# Notes on Geometry and Topology

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# Introduction

These are mostly my learning notes on a variety of topics in differential geometry and topology.

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# Chapter 1

# Smooth Manifolds and Submanifolds

## 1.1 Topological Manifold and Smooth Structure

The goal of Chapter 1 is to define the central object of modern geometry, the smooth manifold. To define a smooth manifold, we first need to study a more basic case called a topological manifold, which is a special type of topological space that locally resembles Euclidean space.

The key to defining a topological manifold is the following property: locally Euclidean.

**Definition 1.1** (Locally Euclidean)**.** A topological space $X$ is said to be locally Euclidean iff $\forall x \in X$, there is some open neighborhood $x \in U \subseteq X$, such that we can find a map $\varphi : U \to V \subseteq \mathbb{R}^n$ that is a homeomorphism.

**Definition 1.2** (Topological Manifolds)**.** A topological manifold is an $n$-dimensional topological space $M$ iff $M$ is second countable, Hausdorff, and locally Euclidean to some fixed $\mathbb{R}^n$. A chart is a pair $(U_\alpha, \varphi_\alpha)$ that $U_\alpha \subseteq M$ is a open subset and $\varphi_\alpha : U_\alpha \to \phi(U_\alpha) \subset \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. The set of charts that can cover the entire manifold $M$ is called an atlas.

From my previous experiences, it seems that it is often more important to have a taste of what is not a topological manifold, so here are some examples:

**Example 1.1.** *If we consider the set $S = (\mathbb{R} \setminus \{0\}) \cup \{A, B\}$, with the topology defined by the following laws:*

- *In $\mathbb{R} \setminus \{0\}$, take the subset topology inherit from $\mathbb{R}$.*
- *The open set that contains $A$ (or $B$), for some $c, d \in \mathbb{R}_{\geq 0}$ consider the set $I_A(c, d) = (-c, 0) \cup (0, d) \cup \{A\}$ and the basis is given by $\mathcal{B} = \{I_A(c, d) : c, d \in \mathbb{R}_{\geq 0}\} \cup \{I_B(c, d) : c, d \in \mathbb{R}_{\geq 0}\} \cup \{(a, b) \subseteq \mathbb{R} : 0 \notin (a, b)\}$*

*This set is locally Euclidean, second countable, but not Hausdorff.*

*Proof.* It is quite obvious that the set is locally Euclidean, since if we consider $I \subseteq S$ such that $A, B \notin I$, then $I \subseteq \mathbb{R}$. For some open set containing $A$ or $B$, we just take $(I \setminus \{A, B\}) \subseteq \mathbb{R}$ and take $A, B \mapsto 0$. The continuity of this map is obvious. In addition, the second countability of the topological space $S$ is also easy to prove. However, consider any open neighborhood of $A$ and $B$, $I_A(a, b)$ and $I_B(c, d)$, then there will always intersect. Let $\alpha := \max\{a, c\}$ and $\beta := \min\{b, d\}$, then $I_A(a, b) \cap I_B(c, d) = (\alpha, 0) \cup (0, \beta)$ always nonempty. Thus, points $A$ and $B$ are not disjoint, and the space $S$ is not Hausdorff. $\square$

**Example 1.2.** *As a counterexample of a topological space that is locally Euclidean, Hausdorff, but not second-countable, consider the uncountable index set $I$ and the topological space*

$$X = \coprod_{i \in I} S_i := \{(x, i) \mid x \in S^1, i \in I\}$$

*with the coproduct topology. This space is not second-countable. Recall that the coproduct has a universal property that the following diagram commutes with $f : Y \to X_1 \coprod X_2$ is unique:*

$$\begin{tikzcd} & Y & \\ X_1 \arrow[ur, "f_1"] \arrow[r, "i_1"'] & X_1 \coprod X_2 \arrow[u, dashed, "\exists! f"'] & X_2 \arrow[ul, "f_2"'] \arrow[l, "i_2"] \end{tikzcd}$$

*and the coproduct topology is defined to be the coarsest topology such that $i_1$, $i_2$ are continuous and $f$ is continuous if $f_1$, $f_2$ are continuous. Then, $U \in X$ is open if $i_k^{-1}(U)$ is open in $S^1$.*

*Proof.* Suppose $X$ has a topological basis $\exists \mathcal{B}$. For any $k \in I$, $S_k^1$ denotes the $i$-th 1-sphere in the coproduct. Then, since $S_k^1 \subseteq X$ such that $p_k \in B_k \subseteq S_k^1$. Notice that $\forall k \neq l : S_k^1 \cap S_l^1 = \emptyset$, thus, for any $S_i^1$, we can take $B_i$ and each two of the $B_i$ with different indices are disjoint. Then we can take the map $I \to \mathcal{B}$ sends each $i$ to $B_i$, which means $|B| \geq |I|$ is uncountable. Thus $X$ is not second countable. □

The topological manifold only allows the continuity ($C^0$) to be defined, which is not enough for calculus. To define calculus on a smooth manifold, which is a generalization of smooth curves and surfaces in Euclidean space, we need to define a smooth structure.

**Definition 1.3** ($C^k$-Structure)**.** A $C^k$-structure on a manifold is an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ for some indexes set $I$, and satisfies the following properties:

- $\{U_\alpha \mid \alpha \in I\}$ covers $M$, $\bigcup_\alpha U_\alpha = M$.
- For any $\alpha, \beta \in I$, $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.
- The collection $\mathcal{A}$ is maximal: $\forall \alpha \in I$, if charts $(U, \varphi)$ and $(U_\alpha, \varphi_\alpha)$ are either $U \cap U_\alpha = \emptyset$, or $\varphi \circ \varphi_\alpha$ is a diffeomorphism, then $(U, \varphi) \in \mathcal{A}$

A smooth manifold $(M, \mathcal{O}_M, \mathcal{A})$ is a topological manifold $(M, \mathcal{O}_M)$ with a smooth structure $\mathcal{A}$ on it. For example, consider the following sets:

**Example 1.3.** *As a topological manifold, $\mathbb{R}^n$ can be constructed by a single chart $(\mathbb{R}^n, i)$, and the smoothness is trivial. However, as a critical counterexample to the uniqueness of the smooth structure, we can take another chart $(\mathbb{R}^n, \varphi)$, where $\varphi(u) = u^3$ is indeed a diffeomorphism. The two smooth structures above are not compatible.*

*Proof.* Consider the map $\varphi \circ \mathrm{id} = \varphi$ is smooth. However, the inverse map $(\varphi \circ \mathrm{id})^{-1} = \mathrm{id}^{-1} \circ \varphi^{-1} = \varphi^{-1}$, which is defined by $\varphi^{-1}(v) = v^{1/3}$ is not globally smooth since $(\varphi^{-1})'(v) = v^{-2/3}/3$, which is undefined at the origin. □

**Example 1.4** (Unite Circle $S^1$)**.** *A classical example of a smooth manifold is the unit circle $S^1$, take*

$$S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \simeq \{\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in [0, 2\pi]\}$$

*As a topological subset of $\mathbb{R}^2$, consider the following chats:*

$$U_1 = S^1 - \{(1, 0)\} \simeq \{\mathrm{e}^{\mathrm{i}\theta} \mid \theta \in (0, 2\pi)\}$$
$$U_2 = S^1 - \{(-1, 0)\} \simeq \{\mathrm{e}^{\mathrm{i}\eta} \mid \eta \in (\pi, 3\pi)\}$$

*Then, a defined homeomorphism*

$$\varphi_1 : U_1 \to (0, 2\pi), \quad \varphi_2 : U_2 \to (\pi, 3\pi)$$

*where $\varphi_1$ and $\varphi_2$ are homeomorphism. The transformation map is given by*

$$\varphi_2 \circ \varphi_1^{-1} : (0, \pi) \cup (\pi, 2\pi) \to (\pi, 2\pi) \cup (2\pi, 3\pi)$$

*which is defined by*

$$\varphi_2 \circ \varphi_1^{-1}(\theta) = \begin{cases} \theta + 2\pi, & \theta \in (0, \pi) \\ \theta, & \theta \in (\pi, 2\pi) \end{cases}$$

*is smooth, and also we can also check $\varphi_1 \circ \varphi_2^{-1}$ in the same way. Thus, $\mathcal{A} = \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is a smooth structure, and $S^1$ is a smooth manifold.*

Another important example of a smooth manifold is the product manifold, which is given by the following proposition

**Proposition 1.1** (Product Manifold)**.** *$M$ and $N$ are smooth manifolds with dimension $m$ and $n$ $\iff$ $M \times N$ is smooth manifold, and* $\dim M \times N = m + n$*.*

*Proof.* The Hausdorff and second countable properties are preserved in the Cartesian product, and the Hausdorff and second countable properties on a superset can be directly extended to a subset; the proof is in Appendix A.
($\Rightarrow$) If $M$ and $N$ are orientable $C^\infty$-manifolds with dimension $m$ and $n$, then we can take the orientable smooth structures $\mathcal{A}_M = \{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$ and $\mathcal{A}_N = \{(V_\beta, \psi_\beta) \mid \beta \in J\}$. Consider the inherent smooth structure $\mathcal{A}_{M\times N} = \{(U \times V, \varphi) \mid U \in \mathcal{A}_M, V \in \mathcal{A}_N\}$ and the coordinate map is given by

$$\forall (p, q) \in M \times N : \varphi(p, q) = (\phi^1(p), \dots, \phi^m(p), \psi^1(q), \dots, \psi^n(q))$$

Since $\phi : M \to \mathbb{R}^m$ and $\psi : N \to \mathbb{R}^n$ are homomorphisms onto their image, the coordinate map given above on $M \times N$ is a homomorphism onto its image $\varphi(U \times V) = \phi(U) \times \psi(V) \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Then we have to check the smoothness of the manifold, the transition map is given by

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x) = \left(\phi_\alpha \circ \phi_\beta^{-1}(x^1, \dots, x^m); \psi_\alpha \circ \psi_\beta^{-1}(x^{m+1}, \dots, x^{m+n})\right)$$

By the smoothness of $\phi$ and $\psi$, the transition map $\varphi_\alpha \circ \varphi_\beta^{-1}$ is smooth. Thus, $M \times N$ is a smooth manifold with dimension $m + n$.
($\Leftarrow$) In the similar way, $\varphi_\alpha \circ \varphi_\beta^{-1}(x) = (y^1(x), \dots, y^{m+n}(x)) \in C^\infty(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$ indicates that each of its component is a smooth function on $\mathbb{R}^{m+n}$, which leads to the smooth structure on $M$ and $N$ separately. □

The smooth manifold is the generalization of a well-behaved subset of $\mathbb{R}^n$, so it is natural to talk about the smoothness of the map

**Definition 1.4** ($C^k$-Map)**.** For some smooth manifold $M$ and $N$ of dimension $m$ and $n$, a continuous map $f : M \to N$ is said to be $C^k$ iff the local coordinate representation of chart $(U, \varphi)$ on $M$ and $(V, \psi)$ on $N$

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is $C^k$ as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.

**Proposition 1.2** (Composition of $C^k$ Maps)**.** *The composition of finite $C^k$ maps $f_1, \dots, f_n$ is still a $C^k$-map.*

*Proof.* Because continuity and differentiability are defined chart-wise and composition is a local operation, it suffices to consider maps between open subsets of Euclidean spaces. Let

$$f : V \subset \mathbb{R}^n \to \mathbb{R}^m, \qquad g : U \subset \mathbb{R}^p \to V$$

1. If $f$ and $g$ are continuous, then for any open set $O \subset \mathbb{R}^m$,

$$(f \circ g)^{-1}(O) = g^{-1}\big(f^{-1}(O)\big),$$

which is open because $f^{-1}(O)$ is open and $g$ is continuous. Thus $f \circ g$ is continuous, i.e., lies in $C^0$.

2. Fix $k \geq 1$ and assume that if $f, g \in C^{k-1}$, then $f \circ g \in C^{k-1}$.

3. Since $f, g \in C^k$, they are $C^1$. For each $x \in U$,

$$D(f \circ g)(x) = Df\big(g(x)\big) \circ Dg(x)$$

The map $x \mapsto Df(g(x))$ is the composition of $g$ (class $C^k$) with $Df$ (class $C^{k-1}$), hence is $C^{k-1}$ by the induction hypothesis. Likewise $x \mapsto Dg(x)$ is $C^{k-1}$. Since both $Dg$ and $Df$ are $C^{k-1}$, by the hypothesis, $D(f \circ g)(x) = D_{g(x)}f(g(x)) \circ D_x g(x)$ is a $C^{k-1}$ map.

Therefore $f \circ g \in C^k$. Together with the base case, induction on $k$ completes the proof. ☐

**Definition 1.5** (Diffeomorphism)**.** A map $f : M \to N$ is said to be a diffeomorphism between smooth manifold $M$ and $N$ iff

1. $f$ is bijective
2. $f$ and $f^{-1}$ are both smooth

Consider smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ on a $n$-dimensional manifold $M$, the coordinate transformation is given by $\varphi_\beta \circ \varphi_\alpha^{-1}(x) = (y^1, \dots, y^m)$, where $x = (x^1, \dots, x^n) \in \varphi_\alpha(U_\alpha \cap U_\beta)$, the Jacobi matrix is given by

$$J(\varphi_\beta \circ \varphi_\alpha^{-1}) = \left(\frac{\partial y^i}{\partial x^j}\right)_{1 \leq i,j \leq n}$$

And the Jacobian is

$$\det J(\varphi_\beta \circ \varphi_\alpha^{-1})(p) = \frac{\partial(y^1, \dots, y^n)}{\partial(x^1, \dots, x^n)}(\varphi_\alpha(p))$$

With the Jacobian, the orientability of a smooth manifold can be defined as follows.

**Definition 1.6** (Orientability of Manifolds)**.** Let $M$ be a smooth manifold with (at least $C^1$) cover $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$, $M$ is orientable iff

$$\forall \alpha, \beta \in I : \big(U_\alpha \cap U_\beta \neq \emptyset \implies J(\varphi_\alpha \circ \varphi_\beta^{-1}) > 0\big)$$

And $\mathcal{A}$ is said to be an orientable cover. If orientable covers do not exist, then $M$ is not orientable.

An obvious fact about the orientation is the following proposition:

**Proposition 1.3.** *Smooth manifolds $M$ and $N$ orientable $\iff$ smooth manifold $M \times N$ orientable.*

*Proof.* We defined the smooth structure on $M \times N$ to be induced by the smooth structure on $M$ and $N$; the smooth structure and transition map are given by the following equations:

$$\begin{aligned}
&\mathcal{A}_{M \times N} = \{(U_\alpha \times V_\beta, \varphi_{(\alpha,\beta)} = (\phi_\alpha, \psi_\beta)) \mid (\alpha, \beta) \in I \times J\} \\
&\varphi_{(\alpha_1,\beta_1)} \circ \varphi^{-1}_{(\alpha_2,\beta_2)}(x) = \big(\phi_{\alpha_1} \circ \phi^{-1}_{\alpha_2}(x^1, \dots, x^m), \psi_{\beta_1} \circ \psi^{-1}_{\beta_2}(x^{m+1}, \dots, x^{m+n})\big)
\end{aligned}$$

And the Jacob matrix is defined by

$$J\big(\varphi_{(\alpha_1,\beta_1)} \circ \varphi^{-1}_{(\alpha_2,\beta_2)}\big)(x) = \begin{pmatrix} \partial_j(\phi_{\alpha_1} \circ \phi^{-1}_{\alpha_2})^i(x) & \\ & \partial_j(\psi_{\beta_1} \circ \psi^{-1}_{\beta_2})^i(x) \end{pmatrix}$$

By the orientability of $M$ and $N$, we can always choose the open cover to make the following Jacobian positive

$$\det J(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})(x) = \det \left( \frac{\partial(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})^i}{\partial x^j}(x) \right)_{m\times m} > 0$$
$$\det J(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})(x) = \det \left( \frac{\partial(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})^i}{\partial x^j}(x) \right)_{n\times n} > 0$$

Thus, the Jacobian of the corresponding chart transition map on $M \times N$

$$\det J\big(\varphi_{(\alpha_1,\beta_1)} \circ \varphi^{-1}_{(\alpha_2,\beta_2)}\big)(x) = \det J(\phi_{\alpha_1} \circ \phi_{\alpha_2}^{-1})(x) \cdot \det J(\psi_{\beta_1} \circ \psi_{\beta_2}^{-1})(x) > 0$$

Thus, there exists an orientable atlas on $M \times N$; $M \times N$ is orientable. $\square$

**Example 1.5** (Real Projective Space $\mathbb{RP}^n$)**.** *The real projective space is given by* $\mathbb{RP}^n := (\mathbb{R}^{n+1} - \{0\})/\sim$, *where the equivalence relation is given by*

$$\forall x, y \in (\mathbb{R}^{n+1} \setminus \{0\}) : \exists \lambda \in \mathbb{R} \setminus \{0\} : (x \sim y \iff x = \lambda y)$$

*Consider the quotient map* $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$*, the topology on* $\mathbb{RP}^n$ *is induced by the quotient map, i.e.*

$$V \subseteq \mathbb{RP}^n \text{ open} \iff \pi^{-1}(V) \text{ is open in } \mathbb{R}^{n+1}$$

*It is obvious that the quotient topology ensures that* $\pi$ *is continuous and surjective, and thus* $\mathbb{RP}^n$ *is second countable and Hausdorff. Real projective space can also be written as a quotient space of* $S^n/\sim$*, where* $\forall x, y \in S^n$

$$x \sim y \iff x = -y$$

*The chart on* $\mathbb{RP}^n$ *is given by*

$$U_k = \{[x] \in \mathbb{RP}^n \mid x = (x^1, \dots, x^{n+1}), x^k \neq 0\}, \quad \text{where } [x^1 : \dots : x^k] \sim \left[\frac{x^1}{x^{n+1}} : \dots : \frac{x^{k-1}}{x^k} : 1 : \frac{x^{k+1}}{x^k} : \dots : \frac{x^{n+1}}{x^k}\right]$$

*The overlapping region of two charts with* $k \neq l$ *is just*

$$U_k \cap U_l = \{[x] = [x^1 : \dots : x^{n+1}] \mid x^k, x^l \neq 0\}$$

*By definition* $\mathbb{RP}^n = \bigcup_{k=1}^{n+1} U_k$*, the coordinate map is given by* $\varphi_k : U_k \to \mathbb{R}^n$

$$\varphi_k([x]) = (\eta^1, \eta^2, \dots, \eta^n)$$

*where* $\eta^i = x^i/x^k$ *if* $i < k$ *and* $\eta^i = x^{i+1}/x^k$ *if* $i \geq k$*. To compute the transition map, denote* ${}_j\xi^i = x^i/x^j$*, then*

$$\varphi_l \circ \varphi_k^{-1}({}_k\xi^1, \dots, {}_k\xi^{k-1}, {}_k\xi^{k+1}, \dots, {}_k\xi^{n+1}) = ({}_l\xi^1, \dots, {}_l\xi^{l-1}, {}_l\xi^{l+1}, \dots, {}_l\xi^{n+1})$$

*Since the coordinate has the relation*

$$\begin{aligned} {}_l\xi^h &= x^h/x^l = \left(\frac{x^h}{x^k}\right) / \left(\frac{x^l}{x^k}\right) = {}_k\xi^h / {}_k\xi^l, \quad h \neq l, k \\ {}_l\xi^k &= x^k/x^l = ({}_k\xi^l)^{-1} \end{aligned}$$

*By the fact that* $x^l$ *and* $x^k$ *are nonzero, then it is obvious that* $\varphi_l \circ \varphi_k^{-1} : \varphi_k(U_k \cap U_l) \to \varphi_l(U_l \cap U_k)$ *is a smooth map. Since this statement is general,* $U_k$ *and* $U_l$ *are smoothly compatible* $\forall k, l$*. Thus,* $\mathbb{RP}^n$ *is a smooth manifold.*

The generalization of the projective space is called the Grassmannian, which is also a smooth manifold.

**Example 1.6** (Grassmannian). *Given an finite dimensional vector space $V$ such that* $\dim V = n$, $n \geq k$, *then the Grassmannian on $V$ is given by*

$$\mathrm{Gr}_k(V) := \{W \subseteq V \text{ linear subspace} \mid \dim W = k\}$$

**Remark 1.1.** *The Grassmannian can reduce to projective space* $\mathrm{Gr}_1(\mathbb{R}^{n+1}) \cong \mathrm{Gr}_n(\mathbb{R}^{n+1}) \cong \mathbb{RP}^n$.

**Theorem 1.7.** Grassmannian $\mathrm{Gr}_k(V)$ wirh $V$ being $n$-dimensional $\mathbb{R}$-vector space is a $(n-k)k$ dimensional $C^\infty$-manifold.

*Proof.* WOLG, take $V = \mathbb{R}^n$. It is an obvious fact that the Grassmannian is a quotient manifold from the set of $k$-frames in $V$, denoted as

$$\mathrm{Fr}_k(\mathbb{R}^n) := \{(v_1, \dots, v_k) \in \mathbb{R}^{nk} \mid (v_1, \dots, v_k) \text{ linear independent}\} \cong \{F \in M_{n\times k}(\mathbb{R}) \mid \operatorname{rank} F = k\}$$

Grassmannian has the topology as a quotient topology from $\mathrm{Fr}_k(\mathbb{R}^n)$ with the equivland class $\forall F_1, F_2 \in \mathrm{Fr}_k(\mathbb{R}^n)$

$$F_1 \sim F_2 \iff \exists M \in \mathrm{GL}_k(\mathbb{R}) : F_1 = F_2 M$$

i.e., $\mathrm{Gr}_k(\mathbb{R}^n) \cong \mathrm{Fr}_k(\mathbb{R}^n)/\mathrm{GL}_k(\mathbb{R}^n)$. Thus, the topology on the Grassmannian is defined by the quotient map $\pi : \mathrm{Fr}_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$

$$U \subseteq \mathrm{Gr}_k(\mathbb{R}^n) \text{ is open} \iff \pi^{-1}(U) \text{ is open.}$$

Open sets in the $k$-frame are given by $U_I = \{F \in \mathrm{Fr}_k(\mathbb{R}^n) \mid \det F_I \neq 0\}$. Where $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ such that $i_a \neq i_b$ $\forall a \neq b$, and $F_I \in M_k(\mathbb{R})$ is defined by $(F_I)_{ab} = F_{i_a b}$, also, since $U_I$ is invariant under $\mathrm{GL}_k(\mathbb{R})$-right action $\rho : \mathrm{Fr}_k(\mathbb{R}^n) \times \mathrm{GL}_k(\mathbb{R}) \to \mathrm{Fr}_k(\mathbb{R}^n)$, $\rho(A, M) = AM$. Thus, $U_I$ can be viewed as the open set in $\mathrm{Gr}_k(\mathbb{R}^n)$.
Take the local coordinate of each chart $U_I$ be

$$\begin{aligned} \phi_I : U_I &\to M_{(n-k)\times k}(\mathbb{R}) \cong \mathbb{R}^{(n-k)k} \\ F_I &\to F_{I^c} \end{aligned}$$

As an example, consider $n = 4$, $k = 2$. Take $I = (2, 3)$ (i.e., $I^c = (1, 4)$), then the frame and the coordinate are given by

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \\ f_{41} & f_{42} \end{pmatrix} \sim A = FM = \begin{pmatrix} * & * \\ 1 & 0 \\ 0 & 1 \\ * & * \end{pmatrix}, \quad \begin{aligned} F_I &= \begin{pmatrix} f_{21} & f_{22} \\ f_{31} & f_{32} \end{pmatrix}, \ M = (F_I)^{-1} \\ \phi_I(F) &= F_{I^c} = \begin{pmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{pmatrix} \end{aligned}$$

Where $\phi_I(F) = F_{I^c}$ up to the fixed identification $M_{(n-k)\times k}(\mathbb{R}) \cong \mathbb{R}^{(n-k)k}$. For $M \in M_{(n-k)\times k}(\mathbb{R})$, let $A^{(J)}(M) \in M_{n\times k}(\mathbb{R})$ be the matrix whose $J$-rows equal $I_k$ and whose $J^c$-rows equal $M$ (in order). Set

$$B_{I\leftarrow J}(M) := \big(A^{(J)}(M)\big)_I, \qquad C_{I\leftarrow J}(M) := \big(A^{(J)}(M)\big)_{I^c}$$

Then, on the intersection domain

$$U_I \cap U_J = \{\, M \in M_{(n-k)\times k}(\mathbb{R}) \mid \det B_{I\leftarrow J}(M) \neq 0 \,\}$$

The transition map is then given by

$$\phi_I \circ \phi_J^{-1}(M) = C_{I\leftarrow J}(M)\big(B_{I\leftarrow J}(M)\big)^{-1}$$

Since the transition map is linear, the Grassmannian is a $C^\infty$-map. $\square$

For the next proposition about orientability, we need to introduce a topological operation first

**Definition 1.8** (The Connected Sum)**.** Let $M_1$ and $M_2$ be connected $n$-dimensional smooth manifolds, take points $p_1 \in M_1$ and $p_2 \in M_2$. Take local coordinate systems on $M_1$ and $M_2$, denotes as $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ such that $\varphi_1(p_1) = \varphi_2(p_2) = 0 \in \mathbb{R}^n$, and

$$\varphi_1(U_1) = \varphi_2(U_2) = B_2(0) = \Big\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n}(x^i)^2 < 4\Big\}$$

Denote the set

$$A(1/2, 2) = B_2(0) - \overline{B_{1/2}(0)} = \Big\{x \in \mathbb{R}^n \mid \frac{1}{4} < \sum_{i=1}^{n}(x^i)^2 < 4\Big\}$$

and its preimage are $V_1 = \varphi_1^{-1}(A(1/2, 2))$, $V_2 = \varphi_2^{-1}(A(1/2, 2))$. Consider the map

$$\phi : A(1/2, 2) \to A(1/2, 2), \phi(x) = x\Big(\sum_{i=1}^{n}(x^i)^2\Big)^{-1}$$

The following lemma is significant:

**Lemma.** *$\phi$ is a diffeomorphism.*

*Proof.* By the smoothness and bijectivity of $f(x) = x$ and $f(x) = 1/|x|$ when $x > 0$, the lemma is obvious. □

By the lemma above, by the smoothness of the manifolds $M_1$ and $M_2$, the map $\varphi_2^{-1} \circ \phi \circ \varphi_1 : V_1 \to V_2$ is also a diffeomorphism. Consider the quotient space:

$$M_1 \# M_2 = \big(M_1 - \varphi_1^{-1}(\overline{B_{1/2}(0)})\big) \sqcup \big(M_2 - \varphi_2^{-1}(\overline{B_{1/2}(0)})\big)/\sim$$

Where the quotient is being defined based on the map $\varphi_2^{-1} \circ \phi \circ \varphi_1$

$$\forall x \in V_1 : \forall y \in V_2 : \big(x \sim y \iff y = \varphi_2^{-1} \circ \phi \circ \varphi_1(x)\big)$$

The smooth manifold $M_1 \# M_2$ is called the connected sum of $M_1$ and $M_2$.

**Proposition 1.4.** *The connected sum of two orientable $n$-dimensional $C^\infty$-manifolds is still orientable.*

*Proof.* Consider connected smooth manifolds $M_1$ and $M_2$ in the definition of connected sum above, suppose $M_1$ and $M_2$ are both orientable. By the given definition of the connected sum of manifolds, the only place that needs to be examined is the open set that includes the quotient part. For arbitrary open sets $V_1$ and $V_2$ that contain the quotient part of the manifold, the chart transition map is given by

$$\varphi_2 \circ (\varphi_2^{-1} \circ \phi \circ \varphi_1) \circ \varphi_1^{-1} = \phi$$

which we already know is a diffeomorphism, since $M_2$ is given to be orientable, it is safe to change a coordinate map by composition with a reflection transformation

$$\hat{\varphi}_2 = \mathcal{P} \circ \varphi_2(x), \quad \mathcal{P}x = (-x^1, \ldots, x^n)$$

Where the chart transition map can be expanded in the new chart as $\hat{\varphi}_2 \circ (\varphi_2^{-1} \circ \phi \circ \varphi_1) \circ \varphi_1^{-1}$, the Jacobian of the transition map is then given by

$$\det J(\hat{\varphi}_2 \circ \varphi_2^{-1} \circ \phi \circ \varphi_1 \circ \varphi_1^{-1}) = \det J(\mathcal{P} \circ \phi) = -\det J(\phi)$$

Take $r^2 = \|x\|^2 = \sum_i (x_i)^2$, then $\phi(x) = x/r^2$, then we can calculate the partial derivative

$$J_{ij}(\phi) = \frac{\partial \phi^i}{\partial x^j} = \delta^i{}_j r^{-2} + x^i \frac{\partial r^{-2}}{\partial x^j} = \delta^i{}_j r^{-2} - 2x^i x_j r^{-4}$$

which, in matrix notation, is

$$J(\phi) = (r^{-2}I - 2r^{-4}xx^T)$$

And the Jacobian is just the determinant of this matrix

$$\det J(\phi) = (r^{-2})^n \det\bigl(I - 2r^{-2}xx^T\bigr)$$

To compute the determinant, we will need a lemma

**Lemma** (Sylvester Identity of Determinants)**.** *If $A \in M_{m*n}(\mathbb{R})$, and $B \in M_{n*m}(\mathbb{R})$, then*

$$\det(I + AB) = \det(I + BA)$$

*Proof.* Consider the following block matrix in $(m + k) \times (m + k)$,

$$M = \begin{pmatrix} I_m & A \\ B & I_k \end{pmatrix}$$

By elementary transformation, we can get

$$\det M = \det \begin{pmatrix} I_m - AB & O \\ B & I_k \end{pmatrix} = \det \begin{pmatrix} I_m & A \\ O & I_k - BA \end{pmatrix}$$

Which proves the lemma $\det(I_m - AB) = \det(I_k - BA)$ □

With the lemma, $\det J(\phi) = r^{-2n}(1 - 2r^{-2}\|x\|^2) = -r^{-2n} < 0$. Then, $\det(\mathcal{P} \circ \phi) = r^{-2n} > 0$. This means $M_1 \# M_2$ is orientable. □

The orientation of smooth manifolds can also be defined by differential forms. In Chapter 2, we will prove the equivalence of the two definitions. In the last part of this section, we will introduce the connectivity of manifolds and their relation with orientation.

**Proposition 1.5.** *If $M$ is a connected topological manifold, then $M$ is path-connected.*

*Proof.* Note that $M$ is connected, which indicates that there do not exist any open sets $U, V \in \mathcal{O}_M$ like $M = U \sqcup V$. Locally, take $p \in M$ and the chart $(U, \varphi)$ contains $p$. By the definition of a chart,

$$\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$$

By the definition of open sets, we can take $B_r(\varphi(p)) \subseteq \varphi(U)$. Let $V = \varphi^{-1}\bigl(B_r(\varphi(p))\bigr)$. By the given coordinate map, $V$ is open and path-connected as an inherent property of Euclidean space. Thus, $M$ is locally path-connected.
Thus, for some $p \in M$, it is sufficient to take a branch connected to a path not empty $C_p \subseteq M$. By path-connectivity, $\forall q \in C_p : \exists V \in \mathcal{O}_M$ such that $q \in V \subseteq C_p$, makes $C_p$ open.

**Lemma.** *Let $X$ be a locally path-connected topological space, and the path-connected branch containing $x \in X$ is given by*

$$C_x = \bigl\{y \in X \mid \exists \gamma \in C^0\bigl([0, 1], X\bigr), \gamma(0) = x, \gamma(1) = y\bigr\}$$

*Then, $\forall x, y \in X$ either $C_x = C_y$ or $C_x \cup C_y = \emptyset$ and $\bigcup_{x \in X} C_x = X$, i.e., the locally path-connected branch constructs an equivalence class.*

*Proof.* We need to check reflexivity, symmetry, and transitivity to prove that the relation above is an equivalence relation. First, define the relation as

$$\forall x, y \in X : \Bigl(x \sim y \iff \exists \gamma \in C^0\bigl([0, 1], X\bigr) : \gamma(0) = x, \gamma(1) = y\Bigr)$$

1. (Reflexivity) $x \sim x$ by the constant map $\forall a \in [0, 1] : \gamma(a) = x$.

2. (Symmetry) If $x \sim y$, i.e., $\gamma : [0,1] \to X$ connected $x$ and $y$, then $\forall t \in [0,1] : \bar{\gamma}(t) = \gamma(1-t)$ ensures that $y \sim x$.
3. (Transitivity) Suppose for $x, y, z \in X$, $x \sim y$ by path $\gamma_1$ and $y \sim z$ by path $\gamma_2$, then consider

$$\gamma(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le 1/2 \\ \gamma_2(2t-1), & 1/2 \le t \le 1 \end{cases}$$

thus, $(x \sim y) \wedge (y \sim z) \implies (x \sim z)$.

Thus, $\sim$ is an equivalence relation, and the lemma has been proved. □

Thus, if $C_p \neq M$, then $\exists p \neq q \in M : C_p \sqcup C_q = M$, which disobeys the connectivity of $M$. Thus, $C_p = M$, $M$ is a path-connected set. □

**Definition 1.9** (Orientation)**.** Let $M$ be an orientable smooth manifold, and $\mathcal{D}$ be an orientable cover. If every chart that is compatible (in the sense of orientation) with $(U, \varphi) \in \mathcal{D}$ is in $\mathcal{D}$, then $\mathcal{D}$ is said to be an orientation.

**Proposition 1.6.** *Connected orientable smooth manifolds always have exactly two orientations.*

*Proof.* Let $M$ be the connected orientable smooth manifold, take an orientable atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta) : \beta \in J\}$. We take $(U_\alpha, \phi_\alpha) \in \mathcal{A}$ and $(V_\beta, \psi_\beta) \in \mathcal{B}$, such that $U_\alpha \cap V_\beta \neq \emptyset$, and $\forall p \in U_\alpha \cap V_\beta$:

$$f(p) = \frac{J(\phi_\alpha \circ \psi_\beta^{-1})}{|J(\phi_\alpha \circ \psi_\beta^{-1})|}(p)$$

Since $A$ and $B$ are both orientable atlas, $f$ can be either 1 or $-1$, which indicates $\mathcal{B} = \mathcal{A}$ or $\mathcal{B} = \mathcal{A}^-$ □

**Example 1.7** (Orientability of Real Projective Space)**.** *By the previous example 1.5 of projective spaces, the transition map of* $\mathbb{R}\mathrm{P}^n$ *is given by*

$$\varphi_l \circ \varphi_k^{-1}({}_k\xi^1, \dots, {}_k\xi^{k-1}, {}_k\xi^{k+1}, \dots, {}_k\xi^{n+1}) = ({}_l\xi^1, \dots, {}_l\xi^{l-1}, {}_l\xi^{l+1}, \dots, {}_l\xi^{n+1})$$

*where* ${}_j\xi^i = x^i/x^j$*. Thus, for* $h \neq k, l$

$$\frac{\partial_l \xi^h}{\partial_k \xi^\beta} = \begin{cases} \frac{1}{{}_k\xi^l}, & \beta = h \\ -\frac{{}_k\xi^h}{({}_k\xi^l)^2}, & \beta = l \\ 0, & \text{otherwise} \end{cases}$$

*And for* $h = k$

$$\frac{\partial_l \xi^h}{\partial_k \xi^\beta} = \begin{cases} -\frac{1}{({}_k\xi^l)^2}, & \beta = l \\ 0, & \text{otherwise} \end{cases}$$

*With proper coordinate transformation (exchange the order of coordinates) that moves* $\beta = l$ *to the last column and* $h = k$ *to the last row, the upper-triangular form of the Jacobi matrix is given by*

$$J(\varphi_l \circ \varphi_k^{-1}) = \begin{pmatrix} \frac{1}{{}_k\xi^l} I_{n-1} & \left[\frac{-{}_k\xi^h}{({}_k\xi^l)^2}\right]_{h \neq k,l} \\ 0 & -\frac{1}{({}_k\xi^l)^2} \end{pmatrix}$$

*Thus, the determinant is given by*

$$\det J(\varphi_l \circ \varphi_k^{-1}) = ({}_k\xi^l)^{-n-1}$$

*Notice that if* $n+1$ *(i.e.* $n$ *is odd) is even, then* $({}_k\xi^l)^{-n-1}$ *is always positive, and when* $n$ *is even, the* $({}_k\xi^l)^{-n-1}$ *does not have a certain sign. Thus,* $\mathbb{R}\mathrm{P}^n$ *orientable is* $n$ *is odd.*

## 1.2 Submanifolds

The study of submanifolds is naturally introduced from the concept of a subset and the rank of a map.

**Example 1.8** (Invertible Linear Map). *Consider the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, from undergraduate linear algebra, $A$ is invertible iff $\det A \neq 0$; we also say that the linear map is "full rank". From the perspective of smooth manifolds, we can extend the concept of full rank to the map on manifolds.*
*Any linear map on $\mathbb{R}^n$ can be written in matrix form, which can also be viewed as the elements in $\mathbb{R}^{n^2}$, on which we can induce a norm. For $A : \mathbb{R}^n \to \mathbb{R}^n$, the $L^2$-norm on $\mathbb{R}$ can be defined by*

$$\begin{aligned} \|\cdot\| : &\mathbb{R}^n \to [0,\infty) \\ &x \mapsto \|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \end{aligned}$$

*Then, we can induce the operator norm in the following way:*

**Definition 1.10** ((Linear) Operator Norm). With given vector norm $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$, the linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ has norm:

$$\|A\| := \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

*Then we have the following proposition*

**Proposition 1.7.** *The linear map $B : \mathbb{R}^n \to \mathbb{R}^n$ satisfies that $\|B\| < 1$, then $I_n - B$ is invertible.*

*Proof.* To prove the bijectivity, in this case, we only need to check the kernel of the linear map $I_n - B$. Consider the equation $(I_n - B)x = 0$, then $x = Bx$. Thus, the 2-norm is given by

$$\|x\| = \|Bx\| \leq \|B\|\|x\|$$

Which indicates $(1 - \|B\|)\|x\| \leq 0$. Known that $1 - \|B\| > 0$, then

$$(I_n - B)x = 0 \iff x = 0$$

Which means $\ker(I_n - B) = \{0\}$. Since $B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$, $B$ is bijective. $\square$

A significant observation from the example above is that the identity matrix is still nondegenerate after a small perturbation. Thus, we have the following generalization.

**Definition 1.11** (Rank of $C^k$-Maps). Let $f : M \to N$ to be a $C^k$-map $(k \geq 1)$ between smooth manifolds, let $p \in M$ and $q = f(p) \in N$. Taking local maps $(U, \varphi)$ on $M$ and $(V, \psi)$ on $N$, the rank of map $f$ at point $p$ is given by

$$\mathrm{rank}_p f := \mathrm{rank}\, J(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}$$

**Proposition 1.8.** *The definition of rank does not depend on the choice of coordinates.*

*Proof.* Consider the chart transition map, $\Theta := \tilde{\varphi} \circ \varphi^{-1}$ and $\Phi : \tilde{\psi} \circ \psi^{-1}$, by definition, $\Theta : \varphi(\tilde{U} \cap U) \to \tilde{\varphi}(\tilde{U} \cap U)$ and $\Phi : \psi(\tilde{V} \cap V) \to \tilde{\psi}(\tilde{V} \cap V)$ are $C^k$-diffeomorphisms $(k \geq 1)$. Thus, consider $U, \tilde{U} \subseteq M$, $V, \tilde{V} \subseteq N$ and $\dim M = m$, $\dim N = n$.

$$P := J(\Theta) = \Big(\frac{\partial(\tilde{\varphi} \circ \varphi)^i}{\partial x^j}\Big)_{1\leq i,j\leq m}, \quad Q := J(\Phi) = \Big(\frac{\partial(\tilde{\psi} \circ \psi)^i}{\partial y^j}\Big)_{1\leq i,j\leq n}$$

are invertible matrices. Then, consider the local coordinate representation: $\tilde{f} = \psi \circ f \circ \varphi$ and $\tilde{\psi} \circ f \circ \tilde{\varphi} = \Phi \circ \tilde{f} \circ \Theta^{-1}$, we can compute the rank of the Jacobian:

$$\operatorname{rank} J(\Phi \circ \tilde{f} \circ \Theta^{-1})_{\varphi(p)} = \operatorname{rank}(PJ(\tilde{f})Q)_{\varphi(p)} = \operatorname{rank} J(\psi \circ f \circ \varphi)_{\varphi(p)} = \operatorname{rank}_p f$$

Thus, the rank of a function at a point $\operatorname{rank}_p f$ is well defined under coordinate transformations. □

An important way to define a submanifold is to consider the image (preimage) of a smooth map, formally via the inverse function theorem (IFT). The IFT on smooth manifolds is directly induced by the IFT on Euclidean spaces.

**Theorem 1.12** (Inverse Function Theorem)**.** Let $f : M^n \to N^n$ be the $C^k$-map between $C^\infty$-manifold $M$ and $N$, if $\operatorname{rank}_p f = n$, then there exists an open neighborhood $U \subseteq M$ that contains $p$ and $V \subseteq N$ contains $q = f(p)$, such that $f|_U : U \to V$ is a $C^k$-diffeomorphism.

*Proof.* As a local constraint to differentiable functions on $C^\infty$-manifolds, it is sufficient to take $M = N = \mathbb{R}^n$, and take $p = 0 \in \mathbb{R}^m$. Since $\operatorname{rank}_0 F = n$, it is sufficient to consider the Jacobi matrix of the map $J(f)_0 = I_n$ by the composition of a linear map, which will not change the result. Thus, at the origin $p = 0$, the map $F$ is a perturbation of the identity map. Let

$$\forall x \in \mathbb{R}^n : \; g = f(x) - x$$

Then, $J(g)_0 = J(f)_0 - I_n = 0$, then $\exists \epsilon > 0$,

$$\forall x \in \overline{B_\epsilon(0)} : \; \|J(g)_x\| \leq \frac{1}{2}$$

And by the mean value theorem

$$\forall x_1, x_2 \in \overline{B_\epsilon(0)} : \; \|g(x_1) - g(x_2)\| \leq \|J(g)_\xi\| \|x_1 - x_2\| \leq \frac{1}{2}\|x_1 - x_2\|$$

Which indicates that $g$ is 1/2-Lipschitz continuous, $\forall \xi \in B_\epsilon(0)$, solving the equation $y = f(x)$ is just finding the fixed point of $g_y(x) = x - (f(x) - y)$. The function is also 1/2-Lipschitz, since $g_y(x_1) - g_y(x_2) = g(x_1) - g(x_2)$. By the Banach fixed-point theorem, $\exists x_y \in B_\epsilon(0) : g_y(x_y) = x_y + y - f(x_y) = x_y$, i.e., $f(x_y) = y$ has a unique solution $x_y = h(y)$ such that

$$f(h(y)) = y \iff y - g(h(y)) = h(y)$$

Given $V = B_{\epsilon/2}(0)$ and $U = f^{-1}(V) \cap B_\epsilon(0)$, to prove that $h : V \to U$ is a $C^k$-diffeomorphism, we only need to check the continuity and $C^k$ of the map.

- $h$ is continuous. Since $\forall y_1, y_2 \in B_\epsilon(0)$

$$\begin{aligned} \|h(y_1) - h(y_2)\| &\leq \|y_1 - y_2\| - \|g(h(y_1)) - g(h(y_2))\| \\ &\leq \|y_1 - y_2\| - \frac{1}{2}\|h(y_1) - h(y_2)\| \end{aligned}$$

  Thus, $\|h(y_1) - h(y_2)\| \leq \frac{2}{3}\|y_1 - y_2\|$ is Lipschitz. Thus, $h$ is continuous.

- $h$ is $C^k$. Firx a base point $y_0 \in V$, given any $y \in V$,

$$\begin{aligned} h(y) - h(y_0) &= (y - y_0) + \big(g(h(y)) - g(h(y_0))\big) \\ &= (y - y_0) + J(g)_{h(y_0)}\big(h(y) - h(y_0)\big) + O(\|h(y) - h(y_0)\|) \\ \implies y - y_0 &= \big[I_n - J(g)_{h(y_0)}\big](h(y) - h(y_0)) + O(\|h(y) - h(y_0)\|) \end{aligned}$$

  Which implies that the Jacobian exists and $Jh(y_0) = I_n - J(g)_{h(y_0)} = (Jf(h(y)_0))^{-1}$. Repeat this procedure, by the $C^k$-differentiability of $f$, $h$ is a $C^k$-map.

Thus, we proved that $f$ is locally a $C^k$-diffeomorphism. $\square$

Based on the IFT, it is sufficient to define the following behavior of maps, which provides us with a nice hint on how to study the submanifold.

**Definition 1.13** (Immersion, embedding, and submersion)**.** Let $f : M^m \to N^n$ be a $C^k$-map between differentiable manifolds. If $\operatorname{rank}_p f = m \ \forall p \in M$, $f$ is said to be a $C^k$-immersion. If $f : M \to f(M)$ is a $C^k$-immersion and also a homeomorphism ($f(M) \subseteq N$ has the subset topology of $N$), then $f$ is a $C^k$-embedding. If $\operatorname{rank}_p f = n \ \forall p \in M$, then $f$ is a $C^k$-submersion.

Here are some examples:

**Example 1.9** (An immersion that is not embedding (not injective))**.** *Consider the map $f : \mathbb{R} \to \mathbb{R}^2$ such that*

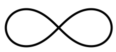
$$f(\theta) = (\cos\theta, \sin\theta)$$

*It is easy to check the smoothness and* $\operatorname{rank} f = 1$*. However, since $f(\theta) = f(\theta + 2n\pi)$, $f$ is not injective, and thus, not an embedding.*

**Example 1.10** (An injective immersion that is not embedding)**.** *Consider the map $f : \mathbb{R} \to \mathbb{R}^2$ defined by*

$$f(t) = \left(\frac{t^3+t}{t^4+1}, \frac{t^3-t}{t^4+1}\right) \quad \forall t \in \mathbb{R}$$

*$f$ is injective and* $\operatorname{rank} f = 1$*. However, it is not a submersion since the image of the map is the lemniscate shown in the following figure:*



*Which is clearly not homeomorphic to $\mathbb{R}$ and thus, not diffeomrorphic to $\mathbb{R}$*

**Example 1.11** (Embedding of torus)**.** *The torus is given by $T^2 = S^1 \times S^1$. Consider the map $f : T^2 \to \mathbb{R}^3$ defined by taking $R > r > 0$ and*

$$f(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\phi}) = \big((R - r\cos\phi)\cos\theta,\ (R - r\cos\phi)\sin\theta,\ r\sin\theta\big), \quad \forall \theta, \phi \in [0, 2\pi]$$

*is a smooth embedding of $T^2$ into $\mathbb{R}^3$.*

**Theorem 1.14** (Local Form of Immersion)**.** Let $f : M^m \to N^n$ be an immersion between differentiable manifolds, $f(p) = q$. Then, exist charts $(U, \varphi)$ on $M$ contains $p$ and $(V, \psi)$ on $N$ contains $q$ such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ given by

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0)$$

*Proof.* Since the result is local, we can consider only the case that $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ $(m < n)$. Then, let the component form of the immersion is given by $f(x) = (f_1(x), \dots, f_n(x)) \ \forall x \in \mathbb{R}^m$, consider the Jacobian of the immersion (which is a $n \times m$ matrix with rank $m$)

$$J(f)_x = \frac{\partial(f_1, \dots, f_n)}{\partial(x^1, \dots, x^m)}(x)$$

By Gauss reduction, we can take the $m \times m$ component nondegenerate near $x = 0$

$$\left|\frac{\partial(f_1, \dots, f_m)}{\partial(x^1, \dots, x^m)}\right|(x) \neq 0$$

And the map defined by $g(x^1,\dots,x^n) = (f_1,\dots,f_m,f_{m+1}-x^{m+1},\dots,f_n-x^n)$ has Jacobian

$$J(g) = \begin{pmatrix} \left[\frac{\partial f_i}{\partial x^j}\right]_{m\times m} & 0 \\ * & I_{n-m} \end{pmatrix}$$

It is clear that this Jacobian is nondegenerate near $x=0$. By the inverse function theorem, $\exists U', V$ charts such that $g|_{U'} : U \to V$, then $\exists \psi : V \to U'$ be the inverse of $g|_{U'} \circ \psi = \mathrm{id}_V$. Let

$$U = \{(x^1,\dots,x^m) \in \mathbb{R}^m \mid (x^1,\dots,x^m,0,\dots,0) \in U'\}$$

Then, $f|_U : U \to V$ has the local coordinate representation with the local coordinate $(U,\varphi)$

$$\psi \circ f(x^1,\dots,x^m) = \psi(f_1(x),\dots,f_n(x)) = (x^1,\dots,x^m,0,\dots,0)$$

which proves the theorem. □

**Corollary.** *Any immersion is locally embedded.*

The corollary is directly from the local form. Since the results show that immersions can be expressed (using proper coordinates) in local level sets of coordinate functions.

**Theorem 1.15** (Local Form of Submersion)**.** Let $f : M^m \to N^n$ be a submersion between differentiable manifolds, $f(p) = q$. Then, exist charts $(U,\varphi)$ on $M$ contains $p$ and $(V,\psi)$ on $N$ contains $q$ such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ given by

$$\psi \circ f \circ \varphi^{-1}(x^1,\dots,x^m) = (x^1,\dots,x^n)$$

*Proof.* It is still sufficient to take $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ $(m > n)$, and $f(x) = (f_1(x),\dots,f_n(x))$. Since $f$ is a submersion, $\operatorname{rank} f(x) = n$. Thus,

$$J(f)_x = \frac{\partial(f_1,\dots,f_n)}{\partial(x^1,\dots,x^m)}(x) \text{ is a } n\times m \text{ matrix with rank } n$$

That means, using Gauss reduction, the full rank block Jacobian can be written as

$$\left|\frac{\partial(f_1,\dots,f_n)}{\partial(x^1,\dots,x^n)}\right| \neq 0$$

Then, take the following construction: consider the function defined by $g(x^1,\dots,x^m) = (f_1,\dots,f_n,x^{n+1},\dots,x^m)$

$$J(g) = \begin{pmatrix} \left[\frac{\partial f_i}{\partial x^j}\right]_{n\times n} & * \\ 0 & I_{m-n} \end{pmatrix}$$

Thus, by the inverse function theorem, the function $g$ is a local diffeomorphism. Let $U'$ be the chart that $g|_{U'} : U' \to V$ invertible, $\psi(f_1,\dots,f_n,x^{n+1},x^m) = (x^1,\dots,x^m)$ be the local inverse. Then, take $\forall x \in V$ and $\psi$ as the local coordinate map on $V$

$$\psi \circ f(x^1,\dots,x^m) = (x^1,\dots,x^n)$$

Which proves the theorem. □

Similarly, we have the following corollary

**Corollary.** *Submersion must locally be an open map (and quotient map, if surjective).*

**Definition 1.16** (Immersion/Regular (Embedded) Submanifold)**.** Consider $M^m, N^n$ be smooth manifolds. If $i : M \hookrightarrow N$ is an injective immersion, then $M$ (with the induced topology by $i$) is the immersion submanifold of $N$; if $i$ is an embedding, then $M$ is the regular submanifold (or embedded submanifold) of $N$.

In the folloing example, let $M^m$, $N^n$ be smooth manifolds, $(U, \varphi = (x^1, \ldots, x^m))$ and $(V, \psi = (y^1, \ldots, y^n))$ be coodrdinate maps.

**Example 1.12** (Graphs are regular submanifolds)**.** *Consider the smooth map $f : M^m \to N^n$, the graph of $f$ is defined by*

$$\Gamma_f := \{(m, n) \in M \times N \mid n = f(m)\}$$

*The Graph of $f$ is a $m$ dimensional regular submanifolds of $M \times N$. Let the local coordinate representation of $f$ be $g = \psi \circ f \circ \varphi^{-1}$, the Jacobian of the inclusion map $\Gamma_f \overset{i}{\hookrightarrow} M \times N$ in local coordinate should be a $(m+n) \times n$ dimensional matrix*

$$J(i) = \begin{pmatrix} I_m \\ J(g) \end{pmatrix}$$

*Where $J(g)$ is the Jacobian of $g$ in dimension $n \times m$. It is obvious that $\operatorname{rank} i = n$, which means the inclusion map is an immersion. To prove it is an embedding, it is sufficient to prove that the projection map $p_M : M \times N \to M$ that sends $(p, q) \mapsto p$ is the continuous inverse from $\Gamma_f$ to $M$, which means that $i$ is an embedding.*

**Remark 1.2.** *The immersion submanifolds in $N$ do not require a subset topology from $N$. A counterexample is that the Example 1.10 is an immersion submanifold of $\mathbb{R}^2$, but is not a manifold if it contains a subset topology from $\mathbb{R}^2$.*

**Example 1.13** (Immersion Submanifold that Dense in $T^2$)**.** *Consider the inclusion map $i : \mathbb{R} \to T^2 = S^1 \times S^1$ defines by*

$$i(t) = (e^{2\pi \mathrm{i} t}, e^{2\pi\alpha \mathrm{i} t})$$

*for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The image $i(\mathbb{R})$ is dense in $T^2$ and thus, the inclusion map is not proper when the image is in the subset topology induced from $T^2$. That means the image cannot be a regular manifold in $T^2$.*

If $\pi : M \to N$ is some continuous map, a smooth local section $\sigma : U \to M$ of $\pi$ is a right inverse of $\pi$ in some open neighborhood $U$ of $\pi(p)$ in $N$, i.e., $\pi \circ \sigma = \mathrm{id}|_U$. Based on the local section, the submersion also has the following property:

**Theorem 1.17** (Local Section Theorem)**.** Suppose $M$ and $N$ are smooth manifolds and $\pi : M \to N$ is a smooth map. Then it is a smooth submersion if and only if every point of M is in the image of a smooth local section of $\pi$.

*Proof.* First, suppose $\pi : M \to N$ is a submersion, then, by the local coordinate representation of the submersion, there exists a local coordinate $(U, \varphi)$ such that

$$\psi \circ \pi \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$$

Then, we can find the local inverse $\sigma : V \subseteq N \to M$ with the local coordinate representation

$$\sigma(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$$

Since the choice of $p \in M$ is arbitrary, and we can always find a coordinate $\varphi$ that maps $p$ to $(0, \ldots, 0)$ and $\varphi(U) = B_\epsilon(0)$, every point of $M$ is in the image of a smooth local section of $\pi$.

Conversely, suppose $\forall p \in N : V \subseteq N$ is an open neighborhood of $p$, there exists $\sigma : V \subseteq N \to M$ such that $\pi \circ \sigma = \mathrm{id}|_V$. Then the Jacobian of the map is

$$\begin{aligned} J(\psi \circ \pi \circ \sigma \circ \psi^{-1}) &= J(\psi \circ \pi \circ \varphi^{-1}) J(\varphi \circ \sigma \circ \psi^{-1}) \\ &= I_n, \quad \text{which is in rank } n \end{aligned}$$

Then, $J(\psi \circ \pi \circ \varphi^{-1})$ is in rank $n$, and thus, $\forall p \in M : \mathrm{rank}_p \pi = n$, which means $\pi$ is a immersion. ☐

We will see an important example of submersion in this type, called the fiber bundle.

**Theorem 1.18** (The Chateceristic of Regular Submanifold)**.** Consider smooth manifolds $M^m$ and $N^n$. $M$ is the regular submanifold of $N \iff M$ is the topological subspace of $N$ and $\forall p \in M$, exists a local chart $U$ contains $p$ with coordinate map $\varphi = (x^1, \dots, x^n)$ such that

$$M \cap U = \{q \in U \mid x^i(q) = 0 \ \forall i = m+1, \dots, n\}$$

*Proof.* ($\Rightarrow$) Suppose $M$ is the regular submanifold of $N$. Then, the inclusion map $M \overset{i}{\hookrightarrow} N$ is an immersion. By the local coordinate form of immersion, we can find the local coordinate $(U, \varphi)$ and $(V, \psi)$ on $M$ and $N$

$$\begin{aligned} \psi \circ i \circ \varphi^{-1} &: \varphi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n \\ &\varphi(p) = (x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0) \end{aligned}$$

Then it is natural to defined the chart $(\tilde{U}, \tilde{\varphi})$ on $N$ that

$$\tilde{U} := \{q \in V \mid (x^1(q), \dots, x^m(q)) \in \varphi(U)\}$$

Where $\psi(q) = (x^1(q), \dots, x^n(q))$. Then we can check the coordinate representation on $M \cap \tilde{U}$. Let $q = i(p)$ for some $p \in U$. Thus

$$\psi(q) = \psi \circ i(p) = (x^1(q), \dots, x^m(q), 0, \dots, 0)$$

And thus $x^{m+1}(q) = \cdots = x^n(q) = 0$. Then, if we take $q \in \tilde{U}$ and $x^{m+1}(q) = \cdots = x^n(q) = 0$. Denote

$$u = (u^1, \dots, u^m) := (x^1(q), \dots, x^m(q)) \in \varphi(U)$$

Then $p = \varphi^{-1}(u^1, \dots, u^m) \in U$, and thus

$$\psi(q) = \psi \circ i \circ \varphi^{-1}(u^1, \dots, u^m) = (u^1, \dots, u^m, 0, \dots, 0)$$

Thus, $q \in M$, and $M \cap \tilde{U} = \{q \in \tilde{U} \mid x^i(q) = 0, m+1 \leq i \leq n\}$.
($\Leftarrow$) It is easy to show that the inclusion map satisfies the assumption in the theorem is a embedding. ☐

With the characteristics of a regular submanifold, we can prove the following theorem.

**Theorem 1.19** (Constant Rank Theorem)**.** Let $M^m$ and $N^n$ be smooth manifolds. $f : M^m \to N^n$ is $C^\infty$-map, if $\mathrm{rank}_p f = k \ \forall p \in M$. Then, $\forall p, q \in M, N$, there is a pair of locoal coordinates $(U, \varphi)$ and $(V, \psi)$ such that $f$ has local coordinate form

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

*Proof.* Since the claim is local, we can let $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ and

$$f(x^1, \dots, x^m) = (f_1(x), \dots, f_n(x))$$

And similar to the previous proofs, we can assume the matrix

$$M_f = \left[\frac{\partial f_i}{\partial x}\right]_{1\leq i,j\leq k}$$

has ranl $k$, i.e. $M_f$ is the largest invertible block in $J(f)$. The IFT inspires us to define the function

$$\varphi(x^1,\ldots,x^m) = (f_1,\ldots,f_k(x),x^{k+1},\ldots,x^m)$$

The Jacobian of $\varphi$ is given by

$$J(\varphi) = \begin{pmatrix} M_f & * \\ 0 & I_{m-k}\end{pmatrix}$$

which is nondegenerate at the origin, and implies that we can use IFT in an open neighborhood. Consider open neighborhood $U, V \subseteq \mathbb{R}^m$ of $0 \in \mathbb{R}^m$ such that $\varphi|_U : U \to V$ has local inverse $\varphi^{-1} : V \to U$. Then, take $\varphi$ as a local coordinate map and

$$F(x) = f\circ\varphi^{-1}(x^1,\ldots,x^m) = (x^1,\ldots,x^k,F^{k+1},\ldots,F^n)$$

Since $\operatorname{rank} f \equiv k$,

$$\frac{\partial F^i}{\partial x^j} = 0, \quad k+1\leq i\leq n,\ k+1\leq j\leq m$$

Which implies the function $F^i = F^i(x^1,\ldots,x^k)$. Then, defined $x = (x^1,\ldots,x^k)$, consider the coordinate transformation such that

$$\psi(x^1,\ldots,x^n) = \big(x^1,\ldots,x^k,x^{k+1}-F^{k+1}(x),\ldots,x^n-F^n(x)\big)$$

With this coordinate map

$$\begin{aligned}\psi\circ f\circ\varphi^{-1}(x^1,\ldots,x^m) &= \big(x^1,\ldots,x^k,(F^{k+1}-F^{k+1})(x),\ldots,(F^n-F^n)(x)\big)\\ &= \big(x^1,\ldots,x^k,0,\ldots,0\big)\end{aligned}$$

In this way, we proved the theorem. □

## 1.3 Tangent Spaces

As a locally Euclidean topological space, the open coordinate charts are homeomorphic to open sets in $\mathbb{R}^n$. Recall that in multivariable calculus, the key property is that every map $f:\mathbb{R}^m\to\mathbb{R}^n$ is locally linearized at a small neighborhood $p$ as a linear transformation, or "matrix". More precisely, a map $f:\mathbb{R}^m\to\mathbb{R}^n$ is said to be differentable at $a\in\mathbb{R}^m$ if

$$\exists! L\in \mathrm{Hom}_{\mathrm{Vect}}(\mathbb{R}^m,\mathbb{R}^n) : \lim_{h\to 0}\frac{\|f(a+h)-f(a)-Lh\|}{\|h\|} = 0$$

Which means the global differentiability is just saying $f(x+h)-f(x) = Lh+\alpha(x,h)$, where $\forall x\in\mathbb{R}^n$ : $\alpha(x,h) = o(h)$. To construct a similar (locally) linear structure on $C^\infty$-manifolds, we need the tangent space associated to each point on the manifold. In the following passage, $C^\infty(M)$ is the global smooth function on $M$ and $C_p^\infty$ is the function germ.

**Definition 1.20** ($C^\infty$-Function Germ at $p$)**.** The function germ is defined by $C_p^\infty = C^\infty(M)/\sim_p$, where the equivlance relation is given by

$$\forall f,g\in C^\infty(M) : f\sim_p g \iff \exists U_p\subseteq M : f|_{U_p} = g|_{U_p}$$

Where $U_p$ is an open neighborhood of $p$.

**Definition 1.21** (Tangent Vector)**.** Let $M$ be a smooth manifold, fix a point $p \in M$. A tangent vector at point $p$ is a $\mathbb{R}$-linear map $X_p : C_p^\infty \to \mathbb{R}$ such that $\forall f, g \in C_p^\infty$

$$X_p(f \cdot g) = f(p)X_p(g) + X_p(f)g(p)$$

The space of all vector spaces at $p$ is called the tangent space at $p$, denoted as $T_pM$.

The purpose of this definition is to define the differential of a smooth function on a manifold. In Euclidean space

$$\boxed{\text{Vectors } v_p = (v^1, \ldots, v^n)_p} \xleftrightarrow{\text{One to one correspondence}} \boxed{\text{Derivators } D_p = v^1\partial_1|_p + \cdots + v^n\partial_n|_p}$$

each vector $v$ has a corresponding directional derivativor $v \cdot \nabla$. The derivativor (algebraic) definition of a tangent vector is just to use the correspondence relation above, which uses the linearity and Leibniz law to define the tangent vector.

From the definition, the following properties are easy to check:

**Proposition 1.9** (Properties of Tangent Vectors)**.** *Let $M$ be a smooth manifold, $p \in M$. $\forall v_p \in T_pM$, the following statements are true*

1. $\forall f \in C^\infty(M) : f = \text{const} \implies v_p(f) = 0$
2. $\forall f, g \in C^\infty(M) : f(p) = g(p) = 0 \implies v_p(f \cdot g) = 0$

The proof is simply to apply the linearity and Leibniz law.

Other than the algebraic definition, there is a more geometrical way to define a tangent vector using smooth curves on a manifold $\gamma : I \to M$, where $I = (-\epsilon, \epsilon)$ for some $\epsilon > 0$.

**Definition 1.22** (Tangent Vector as Velocity of Curves)**.** Let $M$ be a smooth manifolds, $\gamma : I \to M$, a tangent vector along $\gamma$ is defined by

$$X(f) := \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0} \in T_pM$$

It is easy to check that $X$ is linear and Leibniz. We also need to show that the tangent vector in curve definition is well defined on function germs, i.e.

**Proposition 1.10** (Well-Definedness of Tangent Vectors)**.** *Let $M$ be a smooth manifold, $p \in M$. $f, g \in C^\infty(M)$ where $U_p$ is an open neighborhood of $p$. Then $\forall X_p \in T_pM$*

$$f|_{U_p} = g|_{U_p} \implies X_p(f) = X_p(g)$$

*Proof.* By the definition of tangent vector using curves, take $I_\epsilon := (-\epsilon, \epsilon)$ for some $\epsilon > 0$, and the curve $\gamma : I_\epsilon \to M$ such that $\gamma(0) = p$

$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0}$$

Since $f|_{U_p} = g|_{U_p}$, it is sufficient to take $0 < \epsilon' < \epsilon$ such that $\gamma(I_{\epsilon'}) \subseteq U_p$ and thus

$$X_p(f) = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma(t)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} g \circ \gamma(t)\Big|_{t=0} = X_p(g)$$

This proves the well-definedness. $\square$

[**Derek:** To be complete]

## 1.4 The Essential Fact of Lie Groups and Lie Algebra

# Chapter 2

# Fundamental Facts related to Functions on Manifolds

## 2.1 Partition of Unity

## 2.2 Sard's Theorem

**Definition 2.1** (Regular/Critical Points)**.** For $C^\infty$-map $f : M \to N$, $p \in M$ is a regular point if $f_{*,p}$ is surjective, otherwise, $p$ is a critical point.

## 2.3 Whitney Embedding Theorem

## 2.4 Tubular Neighborhood Theorem

## 2.5 Transversality

# Chapter 3

# Tangent Bundle and Calculus on Manifold

## 3.1 Tangent and Cotangent Bundles

## 3.2 Integration Curve and Flow of Vector Fields

## 3.3 Integrability Theorem

## 3.4 Vector Bundles

## 3.5 Differential Forms and Exterior Algebra

## 3.6 Tensor Bundles

## 3.7 Manifolds with Boundary

## 3.8 Stokes Formula

## Chapter 4

# de Rham Theory and Poincaré Lemma

### 4.1 Poincaré Lemma

### 4.2 The de Rham Cohomology Groups

### 4.3 Homotopy Invariance and Mayer-Vietoris Sequence

### 4.4 The de Rham Theorem

### 4.5 Poincaré Duality

# Part II

# A First Step to Geometry and Topology of Manifolds

# Chapter 5

# The Degree of Mapping and Intersecting Index of Submanifolds

## 5.1 The Concept of Homotopy

## 5.2 The Degree of Map

## 5.3 Poincaré–Hopf Theorem

## 5.4 Thom Class and Intersection Index

# Chapter 6

# Fundamental Groups and Covering Spaces of Manifolds

## 6.1 Fundamental Groups

## 6.2 Van Kampen Theorem

## 6.3 Covering Spaces and its Classification

## 6.4 Deck Transformations and Group Actions

## 6.5 Computation of Some Classical Cases, Liftings

# Chapter 7

# First Step to Geometry on Manifolds

## 7.1 Riemannian Manifolds as Metric Spaces

## 7.2 Connections

## 7.3 Geodisics and Jacobi Fields

## 7.4 Curvature

## 7.5 Comparison Theory

## 7.6 Geometry of Submanifolds

## 7.7 Homogeneous Space

## 7.8 Hodge Theory and Harmonic Forms

# Chapter 8

# Additional Topics on the Variation Problems

This chapter lies somewhat out of the book's main line of development, and may be omitted in a first reading

# Part III

# More Advanced Topics on Geometry and Topology

# Chapter 9

# Geometry on Smooth Fiber Bundles and Chern-Weil Theory

## 9.1 More about Lie Groups

## 9.2 Fiber Bundles and Principle Bundles

## 9.3 Connection and Curvature on Principle Bundles

## 9.4 Chern-Weil Theorem

## 9.5 Characteristic Classes

## 9.6 Bott Vanishing Theorem for Foliations

## 9.7 Bott and Duistermaat-Heckman Formulas

## 9.8 Gauss-Bonnet-Chern Theorem

## Chapter 10

# Introduction to Gauge Theory

### 10.1 Spinors and Dirac Operator

### 10.2 Linear Elliptic Operators on Manifolds

### 10.3 Fredholm Maps

### 10.4 The Seiberg–Witten Gauge Theory

This chapter lies somewhat out of the book's main line of development, and may be omitted in a first reading

# Chapter 11

# Introduction to Dynamical Systems on Manifolds

## Chapter 12

# More Algebraic Topology

# Chapter 13

# Introduction to Complex Geometry

# Chapter 14

# Introduction to Symplectic Geometry and Quantizations

# Chapter 15

# A Breif Introduction to TQFT

# Part IV

# Appendix

# Appendix A

# Review on General Topology

The study of modern geometry focuses on connecting the (local) geometrical quantities (e.g., curvature, length, volume, etc.) and the (global) topological properties (genus, Euler's characteristics, fundamental groups, etc.). Often, this connection is given by an integral or some other global operation. Furthermore, the study of geometry focuses on a concept called "topological manifolds", which is a topological space with some "good properties". Thus, it is useful to have a brief review of the concepts in general topology.

**Definition A.1** (Topological Space, Open and Closed Sets)**.** A topological space is a tuple $(\mathcal{T}, \mathcal{O})$, where $\mathcal{T}$ is a nonempty (ZFC) set and the additional structure is the topology $\mathcal{O} \subseteq \mathcal{P}(\mathcal{T})$. The element of topology $\mathcal{O}$ is defined to be an open subset of $\mathcal{T}$, which satisfy:

- $\mathcal{T}, \emptyset \in \mathcal{O}$
- $\forall U, V \in \mathcal{O} : U \cap V \in \mathcal{O}$
- $\forall \{U_\alpha\}_{\alpha \in I} \subseteq \mathcal{O} : \bigcup_{\alpha \in I} U_\alpha \in \mathcal{O}$

If $A \subseteq \mathcal{T}$ is open, then $\mathcal{T} \backslash A$ is defined to be a closed set. Without the discussion of topology, we can denote the topological space as $X$.

**Proposition A.1.** *For any topological space $(X, \mathcal{O}_X)$, $X$ itself and $\emptyset$ are both open and closed.*

*Proof.* The proof is simple. Notice that the definition of topology requires $X, \emptyset \in \mathcal{O}_X$, i.e., both $X$ and $\emptyset$ are open. Also, $X \backslash X = \emptyset$ is closed, and $X \backslash \emptyset$ is closed. □

**Remark A.1.** *For a topological space $(X, \mathcal{O}_X)$,*

- *The definition of topological spaces indicates that the infinite intersection of open sets may not be open, which is reasonable in Euclidean space since the infinite intersection of $\{(-1/n, 1/n) : n \in \mathbb{N}\}$ it s a single point $\{0\}$ that is closed.*
- *Not all subsets of $X$ can be classified as "open" or "closed", i.e., there can be a set that is neither open nor closed.*
- *For some topological space (not a connected space, actually), there can be a subset other than $X$ and $\emptyset$ that is both open and closed.*

It is quite obvious for readers familiar with real analysis that this is a generalization of open sets in $\mathbb{R}^n$. We use the universal properties of open sets in $\mathbb{R}^n$ as the definition of open sets in more general topological spaces. Here are some examples:

**Example A.1.** *Consider sets* $S = \{1, 2, 3, 4, 5, 6\}$*,*

- $\big\{\{1\}, \{2\}, \{6\}, \{1, 2\}, \{1, 6\}, \{2, 6\}, \{1, 2, 6\}\big\}$ *is a topology on* $S$*.*
- $\big\{\{1\}, \{2\}, \{5\}, \{1, 2, 5\}\big\}$ *is not a topology on* $S$*.*

**Example A.2.** *For any nonempty (ZFC) set* $S$*, two trivial topologies can be given*

- $\mathcal{O}_{chaotic} = \{\emptyset, S\}$ *is the chaotic topology on* $S$*.*
- $\mathcal{O}_{descrete} = \mathcal{P}(S)$ *is the discrete topology on* $A$*.*

For finite sets $|M| \geq 1$, the topology that can be established is given by the following table:

| Cardinality $\vert M\vert < 1$ | Numbers of Topology |
|---|---|
| 1 | 1 |
| 2 | 4 |
| 3 | 29 |
| 4 | 355 |
| 5 | 6942 |
| 6 | 209527 |
| $\vdots$ | $\vdots$ |

The most "basic" form of open sets is the open neighborhood:

**Definition A.2** (Neighborhood)**.** For some topological space $X$ and the point $x \in X$, $V \subseteq X$ is a neighborhood of $x$ if $\exists U \in \mathcal{O}_X : x \in U \subseteq V$. If $V$ is open (closed), then it is an open (closed) neighborhood of $x$.

**Remark A.2.** *As a remark, it is important to mention that the neighborhood does not necessarily have to be "small". As an example, since any neighborhood* $U$ *satisfies* $x \in U \subset X$*, then* $X$ *is both an open and a closed neighborhood of any point* $x \in X$*.*

**Proposition A.2.** *Let* $X$ *be a topological space, and* $x \in X$ *is a point. For some* $V \subseteq X$*, the following statements are equivlant:*

1. $V$ *is the open neighborhood of* $x$*.*
2. $V$ *is open, and* $x \in V$*.*

*Proof.* ($1 \Rightarrow 2$) By the definition of open neighborhood, we know the following information: $V$ is open (the requirement of "open" neighborhood) and $\exists U \in \mathcal{O}_X : x \in U \subseteq V \implies x \in V$, which proves 2
($2 \Rightarrow 1$) If $V$ is open and $x \in V$, then just take $U = V$ and $x \in V \subseteq V$, which makes $V$ being an open neighborhood of $x$. $\square$

With the concept of neighborhood being introduced, we can easily distinguish whether a set is open or not.

**Theorem A.3** (Determination of Open Sets)**.** Let $X$ be a topological space, $U \subseteq X$ is nonempty subset of $X$, the following proposition are equivalent:

1. $U$ is open.
2. $\forall x \in U$, $\exists V \subseteq U$ such that $V$ is a open neighborhood of $x$.

*Proof.* ($1 \Rightarrow 2$) For some open set $U$, take arbitrary $x \in U$, then $U$ itself is a open neighborhood of $x$.
($2 \Rightarrow 1$) By the given condition, $\forall x \in U$ take the corresponding open neighborhood $V_x \subseteq U$

$$\bigcup_{x \in U} V_x = U$$

Since $\forall x \in U : V_x$ are open, then, by the axiom of topological space, any union of open sets is open. Thus, $U$ is open. ☐

To describe the topology structure on any set (often uncountable infinite sets, like surfaces in $\mathbb{R}^n$), it is useful to discuss some generating sets of the topology, called a (topological) basis. We often choose some of the most "representative" open sets in the topology to form a topological basis.

**Definition A.4** (Topological Basis)**.** Let $(X, \mathcal{O}_X)$ be a topological space. For some $\mathcal{B} \in \mathcal{O}_X$, $\mathcal{B}$ is said to be a (topological) basis of the topological space iff

$$\forall U \in \mathcal{O}_X : \forall x \in U : \exists B \in \mathcal{B} : x \in B \subseteq U$$

The definition of basis has an equivalent statement:

**Proposition A.3** (Equivalent Description of Topological Basis)**.** *For some set contains open sets $\mathcal{B} \in \mathcal{O}_X$ in the topological space $(X, \mathcal{O}_X)$, the following statements are equivalent:*

1. *$\mathcal{B}$ is the basis of $X$*
2. *$\forall U \in \mathcal{O}_X : \exists \mathcal{B}' \subseteq \mathcal{B} : U = \bigcup_{B \in \mathcal{B}'} B$*

*Proof.* ($1 \Rightarrow 2$) For any open set $U$, for any $x \in U$. By the definition of basis, we can always find $x \in B_x \in \mathcal{B}$ such that $B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. WE can just take $\mathcal{B}' = \{B_x\}_{x \in U} \subseteq \mathcal{B}$, which proves the second statement.
($2 \Rightarrow 1$) By the given condition that $\forall U \in \mathcal{O}_X : U = \bigcup_{B \in \mathcal{B}'} B$, where $\mathcal{B}' \subseteq \mathcal{B}$. Thus, there must be some set $B \in \mathcal{B}'$ that $x \in B \in \mathcal{B}$, which is the definition of basis. ☐

**Example A.3.** *The natural topology on $\mathbb{R}^n$ is generated by the following basis:*

$$\mathcal{B} = \{B(x, r) | x \in \mathbb{Q}^n, r \in \mathbb{Q}_{\geq 0}\}, \quad B(x, r) = \{y \in \mathbb{R}^n | x \in \mathbb{R}^n, d(x, y) < r\}$$

*which is the topology we used for most metric spaces.*

**Definition A.5** (Second Countable, $A_2$)**.** A topological space is said to be second countable iff it can be generated by a countable basis.

By the given example of a topological basis, an obvious fact is that $\mathbb{R}^n$ is second countable. A further result is that any metric space is second countable. The topological basis of a topological space has the following properties:

**Proposition A.4** (Properties of Basis)**.** *Let $\mathcal{B} \subset \mathcal{O}_X$ is a basis of topological space $(X, \mathcal{O}_X)$, then $\mathcal{B}$ has the following properties:*

1. *$\forall x \in X : \exists B \in \mathcal{B} : x \in B$*
2. *$\forall B_1, B_2 \in \mathcal{B} : \forall x \in B_1 \cap B_2 : \exists B \in \mathcal{B} : x \in B \subseteq B_1 \cap B_2$*

*Proof.* The first property is directly from the definition of basis. Here, we prove the second property only. ☐

[**Derek:** To be proved.]

With the properties above, an important technique is to use the basis defined topology on a set. DZ To be finished.

With the definition of topology and its generating set, we need a better explanation of the motivation for the definition of topology.

> Topology is the minimum structure to define the continuity of the map.

As a generalization of the open set and continuity of functions in Euclidean space $\mathbb{R}^n$. We can have the following definition:

**Definition A.6** (Continuity of functions)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces, the map $f : X \to Y$ is continuous if

$$\forall V \in \mathcal{O}_Y : f^{-1}(V) \in \mathcal{O}_X$$

A necessary step to check the well-definedness of continuity is to consider the $\mathbb{R}^n$ and the natural topology of it.

**Proposition A.5** (Continuity on $\mathbb{R}^n$ and Topological Continuity)**.** *We take the topological space $X = \mathbb{R}^n$ and the standard topology in Euclidean space. The continuity of real functions in analysis and topological continuity are equivalent.*

Appendix B

# Linear Algebra

Appendix C

# Fundamental Result of Real Analysis

Appendix D

# Functional Analysis

# Appendix E

# A Berif Introduction to Category