CS364A Exercise Set 2

Deming Chen cdm@pku.edu.cn

November 1, 2022

Exercise 9

In a single-item auction, the feasible allocation set X is the set of 0-1 vectors that have at most one 1. We need show that besides not giving all bidders the item, always awarding the good to the highest bidder and charging losers 0 is unique monotone allocation rule $\mathbf{x} : \mathbf{b} \to X$ with permutation symmetry (i.e. swapping two bidder's bids results in swapping their allocation).

Assume there are n bidders ($n \geqslant 2$). If for some $\mathbf{b}$ (assuming $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$), the good is not given to the first bidder, but the ith bidder ($i > 1$). Increasing b_i to b_1 when fixing other bids, x_i will still be 1 (because of monotonicity), so other bidders' allocation stay 0. Then decrease b_1 to initial b_i, and x_1 stay 0. The procedure described above is just swapping b_1 and b_i, but x_1 and x_i does not swap, which contradicts the permutation symmetry.

Furthermore, permutation symmetry is essential. Consider an allocation rule that gives the good to the bidder with the highest b_i/i. This rule is also monotone, but does not make sense.

Exercise 10

The "sandwich inequality" $\forall\ 0 \leqslant y < z$:

$$z \cdot [x(y) - x(z)] \leqslant p(y) - p(z) \leqslant y \cdot [x(y) - x(z)]. \tag{1}$$

It implies that $(y - z) \cdot [x(y) - x(z)] \geqslant 0$. That is, if an allocation rule is implementable, it is monotone.

Exercise 11

Suppose that $x(z)$ is differentiable, so that

$$p(z) = \int_0^z w x'(w) \mathrm{d}w. \tag{2}$$

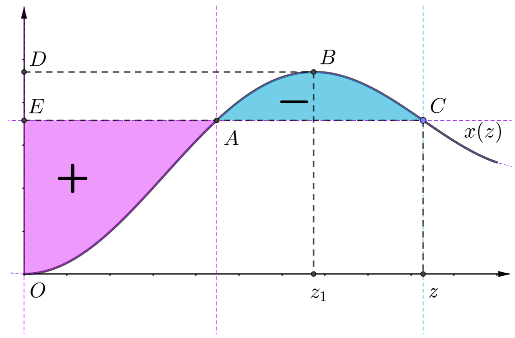
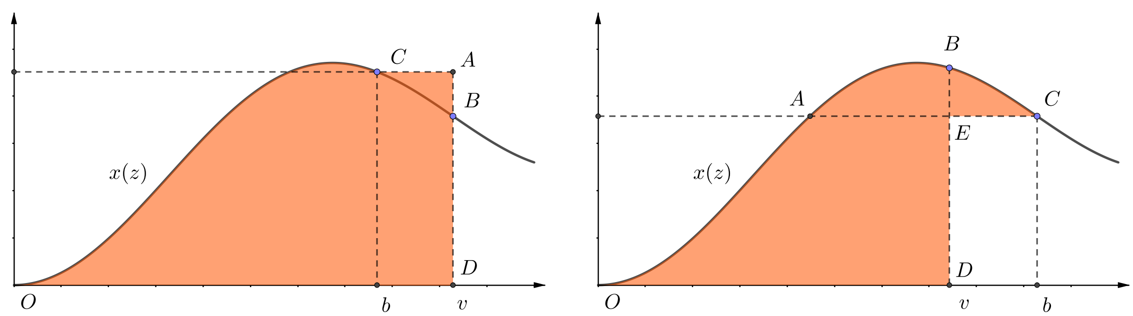


Figure 1: A bidder's payment when the allocation rule is not monotone. Fixing other bidders' bids, $x(z) = x_i(\cdot, \mathbf{b}_{-i})$. $p(z)$ is equal to the area of the purple region minus the area of the blue region.

If $x(z)$ is not monotone, as shown in Figure 1,

$$\begin{aligned} p(z) &= \int_0^z wx'(w)\mathrm{d}w \\ &= \int_0^{z_1} wx'(w)\mathrm{d}w + \int_{z_1}^{z} wx'(w)\mathrm{d}w \\ &= S_{OABD} - S_{ECBD} \\ &= S_{OAE} - S_{ABC}. \end{aligned} \tag{3}$$

Suppose the bidder's valuation is v. If $v > b$ but $x(v) < x(b)$, as shown in Figure 2 (a). The bidder's utility is $u(b) = S_{OCAD}$, which is larger than the utility he or her get when bidding truthfully $u(v) = S_{OCBD}$. If $v < b$ but $x(v) > x(b)$, as shown in Figure 2 (b). The bidder's utility is $u(b) = S_{OABCED}$, which is larger than the utility he or her get when bidding truthfully $u(v) = S_{OABD}$.



(a) underbidding

(b) overbidding

Figure 2: The bidder's utility if he or she does not bid truthfully when $x(z)$ is not monotone. The area of the orange region equals to the bidder's utility of bid b.

Exercise 12

Assume $x_i(\cdot, \mathbf{b}_{-i}) = x(\cdot)$ a monotone and piecewise differentiable function. A jump Δx at z means

$$\left.\frac{\partial x_i}{\partial b_i}\right|_{b_i=z} = \Delta x \cdot \delta(b_i - z)$$

where $\delta(\cdot)$ is the Dirac-δ function. The payment

$$p(z) = \int_0^z w x'(w) \mathrm{d}w. \tag{4}$$

Since $x(z)$ is non-decreasing, $x'(z) \geqslant 0$. No matter whether $z < v$ or $z \geqslant v$,

$$\int_z^v (v - w) \cdot x'(w) \mathrm{d}w \geqslant 0. \tag{5}$$

So,

$$\int_z^v w \cdot x'(w) \mathrm{d}w \leqslant \int_z^v v \cdot x'(w) \mathrm{d}w = v[x(v) - x(z)]. \tag{6}$$

Then,

$$v \cdot x(z) - \int_0^z w \cdot x'(w) \mathrm{d}w \leqslant v \cdot x(v) - \int_0^v w \cdot x'(w) \mathrm{d}w, \tag{7}$$

which implies that $v \cdot x(z) - p(z) \leqslant v \cdot x(v) - p(v)$. As a result, a bidder can maximize their utility by bidding their valuation truthfully.

Exercise 13

Assume truthful bidding. The surplus-maximizing allocation rule is

$$\mathbf{x}(\mathbf{b}) = \underset{(x_1, x_2, \cdots, x_n) \in X}{\operatorname{argmax}} \sum_{i=1}^n b_i \beta_i x_i, \tag{8}$$

where $(x_1, x_2, \cdots, x_n)$ is a permutation of $(\alpha_1, \alpha_2, \cdots, \alpha_k, 0, \cdots, 0)$ The allocation rule is to assign the jth highest slot to the bidder with jth highest $b\beta$.

For bidder i, fix other bidders' bids b_{-i}. Raising b_i is equivalent to raising $b_i\beta_i$. It only increase bidder i's position in the sorted order of bids, which can only net he or she a higher slot. So, this allocation rule is monotone.

Sort and re-index the bidders so that $b_1\beta_1 \geqslant b_2\beta_2 \geqslant \cdots \geqslant b_n\beta_n$. the per-click payment of bidder i is

$$p_i(\mathbf{b}) = \sum_{l=i}^k \frac{b_{l+1}(\alpha_l \beta_i - \alpha_{l+1}\beta_i)}{\alpha_i \beta_i} = \sum_{l=i}^k \frac{b_{l+1}(\alpha_l - \alpha_{l+1})}{\alpha_i}. \tag{9}$$

Exercise 14

For $\mathbf{b} = (b_1, b_2, \cdots, b_n)$, assume the surplus-maximizing allocation is $(x_1^*, x_2^*, \cdots, x_n^*)$. Without losing of generality, let's increase b_1 by $\Delta b > 0$. Now $\mathbf{b}' = (b_1 + \Delta b, b_2, \cdots, b_n)$, and the surplus-maximizing allocation is $(x_1', x_2', \cdots, x_n')$.

Suppose $x_1' < x_1^*$. Since $(x_1', x_2', \cdots, x_n')$ maximize the surplus,

$$(b_1 + \Delta b)x_1' + b_2 x_2' + \cdots + b_n x_n' \geqslant (b_1 + \Delta b)x_1^* + b_2 x_2^* + \cdots + b_n x_n^*. \tag{10}$$

Add $-\Delta b x_1' > -\Delta b x_1^*$ to inequality (10):

$$b_1 x_1' + b_2 x_2' + \cdots + b_n x_n' > b_1 x_1^* + b_2 x_2^* + \cdots + b_n x_n^*, \tag{11}$$

which implies that $(x_1^*, x_2^*, \cdots, x_n^*)$ is not the optimal allocation initially, a contradiction.

Therefore, the surplus-maximizing allocation rule is monotone.

Exercise 15

Given a winner set S, the social surplus

$$V(\mathbf{b}) = \sum_{i \in S} b_i$$

is a continuous function of $\mathbf{b}$. Since "max" is also a continuous function, the maximal social surplus

$$V^*(\mathbf{b}) = \max_S \sum_{i \in S} b_i \tag{12}$$

is continuous.

Fix other bidders' bids $\mathbf{b}_{-i}$ and assume they bid truthfully. $V^*(b_i, \mathbf{b}_{-i})$ is also a continuous function. Let b_i^* denote bidder i's critical bid. If $b_i = b_i^* - \varepsilon$, where ε is a very small positive real number, bidder i will loss. In this case, $V^*(b_i^* - \varepsilon, \mathbf{b}_{-i})$ is just maximum surplus of a feasible set that excludes i. If $b_i = b_i^* + \varepsilon$,

$$V^* = b_i^* + \varepsilon + \sum_{j \in S^* \setminus \{i\}} v_j. \tag{13}$$

When $\varepsilon \to 0$, $V^*(b_i^* - \varepsilon, \mathbf{b}_{-i})$ and $V^*(b_i^* + \varepsilon, \mathbf{b}_{-i})$ should approach the same value, which means that b_i^* equals to the difference between the maximum surplus of a feasible set that excludes i and the surplus of the bidders other than i in the chosen outcome S^*.

Exercise 16

For bidder i, the maximum surplus of a feasible set that excludes i is

$$V_{-i} = \sum_{j \neq i} b_j \cdot x_j(0, \mathbf{b}_{-i}). \tag{14}$$

The surplus of the bidders other than i in the chosen outcome is

$$V_{-i}^* = \sum_{j \neq i} b_j \cdot x_j(\mathbf{b}). \tag{15}$$

According to the previous exercise, the payment of bidder i is $(V_{-i} - V_{-i}^*)$.

Exercise 18

In a knapsack auction, a bidder either wins (getting w_i) or loses (getting nothing). To prove that an allocation rule is monotone, we just need to prove that given bids of other bidders $\mathbf{b}_{-i}$, for some b_i, if $x_i(b_i, \mathbf{b}_{-i}) = 1$, then for any $b > b_i$, $x_i(b, \mathbf{b}_{-i}) = 1$.

If b_i is the highest among all bids, and i gets the goods because i's surplus is greater than that of the step (2) solution, then increasing b_i can also makes b_i the highest bid.

If i get the good because of step (2), then increasing b_i can makes b_i/w_i larger, therefore $x_i(b, \mathbf{b}_{-i}) = 1$.