Derivation of Conformally Flat Metric for Generalized Schwarzschild-like Solutions

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Why Study Isotropic Coordinates

Current cosmological observations indicate that our universe is undergoing accelerated expansion, driven by dark energy and other possible non-trivial energy components. As a key prediction of general relativity, black holes may interact with the large-scale cosmological expansion. Therefore, studying black hole solutions in the presence of a positive cosmological constant or other dark energy-like fluids is of significant theoretical and observational interest.

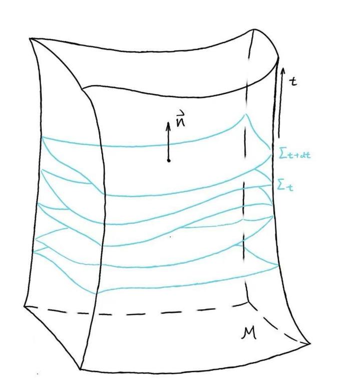

Figure 1. The figure shows the 3+1 decomposition, where Σ_t represents the 3-dimensional spatial section of spacetime at time t.

Such black holes can be described by the Kiselev spacetime, which represents a black hole surrounded by a generic anisotropic perfect fluid. In general, the nonlinear dynamics and strong gravity of black holes cannot be modeled analytically, necessitating numerical relativity simulations. These simulations solve Einstein's equations using the (3+1) decomposition, where the metric evolution is computed in three spatial and one temporal coordinate within classical general relativity. However, incorporating a cosmological background makes this decomposition non-trivial.

A straightforward approach to address this issue is to transition to isotropic coordinates in the background metric. In our project, we develop a general method for transforming Kiselev solutions into isotropic coordinates, extending the standard Schwarzschild metric to account for the influence of surrounding fluids.

Spherically Symmetric Spacetime & Shcwarzschild-like Metric

In this and all the following sections, we will use geometrical units in which $G = c = 1$.
Any study of static spherically symmetric solutions of Einstein's equations can be started by the following general form of metric [1, 2, 3]:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{1}$$

Where

- $\mathrm{d}\Omega = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is the angular term, which shows the spherically symmetric nature of the spacetime.
- $-g_{00} = g_{11}^{-1} = f(r)$ is the metric function defined on the spacetime manifold $\mathcal{M}$. It contains critical information about the black hole (mass M, charge Q, etc.). For the Schwarzschild solution describing a neutral static black hole in vacuum:

$$f(r) = 1 - \frac{2M}{r}$$

And for the Reissner-Nordstrom solution describing a charged static black hole:

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

The mathematical singularity of this metric at $r = r_g = 2M$ leads to the concept of the event horizon. A change of coordinates can remove the singularity at the event horizon.

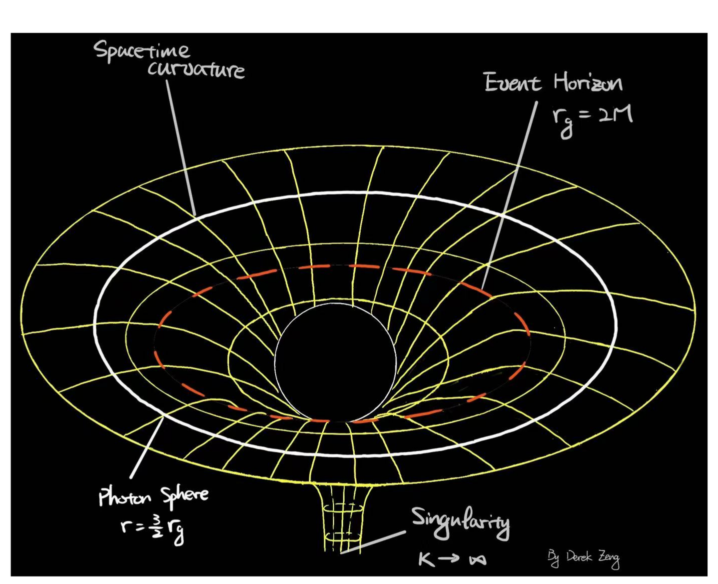

Figure 2. The Schwarzschild Black Hole

The figure on the left is an informal illustration of a constant-time, constant-radius slice of Schwarzschild spacetime, which is a 2-sphere of areal radius r.
The intrinsic (Gaussian) curvature of this 2-sphere is

$$K_{\mathrm{G}} = \frac{1}{r^2}$$

which diverges as $r \to 0$, indicating a genuine curvature singularity at the centre.
A similar divergence occurs for any static spherically symmetric black hole solution in 4-spacetime.

Finally, we need to justify the term "Schwarzschild-like metric". In this poster, this term refers to the static spherically symmetric spacetime metric.

Isotropic Coordinates & General Formula of the Transformation

The isotropic coordinates refer to a coordinate that is only meaningful outside of the event horizon of a black hole in conformally flat spacetime [2], in which the metric has the form:

$$\begin{aligned}\mathrm{d}s^2 &= -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \\ &= -F(\rho)\mathrm{d}t^2 + G(\rho)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\end{aligned} \tag{2}$$

Where the spatial part of the metric is "conformally flat", which means that there exists $f : U \to \mathbb{R}$ such that $g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = fg'_{ij}\mathrm{d}y^i \otimes \mathrm{d}y^j$, $f \in C^\infty(U)$. Our goal is to find an explicit transformation $r \mapsto \rho$ that transforms the metric (1) into the isotropic coordinates. The key to get the transformation is to notice that it is completely equivariant to write the spacial part of the solution in a spherical coordinate with a radial coordinate ρ:

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^2 \tag{3}$$

Then, naively comparing two forms (2) of Schwarzschild-like metric in different coordinates, we obtain $G(\rho) = r(\rho)^2/\rho^2$ and the following differential equation:

$$\frac{\mathrm{d}r}{\mathrm{d}\rho} = \frac{r}{\rho}\sqrt{f(r)} \tag{4}$$

The general solution of this equation is given by

$$\rho = \rho_0 \exp\left(\int \mathrm{d}r \frac{1}{r\sqrt{f(r)}}\right) \tag{5}$$

which defines the explicit form of coordinate transformation to isotropic coordinates.

Schwarzschild & Riessner-Nordstrom Metric in Isotropic Coordinates

By the result of the last section, we can immediately compute the Schwarzschild and Reissner-Nordstrom metrics in isotropic coordinates. For Schwarzschild metric:

$$\rho(r) = (\sqrt{r} + \sqrt{r + 2M})^2, \quad r(\rho) = \rho\left(1 + \frac{M}{2\rho}\right)^2 \tag{6}$$

Where M is the mass of the black hole. The corresponding 1-form is

$$\mathrm{d}r = \left(1 - \frac{M^2}{4\rho^2}\right)\mathrm{d}\rho \tag{7}$$

The Schwarzschild metric in isotropic coordinates is thus

$$\mathrm{d}s^2 = -\left(\frac{1 - \frac{M}{2\rho}}{1 + \frac{M}{2\rho}}\right)^2 \mathrm{d}t^2 + \left(1 - \frac{M}{2\rho}\right)^4 (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{8}$$

For the Riessner-Nordstrom metric, the coordinate transformation is given by

$$\rho(r) = r - M + \sqrt{Q^2 - 2Mr + r^2}, \quad r(\rho) = \frac{M^2 - Q^2 + 2M\rho + \rho^2}{2\rho} \tag{9}$$

Where M is the mass of the black hole and Q is the charge of the black hole. Note that the coordinate transformation will require that $M^2 \geq Q^2$, which means the black hole is not extremely charged. The 1-form of the radius variable is

$$\mathrm{d}r = \left(1 - \frac{M^2 - Q^2}{4\rho^2}\right)\mathrm{d}\rho, \tag{10}$$

which leads to the Reissner-Nordstrom Metric in Isotropic coordinates

$$\mathrm{d}s^2 = -\left(\frac{1 - \frac{M^2 - Q^2}{4\rho^2}}{1 + \frac{M}{\rho} + \frac{M^2 - Q^2}{4\rho^2}}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{M}{\rho} + \frac{M^2 - Q^2}{4\rho^2}\right)^2 (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{11}$$

Kottler (Schwarzschild-de Sitter) Metric in Isotropic Coordinates

The Kottler solution of Einstein's equations describes the neutral spherically symmetric black hole in the spacetime with nonzero cosmological constant, which is given by :

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega \tag{12}$$

The corresponding differential equation is:

$$\frac{\mathrm{d}r}{\mathrm{d}\rho} = \frac{r}{\rho}\sqrt{1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2} \tag{13}$$

The solution can be written as the Legendre incomplete elliptic integral of the first kind by factoring the function:

$$1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 = -\frac{\Lambda}{3r}(r - r_1)(r - r_2)(r - r_3)$$

Where r_i, $i = 1, 2, 3$ are roots of $-\frac{\Lambda}{3}r^3 + r^2 - 2M = 0$, in which $r_1 > r_2 > r_3$. The coordinate transformation is given by:

$$\rho(r) = \rho_0 \exp\left(-\sqrt{\frac{\Lambda}{3}}\frac{2}{\sqrt{r_1 - r_2}}F\left(\arcsin\sqrt{\frac{r - r_1}{r_1 - r_2}}, k\right)\right) \tag{14}$$

$$r(\rho) = r_1 + (r_1 - r_2)\mathrm{sn}^2\left(\frac{1}{2}\sqrt{\frac{\Lambda(r_1 - r_3)}{3}}\ln\frac{\rho}{\rho_0}, k\right) \tag{15}$$

Where the parameter $k = \sqrt{\frac{r_1 - r_2}{r_1 - r_3}}$, $F(\phi, k)$ is the Legendre incomplete elliptic integral of the first kind, and $\mathrm{sn}(\psi, k)$ is the elliptic sine function. Substituting the coordinate transformation into the metric, the Kottler metric in isotropic coordinates is given by

$$\mathrm{d}s^2 = -\left[1 - \frac{2M}{r(\rho)} - \frac{\Lambda}{3}r(\rho)^2\right]\mathrm{d}t^2 + \left[\frac{r(\rho)}{2\rho}\right]^4 (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{16}$$

Discussion and Future Studies

In this work, we focus on expressing Schwarzschild-like solutions, including the Kottler (Schwarzschild-de Sitter) family, in isotropic coordinates. Our primary motivation is rooted in numerical relativity, where the (3+1) decomposition is commonly used to evolve the metric in time. One challenge in such simulations is preparing conformally flat initial data that smoothly extends to encompass both near-horizon regions and cosmological scales. By transforming the Kottler solution—and more general solutions within the Kiselev class—into isotropic coordinates, we can potentially streamline the construction of these initial data sets.
Because the (3+1) decomposition naturally splits spacetime into spatial hypersurfaces and a time direction, having a conformally flat spatial metric component is often advantageous for both theoretical analyses and numerical codes (e.g., using the BSSN formulations). The explicit isotropic forms derived here provide a direct way to identify conformal factors and other significant parameters when setting up boundary conditions or matching asymptotic regions.
While we have shown explicit transformations for Schwarzschild, Reissner-Nordstrom, and Schwarzschild-de Sitter spacetimes, a key direction is the more general Kiselev metric (with various fluid equation-of-state parameters) [4]. We will analytically evaluate the coordinate transformation in Kiselev spacetime, given by:

$$\rho(r) = \rho_0 \exp\left(\int \mathrm{d}r \frac{1}{r}\left[1 - \frac{r_g}{r} + \sum_n \left(\frac{r_n}{r}\right)^{3w_n + 1}\right]^{-1/2}\right) \tag{17}$$

In this way, we can extend our isotropic-coordinate approach to these anisotropic perfect-fluid solutions, which will allow numerical relativity studies to incorporate exotic dark energy-like components or quintessence fields in a similarly systematic way.

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