CS364A Exercise Set 3

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Exercise 20

The probability density function of two bidders is

$$f(z) = 1, z \in [0,1].$$

(a) With no reserve, the expected revenue obtained by the Vickrey auction is

$$\begin{aligned}
r &= \iint \min\{v_1, v_2\} f(v_1) f(v_2) \mathrm{d}v_1 \mathrm{d}v_2 \\
&= \int_{v_1=0}^{1} \int_{v_2=0}^{1} \min\{v_1, v_2\} \mathrm{d}v_1 \mathrm{d}v_2 \\
&= \int_{v_1=0}^{1} \int_{v_2=0}^{v_1} v_2 \mathrm{d}v_1 \mathrm{d}v_2 + \int_{v_1=0}^{1} \int_{v_2=v_1}^{1} v_1 \mathrm{d}v_1 \mathrm{d}v_2 \\
&= \int_0^1 \frac{1}{2} v_1^2 \mathrm{d}v_1 + \int_0^1 v_1(1 - v_1) \mathrm{d}v_1 \\
&= \int_0^1 (v_1 - \frac{1}{2} v_1^2) \mathrm{d}v_1 \\
&= \frac{1}{3}.
\end{aligned} \tag{1}$$

(b) With reserve 1/2, the expected revenue obtained by the Vickrey auction is

$$\begin{aligned}
r =& \int_{v_1=\frac{1}{2}}^{1} \int_{v_2=\frac{1}{2}}^{1} \min\{v_1, v_2\} f(v_1) f(v_2) \mathrm{d}v_1 \mathrm{d}v_2 + \int_{v_1=\frac{1}{2}}^{1} \int_{v_2=0}^{\frac{1}{2}} \frac{1}{2} f(v_1) f(v_2) \mathrm{d}v_1 \mathrm{d}v_2 \\
&+ \int_{v_1=0}^{\frac{1}{2}} \int_{v_2=\frac{1}{2}}^{0} \frac{1}{2} f(v_1) f(v_2) \mathrm{d}v_1 \mathrm{d}v_2 \\
=& \int_{v_1=\frac{1}{2}}^{1} \int_{v_2=\frac{1}{2}}^{v_1} v_2 \mathrm{d}v_1 \mathrm{d}v_2 + \int_{v_1=\frac{1}{2}}^{1} \int_{v_2=v_1}^{1} v_1 \mathrm{d}v_1 \mathrm{d}v_2 + 2 \times (1 - \frac{1}{2}) \times \frac{1}{2} \times \frac{1}{2} \\
=& \int_{\frac{1}{2}}^{1} (\frac{1}{2} v_1^2 - \frac{1}{8}) \mathrm{d}v_1 + \int_{\frac{1}{2}}^{1} v_1(1 - v_1) \mathrm{d}v_1 + \frac{1}{4} \\
=& \int_{\frac{1}{2}}^{1} (v_1 - \frac{1}{2} v_1^2 - \frac{1}{8}) \mathrm{d}v_1 + \frac{1}{4} \\
=& \frac{5}{12}
\end{aligned} \tag{2}$$

Exercise 21

The virtual valuation function is

$$\phi(v) = v - \frac{1 - F(v)}{f(v)}. \tag{3}$$

(a) For the uniform distribution on $[0, a]$ with $a > 0$, $F(v) = v/a$,$f(v) = 1/a$. The virtual valuation function is

$$\phi(v) = v - \frac{1 - v/a}{1/a} = 2v - a. \tag{4}$$

$\phi(v)$ is strictly increasing, so the uniform distribution on $[0, a]$ is regular.

(b) For the exponential distribution with rate $\lambda > 0$, $F(v) = 1 - e^{-\lambda v}$, $f(v) = \lambda e^{-\lambda v}$. The virtual valuation function is

$$\phi(v) = v - \frac{1 - (1 - e^{-\lambda v})}{\lambda e^{-\lambda v}} = v - \frac{1}{\lambda}. \tag{5}$$

$\phi(v)$ is strictly increasing, so the exponential distribution on $[0, a]$ is regular.

(c) For the distribution given by $F(v) = 1 - 1/(v + 1)^c$ on $[0, +\infty)$, where $c > 0$ is some constant, $f(v) = c/(1 + v)^{1+c}$. The virtual valuation function is

$$\phi(v) = v - \frac{1/(v + 1)^c}{c/(1 + v)^{c+1}} = (1 - \frac{1}{c})v - \frac{1}{c}. \tag{6}$$

Only when $c > 1$ is $\phi(v)$ increasing strictly. So, the distribution given by $F(v)$ is regular if and only if $c > 1$.

Exercise 22

When $c = 1$, $\phi(v) = -1$. Since the virtual valuation is always negative, the expected virtual value is non-positive. However, the expected revenue must be non-negative. More specifically, if one bidder gets the item, the expect revenue is 0 (because all bidders pay 0), but the expected virtual valuation is -1.

The reason of this contradiction from **step-3** in Lecture 5. Assume bidder i gets the goods. Other bidders expected payment and expected virtual valuation is 0. But for bidder i, he or she will get the item no matter how much the bid is. In **step-3**, we have

$$\int_0^{v_{max}} (1 - F_i(z)) \cdot z \cdot x_i'(z, \mathbf{v}_{-i}) \mathrm{d}z = (1 - F_i(z)) \cdot z \cdot x_i(z, \mathbf{v}_{-i})\Big|_0^{v_{max}} + \cdots . \tag{7}$$

We claimed that, in Lecture 5, when $z = v_{max}$, $F_i(z) = 1$, so $(1 - F_i(z)) \cdot z \cdot x_i(z, \mathbf{v}_{-i}) = 0$. However, when the distribution is not bounded, $v_{max} = +\infty$, then

$$\lim_{z \to +\infty} (1 - F_i(z)) \cdot z = \lim_{z \to +\infty} \frac{z}{1 + z} = 1 \neq 0. \tag{8}$$

Exercise 23

The allocation rule is to give the items to the highest k bidders whose bids are higher than the reserve price. More specifically, if less than k bidders bids more than the reserve price, those bidders pay the reserve price each; otherwise, the highest k bidders pay the reserve price or the $k + 1$ highest bid, whichever is larger.

The reserve price depends on F, but not on k or n.

Exercise 24

The allocation rule is that, for bidders whose bids are larger than the reserved price, the highest one gets α_1, the second highest one gets α_2, etc, until all slots are allocated, or no more bidder bids larger than the reserved price.

To describe the payment rule more briefly, assume $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$. If at least k bidders bid more than the reserved price, the winners' payment is

$$p_i(\mathbf{b}) = \sum_{l=i}^{k} b_{l+1}(\alpha_l - \alpha_{l+1}), \tag{9}$$

as an auction without reserve price. If the number of bidders whose bids are larger than the reserved price is less than k (denote it m), the winners' payment is

$$p_i(\mathbf{b}) = \sum_{l=i}^{m} b_{l+1}(\alpha_l - \alpha_{l+1}) + r \cdot \alpha_{m+1}, i = 1, 2, \cdots, m, \tag{10}$$

where r is the reserve price. In a word, the auction with a reserve price is the same as the auction without any reserve price added k bidders whose bids are equal to the reserve price.

The reserve price depends on F, but not on k or n.

Exercise 25

The proof is similar to Exercise 14.

For $\mathbf{b} = (b_1, b_2, \cdots, b_n)$, assume that the virtual surplus-maximizing allocation is $(x_1^*, x_2^*, \cdots, x_n^*)$. Without losing of generality, let's increase b_1 by $\Delta b > 0$, and $\phi_1(b_1 + \Delta b) = \phi_1(b_1) + \Delta\phi$. Now $\mathbf{b}' = (b_1 + \Delta b, b_2, \cdots, b_n)$, and the virtual surplus-maximizing allocation is $(x_1', x_2', \cdots, x_n')$.

Suppose $x_1' < x_1^*$. Since $(x_1', x_2', \cdots, x_n')$ maximize the virtual surplus,

$$[\phi_1(b_1)+\Delta\phi]x_1' + \phi_2(b_2)x_2' + \cdots + \phi_n(b_n)x_n' \geqslant [\phi_1(b_1)+\Delta\phi]x_1^* + \phi_2(b_2)x_2^* + \cdots + \phi_n(b_n)x_n^*. \tag{11}$$

Add $-\Delta\phi x_1' > -\Delta\phi x_1^*$ to inequality (11):

$$\phi_1(b_1)x_1' + \phi_2(b_2)x_2' + \cdots + \phi_n(b_n)x_n' > \phi_1(b_1)x_1^* + \phi_2(b_2)x_2^* + \cdots + \phi_n(b_n)x_n^*, \tag{12}$$

which implies that $(x_1^*, x_2^*, \cdots, x_n^*)$ is not the optimal allocation initially, a contradiction.

Therefore, the virtual surplus-maximizing allocation rule is monotone.

Exercise 26

(a) Without losing of generality, assume $\phi_1(v_1) \geqslant \phi_2(v_2) \geqslant \cdots \geqslant \phi_n(v_n)$. If there is a winner ($\phi_1(v_1) \geqslant 0$), his or her payment is $\phi_1^{-1}(\max\{0, \phi_2(v_2)\})$.

(b) Say 2 bidders' valuations are from uniform distribution $[0, 1]$ and $[0, 2]$, respectively. From Exercise 21,

$$\phi_1(v) = 2v - 1$$
$$\phi_2(v) = 2v - 2.$$

Assume bidder 1 bids 0.7, bidder 2 bids 1.1, then $\phi_1(0.7) = 0.4$, but $\phi_2(1.1) = 0.2$. In the optimal auction, bidder 1 will win the auction, though $0 < b_1 < b_2$.

(c) If bidder 2 wins, he or she need to pay 1.2, which is larger than his or her bid. This will encourage the bidders to underbid.

Exercise 26

Before proving the conclusion, we prove a lemma: for n positive numbers $x_1, x_2, \cdots, x_n$,

$$\begin{aligned}(n-1)\max\{x_1, x_2, \cdots, x_n\} \leqslant &\max\{x_2, x_3, \cdots, x_n\}\\ &+\max\{x_1, x_3, \cdots, x_n\}\\ &+\cdots+\max\{x_1, x_2, \cdots, x_{n-1}\}\end{aligned} \tag{13}$$

To prove it, Without losing of generality, assume that $\max\{x_1, x_2, \cdots, x_n\} = x_1$, so

$$RHS = (n-1)\cdot x_1 + \max\{x_2, x_3, \cdots, x_n\} \geqslant (n-1)\cdot x_1 = LHS. \tag{14}$$

In a single-item auction, the expected revenue in the optimal auction over n bidders is

$$r_n = \mathbb{E}_{\mathbf{v}}[\max\{\phi_1(v_1)^+, \phi_2(v_2)^+, \cdots, \phi_n(v_n)^+\}] \tag{15}$$

where $\phi_i(v_i)^+ = \max\{\phi_i(v_i), 0\}, i = 1, 2, \cdots, n$. Since $v_1, v_2, \cdots, v_n$ are drawn from i.i.d, the expected revenue in the optimal auction over $(n-1)$ bidders is

$$\begin{aligned}r_{n-1} &= \mathbb{E}_{\mathbf{v}_{-1}}[\max\{\phi_2(v_2)^+, \phi_3(v_3)^+, \cdots, \phi_n(v_n)^+\}]\\ &= \mathbb{E}_{\mathbf{v}_{-2}}[\max\{\phi_1(v_1)^+, \phi_3(v_3)^+, \cdots, \phi_n(v_n)^+\}]\\ &= \cdots\\ &= \mathbb{E}_{\mathbf{v}_{-n}}[\max\{\phi_1(v_1)^+, \phi_2(v_2)^+, \cdots, \phi_{n-1}(v_{n-1})^+\}]\end{aligned} \tag{16}$$

By inequality (13),

$$\begin{aligned}(n-1)r_n =&(n-1)\mathbb{E}_{\mathbf{v}}[\max\{\phi_1(v_1)^+, \phi_2(v_2)^+, \cdots, \phi_n(v_n)^+\}]\\ \leqslant&\mathbb{E}_{\mathbf{v}}\max\{\phi_2(v_2)^+, \phi_3(v_3)^+, \cdots, \phi_n(v_n)^+\}]\\ &+\mathbb{E}_{\mathbf{v}}[\max\{\phi_1(v_1)^+, \phi_3(v_3)^+, \cdots, \phi_n(v_n)^+\}]\\ &+\cdots+\mathbb{E}_{\mathbf{v}}[\max\{\phi_1(v_1)^+, \phi_2(v_2)^+, \cdots, \phi_{n-1}(v_{n-1})^+\}]\\ =&nr_{n-1}.\end{aligned} \tag{17}$$

So, $r_{n-1} \geqslant (n-1)r_n/n$. By the Bulow-Klemperer theorem, the expected revenue of the Vickrey auction with n bidders is at least r_{n-1}. Therefore, the expected revenue of the Vickrey auction (with no reserve) is at least $(n-1)/n$ times that of the optimal auction (with the same number n of bidders).