Notes on MATH 535 Fall 2025

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Contents

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Lecture 1: Topology on Metric Spaces

Definition 1 (Euclidean distance)**.** The Euclidean distance $\forall x, y \in \mathbb{R}^n$ is given by

$$d(x,y) := \Big(\sum_{i=1}^{n}(x_i - y_i)^2\Big)^{1/2} \tag{0.1}$$

The generalization of the idea of distance in $\mathbb{R}$ gives the general definition of metric space.

Definition 2 (Metric space)**.** A set X is said to be a metric space if there exists a map $d : X \times X \to \mathbb{R}$ such that $\forall x, y, z \in X$

1. (Symmetry) $d(x,y) = d(y,x)$
2. (Nondegenrate) $d(x,y) \geq 0$, and $d(x,y) = 0 \iff x = y$
3. (Triangule inequality) $d(x,y) + d(y,z) \geq d(x,z)$

The map d is said to be a metric on X

Definition 3 (Open ball)**.** The open ball in the metric space (X,d) of radius r at $x \in X$ is $B_r := \{y \in X | d(x,y) < r\}$.

For open balls, we have the following lemma.

Lemma. *Let (X,d) be a metric space for some $x \in X$ and $r \in [0,\infty)$, consider open ball $B_r(x)$. Then $\forall y \in B : \exists \rho > 0 : B_\rho(y) \subseteq B_r(x)$.*

Proof. Since $y \in B_r(x)$, $d(x,y) \leq r$. Let $\rho = r - d(x,y) \geq 0$. Then take arbitrarily $z \in B_\rho(y)$, $d(y,z) \leq \rho = r - d(x,y)$, which implies $d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \rho = r$. Thus, $z \in B_r(x)$. Thus, $B_\rho(y) \subseteq B_r(x)$. $\square$

The lemma inspires the following definition.

Definition 4 (Open set in metric space)**.** A subset $U \subseteq X$ is open if $\forall x \in U : \exists r > 0 : B_r(x) \subseteq U$.

An obvious result of this definition is that $\emptyset$, X, and any open ball in X are open.

Lemma. *Let (X,d) be a metric space, we have the following results:*

1. *If $U, V \subseteq X$ is open, then $U \cap V$ is open.*
2. *If $\{U_\alpha\}_{\alpha \in A}$ is an arbitrary collection of open sets, then $\bigcup_{\alpha \in A} U_\alpha$ is open.*

Proof. HW1 Problem 1. $\square$

The properties of these properties can be generalized to the following. idea of an abstract topology.

Lecture 2: Topology

Definition 5 (Topological space)**.** A topological space $(X, \mathcal{T}_X)$ is a set X with a topology $\mathcal{T}_X \subseteq \mathcal{P}(X)$ such that

1. $\emptyset, X \in \mathcal{T}_X$
2. For some index set I and $\alpha \in I$, $\forall U_\alpha \in \mathcal{T}_X : \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}_X$
3. $\forall U, V \in \mathcal{T}_X : U \cap V \in \mathcal{T}_X$

The element of $\mathcal{T}_X$ is called opensets.

Definition 6 (Claosed Sets)**.** A subset $C \subseteq X$ of the topological space $(X, \mathcal{T})$ is said to be closed if $X \backslash C \in \mathcal{T}$.

Remark. *A set can be open and closed at the same time, and can also be neither open nor closed.*

Metrc on any metrc space induces a topology which contains all open sets in the sense of a metric space. An important example is the standard topology in Euclidean space:

Definition 7 (Standard Topology on $\mathbb{R}$)**.** The standard topology $\mathcal{T}_{\text{standard}}$ on $\mathbb{R}^n$ is topology induced by the Euclidean metric.

Also, for any nonempty set X has two trivial topologies.

Definition 8 (Trivial Topologies)**.** The trivial topology on the nonempty set X is given by

- The indiscrete topology of X is defined by $\mathcal{T}_{\min} := \{\emptyset, X\}$.
- The discrete topology of X is defined by $\mathcal{T}_{\max} := \mathcal{P}(X)$.

Lemma. *The discrete topology is the metric topology given by the metric $d : X \times X \to [0, +\infty)$,*

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases} \tag{0.2}$$

Proof. Check the open ball given by this metric: $B_1(x) := \{y \in X | d(x, y) < 1\} = \{x\}$. Thus, it is obvious that the metric gives the discrete topology. $\square$

Definition 9 (Hausdorff)**.** A topological space $(X, \mathcal{T})$ is Hausdorff if $\forall x, y \in X : \exists U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Lemma. *All metric spaces are Hausdorff.*

Proof. Take $x, y \in X$ such that $x \neq y$, then $r = d(x, y) > 0$. Take $\rho = \frac{r}{2}$. If $B_\rho(x) \cap B_\rho(y) \neq \emptyset$, $\exists z \in X$ such that $d(x, z) < r/2$ and $d(y, z) < r/2$. However, that indecates $d(x, z) + d(y, z) < d(x, y)$, which controdicts with triangle inequality. Thus, $B_\rho(x) \cap B_\rho(y) = \emptyset$ $\square$

Remark. *Not all topologies can be induced by a metric. A counterexample is given by the following topology called the cofinite topology on $\mathbb{R}$: $\mathcal{T} := \{\emptyset\} \cup \{U \subseteq \mathbb{R}\}$ such that $\mathbb{R} \backslash U$ is finite. Check this is a topology:*

- *$\emptyset \in \mathcal{T}$ and $\mathbb{R} \in \mathcal{T}$ since $\mathbb{R}\backslash\mathbb{R} = \emptyset$ is finite*
- *$\forall U, V \in \mathcal{T}$, $\mathbb{R}\backslash(U \cap V) = (\mathbb{R}\backslash U) \cup (\mathbb{R}\backslash V)$. Since both U and V are finite, $\mathbb{R}\backslash(U \cap V)$ is finite. So $U \cap V \in \mathcal{T}$*
- *If $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, $U_\alpha \neq \emptyset$.*
$$\mathbb{R}\backslash\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcap_{\alpha \in A} (\mathbb{R}\backslash U_\alpha)$$
Since the intersection of finite sets are always finite, $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

Thus, $\mathcal{T}$ is a topology.

Lemma. *No metric on $\mathbb{R}$ gives a cofinite topology $\mathcal{T}$*

Proof. The space is not Hausdorff. □

Theorem 1 (Defined Topology via Closed Sets)**.** We can defined the topology on set X vis the collection of all closed subsets in X, denote as $\mathcal{T}^c$, which satisfy:

- $\emptyset, X \in \mathcal{T}^c$
- $\forall C_1, C_2 \in \mathcal{T}^c \implies C_1 \cap C_2 \in \mathcal{T}^c$
- $\forall \{C_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}^c : \bigcap_{\alpha \in A} C_\alpha \in \mathcal{T}^c$

Proof. HW1. □

Example. *There are two important counterexamples to show the infinite intersection of open sets and the infinite union of closed sets may not be open/closed.*

- $\bigcap_{n \in N} \left(\frac{1}{n}, \frac{1}{n}\right) = \{0\}$
- $\bigcup_{n \in N} \left[\frac{1}{n}, 2 - \frac{1}{n}\right] = (0, 2)$

Lecture 3: Continuous Map

Recall that $f:\mathbb{R}\to\mathbb{R}$ is continuous if

$$\forall\epsilon>0:\exists\delta>0:\big(|x-x_0|<\delta\implies|f(x)-f(x_0)|<\epsilon\big)$$

And the absolute value is a metric on $\mathbb{R}$. For more general metric spaces, we have the following definition:

Definition 10 (Continuous Map between Metric Spaces)**.** let (X,d_X) and (Y,d_Y) be metric spaces, then the map $f:X\to Y$ is continuous if

$$\forall\epsilon>0:\exists\delta>0:\big(d(x,x_0)<\delta\implies d(f(x),f(x_0))<\epsilon\big)$$

Definition 11 (Continuous Map between Topological Spaces)**.** Let $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ be topological spaces, then the map $f:X\to Y$ is continuous if $\forall U\in\mathcal{T}_Y:f^{-1}(U)\in\mathcal{T}_X$

Lemma. *The definitions of continuity in metric space are equivalent to $B_\epsilon(x_0)\subseteq f^{-1}\big(B_\epsilon(f(x_0))\big)$.*

Proof. For a continuous map $f:X\to Y$, by the definition of continuous in a metric space, $x\in B_\delta(x_0)\iff f(x)\in B_\epsilon\big(f(x_0)\big)$, then $x\in f^{-1}\big(B_\epsilon\big(f(x_0)\big)\big)$. □

Theorem 2. The two definitions of continuity above are consistent.

Proof. (Metric Continuous ⇒ Topological Continuous) If $f:X\to Y$ is metric continuous, then suppose $U\in\mathcal{T}_x$, $f^{-1}(U)\neq\emptyset$, $f^{-1}(U)$ is an open set. By the fact that U is open, $\forall f(x_0)\in U$, $\exists\epsilon>0:B_\epsilon(f(x_0))\subseteq U$. Since f is metric continuous, by the lemma, $B_\epsilon(x_0)\subseteq f^{-1}\big(B_\epsilon(f(x_0))\big)\subseteq f^{-1}(U)$. Thus, $f^{-1}(U)$ continuous, f is topological continuous.
(Metric Continuous ⇐ Topological Continuous) Suppose $\forall U\in\mathcal{T}_Y$, $f^{-1}(U)\in\mathcal{T}_x$. Let $x_0\in X$, $\forall\epsilon>0$, then $B_\epsilon(f(x_0))\subseteq Y$ is open. By assumption $f^{-1}\big(B_\epsilon(f(x_0))\big)$ is open in X, which implies $\forall x_0\in f^{-1}\big(B_\epsilon(f(x_0))\big)$: $\exists\delta>0:B_\delta(x_0)\subseteq f^{-1}\big(B_\epsilon(f(x_0))\big)$. By the lemma, f is metric continuous. □

Now, suppose X has two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$.

Definition 12 (Coaser Topology and Finer Topology)**.** If $\mathcal{T}_1\subseteq\mathcal{T}_2$,

- $\mathcal{T}_1$ is smaller/coaser/weaker than $\mathcal{T}_2$.
- $\mathcal{T}_2$ is bigger/finer/stronger than $\mathcal{T}_1$.

Lemma. *Let $(X,\mathcal{T}_X)$ be a topological space, $Y\subseteq X$. Then $\mathcal{T}_Y:=\{U\subset Y|\exists\tilde{U}\in\mathcal{T}_X,U=Y\cap\tilde{U}\}$ is the coarsest topology on Y such that the inclusion map $i:Y\to X$, $i(y)=y$ is continuous.*

Proof. First, we check the this is indeed a topology.

- $\emptyset=Y\cap\emptyset\in\mathcal{T}_\mathcal{Y}$, and $Y=Y\cap X\in\mathcal{T}_Y$
- $\forall U,V\in\mathcal{T}_Y:\exists\bar{U},\bar{V}\in\mathcal{T}_X:U=Y\cap\bar{U},V=Y\cap\bar{V}$ which implies

  $$U\cap V=(Y\cap\bar{U})\cap(Y\cap\bar{V})=Y\cap(\bar{U}\cap\bar{V})$$

  since $\bar{U},\bar{V}\in\mathcal{T}_X$, $U\cap V\in\mathcal{T}_Y$.

- Given $\{U_\alpha\}_{\alpha\in A} \subseteq \mathcal{T}_Y : \forall \alpha \in A : \exists \bar{U}_\alpha \in \mathcal{T}_X : U_\alpha = Y \cap \bar{U}_\alpha$, then

$$\bigcup_{\alpha\in A} U_\alpha = \bigcup_{\alpha\ \in A} (\bar{U}_\alpha \cap Y) = \Big(\bigcup_{\alpha\ \in A} \bar{U}_\alpha\Big) \cap Y$$

  by the fact that $\bigcup_{\alpha\ \in A} \bar{U}_\alpha \in \mathcal{T}_X$, $\bigcup_{\alpha\in A} U_\alpha \in \mathcal{T}_Y$.

Thus, the subset topology $\mathcal{T}_Y$ is indeed a topology. Then, we need to show that it is the smallest topology that makes the inclusion map $i = \mathrm{Id}\big|_Y : Y \to X$ to be continuous. [***Derek: finish the proof***] $\square$

Lecture 4: Topological Basis

Definition 13 (Topological Basis)**.** Given a topological space $(X, \mathcal{T}_X)$, a basis is a subset $\mathcal{B} \subseteq \mathcal{T}_X$ such that

$$\forall U \in \mathcal{T}_X : \exists \{B_i\}_{i \in I} \subseteq \mathcal{B} : U = \bigcup_{i \in I} B_i$$

Example. *An basis for $\mathbb{R}^n$ with standard topology, is the set of all open balls in $\mathbb{R}^n$.*

Lemma. *Consider maps between topological spaces $f : X \to Y$ and $g : Y \to Z$. f, g continuous implies $g \circ f$ continuous.*

Proof. By the continuity of g, $\forall U \subseteq Z$ open, $g^{-1}(U) \subseteq Y$ open; and by the continuity of f, since $g^{-1}(U)$ is open, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \subseteq X$ is open. Thus, $f \circ g$ is continuous. □

Lemma. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, $\mathcal{B}$ is a basis for $\mathcal{T}_Y$. Then the following propositions are equivalent:*

1. *$f : X \to Y$ continuous.*
2. *$\forall B \in \mathcal{B} : f^{-1}(B) \in \mathcal{T}_X$*

Proof. $(1 \Rightarrow 2)$ Trivial.
$(2 \Rightarrow 1)$ Suppose $\forall B \in \mathcal{B} : f^{-1}(B)$ open, then for any $U \in \mathcal{T}_Y : \exists \{B_i\}_{i \in I} \subset \mathcal{B} : U = \bigcup_{i \in I} B_i$

$$f^{-1}(U) = \bigcup_{i \in I} f^{-1}(B) \in \mathcal{T}_X$$

Thus, $f : X \to Y$ continuous. □

Theorem 3. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfy the following conditions

1. $\bigcup \mathcal{B} = X$
2. $\forall B_1, B_2 \in \mathcal{B}$ then $B_1 \cap B_2$ is a union of elements of $\mathcal{B}$

Then, $\mathcal{T} := \{U \subseteq X | U = \bigcup \mathcal{A}\}$ for some $\mathcal{A} \subseteq \mathcal{B}$ is a topology on X generated by the basis $\mathcal{B}$.

Proof. To ensure $\mathcal{B}$ is a basis, we first need to check that $\mathcal{T}$ is a topology.

- $\emptyset = \bigcup \emptyset \in \mathcal{T}$, $X = \bigcup \mathcal{B} \in \mathcal{T}$
- $\forall U, V \in \mathcal{T} : \exists \mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{B} : U = \bigcup \mathcal{B}_1, V = \bigcup \mathcal{B}_2$. Then

  $$U \cap V = \bigcup (\mathcal{B}_1 \cap \mathcal{B}_2)$$

  By the second condition, it is still being the union of element in $\mathcal{B}$. Thus $U \cap V \in \mathcal{T}$
- $\forall \{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T} : \forall U_\alpha \in \{U_\alpha\}_{\alpha \in A} : \exists \mathcal{B}_\alpha :\subseteq \mathcal{B}$. Then

  $$\bigcup_{\alpha \in A} U_\alpha = \bigcup \Big(\bigcup_{\alpha \in A} \mathcal{B}_\alpha \Big)$$

  By the second assuption, $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

Thus, $\mathcal{T}$ is indeed a topology and generated by the basis $\mathcal{B}$. $\square$

Definition 14 (Subbasis)**.** Let $(X, \mathcal{T})$ be a topological space, the subset $S \subseteq \mathcal{T}$ is a subbasis of $\mathcal{T}$ if $\forall U \in \mathcal{T}$ is a union of finite intersections of elements in S. Then,

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{k} s_{a_i} | k \in \mathbb{N}, s_{a_i} \in S \right\}$$

is a basis of $\mathcal{T}$.

Corollary 1. *Let X be a set, $S \subseteq \mathcal{P}(X)$. If $\bigcup S = X$. Then exists a unique coarsest topology $\mathcal{T}_S$ that contains S such that S is the subbasis of $\mathcal{T}_S$.*

Lecture 5-6: Homeomorphisms, Product/Coproduct Topology

Definition 15 (Homeomorphism)**.** A continuous map $f : X \to Y$ is a homeomorphism if f is bijective and has a continuous inverse $f^{-1} : Y \to X$. The topological spaces X, Y are said to be homeomorphic.

Remark. *f continuous does not imply f^{-1} continuous. An example is given by taking $I = (0, 2\pi]$ with the inherent topology from $\mathbb{R}$ and considering the map*

$$\begin{aligned} f : [0, 2\pi) &\to S^1 \\ \theta &\mapsto (\cos\theta, \sin\theta) \end{aligned}$$

is continuous (we already proved that the inherent topology is the coarsest topology such that the inclusion map is continuous). However, the inverse is not continuous since the preimage of $[0, \pi/2)$ is the curve between $(1, 0)$ (included) and $(0, 1)$ (not included). This arc is not open on S^1.

Definition 16 (Categorical Product)**.** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y$ be topological spaces, then given any maps $f_X : Z \to X$, $f_Y : Z \to Y$ on set Z, $f : Z \to X \times Y$ is the unique map such that we consider the projection

$$\begin{aligned} p_X : X \times Y &\to Y, \\ (x, y) &\mapsto x \end{aligned} \qquad\qquad \begin{aligned} p_Y : X \times Y &\to Y \\ (x, y) &\mapsto y \end{aligned}$$

we have $p_x \circ f = f_X$ and $p_Y \circ f = f_Y$, i.e. the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2 \end{tikzcd}$$

Definition 17 (Product Topology)**.** The product topology $\mathcal{T}_{X \times Y}$ is the coarsest topology on $X \times Y$ that

- p_X, p_Y continuous
- For topological space $(Z, \mathcal{T}_Z)$, and the following continuous map

$$f_X : (Z, \mathcal{T}_Z) \to (X, \mathcal{T}_X) \tag{0.3}$$
$$f_Y : (Z, \mathcal{T}_Z) \to (Y, \mathcal{T}_Y) \tag{0.4}$$

  The unique map $f = (f_X, f_Y)$ is continuous.

The construction of product topology is the following process. Notice that the preimage of the projection is given by

$$\forall U \in \mathcal{T}_X : p_X^{-1}(U) = U \times Y$$
$$\forall V \in \mathcal{T}_Y : p_Y^{-1}(V) = X \times V$$

Thus, a subbasis of the space is given by

$$S = \{U \times Y | U \in \mathcal{T}_X\} \cup \{X \times V | V \in \mathcal{T}_Y\}$$

Then a basis is given by

$$\mathcal{B} = \{U \times V | U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

Proposition 1. *Under the topology given by the basis above, the projection p_X, p_Y and the unique map $f = (f_X, f_Y)$ are continuous for any continuous map f_X, f_Y.*

Proof. [**Derek: finish the proof**] □

Definition 18 (Arbitrary Product Space)**.** Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of sets, there exists a set $\prod_{\alpha \in A} X_\alpha$ and a collection of maps $\{p_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta | \beta \in A\}$ such that for all $Z \in$ **Set** and any collection $\{f_\beta : Z \to X_\beta | \beta \in A\}$, $\exists! f : Z \to \prod_{\alpha \in A} X_\alpha$ such that $p_\beta \circ f = f_\beta$.

To construct a topology on this arbitrary product, take

$$\prod_{\alpha \in A} X_\alpha = \{g : A \to \bigcup_{\alpha \in A} X_\alpha | g(\alpha) \in X_\alpha\}$$

and $p_\beta(g) = g(\beta)$. In general, take $g_\alpha := g(\alpha)$, $\{g_\alpha\}_{\alpha \in A} = g \in \prod_{\alpha \in A} X_\alpha$

Theorem 4 (Product Topology)**.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ be a collection of topological spaces, then $\exists \mathcal{T}_{\text{prod}}$ be a topology on $\prod_{\alpha \in A} X_\alpha$ such that

- $p_\beta : (\prod_{\alpha \in A} X_\alpha, \mathcal{T}_{\text{prod}}) \to (X_\beta, \mathcal{T}_\beta)$ is continuous for any $\beta \in A$
- for any topological space $(Z, \mathcal{T}_Z)$ and the collection of continuous maps $\{f_\alpha : (Z, \mathcal{T}_Z) \to (X_\alpha, \mathcal{T}_\alpha)\}$, the unique map
$$f : (Z, \mathcal{T}_Z) \to \Big(\prod_{\alpha \in A} X_\alpha, \mathcal{T}_{\text{prob}}\Big)$$
is continuous.

Proof. Let $S = \{p_\alpha^{-1}(U_\alpha) | \alpha \in A, U_\alpha \in \mathcal{T}_\alpha\}$, the following claims are true

- $(x_\alpha)_{\alpha \in A} \in p_\beta^{-1}(U_\alpha) \iff x_\beta \in U \quad \forall \beta \in A$
- $\bigcup S = \prod_{\alpha \in A} X_\alpha$

Thus, S is a subbasis of a topology $\mathcal{T}_{\text{prod}}$. Then, given $\alpha_q, \dots, \alpha_n \in A$, $U_\alpha \in \mathcal{T}_\alpha$, $\forall i \in \{1, \dots, n\}$, the set of

$$\bigcap_i p_{\alpha_i}^{-1}(U_{\alpha_i})$$

with $\big\{X_\alpha | \alpha \in A, X_{\alpha_i} \in U_{\alpha_i}\big\}$ is a basis of $\mathcal{T}$. Then, $\forall \beta \in A$, p_β is obviously continuous. With the topological space $(Z, \mathcal{T}_Z)$ and the collection of continuous maps $\{f_\alpha : (Z, \mathcal{T}_Z) \to (X_\alpha, \mathcal{T}_\alpha)\}$, the

$$\begin{aligned} f^{-1}\big(p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})\big) &= \bigcap_{i=1}^n f^{-1}\big(p_{\alpha_i}^{-1}(U_{\alpha_i})\big) \\ &= \bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_{\alpha_i}) \quad \text{is open} \end{aligned}$$

Thus, the function f is continuous. □

With reversing all arrows in the definition of product (taking the dual category), we obtain the coproduct.

Definition 19 (Coproduct/Disjoint Union)**.** The coproduct of X_1, X_2 is defined by the following commutative diagram:

$$\begin{tikzcd} & Y & \\ X_1 \arrow[ur, "f_1"] \arrow[r, "i_1"'] & X_1 \coprod X_2 \arrow[u, dashed, "f"] & X_2 \arrow[ul, "f_2"'] \arrow[l, "i_2"] \end{tikzcd}$$

An explicit construction is given by $\coprod_{\alpha \in A} X_\alpha = \bigcup_{\alpha \in A}(X_\alpha \times \{\alpha\})$, where the inclusion map is given by $i_\alpha(x) = (x, \alpha) \forall \alpha \in X_\alpha$.

$$\begin{aligned} f : \bigcup_{\alpha \in A} (X_\alpha \times \{\alpha\}) &\to Z \\ (x, \alpha) &\mapsto f_\alpha(x) \end{aligned}$$

Definition 20 (Coproduct Topology)**.** The coproduct topology is given by letting $U \cap X_\alpha$ open in $X_\alpha \forall \alpha \in A$. The topology is then given by

$$S = \Big\{ \coprod_{\alpha \in A} U_\alpha | U_\alpha \in X_\alpha \text{ open} \Big\}$$

Theorem 5. Under coproduct topology, inclusion maps are continuous and for the collection of continuous maps. $\{h_\alpha : X_\alpha \to Z\}$, the induced map

$$h : \Big(\coprod_{\alpha \in A} X_\alpha, \mathcal{T}_{\text{coprod}} \Big) \to (Z, \mathcal{T}_Z)$$

such that $h \circ i_\alpha = h_\alpha$.

Lectur 7: Open and Closed Maps, Quotient Topology

Definition 21 (Open/Closed Maps)**.** The continuous map $f : X \to Y$ is open (closed) if the image of open (closed) sets is open (closed)

Theorem 6. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ is a collection of topological spaces, then $\forall \beta \in A$

$$p_\beta \prod_{\alpha \in A} X_\alpha \to X_\eta$$

is an open map.

We shall prove the theorem using the following lemma:

Lemma. *The continuous map $f : X \to Y$ is open if for a basis $\mathcal{B}_X \subset \mathcal{T}_X$, $\forall B \in \mathcal{B}_X$, $f(B) \in \mathcal{T}_Y$*

Proof. The proof is simple. □

Then, we can prove the theorem.

Proof. As a basis, consider $\mathcal{B} = \{p_{\alpha_q}^{-1}(U_{\alpha_1}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n}) | \forall U_{\alpha_i} \in \mathcal{T}_{\alpha_1}\}$. With $(x_\alpha)_{\alpha \in A}$, $p_\beta\big((x_\alpha)_{\alpha \in A}\big) = x_\beta$. Thus,

$$p_\beta\Big(\bigcap_{1 \le i \le r} p_\alpha^{-1}(U_{\alpha_i})\Big) = \bigcap_{1 \le i \le r} U_i \text{ open}$$

Thus, the projection is an open map. □

However, in general, the projection is not closed.

A quotient map is on the equivalence relation $\sim$ on X is given by

$$\begin{aligned} q : X &\to X/\sim \\ x &\mapsto [x] \end{aligned}$$

and for any $f : X \to Z$, if $x \sim x' \implies f(x) = f(x')$, there exists a unique map $\bar{f} : X/\sim \to Z$ such that $\bar{f}([x]) = f(x)$.

Definition 22 (Quotient Topology)**.** Let $(X, \mathcal{T}_X)$ be a topological space, a quotient topology is the topology such that

- The quotient map $q : X \to X/\sim$ is continuous
- For any continuous map $f : X \to Z$, the unique map $\bar{f}$ is continuous.

Here is a construction of quotient topology. Consider the following construction:

$$\mathcal{T}_{\text{quot}} = \big\{U \subseteq X/\sim | q^{-1}(U) \in \mathcal{T}_X\big\}$$

Proposition 2. *$\mathcal{T}_{\text{quot}}$ is a topology on $X/\sim$*

Proof. First of all, $\emptyset, X/\sim \in \mathcal{T}_{\text{quot}}$ is obvious. For $U, V \in \mathcal{T}_{\text{quot}}$, i.e., $q^{-1}(U), q^{-1}(V) \in \mathcal{T}_X$. By the fact $q^{-1}(U \cap V) = q^{-1}(U) \cap q^{-1}(V) \in \mathcal{T}_X$, $U \cap V \in \mathcal{T}_{\text{quot}}$. In the same way, $\mathcal{T}_{\text{quot}}$ also meets the union property. □

Proposition 3. *The construction satisfies the universal property.*

Proof. $f : X \to Z$ continuous, constant on equivalent class. Then, $f = \bar{f} \circ q$ and $\forall W \subseteq Z$ open,

$$f^{-1}(W) = q^{-1}(f^{-1}(W))$$

since f continuous, q continuous by construction, $\bar{f}$ is continuous. $\square$

A generalization is given by considering the topological space $(X, \mathcal{T}_X)$,

$$q : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

such that $\forall U \subset Y : q^{-1}(U)$ is open, then $U \in \mathcal{T}_Y$ (q continuous, as a corollary).

Definition 23 (Level Sets of q)**.** A level set (fiber) of q is $q^{-1}(Y)$ for $y \in Y$.

Lecture 8: Limit Points and Sequences

Definition 24 (Neighborhood)**.** Let $(X, \mathcal{T}_X)$ be a topological space. A neighborhood of point $p \in X$ is a set $N \subseteq X$ that contains x such that $\exists U \in \mathcal{T}_X$ with $x \in U \subseteq N \subseteq X$.

Definition 25 (Limite Points)**.** Let $(X, \mathcal{T}_X)$ be a topological space, $A \subseteq X$. $x \in X$ is a limit point of A if $\forall N$ neighborhood of x, $N \cap A \backslash \{x\} \neq \emptyset$.

$$A' := \{\text{Limit points of } A\}$$

As an example, $A = [0,1) \cup \{2\}$, every point in $[0,1]$ is a limit point of A. However, 2 is not a limit point of A.

Definition 26 (Closure)**.** Let $A \subseteq X$, the closure of A is the smallest closed subset in X containing A, denoted as $\bar{A}$.

Lemma. *$\bar{A}$ exists and is unique.*

Proof. If C and D are both closures of A, then C, D are both closed sets containing A. Thus, $C \subseteq D$ and $D \subseteq C$, which means $C = D$.
Let $\mathcal{C}$ be the set of all closed sets containing A,

$$\bar{A} = \bigcap_{C \in \mathcal{C}} C$$

it is obvious that $\bar{A}$ is the closure. □

It is obvious that if A is closed, then $\bar{A} = A$.

Lemma. *The closure of a set $A \subseteq X$ always be*

$$\bar{A} = A \cup A'$$

where A' is the set of the limit points of A.

Proof. (HW 3) $x \notin A \cup A' \iff \exists N_x : N_x \cap A = \emptyset$, where N_x is a neighborhood of x.
With the proposition above, to prove the lemma, it is sufficient to prove that $x \notin \bar{A} \iff \exists N_x : N_x \cap A = \emptyset$, where N_x.
($\Rightarrow$) $x \notin \bar{A}$ then $x \in X \backslash \bar{A}$. Let $N_x = X \backslash \bar{A}$ and $N_x \cap A = \emptyset$, thus x is not a limit point of A.
($\Leftarrow$) Suppose $\exists N_x$ neighborhood of x such that $N_x \cap A = \emptyset$, then $\exists U$ open in X such that $x \in U \subseteq N_x$ and $A \subset X \backslash U$, that means $\bar{A} \subseteq X \backslash U$. Since $x \in U$, $x \notin \bar{A}$. □

Definition 27 (Sequence)**.** A sequence $\{x_n\}_{n \in \mathbb{N}}$ is a map $\mathbb{N} \to X$ and $n \mapsto x_n$. $\{x_n\}$ converges to $y \in X$ if $\forall W$ neighborhood of y, $\exists N \in \mathbb{N} : (n \geq N \implies x_n \in W)$, we say that y is the limit of the sequence $\{x_n\}$, denoted as $x_n \to y$.

Remark. *Generally, the following propositions are not true:*

- *Limits of convergent sequences are unoique*
  A conterexample is given by $X = \{a, b, c\}$ and $\mathcal{T} = \big\{\{a,b\}, \{c,b\}, \{b\}, \emptyset, X\big\}$. Consider $x_n = b \forall n \in N$. However, every open sets in X containing a are all contains b. Thus x_n always in a neighborhood of a. Thus, $x_n \to a$.

- $\forall A \subseteq X : \forall y \in \bar{A} : \exists \{a_n\}$ *sequence in* A *such that* $a_n \to y$.
  Another conterexample is that $\mathbb{R}^{\mathbb{N}}$, *the space of all sequence in* $\mathbb{R}$ *with box topology generate by* $\mathcal{B} := \{\prod_{i\in\mathbb{N}} U_i | U_i \subseteq \mathbb{R}$ *open*$\}$. *Consider the sequence that* $n \mapsto 0$. *Since* $0 \in U_i \subset \mathbb{R}$ *open,* $\exists \epsilon_i > 0 : (-\epsilon_i, \epsilon_i) \subseteq U_i$ *which means*
  $$\prod_i (-\epsilon_i, \epsilon_i) \subseteq U_i$$
  since $(-\epsilon_i, \epsilon_i) \cap (0, +\infty) \neq \emptyset$. *Given* W *be any neighborhood of* 0 *in box topology,*
  $$\emptyset \neq U \cap A \subseteq W \cap A \implies 0 \in \bar{A}$$
  Given any $\{a_n\} \subseteq A$, $a_n = \{a_{n,k}\}_{k\in\mathbb{N}}$.
  $$U = \prod_{n\in\mathbb{N}} a_{n,n}$$
  since $a_{n,n} \notin (-a_{n,n}, a_{n,n})$, $a_n \notin U \, \forall n$, *then no sequence converges to* 0.

Proposition 4. *Suppose* $A \subseteq X$ *subset,* $\{a_n\}$ *is a sequence with* $a_n \to y$. *Then* $y \in \bar{A}$.

Proof. Since $x_n \to y$, then $\exists N \in \mathbb{N} : x_N \in W$, where W is a neighborhood of y. Since W contains $x_N \in A$, $W \cap A \neq \emptyset$. □

Definition 28 (Neighborhood Basis)**.** A neighborhood basis $\mathcal{B}_x$ at point $x \in X$ is a collection of neighborhoods of x, such that $\forall N \subseteq X$ is a neighborhood of x, $\exists B \in \mathcal{B}_x : B \subseteq N$

Definition 29 (First-Countable)**.** X is first countable if every point has a countable neighborhood basis.

Proposition 5. *Metric spaces are all first countable.*

Proposition 6. X *satisfies* $\forall A \subseteq X : \forall y \in \bar{A} : \exists \{a_n\}$ *sequence in* A *such that* $a_n \to y$ *if and only if* X *first-countable*

Proof. Consider the countable neighborhood basis $\{N_i\}_{i\in\mathbb{N}}$ of $y \in X$, replace N_1 with $N_1 \cap \cdots \cap N_i$. Then $N_1 \supseteq N_2 \supseteq \cdots$ and since $y \in \bar{A}$, $N_i \cap \bar{A} \neq \emptyset \forall i$. Choose $a_i \in N_i \cap \bar{A}$m then it is obvious that $a_i \to y$. □

Lecture 9: Interier and Boundary

Definition 30 (Interior)**.** Let X be a topological space, int$A$ is the largest open set contained by A. In other words, $U \subseteq A$ open implies $U \subseteq \text{int}A$

Proposition 7. *The interior exists and is unique for any subset of X.*

Proof. Same with closure. □

Proposition 8. *For any set $A \subseteq X$*

- $X \backslash \text{int}A = \overline{X \backslash A}$
- $X \backslash \overline{A} = X \backslash \text{int}A$

Proof. HW □

Proposition 9. *For any set $A \subseteq X$*

- $A \subseteq B \implies \text{int}A \subseteq \text{int}B,\ \overline{A} \subset \overline{B}$
- $\overline{\overline{A}} = \overline{A},\ \text{int}(\text{int}A) = \text{int}A$
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$
- $\text{int}(A \cap B) = \text{int}A \cap \text{int}B$

Proof. Obvious. □

A counterexample of $\text{int}A \cup \text{int}B \neq \text{int}(A \cup B)$. Let $A = [0, 1]$ and $B = [1, 2]$

$$\text{int}A \cup \text{int}B = (0, 1) \cup (0, 2)$$
$$\text{int}(A \cup B) = (0, 2)$$

Definition 31 (Boundary)**.** The boundary of $A \subseteq X$ is given by

$$\partial A = \overline{A} \cap \overline{X \backslash A}$$

An example is given by $X = \mathbb{R}$, the boundary of rational numbers is given by

$$\partial \mathbb{Q} = \overline{\mathbb{Q}} \cap \overline{\mathbb{R} \backslash \mathbb{Q}} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$$

Proposition 10. *Let $A \subseteq X$, then*

- $\overline{A} = A \cup \partial A$
- $\text{int}A = A \backslash \partial A$
- $X = \text{int}A \cup \partial A \cup \text{int}(X \backslash A)$

Proof. Follows from set theory. □

Lecture 10: Limit of Nets

Definition 32 (Preorder Sets)**.** A preorder on a set Λ is a relation $\leq$ such that

- Reflexive $\lambda \leq \lambda \ \forall \lambda \in \Lambda$
- Transitive $\lambda_1 \leq \lambda_2, \lambda_2 \leq \lambda_3 \implies \lambda_1 \leq \lambda_3$

Remark. *It is sufficient to have $\lambda \leq \mu$ and $\mu \leq \lambda$ such that $\mu \neq \lambda$ in this case.*

Definition 33 (Directed Set)**.** A directed set is a preorder $(\Lambda, \leq)$ such that $\forall \lambda_1, \lambda_2 \in \Lambda : \exists \lambda_3 \in \Lambda : \lambda_1 \leq \lambda_3, \lambda_2 \leq \lambda_3$

Definition 34 (Net)**.** A net $(x_\lambda)_{\lambda \in \Lambda}$ in a set X is a function $x : \Lambda \to X$ where Λ is a directed set.

Definition 35 (Convergent)**.** A net $(x_\lambda)_{\lambda \in \Lambda}$ in a topological space converges to $y \in X$ if $\forall W \subseteq X$ is the neighborhood of y, $\exists \lambda_0 \in \Lambda$ such that $\lambda_0 \leq \lambda \implies x_\lambda \in W$, denoted by $x_\lambda \to y$. x_λ converges if it has a limit.

Proposition 11. *Let $A \subseteq X$ and $y \in \overline{A}$ if and only if $\exists (x_\lambda)$ net in A such that $x_\lambda \to y$.*

Proof. ($\Leftarrow$) $z \in \overline{A}$ iff $\forall N_z$ neighborhood of z, $N_z \cap A \neq \emptyset$. Suppose $x_n \to y$ in $\forall \lambda : x_\lambda \in A$, since $\forall N_y$ neighborhood of y, $W \cap A \supseteq W \cap \{x_\lambda | \lambda \in A\} \neq \emptyset$, then $y \in \overline{A}$.
($\Rightarrow$) Suppose $y \in \overline{A}$, then let Λ be the set of the neighborhoods of y ordered by the reverse inclusion, i.e., $N_1 \leq N_2$ iff $N_1 \supseteq N_2$. Then, simply by the definition of closure, $\forall N \in \Lambda$, $N \cap A \neq \emptyset$. For each $N \in \Lambda$, choose $x_N \in N \cap A$. If W is a neighborhood of y, take $N \in \Lambda$ such that $x_N \in N \subseteq W$, $N \geq W$, thus, $x_N \to y$. ☐

Proposition 12. *$f : X \to Y$ is continuous $\iff$ $\forall (x_\lambda)$ nets such that $x_\lambda \to w$, then $f(x_\lambda) \to f(w)$ in y.*

Proof. ($\Rightarrow$) Suppose $x_\lambda \to w$, $f : X \to Y$ continuous. Let U be a neighborhood of $f(w)$ in Y, then $f^{-1}(U)$ is a neighborhood of w in X. By the definition of convergence, $\exists \lambda_0 \in \Lambda : \lambda_0 \leq \lambda \implies x_\lambda \in f^{-1}(U)$, then $f(x_\lambda) \in U$. However, that means $f(x_\lambda) \to f(w)$.
($\Leftarrow$) We shall prove by contradiction. Suppose f is not continuous, $\exists V \subseteq Y$ open such that $K = f^{-1}(V)$ is not open. Since K is not open, $K \backslash \text{int} K$ is not empty. Let $\Lambda = \{\text{Neighborhood of } w : N_1 \leq N_2 \iff N_1 \supseteq N_2\}$. Since $w \in K \backslash \text{int} K$, $\forall U \in \Lambda : U \backslash K \neq \emptyset$. Given $U \in \Lambda$, choose $x_U \in U \backslash K = U \backslash f^{-1}(U)$, then $f(x_u) \notin V$ but $f(w) \in V$. Thus, $f(x_U) \not\to f(w)$. If N is a neighborhood of W, take $\text{int} N$ that is open and is also an element in Λ. $\forall U \in \Lambda$ such that $\text{int} N \leq U$, then $x_U \in U \subseteq N$, which means $x_U \to w$. ☐

Proposition 13. *The limit of nets in X is unique $\iff$ X is Hausdorff.*

Proof. ($\Rightarrow$) Suppose X is Hausdorff, consider net (x_λ) be a net in X and $x_\lambda \to a$ as well as $x_\lambda \to b$. Suppose $a \neq b$, then $\exists U, V \subseteq X$ be open sets containing a, b, and $U \cap V = \emptyset$. However, $\exists \lambda_1, \lambda_2 \in \Lambda$ such that $\lambda \geq \lambda_1 \implies x_\lambda \in U$ and $\lambda_2 \implies x_\lambda \in V$, that means $\exists \lambda_3 \in \Lambda$ that is greater than both λ_1 and λ_2, then $x_{\lambda_3} \in U \cap V$, contradicting Hausdorff.
($\Leftarrow$) We shall prove bu contrapositive. Suppose X is not Hausdorff, then $\exists a, b \in X$ such that any $U, V \subseteq X$ neighborhood of a, b are not disjoint. Take the set

$$\Lambda = \{(U, V) \in \mathcal{T}_X \times \mathcal{T}_X | a \in U, b \in V, U \cap V \neq \emptyset\}$$

then we can pick $x_{U,V} \in U \cap V$, take $(W, O) \in \Lambda : \exists (U, V) \in \Lambda : (W, O) \leq (U, V)$ and $x_{U,V} \in U \cap V$. Thus, $x \to a$ and $x \to b$, the limit is not unique. ☐

Lecture 11-12: Compactness

Definition 36 (Cover)**.** Let X be a topological space. A cover of X is a collection of subsets of X, $\{U_\alpha\}_{\alpha\in A}$, such that $\bigcup_{\alpha\in A} = X$. An open cover is a cover such that all the elements are open. A subcover is a subset of the cover that still covers X.

Definition 37 (Compactness)**.** X is compact $\iff$ if every open cover has a finite subcover.

Proposition 14. *$Y \subseteq X$ is compact $\iff$ $\forall\{U_\alpha\}_{\alpha\in A}$ collection of open subsets of X such that $\bigcup_\alpha U_\alpha \supseteq Y$, $\exists n > 0 : a_1, \ldots, a_n \in A : U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \supseteq Y$.*

Proof. ($\Rightarrow$) By the given setting, suppose Y is compact, an open cover of Y in subset topology is $\{Y \cap U_\alpha\}_{\alpha\in A}$. Thus,

$$Y = (Y \cap U_{\alpha_1}) \cup \cdots \cup (Y \cap U_{\alpha_n}) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

gives a finite subcover of Y.
($\Rightarrow$) Suppose $Y \subseteq \bigcup\{V_\alpha\}_{\alpha\in A}$ where $\{V_\alpha\}_{\alpha\in A}$ is the collection of open sets given in the setting, then take $U_\alpha = Y \cap V_\alpha$ and it is easy to see that

$$Y = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$$

Thus, Y is compact. ☐

Proposition 15. *The image of a compact set under a continuous map is compact.*

Proof. Let X compact, $f : X \to Y$ continuous. Let $\{U_\alpha\}_{\alpha\in A}$ be a collection of open sets such that $\bigcup_{\alpha\in A} U_\alpha \supseteq f(X)$. By the continuity, $f^{-1}(U_\alpha) = V_\alpha$ is open and

$$\bigcup_{\alpha\in A} U_\alpha \supseteq f(X) \implies \bigcup_{\alpha\in A} V_\alpha = X$$

i.e. $\{V_\alpha\}_{\alpha\in A}$ is an open cover of X. By the compactness of X, we can find some finite subcover $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$. The image of the subcover also covers the image of X. Thus, by the previous lemma, $f(X)$ is compact. ☐

Proposition 16. *X compact, $C \subseteq X$ closed $\implies$ C compact*

Proof. HW4. ☐

Note that the converse of this proposition is false.

Proposition 17. *A compact subset of a Hausdorff space is always closed.*

Proof. Let X be Hausdorff and $C \subseteq X$ compact. To show that C is closed, it is sufficient to show that $C = \bar{C}$, i.e., $\forall x \in X\backslash C$: $\exists$ open neighborhood $U \ni x : U \cap C = \emptyset$. $\forall c \in C$, $\exists U_c \ni x$ and $V_c \ni c$ be open neighborhoods of x and c such that $U_c \cap V_c = \emptyset$. Then, $C \subseteq \bigcup_{c\in C} V_c$. Since C is compact, we have a finite subcover $C \subseteq V_{c_1} \cup \cdots \cup V_{c_n}$ for some $n \in \mathbb{N}$. Then,

$$\left(\bigcap_{i=i}^{n} U_{c_i}\right) \cap V_{c_k} = \emptyset$$

Then, $\bigcap_{i=i}^{n} U_{c_i}$ is an open neighborhood of x that is disjoint with C. ☐

A fact is that $[0, 1] \in \mathbb{R}$ is compact, this simple fact gives several strong consequences that

- $S^1 \cong \mathbb{R}/\mathbb{Z}$ is compact

  Proof. Consider $i : [0,1] \hookrightarrow \mathbb{R}$ and $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, are both continuous, and $\pi \circ i$ is surjective. Thus, by the lemma, $\mathbb{R}/\mathbb{Z}$ is compact.
  Then, we need to prove that $S^1 \cong \mathbb{R}/\mathbb{Z}$. Consider $f : \mathbb{R} \to S^1$ such that $f(x) = \mathrm{e}^{2\pi \mathrm{i} x}$. Since

  $$\mathrm{e}^{2\pi \mathrm{i} x_1} = \mathrm{e}^{2\pi \mathrm{i} x_2} \implies x_1 - x_2 \in \mathbb{Z}$$

  The map f has the same fiber of quotient map. Thus, $S^1 \cong \mathbb{R}/\mathbb{Z}$. $\square$

- $[a,b] \in \mathbb{R}$ ia compact

Then we prove that $[0,1]$ is compact

Theorem 7 (Meine-Bolzano-Weierstrass)**.** $[0,1]$ is compact

Proof. We shall prove by contradiction. Suppose $[0,1]$ is not compact, then exists an open cover $\{U_\alpha\}_{\alpha \in A}$ has no finite subcover. Then, either $[0,1/2]$ or $[1/2,1]$ cannot be covered by finitely many elements in $\{U_\alpha\}_{\alpha \in A}$. W.O.L.G, let $I_0 = [1/2,1]$ cannot be covered by finitely many elements in the cover. Repeating this process, we generate a nested interval sequence

$$I_0 \supseteq I_1 = [a_1, b_1] \supseteq I_2 = [a_2, b_2] \supseteq \cdots$$

Which I_n has length $1/2^n$, which indicates that the upper and lower bounds of the sequence converge. $a_n \to a$, $b_n \to b$, since the length of the sequence $\lim_{n\to\infty} 1/2^n = 0$, $a = b$. Thus,

$$\bigcap_{n=1}^{\infty} I_n = c$$

Since $[0,1]$ is covered by $\{U_\alpha\}_{\alpha \in A}$, $\exists \beta \in A : c \in U_\beta$. Since U_β open, then $\exists \epsilon > 0 : (c - \epsilon, c + \epsilon) \subseteq U_\beta$. Sicne $[a_n, b_n] \to \{c\}$, $\exists N \in \mathbb{N} : [a_N, b_N] \subseteq (c - \epsilon, c + \epsilon)$. Thus, we reach the contradiction that $[a_n, b_n]$ cannot be covered by finite many elements in $\{U_\alpha\}_{\alpha \in A}$. Hence, we proved that $[0,1]$ is compact. $\square$

Lemma (Tube)**.** *Let X and Y be topological spaces, where Y is compact. $x_0 \in X$, $U \subseteq X \times Y$ with $\{x_0\} \times Y \subseteq U$. Then $\exists V \ni x_0$ open neighborhood of x_0 such that $V \times Y \subseteq U$.*

Proof. $\forall y \in Y$, $\exists V_y$ open nerghborhood of x, W_y open neighborhood of y, such that $V_y \times W_y \subseteq U$. $\{W_y\}_{y \in Y}$ is an open cover of Y, since Y compact, then it si sufficent to pick $y_1, \ldots, y_n \in Y$ sucht ath $W_{y_1} \cup \cdots \cup W_{y_n} = Y$. Since $V_{y_1} \cap \cdots \cap V_{y_n} = V$ contains x is a finite intersection of open sets, which is still open. Thus, $V \times Y \subseteq U$ $\square$

Corollary 2. *The product of two compact sets is compact.*

Inductively, a finite product of compact sets is compact.

Definition 38 (Bounded)**.** $X \subseteq \mathbb{R}^n$ is bounded if $\exists M > 0$ such that $X \subseteq [-M, M]^n$.

Consequently, the compactness in $\mathbb{R}^n$ is fulkly discribed by the following theorem:

Theorem 8 (Heine-Borel)**.** In $(\mathbb{R}^n, \mathcal{T}_{\text{standard}})$, a subset $K \subseteq \mathbb{R}^n$ is compact $\iff$ K is closed and bounded.

Proof. ($\Rightarrow$) suppose $K \subseteq \mathbb{R}^n$ is compact. Then since $\mathbb{R}^n$ Hausdorff, K is closed (HW4). Since

$$\mathbb{R}^n = \bigcup_{k>0} B_k(0)$$

and K is compact, K has been covered by only a finite number of open balls. Thus, take the maximum radius ρ, K is bounded in $[-\rho, \rho]^n$.
($\Leftarrow$) K bounded, $K \subseteq [-M, M]^n$, since K is closed in $\mathbb{R}^n$m K is closed in $[-M, M]^n$ with subset topology. Thus, by the fact that closed sets in a compact space are compact, $[-M, M]^n$ compact $\implies$ K compact. $\square$

Proposition 18. *A real-valued continuous function on a compact set X achieves a* max *and a* min *.*

Proof. By continuity, $f(X)$ is compact, then $f(X)$ is closed and bounded. Then, $\alpha = \inf f(X)$ and $\alpha = \sup f(X)$ exists. Since $f(X)$ closed, $\alpha, \beta \in f(X)$. $\square$

Lecture 13: Compactness and Net Converges

Definition 39 (Finite Intersection Property)**.** $\{C_\alpha\}_{\alpha\in A}$ is a collection of subsets of X, it has the finite intersection property (FIP) iff $\forall \alpha_1,\dots,\alpha_n \in A : \bigcap_{i=1}^n C_{\alpha_i} \neq \emptyset$.

Proposition 19. *X is compact $\iff$ for every collection of closed sets $\{C_\alpha\}_{\alpha\in A}$ with the finite intersection property (that is, $\bigcap_{\alpha\in F} C_\alpha \neq \emptyset$ for every finite $F \subseteq A$), one has $\bigcap_{\alpha\in A} C_\alpha \neq \emptyset$.*

Proof. ($\Rightarrow$) Assume X is compact and let $\{C_\alpha\}_{\alpha\in A}$ be closed sets with the finite intersection property. Suppose, for contradiction, that $\bigcap_{\alpha\in A} C_\alpha = \emptyset$. Then the open sets $U_\alpha = X \setminus C_\alpha$ cover X. By compactness there exist $\alpha_1,\dots,\alpha_n$ such that $X = \bigcup_{i=1}^n U_{\alpha_i}$. Hence $\bigcap_{i=1}^n C_{\alpha_i} = X \setminus \bigcup_{i=1}^n U_{\alpha_i} = \emptyset$, contradicting the finite intersection property. Therefore $\bigcap_{\alpha\in A} C_\alpha \neq \emptyset$.
($\Leftarrow$) Assume that every collection of closed sets with the finite intersection property has a nonempty intersection. Let $\{U_i\}_{i\in I}$ be an open cover of X. If it had no finite subcover, then for every finite $F \subseteq I$, $\bigcup_{i\in F} U_i \neq X$, so $\bigcap_{i\in F}\big(X \setminus U_i\big) \neq \emptyset$. Set $C_i = X \setminus U_i$. Each C_i is closed, and $\{C_i\}_{i\in I}$ has the finite intersection property. By the hypothesis, $\bigcap_{i\in I} C_i \neq \emptyset$, which means $X \setminus \bigcup_{i\in I} U_i \neq \emptyset$, contradicting that $\{U_i\}_{i\in I}$ covers X. Therefore, there is a finite subcover, and X is compact. □

Definition 40 (Cluster Point)**.** Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X, then $p \in X$ is a cluster point if $\forall W$ neigborhood of p, $\forall \lambda_0 \in \Lambda : \exists \lambda \in \Lambda$ with $\lambda_0 < \lambda$ and $x_\lambda \in W$.

An example is that $X = \mathbb{R}$, $x_n = (-1)^n$, then $p = -1, 1$ be both cluster points.

Definition 41 (Subnet)**.** A subnet $(x_{\lambda_\mu})_{\mu\in M}$, $\varphi : M \to \Lambda$, $\varphi(\mu) = \lambda_\mu$ satisfies

- $\mu_1 \leq \mu_2 \implies \lambda_{\mu_1} \leq \lambda_{\mu_2}$
- $\forall \lambda \in \Lambda : \exists \mu \in M : \lambda \leq \lambda_\mu$

Proposition 20. *$p \in X$ is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$ if there exists a subnet that converges to p*

Proof. □

Definition 42 (λ_0-Tail)**.** The λ_0-tail of a net $(x_\lambda)_{\lambda\in\Lambda}$ is $\Gamma_{\lambda_0} := \{x_\lambda | \lambda_0 \leq \lambda\}$

Proposition 21. *$\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ has FIP*

Proof. □

Theorem 9. X is compact $\iff$ every net in X has cluster point.

Proof. ($\Rightarrow$) Suppose $(x_\lambda)_{\lambda\in\Lambda}$ is a net in compact X. Consider $\{\Gamma_\lambda\}_{\lambda\in\Lambda}$ has FIP, so does $\{\overline{\Gamma_\lambda}\}_{\lambda\in\Lambda}$. Since X compact, then $\bigcap_{\lambda\in\Lambda} \overline{\Gamma_\lambda} \neq \emptyset$. Take $p \in \bigcap_{\lambda\in\Lambda} \overline{\Gamma_\lambda} \neq \emptyset$, then $\forall \lambda \in \Lambda : p \in \overline{\Gamma_\lambda}$ Thus, $\forall W$ neighborhood of p, $\forall \lambda \in \Lambda$, $W \cap \Gamma_\lambda = \emptyset$. TGya, $\forall \lambda \in \Lambda \exists \lambda' \in \Lambda : \lambda \leq \lambda' : x_{\lambda'} \in W$. Which means p is a cluster point.
($\Leftarrow$) Suppose any net in X has a cluster point. $\mathcal{C}$ be the collection of closed subsets with FIP. Then take $\mathcal{G}$ to be the collection of finite intersections of $\mathcal{C}$. then $\forall G \in \mathcal{G}$ is nonempty. Take $x_G \subseteq G$, get a net $(x_G)_{G\in\mathcal{G}}$, this net has cluster point p. Then $\forall E$ neighborhood of p, $\forall G \in \mathcal{G} : \exists G' \in \mathcal{G} : G \leq G'$ (i.e., $G' \subseteq G$) and $x_{G'} \in W$. Then, $G \cap W \supseteq G' \supseteq W \ni x_G$ and thus, $G \cap W \neq \emptyset$ and $p \in \bar{G} = G$ (G closed). Thus $p \in \bigcap_{G\in\mathcal{G}} G \subseteq \bigcap_{C\in\mathcal{C}} C \neq \emptyset$. □

Lecture 14: Compactness of Metric Space

Definition 43. Let (X, d) be a metric space, $Y \subseteq X$. The diameter of Y is defined by

$$\operatorname{diam}(Y) := \sup\{d(y_1, y_2) | y_1, t_2 \in Y\}$$

Proposition 22 (Lebeague Lemma). *Let $\{U_\alpha\}_{\alpha \in A}$ be a open cover of a compact metric space (X, d). Then $\exists \delta > 0$ such that $\forall Y \subseteq X$, $\operatorname{diam}(Y) < \delta \implies \exists \alpha^* \in A : Y \subseteq U_{\alpha^*}$.*

Proof. For all $x \in X$, $\exists \alpha(x)$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ open, $\exists B_{2\epsilon(x)}(x) \subseteq U_{\alpha(x)}$. Then $X = \bigcup_{x \in X} B_{2\epsilon(x)}(x)$, which generate a open cover of X. X compact $\implies \exists\{x_1, \ldots, x_n\} \subseteq X : X = B_{2\epsilon(x_1)}(x_1) \cup \cdots \cup B_{2\epsilon(x_n)}(x_n)$. Let $\delta = \min\{\epsilon(x_i) | i = 1, \ldots, n\}$. Sippose $Y \subseteq X$ such that $\operatorname{diam}(Y) < \delta$. Claim that $Y \subseteq U_{\alpha(x_i)}$. $\exists 1 \leq i \leq n : Y \cap B_{\epsilon(x_i)}(x_i) = \emptyset$. Take $U_{\alpha(x_i)}$, $\forall y \in Y$, $d(y_0, y) \leq \operatorname{diam}(Y) \leq \delta \leq \epsilon(x_i)$. Thus, $d(x_i, y) \leq d(x_i, y_0) + d(y_0, y) \leq 2\epsilon(x)$. Thus, $Y \subseteq B_{2\epsilon(x_i)}(x_i) \subseteq U_{\alpha(x_i)}$. $\square$

Definition 44 (Cauchy Sequence). A sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy if $\forall \epsilon > 0 : \exists N \in \mathbb{N} : (n, m > N \implies d(x_n, x_m) < \epsilon)$

Definition 45 (Completeness). A metric space (X, d) is complete if every Cauchy sequence converges.

Definition 46 (Totally Bounded). (X, d) totally bounded if $\forall r \in [0, +\infty) : X = \bigcup_{i=1}^{n} B_r(x_i)$.

Theorem 10. (X, d) is a metric, then the following propositions are equivalent:

1. $(X, \mathcal{T}_d)$ compact
2. Every sequence has a convergent subsequence in (X, d)
3. (X, d) is complete and totally bounded.

Proof. $(1 \Rightarrow 2)$ Suppose $(X, \mathcal{T}_d)$ is compact but $\forall y \in X : \exists U_y$ open neighborhood of y such that $x_n \in U_y$ for only finitely many $n \in N$. $\{U_y\}_{y \in X}$ is an open cover of X, it has a finite sub cover. Thus, $\mathbb{N}$ is finite since that indicates x_n has finite terms in X, which leads to a contradiction.
$(2 \Rightarrow 3)$ Suppose $(x_k)_{k \in N}$ is Cauchy, then $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence converges to x_∞. Thus, $x_n \to x_\infty$. Thus, X is complete. Suppose $\exists \epsilon > 0$ such that X cannot be covered by finitely many ϵ-balls. Then, for some $x_0 \in X$,

$$\exists x_1 \in X \backslash B_\epsilon(x_0) : \exists x_2 \in X \backslash (\bigcup_{i=0}^{1} B_\epsilon(x_i)) : \cdots : \exists x_n \in X \backslash (\bigcup_{i=0}^{n-1} B_\epsilon(x_i)) : \cdots$$

The sequence x_i defined in this way has no convergent subsequence.
$(3 \Rightarrow 1)$ Suppose X is complete and totally bounded, suppose the open cover $\{U_\alpha\}_{\alpha \in A}$ has no finite subcover. Since X is totally bounded, $\exists$ a cover by finitely many balls of radius 1. Then, for some $x_0 \in X$, $\exists x_1$ such taht $B_{1/2}(x_1) \cap B_1(x_0) = \emptyset$. Repeat this process, we get a sequence $(x_n)_{n \in \mathbb{N}}$ such taht $B_{1/2^n}(x_n) \cap B_{1/2^{n-1}}(x_{n-1}) = \emptyset$. That means,

$$\begin{aligned} d(x_n, x_n + k) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{x-k+1}, x_{n-k}) \\ &\leq \frac{1}{2^{n-1}} \Big(1 + \frac{1}{2} + \cdots + \frac{1}{2^k}\Big) \leq \frac{1}{2^{n-2}} \end{aligned}$$

Which means $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since X complete, $x_n \to y \in X$, which implies $y \in U_{\alpha_0} \in \{U_\alpha\}_{\alpha\in A}$. Thus, $\exists r > 0 : B_r(y) \subseteq U_{\alpha_0}$. Since $x_n \to y$, $x_n \in B_n(y) \implies B_{1/2^n} \subseteq U_{\alpha_0}$ which leads to a contradiction. $\square$

Lecture 15-16: Compactness of Product Space (Tychonoff Theorem)

Definition 47 (Filter)**.** A filter on X (set) is a nonempty collection $\mathcal{F}$ of subsets on X such that

- $\emptyset \notin \mathcal{F}$
- $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$
- $A \in \mathcal{F} : A \subseteq B \implies B \in \mathcal{F}$

The filter on X is just a collection of "large subsets" in X. Here is an example: if X is a nonempty set, $\{X\}$ is a filter.
If X is a topological space, $x_0 \in X$, then $N_{x_0} := \{\text{All neighborhoods of } X\}$ is a filter called the neighborhood filter.
Any filter has FIP.

Definition 48 (Filter Base)**.** A nonempty subset $\mathcal{B} \subseteq \mathcal{F}$ is a filter base for $\mathcal{F}$ if $\forall F \in \mathcal{F} : \exists B \in \mathcal{B} : B \subseteq F$.

The neighborhood basis $\mathcal{B}_x$ of x is a filter basis of the neighborhood filter N_x.

Proposition 23. *$\mathcal{B} \in \mathcal{F}$ is a filter base if:*

- $\mathcal{B} \neq \emptyset$
- $\forall B \in \mathcal{B} : B \neq \emptyset$
- $\forall B_1, B_2 \in \mathcal{B} : \exists B_3 \in \mathcal{B} : B_3 \subseteq B_1 \cap B_2$

Proposition 24. *$\mathcal{B} \subseteq \mathcal{P}(X)$ be a filter basis $\mathcal{B} \neq \emptyset$. If $\forall B_1, B_2 \in \mathcal{B}$, if $B_1 \cap B_2 \neq \emptyset : \exists B_3 \in \mathcal{B} \in B_1 \cap B_2$. Then $\mathcal{F} = \{F \subseteq X | \exists B \in \mathcal{B} \text{ such that } V \subseteq F\}$ is a filter, and $\mathcal{B}$ is a filter base.*

Proof. HW5 □

Corollary 3. *Let X be a set, $\mathcal{S} \in \mathcal{P}(X)$ is a nonempty collection of subsets of X with FIP, then $\mathcal{B} = \{\text{finite intersection of sets in } \mathcal{S}\}$. Satisfies the proposition above, and $\exists \mathcal{F}$ filter such that $\mathcal{S} \subseteq \mathcal{F}$*

Definition 49. The smallest filter that contains $\mathcal{S}$ is the filter generates by $\mathcal{S}$

Lemma. *Let $f : X \to Y$ be a function, and $\mathcal{F}$ be a filter in X. Then $f_*\mathcal{F} = \{A \subseteq Y | f^{-1}(A) \in \mathcal{F}\}$ is a filter in Y*

Proof. Easy. □

Definition 50 (Filter Converges)**.** Let X be a topological space, $\mathcal{F}$ filter on X, $x \in X$. $\mathcal{F}$ converges to x ($\mathcal{F} \to x$) if $\forall W$ neighborhood of x, $W \in \mathcal{F}$

Proposition 25. *$f : X \to Y$ continuous, $\mathcal{F} \to x$ filter. Then $f_*\mathcal{F} \to f(x)$*

Proof. Easy □

Definition 51 (Ultrafilter)**.** A filter $\mathcal{U}$ on X is an ultrafilter if $\forall \mathcal{F}$ filter on X, $\mathcal{F} \subseteq \mathcal{U} \implies \mathcal{U} = \mathcal{F}$ (i.e., $\mathcal{F}$ is a maximum filter).

To show the existance of ultrafilter, we need the axiom of choice.

Definition 52 (Chain)**.** Let $(S, \leq)$ be a partially ordered set, $A \subseteq S$ is a chain in S if $(A, \leq)$ is totally ordered. An upper bound of $A \subset S$ is an element $d \in S$ such that $\forall a \in A : a \leq d$.

Theorem 11 (Zorn's Lemma)**.** Let $(S, \leq)$ be a partially ordered set. Suppose every chain in S has an upper bound, then S has a maximal element.

Proof. IDK □

Theorem 12 (Existance of Ultrafilter)**.** Every filter $\mathcal{F}_0$ on X is contained by an ultrafilter.

Proof. Let $S = \{\mathcal{F} \text{ filters on } X \,|\mathcal{F}_0 \subseteq \mathcal{F}\}$. S is partially ordered by inclusion. Then, it is enough to show that $\mathcal{F} = \bigcup_{\alpha \in A} \mathcal{F}_\alpha$ the arbitrary union of filters is still a filter.

- $\emptyset \notin \mathcal{F}_\alpha \implies \mathcal{F}$
- $\forall A, B \in \mathcal{F}$, then $\exists \alpha_1, \alpha_2 \in A$ such that $A \in \mathcal{F}_{\alpha_1}$ and $B \in \mathcal{F}_{\alpha_2}$. $\{\mathcal{F}_\alpha\}$ is a chain, suppose $\mathcal{F}_{\alpha_1} \subseteq \mathcal{F}_{\alpha_2}$, then $A \cap B \in \mathcal{F}_{\alpha_2} \subseteq \mathcal{F}$
- If $A \in F_\alpha$, $A \subseteq B$, then $B \in \mathcal{F}_\alpha \subseteq \mathcal{F}$.

Thus, $\mathcal{F}$ is a filter and ultrafilter exists. □

Proposition 26. *$\mathcal{U}$ is an ultrafilter on X $\iff$ $\forall A \subseteq X : A \in \mathcal{U}$ or $X \backslash A \in \mathcal{U}$*

Proof. ($\Rightarrow$) If $A \in X$, $A \notin U$, then consider $\mathcal{U} \cup \{A\}$, since it is not a filter ($\mathcal{U}$ maximal), then $\mathcal{U} \cup \{A\}$ has no FIP. Which mean $\exists B \in \mathcal{U} : A \cap B = \emptyset$. Thus, $B \subseteq X \backslash A \implies X \backslash A \in \mathcal{U}$.
Suppose $A \subseteq U$ and $X \backslash A \in U$ and U maximal, thus, $A \cap (X \backslash A) = \emptyset \in \mathcal{U}$, which leads to the contradiction. □

Proposition 27. *$\mathcal{U}$ ultrafilter on X. Suppose $X = Y_1 \cup \cdots \cup Y_n$ for some n, $Y_1, \ldots, Y_n \subseteq X$. Then $\exists k : Y_k \in \mathcal{U}$*

Proof. Prove by contradiction. Suppose $Y_1, \ldots, Y_n \notin \mathcal{U}$. Then $X \backslash Y_i \in U$ $\forall i$. Thus,

$$X \backslash \Big(\bigcup_{i=1}^{n} Y_i \Big) = \emptyset \in \mathcal{U}$$

Which leads to a contradiction. □

Proposition 28. *X is compact $\iff$ Every ultrafilter converges.*

Proof. ($\Rightarrow$) Suppose X is compact, suppose $\forall x \in X : \exists U_x \ni x$ be an open neighborhood of x such that for an ultrafilter $\mathcal{U}$, U_x is not contained in the elements of $\mathcal{U}$. Suppose $\{U_x\}$ is the set of open neighborhood of point $x \in X$, then we can find finite $\{U_{x_1}, \ldots, U_{x_n}\}$ be a finite subcover (by the conpactness). Then the previous statement given that there is some U_{x_k} contained by the element of $\mathcal{U}$, which leads to a contradiction.
($\Leftarrow$) Suppose every ultrafilter converges in X. Let $\mathcal{C}$ be a collection of closed sets with FIP. Then it is sufficient to prove the statement by showing that $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$. A finite intersection of elements in $\mathcal{C}$ forms a filter $\mathcal{F}$

such that $\mathcal{C} \subseteq \mathcal{F}$. Then there is an ultrafilter $\mathcal{U}$ with $\mathcal{C} \subseteq \mathcal{U}$. By assumption, $\mathcal{U} \to x$ for some $x \in X$. Thus, $\forall W$ open neighborhood of x, $W \in \mathcal{U}$, then since C and W both in the ultrafilter, $C \cap W \in \mathcal{U}$, thus $C \cap W \neq \emptyset$. Which means $\forall C \in \mathcal{C} : x \in C$. ☐

Lemma. *Let $X = \prod_{\alpha \in A} X_\alpha$ and $\mathcal{F}$ is a filter on X. Then $\mathcal{F}$ converges $\iff$ $\pi_{\alpha *}\mathcal{F}$ converges in X_α $\forall \alpha$.*

Proof. ($\Rightarrow$) By the continuity of π_α, it is easy to show.
($\Leftarrow$) SIppose $\pi_{\alpha *}\mathcal{F} \to \pi_\alpha(x)$ $\forall \alpha$. Let W be a neighborhood of $x \in X$. $\exists V$ neighborhood of x in X with $X \subseteq W$ such that $V = \bigcap_{i=1}^{k} \pi_{\alpha_i}^{-1}(V_i)$, where $V_i \subseteq X_{\alpha_i}$. Thus, $\pi_{\alpha_1}(x) \in V_i$. By the definition of converges, then $V_i \in \pi_{\alpha *}\mathcal{F}$ $\forall i \implies \pi_{\alpha_i}^{-1}(V_i) \in \mathcal{F}$ $\forall i$.
$\implies V \in \mathcal{F}$. By definition of filter, $W \in \mathcal{F}$, thus, $\mathcal{F} \to x$. ☐

Lemma. *$f_i : X \to Y$ continuous, $\mathcal{U}$ ultrafilter on X, $f_* U$ ultrafilter on Y*

Proof. $A \subseteq Y$ be arbitrary nonempty, then either $f^{-1}(A)$ or $f^{-1}(X \backslash A)$ in $\mathcal{U}$. Thus, the image satisfies the same relation. ☐

Theorem 13 (Tychonoff Theorem)**.** $\{X_\alpha\}_{\alpha \in A}$ familty of comptact sets. Then $X = \prod_{\alpha \in A} X_\alpha$ comptact.

Proof. Let $\mathcal{U}$ be an ultrafilter on X, then $\pi_{\alpha *}\mathcal{U}$ is an ultrafilter on X_α. Since X_α comptact, then $\pi_{\alpha *}\mathcal{U}$ convergent on X_α. Thus, $\mathcal{U}$ converges on X. ☐

Lecture 17-18: Connectedness and Path Connectedness

Definition 53 (Connected Space)**.** X is connected if $X = U \cup V$, $U \cap V = \emptyset$ and open $\implies U = \emptyset$ or $V = \emptyset$.
Equivalently, X connected $\iff$ X has not clopen subsets except X and $\emptyset$.

Example: $(\mathbb{Q}, \mathcal{T}_{\text{subspace}})$ is not connected.

Theorem 14. $[0, 1]$ is connected.

Proof. Suppose $[0, 1] = U \cup V$ with U, V open and disjoint. W.O.L.G, $0 \in U$. Let $S = \{x \in [0, 1] | [0, x] \subseteq U\}$. U open and $0 \in U$, i.e., $\exists r > 0 : [0, r) \subseteq U$, then $[0, r/2] \in S$ and S is nonempty. Let $c = \sup S$. If $y \in [0, c)$, then $y \leq c = \sup S$ and $\exists y \leq z < \sup S$ such that $[0, y] \subseteq [o, z] \subseteq U$ which means $y \in X$. Moreover, if $c \notin U$, then $c \in V$, then $\exists \delta > 0 : (c - \delta, c + \delta) \subseteq V$, thus, $(c - \delta)/2 \in [0, c) \subseteq V$ and $(c - \delta)/2 \in V$ contradict with $U \cap V = \emptyset$. Thus, $[0, c] \subseteq U$. If $c \neq 1$, then $(c - \epsilon, c + \epsilon) \subseteq U$, then $c + \epsilon/2 \in S$ contradict with $c = \sup S$. Thus, $c = 1$ and $U = [0, 1]$, $V = \emptyset$. □

Proposition 29. *X connected $\iff$ $\forall Y$ discrete space, any continuous function $f : X \to Y$ is constant*

Proof. $(\Rightarrow)$ With X connected and Y discrete, then $\forall y \in Y : f^{-1}(y)$ is both open and closed. if $f^{-1}(y) \neq \emptyset$, then $f^{-1}(y) = X$.
$(\Leftarrow)$ Suppose X is not continuous, then $\exists U, V$ disjoint clopen subsets of X such that $X = U \cup V$. Then, $f : X \to \{-1, 1\}$ such that $f(U) = -1$ and $f(V) = 1$ is continuous. □

Proposition 30. *$f : X \to Z$ continuous and X is connected, then $f(X)$ connected with subspace topology in Z.*

Proof. Let Y be descrete, $g : f(X) \to Y$ continuous. Then $g \circ f : X \to Y$. Since X is connected, then $g \circ f$ is constant and thus $g : f(X) \to Y$ is constant and by the previous proposition, $f(X)$ is connected. □

Proposition 31. *$\{X_i\}_{i \in I}$ is a collection of connected subset of Z such that $X_i \cap X_j \neq \emptyset$ $\forall i, j \in I$. Then*

$$X = \bigcup_{i \in I} X_i$$

Is connected.

Proof. Take Y discrete, then take $g : X \to Y$ continuous. Since tach X_i connected, $g|_{X_i}$ is constant. Sicne $X_i \cap X_j \neq \emptyset$ $\forall i, j \in I$, then g is contant. By the previous proposition, X is connected. □

Corollary 4. *$\mathbb{R}$ is connected.*

Proposition 32. *Suppose $A \subseteq X$ is connected, then $A \subseteq E \subseteq \bar{A}$. Then E is connected.*

Proof. Suppose $V, W \subseteq F$, $E = V \cup W$ and V, W disjoint closed.Since V, W closed, then $\exists \tilde{V}, \tilde{W}$ closed in X with $V = \tilde{V} \cap E$ and $W = \tilde{W} \cap E$. Then $A \cap V = A \cap (\tilde{V} \cap E) = A \cap \tilde{V}$, similarly, $A \cap W = A \cap \tilde{W}$. Thus, $A \cap V$, $A \cap W$ are closed. Also, $A = A \cap E = (A \cap V) \cup (A \cap W)$ is the disjoint union of two closed sets. Since A connected, then $A \cap W = \emptyset$. Then $A = A \subseteq \tilde{V}$, since $\tilde{V}$ closed, then $\bar{A} \subseteq \tilde{V}$, then, $E \subseteq \tilde{V}$. Hence,

$$W = E \cap \tilde{W} = (E \cap \tilde{V}) \cap (E \cap \tilde{W}) = V \cap W = 0$$

Thus, E is connected. □

Theorem 15 (Intermediate Value). X connected, $f : X \to \mathbb{R}$ continuous. $\forall a, b \in X : f(a) < f(b)$. Then $\forall c \in (f(a), f(b)) : \exists x \in X : f(x) = c$

Proof. By contradiction. $\exists c \in (f(a), f(b)) : \nexists x \in X : f(x) = c$. Then $X = f^{-1}(\mathbb{R}) = f^{-1}(\mathbb{R}\backslash\{c\})$ and by continuoity, $f^{-1}((-\infty, c)), f^{-1}((c, +\infty))$ are two open, disjoint subsets whose union is X, which contradicts with the connectedness of X. ☐

Consider the equivlance relation $\sim$ such that

$$x \sim y \iff \exists A \subseteq X : x, y \in A \text{ and } A \text{ connected}$$

This is obviously an equivalent relation.

Definition 54 (Connected Components). A connected component is an element of $X/\sim$

Remark. *Connected components does not need be open.*

Proposition 33. *The following statements are true*

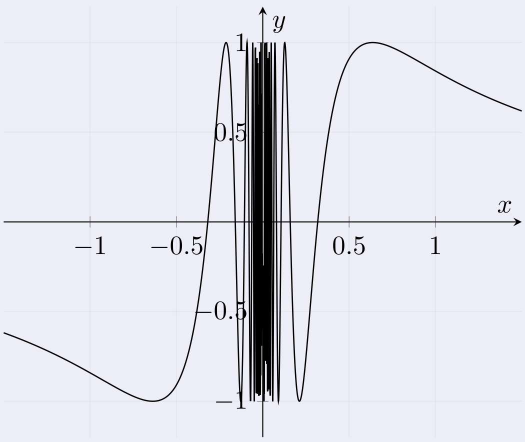
- *Connected components are connected and closed.*
- $Y \subseteq X$, Y *connected* $\implies \exists C$ *connected component such that* $Y \subseteq C$.

Proof. Easy. ☐

Definition 55 (Path Connected). A path is a continuous map $\gamma : [0, 1] \to X$, it is from x to y if $\gamma(0) = x$ and $\gamma(1) = y$. X is path-connected if

$$\forall x, y \in X : \exists \gamma : I \to X \text{ path from } x \text{ to } y$$

Theorem 16. Conder the map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(x) = (x, \sin(1/x))$. Take $A = f(\mathbb{R})$, then $\bar{A}$ is connected but not path connected.



Proof. Suppose $\bar{A}$ is path connected, then $\exists \gamma : [0, 1] \to A$ continuous such that $\gamma(0) = (0, 0)$, $\gamma(1) \in A$. $\bar{A}\backslash A = \{0\} \times [-1, 1]$ is closed in $\mathbb{R}^2 \implies \gamma^{-1}(\bar{A}\backslash A)$ closed in $[0, 1]$. Take $d = \sup(\gamma^{-1}(\bar{A}\backslash A))$, since

$d \in \gamma(\bar{A} \backslash A)$, $\gamma(d, 1) \in A$. Now, take $\gamma(t) = (x(t), y(t))$ for some continuous x, y. For $t > d$, $\gamma(t) \in A$, i.e., $y(t) = \sin(1/x(t))$. $(x(d), y(d)) \in \bar{A} \backslash A$ implies $x(d) = 0$. Since $x(t) \to x(d)$ when $x \to d$, take sequence $t_n \in [d, 1]$ with $t_n \to d$, then it is sufficient to take $x(t_n) = 1/(\pi/2 + \pi n)$ for n sufficiently large. With the choice of x, the corresponding y coordinates is just $y(t_n) = \sin(\pi/2 + n\pi) = (-1)^n$ and the continutiy implies $\lim_{n\to\infty} y(t_n)$ converges, which leads to contradiction. □

Proposition 34. *Path connected space are connected.*

Proof. Suppose X path connected, then $\forall x \in X : \forall y \in X : \exists \gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. By the connectedness of $[0, 1]$, $\gamma([0, 1])$ connected. By the fact that the choice of x, y are arbitrary in X, X is connected. □

Definition 56 (Locally Path-Connected)**.** X is locally path-connected if $\forall x \in X$, $\exists U \subseteq X$ neighborhood of x, there is a path connected neighborhood V of x such taht $x \in V \subseteq U$.

Given the relation

$$x \sim y \iff \text{exists path between } x \text{ and } y.$$

Definition 57. The element of $X/\sim$ is called path component.

Proposition 35. *X locally path connected, then the path components are open.*

Proof. Take $x \in X$, $P = \{y \in X | \text{exist path connecting } x \text{ and } y\}$. Then by the locally path connecteddness, $\exists V_y$ path connected neighborhood of y such that $\forall z \in V_y : z \in P$ and thus, $V_y \in P$. So P is open in X. □

Theorem 17. X connected, locally path-connected $\iff$ X is path connected.

Proof. Let P be path component of X. If $P \neq X$, X has other path components that has union Q, since path components are open, both P, Q clopen can non empty. Thus, X not connected, which cause contradiction. □