On the Classification of V_4 Central Extension by $\mathbb{Z}_2$

Derek Zeng*

December 11, 2025

Abstract

We classified the central extension given by the short exact sequence $1 \to \mathbb{Z}_2 \hookrightarrow G \to V_4 \to 1$ by explicitly classifying the 2-cohomology class in $H^2(V_4, Z)$. The result shows that the group order $|G| \equiv 8$ and the maximum order of group elements in G is 4. We also shows that G can only be one of $(\mathbb{Z}_2)^{\times 3}$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, D_8, or Q_8.

Contents

1 Introduction and Notation

In this note we study central extensions of the Klein four-group

$$V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

by a cyclic group of order two. Concretely, we consider all short exact sequences

$$1 \longrightarrow \mathbb{Z}_2 \xhookrightarrow{\;i\;} G \xrightarrow{\;\pi\;} V_4 \longrightarrow 1$$

such that the image $i(\mathbb{Z}_2)$ lies in the center $Z(G)$ of G. Our goal is to classify, up to isomorphism, all groups G which arise in this way.

From the general theory of group extensions, central extensions of a group Q by an abelian group A (equipped with a Q-module structure) are classified by the second group cohomology $H^2(Q, A)$. In our situation,

$$Q = V_4, \qquad A \cong \mathbb{Z}_2,$$

and the V_4-action on A is trivial because A embeds into the center of G. Thus the relevant object is the cohomology group

$$H^2(V_4, Z),$$

---
*derek6@illinois.edu

where $Z \cong \mathbb{Z}_2$ denotes the coefficient group written additively. Each cohomology class corresponds to an equivalence class of central extensions

$$1 \to Z \to G \to V_4 \to 1,$$

and different representatives of a fixed cohomology class yield isomorphic extensions (via an isomorphism which is the identity on the kernel and on the quotient).

Instead of relying only on abstract classification results, we explicitly compute $H^2(V_4, Z)$ by describing all normalized 2-cocycles

$$f \colon V_4 \times V_4 \to Z$$

and their coboundaries. This concrete approach makes it possible to read off directly the multiplication law in the corresponding extension G_f, and to identify G_f with familiar groups of order 8. The main result of this note shows that:

- the order of any such extension G is $|G| = 8$;

- every element of G has order dividing 4; and

- up to isomorphism, G must be one of

$$(\mathbb{Z}_2)^{\times 3}, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad D_8, \quad Q_8.$$

The proof proceeds by identifying $H^2(V_4, Z)$ with a 3-dimensional $\mathbb{Z}_2$-vector space, choosing convenient normalized cocycle representatives, and then analyzing the resulting relations in G_f.

Notation

We now fix the basic notation and conventions used throughout the paper.

- All groups are written multiplicatively unless otherwise stated. The identity element of a group G is denoted by 1, and its center by $Z(G)$.

- For $n \in \mathbb{N}$, $\mathbb{Z}_n$ denotes the cyclic group of order n. In particular, we freely identify

$$\mathbb{Z}_2 \cong \{0, 1\}$$

  with addition modulo 2 when it appears as a coefficient group in cohomology.

- The Klein four-group is

$$V_4 := \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \{0, a, b, a + b\},$$

  written additively with

$$a + a = b + b = 0, \qquad a + b = b + a.$$

  When convenient, we also use multiplicative notation

$$V_4 = \{1, x, y, xy\}$$

  with $x^2 = y^2 = 1$ and $xy = yx$.

- The kernel of the extension will always be denoted by

$$Z \cong \mathbb{Z}_2,$$

  written additively when we regard it as a coefficient module in group cohomology, and multiplicatively (as $\{1, z\}$) when we regard it as a central subgroup of G.

- A *central extension* of a group Q by an abelian group A is a short exact sequence

$$1 \longrightarrow A \overset{i}{\lhook\joinrel\longrightarrow} G \overset{\pi}{\longrightarrow} Q \longrightarrow 1$$

  with $i(A) \subseteq Z(G)$. In this note we always take $Q = V_4$ and $A \cong \mathbb{Z}_2$ with trivial V_4-action.

- For $g, h \in G$, we write

$$[g, h] := g^{-1}h^{-1}gh$$

  for the commutator. The commutator subgroup is denoted by $[G, G]$, and the abelianization by $G/[G, G]$.

- The dihedral group of order 8 is

$$D_8 := \langle r, s \mid r^4 = 1,\ s^2 = 1,\ srs = r^{-1} \rangle,$$

  and the quaternion group is

$$Q_8 := \{\pm 1, \pm i, \pm j, \pm k\},$$

  with the usual relations $i^2 = j^2 = k^2 = ijk = -1$.

- For group cohomology, we regard $Z \cong \mathbb{Z}_2$ as a V_4-module via the trivial action. The groups of cochains, coboundaries, and cocycles, together with the coboundary operators δ^n, will be introduced in detail in Section 3 when needed for the computation of $H^2(V_4, Z)$.

2   Main Theorem

Based on all concepts we mentioned above, we have the main theorem of this note:

Theorem 1 (Classification of V_4 Central Extension by $\mathbb{Z}_2$). Fix a short exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \overset{i}{\lhook\joinrel\longrightarrow} G \overset{\pi}{\longrightarrow} V_4 \longrightarrow 1$$

where $i(\mathbb{Z}_2) \subseteq Z(G)$. The following statements holds:

1. $|G| = 8$ and $\forall g \in G : g^4 = 1$.

2. If G is an Abelian group, then

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \text{or} \quad G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$$

3. If G is a non-Abelian group, then

$$G \cong Q_8, \quad \text{or} \quad G \cong D_8$$

---

3   Preparation of the Proof

To prove the theorem, we need to apply the technique of the group cohomlogy:
In the following passage, we denote

$$V_4 := \{0, a, b, a + b\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

where $a + a = b + b = 0$ and $a + b = b + a$.
Let $Z \cong \mathbb{Z}_2 \cong (\mathbb{Z}_2, +)$ with a trivial V_4 action (i.e. $\rho = \text{proj}_2 : V_4 \times Z \to Z$ is the projection map) and

$C^n(V_4, Z) := \text{Map}\big((V_4)^{\times n}, Z\big)$ be an abelian addition group, equipped with the chain map $\forall f \in C^n(V_4, Z)$: $\delta^n : C^n(V_4, Z) \to C^{n+1}(V_4, Z)$

$$(\delta^n f)(x_1, \ldots, x_{n+1}) = f(x_2, \ldots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(\ldots, x_i + x_{i+1}, \ldots) + (-1)^{n+1} f(x_1, \ldots, x_n)$$

One can show that $\delta^n \circ \delta^{n-1} \equiv 0$.

Remark. *Here, the function takes value in $Z \cong \mathbb{Z}_2$, which allow us to use $-a$ as the inverse (in the sense of group element) of a, and we shall really ignore the sign since $-1 \equiv 1 \pmod{2}$. Note that in general, we shall not assume there is an $\mathbb{R}$-algebra structure on the cochain complex.*

The following diagram gives an cochain complex (i.e. $\text{im}(\delta^n) \subseteq \ker(\delta^{n+1})$)

$$\cdots \xrightarrow{\delta^{n-1}} C^n(V_4, Z) \xrightarrow{\delta^n} C^{n+1}(V_4, Z) \xrightarrow{\delta^{n+1}} \cdots$$

Then, we shall defined the concept the cocycle and coboundary:

Definition 2 (Cocycle, Coboundary, and Cohomology). The n-cocycle is defined by

$$Z^n(V_4, Z) := \ker(\delta^n) = \{f \in C^n(V_4, Z) \mid \delta^n f = 0\}$$

The n-coboundaries is defined by

$$B^n(V_4, Z) := \text{im}(\delta^{n-1}) = \{f \in C^n(V_4, Z) \mid \exists g \in C^{n-1}(V_4, Z) : f = \delta^{n-1} g\}$$

The n-th order Group Cohomology is the quotient group given by

$$H^n(V_4, Z) := \frac{Z^n(V_4, Z)}{B^n(V_4, Z)}$$

There is a concrete condition to check if something is a 2-cocycle:

Lemma (2-Cocycle Condition with Trivial Action). *Let $Z \cong (\mathbb{Z}_2, +)$ with trivial V_4-action, and let $f \in C^2(V_4, Z) = \text{Map}(V_4 \times V_4, Z)$. Then f is a 2-cocycle (i.e. $f \in Z^2(V_4, Z)$) if and only if for all $x, y, z \in V_4$ we have*

$$f(y, z) + f(x + y, z) + f(x, y + z) + f(x, y) = 0 \in Z.$$

Proof. By definition we have

$$(\delta^2 f)(x, y, z) = f(x_2, x_3) - f(x_1 + x_2, x_3) + f(x_1, x_2 + x_3) - f(x_1, x_2)$$

for all $(x_1, x_2, x_3) = (x, y, z) \in V_4^3$. Since $Z \cong \mathbb{Z}_2$ is written additively and $-u = u$ for every $u \in Z$, the formula becomes

$$(\delta^2 f)(x, y, z) = f(y, z) + f(x + y, z) + f(x, y + z) + f(x, y).$$

Thus $\delta^2 f = 0$ holds if and only if the displayed equality holds for all $x, y, z \in V_4$. $\qquad\square$

Let $Z = i(\mathbb{Z}_2) \cong (\mathbb{Z}_2, +)$ be an abelian group with trivial V_4-action. For every short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \overset{i}{\hookrightarrow} G \overset{\pi}{\longrightarrow} V_4 \longrightarrow 1$$

with $i(Z) \subseteq Z(G)$, choose a set-theoretic section $s : V_4 \to G$ such that $\pi \circ s = \text{id}_{V_4}$ and $s(0) = 1$ (identity in G). Then there exists a unique 2-cocycle $f \in Z^2(V_4, Z)$ such that

$$s(x)s(y) = i\big(f(x, y)\big) s(x + y), \quad \forall x, y \in V_4.$$

Conversely, for every normalized 2-cocycle $f \in Z^2(V_4, Z)$, the set $G_f := Z \times V_4$ equipped with multiplication

$$(\alpha, x) \cdot (\beta, y) := (\alpha + \beta + f(x, y), \, x + y)$$

is a group, and fits into a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \stackrel{i}{\hookrightarrow} G_f \stackrel{\pi}{\longrightarrow} V_4 \longrightarrow 1$$

where $i(\alpha) = (\alpha, 0)$ and $\pi(\alpha, x) = x$. Two such groups G_f and $G_{f'}$ are isomorphic via an isomorphism which is the identity on Z and V_4 if and only if f and f' represent the same class in $H^2(V_4, Z)$. Thus, we have reach the following lemma

Lemma (Classification of Central Extension). *The central extension G of the short exact sequence of groups is classified by the equivalence class in $H^2(V_4, Z)$.*

In order to prove the main theorem, we shall simplified to computation of general 2-cocycle to "normalized 2-cocycle"

Definition 3 (Normalized 2-Cocycle). The 2-cocycle $f \in Z^2(V_4, Z)$ is a normalized 2-cocycle if

$$f(0, x) = f(x, 0) = 0 \in Z \quad \forall x \in V_4$$

We shall claim that it is equivalent to consider only normalized 2-cocycle.

Lemma (Represent Cohomology Class using Normalized Cocycle). *For any 2-cocycle $\tilde{f} \in Z^2(V_4, Z)$, there is $\delta^1 g \in B^2(V_4, Z)$ such that*

$$f = \tilde{f} + \delta^1 g$$

is a normalized 2-cocycle.

Proof. Since the 2-cocycles are Z-valued functions, take $(x, y, z) = (x, 0, 0)$ and $(x, y, z) = (0, 0, z)$ one will obtain

$$\tilde{f}(x, 0) = \tilde{f}(0, z) = \tilde{f}(0, 0) \; \forall x, z \in V_4$$

We shall defined $\tilde{f}(0, 0) = c$. Then we shall take $g : V_4 \to Z$ defined by $g(0) = -c$ and $g(x) = 0 \; \forall x \neq 0$. $\forall x \in V_4$

$$(\delta^1 g)(x, 0) = g(0) - g(x + 0) + g(x) = -c \tag{3.1}$$
$$(\delta^1 g)(0, x) = g(x) - g(x + 0) + g(0) = -c \tag{3.2}$$

Thus, with $f := \tilde{f} + \delta^1 g$, $f(x, 0) = f(0, x) = 0$. $\qquad\square$

The normalized 2-cocycle is much more easier to compute. Since we take that $f(x, 0) = f(0, x) = f(0, 0) = 0 \in \mathbb{Z}$ and $V_4 = \{0, a, b, a + b\}$, there remains the following non-trivial values:

$$p := f(a, a), \quad q := f(b, b), \quad r := f(a, b)$$
$$s := f(b, a), \quad t_1 := f(a, a + b), \quad u_1 := f(b, a + b)$$
$$t_2 := f(a + b, a), \quad u_2 := f(a + b, b), \quad v := f(a + b, a + b)$$

They are all elements in $Z \cong \mathbb{Z}_2$. By 2-cocycle condition:

1. $(\delta^2 f)(a, a, b) = f(a, b) + f(0, b) + f(a, a + b) + f(a, a) = 0 \implies t_1 = p + r.$

2. $(\delta^2 f)(b, b, a) = f(b, a) + f(0, a) + f(b, b + a) + f(b, b) = 0 \implies u_1 = q + s.$

5

3. $(\delta^2 f)(a, b, b) = f(b, b) + f(a + b, b) + f(a, 0) + f(a, b) = 0 \implies u_2 = q + r$.

4. $(\delta^2 f)(b, a, a) = f(a, a) + f(b + a, 0) + f(b, 0) + f(b, a) = 0 \implies t_2 = p + s$.

5. $(\delta^2 f)(a, b, a + b) = f(b, a + b) + f(a + b, a + b) + f(a, 0) + f(a, b) = 0 \implies v = u_1 + r = q + s + r$.

Thus, all nine values been determined by $(p, q, r, s) \in (\mathbb{Z}_2)^{\times 4}$, i.e., $Z^2(V_4, Z)$ is a 4-dimensional $\mathbb{Z}_2$-vector space.

To compute $H^2(V_4, Z)$, we also need to compute the 2-coboundary. With normalized $g : V_4 \to Z$, $b(0) = 0$. We denote

$$x := g(a), \quad y := g(b), \quad z := g(a + b)$$

Then, 2-coboundary is given by

$$(\delta^1 g)(u, v) = g(u) + g(v) + g(u + v)$$

Then, by explicit computation,

1. $(\delta^1 g)(a, a) = (\delta^1 g)(b, b) = 0$.

2. $(\delta^1 g)(a, b) = g(a) + g(b) + g(a + b) = x + y + z$, which implies $r \mapsto r + (x + y + z)$ in the action of $B^2(V_4, Z)$.

3. $(\delta^1 g)(b, a) = g(b) + g(a) + g(b + a) = x + y + z$, which implies $s \mapsto s + (x + y + z)$ in the action of $B^2(V_4, Z)$.

It is obvious that $t := r + s \sim r + s + 2(x + y + z)$ is been preserved in the action, since $2 \equiv 0$ in $\mathbb{Z}_2$. That means we shall take the $x, y, z \in \mathbb{Z}_2$ such that $r + (x + y + z) = 0$, i.e., $x + y + z = r$, and in the same action, $s \mapsto s + r$. Hence, we reach the lemma that

Lemma (Second Cohomology Group). *Every cohomology class can be represent by $(p, q, t) \in (\mathbb{Z}_2)^{\times 3}$, and thus, the second group cohomology group $H^2(V_4, Z) \cong (\mathbb{Z}_2)^{\times 3}$.*

We shall now prove the main theorem.

4   Proof of the Main Theorem

Fix a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \xhookrightarrow{i} G \xrightarrow{\pi} V_4 \longrightarrow 1$$

with $Z := i(\mathbb{Z}_2) = \{1, z\} \subseteq Z(G)$ a central subgroup of order 2.

(1) Exactness of the sequence gives $\ker \pi = Z \cong \mathbb{Z}_2$ and $\pi(G) = V_4$, so the First Isomorphism Theorem yields

$$|G| = |\ker \pi| \cdot |\pi(G)| = 2 \cdot 4 = 8.$$

For any $g \in G$, its image $\tilde{g} := \pi(g) \in V_4$ satisfies $\tilde{g}^2 = 1$, hence $\pi(g^2) = 1$ and therefore $g^2 \in \ker \pi = Z$. Since $Z \cong \mathbb{Z}_2$, every element of Z has square 1, so $g^4 = 1$.

In the rest of the proof we fix the additive notation

$$V_4 = \{0, a, b, a + b\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

with $a + a = b + b = 0$ and $a + b = b + a$. We also identify $Z \cong (\mathbb{Z}_2, +)$.

(Cohomological description). By the general correspondence for central extensions, our short exact sequence arises from a normalized 2-cocycle $f \in Z^2(V_4, Z)$ via the group

$$G \cong G_f := Z \times V_4$$

with multiplication

$$(\alpha, x) \cdot (\beta, y) = (\alpha + \beta + f(x, y), x + y).$$

Two such cocycles differing by a 2-coboundary yield isomorphic extensions (via an isomorphism that is the identity on Z and V_4), so we lose no generality by assuming f is in a fixed representative form for its cohomology class in $H^2(V_4, Z)$.

As computed previously, every normalized 2-cocycle f is determined by four parameters

$$p := f(a, a), \quad q := f(b, b), \quad r := f(a, b), \quad s := f(b, a)$$

in $\mathbb{Z}_2$, and the 2-cocycle condition forces all other values of f to be linear combinations of these. Furthermore, changing f by a 2-coboundary does not affect p, q and the sum $t := r + s$, while r, s can be shifted simultaneously by the same element of $\mathbb{Z}_2$. Thus each cohomology class in $H^2(V_4, Z)$ admits a unique normalized representative with

$$f(a, a) = p, \quad f(b, b) = q, \quad f(a, b) = 0, \quad f(b, a) = t, \quad \text{for some } (p, q, t) \in (\mathbb{Z}_2)^{\times 3}$$

We denote this representative by $f_{p,q,t}$.

Let G be the group corresponding to $f_{p,q,t}$. Writing

$$z := (1, 0) \in G, \qquad a := (0, a), \quad b := (0, b)$$

one checks directly that z is central of order 2, and that

$$a^2 = z^p, \qquad b^2 = z^q, \qquad ab = z^t ba$$

In particular,

$$[a, b] = a^{-1} b^{-1} ab = z^t$$

(2) The Abelian case. The group G is Abelian if and only if $[a, b] = 1$, i.e. $t = 0$, which forces $ab = ba$.

If furthermore $p = q = 0$, then $a^2 = b^2 = 1$, hence every element of G has order 1 or 2, and since $|G| = 8$ we must have

$$G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

If $t = 0$ but $(p, q) \neq (0, 0)$, say $p = 1$ and $q = 0$ (the other case is symmetric), then $a^2 = z \neq 1$ and $b^2 = 1$, so a has order 4 and b has order 2. Because G is Abelian and z is central, the subgroup generated by a is cyclic of order 4, and together with b we obtain an Abelian group of order 8 in which there exists an element of order 4 but no element of order 8 (by (1)). Thus G is not isomorphic to C_8, and the classification of Abelian groups of order 8 implies

$$G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$$

This proves (2).

(3) The non-Abelian case. Now assume that G is non-Abelian. Then necessarily $t = 1$, so

$$[a, b] = z \neq 1, \quad ab = zba$$

We still have $a^2 = z^p$ and $b^2 = z^q$ with $p, q \in \{0, 1\}$.

Case (3a): $(p, q) = (0, 0)$. Here $a^2 = b^2 = 1$ and $ab = zba$, so

$$(ab)^2 = a^2 b^2 [b, a] = 1 \cdot 1 \cdot z = z$$

hence ab has order 4. Define

$$r := ab, \quad s := a$$

Then $|r| = 4$, $|s| = 2$, and one checks

$$srs = a(ab)a = aaba = ba = r^{-1}$$

Thus G satisfies the standard presentation

$$D_8 = \langle r, s \mid r^4 = 1,\ s^2 = 1,\ srs = r^{-1} \rangle$$

and since $|G| = 8$, we obtain $G \cong D_8$.

Case (3b): $(p, q) = (1, 0)$ *or* $(0, 1)$. These two cases are symmetric under exchanging a and b, so it suffices to treat $(p, q) = (1, 0)$. Then $a^2 = z \neq 1$, $b^2 = 1$, and $ab = zba$.

Put

$$r := a, \quad s := b$$

Then $|r| = 4$, $|s| = 2$, and a short computation using $a^2 = z$ and $ab = zba$ shows that

$$srs = r^{-1}$$

Hence G again satisfies the defining relations of D_8, so $G \cong D_8$.

Case (3c): $(p, q) = (1, 1)$. In this case $a^2 = z = b^2$ and $ab = zba$, so a, b are both of order 4 with the same square z. Define $x := a$, $y := b$.

From $[a, b] = z$ we obtain

$$a^{-1}b^{-1}ab = z$$

This is equivalent to $b^{-1}ab = az$. Since $a^2 = z$ and z is central of order 2, we have $a^{-1} = az$, so

$$b^{-1}ab = a^{-1}$$

Altogether we have

$$x^4 = 1, \quad x^2 = y^2, \quad y^{-1}xy = x^{-1}$$

These are exactly the standard defining relations of the quaternion group

$$Q_8 = \langle x, y \mid x^4 = 1,\ x^2 = y^2,\ y^{-1}xy = x^{-1} \rangle$$

so $G \cong Q_8$.

Summarizing, in the non-Abelian case we obtain $G \cong D_8$ in Cases (3a) and (3b), and $G \cong Q_8$ in Case (3c). This completes the proof of (3) and of the theorem. $\square$