CS364A Exercise Set 1

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Exercise 1

Suggestions for how to modify the Olympic badminton tournament format:

(1) After the "round-robin" phase, determined the opponents of each team in the knockout stage by random drawing.

(2) Awarded to the first team in a group in the "round-robin" phase.

Exercise 3

Two players X and Y play the Rock-Paper-Scissors game. Their payoff matrix is shown below.

X \ Y	R	P	S
R	0 0	1 -1	-1 1
P	-1 1	0 0	1 -1
S	1 -1	-1 1	0 0

We use P_X and P_Y to denote player X and Y's strategy, respectively. More specifically,

$$P_X(R) = p_1,\ P_X(P) = p_2,\ P_X(S) = p_3 = 1 - p_1 - p_2,$$
$$P_Y(R) = q_1,\ P_Y(P) = q_2,\ P_Y(S) = q_3 = 1 - q_1 - q_2,$$

where $p_1 \geqslant 0, p_2 \geqslant 0, p_1 + p_2 \leqslant 1$ and $q_1 \geqslant 0, q_2 \geqslant 0, q_1 + q_2 \leqslant 1$. If they independently select strategies, X's expected payoff is

$$\begin{aligned} u_x(P_X, P_Y) &= p_2q_1 - p_3q_1 - p_1q_2 + p_3q_2 + p_1q_3 - p_2q_3 \\ &= \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \\ 1 & 1 & 3 \end{vmatrix} \\ &= 3p_1q_2 + q_1 + p_2 - 3q_1p_2 - p_1 - q_2 \\ &= (3q_2 - 1)p_1 + (1 - 3q_1)p_2 + q_1 - q_2, \end{aligned} \tag{1}$$

and Y's expected payoff is $u_y(P_X, P_Y) = -u_x(P_X, P_Y) = (3p_2 - 1)q_1 + (1 - 3p_1)q_2 + p_1 - p_2$. Assume there exists a mixed-strategy Nash equilibrium (P_X^*, P_Y^*), which means that for arbitrary P_X, $u_x(P_X^*, P_Y^*) \geqslant u_x(P_X, P_Y^*)$. In other words,

$$\forall p_1, p_2,\ (3q_2^* - 1)p_1^* + (1 - 3q_1^*)p_2^* \geqslant (3q_2^* - 1)p_1 + (1 - 3q_1^*)p_2. \tag{2}$$

Similarly, that for arbitrary P_Y, $u_y(P_X^*, P_Y^*) \geqslant u_y(P_X^*, P_Y)$ implies that

$$\forall q_1, q_2,\ (3p_2^* - 1)q_1^* + (1 - 3p_1^*)q_2^* \geqslant (3p_2^* - 1)q_1 + (1 - 3p_1^*)q_2. \tag{3}$$

In inequality (2), let $p_1 = 0$ and $p_2 = p_2^*$, we have

$$(3q_2^* - 1)p_1^* \geqslant 0; \tag{4}$$

let $p_1 = 1$ and $p_2 = p_2^*$, we have

$$(3q_2^* - 1)(p_1^* - 1) \geqslant 0. \tag{5}$$

In inequality (3), let $q_1 = q_1^*$ and $q_2 = 0$, we have

$$(1 - 3p_1^*)q_2^* \geqslant 0; \tag{6}$$

let $q_1 = q_1^*$ and $q_2 = 1$, we have

$$(1 - 3p_1^*)(q_2^* - 1) \geqslant 0. \tag{7}$$

If $p_1^* = 0$, then inequality (4) and (6) hold. However, inequality (7) implies that $q_2^* = 1$, which contradicts inequality (5). So, $p_1^* > 0$. Similarly, we have $p_1^* < 1$. By inequality (4) and (5), we have $q_2^* = 1/3$. Then by inequality (6) and (7), we have $p_1^* = 1/3$. Similarly, $q_1^* = p_2^* = 1/3$, so that is the only possible mixed-strategy Nash equilibrium.

One can easily verify that $p_1^* = p_2^* = p_3^* = 1/3$ and $q_1^* = q_2^* = q_3^* = 1/3$ is indeed a Nash equilibrium.

Exercise 5

Assuming that third-price auction is dominant-strategy incentive compatible. Let bidder i has a value v_i and bid truthfully ($i = 1, 2, \cdots, k$). Without losing of generality, suppose $v_1 \geqslant v_2 \geqslant \cdots \geqslant v_k$. In that case, the bidder with the second-highest bid (value) will lose this auction, and his or her utility is therefore 0. However, if the bidder with the second-highest bid higher than the bidder with the highest bid, his or her utility is $v_2 - v_3 \geqslant 0$, a contradiction.

As a result, third-price auction is not dominant-strategy incentive compatible.

Exercise 6

If there are k identical copies of a good and $n > k$ bidders, the analog of the second-price auction is to give the items to the k highest bidders and they pay a price equal to the $(k+1)$th highest bid.

For one bidder, suppose his valuation is v, and other bidder bid $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{k-1} \geqslant b_k \geqslant \cdots \geqslant b_{n-1}$. When $v < b_{k-1}$, this bidder's utility is

$$u(b) = \begin{cases} 0,\ \text{if } b < b_{k-1}, \\ v - b_{k-1},\ \text{if } b \geqslant b_{k-1}. \end{cases} \tag{8}$$

In this case, the maximal utility 0 can be achieved by bidding $b = v$. When $v \geqslant b_{k-1}$, this bidder's utility is

$$u(b) = \begin{cases} 0,\ \text{if } b < b_{k-1}, \\ v - b_{k-1},\ \text{if } b \geqslant b_{k-1}. \end{cases} \tag{9}$$

In this case, the maximal utility $(v - b_{k-1})$ can be still achieved by bidding $b = v$.

In both cases, the utility by $b = v$ is non-negative. Therefore, this auction is dominant-strategy incentive compatible.

Exercise 7

The rule is to select the contractor with the smallest reported cost and the payment equals to the second-smallest reported cost among all contractors.

Exercise 8

Assume $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$ so that the social surplus

$$\gamma = \sum_{i=1}^{n} v_i x_i$$

is well-defined. Since

$$\{x_1, x_2, \cdots, x_n\} = \{\alpha_1, \alpha_2, \cdots, \alpha_n\},$$

according to rearrangement inequality, γ reaches its maximum when $v_1 \geqslant v_2 \geqslant \cdots \geqslant v_n$. That is, the social surplus is maximized by assigning the bidder with the jth highest valuation to the jth slot.