Chapter 8

The Disjoint Set ADT

Equivalence Class

1) Definition of Equivalence Class:

Suppose we have a set U={1,2,…,n} of n elements and a set R={(i_1,j_1), (i_2,j_2)…… (i_r,j_r)}of r relations. The relation R is an equivalence relation iff the following conditions are true(symbol'≡' represent the equivalence relation on sets, x,y,z are elements in set) :

- Reflexive $x \equiv x$.
- Symmetric $x \equiv y, y \equiv x$
- Transitive $x \equiv y$ and $y \equiv z$,then $x \equiv z$

Equivalence Class

例如：

判别3个数a, b, c能否构成三角形的三条边？

能构成三角形的等价类：

{ (3,4,5), (4,5,6), (5,6,7), …..}

不能构成三角形的等价类：

{ (1,2,3), (2,3,5), ….. }

Equivalence Class

2) Example:

set s= {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

pairs of equivalence:

(0 4),(3 1),(6 10),(8 9),(7 4),(6 8),(3 5),(2 11)，(11 0)

Initial:{0}， {1}， {2}， {3}， {4}， {5}， {6}， {7}， {8}，
{9}, {10}， {11}

0≡4 {0， 4}， {1}， {2}， {3}， {5}， {6}， {7}， {8}， {9},
{10}， {11}

3 ≡1 {0， 4}， {1， 3}， {2}， {5}， {6}， {7}， {8}， {9}，
{10}, {11}

Equivalence Class

6 ≡10 {0, 4}, {1, 3}, {2}, {5}, {6, 10}, {7}, {8},
{9}, {11}

8 ≡9 {0, 4}, {1, 3}, {2}, {5}, {6, 10}, {7},
{8, 9}, {11}

7 ≡ 4 {0, 4, 7}, {1, 3}, {2}, {5}, {6, 10}, {8, 9}, {11}

6 ≡ 8 {0, 4, 7}, {1, 3}, {2}, {5}, {6, 8, 9, 10}, {11}

3 ≡ 5 {0, 4, 7}, {1, 3, 5}, {2}, {6, 8, 9, 10}, {11}

2 ≡ 11 {0, 4, 7}, {1, 3, 5}, {2, 11}, {6, 8, 9, 10}

11 ≡ 0 {0, 4, 7, 2, 11}, {1, 3, 5}, {6, 8, 9, 10}

Equivalence Class

3) Online equivalence class operation

- **Combine(a,b) : combine the equivalence classes that contains elements a and b into a single class**

- **Find(e) : determine the class that currently contains element e.**

Combine(a,b) is equivalent to

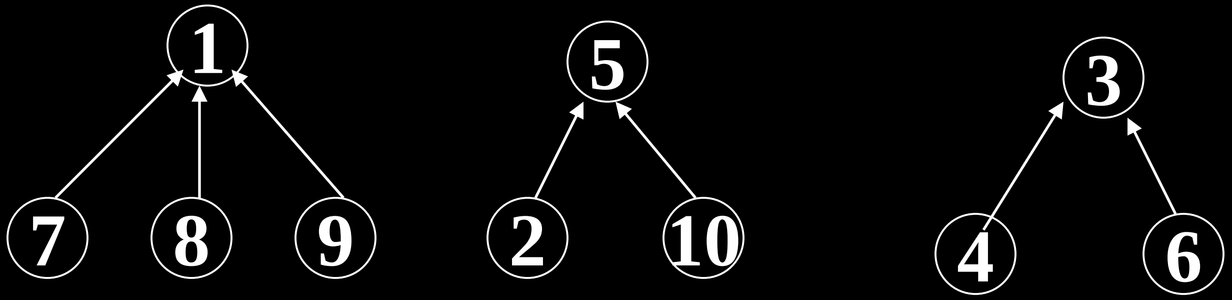
```
i=Find(a); j=Find(b); if(i!=j) Union(i,j);
```

Equivalence Class

4) Tree Representation(Union-Find sets)

Example:

S_1={1,7,8,9}, S_2={5,2,10}, S_3={3,4,6},they all belong to S={1,2,3,……,10}



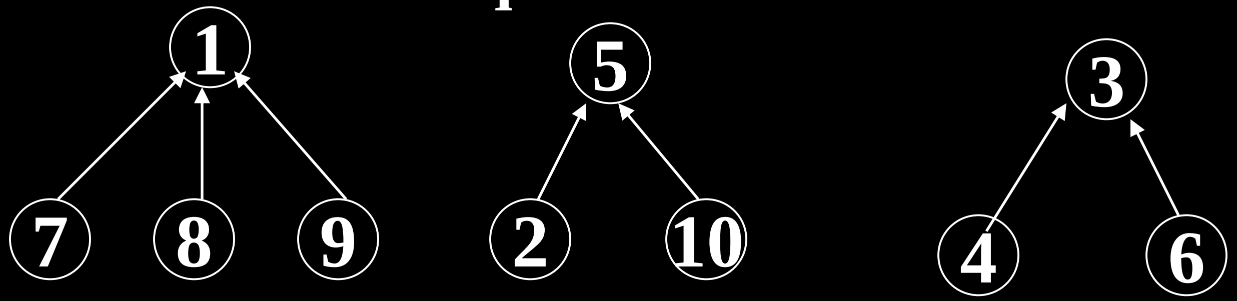
	0	1	2	3	4	5	6	7	8	9	10
parent	▨	0	5	0	3	0	3	1	1	1	5

Equivalence Class

simple tree solution to union-find problem

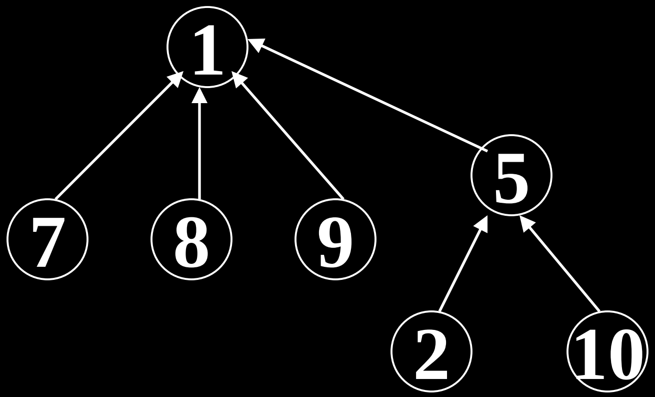
```
   void Initialize(int n)
   {  parent=new int[n+1];
      for(int e=1;e<=n;e++)
         parent[e]=0;
     }

int Find(int e)
  {  while(parent[e])
             e=parent[e];
      return e;
  }
   void Union(int i, int j)
  { parent[j]=i;
  }
```

Equivalence Class



	0	1	2	3	4	5	6	7	8	9	10
parent		0	5	0	3	$1\cancel{0}$	3	1	1	1	5

Union (1,5)



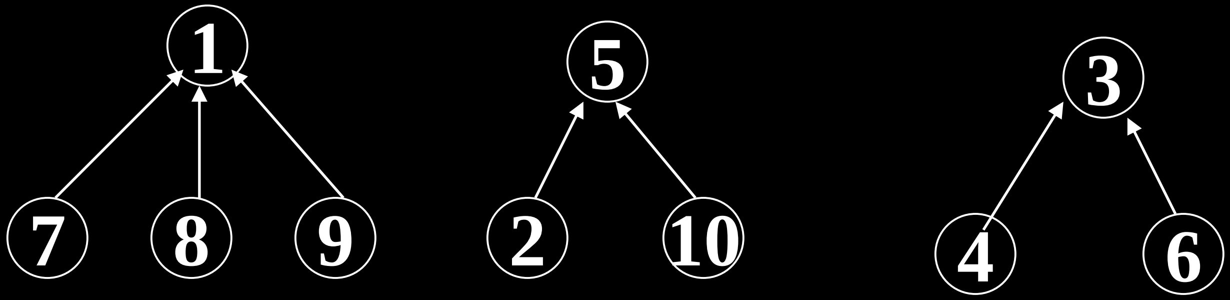
Equivalence Class

Java

```
 public class DisjSets
{  public DisjSets( int numElements )
    public void union( int root1, int root2 )
    public int find( int x )
    private int [ ] s;
}

public DisjSets( int numElements )
{   s = new int [ numElements ];
     for( int i = 0; i < s.length; i++ )
          s[ i ] = -1; //一个根结点
}
```

Equivalence Class

```
 public void union( int root1, int root2 )
{   s[ root2 ] = root1;
}
 public int find( int x )
{    if( s[x ] < 0 )
          return x;
     else
          return find( s[ x ] );
}
```



	0	1	2	3	4	5	6	7	8	9	10
S:		-1	5	-1	3	-1	3	1	1	1	5

Equivalence Class

5) Performance Evaluation

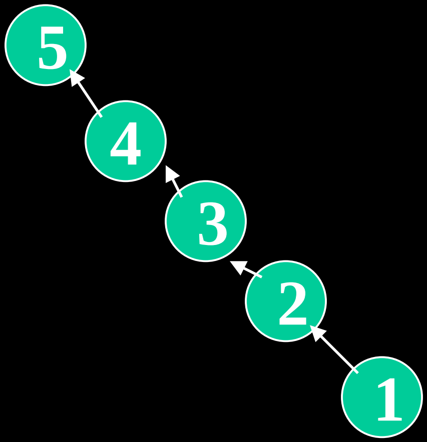
Time complexity: Find-- O(h),

Union-- θ(1)

Assume that u times unions and f times finds are to be performed, f>u,

in the worst case a tree with m elements can have a height of m:

Union(2,1),Union(3,2),Union(4,3),Union(5,4)…



Equivalence Class

6) Performance Enhancement

- ✶**improve Union in order to decrease the time each find take, so that the height of tree will not increase linearly .**
- ✸**.Improvement of *Find* –path compression**

Equivalence Class

✶improve Union

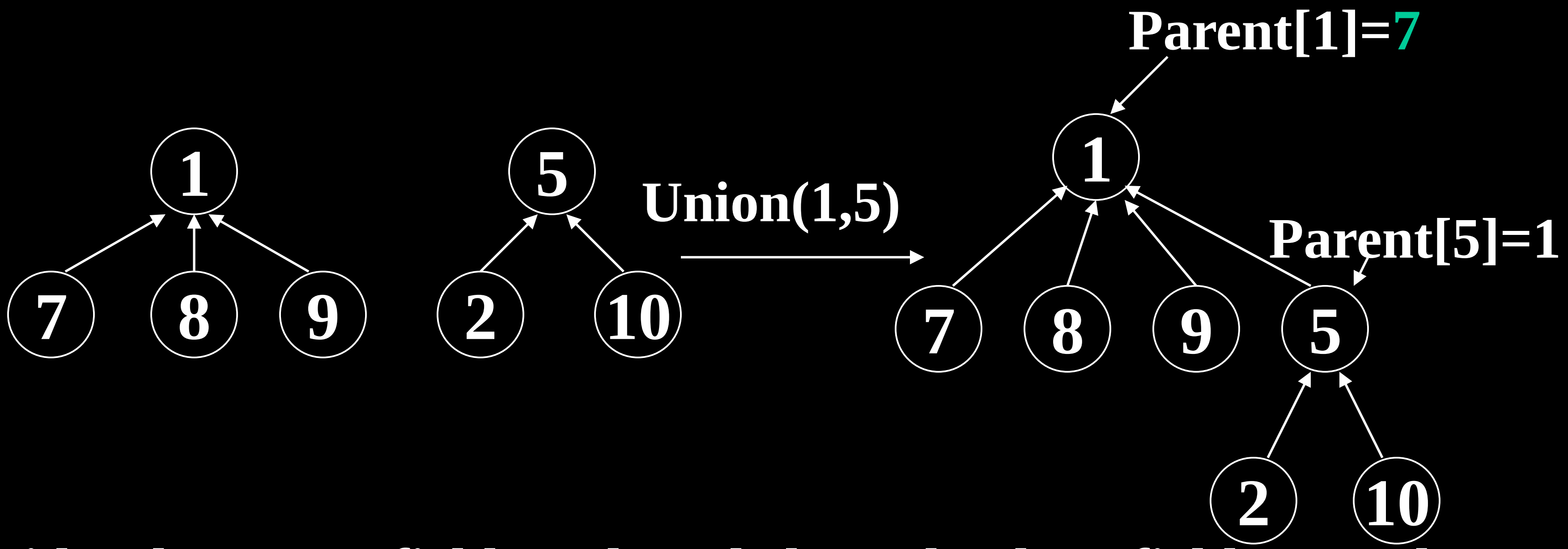
two rules:

- Weight rule: if the number of nodes in tree i is less than the number in tree j, then make j the parent of i; otherwise,make i the parent of j.

- Height rule: if the height of tree i is less than that of tree j, then make j the parent of i; otherwise,make i the parent of j.

Equivalence Class

Let’s discuss the weight rule(C++) :



Besides the *parent* field, each node has <u>a boolean field *root*</u> .The *root* field is <u>true</u> iff the node is presently a root node.The *parent* field of <u>each root node</u> is used to keep a count of the total number of nodes in the tree.

两个数组:　　**Parent,　root**　黑板上具体讲。

Equivalence Class

Union with the weight rule

```
void Initialize(int n)
{ root=new bool[n+1];
  parent=new int[n+1];
  for(int e=1;e<=n;e++)
  { parent[e]=1;
    root[e]=true;
  }
}
int Find(int e)
{ while(!root[e])
      e=parent[e];
  return e;
}
```

Equivalence Class

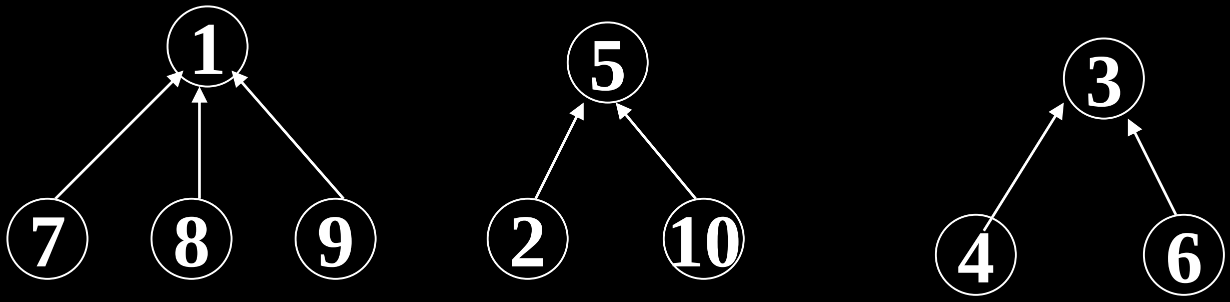
```
void Union(int i, int j)
{  if(parent[i]<parent[j])  //  i becomes subtree of j
      { parent[j]=parent[j]+parent[i];
         root[i]=false;
         parent[i]=j;
       }
    else { parent[i]=parent[i]+parent[j];
             root[j]=false;
             parent[j]=i;
          }
}
```

Equivalence Class

Java(高度规则）

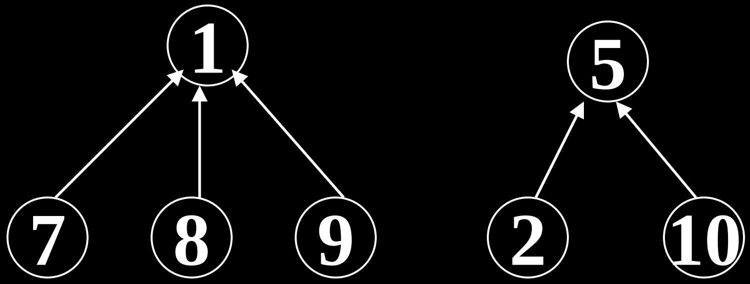
用一个数组来实现，根结点中放负数，而且是代表高度。

```
public void union( int root1, int root2 )
{    if(  s[ root2 ] < s[ root1 ] )
             s[ root1 ] = root2;
      else {   if( s[ root1 ] = = s[ root2 ] )
                      s[ root1 ]--;
                  s[ roo2 ] = root1;} }
```



	0	1	2	3	4	5	6	7	8	9	10
S:		-2	5	-2	3	-2	3	1	1	1	5

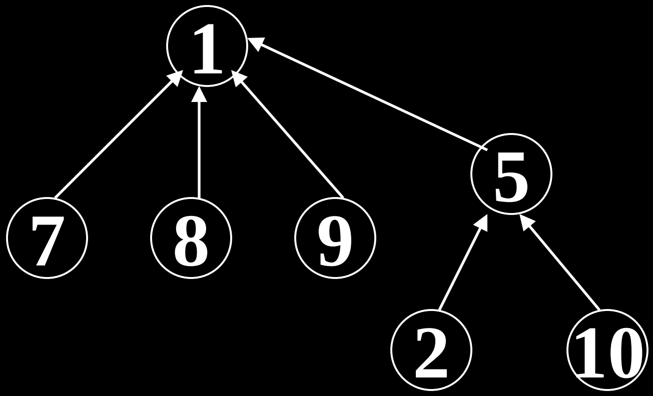
Equivalence Class



union (1, 5)

s[root1]--

s[root2] = root1

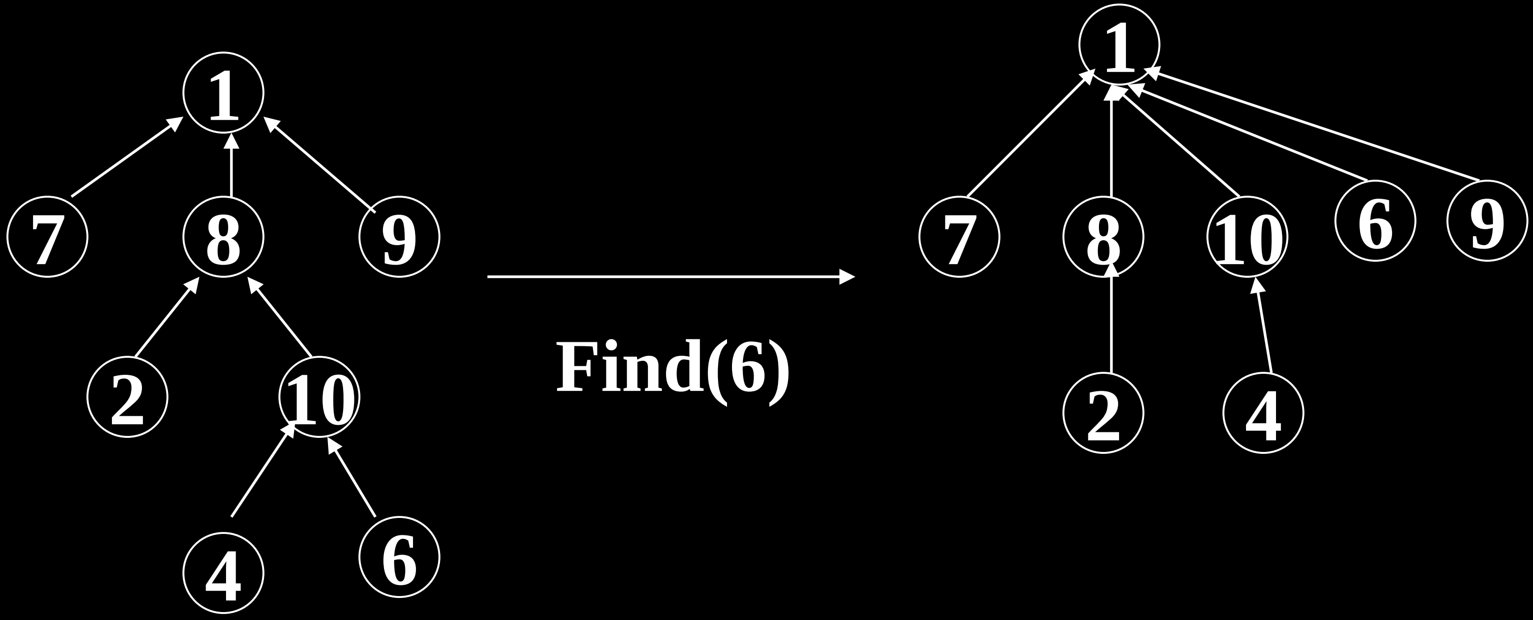


Equivalence Class

✷.Improvement of *Find* –path compression

When processing a equivalence pair, we need to operate *Find* twice, *WeightUnion* once.

Example of improvement:



Equivalence Class

```
int Find( int e) { /* C++  */ }
{  int j=e;
    while(!root[j])  j=parent[j];
    int f=e;
    while(f!=j)
      { int pf=parent[f];   parent[f]=j;  f=pf; }
}
```

Equivalence Class

Java

```
public int find( int x )
{    if( s[ x ] < 0 )
             return x;
      else
            return s[ x ] = find( s[ x ] );
}
```