Arithmetic of Polynomials in Knot Theory

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Introduction

We are interested in the complement space of the figure-eight knot. We equip the complement with a non-Euclidean geometry.

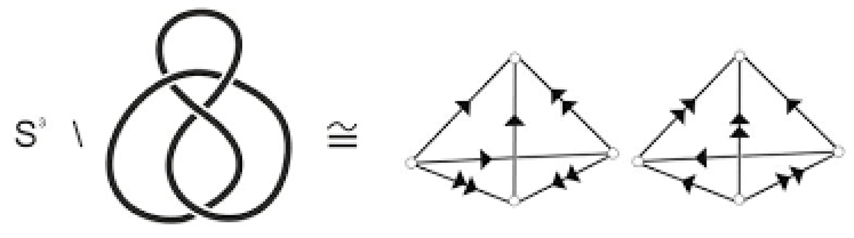


Figure 1: *Homeomorphism of Figure-eight Knot Complement*

We know from Filaseta's paper [1] that the trace field under $-a/b$ Dehn fillings of the complement of the figure-eight knot is $\mathbb{Q}(x+\frac{1}{x})$ where x is the root of the following polynomial,

$$P_{a,b} = x^{4b} \quad -x^{2b} \quad -x^{a} \quad -2 \quad -x^{-a} \quad -x^{-2b} + x^{-4b}$$

Here are two examples of unknot and trefoil along with their corresponding fundamental group of the complement space.

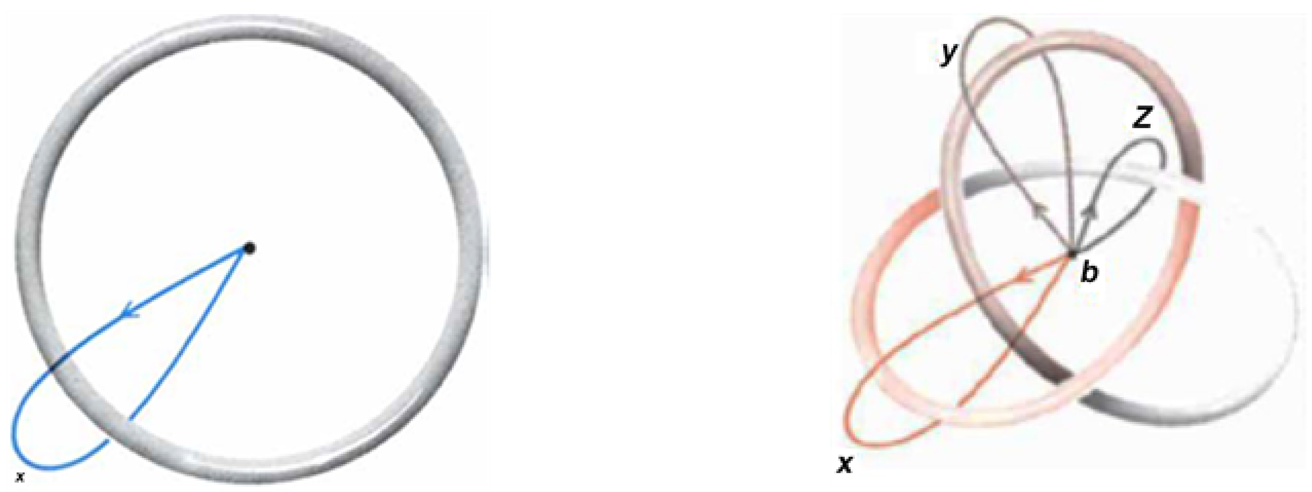


Figure 2: *Left: Unknot, Right: Trefoils*

Goal: A "Dehn filling" is a manifold associated to the figure-eight knot and a rational number $-a/b$. We are interested in the factorization of the polynomial above, which is associated with the trace field of the $(-a/b)$ Dehn filling of the figure-eight knot in both $\mathbb{Q}[x]$ and $\mathbb{F}_2[x]$.

In Rational Field

We begin with the attempts on trying to find out the common factor of the polynomial in $\mathbb{Q}[x]$ for any pair of (a, b). The first theorem shows that a group of cyclotomic factors that commonly exists as factors of $P_{a,b}(x)$:

Theorem 1

Let $a, b, k \in \mathbb{Z}_{>0}$, $\Phi_{2^k}(x) = x^{2^{(k-1)}} + 1$. Then $\Phi_{2^k}(x) \mid P_{a,b}(x)$ if the parameter a, b satisfy

$$\begin{cases} \text{(A)} & 2^{k-1} \mid b \text{ and } a \equiv 2^{k-1} \pmod{2^k}, \\ \text{or (B)} & k \geq 2,\ b \equiv 2^{k-2} \pmod{2^{k-1}} \text{ and } 2^k \mid a. \end{cases}$$

The basic idea of proving Theorem 1 is to check the cyclotomic root $e^{i\pi/2^{k-1}}$ and expand the polynomial using Euler's formula.

This technique is central to our approach: we first use it to prove the conjectures stated above, and then apply it to two analogous cases with odd a, as well as to analyze the relationship between the factors (a, b) and $(a, 0)$.

Theorem 2

- For $P_{a,b}|_{a\equiv 1 \mod 2}$, polynomial $(x+1)^2$ divides $P_{a,b}$
- For $P_{a,b}|_{a\equiv 0 \mod 3}$, polynomial (x^2+x+1) divides $P_{a,b}$
- If a is odd and $a \mid b$, then $g_{a,0} \mid g_{a,b}$ in $\mathbb{Z}[x, x^{-1}]$ (hence also in the rational function field).

Unfortunately, these are the only trivial factors we could find as the quotient was very convoluted.

Using these theorems, we eliminate the trivial factors and obtain the following conjecture on irreducibility over the rational field $\mathbb{Q}$.

Conjecture 1

In $\mathbb{Q}$, consider $a \equiv 1 \pmod 2$, and $\gcd(a, b) = 1$, we observed that,

$$\begin{cases} \frac{P_{a,b}}{(x+1)^2(x^2-x+1)} & \text{is irreducible when } a \equiv 0 \pmod 3 \\ \frac{P_{a,b}}{(x+1)^2} & \text{is irreducible when } a \equiv 1, 2 \pmod 3 \end{cases}$$

We tested a from 1 to 1000 and b from 1 to 200. This conjecture stands the tests.

In GF(2)

Similar to what we do in the rational field, we want to categorize the "behavior" of the same polynomial in $\mathbb{F}_2[x]$ (i.e., with coefficient $\bmod 2$) based on the value of a and b.

Definition 1. We call a polynomial **symmetric** if $p(x) = x^{\deg(p)}p(\frac{1}{x})$. We call a polynomial **asymmetric** otherwise.

Directly from the definition, we can know that the irreducible factor of a symmetrical polynomial can be either:

- Another symmetrical polynomial.
- A pair of asymmetric polynomials with their product being symmetric.

Then, we can fit all the irreducible polynomials we find from conjecture 1 to the following cases. Factorize the polynomials in $\mathbb{F}_2$, we can categorize the results in disjoint sets of three,

Case 1:	All factorized polynomials are symmetric
Case 2:	Some factorized polynomials are symmetric and others are asymmetric.
Case 3:	All factorized polynomials are asymmetric.

We observed the following patterns. For all of the following, we care about $\gcd(a, b) = 1$.

Conjecture 2

$P_{a,b}(x)$ has ≥ 2 asymmetric irreducible factors in $\mathbb{F}_2[x]$ for $b > \frac{a}{4}$, $a \equiv 5, 19 \pmod{24}$

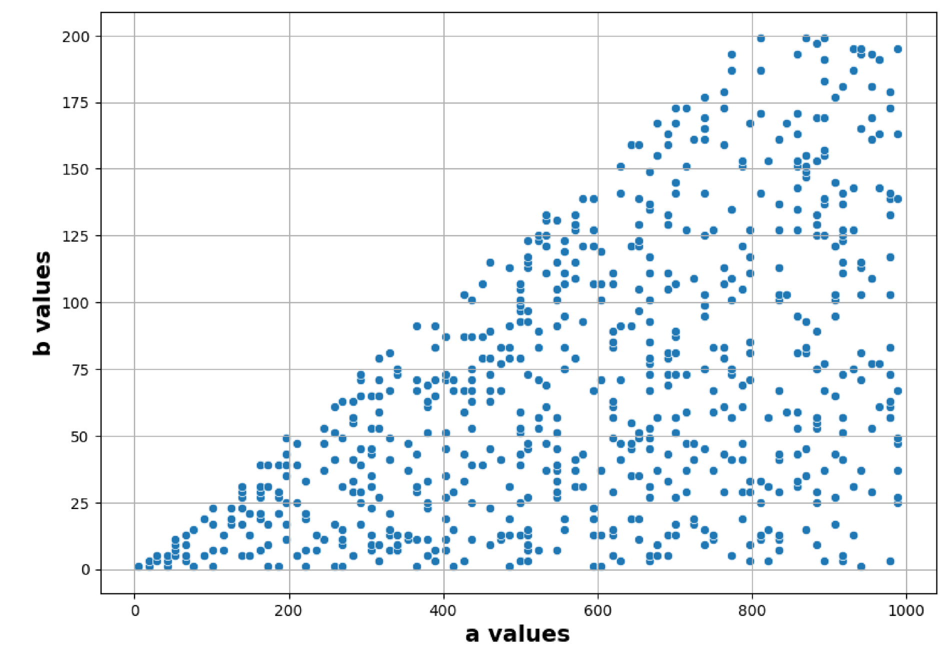


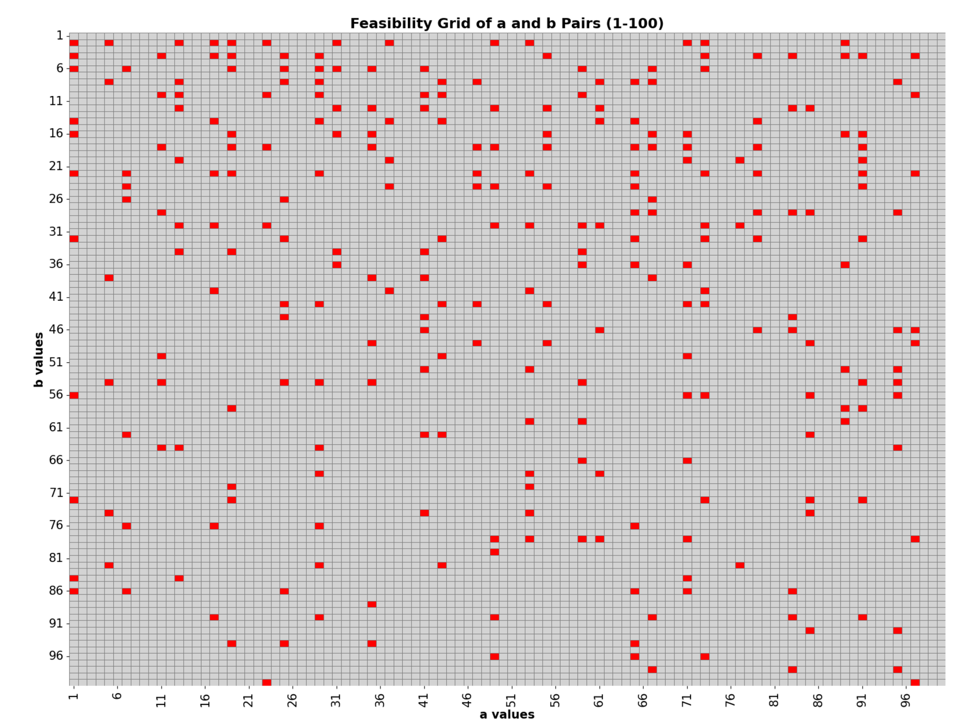
Figure 3: *The Scatter Plot for* $a \equiv 5, 19 \pmod{24}$

We excluded other values a in the Figure 3 to demonstrate the pattern. We did not observe special patterns in case 2. As for case 3, we find restrictions on both a and b's values.

Conjecture 3

Case 3 exists $\iff$ $a \equiv 1 \pmod 2$ and $a \not\equiv 0 \pmod 3$, $b \equiv 0 \pmod 2$

In the following graph, we can see gaps on a divisible by 3, and odd b,



Future Directions

More on Cyclotomic Polynomials

Consider any cyclotomic polynomial $C(x)$. Then, our solution $\hat{x}$, where $C(\hat{x}) = 0$, can be rewritten as $(\cos(\theta), \sin(\theta))$ for some $\theta \in [0, 2\pi)$. Thus, paramterizing $P_{a,b}(\hat{x}) = 0$ directly leads to:

Corollary 6. $\cos(4b\theta) - \cos(2b\theta) - \cos(a\theta) - 1 = 0 \implies \cos^2(2b\theta) - \cos(2b\theta) - \cos(a\theta) - 2 = 0$

Using this equation may allow us to determine in what situations are cyclotomic polynomials factors of or are not factors of $P_{a,b}$. For example, we determine that $x^2 + x + 1$ can never factor $P_{a,b}$.

References

[1] Michael Filaseta, *On the factorization of lacunary polynomials*, Preprint 2022, arXiv:2207.11648.

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