Chapter 4.1

Binary Search Trees,

AVL Trees

B-Trees , B^+Trees

4.1 Binary Search Trees

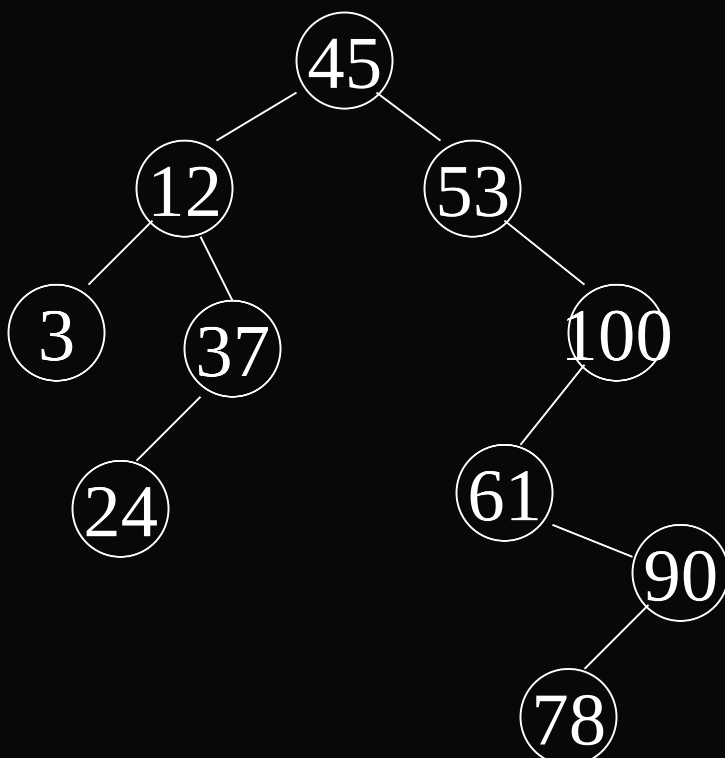
1. **Definition: A binary search tree is a binary tree that may be empty.**

   A nonempty binary search tree satisfies the following properties:

   1) **Every element has a key and no two elements have the same key; therefore,all keys are distinct.**
   2) **The keys(if any)in the left subtree of the root are smaller than the key in the root.**
   3) **The keys(if any)in the right subtree of the root are larger than the key in the root.**
   4) **The left and right subtrees of the root are also binary search trees.**

4.1 Binary Search Trees

Example:



left	**element**	**right**

4.1 Binary Search Trees

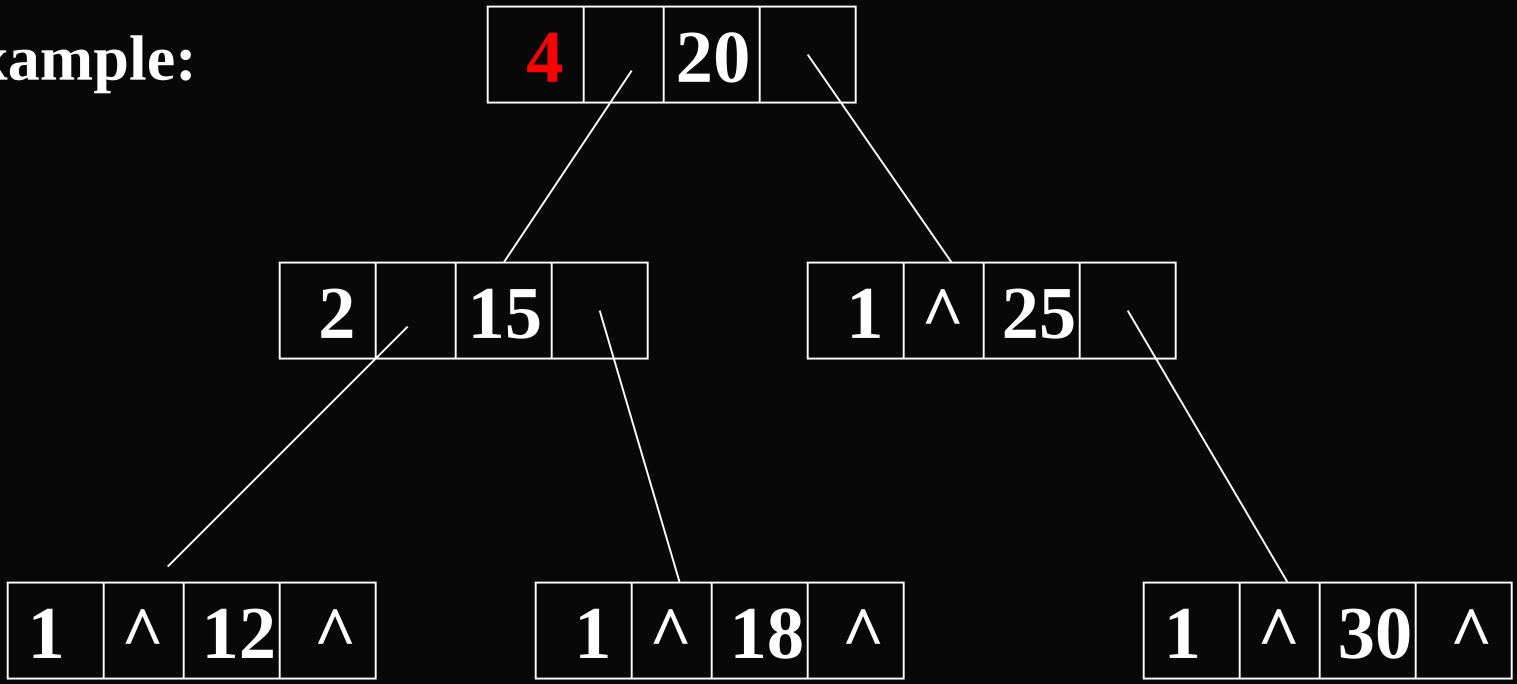
- **An indexed binary search tree is derived from an ordinary binary search tree by adding the field leftSize to each tree node.**
- **Value in Leftsize field=number of the elements in the node's left subtree +1**

leftSize	**left**	**element**	**right**

4.1 Binary Search Trees

Indexed binary search tree

Example:



4.1 Binary Search Trees

2. BinaryNode class

```
class BinaryNode
{   BinaryNode( Comparable theElement )
       { this( theElement, null, null ); }
    BinaryNode( Comparable  theElement,  BinaryNode lt,
                                         BinaryNode rt )
       { element = theElement; left = lt; right = rt; }

     Comparable element;
     BinaryNode  left;
     BinaryNode  right;
}
```

4.1 Binary Search Trees

3. Binary search tree class skeleton

```
public class BinarySearchTree
{   public BinarySearchTree( ) { root = null; }
    public void makeEmpty( ) { root = null; }
    public boolean isEmpty( ) { return root = = null; }

    public Comparable find( Comparable x )
        { return elementAt( find( x, root ) ); }
    public Comparable findMin( )
        { return elementAt( findMin( root ) ); }
    public Comparable findMax( )
        { return elementAt( findMax( root ) ); }
```

4.1 Binary Search Trees

```
public void insert( Comparable x )
  {  root = insert(  x, root ); }
public void remove( Comparable x )
  {root = remove( x, root ); }
public void printTree( )

private BinaryNode root;

private Comparable elementAt( BinaryNode t )
  { return t = = null ? Null : t.element; }
private BinaryNode find( Comparable x, BinaryNode t )
private BinaryNode findMin( BinaryNode t )
private BinaryNode findMax( BinaryNode t )
```

4.1 Binary Search Trees

```
    private BinaryNode insert( Comparable x, BinaryNode t )
    private BinaryNode remove( Comparable x, BinaryNode t )
    private BinaryNode removeMin( BinaryNode t )
    private void printTree( BinaryNode t )
}
```

4.1 Binary Search Trees

4. Find operation for binary search trees

```
//递归实现
private BinaryNode find( Comparable x, BinaryNode t )
{  if( t = = null )
          return null;
   if( x. compareTo( t.element ) < 0 )
          return find( x, t.left );
   else if( x.compareTo( t.element ) > 0 )
          return find( x, t.right );
   else
         return t;  //Match
}
```

4.1 Binary Search Trees

5. Recursive implementation of findMin for binary search trees

```
private BinaryNode findMin( BinaryNode t )
{   if( t = = null )
        return null;
    else if( t.left = = null )
        return t;
    return findMin( t.left );
}
```

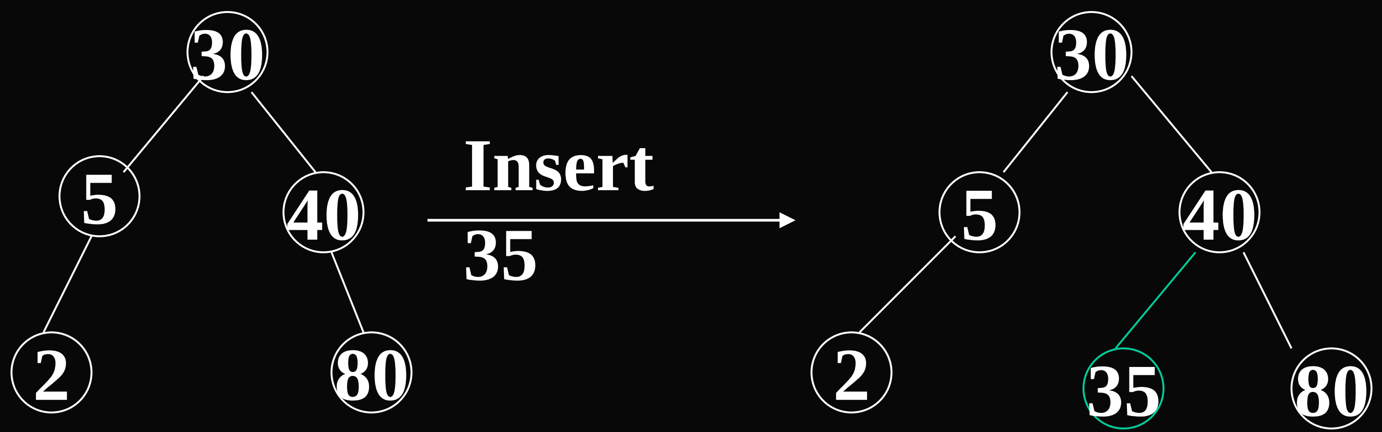
6. Nonrecursive implementation of findMax for binary search trees

```
private BinaryNode findMax( BinaryNode t )
{   if( t != null )
        while( t.right != null )
            t = t.right;

    return t;
}
```

4.1 Binary Search Trees

7. Insertion into a binary search tree



4.1 Binary Search Trees

```
private BinaryNode insert( Comparable x, BinaryNode t )
{   if( t = = null )
        t = new BinaryNode( x, null, null );
    else if( x.compareTo( t.element ) < 0 )
        t.left = insert( x, t.left );
    else if( x.compareTo( t.element ) > 0 )
        t.right = insert( x, t.right );
    else
        ;   //duplicate; do nothing
    return t;
}
```

4.1 Binary Search Trees

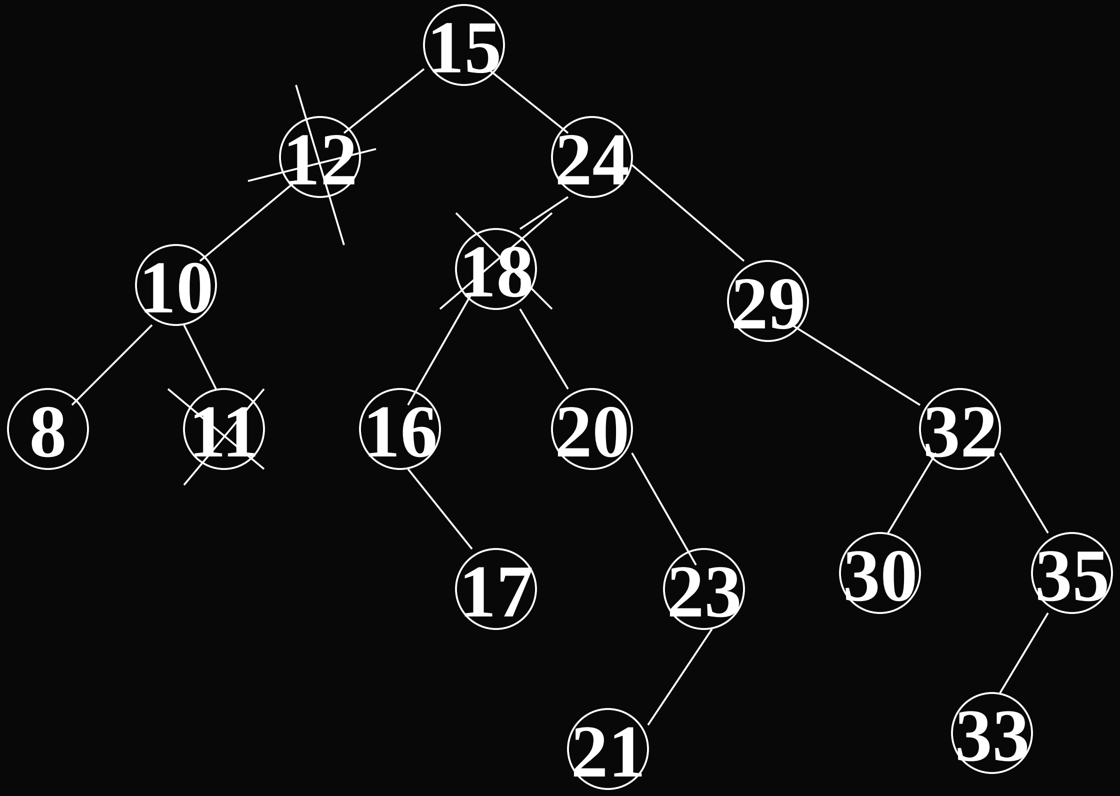
Deletion

It is necessary to adjust the binary search tree after deleting an element, so that the tree remained is still a binary search tree.There is three cases for deleting node p :

- **P is a leaf**
- **P has exactly one nonempty subtree**
- **P has exactly two nonempty subtrees**

4.1 Binary Search Trees

Example: 删除12：用10来取代12



4.1 Binary Search Trees

- **case 1: delete a leaf**
- **case 2: deleted node has exactly one nonempty subtree**
- **case 3: deleted node has exactly two nonempty subtree**
- 用左子树的最大值或者右子树的最小值

We can replace the element to be deleted with either the largest element in the left subtree or the smallest element in the right subtree.

Next step is to delete the largest element in the left subtree or smallest element in the right subtree.

4.1 Binary Search Trees

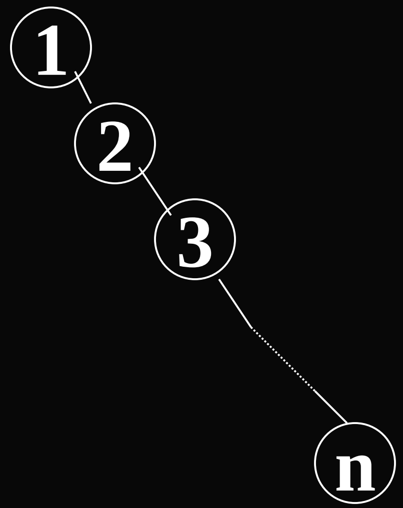
8. Deletion routine for binary search trees

```
 private BinaryNode remove( Comparable x, BinaryNode t )
 {    if( t = = null )
          return t;
      if( x.compareTo( t.element ) < 0 )
          t.left = remove( x, t.left );
      else if( x.compareTo( t.element ) > 0 )
          t.right = remove( x, t.right );
      else if( t.left != null && t.right != null )
           {   t.element = findMin( t.right ).element;
               t.right = remove( t.element , t.right );
           }
      else
           t = ( t.left != null ) ? t.left : t.right;
 }
```

4.1 Binary Search Trees

Height of a binary search tree

- **The height of a binary search tree has influence directly on the time complexity of operations like searching, insertion and deletion.**
- **Worst case: add an ordered elements{1,2,3…n} into an empty binary search tree.**



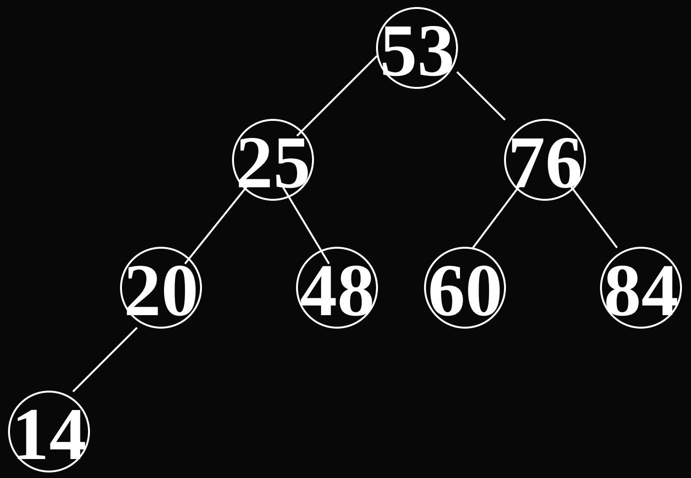
O(n)

4.1 Binary Search Trees

- **Best case and average height:**

  O($\log_2 n$)

Example:{53,25,76,20,48,14,60,84}



4.2 AVL Tree

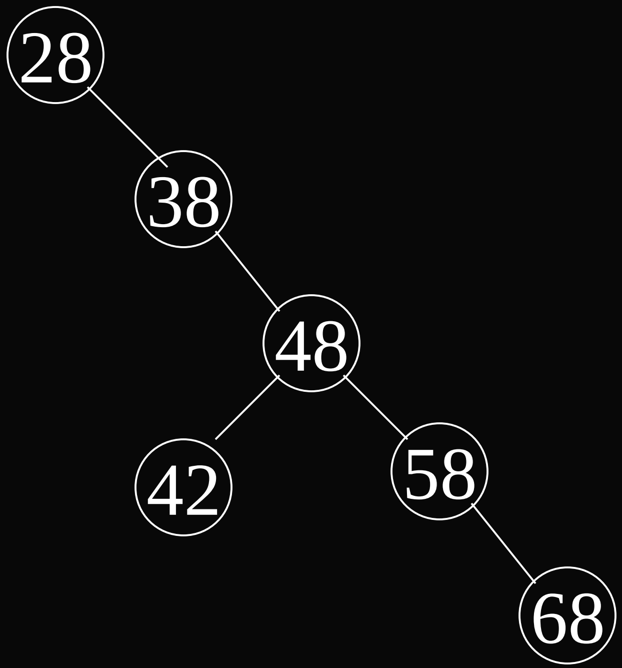
(平衡的二叉搜索树)

The concept of AVL tree was introduced by Russian scientists G.M.Adel’son-Vel’sky and E.M.Landis in 1962.

1. **purpose:**

   the AVL tree was introduced to increase the efficiency of searching a binary search tree, and to decrease the average search length.

4.2 AVL Tree



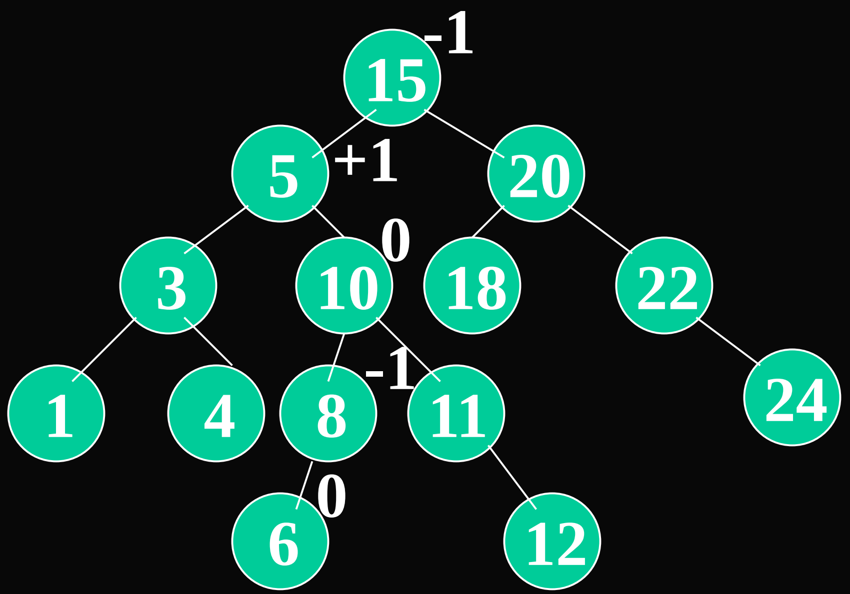
4.2 AVL Tree

2 Definition of an AVL tree:

(1) is a binary search tree

(2) Every node satisfies

$|h_L - h_R| <= 1$ where h_L and h_R are the heights of T_L(left subtree) and T_R(right subtree),respectively.



-1
15
5
+1
20
0
3
10
18
22
-1
1
4
8
11
24
0
6
12

4.2 AVL Tree

- **Height of an tree:**

  the longest path from the root to each leaf node

- **Balance factor *bf(x)* of a node x :**

  height of right subtree of x – height of left subtree of x

Each node:

Left	**data**	**Right**	**balance(height)**

4.2 AVL Tree

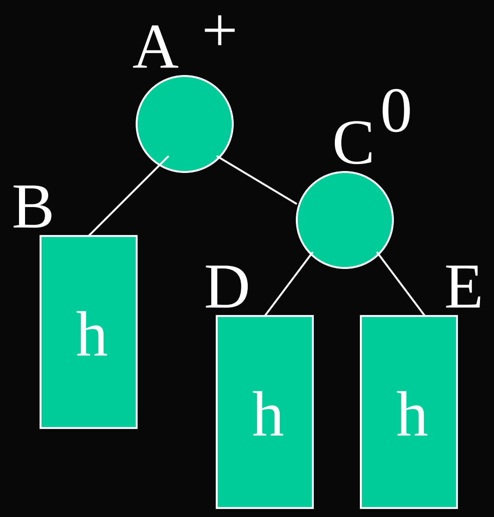
The height of an AVL tree with n elements is $O(\log_2 n)$, so an n-element AVL search tree can be searched in $O(\log_2 n)$ time.

4.2 AVL Tree

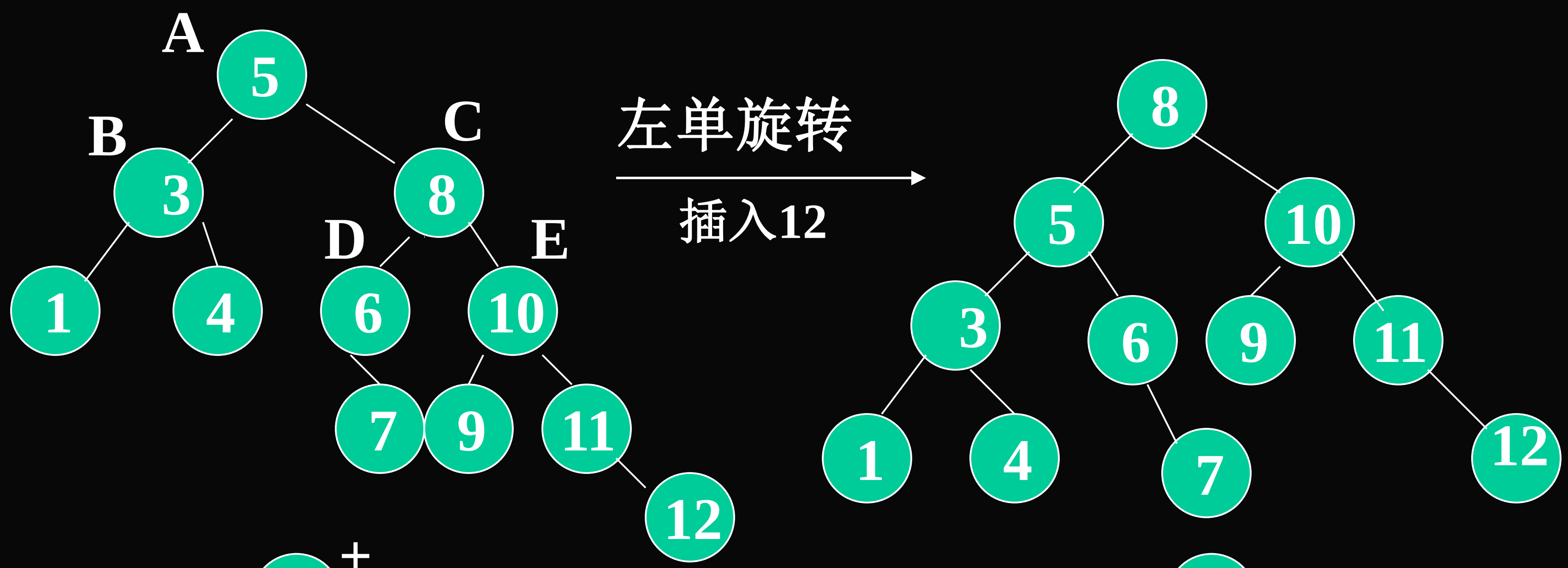
3.inserting into an AVL tree

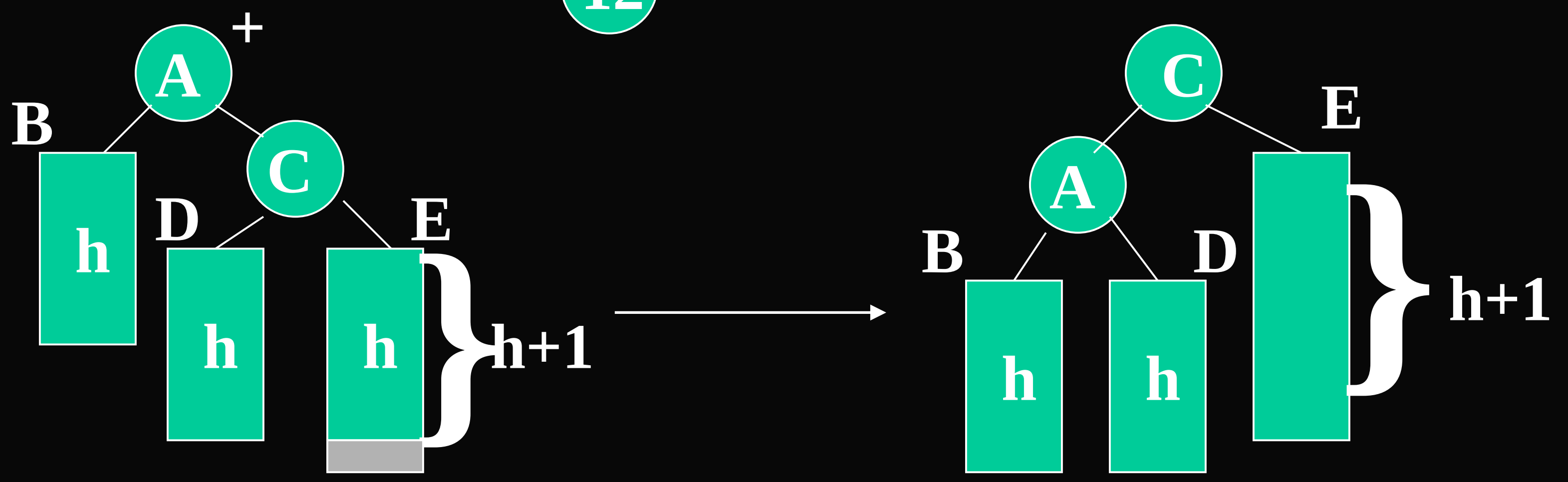
AVL树

- 插入



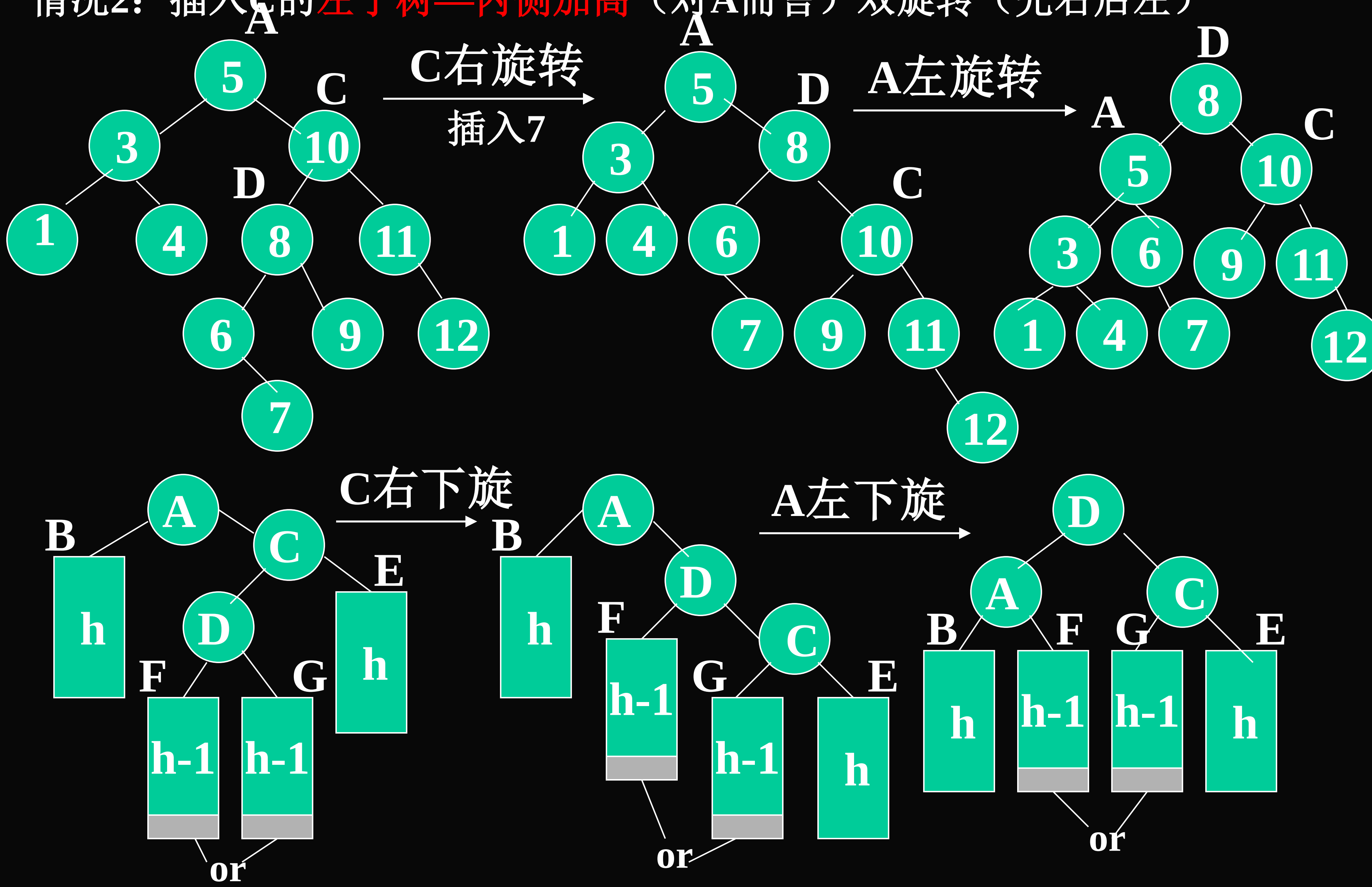
情况1:插入C的右子树__外侧加高(对A而言)　单旋转(左)





调整后:树高不变.原h+2,插入后h+3,调整后h+2,∴不平衡不会向外传递.

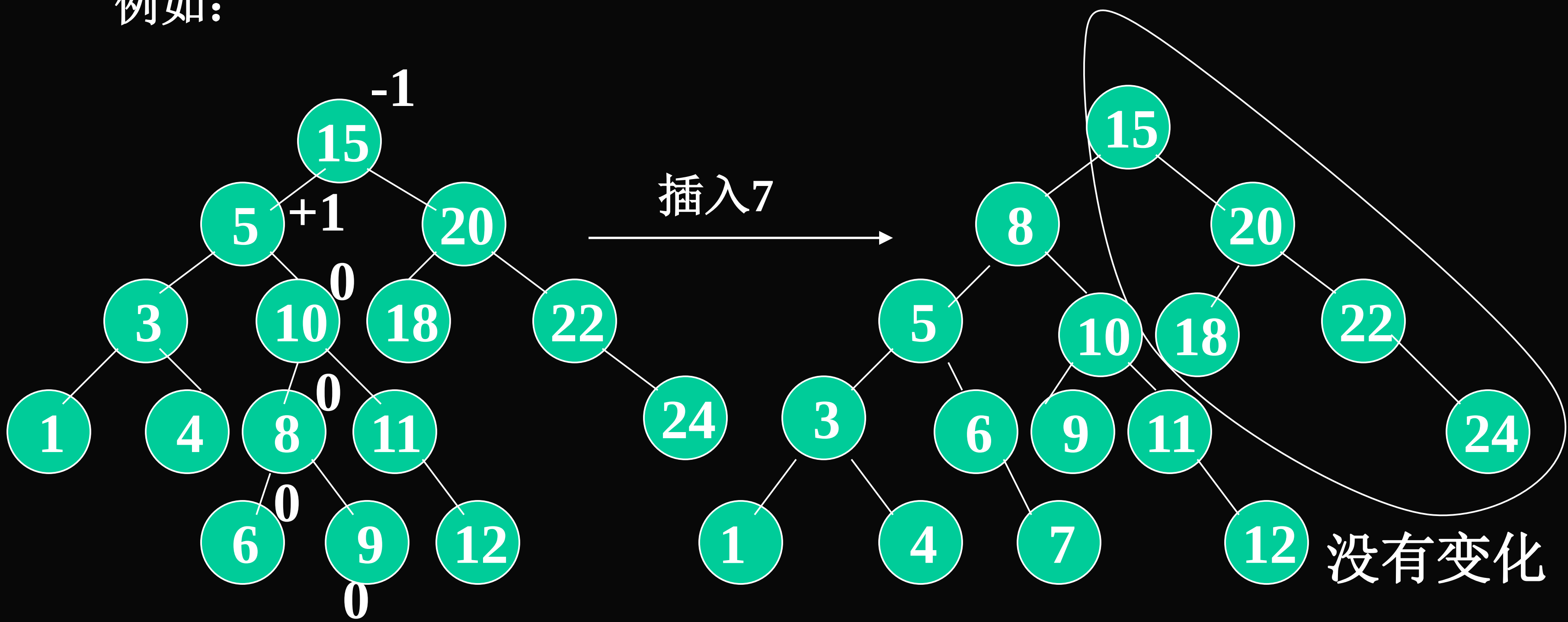
情况2：插入C的左子树—内侧加高（对A而言）双旋转（先右后左）



调整后：树高不变。原h+2,插入后h+3,调整后h+2.

小结一下：以A为根的子树，调整前后，其高度不变，∴ 调整不会影响到以A为根的子树以外的结点。

例如：



*调整只要在包含插入结点的最小不平衡子树中进行,即从根到达插入结点的路径上,离插入结点最近的,并且平衡系数≠0的结点为根的子树。

也可这样讲：插入一个新结点后，需要从插入位置沿通向根的路径回溯，检查各结点左右子树的高度差，如果发现某点高度不平衡则停止回溯。

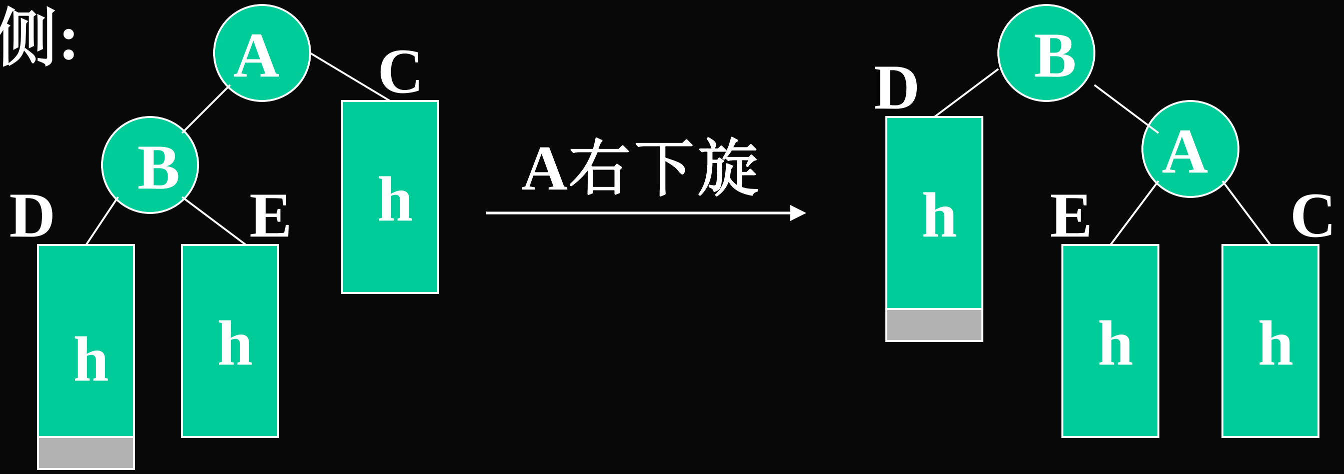
单旋转：外侧—从不平衡结点沿刚才回溯的路径取直接下两层如果三个结点处于一直线A，C，E

双旋转：内侧—从不平衡结点沿刚才回溯的路径取直接下两层如果三个结点处于一折线A，C，D

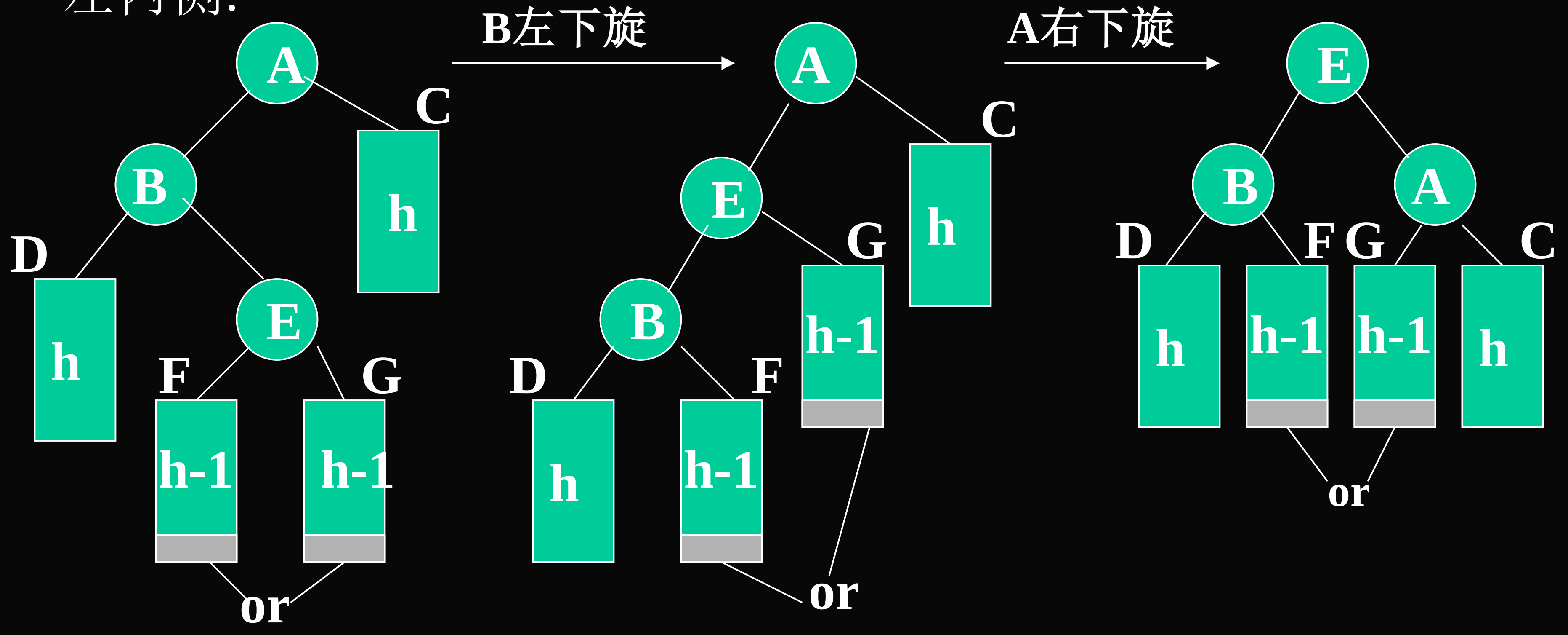
*以上以右外侧，右内侧为例，左外侧，左内侧是对称的。

与前面对称的情况：左外侧，左内侧

左外侧:

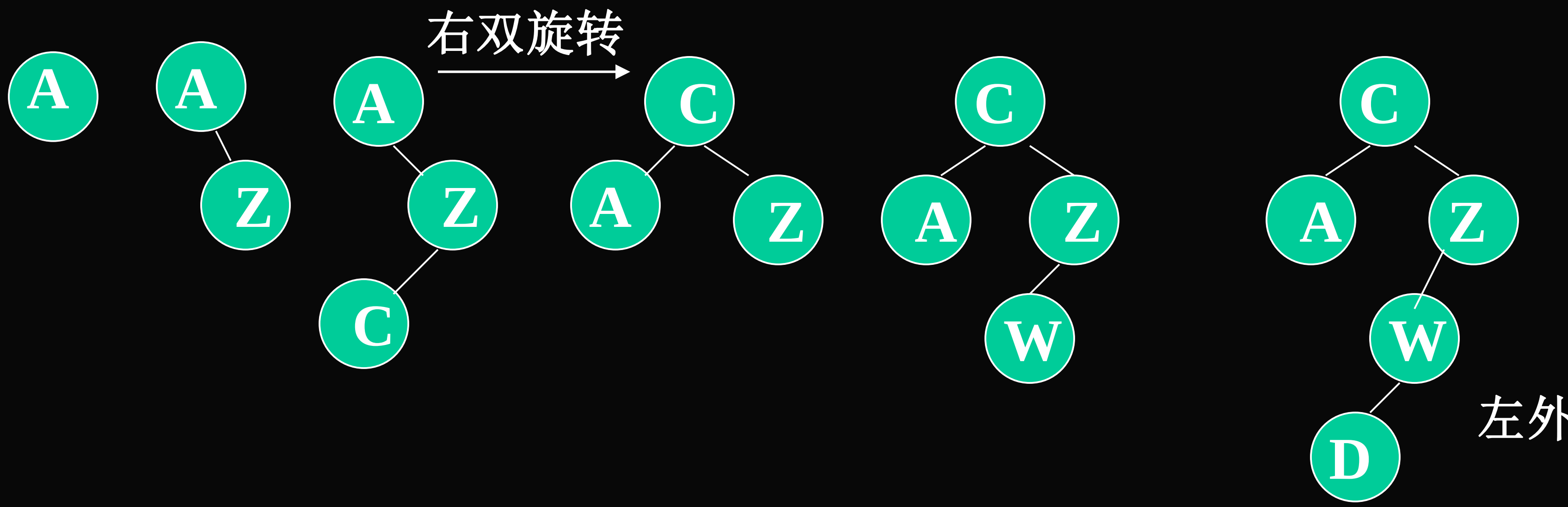


左内侧:

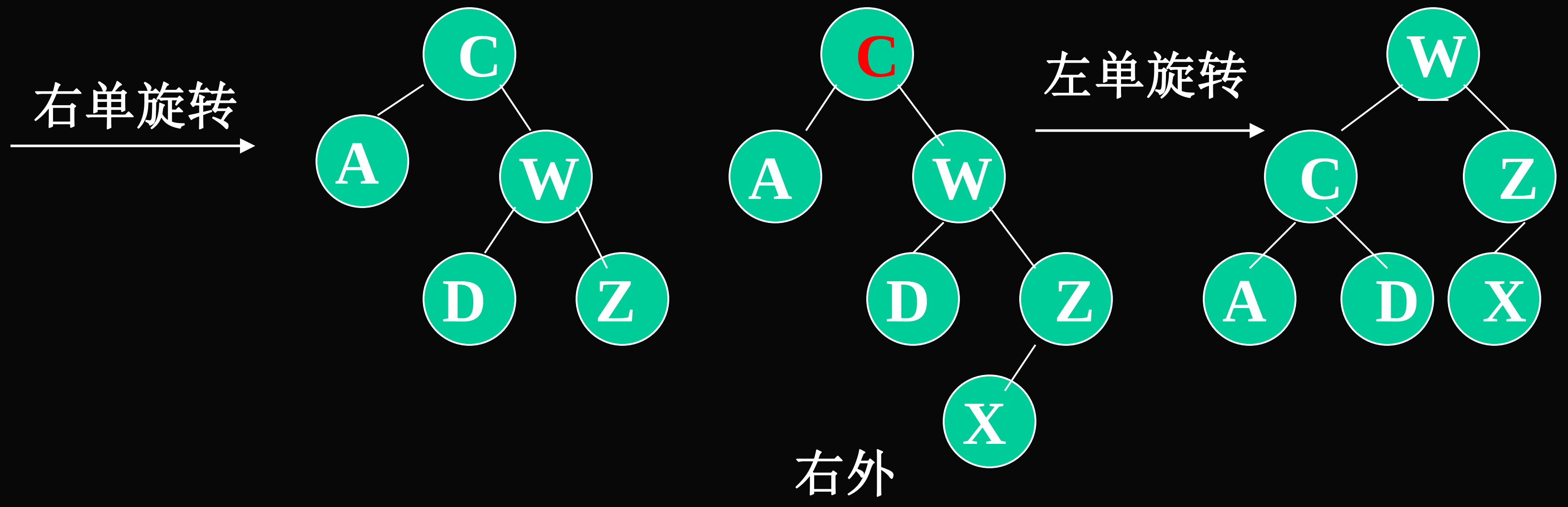

A
B
C
h
D
h
E
F
G
h-1
h-1
or
B左下旋
A
E
C
h
G
h-1
B
D
h
F
h-1
or
A右下旋
E
B
A
D
F
G
C
h
h-1
h-1
h
or

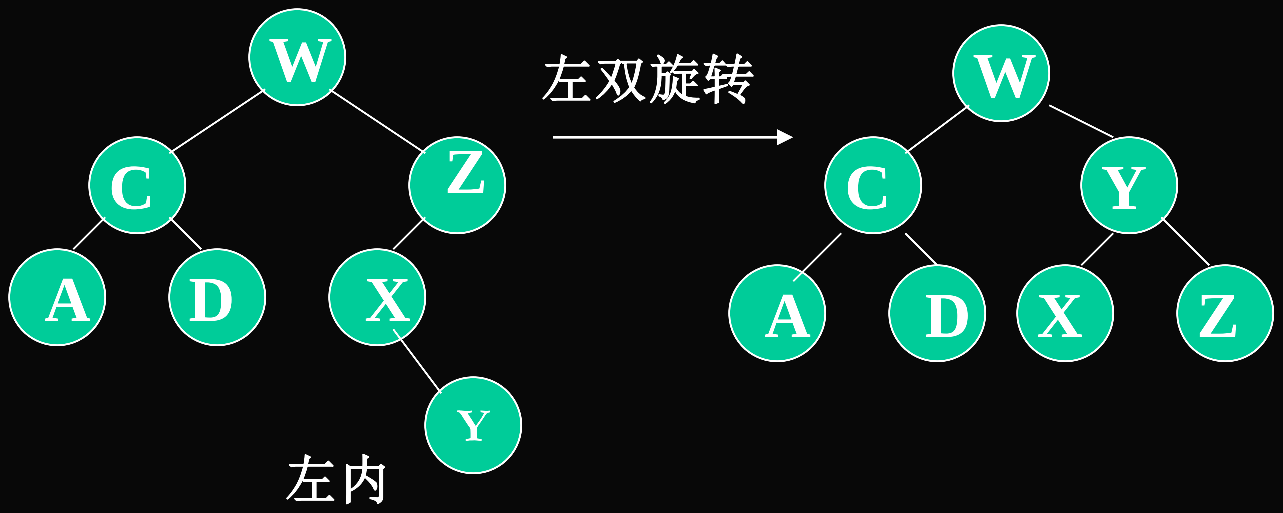
从空的AVL树建树的算法。一个例子：

7个关键码发生四种转动　A，　Z，　C，　W，　D，　X，　Y



右内
折线+右侧


右单旋转
C
A
W
D
Z
C
A
W
D
Z
X
右外
左单旋转
W
C
Z
A
D
X


W
C
Z
A
D
X
Y
左内
左双旋转
W
C
Y
A
D
X
Z

AVL Tree

```
class AVLNode
{   AVLNode( Comparable theElement )
      {  this( theElement, null, null ); }
    AVLNode( Compalable theElement, AVLNode lt, AVLNode rt )
      {  element = theElement; left = lt; right = rt; height = 0; }

    Comparable element;
    AVLNode left;
    AVLNode right;
     int   height;
}

 private static int height( AVLNode t )
{   return t = = null ? –1 : t . height;
}
```

AVL Tree

```
private AVLNode  insert( Comparable x, AVLNode t )
{   if ( t = = null )
        t = new AVLNode( x, null, null );
    else if ( x.compareTo( t.element ) < 0 )
    {  t.left = insert( x, t.left );
        if( height( t.left ) – height( t.right ) = = 2 )
            if( x.compareTo( t.left.element ) < 0 )
                t = rotateWithLeftChild ( t );
            else  t = doubleWithLeftChild( t );
     }
```

AVL Tree

```
else if( x.compareTo( t.element ) > 0 )
{  t.right = insert( x, t.right ) ;
   if( height( t.right ) – height( t.left ) = = 2 )
       if( x.compareTo( t.right.element ) > 0 )
           t = rotateWithRightChild( t );
       else  t = doubleWithRightChild( t );
}
else
   ;
t.height = max( height( t.left ), height( t.right ) ) + 1;
return t;
}
```

AVL Tree

```
private  static  AVLNode  rotateWithLeftChild( AVLNode k2 )
{   AVLNode k1 = k2.left;
    k2.left = k1.right;
    k1.right = k2;
    k2.height = max( height( k2.left ), height( k2.right ) ) + 1 ;
    k1.height = max( height( k1.left ), k2.height ) + 1;
    return k1;
}
private  static  AVLNode  doubleWithLeftChild( AVLNode k3 )
{   k3.left = rotateWithRightChild( k3.left );
    return rotateWithLeftChild( k3 );
}
```

AVL Tree

AVL树的插入:

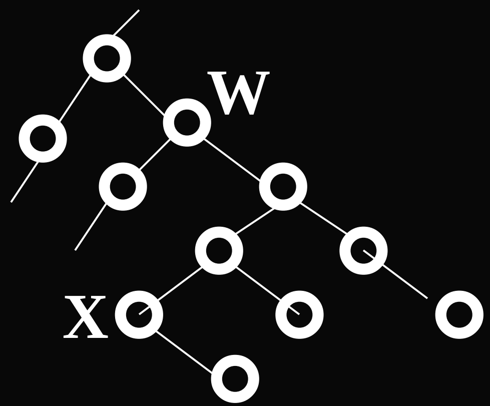
1. 首先要正确地插入
2. 找到有可能发生的最小不平衡子树
3. 判别插入在不平衡子树的外侧还是内侧
4. 根据3的判别结果,再进行单旋还是双旋

4.2 AVL Tree

4.Deletion from an AVL tree

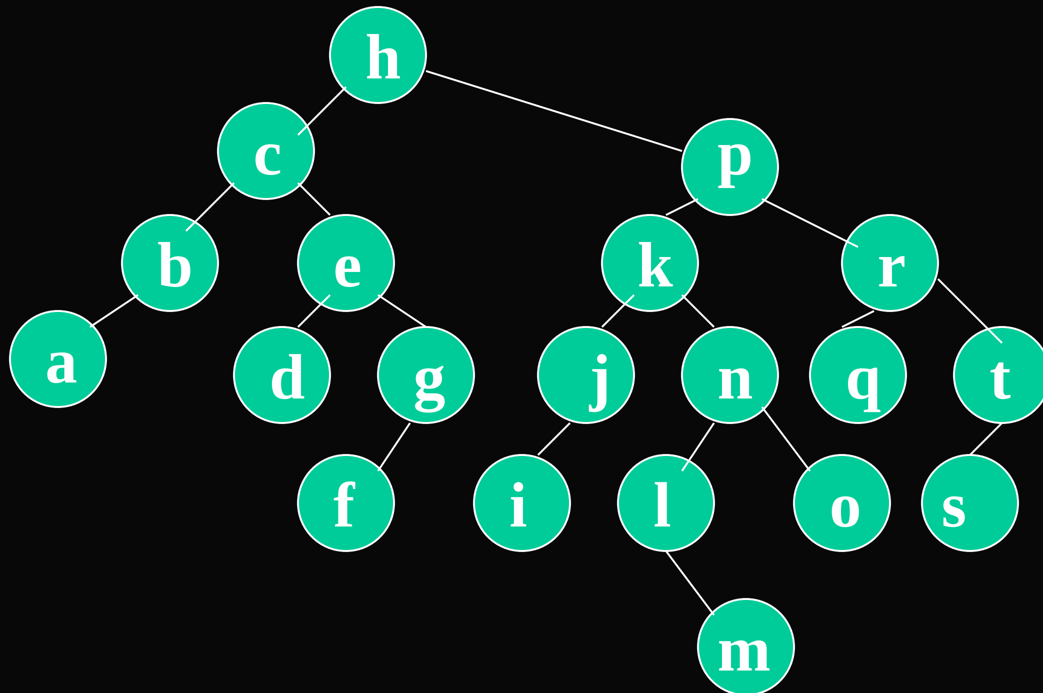
AVL树的删除

方法： 与二叉搜索树的删除方法一样。



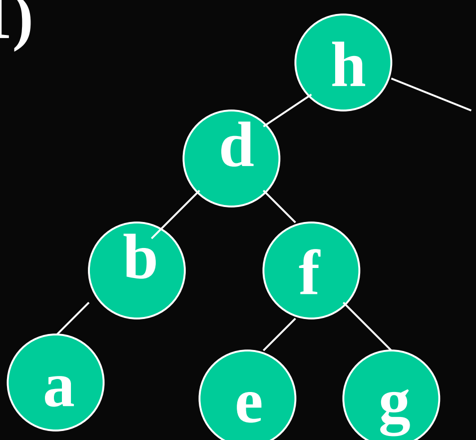
假设被删除结点为W, 它的中序后继为X, 则用X代替W,并删除X. 所不同的是:删除X后,以X为根的子树高度减1,这一高度变化可能影响到从X到根结点上每个结点的平衡因子,因此要进行一系列调整。

例子:



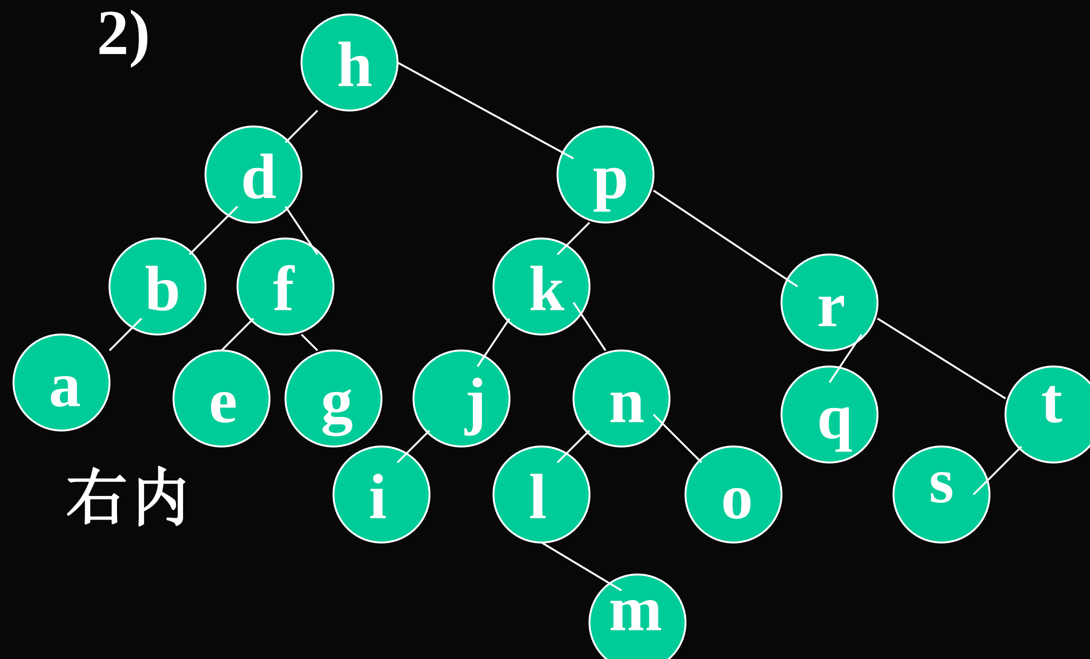
现要删除C

1)



右内

2)



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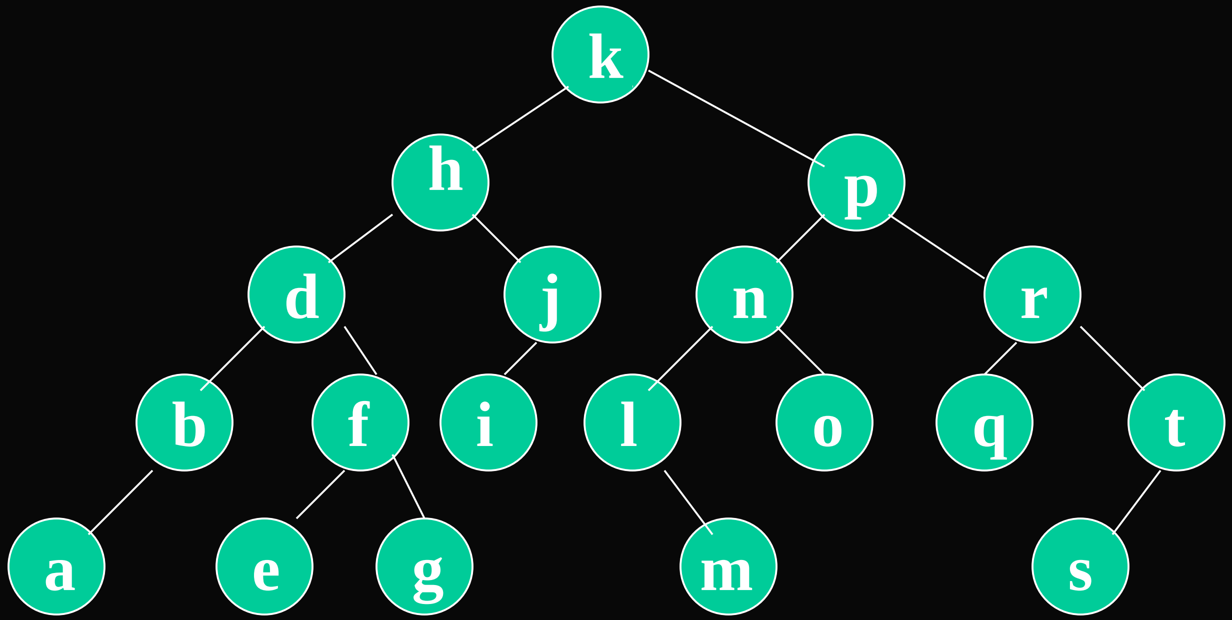


因为删除操作,不平衡要传递，所以设置一个布尔变量**shorter**来指明子树的高度是否被缩短。在每个结点上的操作取决于**shorter**的值和结点的平衡因子，有时还要依赖子女的平衡因子。

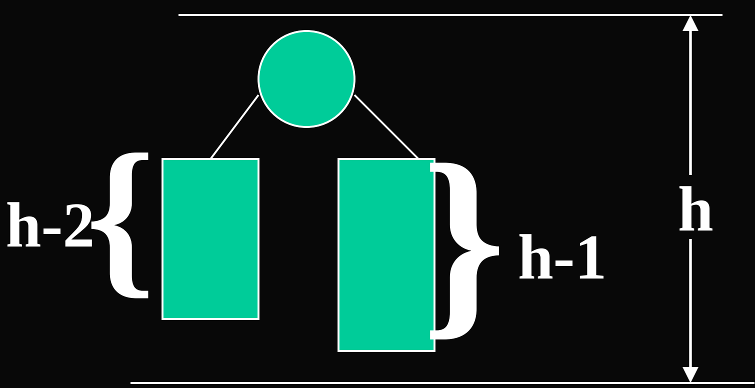
5.算法分析

具有n个结点的平衡二叉树（AVL），进行一次插入或删除的时间最坏情况≦O（$\log_2$ n)

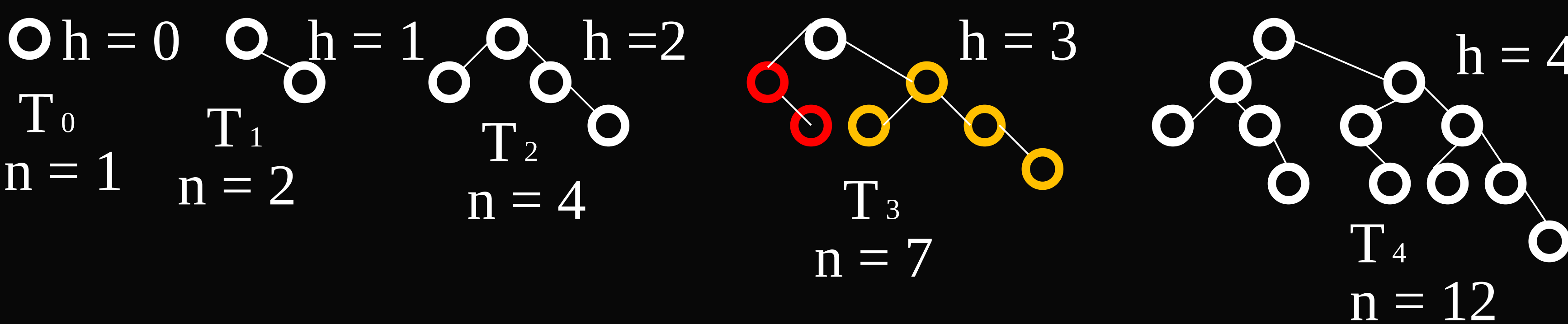
证明：实际上要考虑n个结点的平衡二叉树的最大高度

$$\leqq (3/2)\log_2 (n+1)$$

设T_h为一棵高度为h，且结点个数最少的平衡二叉树。

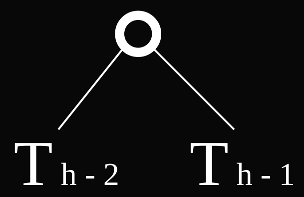


假设右子树高度为h-1
因结点个数最少,∴左子树高度只能是h-2
这两棵左子树,右子树高度分别为h-2, h-1,也一定是结点数最少的:



以上五棵平衡二叉树，又称为Fibonacci树。

也可以这样说一棵高度为h的树，其右子树高度为h-1的Fibonacci树，左子树是高度为h-2的Fibonacci树，即



假设N_h表示一棵高度为h的Fibonacci树的结点个数，则

$$N_h = N_{h-1} + N_{h-2} + 1$$

$N_0 = 1$, $N_1 = 2$, $N_2 = 4$, $N_3 = 7$, $N_4 = 12$， . . .

$N_0 + 1 = 2$, $N_1 + 1 = 3$, $N_2 + 1 = 5$, $N_3 + 1 = 8$, $N_4 + 1 = 13$， . . .

$\therefore$ $N_h + 1$满足费波那契数的定义，并且$N_h + 1 = F_{h+3}$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	. . .
0	1	1	2	3	5	8	. . .

费波那契数F_i满足下列公式

$$F_i = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^i$$

$\because \left|\frac{1-\sqrt{5}}{2}\right| < 1$, $\therefore \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^i$ **相当小**

$$N_h + 1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{h+3} + O(1)$$

∵费波那契数树是具有相同高度的所有平衡二叉树中结点个数最少的

$$n + 1 \geqslant N_h + 1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{h+3} + O(1)$$

$$\therefore\ h \leqslant \frac{1}{\log_2 \frac{1+\sqrt{5}}{2}} \log_2(n+1) + O(1) \approx \frac{3}{2}\log_2(n+1)$$

4.3 B-TREES

1. m-way Search Trees

Definition: An m-way search tree may be empty. If it is not empty, it is a tree that satisfies the following properties:

1) In the corresponding extended search tree(obtained by replacing zero pointer with external nodes), each internal node has up to m children and between 1 and m-1 elements.

2) Every node with p elements has exactly p+1children.

3) Consider any node with p elements:

$$\boxed{C_0\ k_1\ C_1\ k_2\ \ldots\ldots\ k_p\ C_p}$$

$k_1<k_2<\ldots\ldots<k_p$, $c_0,c_1,\ldots\ldots,c_p$ be the p+1 children of the node

4.3 B-TREES

$C_0\ k_1\ C_1\ k_2\ \ldots\ldots\ldots\ k_p\ C_p$

C_0: The elements in the subtree with root c_0 have keys smaller than k_1

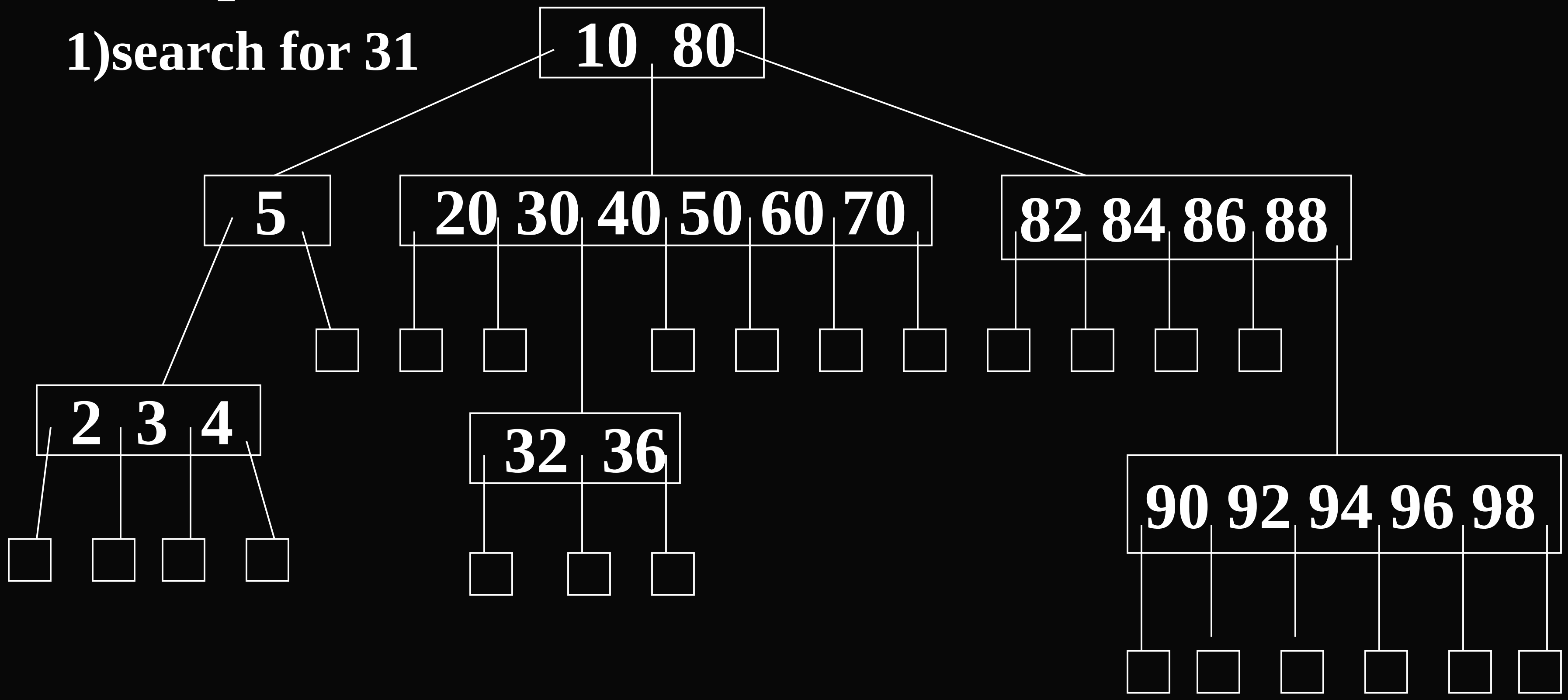
C_p: Elements in the subtree with root c_p have keys larger than k_p

C_i: Elements in the subtree with root c_i have keys larger than k_i but smaller than k_{i+1}, $1<=i<=p$.

4.3 B-TREES

Example:

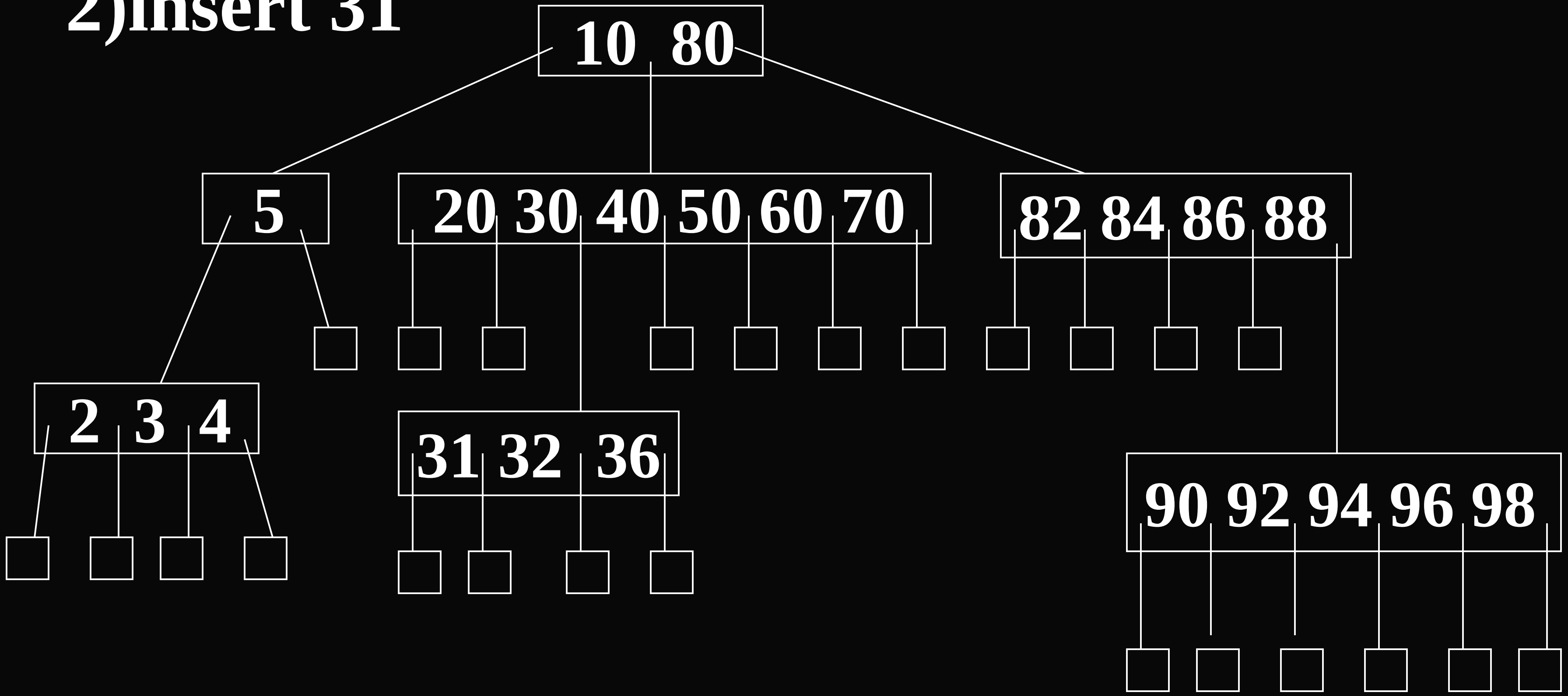
1)search for 31



A seven-way search tree

4.3 B-TREES

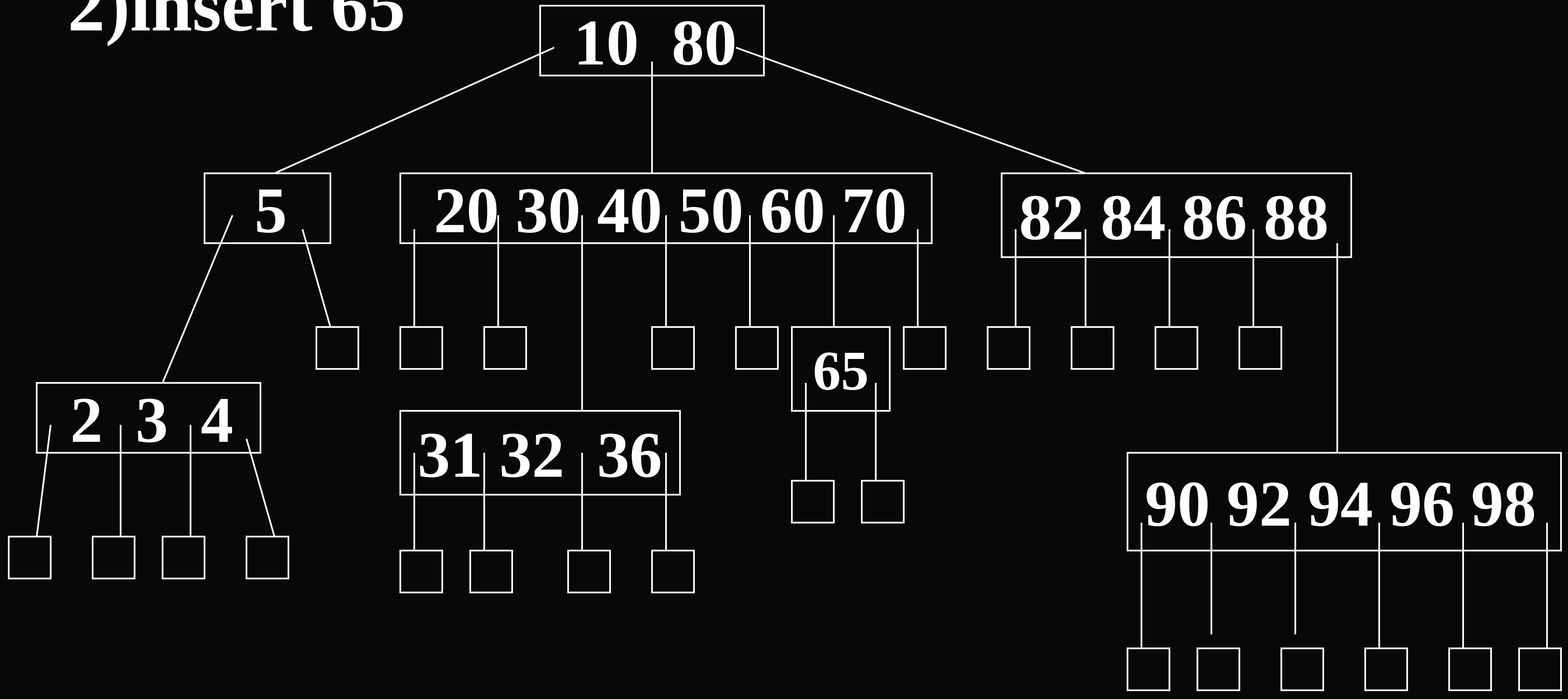
2)insert 31



A seven-way search tree

4.3 B-TREES

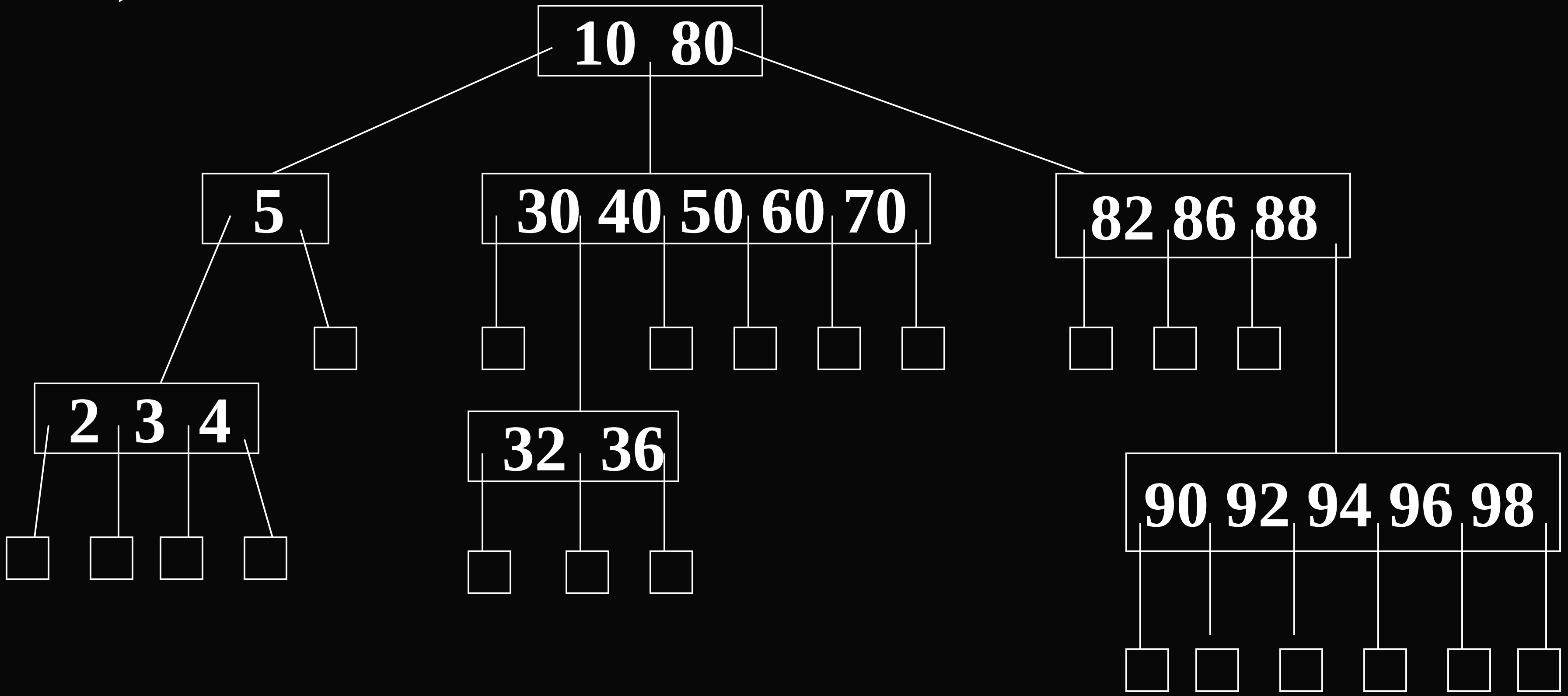
2)insert 65



A seven-way search tree

4.3 B-TREES

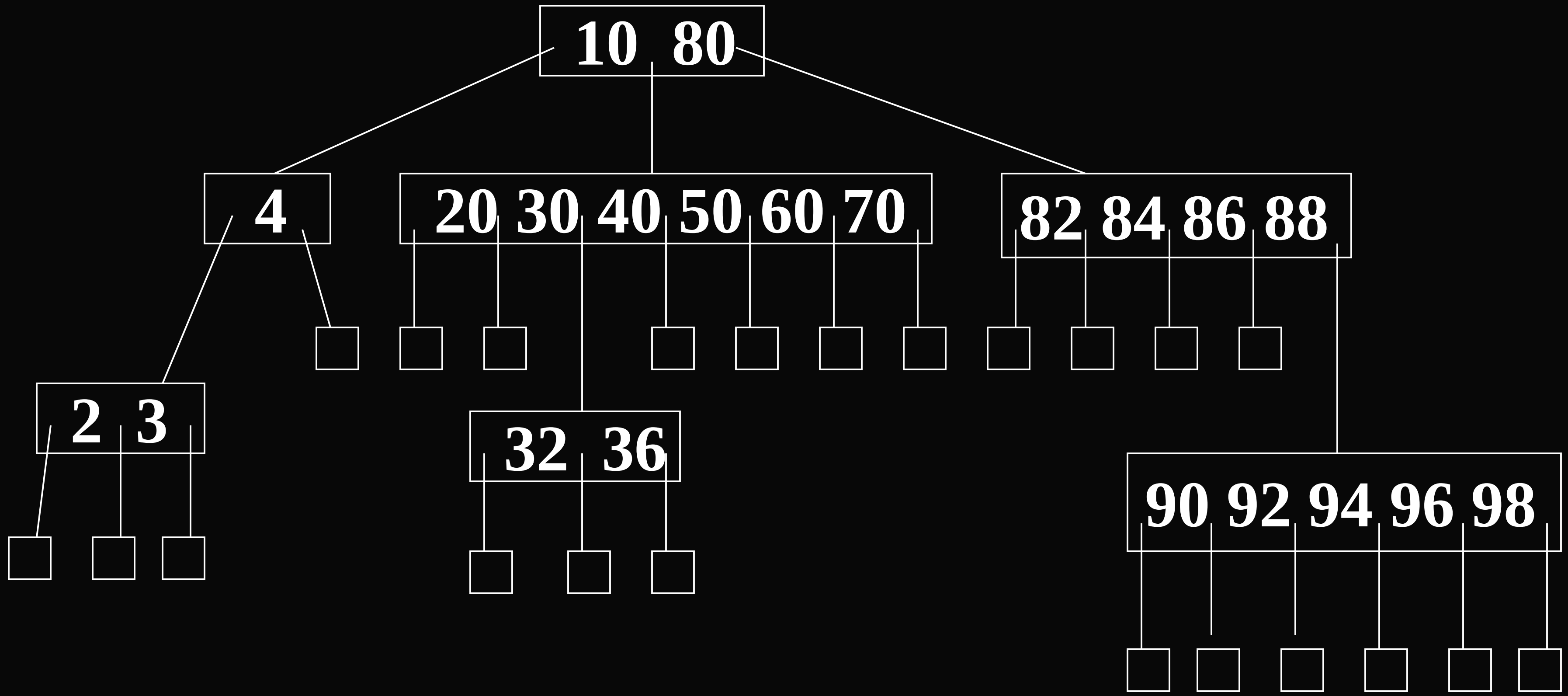
3)Delete 20, 84



A seven-way search tree

4.3 B-TREES

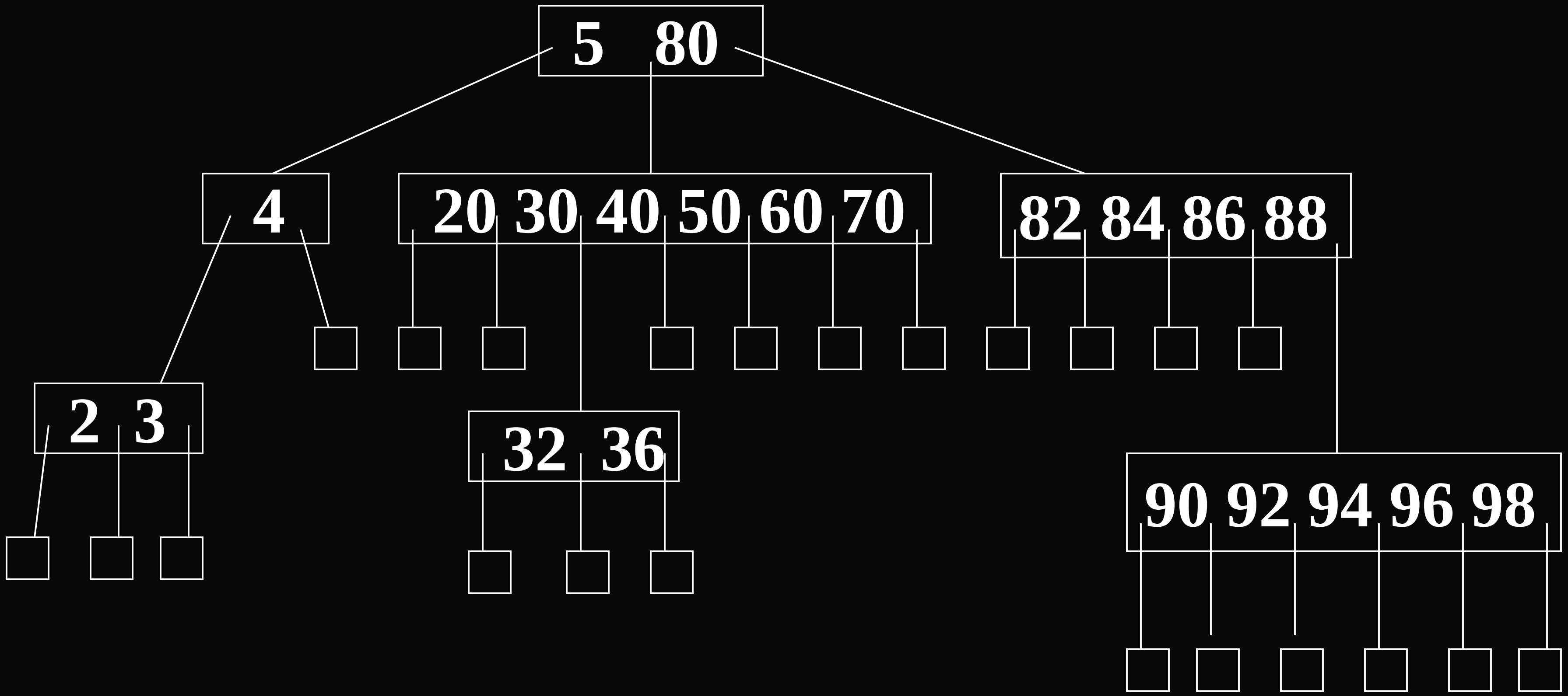
Delete 5, move up 4



A seven-way search tree

4.3 B-TREES

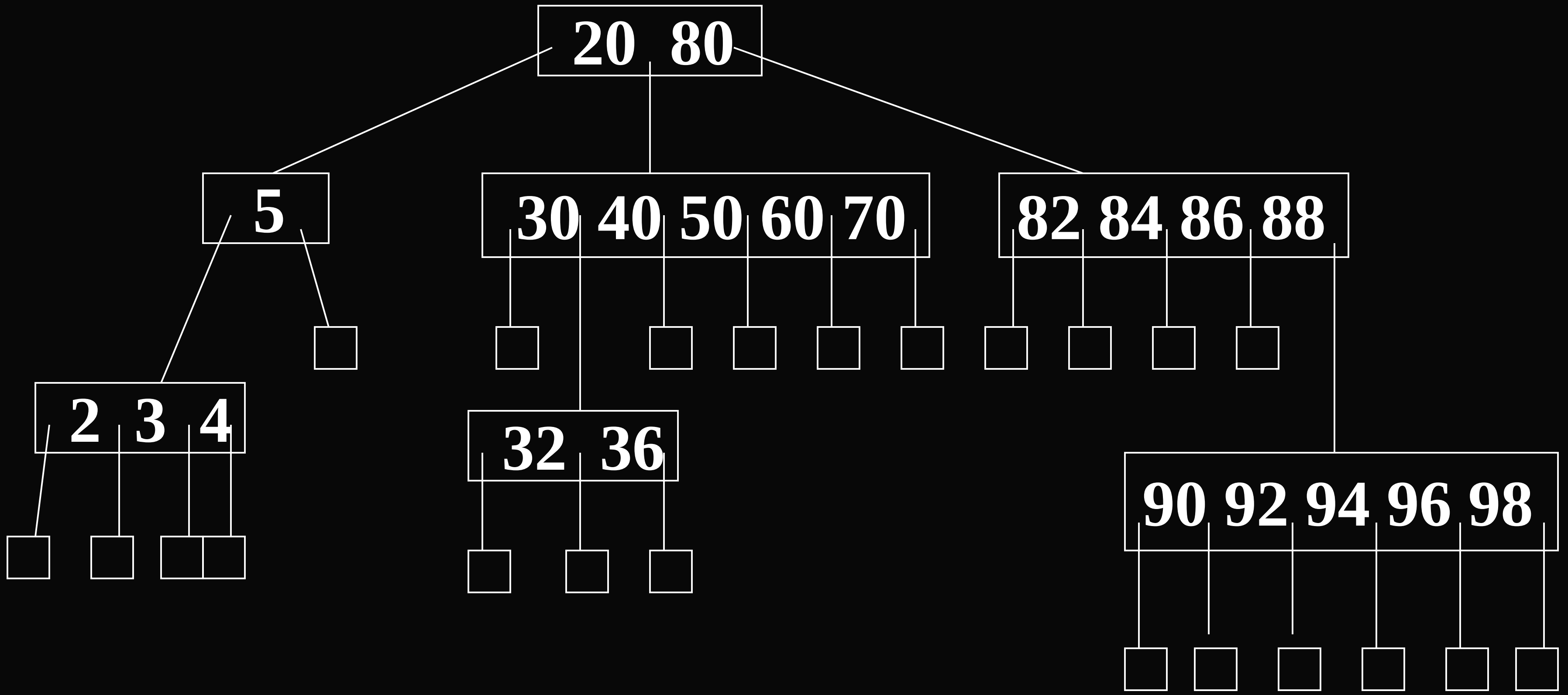
Delete 10: **replace it with the largest element in c_0(5)**



A seven-way search tree

4.3 B-TREES

Delete 10: **replace it with the smallest element in c_1(20)**



A seven-way search tree

4.3 B-TREES

4) Height of an m-way search tree

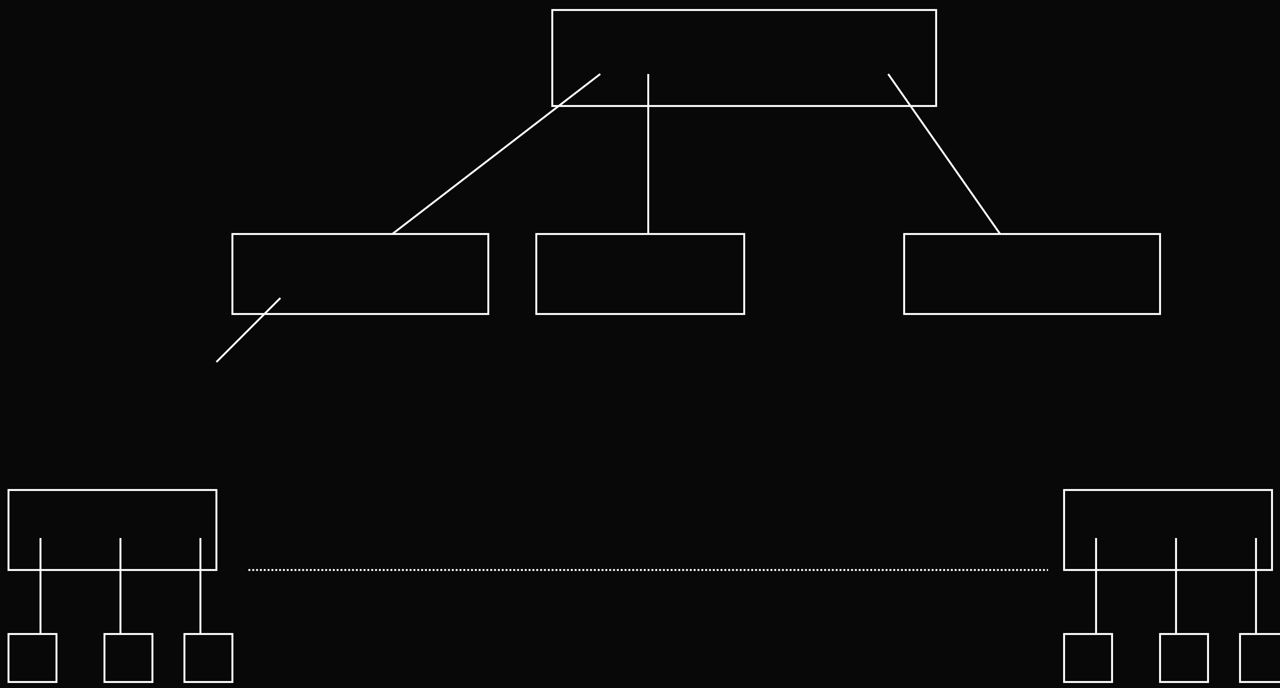
An m-way search tree of height h

may have as few as h elements(one node per level),

as many as m^h-1 elements.

4.3 B-TREES

No of nodes



0 $m^0=1$

1 $m^1=m$

h-1 m^{h-1}

Sum of nodes $\sum_{i=0}^{h-1} m^i=(m^h-1)/(m-1)$

4.3 B-TREES

- **The number of elements in a m-way search tree of height h is between h and m^h-1**
- **The height of a m-way search tree with n elements is between $\log_m(n+1)$ and n**

Example:

height: 5

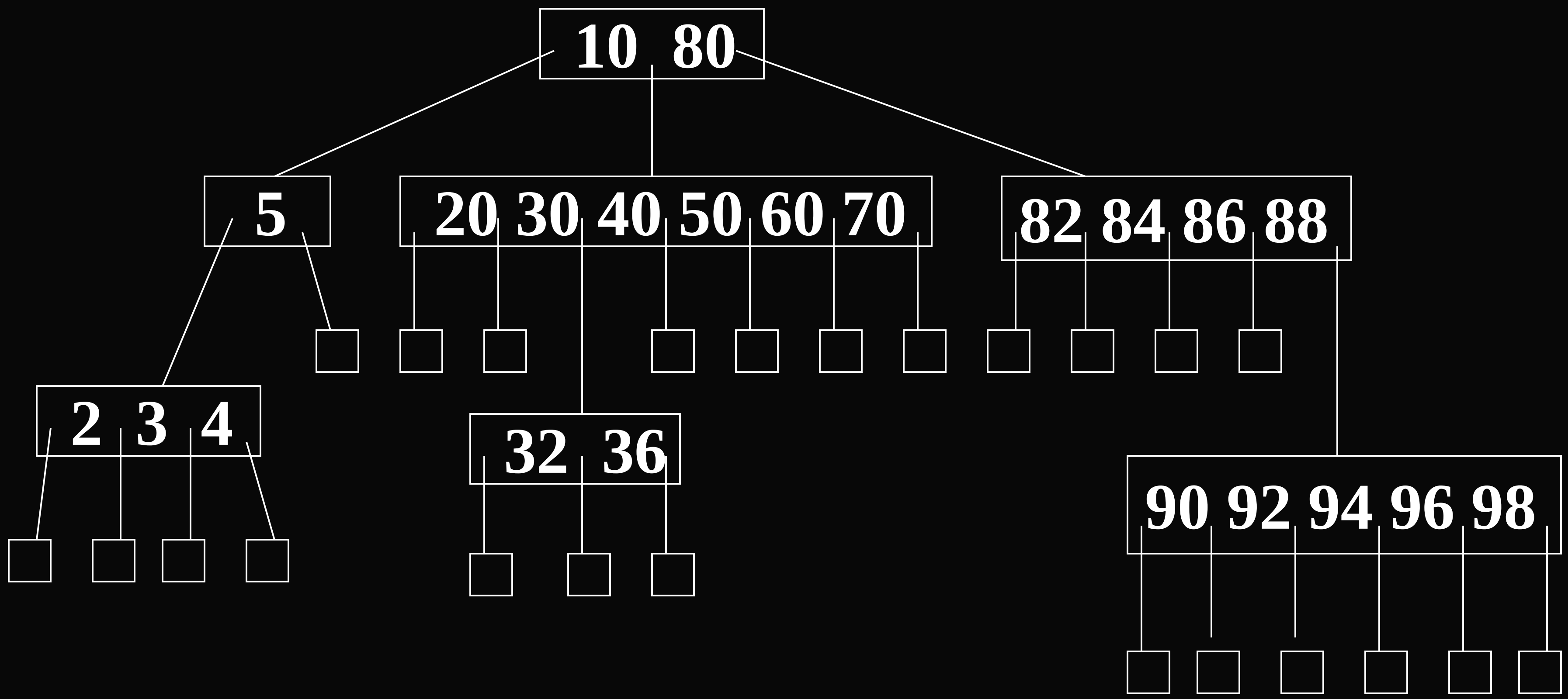
200-way search tree

n :200^5-1 =$32*10^{10}$-1

4.3 B-TREES

二叉搜索树 ----→ 平衡的二叉搜索树 (AVL树)

m路搜索树 ----→ 平衡的m叉搜索树 (B-树）



A seven-way search tree

4.3 B-TREES

2.B-Trees of order m

70年 R.Bayer提出的。

Definition：A B-tree of order m is an m-way search tree.

If the B-tree is not empty, the corresponding extended tree satisfies the following properties:

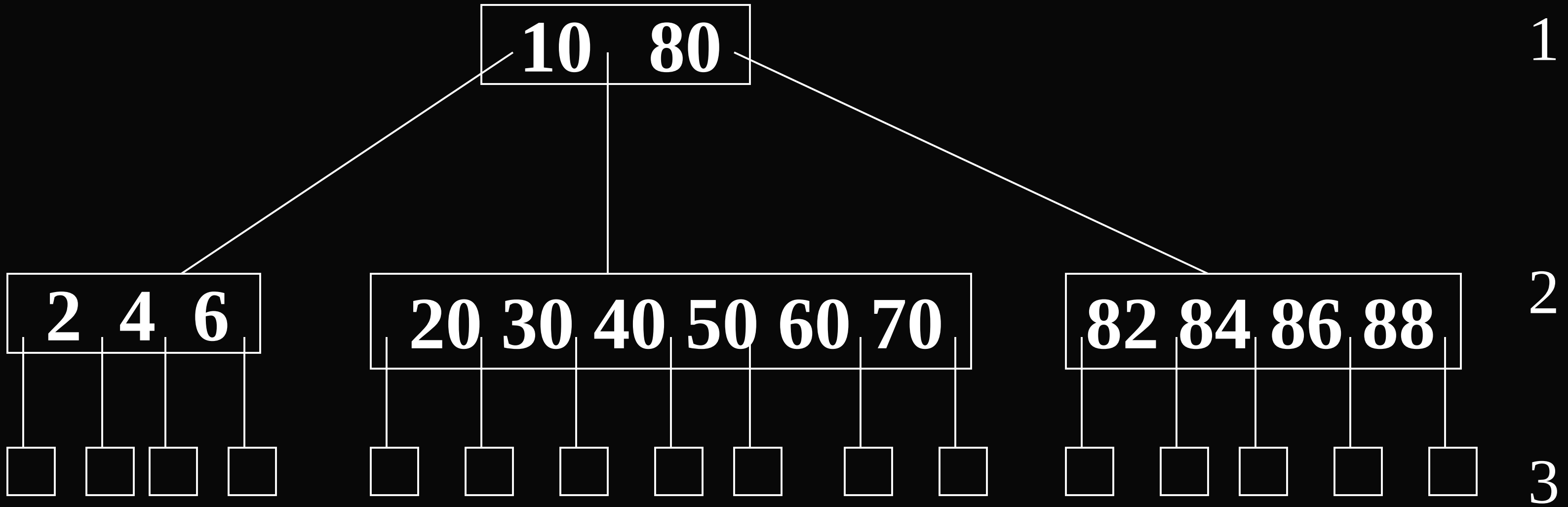
1) the root has at least two children

2) all internal nodes other than the root have at least $\lceil m/2 \rceil$ children

3) all external nodes are at the same level

4.3 B-TREES

example



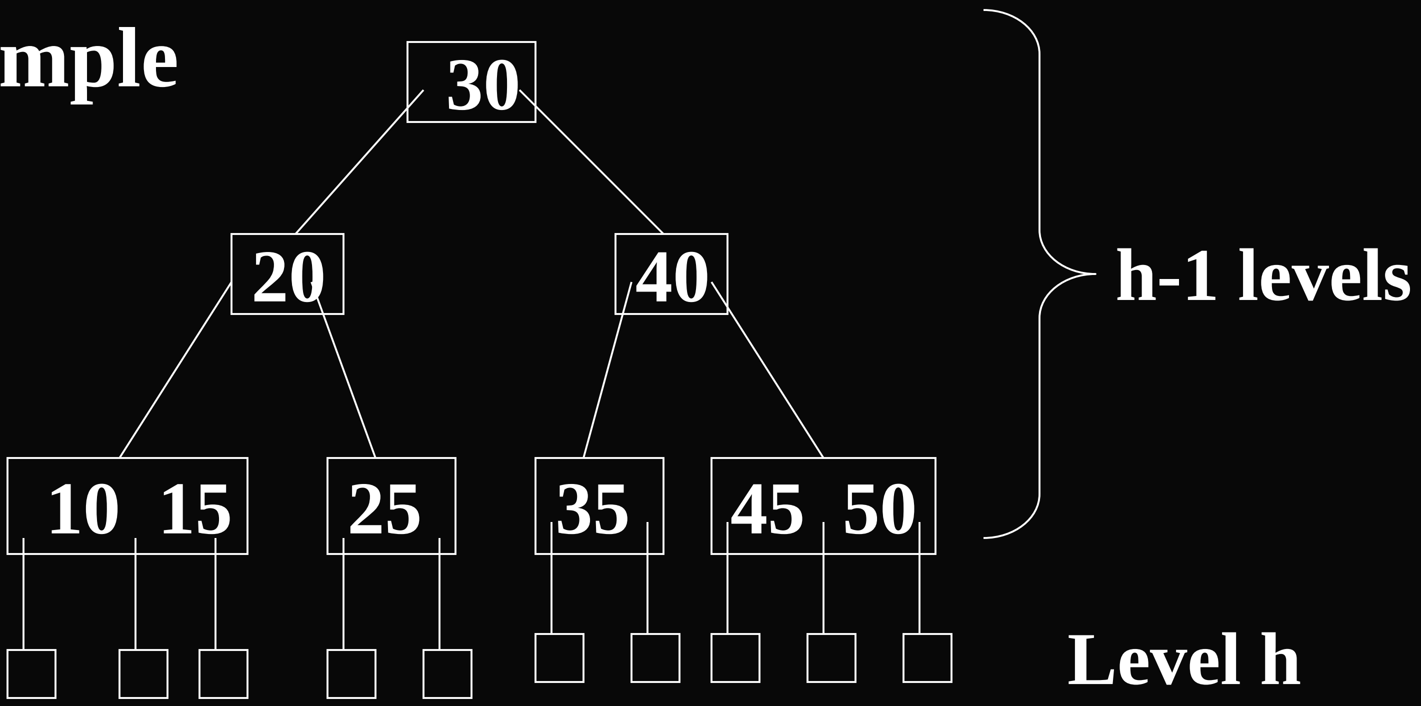
a B-tree of order 7

4.3 B-TREES

- **In a B-tree of order 2, each internal node has at least 2 children, and all external nodes must be on the same level, so a B-tree of order 2 is full binary trees**
- **In a B-tree of order 3(sometimes also called 2-3 tree) , each internal node has 2 or 3 children**

4.3 B-TREES

- **example**



A B-tree of order 3

4.3 B-TREES

B-TREES Properties:

1)all external nodes are on the same level

2)number of external nodes=number of keywords +1

proof:

$b_1=k_0+1$, $b_2=k_1+b_1$, $b_3=k_2+b_2$, ……..,

外部结点$=k_{h-1}+k_{h-2}+\ldots+k_1+k_0+1=n+1$

B_h 是外部节点总数

4.3 B-TREES

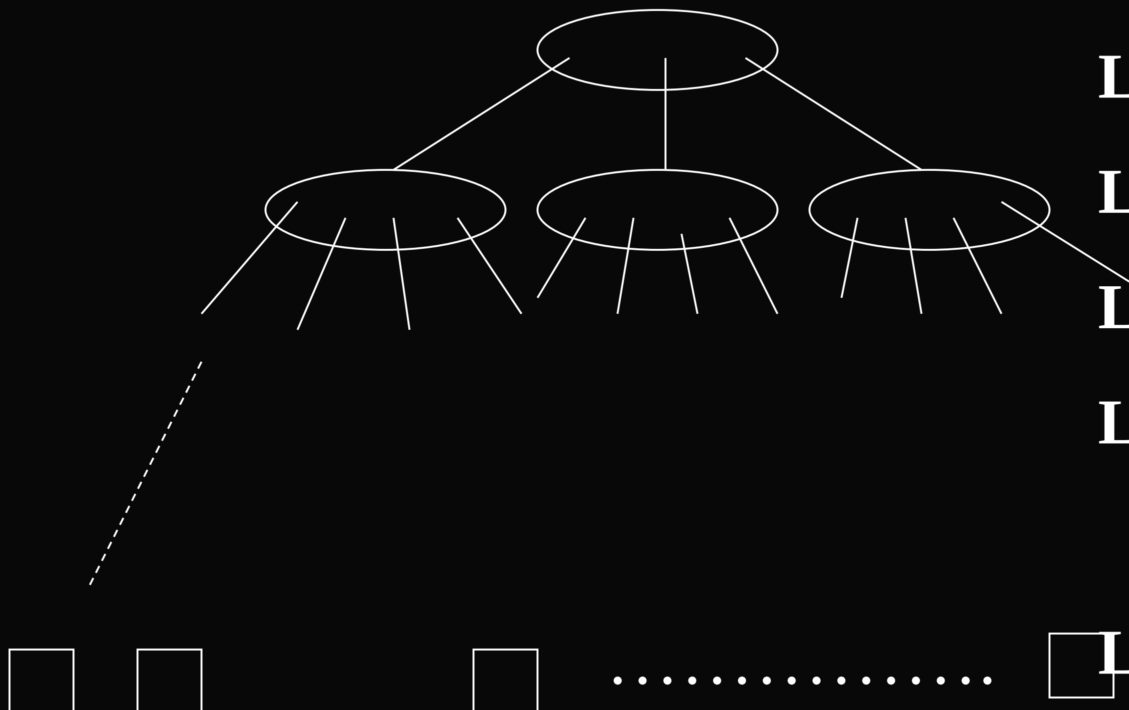
1) Searching a B-Tree

- **A B-tree is searched using the same algorithm as used for an m-way search tree.**
- **Algorithm analysis: the number of disk access is <u>at most</u> h(h is the height of the B-Tree).**

<u>proof:</u> T is a B-Tree of order <u>m</u> with height <u>h</u>, number of elements in T is <u>n</u>, each time we read <u>a node</u> into memory. The <u>n+1</u> external nodes are on level h.

4.3 B-TREES

Number of nodes on the each level of the B-Tree is:



Level 0   1

Level 1 >=2

Level 2 >=2$\lceil m/2 \rceil$

Level 3 >=2$\lceil m/2 \rceil^2$

Level h >= 2$\lceil m/2 \rceil^{h-1}$

4.3 B-TREES

$n+1 >= 2\lceil m/2 \rceil^{h-1}$, $(n+1)/2 >= \lceil m/2 \rceil^{h-1}$,

$h-1 <= \log_{\lceil m/2 \rceil}(n+1)/2$,

$\log_m(n+1) <= h <= 1+\log_{\lceil m/2 \rceil}(n+1)/2$

In the case that each node has m children

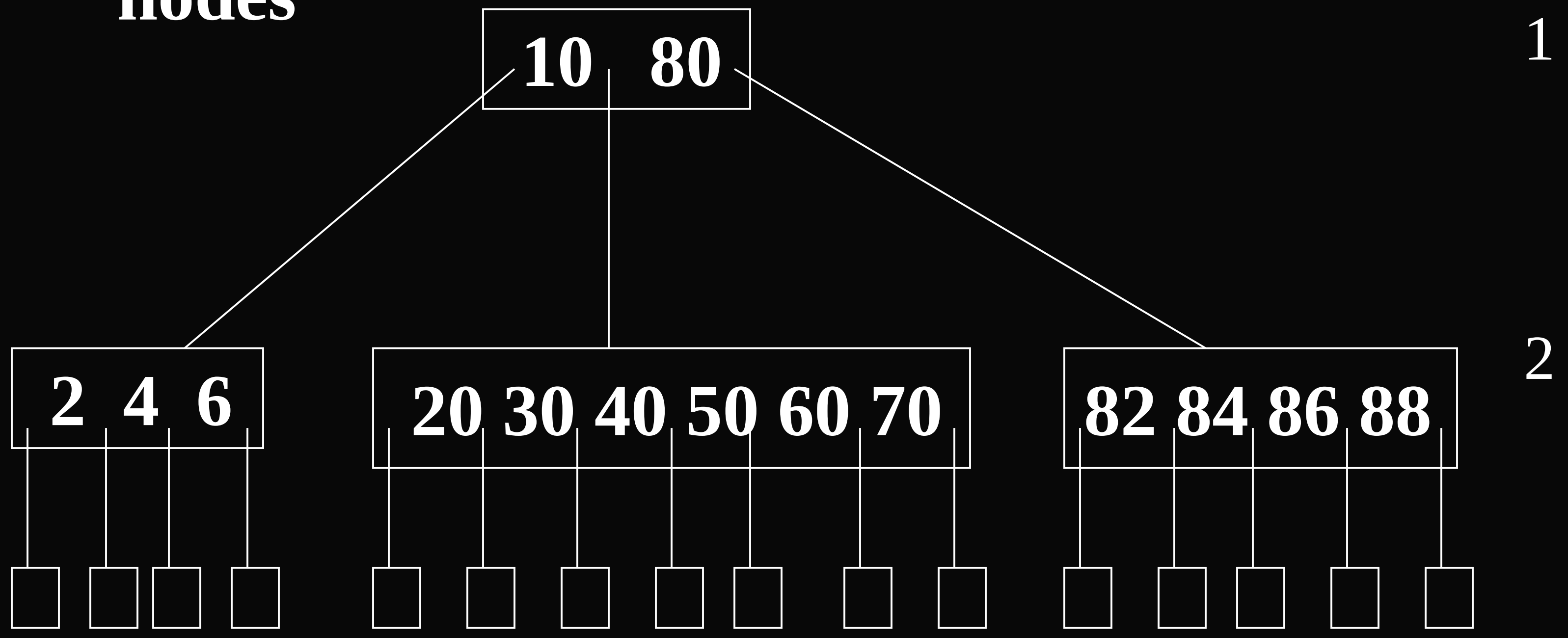
Example: $n=2*10^6$, $m=199$

then $h<=1+\log_{100}(10^2)^3=4$ search one from 200 branches

4.3 B-TREES

2) Inserting into a B-Tree

always happen at one level above the external nodes

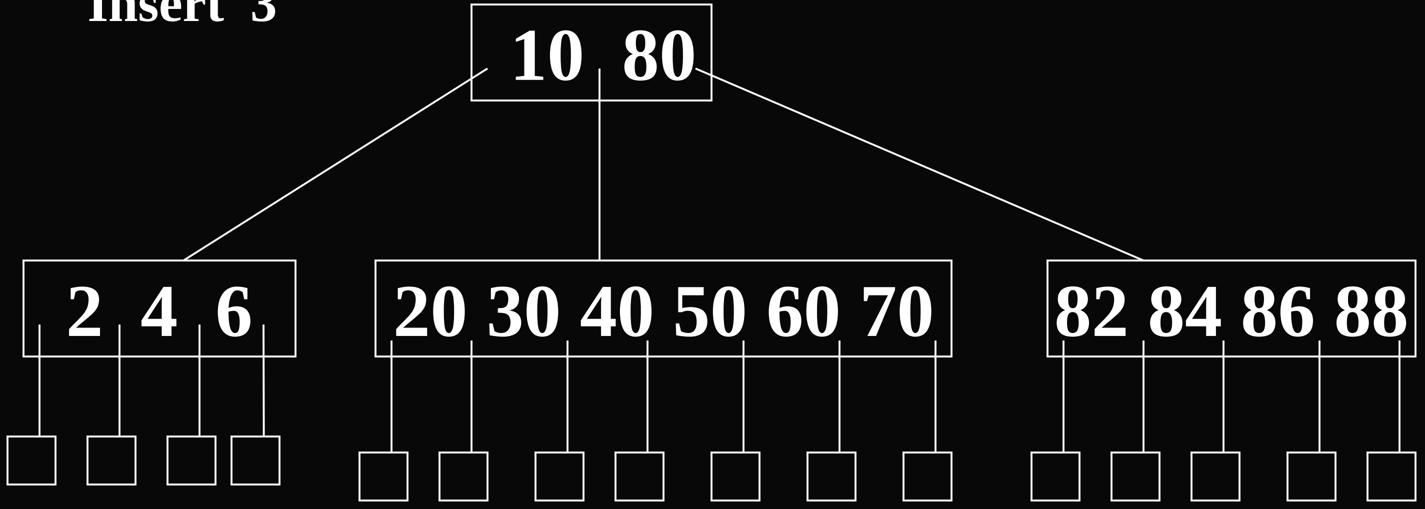


a B-tree of order 7

4.3 B-TREES

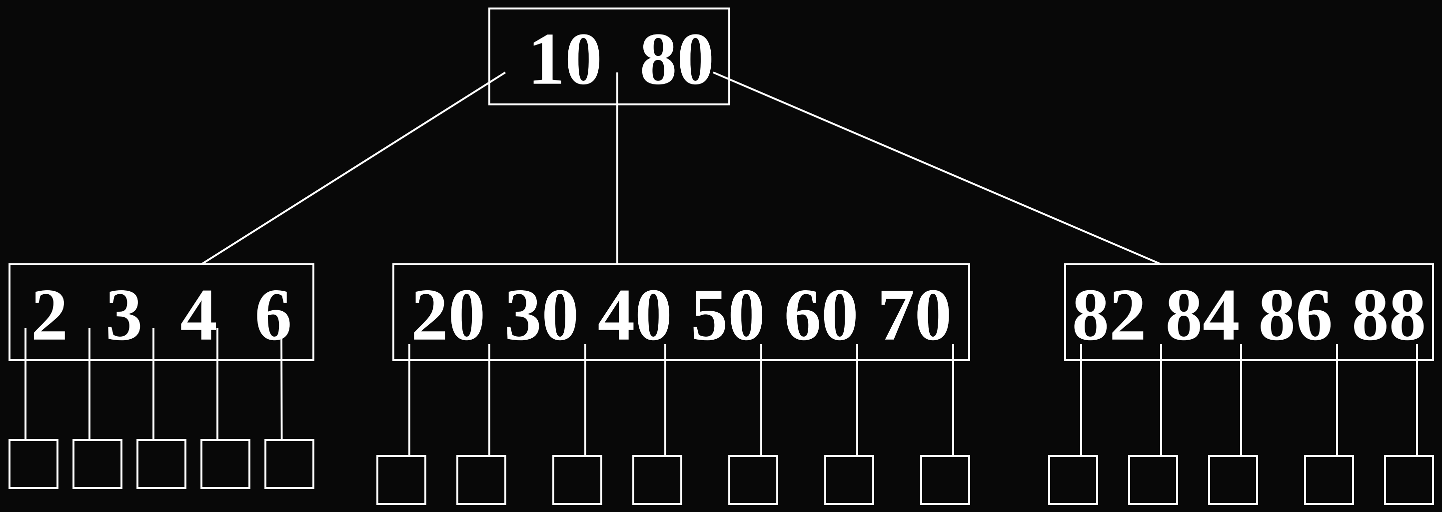
Case 1:number of children in the node<m, insert into the node as ordered

Insert 3



A B-Tree of order 7

4.3 B-TREES



4.3 B-TREES

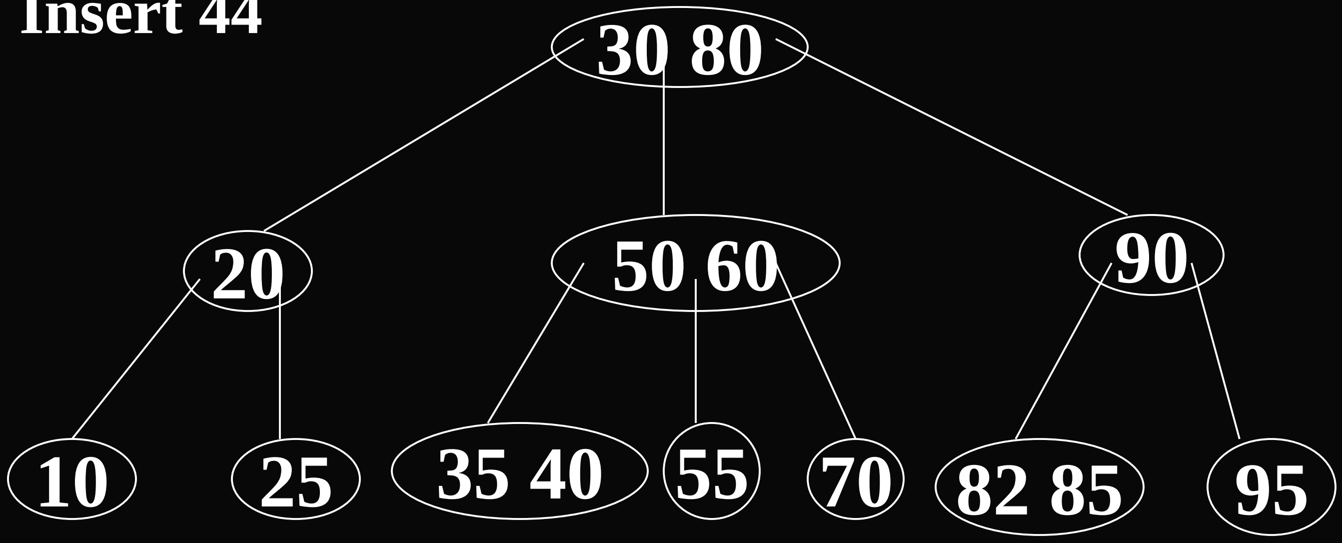
Case 2.

- **Insert into a node with m children (also called a full node), like insert 25 into the B-Tree in the last example, the full node is split into two nodes.**
- **A new pointer will be added to the parent of the full node .**
- **Because $k_{\lceil m/2 \rceil}$ is inserted into parent node, it may cause new split. If the root is split,the height of the tree will increased by 1.**

4.3 B-TREES

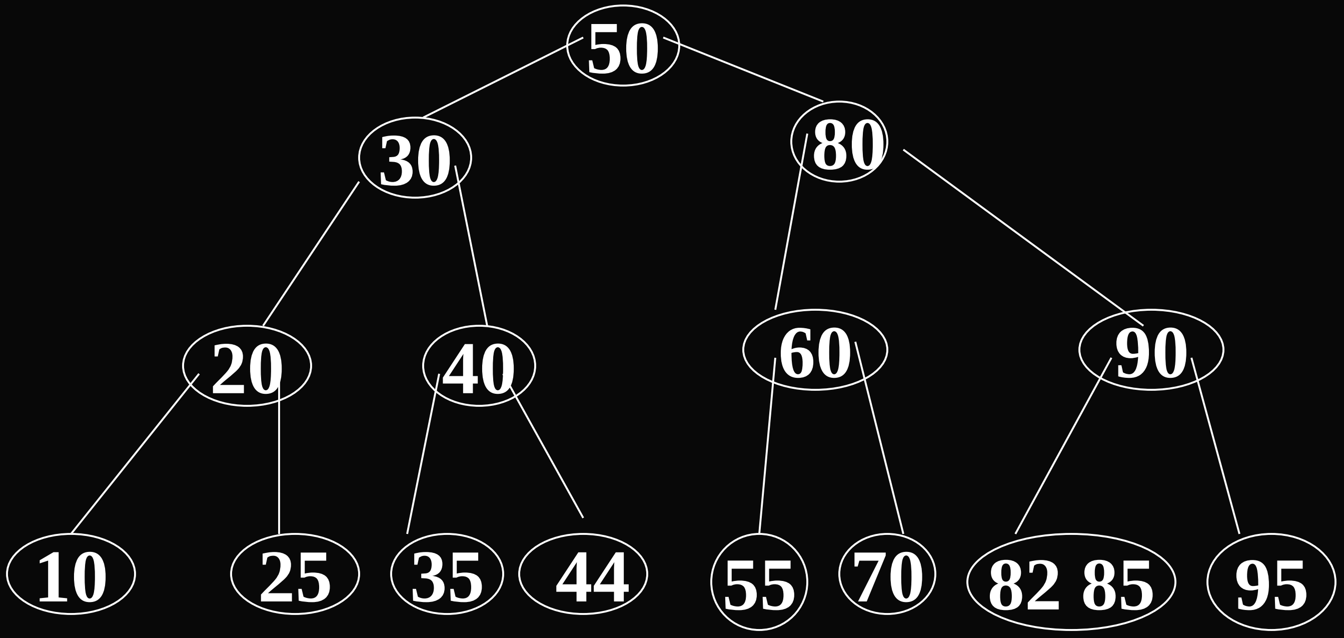
Example:

Insert 44



A B-Tree of order 3

4.3 B-TREES

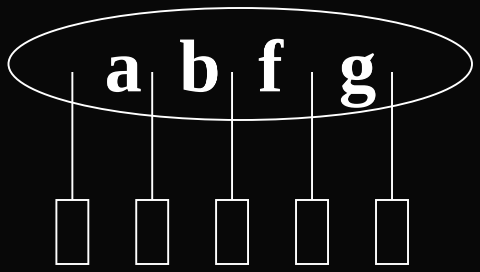


4.3 B-TREES

Another example:

a B-tree of order 5 :

insert k,m,j,e,s,i,r,x,c,……



Algorithm analyses:

If the insert operation causes s node to split, the number of disk access is

h (to read in the nodes on the search path)

+2s (to write out the two split parts of each node that is split)

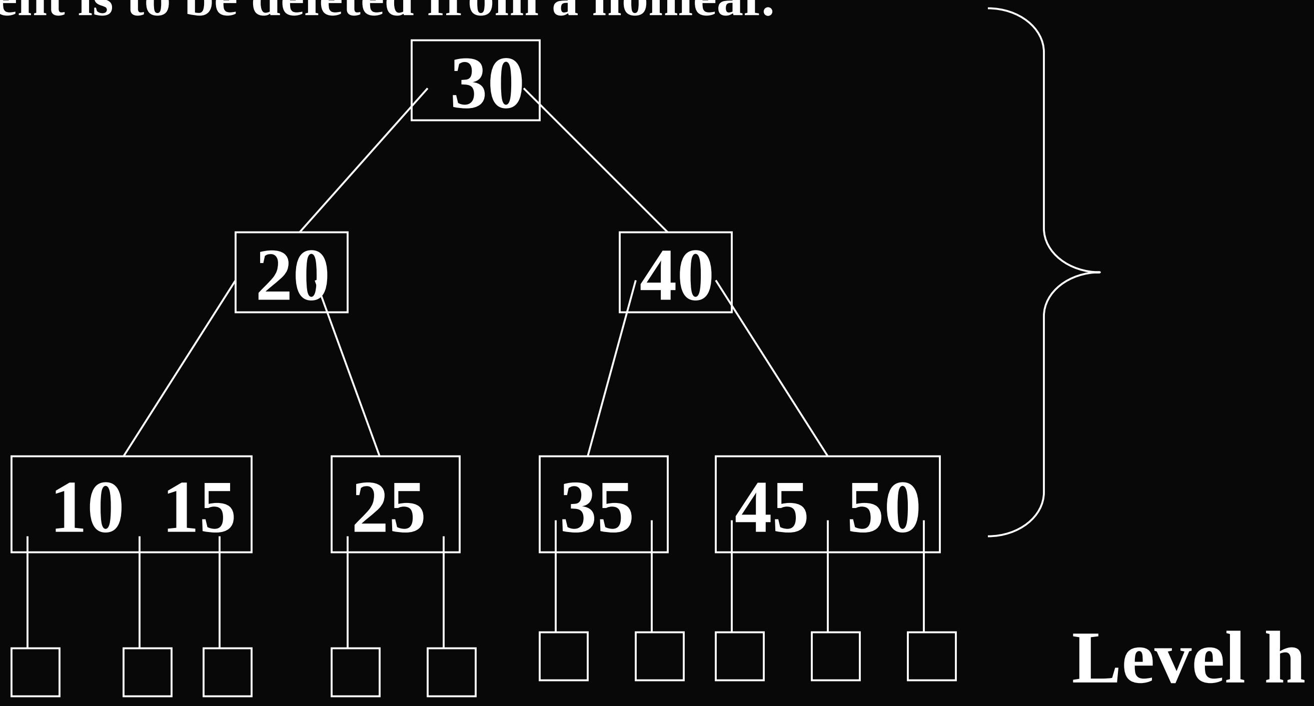
+1 (to write the new node).

4.3 B-TREES

3)deletion from a B-Tree

Two cases:

- **The element to be deleted is in a node whose children are external nodes(i.e.the element is in a leaf)**
- **The element is to be deleted from a nonleaf.**



A B-tree of order 3

4.3 B-TREES

a) the element to be deleted is in a leaf

Case1: delete it directly if it is in a node which has more than $\lceil m/2 \rceil$ children

Case2: if it is in a node which has ⌈m/2⌉children,

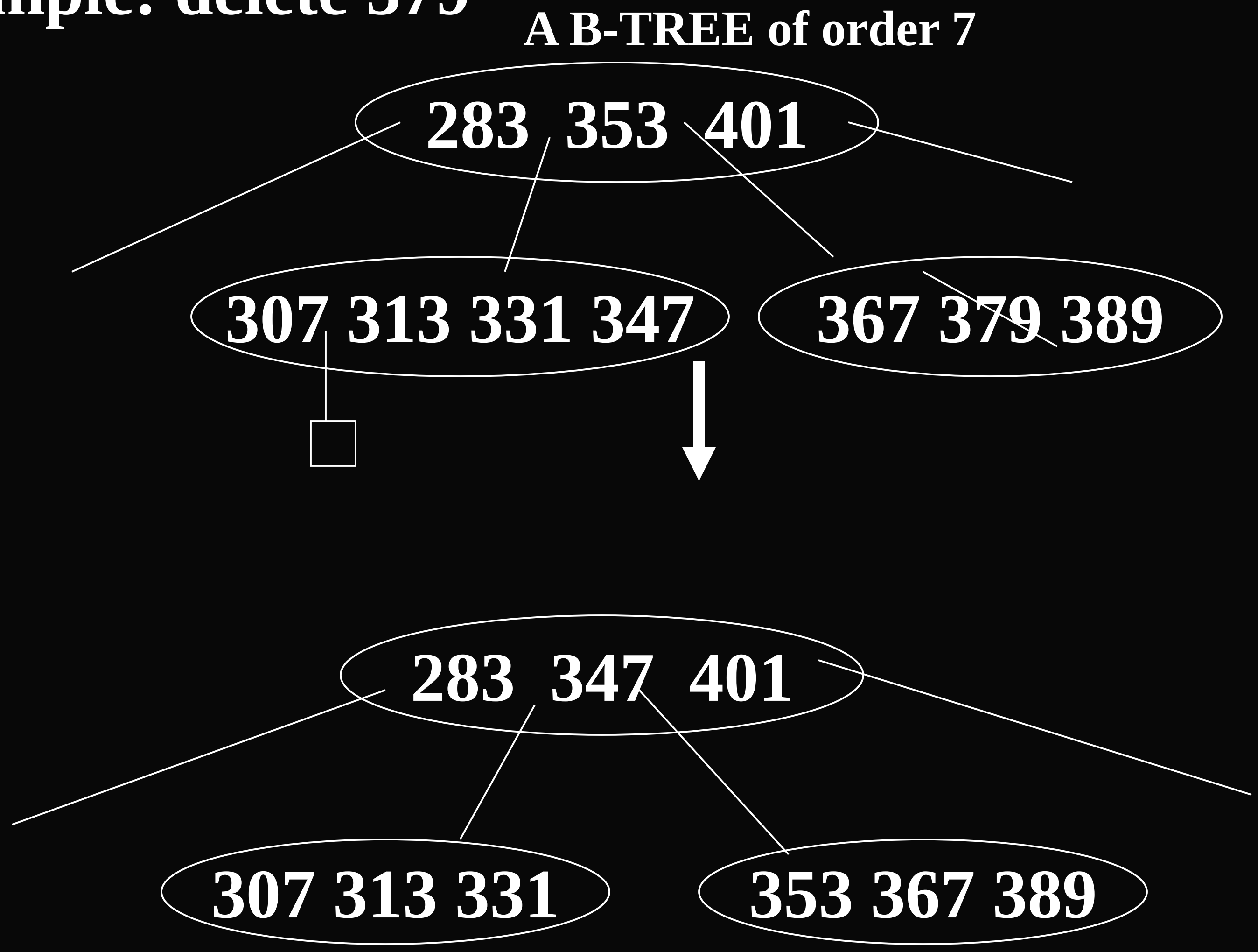
after deletion ,the number of children(⌈m/2⌉-1) is not suitable for a B-Tree

① **borrow an element from the its nearest sibling if can, and do some adjusting.**

能借则借，不能借则合并

4.3 B-TREES

Example: delete 379



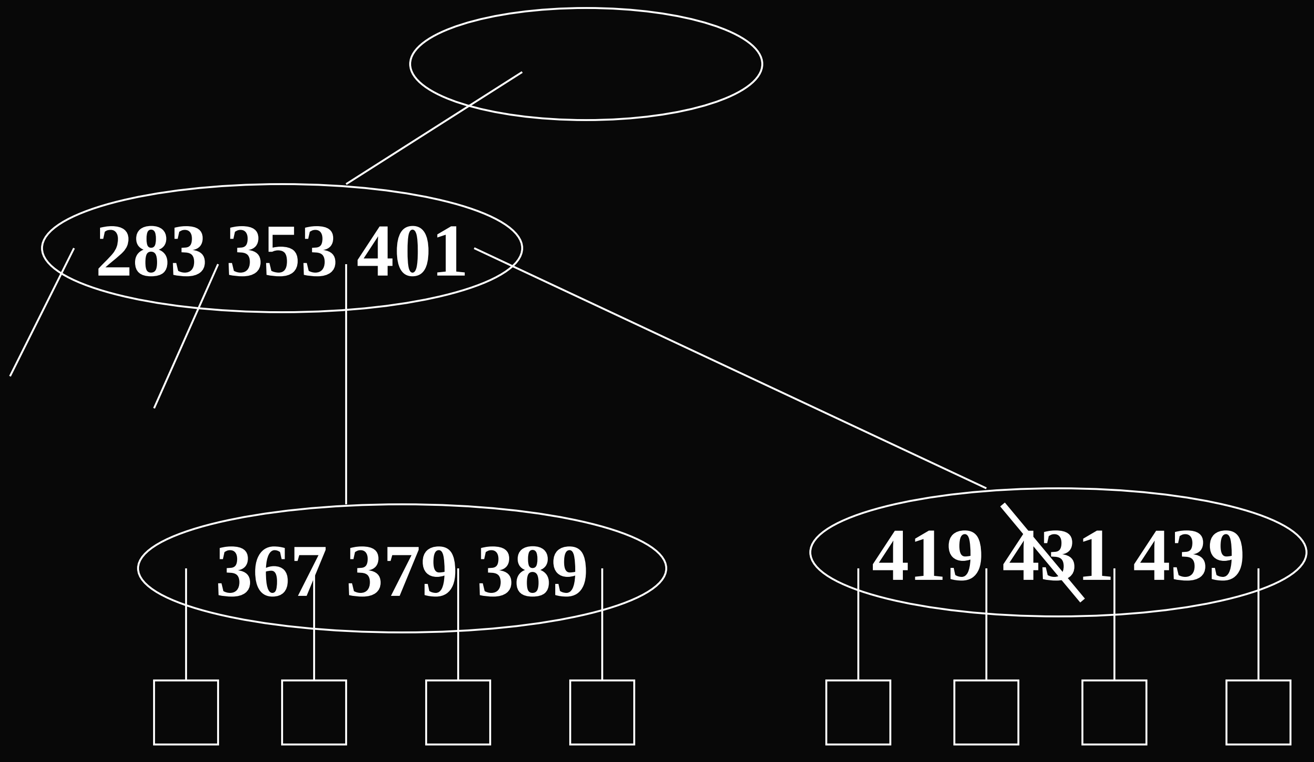
4.3 B-TREES

② **If nearest left or right sibling both only has ⌈m/2⌉ children, then merge them**

- **After deletion ,merge the node and its sibling with the element between them in the parent into a single node**
- **Maybe cause new merge in parent nodes**
- **The height of the tree will deceased by one if root is merged.**

4.3 B-TREES

Example:a B-Tree of order 7 ,delete 431



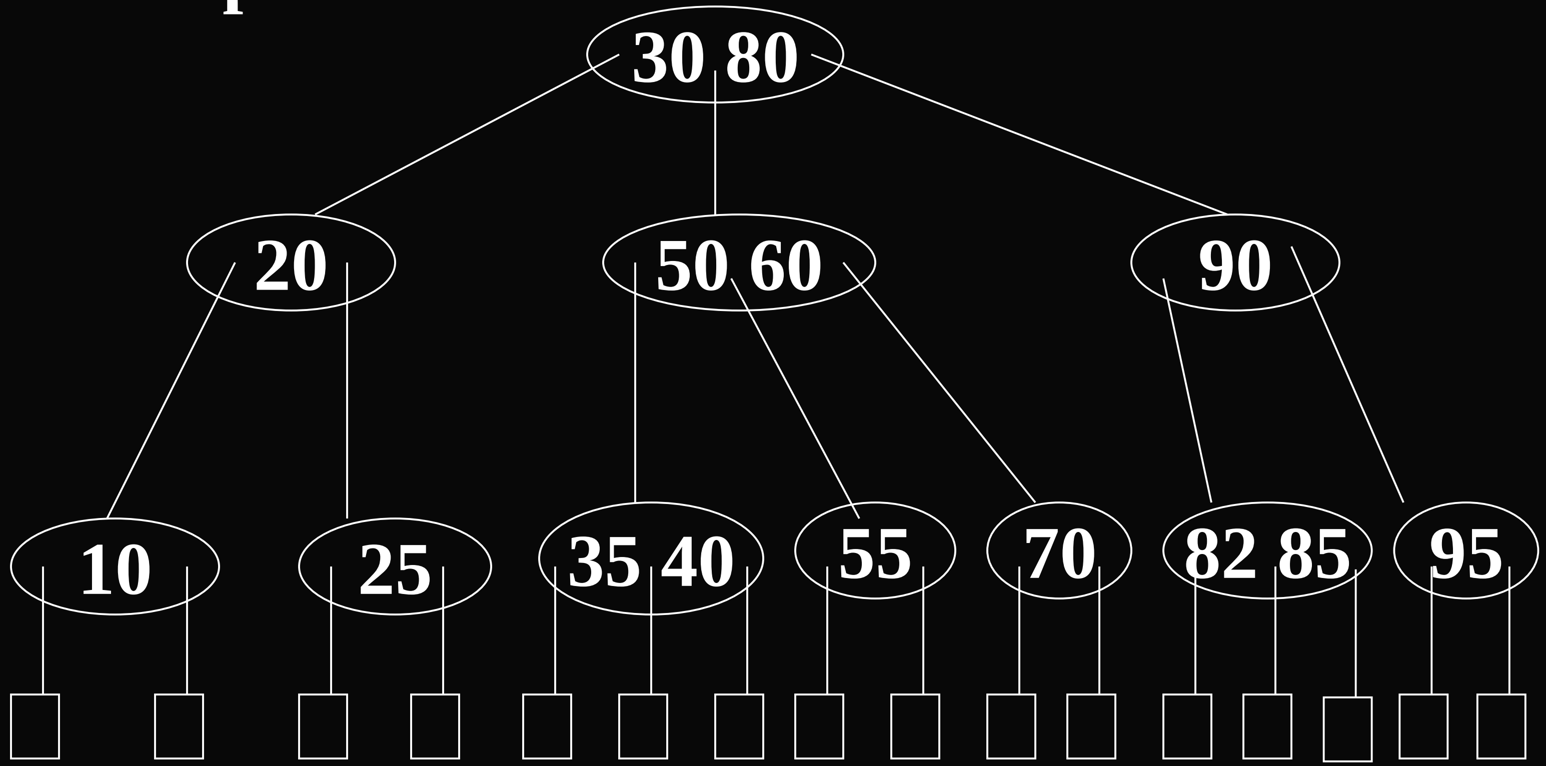
4.3 B-TREES

b)delete a key in a node in the above level

- **Delete it**
- **Replace it with the smallest key in the right subtree or the largest key in the left subtree**
- **Because delete a key in the leaf node , do the adjust mentioned in a)**

4.3 B-TREES

Example:



A B-TREE of order 3

Delete 80, then replace it with 82 or 70, delete 82 or 70 at last

4.3 B-TREES

4)Node structure

$$\boxed{s, c_0, (e_1,c_1),(e_2, c_2)\ldots\ldots(e_s, c_s)}$$

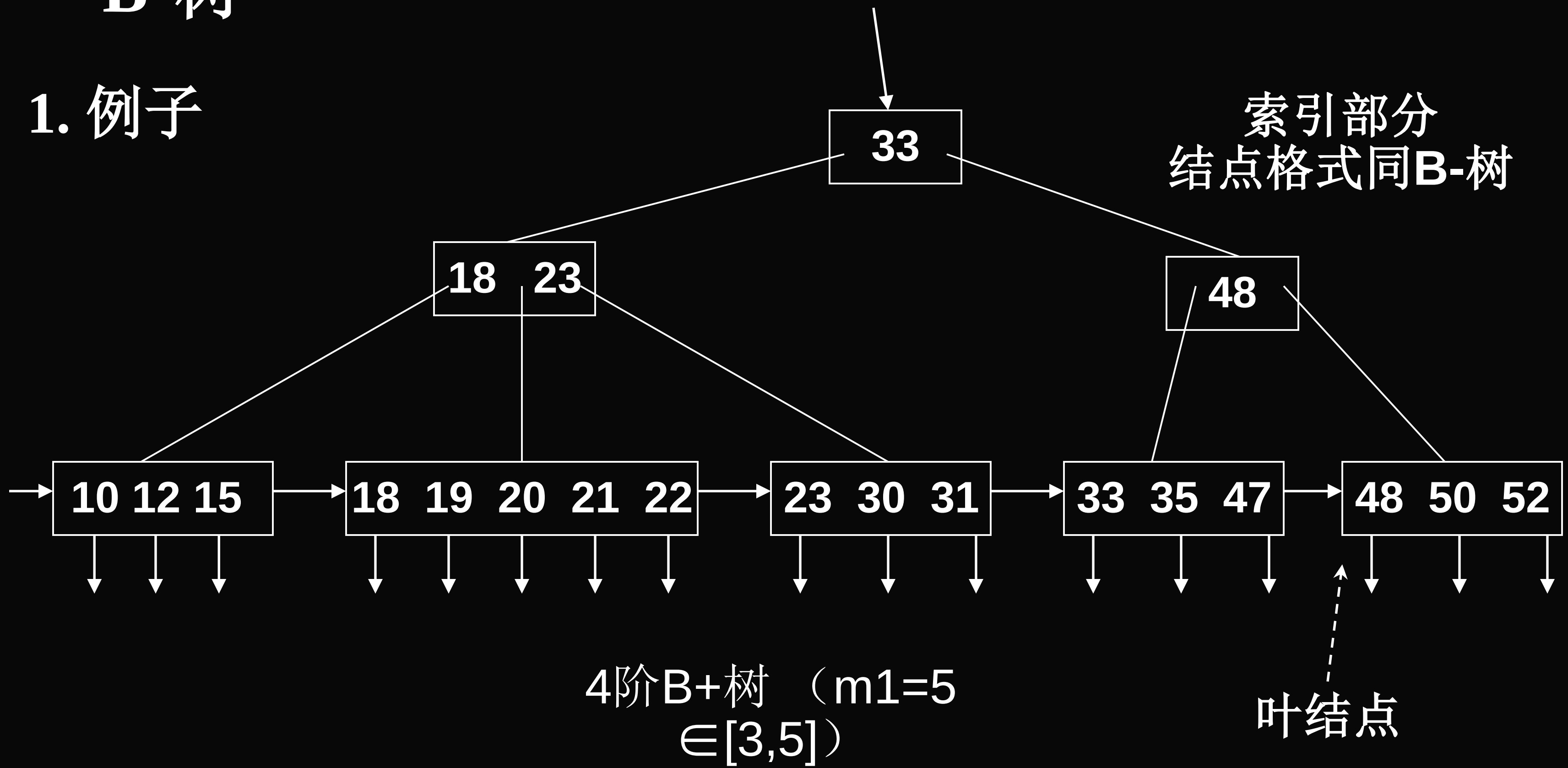
- **S is the number of elements in the node**
- **e_i are the elements in ascending order of key**
- **C_i are children pointers**

4.3 B-TREES

- **Supplements-----B$^+$ tree**

B⁺树

1. 例子



与B-树有所不同：1. 关键码的分布，只分布在叶结点上

2. 叶结点的定义,不一定符合m阶，它依赖于关键码字节数与指针字节数而为m1

2. 定义：

是B-树的一种变形

1）树中每个非叶结点最多有m棵子树

2）根结点（非叶结点）至少有2棵子树

3）除根结点外，每个非叶结点至少有⌈m/2⌉棵子树；有n棵子树的非叶结点有n-1个关键码

4）所有叶结点都处于同一层次上，包含了全部关键码及指向相应数据对象存放地址，关键码按关键码从小到大顺序链接

5）每个叶结点中子树棵树n可以＞m，也可以＜m。假设叶结点可容纳的最大关键码数为m1，则指向对象的指针数也有m1，这时子树棵数n应满足（⌈m1/2⌉，m1 ）

6）根结点本身又是叶结点，则结点格式同叶结点

3. 特点：

有两个头指针：

一个指向B^+树的根结点，可以进行自顶向下的随机搜索；

一个指向关键码最小的叶结点，进行顺序搜索；

4. B^+树的运算：搜索（查找），插入，删除

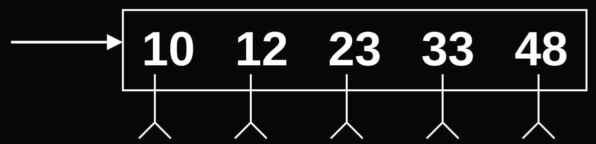
搜索：基本上同B-树，所不同的是一直查到叶结点上的这个关键码为止

插入：仅在叶结点上进行。每插入一关键码，判别子树棵树＞m1，如果大于，则将该结点分裂：

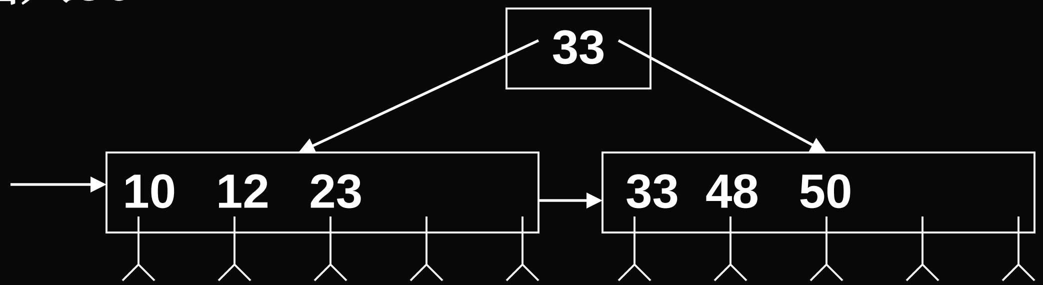
$\lceil (m1+1)/2 \rceil, \lceil (m1+1)/2 \rceil$

问题变为传递到索引结点上可能的分裂，这时上限以m来确定（同B-树）

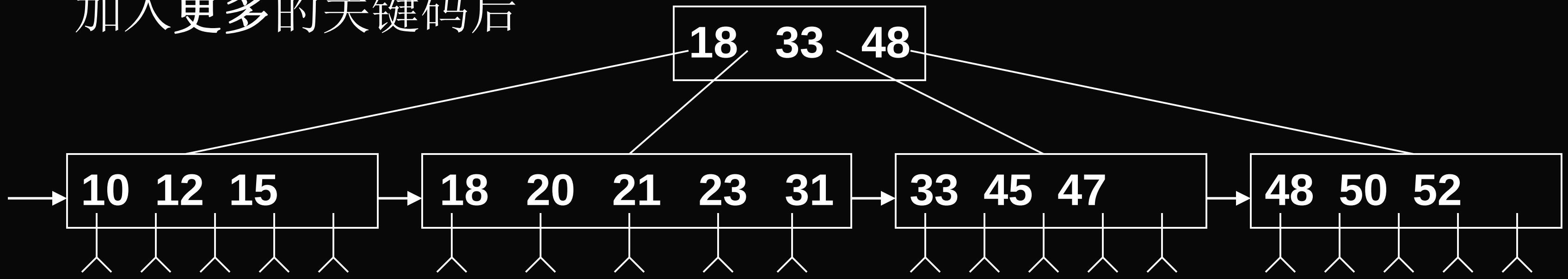
例子： 4阶B+树，叶结点包含5个关键码



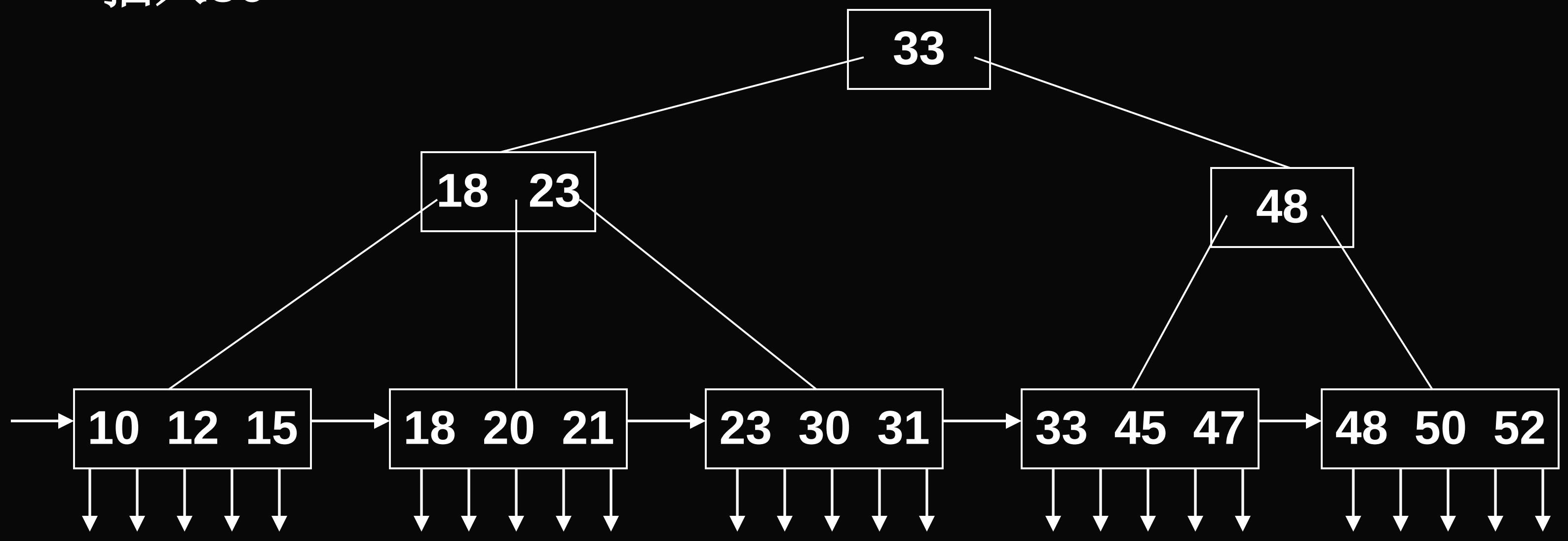
插入50



加入更多的关键码后



插入30



从这里可以看到，索引结构也变了

∴称为动态索引结构

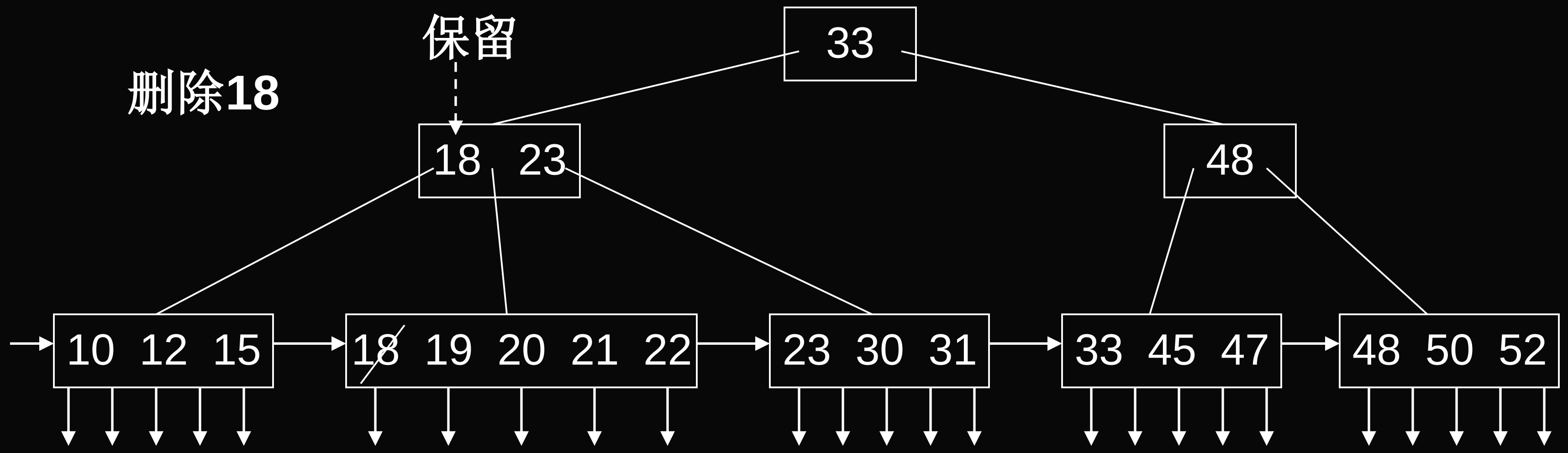
删除：仅在叶结点进行

在叶结点上删除一个关键码后要保证结点中的子树棵数仍然不小于$\lceil m_1/2 \rceil$.

删除操作与B-树类似，但上层索引中的关键码可保留，作为引导搜索的“分界关键码”的作用.

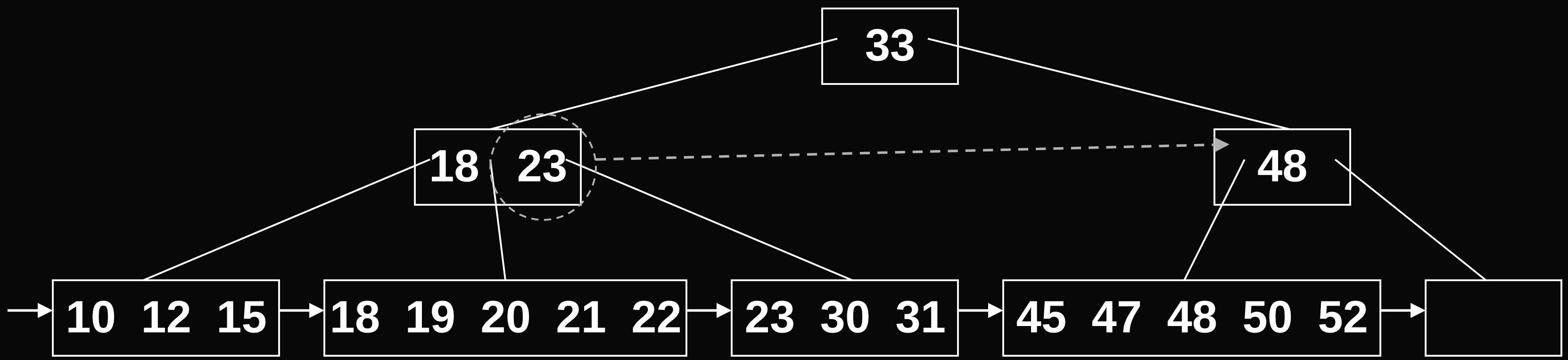
例子：4阶B+树，$m_1=5$

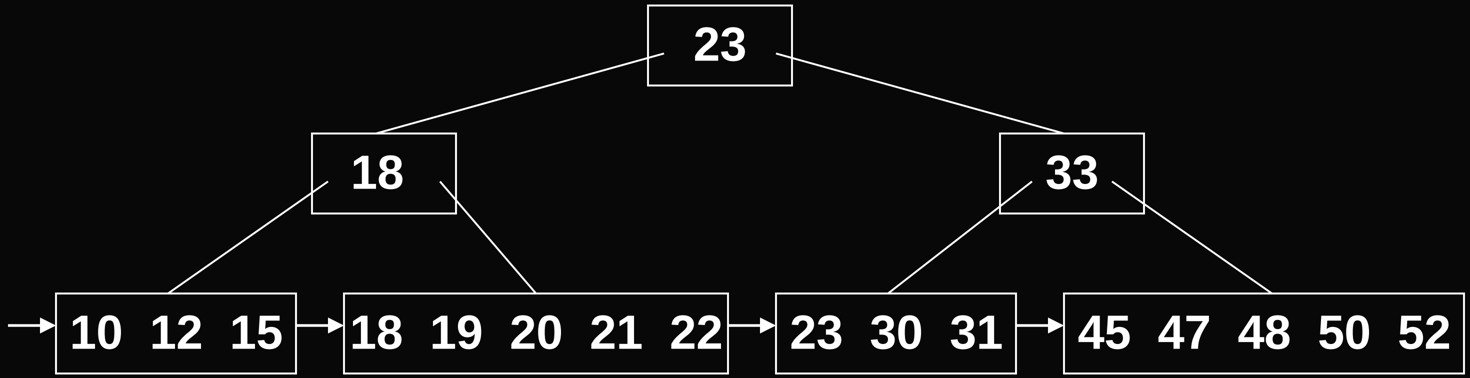
删除**18**



删除后小于下限$\lceil m_1/2 \rceil$，必须做结点的调整或合并工作

在上图中删除12，向右邻借，18借过去后，上层索引改为19；
在上图中删除33，两结点合并。



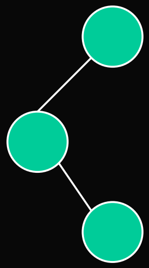


Chapter 4.1

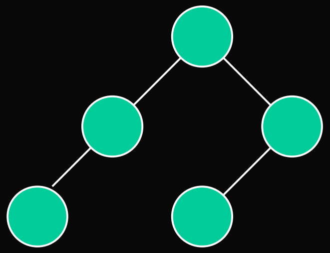
2009年统考题:

6. 下列二叉排序树中，满足平衡二叉树定义的是

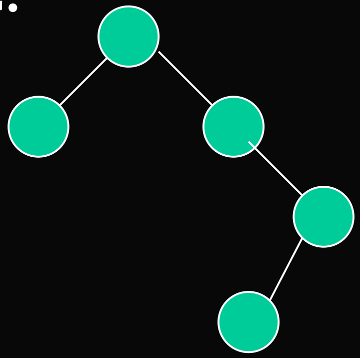
A.



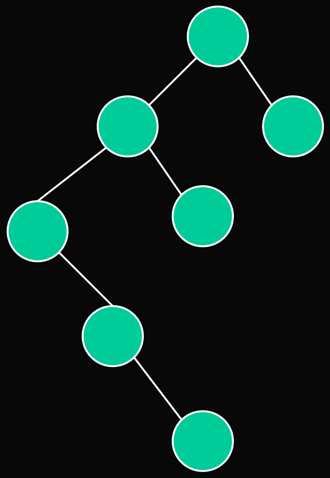
B.



C.



D.



Chapter 4.1

2009年统考题:

7. 下列叙述中, 不符合m阶B 树定义要求的是

A. 根结点最多有m棵子树　　B. 所有叶结点都在同一层上

C. 各结点内关键字均升序或降序排列

D. 叶结点之间通过指针链接

Chapter 4.1

Exercise:

1. a. Show the result of inserting 3, 1, 4, 6, 9, 2, 5, 7 into an initially empty binary search tree.

b. Show the result of deleting the root.

2. 写一递归函数实现在带索引的二叉搜索树（IndexBST)中查找第k个小的元素。

3. 对一棵空的AVL树，分别画出插入关键码为{ 16，3，7，11，9，28，18，14，15}后的AVL树。

4. 设计算法检测一个二叉树是不是一个二叉搜索树.

5. 设有序顺序表中的元素依次为017,094,154,170,275,503,509,512,553,612,677,765,897,908. 试画出对其进行二分法搜索时的判定树, 并计算搜索成功的平均搜索长度。

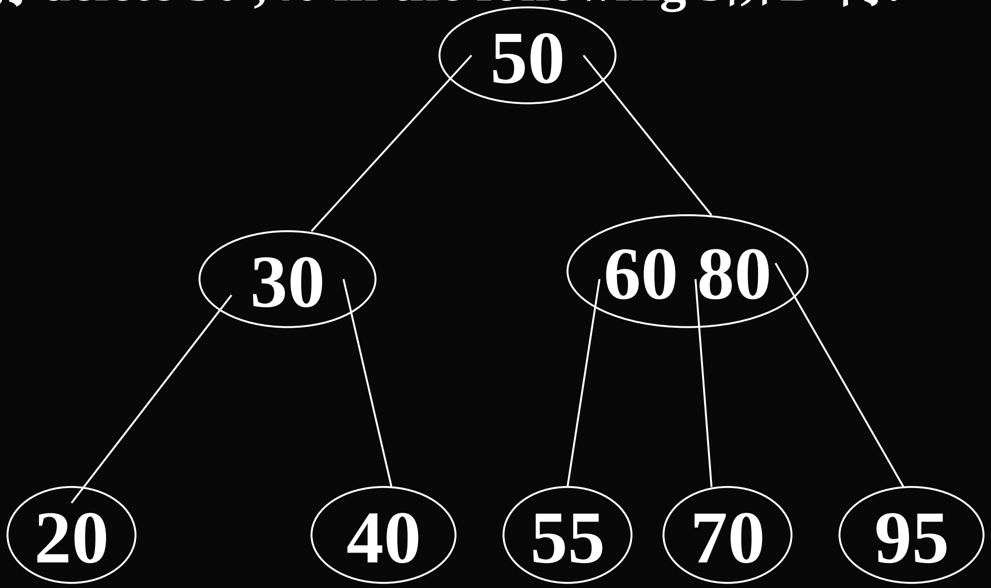
6. 在一棵表示有序集S 的二叉搜索树中, 任意一条从根到叶结点的路径将S分为三部分: 在该结点左边结点中的元素组成集合S1; 在该路径上的结点中的元素组成集合S2; 在该路径右边结点中的元素组成集合S3, S=S1US2US3. 若对于任意的a S1, b S2, c S3, 是否总有a<=b<=c? 为什么?

7. 将关键码DEC, FEB, NOV, OCT, JUL, SEP, AUG, APR, MAR, MAY, JUN, JAN 依次插入到一棵初始为空的AVL 树中, 画出每插入一个关键码后的AVL 树, 并标明平衡旋转的类型.

*8. 对于一个高度为h 的AVL 树, 其最少结点数是多少? 反之, 对于一个有n 个结点的AVL 树, 其最大高度是多少? 最小高度是多少?

Chapter 4.1

9. 分别 delete 50 ,40 in the following 3阶B-树.



10. 分别画出插入65, 15, 40, 30后的3阶B-树。