CS364A Exercise Set 4

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November 6, 2022

Exercise 28

(a) Some luxuries such as "Hermès" might have those properties. Since the cost of sponsored search is low relative to the goods price.

(b)

Exercise 29

Since

$$\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) \geqslant \sum_{j \neq i} b_j(\omega^*), \tag{1}$$

bidder i's payment

$$p_i(\mathbf{b}) = \max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) - \sum_{j \neq i} b_j(\omega^*) \geqslant 0. \tag{2}$$

Bidder i's payment can also be considered as its bid minus a "rebate". Note the definition of ω^* is

$$\omega^* = \operatorname*{argmax}_{\omega \in \Omega} \sum_{j=1}^{n} b_j(\omega). \tag{3}$$

Since $b_i \geqslant 0$,

$$\max_{\omega \in \Omega} \sum_{j=1}^{n} b_j(\omega) \geqslant \max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega), \tag{4}$$

bidder i's payment

$$\begin{aligned}
p_i(\mathbf{b}) =& b_i(\omega^*) - \left[\sum_{j=1}^{n} b_j(\omega^*) - \max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) \right] \\
=& b_i(\omega^*) - \left[\max_{\omega \in \Omega} \sum_{j=1}^{n} b_j(\omega) - \max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) \right] \\
\leqslant& b_i(\omega^*).
\end{aligned} \tag{5}$$

Exercise 30

Each item can be allocated to at most one bidders. Without the loss of generality, each bidder can be allocate at most one item. Construct a weighted bipartite graph $G = (U, V, E)$. Each vertex in U represents a bidder and each vertex in V represents an item. The weight of vertex (s, t)

$$w(s, t) = v_{st}, s \in U, t \in V. \tag{6}$$

Then, the allocation computation becomes a bipartite matching problem, which can be solve in polynomial time.

Exercise 31

When only the first two bidders are present, item A can only be allocated to one of them. The social surplus of allocating AB to bidder #1 and that of allocating A or AB to bidder #2 is the same. The winner needs to pay 1.

When all three bidders are present, allocating A to bidder #2 and B to bidder #3 maximizes the social surplus. Bidder #2's present does not affect bidder #1 and bidder #3's total surplus, which equals 1. So, bidder #2's payment is 0. Similarly, bidder #3's payment is also 0.

In a single-item Vickrey auction, adding an extra bidder can only increase or maintain the second highest bid among all bids. So, adding an extra bidder cannot decrease the revenue.

Exercise 32

Suppose there are 3 bidders #1, #2, #3 and 2 goods A and B. Bidder #1 only wants A:

$$v_1(A) = v_1(AB) = 6, v_1(B) = 0.$$

Bidder #2 only wants B:

$$v_2(B) = v_2(AB) = 6, v_2(A) = 0.$$

For bidder #3, there are synergies between A and B:

$$v_3(A) = v_3(B) = 4, v_3(AB) = 13.$$

If they all bid truthfully, A and B are all given to bidder #3. Bidder #1 and #2 are given no goods. However, if bidder #1 and #2 bid falsely:

$$\begin{aligned} b_1(A) = b_1(AB) = 8, b_1(B) = 0, \\ b_2(B) = b_2(AB) = 8, b_2(A) = 0, \end{aligned} \tag{7}$$

bidder #1 can get A and bidder #2 can get B. Each of them pay 5 for the goods they get, so bidder #1 and #2's utilities are both 1.

This can never happen in the Vickrey auction, since at most one bidder achieve positive utility in the Vickrey auction.

Exercise 33

Suppose there are 2 bidders #1, #2, and 2 goods A and B. For bidder #1, there are synergies between A and B:

$$v_1(A) = v_1(B) = 1, v_1(AB) = 3.$$

Bidder #2 only wants both:

$$v_2(AB) = 5, v_2(A) = v_2(B) = 0.$$

If they all bid truthfully, bidder #2 will get both A and B and pay 3.

However, bidder #2 can submit multiple bids under different names #3 and #4 (if so, there is no #2):

$$b_3(A) = b_3(AB) = 3, b_3(B) = 0,$$
$$b_4(B) = b_4(AB) = 3, b_4(A) = 0.$$

Now, A will be given to bidder #3 and B will be given to bidder #4. Both #3 and #4 need to pay 1, which means the real bidder #2 need to pay only 2.

This cannot happen in the Vickrey auction. In the Vickrey auction, the winner always pay the highest bid among other bidders, no matter whether by bidding truthfully, or by submitting multiple bids.

Exercise 34

For every i and $\mathbf{b}_{-i}$, if i wins with bid b_i, bidder i will never be deleted, so other bidder's scores can never depend on b_i. If bidder i decrease b_i to b_i', it is easy to prove by induction that the result of every iteration stays the same. So, i also wins with bid b_i'.