0 \le t \le \tfrac{1}{2}, \qquad \phi(1) = 1, \qquad \phi \text{ is nondecreasing.}$$

For instance one may take $\phi(t) = \max\{0, 2t - 1\}$.

Let $\mathcal{B} = \{B_1, B_2, \dots\}$ be a countable basis of X.

Step 1: for each basis element $B \in \mathcal{B}$, produce countably many functions. Fix $B \in \mathcal{B}$. For each $x \in B$, apply complete regularity to the closed set $X \setminus B$ and the point $x \in B$: there exists $g_{x,B} \in C(X, [0,1])$ such that

$$g_{x,B}(x) = 0, \qquad g_{x,B}|_{X \setminus B} \equiv 1.$$

Set $f_{x,B} := 1 - g_{x,B}$. Then

$$f_{x,B}(x) = 1, \qquad f_{x,B}|_{X \setminus B} \equiv 0.$$

Define

$$W_{x,B} := f_{x,B}^{-1}\big((\tfrac{1}{2}, 1]\big).$$

Then $W_{x,B}$ is open, contains x, and satisfies $W_{x,B} \subseteq B$ (because $f_{x,B} = 0$ on $X \setminus B$).

Thus $\{W_{x,B}\}_{x \in B}$ is an open cover of B. Since X is second countable, it is Lindelöf by Lemma , hence B is Lindelöf (open subspace). Therefore there exist points $x_{B,1}, x_{B,2}, \dots \in B$ such that

$$B = \bigcup_{n \geq 1} W_{x_{B,n},B}.$$

For each n, define

$$h_{B,n} := \phi \circ (1 - f_{x_{B,n},B}) = \phi \circ g_{x_{B,n},B} \ \in C(X, [0,1]).$$

Then $h_{B,n}|_{X \setminus B} \equiv \phi(1) = 1$, and on $W_{x_{B,n},B}$ one has $f_{x_{B,n},B} > \frac{1}{2}$, hence $g_{x_{B,n},B} < \frac{1}{2}$ and so $h_{B,n} \equiv 0$ on $W_{x_{B,n},B}$.

Step 2: define the global countable family. Set

$$A := \{h_{B,n} : \ B \in \mathcal{B}, \ n \in \mathbb{N}\}.$$

Since $\mathcal{B}$ is countable and each B contributes countably many functions, A is countable.

Step 3: A separates points from closed sets. Let $C \subseteq X$ be closed and let $x \in X \setminus C$. Then $X \setminus C$ is open, so choose $B \in \mathcal{B}$ with

$$x \in B \subseteq X \setminus C.$$

By construction, $x \in B = \bigcup_n W_{x_{B,n},B}$, so choose n with $x \in W_{x_{B,n},B}$. Then $h_{B,n}(x) = 0$ because $h_{B,n} \equiv 0$ on $W_{x_{B,n},B}$. Moreover, since $C \subseteq X \setminus B$, we have $h_{B,n}|_C \equiv 1$. Hence $h_{B,n} \in A$ separates x and C. $\square$

4. Urysohn's metrization theorem

Theorem 24 (Urysohn metrization, Tychonoff form)**.** Let X be a second countable, completely regular T_1 space. Then X is metrizable.

Proof. By Lemma , there exists a countable family $A \subseteq C(X, [0,1])$ separating points from closed sets. By Lemma , the evaluation map

$$e_A : X \to [0,1]^A$$

is an embedding. Since A is countable, $[0,1]^A$ is homeomorphic to $[0,1]^{\mathbb{N}}$ and is metrizable by Lemma . Therefore $e_A(X)$, as a subspace of a metrizable space, is metrizable (restriction of a metric). Since e_A is a homeomorphism $X \cong e_A(X)$, the space X is metrizable.

Explicit metric. Choose an enumeration $A = \{f_1, f_2, \dots\}$ and define

$$d_X(x, y) := \sum_{n=1}^{\infty} 2^{-n} |f_n(x) - f_n(y)|.$$

Because A separates points, d_X is a metric. Moreover, d_X is the pullback of the metric in Lemma under the embedding $x \mapsto (f_n(x))_{n\in\mathbb{N}}$, hence induces the original topology of X. ☐

5. The classical form: T_3 + second countable

Lemma (Regular Lindelöf ⇒ normal)**.** *Let X be regular and Lindelöf. Then X is normal.*

Proof. Let $A, B \subseteq X$ be disjoint closed sets. For each $a \in A$, since X is regular and $a \notin B$, there exists an open set U_a such that

$$a \in U_a, \qquad \overline{U_a} \subseteq X \setminus B.$$

Then $(U_a)_{a\in A}$ is an open cover of A. Since A is closed in a Lindelöf space, it is Lindelöf; hence choose $a_1, a_2, \dots \in A$ such that

$$A \subseteq \bigcup_{n\geq 1} U_{a_n}.$$

Set

$$U := \bigcup_{n\geq 1} U_{a_n}.$$

Then

$$\overline{U} \subseteq \bigcup_{n\geq 1} \overline{U_{a_n}} \subseteq X \setminus B,$$

so $B \subseteq X \setminus \overline{U}$. Thus the open sets U and $V := X \setminus \overline{U}$ satisfy $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$. ☐

Theorem 25 (Urysohn metrization, classical form)**.** If X is second countable and regular (i.e. T_3 in the T_1 convention of Lecture 19), then X is metrizable.

Proof. Second countability implies Lindelöfness (Lemma). Hence, if X is regular and second countable, then X is normal by Lemma .

Let $x \in X$ and let $C \subseteq X$ be closed with $x \notin C$. Since X is T_1, $\{x\}$ is closed. The closed sets $\{x\}$ and C are disjoint, so by Urysohn's lemma (Lecture 20) there exists $f : X \to [0, 1]$ continuous with $f(x) = 0$ and $f|_C \equiv 1$. Thus X is completely regular. Now apply Theorem 24. ☐

Lecture 22: Tietze Extension Theorem

1. Lindelöf spaces

Definition 101 (Lindelöf)**.** A topological space X is *Lindelöf* if every open cover of X admits a countable subcover.

Proposition 82. *Every second countable space is Lindelöf.*

Proof. Let $\mathcal{B} = \{B_1, B_2, \dots\}$ be a countable basis of X, and let $(U_\alpha)_{\alpha \in A}$ be an open cover of X. For each $B_n \in \mathcal{B}$, if there exists $\alpha(n) \in A$ such that $B_n \subseteq U_{\alpha(n)}$, choose one such $\alpha(n)$ and set

$$V_n := U_{\alpha(n)}.$$

(If no such $\alpha(n)$ exists, ignore B_n.) Then the family $\{V_n\}$ is countable.

We claim that $\{V_n\}$ covers X. Indeed, let $x \in X$ and choose $\alpha \in A$ with $x \in U_\alpha$. Since $\mathcal{B}$ is a basis, there exists $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq U_\alpha$. By construction, V_n is defined and satisfies $B_n \subseteq V_n$, hence $x \in V_n$. $\square$

2. Uniform convergence

Definition 102 (Uniform convergence)**.** Let X be a set, (Y, d) a metric space, and let $(f_n)_{n \in \mathbb{N}}$ be maps $f_n : X \to Y$. We say that f_n *converges uniformly* to a map $f : X \to Y$ if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(\forall x \in X) \qquad d\big(f_n(x), f(x)\big) < \varepsilon.$$

Lemma (Uniform limit of continuous maps)**.** *Let X be a topological space and (Y, d) a metric space. Let $(f_n)_{n \in \mathbb{N}} \subseteq C(X, Y)$ and suppose $f_n \to f$ uniformly for some $f : X \to Y$. Then $f \in C(X, Y)$.*

Proof. Fix $x_0 \in X$ and $\varepsilon > 0$. Choose N such that $d(f_N(x), f(x)) < \varepsilon/3$ for all $x \in X$. Since f_N is continuous at x_0, there exists a neighborhood U of x_0 such that

$$x \in U \implies d\big(f_N(x), f_N(x_0)\big) < \varepsilon/3.$$

For $x \in U$, the triangle inequality yields

$$d\big(f(x), f(x_0)\big) \leq d\big(f(x), f_N(x)\big) + d\big(f_N(x), f_N(x_0)\big) + d\big(f_N(x_0), f(x_0)\big) < \varepsilon.$$

Thus f is continuous at x_0, and x_0 was arbitrary. $\square$

3. Tietze extension theorem

Theorem 26 (Tietze extension theorem)**.** Let X be a normal space, let $C \subseteq X$ be closed, and let $f \in C\big(C, [0,1]\big)$. Then there exists $\widetilde{f} \in C\big(X, [0,1]\big)$ such that $\widetilde{f}|_C = f$.

The proof will be deduced from the following approximation step, whose proof uses Urysohn's lemma.

Lemma (One-step approximation)**.** *Let X be normal, let $C \subseteq X$ be closed, and let $h \in C\big(C, [-1,1]\big)$. Then there exists $g \in C\big(X, [-\frac{1}{3}, \frac{1}{3}]\big)$ such that*

$$\big|h(x) - g(x)\big| \leq \frac{2}{3} \qquad (\forall x \in C).$$

Proof. Set
$$A := h^{-1}\Big([-1, -\tfrac{1}{3}]\Big), \qquad B := h^{-1}\Big([\tfrac{1}{3}, 1]\Big).$$
Then A and B are disjoint closed subsets of C. Since C is closed in X, the sets A and B are closed in X as well. By Urysohn's lemma (Lecture 20), there exists a continuous map $u : X \to [0,1]$ such that
$$u|_A \equiv 0, \qquad u|_B \equiv 1.$$
Define
$$g := \frac{2u-1}{3} : X \to \Big[-\frac{1}{3}, \frac{1}{3}\Big].$$
Then $g|_A \equiv -\frac{1}{3}$ and $g|_B \equiv \frac{1}{3}$.
We claim that $|h-g| \le 2/3$ on C. If $x \in A$, then $h(x) \in [-1, -1/3]$ and $g(x) = -1/3$, hence $|h(x)-g(x)| \le 2/3$. If $x \in B$, then $h(x) \in [1/3, 1]$ and $g(x) = 1/3$, hence $|h(x) - g(x)| \le 2/3$. If $x \in C \setminus (A \cup B)$, then $|h(x)| < 1/3$ and $|g(x)| \le 1/3$, so
$$|h(x) - g(x)| \le |h(x)| + |g(x)| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$
□

Proof of Theorem 26. Step 1: reduction to the symmetric interval. Define
$$h := 2f - 1 \in C\big(C, [-1,1]\big).$$
It suffices to extend h to a continuous $\widetilde{h} : X \to [-1,1]$, because then
$$\widetilde{f} := \frac{\widetilde{h}+1}{2} \in C(X, [0,1])$$
satisfies $\widetilde{f}|_C = f$.

Step 2: inductive construction of a uniformly convergent series. We construct, by induction on $n \ge 1$, continuous functions
$$g_n \in C\Big(X, \Big[-\frac{1}{3}\Big(\frac{2}{3}\Big)^{n-1}, \ \frac{1}{3}\Big(\frac{2}{3}\Big)^{n-1}\Big]\Big)$$
such that, if we set partial sums $s_n := \sum_{k=1}^n g_k \in C(X, \mathbb{R})$ and remainders
$$r_n := h - s_n|_C \in C(C, \mathbb{R}),$$
then
$$|r_n(x)| \le \Big(\frac{2}{3}\Big)^n \qquad (\forall x \in C). \tag{0.1}$$
For $n = 0$ set $s_0 := 0$ and $r_0 := h$, so $|r_0| \le 1 = (2/3)^0$. Assume $g_1, \dots, g_n$ have been constructed and (0.1) holds. Then r_n takes values in $\big[-(2/3)^n, (2/3)^n\big]$. Define
$$h_n := \Big(\frac{3}{2}\Big)^n r_n \in C\big(C, [-1,1]\big).$$
Apply Lemma to h_n to obtain $u_{n+1} \in C\big(X, [-1/3, 1/3]\big)$ such that
$$|h_n(x) - u_{n+1}(x)| \le \frac{2}{3} \qquad (\forall x \in C).$$
Set
$$g_{n+1} := \Big(\frac{2}{3}\Big)^n u_{n+1} \in C\Big(X, \Big[-\frac{1}{3}\Big(\frac{2}{3}\Big)^n, \ \frac{1}{3}\Big(\frac{2}{3}\Big)^n\Big]\Big).$$

Then, on C,
$$r_{n+1} = h - s_{n+1}|_C = r_n - g_{n+1}|_C.$$
Multiplying by $(3/2)^n$ gives
$$\left(\frac{3}{2}\right)^n r_{n+1} = \left(\frac{3}{2}\right)^n r_n - u_{n+1} = h_n - u_{n+1},$$
hence
$$\left|\left(\frac{3}{2}\right)^n r_{n+1}(x)\right| = |h_n(x) - u_{n+1}(x)| \le \frac{2}{3}.$$
Therefore $|r_{n+1}(x)| \le (2/3)^{n+1}$ for all $x \in C$, completing the induction.

Step 3: uniform convergence and definition of the extension. For each $n \ge 1$ and each $x \in X$ we have
$$|g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}.$$
Since $\sum_{n\ge1} \frac{1}{3}(2/3)^{n-1} = 1$, the series $\sum_{n\ge1} g_n$ converges uniformly on X. Let
$$\widetilde{h}(x) := \sum_{n=1}^{\infty} g_n(x) \qquad (x \in X).$$
By Lemma , $\widetilde{h}$ is continuous. Moreover,
$$|\widetilde{h}(x)| \le \sum_{n=1}^{\infty} |g_n(x)| \le \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = 1,$$
so $\widetilde{h} : X \to [-1, 1]$.

Step 4: verification on C. For $x \in C$, we have $r_n(x) = h(x) - s_n(x)$, hence
$$h(x) - \sum_{k=1}^{n} g_k(x) = r_n(x).$$
By (0.1), $r_n(x) \to 0$ uniformly on C, hence passing to the limit gives
$$h(x) = \sum_{k=1}^{\infty} g_k(x) = \widetilde{h}(x) \qquad (\forall x \in C).$$
Thus $\widetilde{h}|_C = h$. Finally set $\widetilde{f} = (\widetilde{h} + 1)/2$ as in Step 1, and obtain $\widetilde{f}|_C = f$. □

Remark. *A standard corollary is the bounded real-valued form: if X is normal, $C \subseteq X$ closed, and $f \in C(C, \mathbb{R})$ is bounded, then there exists $\widetilde{f} \in C(X, \mathbb{R})$ extending f with the same bound on X (apply Theorem 26 after an affine rescaling).*

Lecture 23: Locally Compact Hausdorff Spaces and Compactifications

1. Local compactness

Definition 103 (Locally compact)**.** A topological space X is *locally compact* if for every $x \in X$ there exists a neighborhood K of x such that K is compact. (Equivalently: x admits a compact neighborhood.)

Definition 104 (LCH)**.** A space is *locally compact Hausdorff* (abbreviated *LCH*) if it is locally compact and Hausdorff.

Remark. *In a Hausdorff space, every compact subset is closed. In particular, in an LCH space, compact neighborhoods are closed neighborhoods.*

Proposition 83 (Relatively compact open neighborhoods)**.** *Let X be LCH and let $x \in X$. Then for every neighborhood N of x there exists an open neighborhood U of x such that*

$$U \subseteq N \qquad \text{and} \qquad \overline{U} \text{ is compact.}$$

In particular, every neighborhood of x contains a compact neighborhood of x.

Proof. Let N be a neighborhood of x. Choose an open set O with $x \in O \subseteq N$. By local compactness, choose an open set W with $x \in W$ and $\overline{W}$ compact. Set $K := \overline{W}$, which is compact Hausdorff (as a closed subspace of an LCH space), hence regular (Lecture 20). The set $O \cap K$ is an open neighborhood of x in the subspace K, so by regularity of K there exists an open set $U_K \subseteq K$ such that

$$x \in U_K, \qquad \overline{U_K}^{K} \subseteq O \cap K.$$

Write $U_K = K \cap U_0$ for some open $U_0 \subseteq X$, and set

$$U := U_0 \cap W.$$

Then U is open in X, $x \in U$, and $U \subseteq W \subseteq K$. Hence $\overline{U} \subseteq \overline{W} = K$, so $\overline{U}$ is compact. Moreover, since $U \subseteq U_0 \cap K = U_K$, we have

$$\overline{U} \subseteq \overline{U_K}^{K} \subseteq O \cap K \subseteq O \subseteq N.$$

Thus $U \subseteq N$ and $\overline{U}$ is compact; in particular $\overline{U}$ is a compact neighborhood of x contained in N. □

2. Manifolds

Definition 105 (Topological manifold)**.** Fix $n \in \mathbb{N}$. An n-dimensional *topological manifold* is a topological space X satisfying:

1. X is Hausdorff;
2. X is second countable;
3. X is *locally Euclidean of dimension n*: for every $x \in X$ there exists an open neighborhood $U \ni x$ and an open set $V \subseteq \mathbb{R}^n$ such that U is homeomorphic to V.

Remark. *The Hausdorff hypothesis in Definition 105 is essential: "locally Euclidean + second countable" does* not *imply Hausdorff. A standard counterexample is the* line with doubled origin*, which is locally homeomorphic to* $\mathbb{R}$ *and second countable, but not Hausdorff. Thus, manifolds are Hausdorff because we impose it.*

Proposition 84. *Every topological manifold is LCH.*

Proof. A manifold is Hausdorff by definition. Fix $x \in X$ and choose a chart $U \cong V \subseteq \mathbb{R}^n$ around x. Choose an open ball $B \subseteq V$ whose closure $\overline{B}$ is compact in $\mathbb{R}^n$. Its inverse image in X is a neighborhood of x with compact closure, hence provides a compact neighborhood of x. $\square$

3. LCH spaces are Tychonoff

Proposition 85. *Every LCH space is Tychonoff (equivalently: completely regular Hausdorff).*

Proof. Let X be LCH. It is Hausdorff, hence T_1. We prove complete regularity: let $C \subseteq X$ be closed and let $x \in X \setminus C$. Then $X \setminus C$ is an open neighborhood of x. By Proposition 83, there exists an open set U such that

$$x \in U \subseteq X \setminus C, \qquad K := \overline{U} \text{ compact.}$$

Then K is compact Hausdorff, hence normal (Lecture 20), so by Urysohn's lemma (Lecture 20) there exists a continuous map

$$g : K \to [0, 1]$$

such that $g(x) = 0$ and $g|_{K \setminus U} \equiv 1$ (note $K \setminus U$ is closed in K and $x \notin K \setminus U$ because $x \in U$).
Define $f : X \to [0, 1]$ by pasting:

$$f(y) := \begin{cases} g(y), & y \in K, \\ 1, & y \in X \setminus U. \end{cases}$$

This is well-defined because on the overlap $K \cap (X \setminus U) = K \setminus U$ we have $g \equiv 1$. By the pasting lemma (closed sets K and $X \setminus U$ cover X), f is continuous. Moreover $f(x) = g(x) = 0$, and since $C \subseteq X \setminus U$ we have $f|_C \equiv 1$. Thus X is completely regular, hence Tychonoff. $\square$

Corollary. *If X is second countable and LCH, then X is normal and metrizable.*

Proof. By Proposition 85, X is regular (indeed completely regular). Second countable implies Lindelöf (Lecture 22), and regular + Lindelöf implies normal (Lecture 21). Finally, second countable + completely regular + T_1 implies metrizable (Urysohn metrization, Lecture 21). $\square$

Remark. *If X is LCH, then every* open *subspace of X is LCH, and every* closed *subspace of X is LCH. The claim "every subset of an LCH space is LCH" is false in general.*

4. Compactifications

Definition 106 (Compactification)**.** Let X be a Hausdorff space. A *compactification* of X is a pair (Y, j) where Y is a compact Hausdorff space and

$$j : X \hookrightarrow Y$$

is an embedding such that $j(X)$ is dense in Y:

$$\overline{j(X)}^{Y} = Y.$$

Definition 107 (One-point compactification)**.** A compactification (Y, j) of X is a *one-point compactification* if

$$Y \setminus j(X) = \{\infty\}$$

consists of exactly one point.

5. Alexandroff one-point compactification

Theorem 27 (Existence of the one-point compactification)**.** Let X be an LCH space. Define the set

$$X^+ := X \sqcup \{\infty\}$$

(disjoint union of X with a new point ∞). There exists a topology $\mathcal{T}^+$ on X^+ such that:

1. the inclusion $i : X \hookrightarrow X^+$ is an embedding and $i(X) = X^+ \setminus \{\infty\}$;
2. $(X^+, \mathcal{T}^+)$ is compact and Hausdorff;
3. if X is non-compact, then $\overline{i(X)}^{X^+} = X^+$, hence (X^+, i) is a one-point compactification of X;
4. if X is compact, then ∞ is isolated and $X^+ \cong X \sqcup \{\infty\}$ with the coproduct (disjoint union) topology.

The resulting compactification is called the *Alexandroff (one-point) compactification.*

Proof. Define $\mathcal{T}^+$ by declaring a subset $U \subseteq X^+$ to be open if and only if either:

(Type 1) $U \subseteq X$ and U is open in X; or

(Type 2) $\infty \in U$ and $U = (X \setminus C) \cup \{\infty\}$ for some compact set $C \subseteq X$.

We first check that $\mathcal{T}^+$ is a topology. Type 1 opens are closed under arbitrary unions and finite intersections because they are the opens of X. Intersections of type 2 opens satisfy

$$\big((X \setminus C_1) \cup \{\infty\}\big) \cap \big((X \setminus C_2) \cup \{\infty\}\big) = (X \setminus (C_1 \cup C_2)) \cup \{\infty\},$$

and $C_1 \cup C_2$ is compact. Intersections of type 1 and type 2 opens are open in X because, in a Hausdorff space, compact sets are closed; hence $X \setminus C$ is open. Arbitrary unions of type 2 opens are again of type 2 because

$$\bigcup_{j \in J} \big((X \setminus C_j) \cup \{\infty\}\big) = \Big(X \setminus \bigcap_{j \in J} C_j\Big) \cup \{\infty\},$$

and $\bigcap_{j \in J} C_j$ is compact (closed subset of C_{j_0} for any fixed j_0). Also $\varnothing$ is type 1 and X^+ is type 2 with $C = \varnothing$.

(1) The inclusion is an embedding. Every open subset of X is type 1, hence open in X^+, so i is continuous. Conversely, if $U \subseteq X^+$ is open, then $U \cap X$ is open in X: if U is type 1 this is tautological; if $U = (X \setminus C) \cup \{\infty\}$ is type 2, then $U \cap X = X \setminus C$ is open in X because C is compact and X is Hausdorff. Thus the subspace topology on $i(X) = X^+ \setminus \{\infty\}$ coincides with the original topology on X, so i is an embedding.

(2) Compactness. Let $\mathcal{U}$ be an open cover of X^+. Some $U_\infty \in \mathcal{U}$ contains ∞, hence is of type 2:

$$U_\infty = (X \setminus C) \cup \{\infty\}$$

for a compact set $C \subseteq X$. Then $\mathcal{U} \setminus \{U_\infty\}$ is an open cover of C (viewed as a subspace of X). By compactness of C there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $C \subseteq U_1 \cup \cdots \cup U_n$. Then $U_\infty, U_1, \ldots, U_n$ form a finite subcover of X^+, so X^+ is compact.

(3) Hausdorffness. Let $p, q \in X^+$, $p \neq q$. If $p, q \in X$, separate them by disjoint open sets in X (since X is Hausdorff), which are type 1 opens in X^+. If $p = \infty$ and $q = x \in X$, choose (by Proposition 83) an open neighborhood U of x with compact closure $K := \overline{U}$. Then

$$V := (X \setminus K) \cup \{\infty\}$$

is an open neighborhood of ∞ (type 2), and $U \cap V = \varnothing$. Thus X^+ is Hausdorff.

(4) Density and the compact vs. non-compact dichotomy. If X is non-compact and W is any neighborhood of ∞ in X^+, then $W = (X \setminus C) \cup \{\infty\}$ with C compact. If X is non-compact, then $X \setminus C \neq \varnothing$, so $W \cap X \neq \varnothing$. Hence $\infty \in \overline{X}^{X^+}$, so $\overline{i(X)}^{X^+} = X^+$ and (X^+, i) is a one-point compactification.

If X is compact, take $C = X$ in the definition of type 2 opens; then $\{\infty\} = (X \setminus X) \cup \{\infty\}$ is open, so ∞ is isolated and X^+ is the coproduct $X \sqcup \{\infty\}$. $\square$

Remark (Characterization and uniqueness)**.** *If X is non-compact LCH, then the Alexandroff compactification is unique up to unique homeomorphism fixing X: if (Y, j) is any one-point compactification of X, writing $Y \setminus j(X) = \{\infty_Y\}$, there is a unique homeomorphism $\Phi : Y \to X^+$ such that $\Phi \circ j = i$ and $\Phi(\infty_Y) = \infty$.*

Moreover, for non-compact LCH spaces the neighborhoods of ∞ in X^+ are exactly the sets $(X \setminus C) \cup \{\infty\}$ with C compact, so the topology at infinity encodes compact subsets of X.

Remark. *Conversely, if X is Hausdorff and admits a one-point compactification which is Hausdorff, then X must be locally compact. Thus, for Hausdorff spaces,* local compactness is precisely the condition needed to make the one-point compactification Hausdorff.

Lecture 24: Maps between LCH Spaces and One-Point Compactifications

1. Uniqueness of the one-point compactification

Throughout this section, X denotes a non-compact locally compact Hausdorff space.

Proposition 86 (Uniqueness)**.** *Let (Y, f) and (Z, g) be one-point compactifications of X, i.e.*

- *Y and Z are compact Hausdorff spaces,*
- *$f : X \hookrightarrow Y$ and $g : X \hookrightarrow Z$ are embeddings with dense images,*
- *$Y \setminus f(X) = \{y_\infty\}$ and $Z \setminus g(X) = \{z_\infty\}$.*

Then there exists a unique homeomorphism $\varphi : Y \to Z$ such that

$$\varphi \circ f = g, \qquad \varphi(y_\infty) = z_\infty.$$

Equivalently, the diagram

$$\begin{tikzcd} Y \arrow[r, dashed, "\varphi"] & Z \\ & X \arrow[ul, "f"] \arrow[u, "g"'] \end{tikzcd}$$

commutes.

Proof. Uniqueness. Since $Y = f(X) \sqcup \{y_\infty\}$, any map $\varphi : Y \to Z$ satisfying $\varphi \circ f = g$ and $\varphi(y_\infty) = z_\infty$ is uniquely determined:

$$\varphi(f(x)) = g(x) \quad (\forall x \in X), \qquad \varphi(y_\infty) = z_\infty.$$

Existence. Define $\varphi : Y \to Z$ by

$$\varphi(y) := \begin{cases} g(x), & y = f(x) \text{ for some } x \in X, \\ z_\infty, & y = y_\infty. \end{cases}$$

This is well-defined because f is injective. It satisfies $\varphi \circ f = g$ and $\varphi(y_\infty) = z_\infty$.

Bijectivity. Define $\psi : Z \to Y$ similarly by $\psi(g(x)) = f(x)$ and $\psi(z_\infty) = y_\infty$. Then $\psi \circ \varphi = \mathrm{id}_Y$ and $\varphi \circ \psi = \mathrm{id}_Z$, hence φ is bijective.

Continuity of φ. Let $U \subseteq Z$ be open. There are two cases.

Case 1: $z_\infty \notin U$. Then $U \subseteq g(X) = Z \setminus \{z_\infty\}$. Since $g(X)$ is open in Z, the set U is open in the subspace $g(X)$, hence $g^{-1}(U)$ is open in X (because g is an embedding). Therefore

$$\varphi^{-1}(U) = f\big(g^{-1}(U)\big)$$

is open in $f(X)$; since $f(X) = Y \setminus \{y_\infty\}$ is open in Y, it follows that $\varphi^{-1}(U)$ is open in Y.

Case 2: $z_\infty \in U$. Let $K := Z \setminus U$. Then K is closed in the compact space Z, hence compact; moreover $K \subseteq g(X)$. Since $g : X \to g(X)$ is a homeomorphism, the set $C := g^{-1}(K)$ is compact in X, and thus $f(C)$ is compact in Y. Because Y is Hausdorff, $f(C)$ is closed in Y. Now

$$\varphi^{-1}(K) = f(C), \qquad \text{hence} \qquad \varphi^{-1}(U) = Y \setminus f(C)$$

is open in Y.

Thus φ is continuous. Since Y is compact and Z is Hausdorff, any continuous bijection $Y \to Z$ is a homeomorphism. Therefore φ is a homeomorphism. $\square$

Remark. *Proposition 86 expresses the uniqueness (up to unique homeomorphism) of the one-point compactification of a non-compact LCH space. In particular, the Alexandroff compactification constructed in Lecture 23 is* the *one-point compactification (up to canonical homeomorphism).*

2. Proper maps

Definition 108 (Proper map)**.** Let X, Y be topological spaces. A map $f : X \to Y$ is called *proper* if it is continuous and

$$(\forall K \subseteq Y \text{ compact}) \qquad f^{-1}(K) \text{ is compact in } X.$$

Remark. *If Y is Hausdorff and $f : X \to Y$ is proper, then each fiber $f^{-1}(\{y\})$ is compact, and f is a closed map. (These facts are standard and will be used implicitly when convenient.)*

3. Extending maps to one-point compactifications

Let X, Y be LCH spaces. Denote by $X^+ = X \sqcup \{\infty_X\}$ and $Y^+ = Y \sqcup \{\infty_Y\}$ their Alexandroff compactifications (Lecture 23), with the canonical inclusions $i_X : X \hookrightarrow X^+$ and $i_Y : Y \hookrightarrow Y^+$.

Definition 109 (Extension to the Alexandroff compactification)**.** Let $f : X \to Y$ be a map. Define a map

$$f^+ : X^+ \to Y^+$$

by

$$f^+(x) = f(x) \quad (x \in X), \qquad f^+(\infty_X) = \infty_Y.$$

Remark. *The map f^+ is always well-defined as a function, but it need not be continuous. The next theorem characterizes precisely when it is continuous in terms of properness.*

Theorem 28 (Proper maps and continuity at infinity)**.** Let X, Y be LCH spaces and let $f : X \to Y$ be continuous. Then the following are equivalent:

1. f is proper.
2. The extension $f^+ : X^+ \to Y^+$ (Definition 109) is continuous.

Proof. Recall (Lecture 23) that an open set $U \subseteq Y^+$ is of one of the following forms:

(Type 1) $U \subseteq Y$ and U is open in Y;

(Type 2) $\infty_Y \in U$ and $U = (Y \setminus K) \cup \{\infty_Y\}$ for some compact set $K \subseteq Y$.

Similarly for X^+.

(2) $\Rightarrow$ (1): Assume f^+ is continuous and let $K \subseteq Y$ be compact. Then K is also compact in Y^+ (as a closed subset of a Hausdorff compact space). Hence $(f^+)^{-1}(K)$ is compact in X^+. Since $\infty_Y \notin K$ we have $\infty_X \notin (f^+)^{-1}(K)$, and therefore

$$(f^+)^{-1}(K) = f^{-1}(K) \subseteq X.$$

Thus $f^{-1}(K)$ is compact in X^+ and hence compact in the subspace X. Therefore f is proper.

(1) $\Rightarrow$ (2): Assume f is proper. We show that f^+ is continuous by checking preimages of open sets in Y^+.

If $U \subseteq Y^+$ is type 1, say $U \subseteq Y$ open, then

$$(f^+)^{-1}(U) = f^{-1}(U),$$

which is open in X (hence open in X^+).

If $U \subseteq Y^+$ is type 2, write $U = (Y \setminus K) \cup \{\infty_Y\}$ with $K \subseteq Y$ compact. Then

$$(f^+)^{-1}(U) = \bigl(X \setminus f^{-1}(K)\bigr) \cup \{\infty_X\}.$$

By properness, $f^{-1}(K)$ is compact in X; hence $(X \setminus f^{-1}(K)) \cup \{\infty_X\}$ is open in X^+ by the definition of the Alexandroff topology. Thus $(f^+)^{-1}(U)$ is open.

Therefore f^+ is continuous. □

Corollary. *Let X be non-compact LCH and let Y be compact Hausdorff. Then there is no proper map $f : X \to Y$.*

Proof. If f were proper, then $f^{-1}(Y) = X$ would be compact, contradiction. Equivalently, by Theorem 28, the extension $f^+ : X^+ \to Y^+ = Y \sqcup \{\infty_Y\}$ would be continuous, but $\{\infty_Y\}$ is open in Y^+ while $\{\infty_X\}$ is not open in X^+ when X is non-compact. □

Remark. *Theorem 28 is the basic mechanism behind the statement: quotients of LCH spaces by* proper *group actions are well-behaved (in particular Hausdorff). One uses compactification and continuity at infinity to control separation in the quotient.*

Lecture 25: Proper Actions of Topological Groups

1. Topological groups and actions

Definition 110 (Topological group). A *topological group* is a group G equipped with a topology such that the maps

$$m : G \times G \to G, \quad (g, h) \mapsto gh, \qquad \text{and} \qquad \iota : G \to G, \quad g \mapsto g^{-1},$$

are continuous.

Definition 111 (Continuous action). Let G be a topological group and X a topological space. A (left) *continuous action* of G on X is a continuous map

$$\rho : G \times X \to X$$

such that $\rho(e, x) = x$ for all $x \in X$ and $\rho(g, \rho(h, x)) = \rho(gh, x)$ for all $g, h \in G$ and $x \in X$. We write $g \cdot x$ for $\rho(g, x)$ and denote the situation by $G \curvearrowright X$.

Definition 112 (Orbit). Let $G \curvearrowright X$. The *orbit* of $x \in X$ is

$$Gx := \{g \cdot x : \ g \in G\} \subseteq X.$$

Definition 113 (Orbit space and quotient map). Let $G \curvearrowright X$. Define an equivalence relation $\sim$ on X by

$$x \sim y \iff (\exists g \in G) \ y = g \cdot x.$$

The set of equivalence classes is denoted X/G (the *orbit space*). It is endowed with the quotient topology for the canonical surjection

$$\pi : X \to X/G, \qquad x \mapsto [x] = Gx.$$

2. Proper actions

Definition 114 (Proper map). A continuous map $f : U \to V$ is *proper* if for every compact set $K \subseteq V$ the preimage $f^{-1}(K)$ is compact in U.

Definition 115 (Proper action). Let $G \curvearrowright X$ be a continuous action. Define the map

$$\Phi : G \times X \longrightarrow X \times X, \qquad \Phi(g, x) := (g \cdot x, \ x).$$

The action is called *proper* if Φ is a proper map (in the sense of Definition 114).

Remark. *Heuristically, properness says that the set of group elements that move points within a compact region is itself relatively compact. In the LCH setting this is the topological analogue of proper discontinuity in geometric group theory.*

3. Basic properties

Proposition 87. *Let G be compact and let X be Hausdorff. Then every continuous action $G \curvearrowright X$ is proper.*

Proof. Let $\Phi : G \times X \to X \times X$ be as in Definition 115. Let $K \subseteq X \times X$ be compact. We must show $\Phi^{-1}(K)$ is compact in $G \times X$.

Since $X \times X$ is Hausdorff, the set K is closed in $X \times X$, hence $\Phi^{-1}(K)$ is closed in $G \times X$ (continuity of Φ). Let $\pi_2 : X \times X \to X$ be the second projection. Then $\pi_2(K)$ is compact in X. Moreover, if $(g, x) \in \Phi^{-1}(K)$ then $(g \cdot x, x) \in K$, hence $x = \pi_2(g \cdot x, x) \in \pi_2(K)$, so

$$\Phi^{-1}(K) \subseteq G \times \pi_2(K).$$

The product $G \times \pi_2(K)$ is compact (product of compact spaces), and $\Phi^{-1}(K)$ is closed in it; therefore $\Phi^{-1}(K)$ is compact. □

Proposition 88. *Let $G \curvearrowright X$ be a continuous action. For each $g \in G$, define the* translation *(or* action map*)*

$$T_g : X \to X, \qquad T_g(x) := g \cdot x.$$

Then T_g is a homeomorphism with inverse $T_{g^{-1}}$.

Proof. Let $i_g : X \to G \times X$ be the continuous map $x \mapsto (g, x)$. Then $\rho \circ i_g = T_g$, hence T_g is continuous. Since $T_{g^{-1}} \circ T_g = \mathrm{id}_X = T_g \circ T_{g^{-1}}$ by the action axioms, $T_{g^{-1}}$ is the inverse of T_g. By the same argument, $T_{g^{-1}}$ is continuous, hence T_g is a homeomorphism. □

Corollary. *Let $G \curvearrowright X$ be a continuous action and let $U \subseteq X$ be open. Then the* saturation

$$G \cdot U := \{g \cdot u : \ g \in G, \ u \in U\}$$

is open in X.

Proof. Since each T_g is a homeomorphism, $T_g(U)$ is open. Moreover

$$G \cdot U = \bigcup_{g \in G} T_g(U),$$

an arbitrary union of open sets, hence open. □

4. The orbit relation is closed under properness

Assume now that G and X are LCH and the action $G \curvearrowright X$ is proper.

Definition 116 (Orbit equivalence relation)**.** Define

$$R := \{(x_1, x_2) \in X \times X : \ \exists g \in G \text{ with } x_1 = g \cdot x_2\}.$$

Equivalently, $R = (\pi \times \pi)^{-1}(\Delta)$ where Δ is the diagonal in $(X/G) \times (X/G)$, but we will not use this here.

Proposition 89. *Let G, X be LCH and let the action $G \curvearrowright X$ be proper. Then $R \subseteq X \times X$ is closed.*

Proof. Let $\Phi : G\times X \to X\times X$ be given by $\Phi(g,x) = (g\cdot x, x)$. Then Φ is continuous and proper by hypothesis.

In the LCH setting, a proper map is closed: if $F \subseteq G \times X$ is closed and $y \in \overline{\Phi(F)}$, choose a compact neighborhood K of y in $X \times X$; then $\Phi^{-1}(K)$ is compact, hence $F \cap \Phi^{-1}(K)$ is compact. Using nets (or sequences if first countable), one shows $y \in \Phi(F)$. Therefore $\Phi(F)$ is closed.

Applying this to $F = G \times X$ (which is closed in itself) yields that

$$R = \Phi(G \times X)$$

is closed in $X \times X$. ☐

Remark. *If you prefer to avoid the general fact "proper ⇒ closed" in LCH spaces, you can prove closedness of R directly: if $(x_i, y_i) \in R$ converges to (x, y), pick g_i with $x_i = g_i \cdot y_i$. Using local compactness, choose a compact neighborhood K of (x, y) in $X\times X$. Then $(x_i, y_i) \in K$ for i large, hence $(g_i, y_i) \in \Phi^{-1}(K)$. Properness makes $\Phi^{-1}(K)$ compact, so (g_i, y_i) has a convergent subnet with limit (g, y). Continuity gives $x = g \cdot y$, hence $(x, y) \in R$.*

5. A criterion for Hausdorffness of quotients

Proposition 90. *Let X be a topological space and $R \subseteq X \times X$ an equivalence relation. Let $\pi : X \to X/R$ be the quotient map (with the quotient topology). Assume:*

1. *R is closed in $X \times X$;*
2. *π is an open map.*

Then X/R is Hausdorff.

Proof. Let $\pi(x) \neq \pi(y)$, i.e. $(x, y) \notin R$. Since R is closed, its complement $(X \times X) \setminus R$ is open and contains (x, y). Hence there exist open neighborhoods $U \ni x$ and $V \ni y$ in X such that

$$U \times V \subseteq (X \times X) \setminus R.$$

We claim that $\pi(U) \cap \pi(V) = \varnothing$. Suppose not. Then there exists $z \in X$ with $\pi(z) \in \pi(U) \cap \pi(V)$, so there exist $u \in U$ and $v \in V$ such that

$$\pi(u) = \pi(z) = \pi(v),$$

i.e. $u \sim z$ and $v \sim z$, hence $u \sim v$ and $(u, v) \in R$. But $(u, v) \in U \times V \subseteq (X \times X) \setminus R$, contradiction. Thus $\pi(U)$ and $\pi(V)$ are disjoint.

Finally, since π is open, $\pi(U)$ and $\pi(V)$ are open neighborhoods of $\pi(x)$ and $\pi(y)$ in X/R. Therefore X/R is Hausdorff. ☐

6. Proper actions yield Hausdorff orbit spaces

Theorem 29. Let G be an LCH topological group and X an LCH space. If $G \curvearrowright X$ is a proper action, then the orbit space X/G is Hausdorff.

Proof. Let $\pi : X \to X/G$ be the quotient map.

Step 1: the orbit relation is closed. By Proposition 89, the relation

$$R = \{(x_1, x_2) : \exists g \in G,\ x_1 = g \cdot x_2\}$$

is closed in $X \times X$.

Step 2: the quotient map is open. Let $U \subseteq X$ be open. Then $\pi^{-1}(\pi(U))$ is the saturation $G \cdot U$. By Corollary , $G \cdot U$ is open. By definition of the quotient topology, this implies $\pi(U)$ is open in X/G. Thus π is an open map.

Step 3: apply the criterion. Now Proposition 90 applies and yields that X/G is Hausdorff. □

Remark. *The theorem above is the entry point to the general theory of proper group actions, slice theorems, and the structure of orbit spaces. In many geometric situations (e.g. proper actions of Lie groups on manifolds), one can strengthen the conclusion: the quotient is not only Hausdorff, but often inherits local compactness, second countability, and sometimes even manifold/orbifold structure.*

Lecture 26: Stone–Čech Compactification

1. Products and evaluation maps

Notation. *Let I be a set. We write $[0,1]^I := \prod_{i\in I}[0,1]$ for the set of all maps $I \to [0,1]$ endowed with the product topology. For $i \in I$, denote by*

$$\pi_i : [0,1]^I \to [0,1], \qquad \pi_i\big((t_j)_{j\in I}\big) = t_i,$$

the coordinate projection.

Proposition 91 (Pullback on products)**.** *Let $\varphi : M \to N$ be a map between sets. Define*

$$\varphi^* : [0,1]^N \longrightarrow [0,1]^M, \qquad (\varphi^*(u))(m) := u\big(\varphi(m)\big) \quad (m \in M,\ u \in [0,1]^N),$$

i.e. $\varphi^*(u) = u \circ \varphi$. Then φ^* is continuous (for the product topologies).

Proof. Fix $m \in M$. Then

$$\pi_m \circ \varphi^* = \pi_{\varphi(m)}.$$

Since every $\pi_{\varphi(m)}$ is continuous, so is $\pi_m \circ \varphi^*$. As the product topology on $[0,1]^M$ is the initial topology for the family $(\pi_m)_{m\in M}$, this implies φ^* is continuous. □

Notation. *For a topological space X, set*

$$C(X,[0,1]) := \{f : X \to [0,1] \mid f \text{ continuous}\}.$$

Definition 117 (Canonical evaluation map)**.** Let X be a topological space and set $A_X := C(X,[0,1])$. Define

$$\eta_X : X \longrightarrow [0,1]^{A_X}, \qquad \big(\eta_X(x)\big)(f) := f(x) \quad (x \in X,\ f \in A_X).$$

Proposition 92. *Assume that X is completely regular and T_1. Then η_X is an embedding.*

Proof. This is the special case of the embedding lemma from Lecture 21 with $A = A_X = C(X,[0,1])$, which separates points from closed sets by complete regularity. Concretely:

- η_X is continuous because $\pi_f \circ \eta_X = f$ for each $f \in A_X$.
- η_X is injective: if $x \neq y$, then $\{y\}$ is closed (T_1), hence complete regularity gives $f \in A_X$ with $f(x) = 0$, $f(y) = 1$.
- for each closed $C \subseteq X$, the subset $\eta_X(C)$ is closed in $\eta_X(X)$ by the same separation argument; thus $\eta_X^{-1} : \eta_X(X) \to X$ is continuous.

Hence η_X is a homeomorphism onto $\eta_X(X)$ with the subspace topology. □

Remark. *If X is completely regular and T_1, then X is automatically Hausdorff: for $x \neq y$, separate x from the closed set $\{y\}$ by $f : X \to [0,1]$ with $f(x) = 0, f(y) = 1$ and take disjoint open sets $f^{-1}([0,1/2))$ and $f^{-1}((1/2,1])$. Thus the standing hypothesis is equivalent to "X is Tychonoff".*

2. Construction of βX

Definition 118 (Stone–Čech compactification)**.** Let X be completely regular and T_1, and let $\eta_X : X \to [0,1]^{A_X}$ be as in Definition 117. Define

$$\beta X := \overline{\eta_X(X)}^{[0,1]^{A_X}} \subseteq [0,1]^{A_X}.$$

The space βX, together with the map $\eta_X : X \to \beta X$, is called the *Stone–Čech compactification* of X.

Proposition 93. *The space βX is compact and Hausdorff, and $\eta_X(X)$ is dense in βX. Moreover $\eta_X : X \to \beta X$ is an embedding (hence a compactification of X in the sense of Lecture 23).*

Proof. By Tychonoff's theorem, $[0,1]^{A_X}$ is compact; it is Hausdorff because $[0,1]$ is Hausdorff and products of Hausdorff spaces are Hausdorff. Since βX is closed in $[0,1]^{A_X}$, it is compact and Hausdorff. Density of $\eta_X(X)$ in βX is tautological from the definition $\beta X = \overline{\eta_X(X)}$. Finally, η_X is an embedding by Proposition 92, and its image is $\eta_X(X) = \beta X \setminus (\beta X \setminus \eta_X(X))$. $\square$

Remark. *If one replaces $A_X = C(X,[0,1])$ by a smaller separating family $A \subseteq C(X,[0,1])$, one still obtains a compactification*

$$\overline{\eta_A(X)} \subseteq [0,1]^A, \qquad \eta_A(x) = (f(x))_{f\in A},$$

but in general it does not *satisfy the universal mapping property of βX below. The Stone–Čech compactification is the maximal compactification for maps into compact Hausdorff spaces.*

Proposition 94. *If X is compact Hausdorff, then $\eta_X : X \to \beta X$ is a homeomorphism.*

Proof. Since X is compact and βX is Hausdorff, the embedding $\eta_X : X \to \beta X$ has compact image $\eta_X(X)$, hence $\eta_X(X)$ is closed in βX. But $\eta_X(X)$ is dense in βX, so $\eta_X(X) = \beta X$. $\square$

3. Functoriality

Definition 119 (Pullback on $[0,1]$-valued functions)**.** Let $f : X \to Y$ be continuous. Define the pullback

$$f^\sharp : C(Y,[0,1]) \to C(X,[0,1]), \qquad f^\sharp(h) := h \circ f.$$

Proposition 95 (Functorial extension)**.** *Let $f : X \to Y$ be a continuous map between completely regular T_1 spaces. Then there exists a unique continuous map*

$$\beta f : \beta X \to \beta Y$$

such that the diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ \beta X \arrow[r, dashed, "\beta f"'] & \beta Y. \end{tikzcd}$$

Proof. Let $A_X = C(X,[0,1])$ and $A_Y = C(Y,[0,1])$. The pullback map $f^\sharp : A_Y \to A_X$ induces, by Proposition 91, a continuous map

$$(f^\sharp)^* : [0,1]^{A_X} \longrightarrow [0,1]^{A_Y}, \qquad u \longmapsto u \circ f^\sharp.$$

We claim that for each $x \in X$ one has

$$(f^\sharp)^*\big(\eta_X(x)\big) = \eta_Y\big(f(x)\big). \tag{0.2}$$

Indeed, for $h \in A_Y$,

$$\big((f^\sharp)^*(\eta_X(x))\big)(h) = \eta_X(x)\big(f^\sharp(h)\big) = \eta_X(x)(h \circ f) = (h \circ f)(x) = h(f(x)) = \eta_Y(f(x))(h).$$

Thus $(f^\sharp)^*(\eta_X(X)) \subseteq \eta_Y(Y) \subseteq \beta Y$. Since βY is closed in $[0,1]^{A_Y}$ and $(f^\sharp)^*$ is continuous, we obtain

$$(f^\sharp)^*(\beta X) = (f^\sharp)^*\left(\overline{\eta_X(X)}\right) \subseteq \overline{(f^\sharp)^*(\eta_X(X))} \subseteq \overline{\eta_Y(Y)} = \beta Y.$$

Hence we may define βf as the restriction

$$\beta f := (f^\sharp)^*\big|_{\beta X} : \beta X \to \beta Y.$$

Then (0.2) implies $\beta f \circ \eta_X = \eta_Y \circ f$.

Uniqueness: $\eta_X(X)$ is dense in βX and βY is Hausdorff, so a continuous map $\beta X \to \beta Y$ is determined by its restriction to $\eta_X(X)$. □

Remark. *The assignment $X \mapsto \beta X$, $f \mapsto \beta f$ defines a functor from the category of Tychonoff spaces to the category of compact Hausdorff spaces.*

4. Universal property

Theorem 30 (Universal mapping property)**.** Let X be completely regular and T_1, and let K be compact Hausdorff. For every continuous map $k : X \to K$ there exists a unique continuous map

$$\widetilde{k} : \beta X \to K$$

such that

$$\widetilde{k} \circ \eta_X = k.$$

Equivalently, βX is initial among compact Hausdorff spaces receiving a continuous map from X.

Proof. Apply Proposition 95 to the map $k : X \to K$ to obtain $\beta k : \beta X \to \beta K$ with

$$\beta k \circ \eta_X = \eta_K \circ k.$$

Since K is compact Hausdorff, Proposition 94 gives that $\eta_K : K \to \beta K$ is a homeomorphism. Define

$$\widetilde{k} := \eta_K^{-1} \circ \beta k : \beta X \to K.$$

Then $\widetilde{k} \circ \eta_X = k$.

For uniqueness, let $h : \beta X \to K$ be continuous with $h \circ \eta_X = k$. Then h and $\widetilde{k}$ agree on the dense subset $\eta_X(X) \subseteq \beta X$. Since K is Hausdorff, continuous maps into K are determined by their values on a dense subset; hence $h = \widetilde{k}$. □

Remark. *Theorem 30 may be expressed categorically by saying that the functor*

$$\beta : \mathbf{Tych} \longrightarrow \mathbf{CompHaus}$$

is left adjoint to the inclusion **CompHaus** $\hookrightarrow$ **Tych**.

Lecture 27–28: Paracompactness and Partitions of Unity

1. Support and locally finite families

Definition 120 (Support)**.** Let X be a topological space and $f : X \to \mathbb{R}$ a function. The *support* of f is

$$\operatorname{supp}(f) := \overline{\{x \in X : \ f(x) \neq 0\}}.$$

Remark. *If f is continuous, then $f^{-1}(\mathbb{R} \setminus \{0\})$ is open, hence $\operatorname{supp}(f)$ is the closure of an open set. In particular $\operatorname{supp}(f)$ is closed, and $\operatorname{supp}(f) \subseteq X$ is the smallest closed set outside of which f vanishes.*

Definition 121 (Locally finite family)**.** Let X be a topological space and $(A_\lambda)_{\lambda\in\Lambda}$ a family of subsets of X. The family is *locally finite* if for every $x \in X$ there exists a neighborhood N_x of x such that

$$\{\lambda \in \Lambda : \ N_x \cap A_\lambda \neq \varnothing\} \quad \text{is finite.}$$

Definition 122 (Locally finite family of functions)**.** A family of functions $(f_\lambda)_{\lambda\in\Lambda}$ with $f_\lambda : X \to \mathbb{R}$ is called *locally finite* if the family of sets

$$\big(\operatorname{supp}(f_\lambda)\big)_{\lambda\in\Lambda}$$

is locally finite. Equivalently: for every $x \in X$ there exists a neighborhood N_x of x such that $f_\lambda|_{N_x} \equiv 0$ for all but finitely many λ.

Lemma (Local finiteness and continuity of sums)**.** *Let X be a topological space and $(f_\lambda)_{\lambda\in\Lambda}$ a locally finite family of continuous functions $f_\lambda : X \to \mathbb{R}$. Then the pointwise sum*

$$f := \sum_{\lambda\in\Lambda} f_\lambda$$

is well-defined (only finitely many nonzero summands near each point) and defines a continuous function $f : X \to \mathbb{R}$.

Proof. Fix $x \in X$ and choose a neighborhood N of x meeting only finitely many supports, say $\operatorname{supp}(f_{\lambda_1}), \dots, \operatorname{supp}(f_{\lambda_n})$. Then $f_\lambda|_N \equiv 0$ for $\lambda \notin \{\lambda_1, \dots, \lambda_n\}$, hence

$$f|_N = \sum_{i=1}^{n} f_{\lambda_i}|_N,$$

a finite sum of continuous functions; thus f is continuous on N. Since x was arbitrary, f is continuous on X. □

2. Refinements and paracompactness

Definition 123 (Refinement)**.** Let $(U_\alpha)_{\alpha\in A}$ and $(V_\beta)_{\beta\in B}$ be covers of a space X. We say that $(U_\alpha)_{\alpha\in A}$ *refines* $(V_\beta)_{\beta\in B}$ if for every $\alpha \in A$ there exists $\beta \in B$ such that

$$U_\alpha \subseteq V_\beta.$$

Definition 124 (Paracompact)**.** A topological space X is *paracompact* if every open cover of X admits a locally finite open refinement.

Definition 125 (σ-compact)**.** A topological space X is *σ-compact* if there exist compact subsets $K_1, K_2, \dots \subseteq X$ such that

$$X = \bigcup_{n \geq 1} K_n.$$

3. σ-compact LCH spaces are paracompact

Definition 126 (Exhaustion by relatively compact open sets)**.** Let X be a topological space. An *exhaustion* of X is a sequence $(U_n)_{n \geq 1}$ of open subsets of X such that

$$\overline{U_n} \text{ is compact}, \qquad \overline{U_n} \subseteq U_{n+1}, \qquad X = \bigcup_{n \geq 1} U_n.$$

Lemma. *Let X be locally compact, Hausdorff, and σ-compact. Then X admits an exhaustion in the sense of Definition 126.*

Proof. Choose compact sets $K_n \subseteq X$ with $X = \bigcup_{n \geq 1} K_n$. We construct inductively open sets U_n with compact closures such that

$$K_n \cup \overline{U_{n-1}} \subseteq U_n, \qquad \overline{U_n} \text{ compact}, \qquad (n \geq 1),$$

where $U_0 := \varnothing$.

Assume U_{n-1} has been constructed. The set $K_n \cup \overline{U_{n-1}}$ is compact (finite union of compact sets). By local compactness, for each x in this compact set there exists an open neighborhood V_x such that $\overline{V_x}$ is compact. A finite subcover $V_{x_1}, \dots, V_{x_m}$ covers $K_n \cup \overline{U_{n-1}}$. Set

$$U_n := \bigcup_{i=1}^{m} V_{x_i}.$$

Then U_n is open, contains $K_n \cup \overline{U_{n-1}}$, and

$$\overline{U_n} \subseteq \bigcup_{i=1}^{m} \overline{V_{x_i}}$$

is compact. This completes the induction.

Then $\overline{U_n} \subseteq U_{n+1}$ holds by construction, and $X = \bigcup_n U_n$ since $K_n \subseteq U_n$ for all n. □

Theorem 31. Every locally compact, Hausdorff, σ-compact space is paracompact.

Proof. Let X be locally compact Hausdorff and σ-compact. Choose an exhaustion $(U_n)_{n \geq 1}$ as in Lemma . Set $U_0 := \varnothing$ and $U_{-1} := \varnothing$ for convenience.

Let $(A_\alpha)_{\alpha \in A}$ be an open cover of X. For each $n \geq 1$ define the compact *annulus*

$$K_n := \overline{U_n} \setminus U_{n-1}.$$

(Indeed, K_n is closed in the compact set $\overline{U_n}$.)

For each $n \geq 1$ and $\alpha \in A$, define
$$B_{\alpha,n} := A_\alpha \cap \big(U_{n+1} \setminus \overline{U_{n-2}}\big),$$
where $U_{n-2} = \varnothing$ if $n \leq 2$. Then each $B_{\alpha,n}$ is open and $B_{\alpha,n} \subseteq A_\alpha$.

Claim 1: $(B_{\alpha,n})_{\alpha\in A}$ covers K_n. Let $x \in K_n$. Then $x \in \overline{U_n} \subseteq U_{n+1}$. Moreover $\overline{U_{n-2}} \subseteq U_{n-1}$ (by the exhaustion property), and $x \notin U_{n-1}$ (since $x \in \overline{U_n} \setminus U_{n-1}$), hence $x \notin \overline{U_{n-2}}$. Thus
$$x \in U_{n+1} \setminus \overline{U_{n-2}}.$$
Since (A_α) covers X, pick α with $x \in A_\alpha$, then $x \in B_{\alpha,n}$.

By compactness of K_n, there exists a finite subset $A_n \subseteq A$ such that
$$K_n \subseteq \bigcup_{\alpha\in A_n} B_{\alpha,n}.$$
Define the family
$$\mathcal{B} := \{B_{\alpha,n} : \ n \geq 1, \ \alpha \in A_n\}.$$
This is an open cover of $\bigcup_{n\geq 1} K_n = X$ and refines (A_α).

Claim 2: $\mathcal{B}$ is locally finite. Fix $x \in X$ and choose m such that $x \in U_m$ (possible since $\bigcup_m U_m = X$). Then for every $n \geq m+2$ we have $n-2 \geq m$, hence $U_m \subseteq U_{n-2} \subseteq \overline{U_{n-2}}$, so
$$U_m \cap \big(U_{n+1} \setminus \overline{U_{n-2}}\big) = \varnothing,$$
and therefore $U_m \cap B_{\alpha,n} = \varnothing$ for all $\alpha \in A_n$. Thus the neighborhood U_m of x meets only the finitely many families with $n \leq m+1$. Since each A_n is finite, U_m meets only finitely many elements of $\mathcal{B}$.

Therefore $\mathcal{B}$ is a locally finite open refinement of (A_α), and X is paracompact. □

4. Closures of locally finite unions

Proposition 96. *Let $(A_\lambda)_{\lambda\in\Lambda}$ be a locally finite family of subsets of a space X. Then*
$$\overline{\bigcup_{\lambda\in\Lambda} A_\lambda} = \bigcup_{\lambda\in\Lambda} \overline{A_\lambda}.$$

Proof. The inclusion $\bigcup_\lambda \overline{A_\lambda} \subseteq \overline{\bigcup_\lambda A_\lambda}$ always holds.

For the reverse inclusion, let $x \in \overline{\bigcup_\lambda A_\lambda}$. Choose a neighborhood N of x meeting only finitely many $A_{\lambda_1}, \ldots, A_{\lambda_n}$. Then
$$N \cap \bigcup_{\lambda\in\Lambda} A_\lambda = N \cap \bigcup_{i=1}^{n} A_{\lambda_i},$$
so $x \in \overline{\bigcup_{i=1}^n A_{\lambda_i}}$. Since closure commutes with finite unions,
$$\overline{\bigcup_{i=1}^{n} A_{\lambda_i}} = \bigcup_{i=1}^{n} \overline{A_{\lambda_i}} \subseteq \bigcup_{\lambda\in\Lambda} \overline{A_\lambda}.$$
Hence $x \in \bigcup_\lambda \overline{A_\lambda}$. □

Corollary. *If $(F_\lambda)_{\lambda\in\Lambda}$ is a locally finite family of* closed *subsets of X, then $\bigcup_\lambda F_\lambda$ is closed.*

Proof. Apply Proposition 96 and use $\overline{F_\lambda} = F_\lambda$. □

5. Paracompact Hausdorff spaces are normal

Theorem 32. Every paracompact Hausdorff space is normal.

Proof. We proceed in two steps.

Step 1: paracompact Hausdorff $\Rightarrow$ regular. Let X be paracompact Hausdorff, let $x \in X$, and let $C \subseteq X$ be closed with $x \notin C$. For each $y \in C$, since X is Hausdorff, there exist disjoint open sets $U_y \ni x$ and $V_y \ni y$. In particular, $x \notin \overline{V_y}$ because U_y is a neighborhood of x disjoint from V_y.

The family

$$\mathcal{U} := \{X \setminus C\} \cup \{V_y : y \in C\}$$

is an open cover of X. By paracompactness, choose a locally finite open refinement $(W_\alpha)_{\alpha \in A}$ of $\mathcal{U}$. Let

$$A' := \{\alpha \in A : W_\alpha \subseteq V_y \text{ for some } y \in C\}.$$

Then $C \subseteq \bigcup_{\alpha \in A'} W_\alpha$. Moreover, for $\alpha \in A'$, we have $W_\alpha \subseteq V_y$ for some y, hence

$$x \notin \overline{W_\alpha} \subseteq \overline{V_y}.$$

By local finiteness and Proposition 96,

$$\overline{\bigcup_{\alpha \in A'} W_\alpha} = \bigcup_{\alpha \in A'} \overline{W_\alpha},$$

so $x \notin \overline{\bigcup_{\alpha \in A'} W_\alpha}$. Set

$$V := \bigcup_{\alpha \in A'} W_\alpha, \qquad U := X \setminus \overline{V}.$$

Then V is open, contains C, and U is open, contains x, and $U \cap V = \varnothing$. Thus X is regular.

Step 2: regular + paracompact $\Rightarrow$ normal. Let $A, B \subseteq X$ be disjoint closed sets. By regularity, for each $a \in A$ there exists an open neighborhood U_a of a such that

$$\overline{U_a} \cap B = \varnothing \qquad (\text{equivalently, } \overline{U_a} \subseteq X \setminus B).$$

Then $\{U_a\}_{a \in A} \cup \{X \setminus A\}$ is an open cover of X. Choose a locally finite open refinement $(W_\alpha)_{\alpha \in \Gamma}$. Let $\Gamma' \subseteq \Gamma$ be the indices such that $W_\alpha \subseteq U_a$ for some $a \in A$, and set

$$U := \bigcup_{\alpha \in \Gamma'} W_\alpha.$$

Then U is open and contains A. Moreover, for each $\alpha \in \Gamma'$, $\overline{W_\alpha} \subseteq \overline{U_a} \subseteq X \setminus B$ for some a. By Proposition 96,

$$\overline{U} = \overline{\bigcup_{\alpha \in \Gamma'} W_\alpha} = \bigcup_{\alpha \in \Gamma'} \overline{W_\alpha} \subseteq X \setminus B.$$

Hence $\overline{U} \cap B = \varnothing$. Let $V := X \setminus \overline{U}$; then V is open, contains B, and $U \cap V = \varnothing$. Thus X is normal. $\square$

6. Shrinking lemma

Proposition 97 (Shrinking lemma)**.** *Let X be a paracompact Hausdorff space and let $(U_\alpha)_{\alpha \in A}$ be an open cover of X. Then there exists a locally finite open cover $(V_\alpha)_{\alpha \in A}$ of X such that*

$$\overline{V_\alpha} \subseteq U_\alpha \qquad (\forall \alpha \in A),$$

where we allow $V_\alpha = \varnothing$ for some α.

Proof. By Theorem 32, X is regular. For each $x \in X$ choose $\alpha(x) \in A$ such that $x \in U_{\alpha(x)}$. By regularity, there exists an open neighborhood O_x of x such that

$$x \in O_x, \qquad \overline{O_x} \subseteq U_{\alpha(x)}.$$

The family $(O_x)_{x\in X}$ is an open cover of X. By paracompactness, choose a locally finite open refinement $(W_i)_{i\in I}$ of $(O_x)_{x\in X}$. For each $i \in I$ fix $x(i) \in X$ with $W_i \subseteq O_{x(i)}$ and set $\alpha(i) := \alpha(x(i))$. Then

$$\overline{W_i} \subseteq \overline{O_{x(i)}} \subseteq U_{\alpha(i)}.$$

Now define, for each $\alpha \in A$,

$$V_\alpha := \bigcup_{\substack{i\in I \\ \alpha(i)=\alpha}} W_i.$$

Then $(V_\alpha)_{\alpha\in A}$ is an open cover of X (because the W_i cover X). It is locally finite: given $x \in X$, choose a neighborhood meeting only finitely many W_i; then it meets only finitely many V_α. Finally, by Proposition 96 applied to the locally finite family $(W_i)_{i:\alpha(i)=\alpha}$,

$$\overline{V_\alpha} = \overline{\bigcup_{\alpha(i)=\alpha} W_i} = \bigcup_{\alpha(i)=\alpha} \overline{W_i} \subseteq U_\alpha.$$

□

7. Partitions of unity

Definition 127 (Partition of unity)**.** Let X be a topological space and $(U_\alpha)_{\alpha\in A}$ an open cover of X. A *partition of unity subordinate to* (U_α) is a family of continuous functions

$$(\rho_\alpha)_{\alpha\in A} \subseteq C(X, [0,1])$$

such that:

1. the family $(\rho_\alpha)_{\alpha\in A}$ is locally finite (Definition 122);
2. $\operatorname{supp}(\rho_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$;
3. for every $x \in X$ one has
$$\sum_{\alpha\in A} \rho_\alpha(x) = 1,$$
where the sum is well-defined by local finiteness.

Theorem 33 (Existence of partitions of unity)**.** Let X be a paracompact Hausdorff space and $(U_\alpha)_{\alpha\in A}$ an open cover of X. Then there exists a partition of unity $(\rho_\alpha)_{\alpha\in A}$ subordinate to (U_α).

Proof. By Proposition 97, choose a locally finite open cover $(V_\alpha)_{\alpha\in A}$ with $\overline{V_\alpha} \subseteq U_\alpha$. Apply Proposition 97 again to the cover (V_α) to obtain a locally finite open cover $(W_\alpha)_{\alpha\in A}$ with

$$\overline{W_\alpha} \subseteq V_\alpha \qquad (\forall \alpha \in A).$$

(In particular, (W_α) covers X.)

Since X is paracompact Hausdorff, it is normal by Theorem 32. Fix $\alpha \in A$. The closed sets $\overline{W_\alpha}$ and $X \setminus V_\alpha$ are disjoint (because $\overline{W_\alpha} \subseteq V_\alpha$). By Urysohn's lemma (Lecture 20), there exists $\varphi_\alpha \in C(X, [0,1])$ such that

$$\varphi_\alpha|_{\overline{W_\alpha}} \equiv 1, \qquad \varphi_\alpha|_{X \setminus V_\alpha} \equiv 0.$$

Then $\operatorname{supp}(\varphi_\alpha) \subseteq \overline{V_\alpha} \subseteq U_\alpha$.

The family (V_α) is locally finite, hence so is $(\overline{V_\alpha})$, and therefore the family $(\operatorname{supp}(\varphi_\alpha))$ is locally finite. Thus the sum

$$\Phi := \sum_{\alpha \in A} \varphi_\alpha$$

is well-defined and continuous by Lemma . Moreover, since (W_α) covers X, for each $x \in X$ there exists α with $x \in W_\alpha \subseteq \overline{W_\alpha}$, hence $\varphi_\alpha(x) = 1$, so $\Phi(x) \geq 1$ for all $x \in X$.

Define

$$\rho_\alpha(x) := \frac{\varphi_\alpha(x)}{\Phi(x)} \qquad (\alpha \in A,\ x \in X).$$

Then each ρ_α is continuous, takes values in $[0,1]$, and $\operatorname{supp}(\rho_\alpha) \subseteq \operatorname{supp}(\varphi_\alpha) \subseteq U_\alpha$. Finally,

$$\sum_{\alpha \in A} \rho_\alpha(x) = \frac{1}{\Phi(x)} \sum_{\alpha \in A} \varphi_\alpha(x) = 1 \quad (\forall x \in X),$$

and the sum is locally finite because (ρ_α) is locally finite (its supports are contained in the locally finite family $(\operatorname{supp}(\varphi_\alpha))$). Thus $(\rho_\alpha)_{\alpha \in A}$ is a partition of unity subordinate to (U_α). $\square$

Lecture 29: Manifolds

1. Locally Euclidean spaces

Definition 128 (Locally Euclidean of dimension n)**.** Let $n \in \mathbb{N}$. A topological space X is *locally Euclidean of dimension n* if for every $x \in X$ there exist an open neighborhood $U \subseteq X$ of x, an open set $V \subseteq \mathbb{R}^n$, and a homeomorphism $U \simeq V$.

Remark. *If X is Hausdorff and locally Euclidean, then:*

1. *X is locally compact (indeed, $V \subseteq \mathbb{R}^n$ admits relatively compact open neighborhoods, and local compactness is preserved by homeomorphisms);*
2. *X is locally second countable (every open subset of $\mathbb{R}^n$ is second countable, and second countability is inherited by subspaces).*

In particular, a Hausdorff locally Euclidean space is an LCH space and is locally second countable.

2. Second countable LCH spaces are σ-compact

Lemma. *Let X be a second countable locally compact Hausdorff space. Then X is σ-compact.*

Proof. Second countable implies Lindelöf (Lecture 22). For each $x \in X$, by local compactness there exists an open neighborhood U_x such that $\overline{U_x}$ is compact. Then $(U_x)_{x\in X}$ is an open cover of X, hence admits a countable subcover $(U_{x_n})_{n\geq 1}$. Therefore

$$X = \bigcup_{n\geq 1} U_{x_n} \subseteq \bigcup_{n\geq 1} \overline{U_{x_n}},$$

and each $\overline{U_{x_n}}$ is compact. Thus X is σ-compact. $\square$

3. Connected locally compact paracompact spaces are σ-compact

Lemma. *Let $(U_\alpha)_{\alpha\in A}$ be a locally finite family of subsets of a space X. Then every compact set $K \subseteq X$ meets only finitely many U_α.*

Proof. For each $x \in K$ choose a neighborhood N_x meeting only finitely many U_α. The family $(N_x)_{x\in K}$ is an open cover of K; choose a finite subcover $N_{x_1}, \ldots, N_{x_m}$. Then K meets only those U_α that meet one of the N_{x_i}, hence finitely many. $\square$

Proposition 98. *Let X be a connected, locally compact, paracompact Hausdorff space. Then X is σ-compact.*

Proof. By local compactness, for each $x \in X$ there exists an open neighborhood U_x with $\overline{U_x}$ compact. The family $(U_x)_{x\in X}$ is an open cover of X. By paracompactness, there exists a locally finite open refinement $(W_\alpha)_{\alpha\in A}$ of $(U_x)_{x\in X}$. For each α, choose $x(\alpha) \in X$ with $W_\alpha \subseteq U_{x(\alpha)}$; then

$$\overline{W_\alpha} \subseteq \overline{U_{x(\alpha)}} \quad \text{is compact.}$$

Fix a point $x_0 \in X$ and choose α_1 with $x_0 \in W_{\alpha_1}$. Define inductively compact sets $K_n \subseteq X$ by

$$K_1 := \overline{W_{\alpha_1}}, \qquad K_{n+1} := \overline{\bigcup_{\alpha\in A_n} W_\alpha}, \quad A_n := \{\alpha \in A : \ W_\alpha \cap K_n \neq \varnothing\}.$$

Since (W_α) is locally finite and K_n is compact, Lemma implies that A_n is finite. Hence $\bigcup_{\alpha\in A_n} W_\alpha$ is a finite union of open sets with compact closures, so K_{n+1} is compact.

We claim that $K := \bigcup_{n\geq 1} K_n$ is both open and closed in X.

(i) K is open. Let $x \in K$. Then $x \in K_n$ for some n. Choose α with $x \in W_\alpha$ (the W_α cover X). Then $W_\alpha \cap K_n \neq \varnothing$, hence $\alpha \in A_n$, so $W_\alpha \subseteq \bigcup_{\beta\in A_n} W_\beta \subseteq K_{n+1} \subseteq K$. Thus K contains an open neighborhood of each of its points, i.e. K is open.

(ii) K is closed. Let $x \in \overline{K}$ and choose α with $x \in W_\alpha$. Then $W_\alpha \cap K \neq \varnothing$, hence $W_\alpha \cap K_n \neq \varnothing$ for some n. Thus $\alpha \in A_n$, so $x \in W_\alpha \subseteq K_{n+1} \subseteq K$. Hence $\overline{K} \subseteq K$, i.e. K is closed.

Since $x_0 \in K$, the set K is a nonempty clopen subset of the connected space X, hence $K = X$. Therefore

$$X = \bigcup_{n\geq 1} K_n$$

is a countable union of compact sets, so X is σ-compact. ☐

Remark. *Without the connectedness assumption, the argument above shows that in a locally compact paracompact Hausdorff space each* connected component *is σ-compact (apply the construction starting from a point in that component).*

4. σ-compact + locally second countable ⇒ second countable

Definition 129 (Locally second countable)**.** A space X is *locally second countable* if every point $x \in X$ admits a second countable neighborhood. Equivalently, every $x \in X$ has an *open* neighborhood U which is second countable in the subspace topology.

Proposition 99. *Let X be σ-compact and locally second countable. Then X is second countable.*

Proof. Write $X = \bigcup_{n\geq 1} C_n$ with each C_n compact. Fix n. For each $x \in C_n$, choose an *open* second countable neighborhood $U_x \ni x$ (possible by Definition 129). Then $(U_x)_{x\in C_n}$ is an open cover of C_n, hence admits a finite subcover $U_{x_{n,1}}, \dots, U_{x_{n,k_n}}$. Set

$$U_n := \bigcup_{j=1}^{k_n} U_{x_{n,j}}.$$

Then U_n is open in X and is second countable: it is a finite union of open second countable subspaces, hence has a countable base given by the union of the (countable) bases of the $U_{x_{n,j}}$.

Now $X = \bigcup_{n\geq 1} U_n$ is a countable union of open second countable subspaces. For each n, let $\mathcal{B}_n$ be a countable base of U_n; since U_n is open, every element of $\mathcal{B}_n$ is open in X. Then

$$\mathcal{B} := \bigcup_{n\geq 1} \mathcal{B}_n$$

is countable and is a base for X: if $O \subseteq X$ is open and $x \in O$, pick n with $x \in U_n$, then $O \cap U_n$ is open in U_n, so there exists $B \in \mathcal{B}_n$ with $x \in B \subseteq O \cap U_n \subseteq O$. Thus X is second countable. ☐

5. Paracompactness versus second countability for manifolds

Theorem 34. Let X be a connected Hausdorff space which is locally Euclidean of some dimension n. Then the following properties are equivalent:

1. X is second countable;
2. X is σ-compact;
3. X is paracompact.

Proof. By Remark , X is locally compact Hausdorff and locally second countable.

(1) $\Rightarrow$ (2): by Lemma .

(2) $\Rightarrow$ (3): since X is locally compact Hausdorff and σ-compact, it is paracompact (Theorem of Lecture 27: locally compact + σ-compact + Hausdorff $\Rightarrow$ paracompact).

(3) $\Rightarrow$ (2): by Proposition 98.

(2) $\Rightarrow$ (1): by Proposition 99 (using that X is locally second countable).

Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). ☐

Corollary (Equivalent definitions of a connected manifold)**.** *Let X be a connected Hausdorff locally Euclidean space. Then the following are equivalent:*

1. *X is a (connected) topological manifold in the sense:* Hausdorff + locally Euclidean + second countable*;*
2. *X is a (connected) topological manifold in the sense:* Hausdorff + locally Euclidean + paracompact.

Proof. This is Theorem 34 with (1) $\Leftrightarrow$ (3). ☐

Remark. *Connectedness is essential if one wishes to replace "second countable" by "paracompact"* without further hypotheses. *For example, the disjoint union of uncountably many copies of $\mathbb{R}^n$ is Hausdorff, locally Euclidean, and paracompact, but it is not second countable.*

Lecture 30–32: Compact–Open Topology

1. The mapping space and the compact–open topology

Notation. *For topological spaces X, Y, write*

$$C(X,Y) := \{f : X \to Y \mid f \text{ continuous}\}.$$

Definition 130 (Subbasic sets)**.** Let X, Y be topological spaces, let $K \subseteq X$ be compact and $U \subseteq Y$ be open. Define

$$M(K,U) \ := \ \{f \in C(X,Y) \mid f(K) \subseteq U\}.$$

Definition 131 (Compact–open topology)**.** The *compact–open topology* on $C(X,Y)$ is the topology generated by the subbasis

$$\mathcal{S} := \{M(K,U) \mid K \subseteq X \text{ compact}, \ U \subseteq Y \text{ open}\}.$$

We denote the resulting topological space by $C(X,Y)_{\mathrm{co}}$ when confusion is possible.

Lemma. *Let $K_i \subseteq X$ be compact and $U_i \subseteq Y$ be open $(1 \le i \le n)$. Then*

$$\bigcap_{i=1}^{n} M(K_i, U_i) \ = \ M\Big(\bigcup_{i=1}^{n} K_i, \ \bigcap_{i=1}^{n} U_i\Big).$$

Proof. For $f \in C(X,Y)$,

$$f \in \bigcap_{i=1}^{n} M(K_i, U_i) \iff (\forall i) \ f(K_i) \subseteq U_i \iff f\Big(\bigcup_i K_i\Big) \subseteq \bigcap_i U_i,$$

which is equivalent to $f \in M(\cup_i K_i, \cap_i U_i)$. □

Remark. *By Lemma , the family of all finite intersections of sets $M(K,U)$ is a basis of the compact–open topology.*

2. Elementary functoriality

Proposition 100 (Evaluation at a point)**.** *For each $x \in X$, the evaluation map*

$$\mathrm{ev}_x : C(X,Y)_{\mathrm{co}} \to Y, \qquad \mathrm{ev}_x(f) := f(x),$$

is continuous.

Proof. For $U \subseteq Y$ open,

$$\mathrm{ev}_x^{-1}(U) = \{f \in C(X,Y) \mid f(\{x\}) \subseteq U\} = M(\{x\}, U),$$

and $\{x\}$ is compact. □

Proposition 101. *Let X, X', Y, Y' be topological spaces.*

1. *If $h : X' \to X$ is continuous, then the precomposition map*

$$h^* : C(X,Y)_{\mathrm{co}} \to C(X',Y)_{\mathrm{co}}, \qquad f \mapsto f \circ h,$$

is continuous.

2. *If $k : Y \to Y'$ is continuous, then the postcomposition map*

$$k_* : C(X,Y)_{\mathrm{co}} \to C(X,Y')_{\mathrm{co}}, \qquad f \mapsto k \circ f,$$

is continuous.

Proof. (1) Let $K \subseteq X'$ be compact and $U \subseteq Y$ open. Then

$$(h^*)^{-1}(M(K,U)) = \{f \in C(X,Y) \mid (f \circ h)(K) \subseteq U\} = \{f \mid f(h(K)) \subseteq U\} = M(h(K),U),$$

and $h(K)$ is compact since h is continuous.
(2) Let $K \subseteq X$ be compact and $U' \subseteq Y'$ open. Then

$$(k_*)^{-1}(M(K,U')) = \{f \mid (k \circ f)(K) \subseteq U'\} = \{f \mid f(K) \subseteq k^{-1}(U')\} = M(K,k^{-1}(U')),$$

which is open since $k^{-1}(U')$ is open in Y. ☐

Proposition 102. *If Y is Hausdorff, then $C(X,Y)_{\mathrm{co}}$ is Hausdorff.*

Proof. Let $f, g \in C(X,Y)$ with $f \neq g$. Choose $x \in X$ with $f(x) \neq g(x)$. Since Y is Hausdorff, pick disjoint open sets $U, V \subseteq Y$ with $f(x) \in U$ and $g(x) \in V$. Then $M(\{x\},U)$ and $M(\{x\},V)$ are disjoint open neighborhoods of f and g. ☐

3. The evaluation map and local compactness

Definition 132 (Global evaluation map)**.** Define

$$\mathrm{ev} : C(X,Y)_{\mathrm{co}} \times X \longrightarrow Y, \qquad (f,x) \longmapsto f(x).$$

Theorem 35 (Continuity of evaluation)**.** Assume that X is locally compact and Hausdorff. Then the evaluation map $\mathrm{ev} : C(X,Y)_{\mathrm{co}} \times X \to Y$ is continuous.

Proof. Let $(f_0,x_0) \in C(X,Y) \times X$ and let $U \subseteq Y$ be open with $f_0(x_0) \in U$. Set $W := f_0^{-1}(U)$, an open neighborhood of x_0 in X. Since X is locally compact Hausdorff, there exists an open neighborhood V of x_0 such that $\overline{V}$ is compact and

$$\overline{V} \subseteq W = f_0^{-1}(U).$$

Then $f_0(\overline{V}) \subseteq U$, i.e. $f_0 \in M(\overline{V},U)$, and clearly $x_0 \in V$. For $(f,x) \in M(\overline{V},U) \times V$ we have $x \in V \subseteq \overline{V}$ and $f(\overline{V}) \subseteq U$, hence $f(x) \in U$. Thus

$$M(\overline{V},U) \times V \subseteq \mathrm{ev}^{-1}(U),$$

which proves continuity at (f_0,x_0). ☐

Remark. *The hypothesis "X locally compact Hausdorff" is precisely the condition under which the compact–open topology is* admissible*: the canonical evaluation map becomes continuous, and the expected exponential law holds (next subsection).*

4. Exponential law (currying/uncurrying)

Notation. *For spaces X, Y, Z, define the set-theoretic bijection*

$$\mathrm{curry} : C(X \times Y, Z) \longrightarrow \mathrm{Maps}\big(X, C(Y,Z)\big), \qquad F \longmapsto \big(x \mapsto (y \mapsto F(x,y))\big).$$

When curry(F) *takes values in $C(Y,Z)$ and is continuous as a map $X \to C(Y,Z)_{\mathrm{co}}$, we denote it by $\widehat{F} : X \to C(Y,Z)_{\mathrm{co}}$. Conversely, given $g : X \to C(Y,Z)$ we define $\widetilde{g} : X \times Y \to Z$ by $\widetilde{g}(x,y) = g(x)(y)$.*

Theorem 36 (Exponential law)**.** Assume that Y is locally compact Hausdorff. Then the assignment $F \mapsto \widehat{F}$ induces a bijection

$$C(X \times Y, Z) \xrightarrow{\ \sim\ } C\big(X, C(Y,Z)_{\mathrm{co}}\big),$$

and this bijection is a homeomorphism for the compact–open topologies on both sides.

Proof. Step 1: set-level bijection. Let $F : X \times Y \to Z$ be continuous. For each $x \in X$, the partial map $F_x : Y \to Z$, $y \mapsto F(x,y)$ is continuous, hence $F_x \in C(Y,Z)$. Define $\widehat{F} : X \to C(Y,Z)$ by $\widehat{F}(x) := F_x$. Conversely, for $g : X \to C(Y,Z)$ define $\widetilde{g} : X \times Y \to Z$ by $\widetilde{g}(x,y) = g(x)(y)$. These constructions are inverse to each other pointwise.

Step 2: $\widehat{F}$ is continuous (and continuity criterion). Let $K \subseteq X$ be compact, $L \subseteq Y$ compact, and $U \subseteq Z$ open. A subbasic open set of $C(X, C(Y,Z)_{\mathrm{co}})_{\mathrm{co}}$ is of the form

$$M\big(K, M(L,U)\big) = \{g : X \to C(Y,Z) \mid g(K) \subseteq M(L,U)\}.$$

For $F \in C(X \times Y, Z)$, one has

$$\begin{aligned} \widehat{F} \in M\big(K, M(L,U)\big) &\iff (\forall x \in K)\ \widehat{F}(x) \in M(L,U) \\ &\iff (\forall x \in K)\ F(\{x\} \times L) \subseteq U \\ &\iff F(K \times L) \subseteq U \\ &\iff F \in M(K \times L, U), \end{aligned}$$

and $K \times L$ is compact. Thus

$$(\mathrm{curry})^{-1}\big(M(K, M(L,U))\big) = M(K \times L, U),$$

so curry is continuous.

Step 3: continuity of the inverse. Let $g \in C(X, C(Y,Z)_{\mathrm{co}})$. Then $\widetilde{g}$ is the composite

$$X \times Y \xrightarrow{(g,\mathrm{id}_Y)} C(Y,Z)_{\mathrm{co}} \times Y \xrightarrow{\mathrm{ev}} Z,$$

where ev is the evaluation map of Definition 132 for the pair (Y,Z). By Theorem 35 (applied to Y), this evaluation map is continuous since Y is locally compact Hausdorff. Hence $\widetilde{g}$ is continuous, and curry^{-1} maps $C(X, C(Y,Z)_{\mathrm{co}})$ into $C(X \times Y, Z)$.

Moreover, to see that curry^{-1} is continuous as a map between mapping spaces, it suffices to compute preimages of subbasic opens: for $K \subseteq X$ compact, $L \subseteq Y$ compact, $U \subseteq Z$ open,

$$(\mathrm{curry}^{-1})^{-1}\big(M(K \times L, U)\big) = M\big(K, M(L,U)\big),$$

by the equivalence in Step 2. Hence curry^{-1} is continuous.

Therefore curry is a homeomorphism. ∎

5. Continuity of composition

Corollary (Continuity of composition)**.** *Assume that X is locally compact Hausdorff. Then the composition map*

$$\circ : \; C(Y,Z)_{\mathrm{co}} \times C(X,Y)_{\mathrm{co}} \longrightarrow C(X,Z)_{\mathrm{co}}, \qquad (f,g) \longmapsto f \circ g,$$

is continuous.

Proof. Consider the map

$$\Psi : \; C(Y,Z)_{\mathrm{co}} \times C(X,Y)_{\mathrm{co}} \times X \to Z, \qquad (f,g,x) \mapsto f(g(x)).$$

By Theorem 35, the evaluation map $\mathrm{ev}_X : C(X,Y)_{\mathrm{co}} \times X \to Y$ is continuous (since X is locally compact Hausdorff), and likewise $\mathrm{ev}_Y : C(Y,Z)_{\mathrm{co}} \times Y \to Z$ is continuous if Y is locally compact Hausdorff. However, we do not need Y locally compact here: for fixed (f,g) the map $x \mapsto g(x)$ is continuous and f is continuous, so Ψ is continuous as the composite

$$C(Y,Z) \times C(X,Y) \times X \xrightarrow{\mathrm{id} \times \mathrm{ev}_X} C(Y,Z) \times Y \xrightarrow{\mathrm{ev}_Y} Z,$$

where ev_Y is continuous by Proposition 100 on points and by the definition of the compact–open topology on $C(Y,Z)$ (using $K = \{y\}$). A direct subbasis computation is even simpler: for $K \subseteq X$ compact and $U \subseteq Z$ open,

$$(f,g) \in (\circ)^{-1}(M(K,U)) \iff f(g(K)) \subseteq U \iff \big(f \in M(g(K),U)\big),$$

and $g \mapsto g(K)$ is controlled by the compact–open topology. Concretely, one checks that $(\circ)^{-1}(M(K,U))$ is open by rewriting it as a union of basic neighborhoods around (f,g); this is standard and follows from the exponential law with X locally compact. ☐

Remark. *A clean Bourbaki proof uses Theorem 36: for X locally compact Hausdorff, the composition map is the transpose of the continuous map*

$$C(Y,Z) \times C(X,Y) \times X \to Z, \quad (f,g,x) \mapsto f(g(x)),$$

and continuity follows formally from admissibility of the compact–open topology.

6. Metrizability for σ-compact LCH domains (optional)

Definition 133 (Hemicompact)**.** A space X is *hemicompact* if there exists an increasing sequence of compact sets $(K_n)_{n\ge 1}$ such that:

$$X = \bigcup_{n\ge 1} K_n, \qquad \text{and every compact } K \subseteq X \text{ is contained in some } K_n.$$

Remark. *If X is locally compact Hausdorff and σ-compact, then X is hemicompact: one may take an exhaustion (U_n) with $K_n := \overline{U_n}$.*

Proposition 103. *Let X be hemicompact, and let (Y,d) be a metric space. Then the compact–open topology on $C(X,Y)$ is metrizable. More precisely, if (K_n) is a hemicompact sequence, the function*

$$D(f,g) := \sum_{n=1}^{\infty} 2^{-n} \min\Big(1, \sup_{x \in K_n} d\big(f(x),g(x)\big)\Big)$$

defines a metric on $C(X,Y)$ inducing the compact–open topology.

Proof. Standard: each term controls uniform convergence on K_n, and hemicompactness ensures that subbasic conditions $f(K) \subseteq U$ are detected on some K_n. $\square$

Remark. *In the metric case, the compact–open topology coincides with the topology of uniform convergence on compact subsets: a net* (f_λ) *converges to* f *iff* $\sup_{x \in K} d(f_\lambda(x), f(x)) \to 0$ *for every compact* $K \subseteq X$*.*

Lecture 33–34: Profinite Topology

1. Directed sets and inverse systems

Definition 134 (Directed poset)**.** A *directed poset* is a partially ordered set $(I, \leq)$ such that for all $i, j \in I$ there exists $k \in I$ with
$$i \leq k, \qquad j \leq k.$$

Definition 135 (Inverse (projective) system)**.** Let $(I, \leq)$ be a directed poset and let C be a category. An *inverse system* (or *projective system*) indexed by I is the data
$$\big((X_i)_{i\in I},\ (p_{ij} : X_j \to X_i)_{i\leq j}\big)$$
where $X_i \in \mathrm{Ob}(\mathsf{C})$ and $p_{ij} \in \mathrm{Hom}_{\mathsf{C}}(X_j, X_i)$ satisfy:

1. $p_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$;
2. $p_{ik} = p_{ij} \circ p_{jk}$ for all $i \leq j \leq k$.

Definition 136 (Compatible cone)**.** Let $\big((X_i), (p_{ij})\big)$ be an inverse system in C. For an object $Y \in \mathsf{C}$, a family of morphisms $(f_i : Y \to X_i)_{i\in I}$ is *compatible* if
$$p_{ij} \circ f_j = f_i \qquad (i \leq j).$$

Definition 137 (Inverse limit)**.** An *inverse limit* of the inverse system $\big((X_i), (p_{ij})\big)$ is an object $X \in \mathsf{C}$ together with a compatible family $(\pi_i : X \to X_i)_{i\in I}$ such that for every $Y \in \mathsf{C}$ and every compatible family $(f_i : Y \to X_i)$ there exists a unique morphism $\psi : Y \to X$ with $\pi_i \circ \psi = f_i$ for all i.
We denote such an inverse limit (when it exists) by
$$X = \varprojlim_{i\in I} X_i.$$

2. Existence in Set and Top

Proposition 104 (Construction of $\varprojlim$ in **Set** and **Top**)**.** *Let $\big((X_i), (p_{ij})\big)$ be an inverse system of sets (resp. topological spaces). Define*
$$\varprojlim_{i\in I} X_i := \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \ \middle|\ p_{ij}(x_j) = x_i \quad \forall\, i \leq j \right\}.$$
In **Set** *this set with projections $\pi_i((x_j)) := x_i$ is an inverse limit. In* **Top***, equipped with the subspace topology inherited from the product $\prod_i X_i$, it is an inverse limit.*

Proof. Let X be the subset above and $\pi_i : X \to X_i$ the restrictions of the product projections. Compatibility is immediate from the defining relations $p_{ij}(x_j) = x_i$.

Given a compatible family $(f_i : Y \to X_i)$, define $\psi : Y \to \prod_i X_i$ by $\psi(y) := (f_i(y))_{i\in I}$. Compatibility gives $p_{ij}(f_j(y)) = f_i(y)$, hence $\psi(Y) \subseteq X$; thus ψ factors uniquely through a map $Y \to X$. Uniqueness is forced by the requirement $\pi_i \circ \psi = f_i$. In **Top** continuity follows because ψ is continuous into the product and X has the subspace topology. $\square$

Remark. *If each X_i is Hausdorff, then $\varprojlim X_i$ is a closed subspace of $\prod_i X_i$ (it is an intersection of inverse images of diagonals). In particular, if each X_i is compact Hausdorff, then $\varprojlim X_i$ is compact Hausdorff.*

3. Profinite spaces

Definition 138 (Profinite space)**.** A topological space X is *profinite* if X is isomorphic in **Top** to an inverse limit of finite discrete spaces.

Definition 139 (Totally disconnected)**.** A topological space X is *totally disconnected* if every connected subset of X is reduced to a point.

Theorem 37 (Characterization of profinite spaces)**.** For a topological space X the following are equivalent:

1. X is profinite;
2. X is compact, Hausdorff, and totally disconnected.

Proof. (1)⇒(2). Assume $X \simeq \varprojlim X_i$ with each X_i finite discrete. Each X_i is compact Hausdorff and totally disconnected. By Tychonoff, $\prod_i X_i$ is compact Hausdorff and totally disconnected; by Remark , X is a closed subspace of $\prod_i X_i$, hence compact Hausdorff. Total disconnectedness is inherited by subspaces, so X is totally disconnected.

(2)⇒(1). Assume X is compact Hausdorff and totally disconnected.

Step 1 (clopen separations). In a compact Hausdorff space, connected components coincide with quasi-components; hence total disconnectedness implies: for any distinct $x \neq y$ there exists a clopen set $C \subseteq X$ with $x \in C$ and $y \notin C$. Consequently, for every open neighborhood $U \ni x$ there exists a clopen neighborhood C with $x \in C \subseteq U$ (obtain clopen $C_y \ni x$ excluding y for each $y \in X \setminus U$ and use compactness of $X \setminus U$ to take a finite intersection).

Step 2 (finite clopen partitions form a directed system). Let $\mathcal{P}$ be the set of all finite partitions

$$P = \{U_{P,1}, \dots, U_{P,n(P)}\}$$

of X into pairwise disjoint nonempty *clopen* subsets whose union is X. For $P, Q \in \mathcal{P}$, write $P \preceq Q$ if Q refines P (i.e. each part of Q is contained in a unique part of P). Then $(\mathcal{P}, \preceq)$ is directed: a common refinement of P and Q is given by the nonempty intersections of parts of P with parts of Q.

For each $P \in \mathcal{P}$, define the finite discrete set

$$I_P := \{1, \dots, n(P)\},$$

and define $q_P : X \to I_P$ by $q_P(x) = i$ iff $x \in U_{P,i}$. Since the fibers $U_{P,i}$ are clopen, q_P is continuous.

If $P \preceq Q$, define $\pi_{PQ} : I_Q \to I_P$ by the rule: $\pi_{PQ}(j) = i$ if $U_{Q,j} \subseteq U_{P,i}$. Then $\pi_{PP} = \mathrm{id}$ and $\pi_{PR} = \pi_{PQ} \circ \pi_{QR}$ for $P \preceq Q \preceq R$, so $\big((I_P), (\pi_{PQ})\big)$ is an inverse system. Moreover, (q_P) is compatible: $\pi_{PQ} \circ q_Q = q_P$.

Step 3 (identify X with the inverse limit). By the universal property, the compatible family (q_P) induces a unique continuous map

$$\theta : X \longrightarrow \varprojlim_{P \in \mathcal{P}} I_P$$

such that the P-coordinate equals q_P.

Injective. If $x \neq y$, by Step 1 choose a clopen set C with $x \in C$, $y \notin C$. Then $P = \{C, X \setminus C\} \in \mathcal{P}$ and $q_P(x) \neq q_P(y)$, hence $\theta(x) \neq \theta(y)$.

Surjective. Let $(i_P)_{P\in\mathcal{P}} \in \varprojlim I_P$. For each P consider the nonempty clopen set $U_{P,i_P} \subseteq X$. If $P \preceq Q$, compatibility implies $U_{Q,i_Q} \subseteq U_{P,i_P}$. Hence the family $\{U_{P,i_P}\}_{P\in\mathcal{P}}$ has the finite intersection property (in fact it is filtered by reverse inclusion). By compactness of X,

$$\bigcap_{P\in\mathcal{P}} U_{P,i_P} \neq \varnothing.$$

Pick x in this intersection. Then $q_P(x) = i_P$ for all P, hence $\theta(x) = (i_P)$.

Homeomorphism. The space $\varprojlim I_P$ is compact Hausdorff (inverse limit of compact Hausdorff spaces). Since X is compact and θ is a continuous bijection from a compact space onto a Hausdorff space, θ is a homeomorphism.

Thus X is profinite. ☐

Remark. *Compact Hausdorff totally disconnected spaces are also called* Stone spaces. *The Boolean algebra of clopen subsets of X controls the inverse system above (Stone duality).*

4. Profinite groups as inverse limits

Definition 140 (Profinite group)**.** A *profinite group* is a topological group G which is an inverse limit of finite groups endowed with the discrete topology. Equivalently, G is a compact Hausdorff totally disconnected topological group.

Proposition 105. *Let $\big((G_i),(p_{ij})\big)$ be an inverse system of (discrete) groups and group homomorphisms. Then $G := \varprojlim G_i$ is a subgroup of $\prod_i G_i$ (with coordinatewise operations), and with the subspace topology it is a topological group. If each G_i is finite, then G is profinite.*

Proof. The defining relations $p_{ij}(x_j) = x_i$ are preserved under coordinatewise multiplication and inversion because each p_{ij} is a homomorphism, hence $\varprojlim G_i$ is a subgroup of the product. Continuity of multiplication and inversion follows from continuity in the product and the subspace topology. If each G_i is finite discrete, then each G_i is compact Hausdorff totally disconnected; hence so is G. ☐

Proposition 106 (Neighborhood basis, density, and continuity)**.** *Let $G = \varprojlim_{i\in I} G_i$ be an inverse limit of finite (discrete) groups and let $\pi_i : G \to G_i$ be the projections. Then:*

1. *The family $\{\ker(\pi_i)\}_{i\in I}$ is a neighborhood basis of 1_G. In particular, each $\ker(\pi_i)$ is an open normal subgroup of G.*
2. *A subset $D \subseteq G$ is dense in G if and only if $\pi_i(D) = \pi_i(G)$ for all $i \in I$. (If all bonding maps are surjective, then $\pi_i(G) = G_i$ and the condition becomes $\pi_i(D) = G_i$ for all i.)*
3. *Let H be a topological group and $f : H \to G$ a group homomorphism. Then f is continuous if and only if $\pi_i \circ f : H \to G_i$ is continuous for all $i \in I$.*

Proof. (1) Since each G_i is discrete, $\{1\} \subseteq G_i$ is open, hence $\pi_i^{-1}(\{1\}) = \ker(\pi_i)$ is open in G. These sets form a neighborhood basis at 1_G because the topology on G is the initial topology for the projections (π_i).

(2) Since each G_i is finite discrete, the basic open sets in G are finite intersections of sets of the form $\pi_i^{-1}(\{g\})$. Thus D is dense iff it meets every nonempty $\pi_i^{-1}(\{g\})$ with $g \in \pi_i(G)$, which is equivalent to $\pi_i(D) = \pi_i(G)$.

(3) The topology on G is the initial topology for (π_i), so f is continuous iff each $\pi_i \circ f$ is continuous. ☐

5. Profinite topology on a group and profinite completion

Definition 141 (Profinite topology on an abstract group)**.** Let G be a group. Let $\mathcal{N}$ be the set of normal subgroups $N \trianglelefteq G$ of finite index. The *profinite topology* on G is the group topology for which $\mathcal{N}$ is a neighborhood basis of 1_G (equivalently, a basis at $g \in G$ is given by cosets gN with $N \in \mathcal{N}$).

Remark. *The profinite topology on G is Hausdorff if and only if*

$$\bigcap_{N \in \mathcal{N}} N = \{1\},$$

i.e. G is residually finite.

Definition 142 (Profinite completion)**.** Let G be a group and let $\mathcal{N}$ be the directed set of finite-index normal subgroups ordered by reverse inclusion:

$$N \leq M \iff N \supseteq M.$$

For $N \leq M$ define $p_{NM} : G/M \to G/N$ by the natural quotient map. The *profinite completion* of G is the inverse limit

$$\widehat{G} := \varprojlim_{N \in \mathcal{N}} G/N,$$

endowed with the inverse limit topology (hence profinite). The canonical homomorphism

$$\iota_G : G \to \widehat{G}, \qquad g \mapsto (gN)_{N \in \mathcal{N}},$$

is called the *completion map.*

Proposition 107 (Universal property of $\widehat{G}$)**.** *Let G be a group equipped with its profinite topology, and let H be a profinite group. For every continuous homomorphism $\phi : G \to H$ there exists a unique continuous homomorphism*

$$\widehat{\phi} : \widehat{G} \to H$$

such that $\widehat{\phi} \circ \iota_G = \phi$.

Proof. Write $H \simeq \varprojlim_{i \in I} H_i$ with H_i finite discrete and let $\pi_i : H \to H_i$ be the projections. Each $\pi_i \circ \phi : G \to H_i$ is continuous; since H_i is finite discrete, its kernel is an open normal subgroup of finite index in G. Hence $\pi_i \circ \phi$ factors uniquely through G/N for some $N \in \mathcal{N}$. By the universal property of the inverse limit $\widehat{G} = \varprojlim_{N \in \mathcal{N}} G/N$, the compatible family of these factorizations induces a unique continuous homomorphism $\widehat{\phi} : \widehat{G} \to H$ with $\widehat{\phi} \circ \iota_G = \phi$. □

Proposition 108 (Functoriality)**.** *If $\phi : G \to H$ is a group homomorphism, then there exists a unique continuous homomorphism*

$$\widehat{\phi} : \widehat{G} \to \widehat{H}$$

such that $\widehat{\phi} \circ \iota_G = \iota_H \circ \phi$. Moreover, the assignment $G \mapsto \widehat{G}$ is functorial.

Proof. Apply Proposition 107 with $H = \widehat{H}$ and ϕ followed by ι_H. Uniqueness gives functoriality. □

Remark. *The image $\iota_G(G)$ is dense in $\widehat{G}$ (by Proposition 106(2), since it surjects onto each finite quotient).*

6. Examples: $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$

Example (p-adic integers)**.** *Fix a prime p. Consider the inverse system*

$$\cdots \longrightarrow \mathbb{Z}/p^{n+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z}/p\mathbb{Z},$$

where the bonding maps are reduction modulo p^n. The inverse limit

$$\mathbb{Z}_p := \varprojlim_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z}$$

is a profinite ring (hence a profinite group under addition), called the ring of p-adic integers*. Concretely,*

$$\mathbb{Z}_p = \left\{ (x_n)_{n\geq 1} \in \prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z} \;\middle|\; x_{n+1} \equiv x_n \pmod{p^n} \right\}.$$

Example (Profinite integers)**.** *Let $I = \mathbb{N}_{\geq 1}$ partially ordered by divisibility: $m \leq n$ iff $m \mid n$. Consider the inverse system $(\mathbb{Z}/n\mathbb{Z})_{n\in I}$ with bonding maps $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ for $m \mid n$ given by reduction. Then*

$$\widehat{\mathbb{Z}} := \varprojlim_{n\geq 1} \mathbb{Z}/n\mathbb{Z}$$

is a profinite ring, called the ring of profinite integers*.*

Moreover,

$$\widehat{\mathbb{Z}} \;\cong\; \prod_{p \text{ prime}} \mathbb{Z}_p$$

as profinite rings (and hence as profinite groups).

Sketch. For each $n = \prod_p p^{v_p(n)}$, the Chinese remainder theorem gives

$$\mathbb{Z}/n\mathbb{Z} \;\cong\; \prod_p \mathbb{Z}/p^{v_p(n)}\mathbb{Z}$$

(finite product). Passing to inverse limits over n (ordered by divisibility) separates the primes and yields the stated product decomposition. ☐

7. Topological finite generation and strong completeness

Definition 143 (Topologically finitely generated)**.** A profinite group G is *topologically finitely generated* if there exists a finite subset $S \subseteq G$ such that

$$\overline{\langle S\rangle} = G,$$

where $\langle S\rangle$ denotes the abstract subgroup generated by S.

Theorem 38 (Nikolov–Segal (strong completeness))**.** Let G be a topologically finitely generated profinite group and let $H \leq G$ be a subgroup of finite index (as an abstract subgroup). Then H is open (hence closed) in G. Equivalently, every abstract homomorphism from G to a finite group is automatically continuous.

Remark. *This theorem is a deep rigidity statement: for finitely generated profinite groups, the topology is largely determined by the abstract group structure. It is frequently used to justify that "finite index" and "open" may be treated interchangeably in this class.*