Collaborative control in multi-agent system

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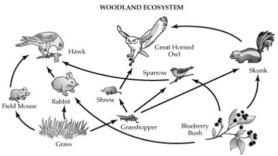
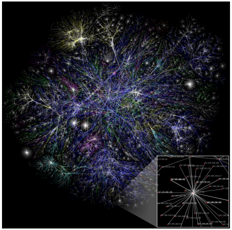
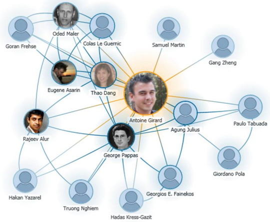
Scientific context

Networks everywhere:

- Biological networks (genetic regulation, ecosystems...)
- Technological networks (internet, sensor networks...)
- Economical networks (production and distribution networks, financial networks...)
- Social networks (scientific collaboration networks, Facebook...)

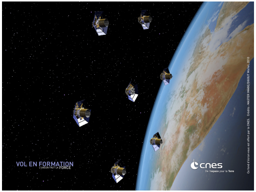






Emerging behaviors in networks

- Distributed decision making in a network.
- Each agent collaborates/negotiates locally with its neighbors in a network.
- The process succeeds if all agents eventually agree globally on some quantity of interest.
- Examples: bird flocks, fish schools, market prices...



What we will see in this lecture

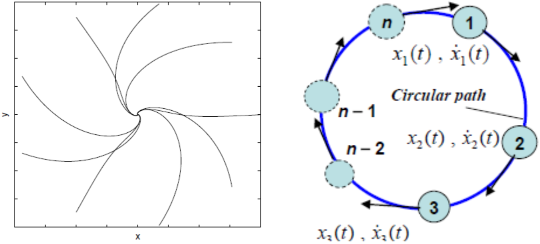
This lecture is not meant to give an exhaustive description of the area...

Instead, we will provide a deeper insight on a small number of representative results.

Main references used while preparing the lecture:

- C. Godsil & G. Royle, *Algebraic Graph Theory*, Springer 2001.
- R. Olfati-Saber, J.A. Fax & R.M. Murray, *Consensus and cooperation in networked multi-agent systems*, Proc. IEEE, 2007.
- V.D. Blondel, J.M. Hendrickx, A. Olshevsky & J.N. Tsitsiklis, *Convergence in multiagent coordination, consensus, and flocking*, Proc. CDC, 2005.
- L. Moreau, *Stability of continuous-time distributed consensus algorithms*, Proc. CDC, 2004.
- S. Martin & A. Girard, *Sufficient conditions for flocking via graph robustness analysis*, Proc. CDC, 2010.
- C. Morarescu & A. Girard, *Opinion dynamics with decaying confidence: application to community detection in graphs*, ArXiv, 2009.

Can a group of agents reach a rendezvous point or realize a formation without a priori knowing the coordinates of their targets?



Can we explain some synchronization behaviors? Movie

Flocking in mobile agents network

Flocking is the behavior exhibited when a group of birds are in flight:

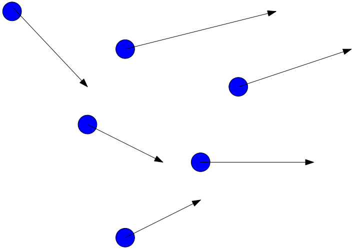
More precisely, a flocking behavior is characterized by three properties:

1. Alignement: the birds have the same velocity.
2. Cohesion: the birds remain together.
3. Separation: there is a minimum distance between birds.

In the following, we focus on the first and second property.

Example: flocking in mobile networks

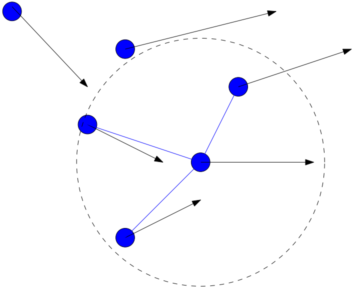
Consider a set of agents willing to move in a common direction:



Agent i is characterized by its position x_i and velocity v_i.

Example: flocking in mobile networks

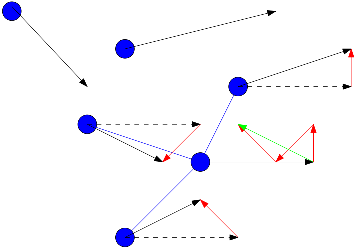
Consider a set of agents willing to move in a common direction:



Agent i has limited communication or sensing capabilities.

Example: flocking in mobile networks

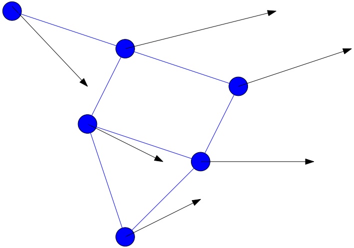
Consider a set of agents willing to move in a common direction:



Agent i tries to align its velocity on its neighbors: $\dot{v}_i = \sum_{j \in N_i} (v_j - v_i)$.

Example: flocking in mobile networks

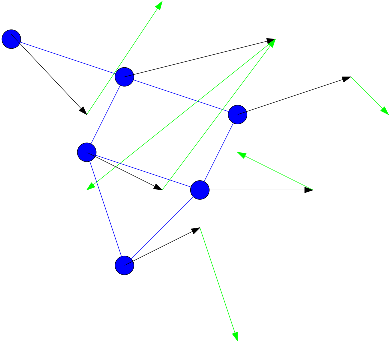
Consider a set of agents willing to move in a common direction:



The communication network is described by a (dynamic) graph.

Example: flocking in mobile networks

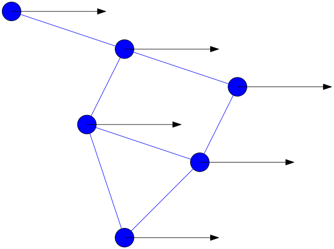
Consider a set of agents willing to move in a common direction:



Global linear dynamics with structure given by the graph: $\dot{v} = -Lv$.

Example: flocking in mobile networks

Consider a set of agents willing to move in a common direction:



Do the agents eventually agree on a common velocity?

Lecture outline

1. Algebraic graph theory
2. Spectral graph theory
3. Consensus algorithms
4. Applications

Lecture outline

1. Algebraic graph theory
    - Basic notions
    - Adjacency matrix
    - Laplacian matrix
    - Normalized Laplacian matrix

Graphs

Definition

A graph is couple $G = (V, E)$ consisting of:

- A finite set of vertices $V = \{1, \ldots, n\}$;
- A set of edges, $E \subseteq V \times V$.

We assume G has no self-loops $(\forall i \in V, (i,i) \notin E)$ and is undirected $(\forall i, j \in V,\ (i,j) \in E \iff (j,i) \in E)$.

Definition

In an undirected graph $G = (V, E)$:

- The neighborhood of a vertex $i \in V$ is the set

$$N_i = \{j \in V |\ (i,j) \in E\}.$$

- The degree of a vertex $i \in V$ is $d_i = |N_i|$.

Graph representation

It is often convenient, interesting, or attractive to represent a graph by a picture, with points for the vertices and lines for the edges, as in the figure below.

Definition

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic iff there exists a bijection $\varphi : V_1 \mapsto V_2$ such that $(i,j) \in E_1 \Leftrightarrow (\varphi(i), \varphi(j)) \in E_2$.

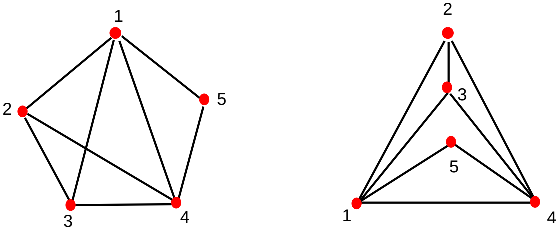
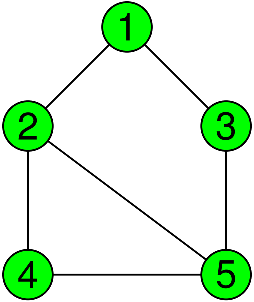


Figure: Two different representations for the same graph.

Example

A simple graph:

$$\begin{aligned} V &= \{1,2,3,4,5\} \\ E &= \{(1,2),(1,3),(2,4),(2,5),(3,5),(4,5)\ldots \\ &\quad (2,1),(3,1),(4,2),(5,2),(5,3),(5,4)\} \end{aligned}$$



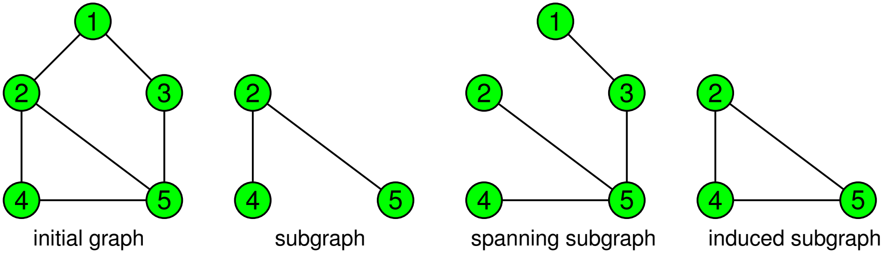
$$N_1 = \{2,3\}, d_1 = 2$$
$$N_2 = \{1,4,5\}, d_2 = 3$$
$$N_3 = \{1,5\}, d_3 = 2$$
$$N_4 = \{2,5\}, d_4 = 2$$
$$N_5 = \{2,3,4\}, d_5 = 3$$

Subgraphs

Definition

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. In addition, if $V' = V$ then G' is a spanning subgraph of G.

The subgraph of G induced by a set of vertices $V' \subseteq V$ is the graph $G' = (V', E')$ where $E' = E \cap V' \times V'$.



Complement graph

Definition

A graph $G = (V, E)$ is called complete iff $(i, j) \in E\ \forall i, j \in V$. The complete graph with n vertices will be denoted by K_n.

If J is the matrix whose components are all 1 and I is the identity matrix, then the adjacency matrix of a complete graph is $J - I$.

Definition

The complement of a graph $G = (V, E)$ is the graph $\bar{G} = (V, \bar{E})$ where $(i, j) \in \bar{E} \Leftrightarrow (i, j) \notin E$. So, if G has n vertices one has $\bar{G} = K_n \setminus G$.

Connectivity notions

Definition

A path in a graph $G = (V, E)$ is a finite sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_p, i_{p+1})$ such that $(i_k, i_{k+1}) \in E$ for all $k \in \{1, \ldots, p\}$.

Definition

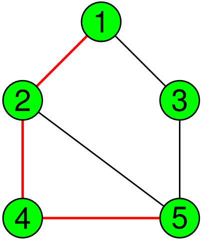
In a graph $G = (V, E)$, two vertices $i, j \in V$ are connected if there exists a path joining i and j (i.e. $i_1 = i$, $i_{p+1} = j$).

G is connected if for all $i, j \in V$, i and j are connected.

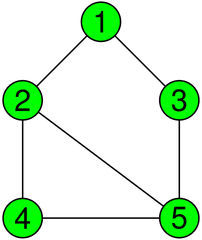
A subset of vertices $V' \subseteq V$ is a connected component of G if:

1. For all $i, j \in V'$, i and j are connected;
2. For all $i \in V'$, for all $j \in V \setminus V'$, i and j are not connected.

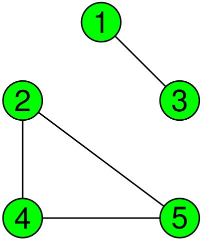
Example



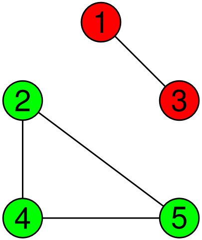
Path joining 1 and 5



Graph is connected



Graph is not connected



Connected components

Symmetric matrices

Definition

A matrix $B = (b_{ij})$ is symmetric iff $b_{ij} = b_{ji}, \ \forall i, j$. In other words $B = B^\top$

Proposition

Let B be a real symmetric matrix.

1. *If u is a right eigenvector of B associated to the eigenvalue λ then u is a left eigenvector of B associated to the eigenvalue λ.*
2. *If u and v are eigenvectors of B with different eigenvalues, then u and v are orthogonal.*
3. *The eigenvalues of a real symmetric matrix B are real numbers.*
4. *There exist matrices T and Γ such that $T^\top T = TT^\top = I$ and $TBT^\top = \Gamma$, where Γ is the diagonal matrix of eigenvalues of B.*

Positive semidefinite matrices

Definition

A matrix B is positive semidefinite if $u^\top Bu \geq 0$ for all vectors u. It is positive definite if it is positive semidefinite and $u^\top Bu = 0$ iff $u = 0$.

In the sequel these terms are used only for symmetric matrices.

Proposition

The matrix B is positive semidefinite iff all its eigenvalues are nonnegative.

Proposition

If B is a positive semidefinite matrix, then there is a matrix F such that $B = F^\top F$.

Adjacency and degree matrices

Definition

The adjacency matrix of a graph $G = (V, E)$ is the $n \times n$ symmetric matrix $A = (a_{ij})$ given for all $i, j \in V$ by:

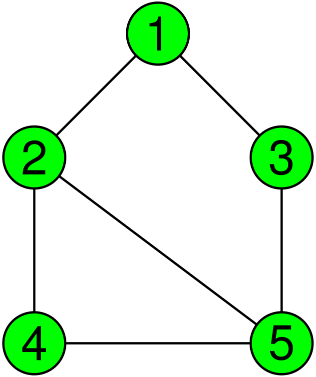
$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Definition

The degree matrix of G is the $n \times n$ diagonal matrix $D = (d_{ij})$ given for all $i, j \in V$ by:

$$d_{ij} = \begin{cases} d_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Example



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Properties

Remark

The adjacency matrix $A(G)$ is not unique.

Lemma

Let $A(G_1)$ and $A(G_2)$ the adjacency matrices associated to the graphs G_1 and G_2. If the eigenvalues of $A(G_1)$ and $A(G_2)$ do not match, then the graphs are not isomorphic.

The converse is not true. Compute for instance the eigenvalues of the graphs represented below.

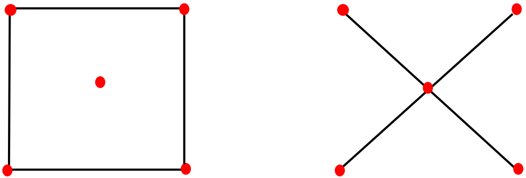


Figure: The spectrum is $\{-2, 0, 0, 0, 2\}$

Estimating the number of paths of length k

Notice that by the rules of matrix multiplication we have

$$\left(A^k\right)_{i,j} = \sum_{i_1, i_2, \ldots, i_{k-1}} A_{i,i_1} A_{i_1,i_2} \ldots A_{i_{k-1},j} \tag{1}$$

Proposition

Let A be the adjacency matrix of G. Then $(A^k)_{i,j}$ is the number of paths of length k starting at vertex i and ending at vertex j.

Exercice

Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ denote the eigenvalues of the adjacency matrix $A(G)$. Show that the following relations hold:

1. $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 0$;
2. $\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2 = 2|E(G)|$;
3. $\lambda_1^3 + \lambda_2^3 + \ldots + \lambda_n^3 = 6|T(G)|$, *where* $|T(G)|$ *represents the number of triangles of the graph* G;

What $\lambda_1^4 + \lambda_2^4 + \ldots + \lambda_n^4$ *counts?*

Eigenvectors & eigenvalues for regular graphs

Definition

A graph in which every vertex has equal valency k is called regular of valency k or $k-$regular. A 3-regular graph is called cubic, and a 4-regular graph is sometimes called quartic.

Proposition

Let G be a k-regular graph on n vertices with eigenvalues $k, \lambda_2, .., \lambda_n$. Then G and its complement G have the same eigenvectors, and the eigenvalues of G are $n-k-1, -1-\lambda_2, ..., -1-\lambda_n$.

Proof.

Remark that $A(\bar{G}) = J - I - A(G)$. Let $\{\mathbf{1}, u_2, \ldots, u_n\}$ be an orthonormal basis of eigenvectors of $A(G)$. Then $\mathbf{1}$ is an eigenvector of $A(\bar{G})$ with the eigenvalue $n-k-1$ and $A(\bar{G})u_k = (J - I - A(G))u_k = (-1-\lambda_k)u_k$. We use here the orthogonality of $\mathbf{1}$ and u_k to obtain $Ju_k = 0$. □

Incidence matrix

Definition

The incidence matrix of a graph $G = (V, E)$ is the $n \times |E|$ matrix $C = (c_{ij})$ defined as follows: each edge $f = (i, j) \in E$ defines a column with n entries, all of them are 0 except the $i^t h$ and $j^t h$ which are 1.

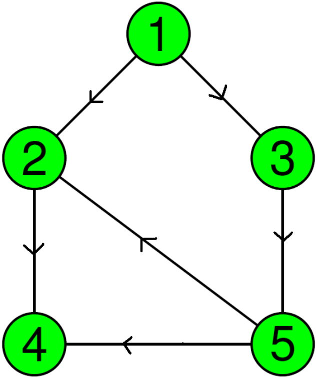
Definition

An orientation of a graph G is an assignment of a direction to each edge.

Definition

The incidence matrix of an oriented graph $G = (V, E)$ is the $n \times |E|$ matrix $C = (c_{ij})$ defined as follows: each edge $f = (i, j) \in E$ with the tail i and the head j defines a column with n entries, all of them are 0 except the $i^t h$ and $j^t h$ which are -1 and 1 respectively.

Example



$$C = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{pmatrix}$$

Properties

Theorem

Let $G = (V, E)$ be a graph with n vertices and c connected components. If σ is an orientation of G and C is the incidence matrix of the oriented graph G^σ, then $\mathrm{rank}(C) = n - c$.

Proof.

Let us define the following subspace $K := \{z \in \mathbb{R}^n \mid z^\top C = 0\}$.

1. For $(i, j) \in E$ and $z \in K$ prove that $z_i = z_j$.
2. Let us consider the connected components of G given by $V_1 = \{1, \dots, n_1\}$, $V_2 = \{n_1 + 1, n_2\}$, $\dots$, $V_c = \{n_{c-1} + 1, \dots, n_c = n\}$. Give an explicit description of K, derive a basis of K, compute $\dim K$.
3. Recall that $\mathrm{rank}(C) = n - \dim K$.

$\square$

Properties

Proposition

If σ is an orientation of $G = (V, E)$ and C is the incidence matrix of G^σ, then $CC^\top = D(G) - A(G)$.

Proof.

1. Let $(i, j) \in E$ and consider (i, j) the k^th edge of G. Compute $C_{h,k}$ for all $h = 1, \ldots, n$.
2. Compute $(CC^\top)_{ij}$
3. Compute $(D(G) - A(G))_{ij}$.

□

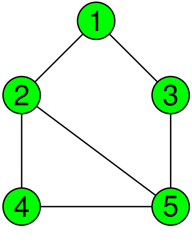
Laplacian matrix

Definition

The Laplacian matrix of a graph $G = (V, E)$ is the $n \times n$ symmetric matrix $L = (l_{ij})$ given for all $i, j \in V$ by:

$$l_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We have $L = D - A$.



$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{pmatrix}$$

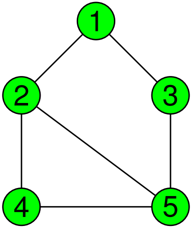
Normalized Laplacian matrix

Definition

The normalized Laplacian matrix of a graph $G = (V, E)$ is the $n \times n$ symmetric matrix $\mathcal{L} = (\ell_{ij})$ given for all $i, j \in V$ by:

$$\ell_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } d_i \neq 0, \\ -1/\sqrt{d_i d_j} & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

If $d_i > 0$ for all $i \in V$, then $\mathcal{L} = I - D^{-1/2} A D^{-1/2} = D^{-1/2} L D^{-1/2}$.



$$\mathcal{L} = \begin{pmatrix} 1 & -\frac{1}{\sqrt{6}} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{\sqrt{6}} & 1 & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{3} \\ -\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & 0 & 1 & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{3} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 1 \end{pmatrix}$$

Fundamental property of the Laplacian matrix

Theorem (Sum of squares property)

Let L be the Laplacian matrix of a graph $G = (V, E)$ then, for all $x \in \mathbb{R}^n$:

$$x^\top L x = \frac{1}{2} \sum_{(i,j)\in E} (x_i - x_j)^2.$$

Proof: For all $x \in \mathbb{R}^n$,

$$\begin{aligned}
x^\top L x &= \sum_{i\in V} x_i \sum_{j\in V} l_{ij} x_j = \sum_{i\in V} x_i \Big(d_i x_i - \sum_{(i,j)\in E} x_j\Big) \\
&= \sum_{i\in V} x_i \sum_{(i,j)\in E} (x_i - x_j) = \sum_{i\in V} \sum_{(i,j)\in E} (x_i^2 - x_i x_j) \\
&= \sum_{(i,j)\in E} (x_i^2 - x_i x_j)
\end{aligned}$$

Fundamental property of the Laplacian matrix

Theorem (Sum of squares property)

Let L be the Laplacian matrix of a graph $G = (V, E)$ then, for all $x \in \mathbb{R}^n$:

$$x^\top Lx = \frac{1}{2} \sum_{(i,j)\in E} (x_i - x_j)^2.$$

Proof: Since whenever $(i, j) \in E$, $(j, i) \in E$ we have

$$\sum_{(i,j)\in E} (x_i^2 - x_i x_j) = \sum_{(i,j)\in E} (x_j^2 - x_i x_j).$$

It follows that

$$x^\top Lx = \frac{1}{2} \sum_{(i,j)\in E} (x_i^2 - 2x_i x_j + x_j^2) = \frac{1}{2} \sum_{(i,j)\in E} (x_i - x_j)^2.$$

Laplacian matrix of a subgraph

Corollary

Let $G = (V, E)$ be a graph and L its Laplacian matrix, let $G' = (V, E')$ be a spanning subgraph of G and L' its Laplacian matrix. Then, for all $x \in \mathbb{R}^n$,

$$x^\top Lx \geq x^\top L'x$$

Proof: By the SOS property,

$$\begin{aligned} x^\top Lx &= \frac{1}{2}\sum_{(i,j)\in E}(x_i - x_j)^2 \\ &= \frac{1}{2}\sum_{(i,j)\in E'}(x_i - x_j)^2 + \frac{1}{2}\sum_{(i,j)\in E\setminus E'}(x_i - x_j)^2 \\ &\geq \frac{1}{2}\sum_{(i,j)\in E'}(x_i - x_j)^2 = x^\top L'x. \end{aligned}$$

Lecture outline

1. Algebraic graph theory
2. Spectral graph theory
    - Courant-Fischer theorem
    - Spectrum of the graph-laplacian
    - Interlacing

Eigenvalues of the Laplacian matrix

Proposition

The Laplacian matrix L of an undirected graph $G = (V, E)$ is symmetric positive. 0 is an eigenvalue of L with associated eigenvector $\mathbf{1}_n$.

Proof: Symmetry is consequence of G being undirected. By the SOS property,

$$\forall x \in \mathbb{R}^n,\ x^T L x = \frac{1}{2} \sum_{(i,j)\in E} (x_i - x_j)^2 \geq 0$$

which gives L positive.
Let $y = L\mathbf{1}_n$, then

$$y_i = \sum_{j\in V} l_{ij} = d_i + \sum_{(i,j)\in E} -1 = d_i - d_i = 0.$$

Eigenvalues of the Laplacian matrix

Proposition

0 is a simple eigenvalue of L if and only if the graph G is connected.

Proof: ($\Longrightarrow$) Assume the graph is not connected, then there exists a connected component of G, $V' \subsetneq V$.
Let $x \in \mathbb{R}^n$, such that $x_i = 1$ if $i \in V'$ and $x_i = 0$ otherwise, let $y = Lx$.
Then,

$$\forall i \in V', \qquad y_i = \sum_{j \in V} l_{ij} x_j = \sum_{j \in V'} l_{ij} = \sum_{j \in V} l_{ij} - \sum_{j \in V \setminus V'} l_{ij} = 0$$

$$\forall i \in V \setminus V', \qquad y_i = \sum_{j \in V} l_{ij} x_j = \sum_{j \in V'} l_{ij} = 0.$$

Then $x \neq \alpha \mathbf{1}_n$ is an eigenvector of L for eigenvalue 0. 0 is not simple.

Eigenvalues of the Laplacian matrix

Proposition

0 is a simple eigenvalue of L if and only if the graph G is connected.

Proof: ($\Longleftarrow$) Assume the graph is connected. Let x such that $Lx = 0$. Then the SOS property gives:

$$\sum_{(i,j)\in E} (x_i - x_j)^2 = x^\top L x = 0.$$

Thus, for all $(i,j) \in E$, $x_i = x_j$. Let $i \neq 1$, since G is connected there exists a path $(i_1, i_2), \ldots, (i_p, i_{p+1})$ joining 1 and i. It follows that

$$x_1 = x_{i_1} = x_{i_2} = \cdots = x_{i_{p+1}} = x_i.$$

The eigenvector x is of the form $\alpha \mathbf{1}_n$. 0 is simple.

Eigenvalues of the Laplacian matrix

Proposition

We denote the eigenvalues of L by $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
Let $\Delta = \max_{i \in V} d_i$, then, for all k, $\lambda_k \leq 2\Delta$.

Proof: Let λ be an eigenvalue of L and x an associated eigenvector. Let $i \in V$, such that for all $j \in V$, $|x_i| \geq |x_j|$. Then,

$$\lambda x_i = \sum_{j \in V} l_{ij} x_j = \sum_{j \in V \setminus \{i\}} l_{ij} x_j + d_i x_i.$$

It follows that

$$|\lambda - d_i| \leq \sum_{j \in V \setminus \{i\}} |l_{ij}| \frac{|x_j|}{|x_i|} \leq \sum_{j \in V \setminus \{i\}} |l_{ij}| = d_i.$$

Then $\lambda \leq 2d_i \leq 2\Delta$.

Spectrum of the Laplacian matrix of the complement

Proposition

We denote the eigenvalues of $L(G)$ *by* $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
Then $L(\bar{G})$ *has the same eigenvectors and its eigenvalues are*
$\lambda_1 = 0, \quad \lambda_i(\bar{G}) = n - \lambda_{n-i+2}, i \geq 2$

Proof. $L(\bar{G}) = nI - J - L(G)$

- $L(\bar{G})\mathbf{1} = 0$;
- If $L(G)u_i = \lambda_i u_i$ then $L(\bar{G})u_i = (n - \lambda_i)u_i$.

Algebraic connectivity

Definition

The second smallest eigenvalue λ_2 of L is referred to as the algebraic connectivity of the graph G.

Theorem

$\lambda_2 > 0$ if and only if G is connected. It satisfies:

$$\lambda_2 = \min_{x \perp \mathbf{1}_n} \frac{x^\top L x}{x^\top x}.$$

Proof: Let $v_1, \ldots, v_n$ be an orthonormal basis of eigenvectors. Then

$$\forall x \perp \mathbf{1}_n,\ x^\top L x = \sum_{k=2}^{n} \lambda_k (v_k^\top x)^2 \geq \lambda_2 \sum_{k=2}^{n} (v_k^\top x)^2 = \lambda_2 x^\top x.$$

Moreover, for $x = v_2$, $v_2^\top L v_2 = \lambda_2$.

Algebraic connectivity of a subgraph

Theorem

Let $G = (V, E)$ be a graph and λ_2 its algebraic connectivity, let $G' = (V, E')$ be a spanning subgraph of G and λ_2' its algebraic connectivity. Then, $\lambda_2 \geq \lambda_2'$.

Proof: Since for all $x \in \mathbb{R}^n$, $x^\top Lx \geq x^\top L'x$, we have

$$\min_{x \perp \mathbf{1}_n} \frac{x^\top Lx}{x^\top x} \geq \min_{x \perp \mathbf{1}_n} \frac{x^\top L'x}{x^\top x}$$

from which follows $\lambda_2 \geq \lambda_2'$.

Remark

Removing (adding) edges to graph can only make the algebraic connectivity decrease (increase).

Normalized Laplacian matrix

Using the fact the $\mathcal{L} = D^{-1/2} L D^{-1/2}$ we can obtain similar results.

Theorem (Sum of squares property)

Let $\mathcal{L}$ be the normalized Laplacian matrix of a graph $G = (V, E)$ then, for all $x \in \mathbb{R}^n$:

$$x^\top \mathcal{L} x = \frac{1}{2} \sum_{(i,j) \in E} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}} \right)^2 .$$

Proposition

The normalized Laplacian matrix L of an undirected graph $G = (V, E)$ is symmetric positive. 0 is an eigenvalue of L with associated eigenvector $D^{1/2}\mathbf{1}_n$.

Normalized Laplacian matrix

Proposition

0 is a simple eigenvalue of $\mathcal{L}$ if and only if the graph G is connected.

Proposition

We denote the eigenvalues of $\mathcal{L}$ by $0 = \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_n$. Then, for all k, $\tilde{\lambda}_k \leq 2$.

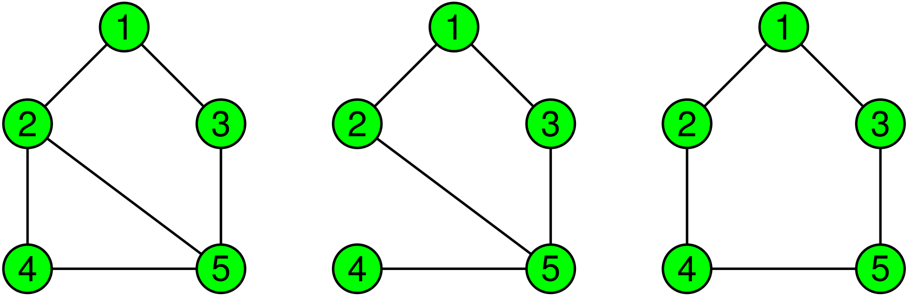
Theorem

$\tilde{\lambda}_2 > 0$ if and only if G is connected. It satisfies:

$$\tilde{\lambda}_2 = \min_{x \perp D^{1/2}\mathbf{1}_n} \frac{x^\top \mathcal{L} x}{x^\top x} = \min_{x \perp D\mathbf{1}_n} \frac{x^\top L x}{x^\top D x}.$$

Example

Second smallest eigenvalues of L and $\mathcal{L}$ for a graph and 2 subgraphs:



$\lambda_2 = 1.38$
$\tilde{\lambda}_2 = 0.67$

$\lambda_2 = 0.83$
$\tilde{\lambda}_2 = 0.59$

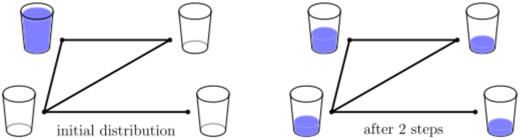
$\lambda_2 = 1.38$
$\tilde{\lambda}_2 = 0.69$

Application: Random walks

A random walk on a graph G can be thought of as a walk where we start at a vertex on the graph and at each step of time pick randomly (in our case uniformly) one of the edges incident to the current vertex and go along that edge to the next vertex. Repeating as often as desired.
As an example, consider the problem of shuffling cards. In this setting the graph is all possible ways to arrange a deck and the edges represent shuffles, i.e., starting with a deck of cards which orderings can be reached using one shuffle. In this case a random walk corresponds to doing a random sequence of shuffles.
One problem of interest for people shuffling cards is how many times do we need to shuffle until the cards are sufficiently random. In this setting random can be taken to mean as saying that knowing the initial configuration of cards before starting the shuffling will not give you any significant information about the current placement of cards (i.e., all of the initial information has been lost).

A watered down approach to random walks

Imagine that we have a cup of full water at our initial vertex, and empty cups everywhere else. At each step we will simultaneously redistribute the water at each vertex to the vertices neighbors, and thus overtime the water should diffuse throughout the graph. In our problem the water represents the probability distribution and the fractional amount of water at a vertex at k steps is the probability that we are at that vertex in k steps.



Problem formulation

Definition

We say that a probability distribution is random if the probability of being at any vertex is proportional to its degree. So, the probability of being at vertex i is equal to $d_i / \sum_{j \in G} d_j$.

Problem

How many steps does it take before we are sufficiently random.

Remark

If A is the adjacency matrix, then $D^{-1}A$ defined by

$(D^{-1}A)_{ij} = \begin{cases} 0 & \text{if } (i,j) \notin E \\ \frac{1}{d_i} & \text{if } (i,j) \in E \end{cases}$ *is a stochastic matrix. That is, $(D^{-1}A)_{ij}$ is the probability that given you are at vertex i you move to vertex j. If x^0 is the initial distribution vector, the probability distribution after k steps will be $x^0(D^{-1}A)^k$*

Connection with graph Laplacian

Remark

We note that

$$D^{-1/2}(I - \mathcal{L})D^{1/2} = D^{-1/2}(D^{-1/2}AD^{-1/2})D^{1/2} = D^{-1}A$$

which say that $D^{-1}A$ and $I - \mathcal{L}$ are similar. Thus, $D^{-1}A$ and $I - \mathcal{L}$ share the same eigenvalues (λ is an eigenvalue of $D^{-1}A$ iff $1 - \lambda$ is an eigenvalue of $\mathcal{L}$).

The previous remark guaranties that 1 is an eigenvalue of $D^{-1}A$ and it is easy to show that the corresponding left eigenvector is $\mathbf{1}D$. Normalizing this vector one gets $\frac{\mathbf{1}D}{\sum_j d_j}$ called the stationary distribution. This is the distribution which we want our random walk to converge to.

Let u_i be an orthonormal set of eigenvectors associated with $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ the eigenvalues of $\mathcal{L}$. As we have seen before, u_i are the eigenvectors associated to the eigenvalues $1 - \lambda_i$ of $D^{-1/2}AD^{-1/2}$. It is easy to check that $u_0 = \mathbf{1}D^{1/2}/\sqrt{\sum_j d_j}$.

Since we have a full set of eigenvalues and orthonormal eigenvectors, we can write:

$$D^{-1/2}AD^{-1/2} = \sum_i (1 - \lambda_i) u_i^\top u_i$$

To check how close our random walk is after k steps to the stationary distribution we use the L^2- norm:

Convergence of random walks

$$||x^0(D^{-1}A)^k - \mathbf{1}D^{1/2}/\sqrt{\sum_j d_j}|| =$$

$$||x^0 D^{-1/2}(D^{-1/2}AD^{-1/2})^k D^{1/2} - \mathbf{1}D^{1/2}/\sqrt{\sum_j d_j}|| =$$

$$||x^0 D^{-1/2}(\sum_i (1-\lambda_i)u_i^\top u_i)^k D^{1/2} - \mathbf{1}D^{1/2}/\sqrt{\sum_j d_j}|| =$$

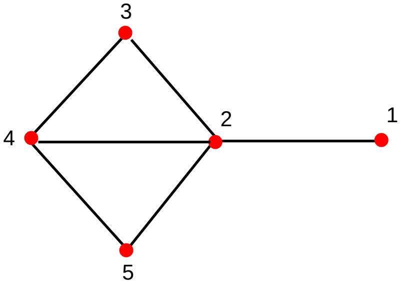
$$||x^0 D^{-1/2}(\sum_i (1-\lambda_i)^k u_i^\top u_i)D^{1/2} - \mathbf{1}D^{1/2}/\sqrt{\sum_j d_j}|| =$$

$$||x^0 D^{-1/2}(\sum_{i\neq 1} (1-\lambda_i)^k u_i^\top u_i)D^{1/2}|| \leq$$

$$\max_{i\neq 1} |1-\lambda_i|^k \frac{\max_i \sqrt{d_i}}{\min_j \sqrt{d_j}}$$

Exercise

Consider the graph in the figure below



For an arbitrary initial distribution x^0 compute how far we are from the stationary distribution after 10 steps.
If we want to be within 10^{-6} of the stationary distribution, how many steps we have to do?

Summary

- We have defined fundamental notions such as graphs, subgraphs, paths, connectivity...
- We have introduced Laplacian and normalized Laplacian matrices:
  - We have shown the SOS property for both types of matrices.
  - We have shown that the connectivity of a graph can be determined by second smallest eigenvalue of these matrices (λ_2 and $\tilde{\lambda}_2$).
- Some results that are valid for the Laplacian matrix cannot be extended to the normalized Laplacian matrix e.g.:

  If G' is a subgraph of G then $\lambda_2' \leq \lambda_2$.

- Though, in practice, the eigenvalue $\tilde{\lambda}_2$ is often a good measure of the connectivity of the graph (less dependent to the size of the graph).

Lecture outline

1. Algebraic graph theory
2. Spectral graph theory
3. Consensus algorithms
    - Discrete time and continuous time
    - Agreement in networks with fixed topology
    - Agreement in networks with dynamic topology
    - Performance of consensus algorithms
    - Consensus under communication time-delays

Consensus algorithms

Consider a set of agents V organized in a network with (possibly dynamic) topology described by an undirected graph $G(t) = (V, E(t))$.

Each agent $i \in V$ has a state $x_i(t) \in \mathbb{R}$ which is updated according to a simple local rule, e.g.

$$\dot{x}_i(t) = \sum_{j \in N_i(t)} (x_j(t) - x_i(t)),$$

or in matrix form

$$\dot{x}(t) = -L(t)x(t).$$

We say that the agents achieve a consensus if

$$\lim_{t \to +\infty} x_1(t) = \cdots = \lim_{t \to +\infty} x_n(t).$$

Various types of consensus algorithms

We will consider the following consensus algorithms:

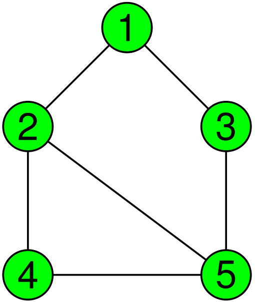
continuous time	discrete time
$\dot{x}_i(t) = \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$	$x_i(t+1) = x_i(t) + \varepsilon \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$
$\dot{x}_i(t) = \frac{1}{d_i(t)} \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$	$x_i(t+1) = x_i(t) + \frac{\varepsilon}{d_i(t)} \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$

Equivalent algorithms in matrix form:

continuous time	discrete time
$\dot{x}(t) = -L(t)x(t)$	$x(t+1) = (I - \varepsilon L(t))x(t)$
$\dot{x}(t) = -D^{-1}(t)L(t)x(t)$	$x(t+1) = (I - \varepsilon D^{-1}(t)L(t))x(t)$

Example

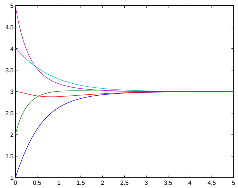
Consider the following network with fixed topology:



We run continuous time consensus algorithms over this network.

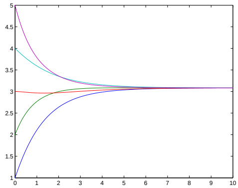
Standard consensus algorithm:

$$\dot{x}(t) = -Lx(t)$$



Normalized consensus algorithm:

$$\dot{x}(t) = -D^{-1}Lx(t)$$



Consensus in networks with fixed topology

We assume that for all $t \in \mathbb{R}$, $G(t) = (V, E)$ and consider the following consensus algorithm:

$$\dot{x}_i(t) = \sum_{j \in N_i} (x_j(t) - x_i(t))$$

or in matrix form: $\dot{x}(t) = -Lx(t)$.

Lemma

The quantity $\mathbf{1}_n^\top x(t) = \sum_{i \in V} x_i(t)$ is invariant.

Proof: Let us compute the derivative

$$\begin{aligned} \frac{d}{dt}\left(\mathbf{1}_n^\top x(t)\right) &= \mathbf{1}_n^\top \dot{x}(t) \\ &= -\mathbf{1}_n^\top (Lx(t)) = -(\mathbf{1}_n^\top L)x(t) = 0. \end{aligned}$$

Consensus value

Proposition

Let $x^* = \frac{1}{n}\mathbf{1}_n^\top x(0)$, *if a consensus is achieved, then*

$$\lim_{t\to+\infty} x(t) = x^*\mathbf{1}_n.$$

Proof: If a consensus is achieved, then for all $i \in V$,

$$\lim_{t\to+\infty} x_i(t) = \lim_{t\to+\infty} \frac{1}{n}\mathbf{1}_n^\top x(t) = \frac{1}{n}\mathbf{1}_n^\top x(0) = x^*.$$

Remark

The consensus value is the average of the initial values. It is independent of the topology of the graph $G = (V, E)$.

Convergence

Theorem

If the graph G is connected, then the consensus is achieved. Moreover,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\| \leq e^{-\lambda_2 t}\,\|x(0) - x^*\mathbf{1}_n\|\,.$$

Proof: Let $V(t) = \|x(t) - x^*\mathbf{1}_n\|^2$. Then,

$$\begin{aligned}\frac{d}{dt}V(t) &= 2\left(x(t) - x^*\mathbf{1}_n\right)^\top \dot{x}(t) = -2\left(x(t) - x^*\mathbf{1}_n\right)^\top Lx(t)\\ &= -2\left(x(t) - x^*\mathbf{1}_n\right)^\top L\left(x(t) - x^*\mathbf{1}_n\right).\end{aligned}$$

Let us remark that

$$\mathbf{1}_n^\top\left(x(t) - x^*\mathbf{1}_n\right) = \mathbf{1}_n^\top x(t) - x^*\mathbf{1}_n^\top\mathbf{1}_n = \mathbf{1}_n^\top x(0) - x^*n = 0$$

which gives $\left(x(t) - x^*\mathbf{1}_n\right) \perp \mathbf{1}_n$.

Convergence

Theorem

If the graph G is connected, then the consensus is achieved. Moreover,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\| \leq e^{-\lambda_2 t}\|x(0) - x^*\mathbf{1}_n\|.$$

Proof: Therefore,

$$(x(t) - x^*\mathbf{1}_n)^\top L(x(t) - x^*\mathbf{1}_n) \geq \lambda_2 \|x(t) - x^*\mathbf{1}_n\|^2$$

which gives

$$\frac{d}{dt}V(t) \leq -2\lambda_2 \|x(t) - x^*\mathbf{1}_n\|^2 = -2\lambda_2 V(t).$$

Thus, $V(t) \leq V(0)e^{-2\lambda_2 t}$. If the graph G is connected then $\lambda_2 > 0$ and the consensus is achieved.

Consensus in discrete time

We assume that for all $t \in \mathbb{N}$, $G(t) = (V, E)$, let $\varepsilon \in (0, \frac{1}{2\Delta})$, we consider the consensus algorithm:

$$\forall i \in V,\; x_i(t+1) = x_i(t) + \varepsilon \sum_{j \in N_i} (x_j(t) - x_i(t))$$

or in matrix form: $x(t+1) = Px(t)$ where $P = I - \varepsilon L$.

Proposition

The quantity $\mathbf{1}_n^\top x(t) = \sum_{i \in V} x_i(t)$ *is invariant. Let* $x^* = \frac{1}{n}\mathbf{1}_n^\top x(0)$, *if a consensus is achieved, then*

$$\lim_{t \to +\infty} x(t) = x^* \mathbf{1}_n.$$

Proof: Let us remark that

$$\mathbf{1}_n^\top x(t+1) = \mathbf{1}_n^\top (I - \varepsilon L) x(t) = \mathbf{1}_n^\top x(t) - \varepsilon \mathbf{1}_n^\top L x(t) = \mathbf{1}_n^\top x(t).$$

Convergence

Lemma

We have $P\mathbf{1}_n = \mathbf{1}_n$ *and for all* $x \perp \mathbf{1}_n$, $\|Px\| \leq (1 - \varepsilon\lambda_2)\|x\|$.

Proof: First, $P\mathbf{1}_n = (I - \varepsilon L)\mathbf{1}_n = \mathbf{1}_n - \varepsilon L\mathbf{1}_n = \mathbf{1}_n$.
Let $x \perp \mathbf{1}_n$ and $v_1, \ldots, v_n$ be an orthonormal basis of eigenvectors of L.
Then,

$$x = \sum_{k=2}^{n} (v_k^\top x) v_k \text{ and } Px = \sum_{k=2}^{n} (v_k^\top x)(1 - \varepsilon\lambda_k) v_k.$$

Thus

$$\|Px\|^2 = \sum_{k=2}^{n} (v_k^\top x)^2 (1 - \varepsilon\lambda_k)^2 \leq \max_{k=2}^{n} (1 - \varepsilon\lambda_k)^2 \|x\|^2.$$

Moreover, from $0 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2\Delta$ and $\varepsilon \in (0, \frac{1}{2\Delta})$, we obtain

$$\max_{k=2}^{n} (1 - \varepsilon\lambda_k)^2 = (1 - \varepsilon\lambda_2)^2 < 1.$$

Convergence

Theorem

If the graph G is connected, then the consensus is achieved. Moreover,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\| \leq (1 - \varepsilon\lambda_2)^t \|x(0) - x^*\mathbf{1}_n\| .$$

Proof: Let $V(t) = \|x(t) - x^*\mathbf{1}_n\|^2$. Then,

$$V(t+1) = \|Px(t) - x^*\mathbf{1}_n\|^2 = \|P(x(t) - x^*\mathbf{1}_n)\|^2$$

Let us remark that

$$\mathbf{1}_n^\top (x(t) - x^*\mathbf{1}_n) = \mathbf{1}_n^\top x(t) - x^*\mathbf{1}_n^\top \mathbf{1}_n = \mathbf{1}_n^\top x(0) - x^* n = 0$$

which gives $(x(t) - x^*\mathbf{1}_n) \perp \mathbf{1}_n$. Then,

$$V(t+1) \leq (1 - \varepsilon\lambda_2)^2 \|x(t) - x^*\mathbf{1}_n\|^2 = (1 - \varepsilon\lambda_2)^2 V(t).$$

Other consensus algorithms

Using the fact the $D^{-1}L = D^{-1/2}\mathcal{L}D^{1/2}$ we obtain similar results for the consensus algorithm:

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j\in N_i}(x_j(t) - x_i(t))$$

or in matrix form: $\dot{x}(t) = -D^{-1}Lx(t)$.

Theorem

If the graph G is connected, then the consensus is achieved. The consensus value is

$$x^* = \frac{\sum_{i\in V} d_i x_i(0)}{\sum_{i\in V} d_i}.$$

Moreover,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\|_D \leq e^{-\tilde{\lambda}_2 t}\|x(0) - x^*\mathbf{1}_n\|_D.$$

For the discrete time case, let $\varepsilon \in (0, \frac{1}{2})$ and consider the consensus algorithm:

$$x_i(t+1) = x_i(t) + \frac{\varepsilon}{d_i} \sum_{j \in N_i} (x_j(t) - x_i(t))$$

or in matrix form: $x(t+1) = (I - \varepsilon D^{-1} L) x(t)$.

Theorem

If the graph G is connected, then the consensus is achieved. The consensus value is

$$x^* = \frac{\sum_{i \in V} d_i x_i(0)}{\sum_{i \in V} d_i}.$$

Moreover,

$$\forall t \geq 0, \; \|x(t) - x^* \mathbf{1}_n\|_D \leq (1 - \varepsilon \tilde{\lambda}_2)^t \, \|x(0) - x^* \mathbf{1}_n\|_D \, .$$

Summary

- We have considered several consensus algorithms in continuous or discrete time for networks with fixed topology.
- For the standard consensus algorithm, the consensus value is independent of the network topology (average of initial values) whereas for the normalized consensus algorithm, it depends on the network topology (weighted average where the weights are the degrees of the vertices of the network).
- Consensus is achieved if the network is connected. The consensus is approached at exponential speed. Moreover, the convergence speed is determined by the second smallest eigenvalue of the Laplacian or normalized Laplacian matrix: the more connected the network the faster the consensus.

Consensus in networks with dynamic topology

We now assume that the graph is time-varying $G(t) = (V, E(t))$ and consider the following consensus algorithm:

$$\dot{x}_i(t) = \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$$

or in matrix form: $\dot{x}(t) = -L(t)x(t)$.
A lot of results are actually similar to the fixed topology case:

Proposition

The quantity $\mathbf{1}_n^\top x(t) = \sum_{i \in V} x_i(t)$ is invariant. Let $x^* = \frac{1}{n}\mathbf{1}_n^\top x(0)$, if a consensus is achieved, then

$$\lim_{t \to +\infty} x(t) = x^* \mathbf{1}_n.$$

Convergence

Theorem

If the graph $G(t)$ is connected for all $t \in \mathbb{R}$, then the consensus is achieved. Moreover, for $\underline{\lambda}_2 \leq \min_{t\in\mathbb{R}^+} \lambda_2(t)$,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\| \leq e^{-\underline{\lambda}_2 t} \|x(0) - x^*\mathbf{1}_n\| .$$

Proof: Let $V(t) = \|x(t) - x^*\mathbf{1}_n\|^2$. Then,

$$\begin{aligned}
\frac{d}{dt}V(t) &= 2\left(x(t) - x^*\mathbf{1}_n\right)^\top \dot{x}(t) = -2\left(x(t) - x^*\mathbf{1}_n\right)^\top Lx(t) \\
&= -2\left(x(t) - x^*\mathbf{1}_n\right)^\top L(t)\left(x(t) - x^*\mathbf{1}_n\right) \\
&\leq -2\lambda_2(t)\|x(t) - x^*\mathbf{1}_n\|^2 \leq -2\underline{\lambda}_2 V(t)
\end{aligned}$$

Thus, $V(t) \leq V(0)e^{-2\underline{\lambda}_2 t}$. If the graph G is connected, we can choose $\underline{\lambda}_2 > 0$ and the consensus is achieved.

Discrete time algorithm

For the discrete time case, let $\varepsilon \in (0, \frac{1}{2(n-1)})$ and consider the consensus algorithm:

$$x_i(t+1) = x_i(t) + \varepsilon \sum_{j \in N_i} (x_j(t) - x_i(t))$$

or in matrix form: $x(t+1) = (I - \varepsilon L)x(t)$.

Theorem

If the graph $G(t)$ is connected for all $t \in \mathbb{R}$, then the consensus is achieved. The consensus value is $x^* = \frac{1}{n}\mathbf{1}_n^\top x(0)$.
Moreover, for $\underline{\lambda}_2 \leq \min_{t \in \mathbb{R}^+} \lambda_2(t)$,

$$\forall t \geq 0, \ \|x(t) - x^* \mathbf{1}_n\| \leq (1 - \varepsilon \underline{\lambda}_2)^t \|x(0) - x^* \mathbf{1}_n\|.$$

Some remarks

- For the standard Laplacian consensus algorithm, it is straightforward to extend the results from fixed topology to dynamic topology under the assumption that the graph $G(t)$ remains connected for all t.
- For the normalized Laplacian consensus algorithm, the results cannot be extended in a straightforward manner. Indeed, even the consensus value is dependent on the graph sequence... Another approach is needed!
- The assumption that the graph remains connected for all time is actually quite strong. Is it possible to relax this assumption ?

A general consensus algorithm

Let us assume that the graph is time-varying $G(t) = (V, E(t))$ and consider the following discrete time consensus algorithm:

$$x_i(t+1) = \sum_{j \in V} p_{ij}(t) x_j(t)$$

under the following assumptions (for some $\alpha > 0$):

Assumption

- $p_{ii}(t) \geq \alpha$, $\forall i \in V$, $t \in \mathbb{N}$.
- $p_{ij}(t) \neq 0$ *if and only if* $(i, j) \in E(t)$, $\forall i, j \in V$, $i \neq j$, $t \in \mathbb{N}$.
- $p_{ij}(t) \in \{0\} \cup [\alpha, 1]$, $\forall i, j \in V$, $i \neq j$, $t \in \mathbb{N}$.
- $\sum_{j \in V} p_{ij}(t) = 1$, $\forall i \in V$, $t \in \mathbb{N}$.

Let us remark that the previous discrete time consensus algorithms satisfy these assumptions.

Convergence analysis

For a subset of agents $V' \subseteq V$, let $m_{V'}(t) = \min_{i \in V'} x_i(t)$ and $M_{V'}(t) = \max_{i \in V'} x_i(t)$.

Proposition

Let $V' \subseteq V$ such that for all $i \in V$, for all $j \in V \setminus V'$, $(i,j) \notin E(t)$. Then,

$$m_{V'}(t+1) \geq m_{V'}(t) \text{ and } M_{V'}(t+1) \leq M_{V'}(t).$$

Proof: Let $i \in V'$, then

$$\begin{aligned} x_i(t+1) &= \sum_{j \in V} p_{ij}(t) x_j(t) = \sum_{j \in V'} p_{ij}(t) x_j(t) \\ &\geq m'_V(t) \sum_{j \in V'} p_{ij}(t) = m_{V'}(t). \end{aligned}$$

Then, $m_{V'}(t+1) \geq m_{V'}(t)$. Similarly, $M_{V'}(t+1) \leq M_{V'}(t)$.

Convergence analysis

Remark

The sequences $M_V(t)$ and $m_V(t)$ are monotonic and bounded, therefore convergent.

The consensus is achieved if and only if

$$\lim_{t\to+\infty} M_V(t) = \lim_{t\to+\infty} m_V(t)$$

or equivalently

$$\lim_{t\to+\infty} M_V(t) - m_V(t) = 0.$$

We will prove this under an assumption of asymptotic connectivity:

Assumption

For all $t \in \mathbb{N}$, the graph $(V, \cup_{s\geq t} E(s))$ is connected.

Convergence analysis

Lemma

For all $t \in \mathbb{N}$ there exists $T \geq t$ such that

$$M_V(T) - m_V(T) \leq (1 - \alpha^n)(M_V(t) - m_V(t)).$$

Proof: Let us remark that for all $s \geq t$, $i \in V$, $m_V(t) \leq x_i(s) \leq M_V(t)$. Consider the following property for $k \in \{1, \ldots, n\}$:

$$P_k : \begin{cases} \exists t_k \geq t,\ V_k \subseteq V, \ \text{ such that } |V_k| = k \text{ and} \\ m_{V_k}(t) \geq m_V(t) + \alpha^k (M_V(t) - m_V(t)). \end{cases}$$

If P_n is true then necessarily $V_n = V$ and since $M_V(t_n) \leq M_V(t)$:

$$\begin{aligned} M_V(t_n) - m_V(t_n) &\leq M_V(t) - m_V(t) - \alpha^n (M_V(t) - m_V(t)) \\ &\leq (1 - \alpha^n)(M_V(t) - m_V(t)). \end{aligned}$$

Convergence analysis

Proof: Let $i_1 \in V$ such that $x_{i_1}(t) = M_V(t)$. Let $t_1 = t$ and $V_1 = \{i_1\}$, then

$$m_{V_1}(t_1) = x_{i_1}(t) = M_V(t) \geq m_V(t) + \alpha(M_V(t) - m_V(t)).$$

Thus, P_1 is true. Assume P_k is true for some $k \in \{1, \dots, n-1\}$.

Let $T_k \geq t_k$ be the first time such that

$$\exists (i_{k+1}, j_{k+1}) \in E(T_k), \text{ for some } i_{k+1} \in V \setminus V_k,\ j_{k+1} \in V_k.$$

For all $t_k \leq s \leq T_k - 1$, $i \in V_k$, $j \in V \setminus V_k$, $(i,j) \notin E(s)$. Then,

$$\forall i \in V_k,\ x_i(T_k) \geq m_{V_k}(T_k) \geq m_{V_k}(t_k) \geq m_V(t) + \alpha^k(M_V(t) - m_V(t)).$$

Convergence analysis

Proof: Then, for all $i \in V_k$,

$$
\begin{aligned}
x_i(T_k+1) - m_V(t) &= \left(\sum_{j\in V} p_{ij}(T_k)x_j(T_k)\right) - m_V(t) \\
&= \sum_{j\in V} p_{ij}(T_k)\left(x_j(T_k) - m_V(t)\right) \\
&\geq p_{ii}(T_k)\left(x_i(T_k) - m_V(t)\right) \\
&\geq \alpha\left(x_i(T_k) - m_V(t)\right) \\
&\geq \alpha\left(m_V(t) + \alpha^k(M_V(t) - m_V(t)) - m_V(t)\right) \\
&\geq \alpha^{k+1}(M_V(t) - m_V(t)).
\end{aligned}
$$

Convergence analysis

Proof: Moreover,

$$
\begin{aligned}
x_{i_{k+1}}(T_k+1) - m_V(t) &= \left(\sum_{j\in V} p_{i_{k+1}j}(T_k)x_j(T_k)\right) - m_V(t) \\
&= \sum_{j\in V} p_{i_{k+1}j}(T_k)\left(x_j(T_k) - m_V(t)\right) \\
&\geq p_{i_{k+1}j_{k+1}}(T_k)\left(x_{j_{k+1}}(T_k) - m_V(t)\right) \\
&\geq \alpha\left(x_{j_{k+1}}(T_k) - m_V(t)\right) \\
&\geq \alpha\left(m_V(t) + \alpha^k(M_V(t) - m_V(t)) - m_V(t)\right) \\
&\geq \alpha^{k+1}(M_V(t) - m_V(t)).
\end{aligned}
$$

Hence, P_{k+1} holds for $t_{k+1} = T_k + 1$ and $V_{k+1} = V_k \cup \{i_{k+1}\}$.
Then, P_n holds.

Convergence analysis

Theorem

If, for all $t \in \mathbb{N}$, the graph $(V, \cup_{s \geq t} E(s))$ is connected, then the consensus is achieved.

Proof: From the previous lemma, there exists an increasing sequence $T_k \in \mathbb{N}$ such that $T_0 = 0$ and

$$\forall k \in \mathbb{N},\ 0 \leq M_V(T_k) - m_V(T_k) \leq (1 - \alpha^n)^k (M_V(0) - m_V(0)).$$

Therefore,

$$\lim_{k \to +\infty} M_V(T_k) - m_V(T_k) = 0.$$

Since in addition $M_V(t)$ and $m_V(t)$ are convergent it necessarily follows that

$$\lim_{t \to +\infty} M_V(t) - m_V(t) = 0.$$

Continuous time consensus

The previous result is not valid for continuous time algorithms.

Consider e.g.:

$$\dot{x}_i(t) = \sum_{j \in N_i(t)} (x_j(t) - x_i(t)),$$

in a network of two agents with dynamic topology $G(t) = (V, E(t))$ where

$$V = \{1, 2\},\ E(t) = \begin{cases} \{(1,2),(2,1)\}, & \text{if } t \in [k, k + 1/2^k) \\ \emptyset, & \text{if } t \in [k + 1/2^k, k + 1) \end{cases},\ k \in \mathbb{N}.$$

Let us remark that for all $t \in \mathbb{R}$, the graph $(V, \cup_{s \geq t} E(s))$ is connected.

Continuous time consensus

It follows that,

$$\dot{x}_2(t) - \dot{x}_1(t) = \begin{cases} -2(x_2(t) - x_1(t)), & \text{if } t \in [k, k + 1/2^k) \\ 0, & \text{if } t \in [k + 1/2^k, k + 1) \end{cases}$$

Then, for all $k \in \mathbb{N}$:

$$\begin{aligned} x_2(k+1) - x_1(k+1) &= x_2(k + 1/2^k) - x_1(k + 1/2^k) \\ &= (x_2(k) - x_1(k))e^{-2/2^k} \\ &= (x_2(0) - x_1(0))e^{-2}e^{-2/2}e^{-2/2^2} \dots e^{-2/2^k} \\ &= (x_2(0) - x_1(0))e^{-2(1+1/2+1/2^2+\dots+1/2^k)} \\ &= (x_2(0) - x_1(0))e^{-4(1-1/2^{k+1})}. \end{aligned}$$

Continuous time consensus

It follows that

$$\lim_{k\to+\infty} x_2(k+1) - x_1(k+1) = (x_2(0) - x_1(0))e^{-4} \neq 0.$$

- The consensus is not achieved despite the fact that for all $t \in \mathbb{R}$, the graph $(V, \cup_{s\geq t} E(s))$ is connected.
- The key observation is that even though the edge $(1, 2)$ appears infinitely often, it is present only for a finite time...
- If one impose a dwell time (when an edge appears it remains present for a duration at least $\tau > 0$), one can avoid this kind of phenomenom.

A general consensus algorithm

Let us assume that the graph is time-varying $G(t) = (V, E(t))$ and consider the following continuous time consensus algorithm:

$$\dot{x}_i(t) = \sum_{j \in V \setminus \{i\}} p_{ij}(t)(x_j(t) - x_i(t))$$

under the following assumptions (for some $\alpha > 0$, $\beta > 0$):

Assumption

- $p_{ij}(t) \neq 0$ *if and only if* $(i, j) \in E(t)$, $\forall i, j \in V$, $i \neq j$, $t \in \mathbb{R}$.
- $p_{ij}(t) \in \{0\} \cup [\alpha, \beta]$, $\forall i, j \in V$, $i \neq j$, $t \in \mathbb{R}$.
- $\sum_{j \in V \setminus \{i\}} p_{ij}(t) \leq \beta$, $\forall i \in V$, $t \in \mathbb{R}$.

Let us remark that the previous continuous time consensus algorithms satisfy these assumptions.

A general consensus algorithm

We add the following dwell time assumption:

Assumption

For all $i, j \in V$, if the edge (i, j) appears at time t, i.e.:

$$(i, j) \in E(t) \text{ and } \exists \varepsilon > 0,\ \forall s \in [t - \varepsilon, t),\ (i, j) \notin E(s)$$

then (i, j) remains at for a duration τ, i.e.:

$$\forall s \in [t, t + \tau],\ (i, j) \in E(s).$$

Then, we have the convergence result:

Theorem

If, for all $t \in \mathbb{R}$, the graph $(V, \cup_{s \geq t} E(s))$ is connected, then the consensus is achieved.

Summary

Discrete time consensus algorithms:

Algorithm	Topology	Consensus if	Convergence rate
$x(t+1) = (I - \varepsilon L)x(t)$	fixed	G is connected	$(1-\varepsilon\lambda_2)^t$
$x(t+1) = (I - \varepsilon D^{-1}L)x(t)$	fixed	G is connected	$(1-\varepsilon\tilde{\lambda}_2)^t$
$x(t+1) = (I - \varepsilon L)x(t)$	dynamic	$G(t)$ is connected for all $t \in \mathbb{R}$	$(1-\varepsilon\underline{\lambda}_2)^t$ with $\underline{\lambda}_2 \leq \min_{t\in\mathbb{N}} \lambda_2(t)$
$x(t+1) = (I - \varepsilon L)x(t)$	dynamic	$(V, \cup_{s\geq t} E(s))$ is connected for all $t \in \mathbb{R}$	-
$x(t+1) = (I - \varepsilon D^{-1}L)x(t)$	dynamic	$(V, \cup_{s\geq t} E(s))$ is connected for all $t \in \mathbb{R}$	-

- Connectivity properties are crucial for convergence of consensus algorithms.
- Some convergence rates are determined by the second smallest eigenvalue of the Laplacian or normalized Laplacian matrix: the more connected the network the faster the consensus.

Summary

Continuous time consensus algorithms:

Algorithm	Topology	Consensus if	Convergence rate
$\dot{x}(t) = -Lx(t)$	fixed	G is connected	$e^{-\lambda_2 t}$
$\dot{x}(t) = -D^{-1}Lx(t)$	fixed	G is connected	$e^{-\tilde{\lambda}_2 t}$
$\dot{x}(t) = -Lx(t)$	dynamic	$G(t)$ is connected for all $t \in \mathbb{R}$	$e^{-\underline{\lambda}_2 t}$ with $\underline{\lambda}_2 \leq \min_{t\in\mathbb{R}^+} \lambda_2(t)$
$\dot{x}(t) = -Lx(t)$	dynamic	Dwell time and $(V, \cup_{s\geq t} E(s))$ is connected for all $t \in \mathbb{R}$	-
$\dot{x}(t) = -D^{-1}Lx(t)$	dynamic	Dwell time and $(V, \cup_{s\geq t} E(s))$ is connected for all $t \in \mathbb{R}$	-

- Results are very similar to those for discrete time algorithms.
- For dynamic topologies, a supplementary dwell time assumption is needed in order to prove that the consensus is achieved.

Networks with communication time-delays

Consider a network of continuous-time integrators with a fixed topology in which the state x_i of the node v_i is transmitted through a communication channel c_{ij} with time-delay τ_{ij} before getting to node v_j. Moreover, we consider all the communication channels generate similar amount of delay: $\tau_{ij} = \tau, \ \forall (i,j) \in E$.

One way to model the system is:

$$\dot{x}_i(t) = \sum_{j \in N_i} a_{i,j}(x_j(t-\tau) - x_i(t-\tau)), \ \forall i$$

The transfer function associated to each edge is $H(s) = e^{-s\tau}$. Thus, in Laplace domain the above equation becomes:

$$sX_i(s) - x_i(0) = \sum_{j \in N_i} a_{i,j} e^{-s\tau}(X_j(s) - X_i(s)), \ \forall i$$

More compactly one gets:

$$X(s) = G_\tau(s)x(0), \quad G_\tau(s) = (sI + Le^{-s\tau})^{-1} \tag{2}$$

Consensus with communication time-delays

Theorem

The protocol (4) globally asymptotically solves the consensus problem if and only if $\tau \in (0, \tau^*)$ with $\tau^* = \frac{\pi}{2\lambda_{max}(L)}$. Moreover, for $\tau = \tau^*$ the system has a globally asymptotically stable oscillatory solution with frequency $\omega = \lambda_{max}(L)$.

Proof We note that despite the nonzero delay τ, $x^* = \frac{1}{n}\mathbf{1}^\top x(t)$ is an invariant quantity. Thus, if the consensus is achieved, the consensus value is the invariant quantity x^*. To establish the stability of (4) we need to find conditions guaranteing that all the zeros of $sI + Le^{-s\tau}$ are on the open left half plane or $s = 0$.
We note that w_k is an eigenvector of $Z_\tau(s) \triangleq sI + Le^{-s\tau}$ iff w_k is an eigenvector of L.

Consensus with communication time-delays

Moreover one has $s = 0$ in the direction of $\mathbf{1}$. For any other eigenvector w_k of L one has

$$Z_\tau(s)w_k = 0 \Leftrightarrow s + e^{-s\tau}\lambda_k = 0 \tag{3}$$

For $\tau = 0$ the system is asymptotically stable and the continuous dependence of solutions of (3) on τ assures that the system becomes unstable when $Z_\tau(j\omega)w_k = 0$. Equivalently one has,

$$j\omega = \lambda_k(\cos(\omega\tau) - j\sin(\omega\tau))$$

leading to $\tau = \frac{\pi}{\lambda_k}$ for $\omega = \lambda_k$. Therefore the first crossing towards instability take place at $\omega = \lambda_{max}(L)$ for $\tau^* = \frac{\pi}{2\lambda_{max}(L)}$

Consensus with communication time-delays

For $\tau = \tau^*$, $G_\tau(s)$ has three poles on the imaginary axis, $s = 0, s = \pm j\lambda_{max}(L)$ and all the other poles are stable. The steady state solution is given by:

$$x_i(t) = a_i + b_i \sin(\lambda_{max}(L)t + \varphi_i), \ \forall i$$

where a_i, b_i, φ_i are constants depending on the initial conditions.

More realistic model

$$\dot{x}_i(t) = \sum_{j \in N_i} a_{i,j}(x_j(t-\tau) - x_i(t)), \ \forall i$$

leading to the following compact form in the Laplace domain:

$$X(s) = G_\tau(s)x(0), \quad G_\tau(s) = (sI + D - Ae^{-s\tau})^{-1} \tag{4}$$

Advanced time-delay systems methodologies allows to study the associated characteristic equation

$$\det(sI + D - Ae^{-s\tau}) = 0$$

This methodologies are out of the scope of this lecture.

Lecture outline

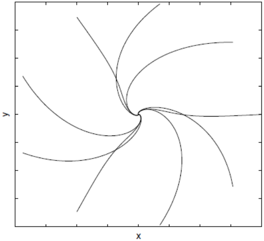
1. Algebraic graph theory
2. Spectral graph theory
3. Consensus algorithms
4. Applications
   - Rendez-vous in space
   - Flocking in mobile networks
   - Opinion dynamics and community detection in social networks
   - Synchronization of coupled oscillators

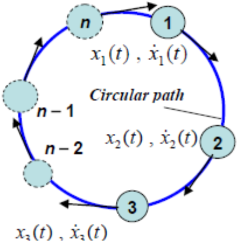
Rendez-vous problem

The *multi-agent rendez-vous problem* is to design local control strategies, one for each agent, which cause all members of the group to meet at single unspecified location.
The problem may be reformulated in order to obtain a specified formation or a uniform distribution in a specified formation.
Different formulations:

- Fixed communication topology (an agent is able to continuously track the position of specified agents in the network).
- Dynamic topology (an agent is able to track the position of agents in a sensing range)

Rendez-vous problem





Synchronous case

The maneuvering times for all agents are all the same length positive value τ_M. The real time axis can be partitioned into a sequence of time intervals $[0, t_1)$, $[t_1, t_2), \ldots, [t_k, t_{k+1}), \ldots$, each of length at least τ_M. During each interval one has a sensing period followed by a maneuvering period of fixed length τ_M. The agents are synchronized in that all are locked during sensing period and all can move only during maneuvering periods.

During the time interval $[t_k, t_{k+1})$ each agent moves taking into account the information obtained during the corresponding sensing period. In other words, the algorithm may be discretized as:

$$x_i(t_{k+1}) = x_i(t_k) + \sum_{j \in N_i(t_k)} u_{i,j}(t_k)$$

where $x_i(t)$ is a vector describing the position of the agent i at the moment t and $u_{i,j}(t)$ is the component of the local control input defining the influence of the position of agent j on the decision of agent i at moment t.

Local control input

As seen in the previous chapter, considering the communication graph is fixed and connected at time 0, a necessary and sufficient condition to assure the rendez-vous is to define the local control input at time t_k by:

$$u_{i,j}(t_k) = \epsilon(x_j(t_k) - x_i(t_k)), \epsilon \in (0, 1/d_{max})$$

This leads to the collective dynamics:

$$x(t_{k+1}) = (Id - \epsilon L)x(t_k) \triangleq Px(t_k)$$

where $x(t) \triangleq (x_1(t), x_2(t), \ldots, x_n(t))$

Implementation

Often (as in the case of SAMI platform), we can control the position of agents via their velocities. Therefore, the above computations are done during the sensing period while during the maneuvering period each agent is driven by the following dynamics:

$$x_i(t_{k+1}) = x_i(t_k) + v_i(k) \cdot \tau_M$$

where $v_i(k)$ is a constant velocity with $|v_i(k)| = \frac{||Px(t_k) - x(t_k)||}{\tau_M}$, having the same orientation as the vector $Px(t_k) - x(t_k)$

Remark

The algorithm and the implementation previously presented assure the rendez-vous even if the communication network is not fixed but $(V, \bigcup_{s>t} E(s))$ is connected for all $t \geq 0$.

Formation problem

Let $\mathcal{F} = (f_1, f_2, \ldots, f_n)$ a formation. Denoting by $y(t)$ the vector $x(t) - \mathcal{F}$ and applying the previous rendez-vous algorithm to y, one obtains $\lim_{t\to\infty} y(t) = y^* \cdot \mathbf{1}$, where y^* is the rendez-vous point.
Therefore,

$$\lim_{t\to\infty} x_i(t) = y^* + f_i \Leftrightarrow \lim_{t\to\infty} |(x_i(t) - f_i) - (x_j(t) - f_j)| = 0$$

which means the formation is achieved asymptotically.

Asynchronous case

The real time axis can be partitioned for each agent into a sequence of time intervals $[0, t_{i,1}),\ [t_{i,1}, t_{i,2}), \ldots, [t_{i,k}, t_{i,k+1}), \ldots$, each of length at most $\tau_D + \tau_M$, where τ_D is greater than τ_M, called *dwell time*.
Each interval $[t_{i,k}, t_{i,k+1})$ consists of a sensing period of fixed length τ_D, during which the agent i is stationary, followed by a maneuvering period of length at most τ_M.
The agents operate asynchronously in the sense that the sequences $t_{i,k}, i \geq 1$ are uncorrelated.
The rendez-vous problem is solved under some supplementary conditions concerning the overlapping of the sensing periods and the registration of neighbors.

Flocking in mobile agents network

Flocking is the behavior exhibited when a group of birds are in flight:



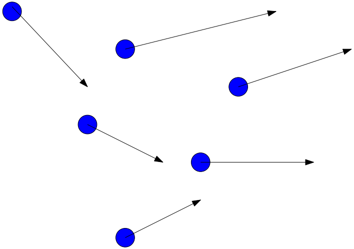
More precisely, a flocking behavior is characterized by three properties:

1. Alignement: the birds have the same velocity.
2. Cohesion: the birds remain together.
3. Separation: there is a minimum distance between birds.

In the following, we focus on the first and second property.

Example: flocking in mobile networks

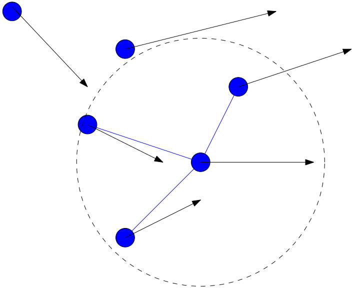
Consider a set of agents willing to move in a common direction:



Agent i is characterized by its position x_i and velocity v_i.

Example: flocking in mobile networks

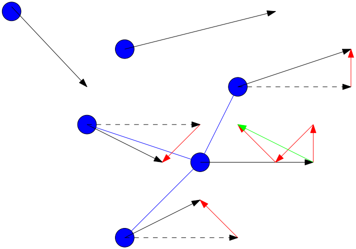
Consider a set of agents willing to move in a common direction:



Agent i has limited communication or sensing capabilities.

Example: flocking in mobile networks

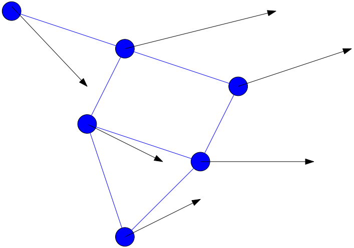
Consider a set of agents willing to move in a common direction:



Agent i tries to align its velocity on its neighbors: $\dot{v}_i = \sum_{j \in N_i} (v_j - v_i)$.

Example: flocking in mobile networks

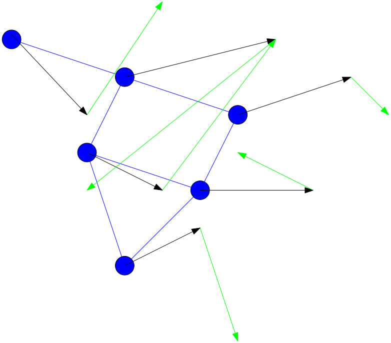
Consider a set of agents willing to move in a common direction:



The communication network is described by a (dynamic) graph.

Example: flocking in mobile networks

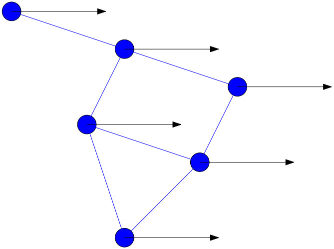
Consider a set of agents willing to move in a common direction:



Global linear dynamics with structure given by the graph: $\dot{v} = -Lv$.

Example: flocking in mobile networks

Consider a set of agents willing to move in a common direction:



Do the agents eventually agree on a common velocity?

A flocking model

- We consider a set of mobile agents $V = \{1, \dots, n\}$: for each $i \in V$, $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$ denote its position and its velocity.
- An agent can communicate only with agents that are sufficiently close: the interaction graph $G(t) = (V, E(t))$ is given by

$$E(t) = \{(i,j) \in V \times V|\ i \neq j \text{ and } \|x_i(t) - x_j(t)\| \leq R\}.$$

- The agents try to agree on their velocity using the continuous time consensus algorithm:

$$\dot{v}_i(t) = \sum_{j \in N_i(t)} (v_j(t) - v_i(t)).$$

A flocking model

The considered model is then:

$$\begin{cases} \dot{x}_i(t) & = & v_i(t) \\ \dot{v}_i(t) & = & \sum_{j \in N_i(t)} (v_j(t) - v_i(t)) \end{cases}$$

We want to determine a set of initial conditions ensuring that a flocking behavior is achieved.

We have already shown the following result:

Theorem

If the graph $G(t)$ is connected for all $t \in \mathbb{R}$, then the consensus is achieved. Moreover, for $\underline{\lambda}_2 \leq \min_{t \in \mathbb{R}^+} \lambda_2(t)$,

$$\forall t \geq 0, \; \|v(t) - \mathbf{1}_n \otimes v^*\| \leq e^{-\underline{\lambda}_2 t} \|v(0) - \mathbf{1}_n \otimes v^*\|$$

with $v^* = \sum_{i \in V} v_i(0)$.

Graph robustness analysis

- We define a notion of robustness for interaction graphs.
- For $i \in V$, let $x_i \in \mathbb{R}^d$ be the position of agent i, let $x = (x_1, \ldots, x_n)$, we define the associated graph $G_x = (V, E_x)$ with

$$E_x = \{(i,j) \in V \times V | \ i \neq j \text{ and } \|x_i - x_j\| \leq R\}.$$

- Let $x^0 \in \mathbb{R}^{nd}$ be a reference configuration. Assuming that G_{x^0} is a connected graph, we are interested in characterizing a neighborhood of x^0 such that for any perturbed configuration y in this neighborhood the graph G_y is connected.
- We introduce a measure of robustness for G_{x^0} which allows us to identify such a neighborhood.

Graph robustness analysis

- Let $(i_1, i_2), \ldots, (i_p, i_{p+1})$ be a path of G_{x^0}, we define the slackening of the path:

$$s((i_1, i_2), \ldots, (i_p, i_{p+1})) = \min_{k=1}^{p}(R - \|x^0_{i_k} - x^0_{i_{k+1}}\|).$$

  If the distances between agents do not change more than $s((i_1, i_2), \ldots, (i_p, i_{p+1}))$ then the path is preserved.

- The path-robustness ρ_{ij} between two agents $i, j \in V$ with $i \neq j$, is the maximal slackening of all paths between i and j:

$$\rho_{ij} = \max_{(i_1,i_2),\ldots,(i_p,i_{p+1}) \in \mathrm{Paths}(i,j)} s((i_1, i_2), \ldots, (i_p, i_{p+1})).$$

  If the distances between agents do not change more than ρ_{ij} there remains a path between i and j. We set $\rho_{ii} = R$.

Graph robustness analysis

- The robustness of the graph $\rho_{G_{x^0}}$ is the minimal path-robustness between all pair of nodes:

$$\rho_{G_{x^0}} = \min_{(i,j)\in V^2} \rho_{ij}$$

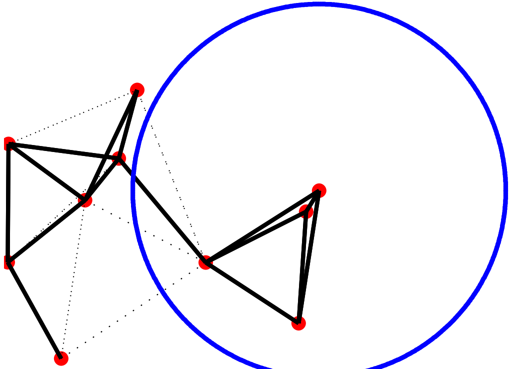
If the distances between agents do not change more than $\rho_{G_{x^0}}$ then the graph remains connected.

- The core robust subgraph of G_{x^0} is the graph $\mathcal{M}(G_{x^0}) = (V, \mathcal{M}(E_{x^0}))$ where

$$\mathcal{M}(E_{x^0}) = \left\{(i,j) \in V \times V|\ i \neq j \text{ and } \|x_i - x_j\| \leq R - \rho_{G_{x^0}}\right\}.$$

Since $\rho_{G_{x^0}} \geq 0$, $\mathcal{M}(G_{x^0})$ is clearly a subgraph of G_{x^0}.

Example of a core robust subgraph:



Graph robustness analysis

Lemma

Let $x^0 \in \mathbb{R}^{nd}$ be a reference configuration such that the graph G_{x^0} is connected. Then, the core robust subgraph $\mathcal{M}(G_{x^0})$ is connected.

Proof: Let $i, j \in V$, then $\rho_{ij} \geq \rho_{G_{x^0}}$. Let $(i_1, i_2), \ldots, (i_p, i_{p+1})$ be a path of G_{x^0} between i and j with maximal slackening, then

$$s((i_1, i_2), \ldots, (i_p, i_{p+1})) = \rho_{ij}.$$

Then, for all $k \in \{1, \ldots, p\}$,

$$R - \|x^0_{i_k} - x^0_{i_{k+1}}\| \geq s((i_1, i_2), \ldots, (i_p, i_{p+1})) \geq \rho_{G_{x^0}}.$$

Therefore, for all $k \in \{1, \ldots, p\}$, $(i_k, i_{k+1}) \in \mathcal{M}(E_{x^0})$.
Then, $(i_1, i_2), \ldots, (i_p, i_{p+1})$ is a path of $\mathcal{M}(G_{x^0})$ between i and j.
Thus, $\mathcal{M}(G_{x^0})$ is connected.

Graph robustness analysis

Proposition

Let $x^0 \in \mathbb{R}^{n\times d}$ be a reference configuration such that the associated graph G_{x^0} is connected. Let $y \in \mathbb{R}^{n\times d}$ be a perturbed configuration such that

$$\|y - x^0\| \leq \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

Then, $\mathcal{M}(G_{x^0})$ is a subgraph of G_y and G_y is connected.

Proof: Let $z = (z_1, \dots, z_n)$ such that $z = y - x^0$.
For all $i, j \in V$, we have $-2z_i^\top z_j \leq \|z_i\|^2 + \|z_j\|^2$. Then,

$$\begin{aligned} \|z_i - z_j\|^2 &= \|z_i\|^2 + \|z_j\|^2 - 2z_i^\top z_j \leq 2(\|z_i\|^2 + \|z_j\|^2) \\ &\leq 2\|z\|^2 = 2\|y - x^0\|^2 \leq \rho^2_{G_{x^0}}. \end{aligned}$$

Then, $\|z_i - z_j\| \leq \rho_{G_{x^0}}$.

Graph robustness analysis

Proposition

Let $x^0 \in \mathbb{R}^{n\times d}$ *be a reference configuration such that the associated graph* G_{x^0} *is connected. Let* $y \in \mathbb{R}^{n\times d}$ *be a perturbed configuration such that*

$$\|y - x^0\| \leq \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

Then, $\mathcal{M}(G_{x^0})$ *is a subgraph of* G_y *and* G_y *is connected.*

Proof: Let $(i,j) \in \mathcal{M}(\mathcal{E}_{x^0})$, then $\|x_i^0 - x_j^0\| \leq R - \rho_{G_{x^0}}$. Thus,

$$\begin{aligned} \|y_i - y_j\| &= \|x_i^0 - x_j^0 + z_i - z_j\| \leq \|x_i^0 - x_j^0\| + \|z_i - z_j\| \\ &\leq R - \rho_{G_{x^0}} + \rho_{G_{x^0}} = R. \end{aligned}$$

Then, $(i,j) \in E_y$. Therefore $\mathcal{M}(G_{x^0})$ is a subgraph of G_y.
Since $\mathcal{M}(G_{x^0})$ is connected, so is G_y.

Graph robustness computation

Algorithm (Computation of the robustness $\rho_{G_{x^0}}$)

// Initialization:

$\forall (i,j) \in V^2,\ \rho_{ij}^0 \leftarrow R - \|x_i^0 - x_j^0\|$;

// Main loop:

for $k \in V$ **do**
  for $i \in V$ **do**
    for $j \in V$ **do**
      $\rho_{ij}^k \leftarrow \max\left(\rho_{ij}^{k-1}, \min\left(\rho_{ik}^{k-1}, \rho_{kj}^{k-1}\right)\right)$;
    end for
  end for
end for

// Computation of robustness:

$\rho_{G_{x^0}} = \min\limits_{(i,j)\in V^2} \rho_{ij}^n$;

Sufficient conditions for flocking

Theorem

Let $x(0) \in \mathbb{R}^{nd}$ be a vector of initial positions of the agents such that the associated graph $G_{x(0)}$ is connected and its robustness $\rho_{G_{x(0)}} > 0$.

Let $v(0) \in \mathbb{R}^{nd}$ be a vector of initial velocities such that

$$\|v(0) - \mathbf{1}_n \otimes v^*\| \leq \frac{\lambda_2^* \rho_{G_{x(0)}}}{\sqrt{2}}$$

where λ_2^* is the algebraic connectivity of $\mathcal{M}(G_{x(0)})$.

Then, for all $t \in \mathbb{R}^+$, $\mathcal{M}(G_{x(0)})$ is a subgraph of $G(t)$. Moreover,

$$\|v(t) - \mathbf{1}_n \otimes v^*\| \leq e^{-\lambda_2^* t} \|v(0) - \mathbf{1}_n \otimes v^*\|.$$

Sufficient conditions for flocking

Proof: Let Π be the set of graphs with n nodes which have $\mathcal{M}(G_{x(0)})$ as a subgraph. Since $G_{x(0)}$ is connected, we have that $\mathcal{M}(G_{x(0)})$ is connected. Therefore, all graphs in Π are connected and since $\mathcal{M}(G_{x(0)}) \in \Pi$,

$$\min_{G \in \Pi} \lambda_2(G) = \lambda_2(\mathcal{M}(G_{x(0)})) = \lambda_2^* > 0.$$

Let us assume that there exists $t > 0$ such that $\mathcal{M}(G_{x(0)})$ is not a subgraph of $G(t)$ (i.e. $G(t) \notin \Pi$). Let

$$t^* = \inf\{t \in \mathbb{R}^+;\ G(t) \notin \Pi\}.$$

For $i \in V$, let $y_i(t) = x_i(t) - v^* t$. Let us remark that for all $i, j \in V$, $y_i(t) - y_j(t) = x_i(t) - x_j(t)$. Therefore, for all $t \in \mathbb{R}^+$, $G(t) = G_{y(t)}$.

Sufficient conditions for flocking

Proof: If $t^* > 0$, it follows that for all $t \in [0, t^*)$

$$\|v(t) - \mathbf{1}_n \otimes v^*\| \leq e^{-\lambda_2^* t} \|v(0) - \mathbf{1}_n \otimes v^*\| \leq e^{-\lambda_2^* t} \frac{\lambda_2^* \rho_{G_{x(0)}}}{\sqrt{2}}.$$

By remarking that

$$y(t) = x^0 + \int_0^t (v(t) - \mathbf{1}_n \otimes v^*) \, ds$$

we have for all $t \in [0, t^*)$

$$\|y(t) - x^0\| \leq \frac{\lambda_2^* \rho_{G_{x^0}}}{\sqrt{2}} \int_0^t e^{-\lambda_2^* s} ds < \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

Then, by continuity of y, there exists $\varepsilon > 0$ such that for all $t \in [0, t^* + \varepsilon]$,

$$\|y(t) - x^0\| \leq \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

Sufficient conditions for flocking

Proof: If $t^* = 0$, since $y(0) = x^0$ and by continuity of y, the same kind of property holds.

Then, we have for all $t \in [0, t^* + \varepsilon]$, $G(t) = G_{y(t)} \in \Pi$. This contradicts the definition of t^*. Therefore, for all $t \in \mathbb{R}^+$, $G(t) \in \Pi$. Thus for all $t \in \mathbb{R}^+$, $\mathcal{M}(G_{x(0)})$ as a subgraph of $G(t)$. This proves the first part of the theorem.

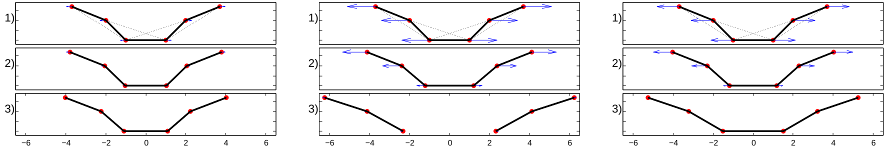
Moreover, it follows that for all $t \in \mathbb{R}^+$, $\lambda_2(t) \geq \lambda_2^*$, and then

$$\|v(t) - \mathbf{1}_n \otimes v^*\| \leq e^{-\lambda_2^* t} \|v(0) - \mathbf{1}_n \otimes v^*\| .$$

Example

Example in a network of 6 agents with communication radius $R = 3.2$:

$$\|v(0) - \mathbf{1}_n \otimes v^*\| = \frac{\lambda_2^* \rho G_{x(0)}}{\sqrt{2}} \qquad \|v(0) - \mathbf{1}_n \otimes v^*\| = \frac{5\lambda_2^* \rho G_{x(0)}}{\sqrt{2}} \qquad \|v(0) - \mathbf{1}_n \otimes v^*\| = \frac{4\lambda_2^* \rho G_{x(0)}}{\sqrt{2}}$$



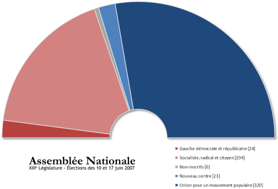
- The bound that we compute is often conservative as shown on the example above.
- However, we can find examples where the bound is tight.

Summary

- We have considered a model of flocking behavior based on communication graphs given by a proximity rule.
- We have established a set of initial conditions for which the flocking behavior is achieved.
- The conditions are only sufficient but can be checked algorithmically.
- The main concepts are those of graph robustness and the associated core robust subgraph that remains for all time ensuring that the communication graph remains connected for all time.

Opinion dynamics in social networks

Opinion dynamics studies the emergence of consensus in social networks:



- In real social networks, it is often the case that a global consensus cannot be reached. Instead, several consensus are reached locally by subsets of agents forming communities.
- Can we propose a model of opinion dynamics reproducing this phenomenom ? Can we learn something on real social networks using this model ?

A model of opinion dynamics

We consider a set of agents $i \in V$ in a network $G = (V, E)$.
Each agent $i \in V$ has an opinion $x_i(t) \in \mathbb{R}$.
We propose a discrete time model of opinion dynamics as follows:

- At each time step, agent i receives the opinion of its neighbors in G.
- Agent i gives confidence only to his neighbor that have an opinion close from its own. Agent i updates its opinion accordingly to its confidence neighborhood.
- Due to loss of patience, the confidence of each agent decreases at each time step: agent i requires that, at each negotiation round, the opinion of agent j moves significantly towards its opinion in order to keep negotiating with j.

A model of opinion dynamics

Formally, the opinion dynamics model is described as follows:

$$x_i(t+1) = x_i(t) + \frac{\varepsilon}{d_i(t)} \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$$

where the interaction graph at time t is $G(t) = (V, E(t))$ with

$$E(t) = \left\{(i,j) \in E | \; \left(|x_i(t) - x_j(t)| \leq R\rho^t\right)\right\}$$

where $\varepsilon \in (0, 1/2)$, $R \geq 0$ and $\rho \in (0, 1)$ are model parameters.

The parameter ρ characterizes the confidence decay of the agents.

Convergence analysis

Proposition

For all $i \in V$, the sequence $(x_i(t))_{t\in\mathbb{N}}$ is convergent. We denote x_i^* its limit. Furthermore, we have for all $t \in \mathbb{N}$,

$$|x_i(t) - x_i^*| \leq \frac{\varepsilon R}{1-\rho}\rho^t.$$

Proof: Let $i \in V$, $t \in \mathbb{N}$, we have

$$\begin{aligned}
|x_i(t+1) - x_i(t)| &= \left| \frac{\varepsilon}{d_i(t)} \sum_{j\in N_i(t)} (x_j(t) - x_i(t)) \right| \\
&\leq \frac{\varepsilon}{d_i(t)} \sum_{j\in N_i(t)} |x_j(t) - x_i(t)| \\
&\leq \frac{\varepsilon}{d_i(t)} \sum_{j\in N_i(t)} R\rho^t = \varepsilon R\rho^t.
\end{aligned}$$

Convergence analysis

Proposition

For all $i \in V$, the sequence $(x_i(t))_{t\in\mathbb{N}}$ is convergent. We denote x_i^* its limit. Furthermore, we have for all $t \in \mathbb{N}$,

$$|x_i(t) - x_i^*| \leq \frac{\varepsilon R}{1-\rho}\rho^t.$$

Proof: Let $t \in \mathbb{N}$, $\tau \in \mathbb{N}$, then

$$\begin{aligned} |x_i(t+\tau) - x_i(t)| &\leq \sum_{k=0}^{\tau-1} |x_i(t+k+1) - x_i(t+k)| \leq \sum_{k=0}^{\tau-1} \varepsilon R \rho^{t+k} \\ &\leq \frac{\varepsilon R}{1-\rho}\rho^t \end{aligned}$$

which shows, since $\rho \in (0,1)$, that the sequence $(x_i(t))_{t\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, it is convergent. The inequality of the proposition is obtained by letting τ go to $+\infty$.

Communities

Definition

A community is subset of agent $C \subseteq V$ such that for all $i, j \in C$, $x_i^* = x_j^*$. The graph of a community C is $G_C = (C, E_C)$ where

$$E_C = \{(i,j) \in E |\ i \in C,\ j \in C\}.$$

The set of communities is $\mathscr{C}$, it is a partition of V. The graph of communities is $G_{\mathscr{C}} = (V, E_{\mathscr{C}})$ where

$$E_{\mathscr{C}} = \{(i,j) \in E |\ x_i^* = x_j^*\}.$$

Let us remark that

$$E_{\mathscr{C}} = \bigcup_{C \in \mathscr{C}} E_C.$$

Characterization of communities

Before giving an algebraic characterization of communities, we need to make a supplementary assumption.

- We have seen that the opinions converge at a speed $O(\rho^t)$.
- In practice, we observe that the convergence is often slightly faster and this motivates the following assumption:

Assumption (Fast convergence)

There exists $\underline{\rho} < \rho$ and $M \geq 0$ such that for all $i \in V$, for all $t \in \mathbb{N}$,

$$|x_i(t) - x_i^*| \leq M\underline{\rho}^t.$$

Characterization of communities

Proposition

There exists $T \in \mathbb{N}$, such that for all $t \geq T$, $E(t) = E_{\mathscr{C}}$.

Proof: $(E(t) \subseteq E_{\mathscr{C}})$

E can be splitted into two subets: E^f consists of edges that disappear in finite time, E^∞ consists of edges that appear infinitely often.
Then, there exists T_1 such that for all $t \geq T_1$, $E(t) \subseteq E^\infty$.

Let $(i,j) \in E^\infty$ then there exists an unbounded sequence of times τ_k such that $(i,j) \in E(\tau_k)$. This gives

$$|x_i(\tau_k) - x_j(\tau_k)| \leq R\rho^t$$

and $x_i^* = x_j^*$. Therefore $(i,j) \in E_{\mathscr{C}}$.
Hence, for all $t \geq T_1$, $E(t) \subseteq E^\infty \subseteq E_{\mathscr{C}}$.

Characterization of communities

Proposition

There exists $T \in \mathbb{N}$, such that for all $t \geq T$, $E(t) = E_{\mathscr{C}}$.

Proof: $(E_{\mathscr{C}} \subseteq E(t))$

Let $(i,j) \in E_{\mathscr{C}}$, then for all $t \in \mathbb{N}$

$$\begin{aligned} |x_i(t) - x_j(t)| &\leq |x_i(t) - x_i^*| + |x_i^* - x_j^*| + |x_j^* - x_j(t)| \\ &\leq |x_i(t) - x_i^*| + |x_j^* - x_j(t)| \leq 2M\underline{\rho}^t. \end{aligned}$$

Since $\underline{\rho} < \rho$, there exists T_2 such that for all $t \geq T_2$, $2M\underline{\rho}^t \leq R\rho^t$. This implies that for all $t \geq T_2$, $(i,j) \in E(t)$.
Hence, for all $t \geq T_2$, $E_{\mathscr{C}} \subseteq E(t)$.

Remark

We have shown that after a finite time, the graph $G(t)$ is fixed.

Characterization of communities

Theorem

For almost all vectors of initial opinions, for all communities $C \in \mathscr{C}$, such that $|C| \geq 2$,

$$\tilde{\lambda}_2(G_C) > (1-\rho)/\varepsilon.$$

Main ideas of the proof:
Let $C \in \mathscr{C}$ and let us assume that $\tilde{\lambda}_2(G_C) \leq (1-\rho)/\varepsilon$.

Let $x_C(t)$ denote the vector of opinions of agents in C. For all $t \geq T$, since $G(t) = G_{\mathscr{C}}$, we have

$$x_C(t+1) = (I - \varepsilon D(G_C)^{-1} L(G_C)) x_C(t).$$

We have seen that the rate of convergence of $x_C(t)$ is $1 - \varepsilon\tilde{\lambda}_2(G_C) \geq \rho$ except if $x_C(T)$ and thus $x(T)$ belongs to a specific subspace of zero measure H_C.

Characterization of communities

Theorem

For almost all vectors of initial opinions, for all communities $C \in \mathscr{C}$, such that $|C| \geq 2$,

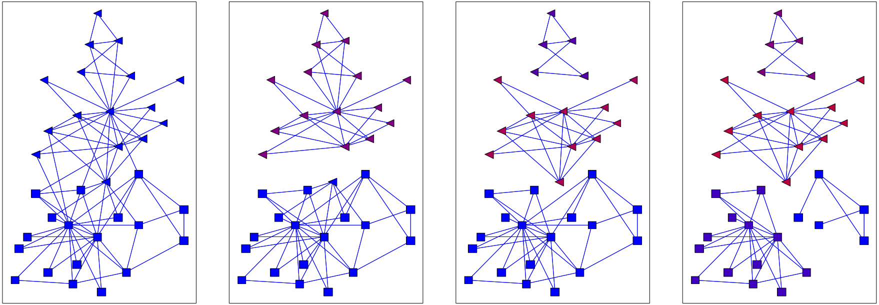
$$\tilde{\lambda}_2(G_C) > (1-\rho)/\varepsilon.$$

Main ideas of the proof:
By assumption the rate of convergence of $x_C(t)$ is smaller than $\underline{\rho} < \rho$. Thus $x(T)$ necessarily belongs to H_C.

Then, by remarking that all matrices $(I - \varepsilon D(t)^{-1} L(t))$ are invertible, we can move backward in time and show that the initial conditions leading to H_C at time T are included in a set of zero measure (consisting of a countable union of subspaces) that is independent of C and T.

Example: karate club network

A social network of 34 agents:



$(1-\rho)/\varepsilon = 0.1$ $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.12$

$(1-\rho)/\varepsilon = 0.2$ $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.25$

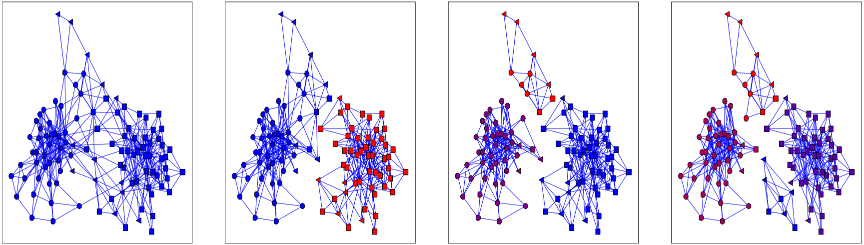
$(1-\rho)/\varepsilon = 0.3$ $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.33$

$(1-\rho)/\varepsilon = 0.4$ $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.57$

The second partition is almost that observed by Zachary in its original study (1973).

Example: book network

A network of 105 books on American politics:



Original network | $(1-\rho)/\varepsilon = 0.1$, $\min_{C\in\mathscr{C}} \tilde{\lambda}_2(G_C) = 0.13$ | $(1-\rho)/\varepsilon = 0.15$, $\min_{C\in\mathscr{C}} \tilde{\lambda}_2(G_C) = 0.18$ | $(1-\rho)/\varepsilon = 0.2$, $\min_{C\in\mathscr{C}} \tilde{\lambda}_2(G_C) = 0.27$

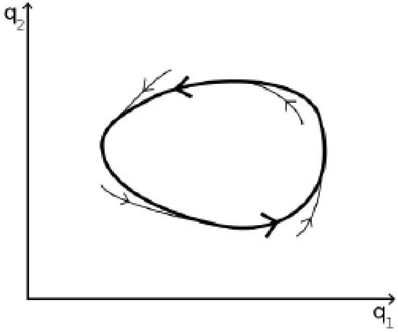
One recovers the information on the political alignment (democrat, republican, centrist) of the books.

Summary

- We have introduced a model of opinion dynamics with decaying confidence.
- In this model, a global consensus may not be achieved and only local agreements can be reached. The agents organize themselves in communities.
- The analysis of the model provided an algebraic characterization of the communities under a fast convergence assumption.
- The experimental results tend to confirm the validity of the algebraic characterization. Moreover, the communities that are obtained on real examples are meaningful and allows to uncover information hidden in the network structure.

Self-sustained oscillators

The state of a pendulum could be described by its position (specified by an angle) and its velocity, giving it a two-dimensional phase space. This means the state of the system at a given time is shown as a certain point in the phase space. As the system evolves, the point moves in some trajectory in the phase space. To describe the motion of an oscillator, then, we can talk about its motion in phase space. Self-sustained oscillators like the ones studied here move in phase space in a special way when left to themselves, they eventually revisit the same points over and over. So the steady-state evolution of a self-sustained oscillator corresponds to some closed curve in phase space. This closed curve is called a limit cycle. Self-sustained oscillations have stable limit cycles, meaning that all trajectories near the limit cycle approach the limit cycle. This is the same as saying that the oscillator is stable to perturbationsafter taking it slightly away from its limit cycle, it will eventually come back to oscillate in the original cycle



For self-sustained oscillators the phase θ is defined so that it grows uniformly in time and gains 2π radians for each trip around the limit cycle. Then each point on the cycle corresponds to a certain value of the phase. Note that the phase is defined to grow uniformly in time, while the system may not evolve uniformly along the limit cycle. Also associated with each oscillator is its natural angular frequency, commonly called ω. This quantity characterizes how quickly the oscillator travels around its limit cycle in the absence of outside influences.

Coupled oscillators

In the 60s Arthur Winfree looked at the behavior of a large collection of interacting limit-cycle oscillators, in an attempt to model collective synchronization in large groups, like the example of the flashing fireflies. He constructed his model by assuming 1) that the oscillators are nearly identical, and 2) that the coupling among oscillators is small. In his model, the rate of change of the phase of an oscillator is determined by a combination of its natural frequency ω_i and the collective state of all of the oscillators combined. Each oscillators sensitivity to the collective rhythm is determined by a function Z, and its own contribution to the collective rhythm is specified by a function X. Thus each oscillator has an equation describing how its phase changes in time:

$$\dot{\theta}_i = \omega_i + \left(\sum_{i=1}^{N} X(\theta_j) \right) Z(\theta_i)$$

Kuramoto oscillators

Kuramoto proved that the long-term dynamics of any system of nearly identical, weakly coupled limit-cycle oscillators can be described by the following equation

$$\dot{\theta}_i = \omega_i + \sum_{i=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), i = 1, \ldots, N \tag{5}$$

where the interaction function Γ_{ij} determines the form of coupling between oscillators i and j. Going further he assumed that each oscillator affected every other oscillator. He further assumed that the interactions were equally weighted and depended only sinusoidally on the phase difference.

$$\dot{\theta}_i = \omega_i + \sum_{i=1}^{N} \frac{K}{N} sin(\theta_j - \theta_i), i = 1, \ldots, N \tag{6}$$

For simplicity in theoretical calculations, the natural frequencies ω_i are generally distributed according to a probability density $g(\omega)$ that is symmetric about some frequency Ω, so that $g(\Omega + \omega) = g(\Omega - \omega)$.

Kuramoto oscillators

Thus, each oscillator tries to run independently at its own frequency, while the coupling tends to synchronize it to all the others. By making a suitable choice of a rotating frame, $\theta_i \mapsto \theta_i - \Omega t$, we can transform (6) to an equivalent system of phase oscillators whose natural frequencies have zero mean. Define the "order" parameters r and ψ as

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

Here r represents the phase-coherence of the population of oscillators, and ψ indicates the average phase. When the coupling is sufficiently weak, the oscillators run incoherently whereas beyond a certain threshold collective synchronization emerges spontaneously.

Transformation

Multiplying by $e^{-i\theta_\ell}$ one gets

$$re^{i(\psi-\theta_\ell)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_\ell)}$$

whose imaginary parts are

$$r\sin(\psi-\theta_\ell) = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j-\theta_\ell)$$

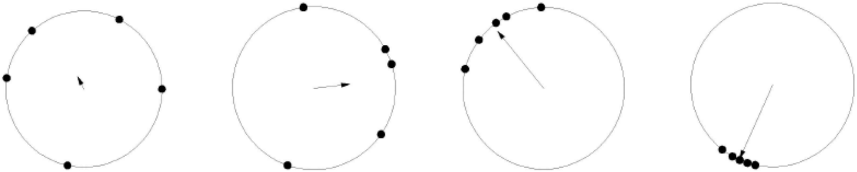
leading at

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi-\theta_i), i = 1,\dots,N$$

Thus the oscillators' equations are no longer explicitly coupled; instead the order parameters govern behavior. A further transformation is usually done, to a rotating frame in which the statistical average of phases over all oscillators is zero. That is, $\psi = 0$. Thus the dynamics becomes:

$$\dot{\theta}_i = \omega_i - Kr\sin(\theta_i), i = 1,\dots,N$$

Synchronization example



(a) $|r| = 0.18$ (b) $|r| = 0.44$ (c) $|r| = 0.91$ (d) $|r| = 0.99$

Movie

Stationary synchronization for $N \to \infty$

For infinitely many oscillators one considers that the density of oscillators at a given phase θ, with given natural frequency ω, at time t is $\rho(\theta, \omega, t)$. In this case the order parameters defined by an arithmetic mean becomes now an average over phase and frequency:

$$re^{i\psi} = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} e^{i\theta} \rho(\theta, \omega, t) g(\omega) d\theta d\omega \qquad (7)$$

This equation illustrates the use of the order parameter to measure oscillator synchronization. In fact, when $K \to 0$ one has $\theta_i \approx \omega_i t + \theta_i(0)$, that is the oscillators rotate at angular frequencies given by their own natural frequencies. Consequently, setting $\theta \approx \omega t$ in (7), we obtain $r \to 0$ as $t \to \infty$. So the oscillators are not synchronized. On the other hand, in the limit of strong coupling, $K \to \infty$, the oscillators become synchronized to their average phase, $\theta_i \approx \psi$, and (7) implies $r \to 1$.

Partial synchronization

For intermediate couplings, $K_C < K < \infty$, part of oscillators are phase-locked ($\dot{\theta}_i = 0$) and part are rotating out of synchrony with the locked oscillators. The extreme cases are:

- *incoherent solution*, $\rho = \frac{1}{2\pi}, r = 0$ corresponding to an angular distribution of oscillators having equal probability in $[-\pi, \pi]$
- *global synchronization* $\theta_i = \psi, r = 1$

A degree of synchronization can be defined for $0 < r < 1$. A typical oscillator running with velocity $v = \omega - Kr\sin(\theta - \psi)$ will become stably locked at an angle such that

$$Kr\sin(\theta - \psi) = \omega, -\pi/2 \leq (\theta - \psi) \leq \pi/2$$

So, oscillators with frequencies $|\omega| > Kr$ cannot be locked.

Partial synchronization

For $\psi = 0$ one gets

$$r = re^{i\psi} = (e^{i\theta})_{lock} + (e^{i\theta})_{drift}$$

$$(e^{i\theta})_{drift} = \int_{-\pi}^{\pi} \int_{|\omega|>Kr} e^{i\theta} \rho(\theta, \omega, t) g(\omega) d\theta d\omega = 0$$

so

$$r = (e^{i\theta})_{lock} = (cos(\theta))_{lock} = \int_{-Kr}^{Kr} cos(\theta(\omega)) g(\omega) d\omega$$

using $\omega = Kr \sin(\theta)$ for locked oscillators and dividing by r one obtains

$$1 = K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^2 \theta g(Kr \sin\theta) d\theta$$

and $r \to 0^+$ leads to the critical point $K_C = \frac{2}{\pi g(0)}$ required to produce partially phase-synchronization.

Short-range models

A natural extension of the Kuramoto model includes short-range interaction effects. In other words each oscillator affects only the oscillators situated in a specific range. The dynamics becomes

$$\dot{\theta}_i = \omega_i + \sum_{j \in N_i} \frac{K}{|N_i|} sin(\theta_j - \theta_i), i = 1, \ldots, N$$

Global synchronization, which implies that all the oscillators are in phase, is rarely seen. Phase locking or partial synchronization is observed more frequently.

For short-range systems one would like to know:

- The existence of a lower critical dimension above which any kind of entrainment is possible. In particular, it is relevant to prove the existence of phase locking and clustering for large enough dimensionality and the differences between both types of synchronization.
- The topological properties of the entrained clusters and the possibility to define a dynamical correlation length describing the typical length

Simplification

Consider the case $\omega_1 = \omega_2 = \ldots = \omega_n$ which may be supposed 0 by introducing a rotating reference frame. Restricting our attention to the angles interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and introducing local coordinates:

$$(-\frac{\pi}{2}, \frac{\pi}{2}) \mapsto \mathbb{R} : \theta_k \mapsto x_k = \tan(\theta_k)$$

the Kuramoto equation becomes:

$$\dot{x}_i = \sum_{j \in N_i} \frac{K}{|N_i|} \frac{\sqrt{1 + x_i^2}}{\sqrt{1 + x_j^2}} (x_j - x_i), \;\; i = 1, \ldots, n$$

Laplacian framework

Thus, the collective dynamics can be expressed as:

$$\dot{x}(t) = -L(t)x(t)$$

where L

$$l_{ij}(t) = \begin{cases} 0 & \text{if } j \notin N_i \\ -\frac{K}{|N_i|}\frac{\sqrt{1+x_i^2}}{\sqrt{1+x_j^2}} & \text{if } j \in N_i \\ \sum_{j\in N_i} \frac{K}{|N_i|}\frac{\sqrt{1+x_i^2}}{\sqrt{1+x_j^2}} & \text{if } j = i \end{cases}$$

Synchronization condition are given in the previous chapter.