CS364A Exercise Set 5

Deming Chen cdm@pku.edu.cn

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Exercise 35

No matter how much these bidders bid, select the winner from all bidders at random, and give the item for free. This auction is evidently DSIC. The expected surplus is

$$Q = \frac{1}{n}\sum_{i=1}^{n} v_i \geqslant \frac{1}{n}\max\{v_i\}. \tag{1}$$

Exercise 36

I don't fully understand the problem.

Exercise 37

Since $x_i \in \{0,1\}$ and $B_i \geqslant 0$, $\min\{b_i x_i, B_i\} = \min\{b_i, B_i\} \cdot x_i$ $(i = 1, 2, \cdots, n)$. So, the truncated welfare

$$V = \sum_{i=1}^{n} \min\{b_i, B_i\} \cdot x_i. \tag{2}$$

We use c_i to denote $\min\{b_i, B_i\}$ for later convenience $(i = 1, 2, \cdots, n)$.

For $\mathbf{b} = (b_1, b_2, \cdots, b_n)$, assume the allocation that maximizes the "truncated welfare" is $(x_1^*, x_2^*, \cdots, x_n^*)$. Without losing of generality, let's increase b_1 by $\Delta b > 0$. Now $c_1' = c_1 + \Delta c$ where $0 \leqslant \Delta c \leqslant \Delta b$. The new allocation is $(x_1', x_2', \cdots, x_n')$.

Suppose $x_1' < x_1^*$. Since $(x_1', x_2', \cdots, x_n')$ maximize the truncated welfare,

$$(c_1 + \Delta c)x_1' + c_2 x_2' + \cdots + c_n x_n' \geqslant c_1' x_1^* + c_2 x_2^* + \cdots + c_n x_n^*. \tag{3}$$

Add $-\Delta c x_1' > -\Delta c x_1^*$ to inequality (3):

$$c_1 x_1' + c_2 x_2' + \cdots + c_n x_n' > c_1 x_1^* + c_2 x_2^* + \cdots + c_n x_n^*, \tag{4}$$

which implies that $(x_1^*, x_2^*, \cdots, x_n^*)$ is not the optimal allocation initially, a contradiction. Therefore, the allocation rule is monotone.

Fix bidder i and other bidders' bids (i.e. fix $\mathbf{c}_{-i}$). Since each bidder gets at most one item, if i wins, i pays its "critical bid" b_i^*, the lowest bid at which it would continue to win. We claim that $b_i^* \leqslant B_i$, otherwise decreasing b_i to B_i does not change c_i, so bidder i can still win.

Exercise 38

The auction in the previous exercise give the goods to the bidder who has the highest value and can afford it.

Exercise 39

The allocation rule proposed in the previous exercise is not DSIC for multi-unit auctions. This mechanism never charges a bidder more than its budget. If m is so large that every bidder can get as many goods as possible within budget. So underbidding enables a bidder to get more goods (since B_i/b_i increases) with no need to pay more (the payment always approximates B_i).

Exercise 40

Consider the following example. There are $m = 2$ identical items and 2 bidders. Their budgets and valuation of one item is shown in Table 1.

Table 1: Bidders' budges and valuations

	#1	#2
B	10	12
v	8	10

If them all bid truthfully, then bidder #2 gets one item at price 5 and bidder #1 gets one item at price 7. However, if bidder #1 reports his budges as 11, then bidder # 2 will get one item when $p = 5.5$. Afterwards, B_2 decreases to 6.5, so bidder #1 will get another item at price $6.5 < 7$.

Exercise 41

We prove the claim by induction on the number of agents n. Initially $n = 1$ and there is nothing to prove. For the inductive step, assume output of the TTCA is unique for all problem with less than n agents. Now, fix n agents and their preference. In the first iteration of TTCA, there may be more than one cycle.

For some cycle in the iteration, after it is deleted, there are less than n agents so the outcome is uniquely defined by the inductive hypothesis. So, the cycle we deleted in the first iteration determines the eventual outcome. We claim that no matter which cycle is deleted in the first iteration, the eventual outcome is the same.

The reason is that if there is only one cycle in the first iteration, we have no other choice. Otherwise, pick 2 cycles arbitrarily, and call them C_1 and C_2. After C_1 is deleted, we can choose which cycle to delete in the second iteration for free, and our choice does not affect the eventual result. Note that C_2 remains, so let's delete C_2 in the second iteration. Similarly, after C_2 is deleted in the first iteration, we can delete C_1 next. Deleting C_1 firstly and C_2 secondly leads to the same outcome as deleting C_2 firstly and C_1 secondly. So the outcome of deleting C_1 at first and C_2 at first is the same. Since C_1 and C_2 are picked arbitrarily, no matter which cycle is deleted in the first iteration, the eventual outcome is the same.

Exercise 42

Let A and B denote the graphs corresponding the scenarios where i reports truthfully and not, respectively. The matching number (the number of edges in a maximum matching) of graph A is no less than the matching number of graph B. There must be some maximum matching in A where i is unmatched, otherwise i can always be matched. These matching without i is also a matching in B, so the matching number of A is the same as the matching number of B.

We first prove that if i does not matched in any maximum matching, reporting a strict subset F_i of its true edge set E_i cannot enable i to be matched in some maximum matching. Let's use A and B to denote the scenarios where i reports truthfully and not, respectively. The matching number of A is strictly larger than the number of edges of any matching where i is matched. Suppose i is matched in some maximum matching in B. This matching is also a matching in A, so the matching number of B is strictly less than that of A, a contradiction.

Secondly, consider the scenario where i is matched in some maximum matching but still unmatched at last. Since the matching number of A is the same as the matching number of B, $M_0^{(B)} \subset M_0^{(A)}$. If $Z_{j+1}^{(A)} = \varnothing$, i.e. $j+1$ is unmatched in any matching $\in M_j^{(A)}$, then $Z_{j+1}^{(B)} = \varnothing$. In this case, $M_{j+1}^{(B)} = M_j^{(B)} \subset M_j^{(A)} = M_{j+1}^{(A)}$. Otherwise, in any matching in $M_{j+1}^{(B)}$, $(j+1)$ is matched, and this matching is also in $M_j^{(B)} \subset M_j^{(A)}$, so it belongs to $M_{j+1}^{(A)}$, which implies $M_{j+1}^{(B)} \subset M_{j+1}^{(A)}$. So, if i is not matched in any matching in $M_n^{(A)}$, it cannot be matched in any matching in $M_n^{(B)}$.

Therefore, no agent can go from unmatched to matched by reporting a strict subset F_i of its true edge set E_i.

Exercise 43

Let's Denote boys $B_1, B_2, \cdots, B_n$, and girls $G_1, G_2, \cdots, G_n$. All boys have the same preference: $G_1 > G_2 > \cdots > G_n$, where $A > B$ means the person prefers A to B. However, the preference of G_i is $B_i > B_{i+1} > \cdots > B_n > B_1 > \cdots > B_{i-1}$. We claim that B_i matches G_i in the end.

We prove the claim by induction on n. Initially, $n = 1$, and the claim is trivial. For the induction step, assume the claim holds for $n-1$. G_1 loves B_1 most and B_1 loves G_1 most, so B_1 is matched up with G_1. Now, we can ignore B_1 and G_1 (every boy proposing to G_1 will be rejected, and B_1 will not propose to other girls). For each boys, $G_2 > G_3 > \cdots > G_n$; for G_i, $B_i > B_{i+1} > \cdots > B_n > B_2 > \cdots > B_{i-1}$ $(i \geqslant 2)$. The remaining matching is the same as the problem with $(n-1)$ initial boys and $(n-1)$ initial girls. By the inductive hypothesis, B_i matches G_i $(i \geqslant 2)$. The claim has been proved.

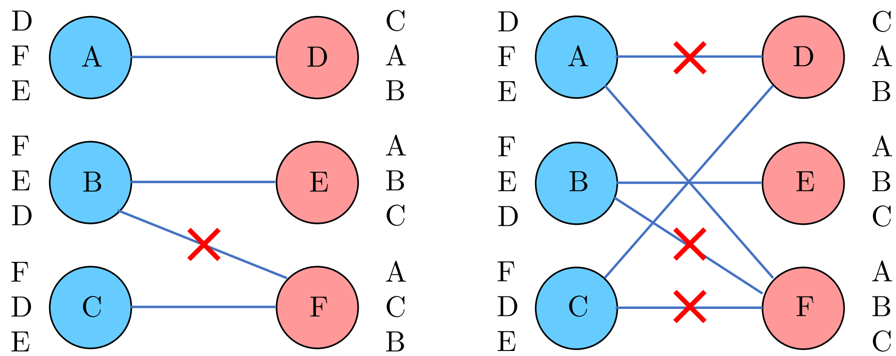
Since each boy systematically worked his way down his preference list, B_i will be rejected by $G_1, G_2, \cdots, G_{i-1}$ before he is accepted by G_i. So, there are

$$\sum_{i=1}^{n}(i-1) = \frac{n(n-1)}{2} \tag{5}$$

rejections in total. The Gale-Shapley algorithm runs for $\Omega(n^2)$ iterations before terminating.

Exercise 44

Consider the following example with 3 boys and 3 girls. Their preference is shown in Figure 1.



(a) F reports truthfully (b) F reports falsely

Figure 1: If F reports falsely, she can be matched with a boy A whom she prefers to C.

If F reports truthfully, then she will accept C's proposal and rejects B's. So, B proposes to E and A proposed to D.

However, F can reports a false preference, as shown in Figure 1 (b). In this case, C proposes to F at first. But then B proposes to F so F rejects C. A proposes to D and then C proposed to D so D rejects C. Then A proposed to F so F rejects B. Now, F receives a better mate.