Understanding Analysis Solutions

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Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Proof AFSOC $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$ such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in lowest reduced terms. Then we can square both sides, yielding $3 = \left(\frac{m}{n}\right)^2 \Longrightarrow 3n^2 = m^2$. Now, we know m^2 is a multiple of 3 and thus m must also. Then, we can write $m = 3k$, and derive

$$\begin{aligned}(\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2\end{aligned}$$

Similar to before, we come to the conclusion that n is a multiple of 3. However, this is a contradiction since m, n are both multiples of 3 and we assumed $\frac{m}{n}$ was in lowest terms. Thus, we conclude $\sqrt{3}$ is irrational.

The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ implies that m is a multiple of 4, since we have

$$4n^2 = m^2 \quad \Rightarrow \quad 2n = m,$$

so we cannot reach our contradiction that m/n is not in lowest terms.

Exercise 1.2.2

(a) False. Consider

$$A_n = \left[0, \frac{1}{n}\right).$$

Then

$$\bigcap_{n=1}^{\infty} A_n = \{0\}.$$

(b) True. Since $\forall i, A_i \subseteq A_1$, $\exists x$ such that $\forall i, x \in A_i$. Therefore, the intersection cannot be empty. Then, every set is finite, and the intersection of any number of finite sets will be finite.

(c) False. Consider $A = \{1, 2\}, B = \{1\}, C = \{2, 3\}$.

$$\{1,2\} \cap \big(\{1\} \cup \{2,3\}\big) = \{1,2\} \neq \big(\{1,2\} \cap \{1\}\big) \cup \{2,3\} = \{1,2,3\}$$

(d) True. Intersection is associative.

(e) True. Intersection is distributive over union.

PROOF We will prove

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.1}$$

by set inclusion.

- Suppose $x \in A \cap (B \cup C)$. By the definition of intersection, we know $x \in A$ and $x \in B \cup C$, the latter which means $x \in B$ or $x \in C$.

  We can consider 2 cases for x,

  1. $x \in B$. Then we know $x \in A$ and $x \in B$, so $x \in A \cap B$ and therefore $x \in (A \cap B) \cup (A \cap C)$
  2. $x \in C$. Symmetric to the case above.

  in all cases, we see $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$, so

  $$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

- Suppose $x \in (A \cap B) \cup (A \cap C)$. Then we have two cases

  1. $x \in A \cap B$. This means $x \in A$ and $x \in B$. If $x \in B$, then $x \in B \cup C$, since $B \subseteq B \cup C$. Putting these facts together, we see $x \in A \cap (B \cup C)$.
  2. $x \in A \cap C$. Symmetric to the case above.

  in all cases, we see $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$, so

  $$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

Exercise 1.2.3

(a) If $x \in (A \cap B)^c$, then we have cases

- $x \in B$ and $x \notin A$. Then $x \notin A$ implies $x \in A^c \Rightarrow x \in A^c \cup B^c$.
- $x \in A$. Symmetric to above.
- $x \notin A$ and $x \notin B$. Then $x \in A^c$ so $x \in A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$, then we have cases

- $x \in A^c$. Then $x \notin A$ so x cannot be in the intersection of A and B, so $x \in (A \cap B)^c$.
- $x \in B^c$. Symmetric to above.

(c) Proof for $(A \cup B)^c = A^c \cap B^c$ pretty similar to above.

Exercise 1.2.4

We are verifying the triangle inequality with a, b.

(a) If a, b have the same sign, then

$$\begin{aligned} |a+b| &= a+b \\ |a|+|b| &= a+b \\ \Rightarrow\ |a+b| &= |a|+|b| \\ \Rightarrow\ |a+b| &\leq |a|+|b| \end{aligned}$$

(b) - $a \geq 0, b < 0$.

$$\begin{aligned} |a+b| &\leq |a| \\ &\leq |a|+|b| \end{aligned}$$

- $a+b \geq 0$. At most one of a, b is negative. If they are both positive, then we have already shown this in part (a). Otherwise, WLOG a is negative. Then

$$\begin{aligned}|a+b| &\leq |b| \\ &\leq |a|+|b|\end{aligned}$$

Exercise 1.2.5

(a) Substitute in $b' = -b$ into the triangle inequality.

(b) Easy to prove directly without using triangle inequality. **TODO**.

A direct proof will look something like:

- If a, b are the same sign, then equality holds
- If a, b are different signs, then if b is negative, then $|a-b| = |a|+|b|$, and if a is negative, then $|a-b| = |a|+|b|$, both of which bound $\big||a|-|b|\big|$.

Exercise 1.2.6

(a) Yes, since $f(A \cap B) = [1,4] = [0,4] \cap [1,16] = f(A) \cap f(B)$. This is by coincidence though, as we will later see. Yes, since $f(A \cup B) = [0,16] = [0,4] \cup [1,16] = f(A) \cup f(B)$.

(b) Choose $A = [-2,0], B = [0,2]$

(c) Suppose $x \in g(A \cap B)$, then $\exists x' \in A \cap B$ such that $g(x') = x$. Since $x' \in A$ and $x' \in B$, we know $x = g(x') \in g(A), g(B)$, so we conclude $x \in g(A) \cap g(B)$.

(d) Equality. **TODO** too lazy to write out the proof. Similar to above.

Exercise 1.2.7

(a) **TODO** I don't think we want to include $x \in \mathbb{I}$...

$$f^{-1}(A) = [0,2] \tag{1.2}$$

$$f^{-1}(B) = [0,1] \tag{1.3}$$

We see $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ is also true.

(b) **TODO**

Exercise 1.2.8

Negating statements. Took some liberties. Also notice that these statements are not necessarily true.

(a) There exists a real number satisfying $a < b$, such that $\forall n \in \mathbb{N}$, $a + 1/n \geq b$.

(b) There exists two distinct real numbers such that there is not a rational number between them.

(c) There exists a natural number $n \in \mathbb{N}$ such that $\sqrt{n}$ is not a natural number nor an irrational number.

(d) There exists a real number $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $n \leq x$.

Exercise 1.2.9

We are given the sequence

$$x_1 = 1, x_{n+1} = \frac{1}{2}x_n + 1 \tag{1.4}$$

and want to show $\forall i \geq 1, x_i < 2$.

We can show this with a direct proof of summation.

An alternative that the book probably wants to see is using **induction**.

- Base Case: $x_1 = 1 < 2$
- Inductive case. Assume $\forall i < n+1, x_i < 2$. Then $x_i/2 + 1 < 2$ since $x_i/2 < 1$.

- By induction our original claim is proved.

Exercise 1.2.10

(a) Similar to Exercise 1.2.9. $y_n < 4$ means $(3/4)y_n < 3$ so $(3/4)y_n + 1 < 4$

(b) In brief,

$$\begin{aligned} y_n &\leq \frac{3}{4}y_n + \frac{1}{4}y_n && (1.5) \\ &< \frac{3}{4}y_n + 1 && \text{(Using } y_n < 4\text{)} \\ &< y_n + 1 && \text{(Sequence definition)} \end{aligned}$$

Exercise 1.2.11

A combinatorial argument is that in order to construct a set, we have 2 choices for every element, to include it or not to. Therefore, we have

$$\prod_{i=1}^{n} 2 = 2^n$$

Exercise 1.2.12

(a) We know that $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$. So if we are trying to show $(A_1 \cup A_2 \cup A_3)^c = (A_1 \cup A_2)^c \cap A_3^c = A_1^c \cap A_2^c \cap A_3^c$. Induction lets us apply the property on smaller parts of our expression.

(b) Induction only proves the property for some $n \in \mathbb{N}$, i.e. some finite n. It is not shown for an infinite n.

(c) **TODO**. Sketch: If x is not in the union of all the A_n, then x cannot be part of any particular A_n either, or else it would be in the union.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) We compute the additive inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned}0+0&\equiv 0\\1+4&\equiv 0\\2+3&\equiv 0\\3+2&\equiv 0\\4+1&\equiv 0\end{aligned}$$

(b) We compute the multiplicative inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned}1\times 1&\equiv 1\\2\times 3&\equiv 1\\3\times 2&\equiv 1\\4\times 4&\equiv 1\end{aligned}$$

(c) $\mathbb{Z}_4$ is not a field because multiplicative inverses do not exist for every single element. For example, 2 multiplied with any number is even, which cannot $\equiv 1 \pmod 4$.

We conjecture that $\mathbb{Z}_n$ always has additive inverses and only has multiplicative inverses if n is prime.

Exercise 1.3.2

We are writing a formal definition for the *infimum* of a set.

(a) $s = \inf A$ means

i) s is a lower bound for A

ii) if b is any lower bound for A, then $b \leq s$

(b) If $s \in \mathbb{R}$ is a lower bound for $A \subseteq \mathbb{R}$, then $s = \inf A$ iff $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$.

Proof ($\Rightarrow$) If $s = \inf A$, then s is the greatest lower bound for A, meaning any $s + \epsilon$ for $\epsilon > 0$ will be greater than some element of A, otherwise $s + \epsilon$ is a greater lower bound and leads to a contradiction that$s \neq \inf A$.

($\Leftarrow$) If $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$, then since s is a lower bound, $\forall b > s$, b will not be a lower bound for A since if, $b > s$, then we can choose $\epsilon = b - s > 0$, and we know that $\exists a \in A$ where $a < s + \epsilon < b$, which means b is not a lower bound. Thus, all lower bounds b must be such that $b \leq s$, and we conclude $s = \inf A$.

Exercise 1.3.3

(a) Since $\inf A$ is a lower bound for A, we know $\inf A \in B$. Now, we need to show $\inf A$ is the supremum of B. $\inf A$ is the least upper bound for B, since if $\exists b \in B, b > \inf A$, then we know that this b is not a lower bound for A, so no such b exists.

(b) There might be a typo in this question. I think the question was meant to read "explain why there is no need to assert that the greatest *lower bound* in the Axiom of Completeness." In this case, the answer would be that the Axiom of Completeness already implies the greatest lower bound property, so there is no need to explicitly state it.

(c) We can take the negative of all elements in A, find $\sup A$, and then negate again to get $\inf A$.

Exercise 1.3.4

If $B \subseteq A$, then

$$\begin{aligned}\sup A = s &\geq a, \forall a \in A\\ s &\geq b, \forall b \in B && \text{(since } B \subseteq A)\\ \Rightarrow\ s &\geq \sup B. && \text{(since } s \text{ is an upper bound for } B)\end{aligned}$$

Exercise 1.3.5

(a)

$$\begin{aligned} & s = \sup(c + A) \\ \Rightarrow & s \text{ is the least upper bound for } c + A \\ \Rightarrow & s - c \text{ is the least upper bound for } A \\ \Rightarrow & s - c = \sup A \\ & s = c + \sup A \end{aligned}$$

(b)

$$\begin{aligned} & s = \sup(cA) \\ & \Rightarrow s \text{ is the least upper bound for } cA \\ & \Rightarrow \frac{s}{c} \text{ is the least upper bound for } A \\ & \Rightarrow \frac{s}{c} = \sup A \\ & s = c \sup A \end{aligned}$$

(c) If $c < 0$, $\sup(cA) = -c \inf(A)$.

Exercise 1.3.6

(a) $\sup : 3; \inf : 1$

(b) $\sup : 1; \inf : 0$

(c) $\sup : \frac{1}{2}; \inf : \frac{1}{3}$

(d) $\sup : 9; \inf : \frac{1}{9}$

Exercise 1.3.7

If $a \geq a', \forall a' \in A$, and $a \in A$, then

$$\forall \epsilon > 0, a - \epsilon < a, \tag{1.6}$$

so a is the least upper bound for A, and $a = \sup A$.

Exercise 1.3.8

Let

$$\epsilon = \sup B - \sup A > 0. \tag{1.7}$$

since $s_b = \sup B$, $\exists b \in B \mid b > s_b - \epsilon/2$. Since $s_b - \frac{\epsilon}{2} > \sup A$, then $b \geq \sup A$, so this $b \in B$ is an upper bound for A.

Exercise 1.3.9

(a) True, take the largest element in the set as the supremum.

(b) False, $\sup(0, 2) = 2$, but $2 > a \in (0, 2)$, but $\sup A = 2 \not< 2 = L$.

(c) False $A = (0, 2), B = [2, 3)$. We have that $\sup A = \inf B$

(d) True.

(e) False, take $A = B = (0, 2)$.

1.4 Consequences of Completeness

Exercise 1.4.1
If $a < 0$, then we have two cases,

1. If $b > 0$, then $a < 0 < b$.
2. If $b = 0$, then we can take $-b, -a$, which satisfies $0 \leq -b < -a$, and apply Theorem 1.4.3.

Exercise 1.4.2
(a) If $a, b \in \mathbb{Q}$, then

$$\begin{aligned} a &= \frac{a_1}{a_2} \\ b &= \frac{b_1}{b_2} \\ \implies a + b &= \frac{a_1 b_2 + a_2 b_1}{a_2 b_2} \in \mathbb{Q} \end{aligned}$$

(b) We can use contradiction,

- AFSOC $a + t \in \mathbb{Q}$. Let $a + t = \frac{m}{n}$. We know $a = \frac{a_1}{a_2}$ since $a \in \mathbb{Q}$, so

$$\begin{aligned} a + t &= \frac{m}{n} \\ t &= \frac{m}{n} - \frac{a_1}{a_2} \in \mathbb{Q}, \end{aligned}$$

which is a contradiction since we are given $t \in \mathbb{Q}$. Therefore, we conclude $a + t \in \mathbb{I}$.

- AFSOC $at \in \mathbb{Q}$. Let $at = \frac{m}{n}$. We know $a = \frac{a_1}{a_2}$ since $a \in \mathbb{Q}$, so

$$\begin{aligned} at &= \frac{m}{n} \\ t &= \frac{m}{n} \cdot \frac{a_2}{a_1} \in \mathbb{Q}, \end{aligned}$$

which is a contradiction since we are given $t \in \mathbb{Q}$. Therefore, we conclude $at \in \mathbb{I}$

(c) $\mathbb{I}$ is not closed under addition or multiplication.

$$\begin{aligned} \left(3 - \sqrt{2}\right) + \left(3 + \sqrt{2}\right) &= 6 \notin \mathbb{I} \\ \left(3 - \sqrt{2}\right) \cdot \left(3 + \sqrt{2}\right) &= 5 \notin \mathbb{I} \end{aligned}$$

Exercise 1.4.3
We can apply Theorem 1.4.3, to find $a < q < b, q \in \mathbb{Q}$, and then subtract an irrational number such as $\sqrt{2}$ to end up at

$$a - \sqrt{2} < q - \sqrt{2} < b - \sqrt{2}, \tag{1.8}$$

where $q - \sqrt{2} \in \mathbb{I}$.

Exercise 1.4.4
Suppose $\exists b$ lower bound such that $b > 0$. Then by Archimedean Property of $\mathbb{R}$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b$, which means b is not a valid lower bound. Thus $b \leq 0$, and 0 is a valid lower bound so the inf is 0.

Exercise 1.4.5
AFSOC $\exists \alpha \in \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$. Then $\alpha > 0$, but by Archimedean property of reals, we have that $\exists n \in \mathbb{N} \mid \frac{1}{n} < \alpha$. Since $\alpha \notin \left(0, \frac{1}{n}\right)$ leads to $\alpha \notin \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$, a contradiction, we conclude the set is empty.

Exercise 1.4.6

(a) If $\alpha^2 > 2$, then

$$\begin{aligned}\left(a - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}\end{aligned}$$

choose $\frac{1}{n_0} < \frac{\alpha^2-2}{2\alpha}$. Then

$$\begin{aligned}\left(a - \frac{1}{n_0}\right)^2 &> \alpha^2 - \frac{2\alpha}{2\alpha}(\alpha^2 - 2) \\ &> 2\end{aligned}$$

but $\alpha - \frac{1}{n_0} < \alpha$, so α is not the least upper bound for the se, so $\alpha \neq \sup T$.

(b) Just replace $\sqrt{2}$ with $\sqrt{b}$ for the proof above.

Exercise 1.4.7

Once we have assigned $g(i) = f(n_i)$, remove $f(n_i)$ from A. Now, there is a new $n_{i+1} = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(1), f(2), \ldots, f(n_i)\}\}$. Assign $g(i+1) = f(n_{i+1})$, and repeat.

Exercise 1.4.8

(a) If both are finite, then their union is finite and trivially countable. If one is finite, then first enumerate elements of the finite set. Then map the rest of $\mathbb{N}$ to the countably infinite set. If both are countably infinite, map one set to odds and the other to evens.

(b) Induction only holds for finite integers, not infinity.

(c) We can arrange each A_n into row n of a $\mathbb{N} \times \mathbb{N}$ matrix. Then, we enumerate by diagonalization.

Exercise 1.4.9

(a) If $A \sim B$, then there is a 1-to-1 mapping. We can just take the inverse of the mapping to derive $B \sim A$.

(b) If we have $f : A \to B$, $g : B \to C$, then we can compose the functions so $g(f(x)) : A \to C$.

Exercise 1.4.10

The set of all finite subsets of $\mathbb{N}$ can be ordered in increasing order by the sum of each subset.

Exercise 1.4.11

(a) $f(x) = (x, 0.5) \in S$

(b) Interweave the decimal expansion of x, y, e.g.

$$f(x, y) = 0.x_1 y_1 x_2 y_2 x_3 y_3 \ldots \tag{1.9}$$

Exercise 1.4.12

(a)

$$\begin{aligned}\sqrt{2} &: x^2 - 2 = 0 \\ \sqrt[3]{2} &: x^3 - 2 = 0\end{aligned}$$

$\sqrt{3} + \sqrt{2}$ is not as trivial, so we will do it out in more steps.

There are two approaches to finding the integer coefficient polynomial. One is to take advantage of symmetry, and derive that

$$\prod (x - (\pm\sqrt{3} \pm \sqrt{2}) \tag{1.10}$$

will work (using loose notation of course). A more general technique is to notice that

$$\begin{aligned} x &= \sqrt{3}+\sqrt{2} \\ x^2 &= 5+2\sqrt{6} \\ (x^2-5)^2 &= 24 \\ x^4-10x^2+1 &= 0. \end{aligned}$$

Notice that this is actually the exact same answer we get in (1.10) if you work it out.

(b) Each element of A_n is a root of a n degree polynomial, which we can represent as an $(n+1)$-tuple of coefficients $\in \mathbb{Z}$. Therefore, $|A_n| = k\left|\mathbb{N}^{n+1}\right| = \left|\mathbb{N}^{n+1}\right|$, which we know is countable.

(c) We proved earlier in Theorem 1.4.13 that a countably infinite union of countable sets is countable. Since there are a countable number of algebraic numbers, and reals are uncountable, we conclude that transcendentals are also uncountable.

Exercise 1.4.13
We are proving the **Schroöder-Bernstein Theorem**, which states if there exist 1-to-1 functions $f : X \to Y$ and $g : Y \to X$, then there exists a 1-to-1, onto function $h : X \to Y$, which implies $X \sim Y$.

(a) By the definition of 1-to-1, there must be a unique $x \in X$ such that $f(x) = y$. A 1-to-1 function maps distinct elements from the domain to distinct elements of the range, so if we take the inverse f^{-1}, it will still be 1-to-1, this time from $Y \to X$.

(b) Possibilities:

- Zero: g^{-1} is not guaranteed to be onto, it may not have an inverse for x.
- Finite: $g^{-1}(x)$ could exist, and similarly for f^{-1}. Once the element doesn't exist in the inverse domain, the chain will stop.
- Infinite: x is in the range of g and the domain of f

(c) We have 2 cases

- The chains are disjoint. Nothing to prove here.
- The chains are not disjoint, i.e. they have one common element. Let us call this element x. We know to the right of x, all the elements in the two chains will be equal. From the left of x, the elements must be equal as well. This is because the inverse chain must be unique starting from x, see part (a). Since all the elements are the same, the chains must be the same as well.

(d) Since we know this chain started with $x \in X$, this y could not have been created from the RHS, otherwise this y would be in the range of f. Therefore, this chain either has infinite or a finite of elements to the right.

- Finite: the chain must start with an element $y \notin Y$ but not in f's range. This is because if we start with $x \in X$, then as mentioned before, all elements $y' \in Y$ will be in f's range. Therefore, if we start with $y \in Y$, it will match the form indicated.
- Infinite: The chain could not have an infinite elements to the left, because then every y must have come from an $f(x')$ for some $x' \in X$.

Therefore, these chains only can have a finite number of elements to the left, and it matches the form indicated.

(e) By the definition of C_x, all the elements of $y \in C_x$ that are $\in Y$ are mapped by f from $x \in X_1$. This means f maps X_1 onto Y_1. Similar logic can be used for g mapping Y_2 onto X_2.

Since we know f is a 1-to-1 function from X to Y, and we just showed it maps X_1 to Y_1. We can conclude that $X_1 \sim Y_1$, since f is a bijection between X_1 and Y_1. We can similarly conclude $X_2 \sim Y_2$

with g. Since X_1, X_2 are a partition of X, since all the chains are disjoint, and similarly for Y_1, Y_2 of Y, and there exists a bijection f, g for $X_1 \sim Y_1$ and $X_2 \sim Y_2$ respectively, we conclude there must be a bijection between X and Y. Therefore, we conclude $X \sim Y$.

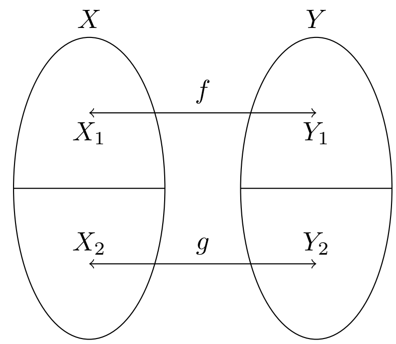

Figure 1.1: f and g mapping X and Y

1.5 Cantor's Theorem

Exercise 1.5.2

(a) Because b_1 differs from $f(1)$ in position 1

(b) b_i differs from $f(i)$ in position i.

(c) We reach a contradiction that we can enumerate all the elements of $(0, 1)$, since we found a real number that isn't enumerated, and thus $(0, 1)$ is uncountable.

Exercise 1.5.3

(a) $\frac{\sqrt{2}}{2} \in (0, 1)$ but is irrational

(b) We can just define our decimal representations to never have an infinite string of 9s.

Exercise 1.5.4

AFSOC S is countable. We will use a diagonalization proof. Then we can enumerate the elements of S using the natural numbers. Now, consider some $s = (s_1, s_2, \dots)$, where

$$s_i = \begin{cases} 0, \text{if } f(i), \text{position } i = 1 \\ 1, \text{otherwise} \end{cases} \tag{1.11}$$

Then since $s \neq f(i)\forall i$, we see $s \notin S$. But this is a contradiction since s only contains elements 0 or 1, and thus should be in S. Thus, we conclude that S is uncountable.

Exercise 1.5.5

(a)

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \tag{1.12}$$

(b) Each element has two choices when constructing a subset of A. To be, or not to be [1], in the set.

Exercise 1.5.6

(a) Many different answers.

$$\begin{gathered} \{(a, \{a\}), (b, \{b\}), (c, \{c\})\} \\ \{(a, \emptyset), (b, \{b\}), (c, \{c\})\} \end{gathered} \tag{1.13}$$

(b)

$$\{(1, \{1\}), (2, \{2\}), (3, \{3\}), (4, \{4\})\}.$$

(c) Because in general, $\left|\mathcal{P}(A)\right| > |A|$ for any set $A \neq \emptyset$. The intuition is that the power set has strictly more elements than A, so A cannot map $\mathcal{P}(A)$ onto.

Exercise 1.5.7

Using the examples found in (1.13).

1. $B = \emptyset$
2. $B = \{a\}$

Exercise 1.5.8

(a) AFSOC $a' \in B$. Then that means $a \notin f(a')$ by the definition of B. But this is a contradiction since $a' \in B = f(a')$.

(b) AFSOC $a' \notin B = f(a')$. Then since $a' \notin f(a')$, by the construction of B, this implies $a' \in B$, but that is a contradiction from our original assumption.

Exercise 1.5.9

(a) This is the same as $\mathbb{N} \times \mathbb{N}$, which is countable.

[1] sorry, had to do it. *Addendum* For context, I took a Shakespeare class in college two semesters prior to when I first wrote this.

(b) Uncountable, since this is essentially constructing the power set of $\mathbb{N}$, and we know $\mathcal{P}(\mathbb{N})$ is uncountable.

(c) Is this question asking for the number of antichains or if there is an antichain with uncountable cardinality?

The latter is obvious, and *no* is the answer since any subset of $\mathbb{N}$ is countable.

If we want to count the number of antichains, we notice that an antichain is essentially a partition of some subset of B. We also notice that every element of $\mathcal{P}(B)$ is also technically a partition, just a partition of size one. This means that the cardinality of the set of antichains is at least the cardinality of $\mathcal{P}(B)$. If $B = \mathbb{N}$, then we know $\mathcal{P}(\mathbb{N})$ is already uncountable, so the set of antichains will also be uncountable.

Chapter 2

Sequences and Series

For the convergence proofs in this chapter, I will lean towards showing how to derive the N that works, rather than just going directly with the proof and supplying a magical N, since I think finding the N is the process that deserves more attention.

2.2 The Limit of a Sequence

Exercise 2.2.1
The proofs are essentially the same, so after the first proof, I'll just give the n that can be used to prove the convergence.

(a) Let $\epsilon > 0$ be arbitrary. Then choose $n \in \mathbb{N}$ such that $n > \frac{1}{\sqrt{6\epsilon}}$. Then

$$\begin{aligned}\left|\frac{1}{6n^2+1}\right| &< \left|\frac{1}{6\frac{1}{6\epsilon}+1}\right| \\ &< \left|\frac{1}{\frac{1}{\epsilon}+1}\right| \\ &< \frac{\epsilon}{\epsilon+1} \\ &< \epsilon\end{aligned}$$

as desired.

(b) Choose $n > \frac{13}{2\epsilon} - \frac{5}{2}$

(c) Choose $n > \frac{4}{\epsilon^2} - 3$

Exercise 2.2.2
Consider the sequence

$$x_n = (-1)^n, n \geq 1. \tag{2.1}$$

Then for $\epsilon > 2$, it is true that $|x_n - 0| < 2, \forall n \geq 1$.

The *vercongent* definition describes a sequence that can be finitely bounded past some n.

Exercise 2.2.3

(a) We have to find one school with a student shorter than 7 feet.

(b) We would have to find a college with a grade that is not A or B.

(c) We just have find a college where a student is shorter than 6 feet.

Exercise 2.2.4
For $\epsilon > \frac{1}{2}$, we can find a suitable N, since we can claim the sequence "converges" to $\frac{1}{2}$. For $\epsilon \leq \frac{1}{2}$, there is no suitable response.

Exercise 2.2.5

(a) $\lim a_n = 0$. Take $n > 1$. Then

$$\begin{aligned} \left| \left[\left[\frac{1}{n} \right]\right] \right| &\leq 0 \\ &< \epsilon. \end{aligned}$$

(b) $\lim a_n = 0$. Take $n > 10$. Then

$$\begin{aligned} \left| \left[\left[\frac{10+n}{2n} \right]\right] \right| &= \left| \left[\left[\frac{5}{n} + \frac{1}{2} \right]\right] \right| \\ &\leq 0 \\ &< \epsilon. \end{aligned}$$

Usually, the sequence converges to some value by getting closer and closer eventually. This means for a smaller ϵ-neighborhood, we have to enumerate more elements, so we need a larger N.

Sometimes, the sequence converges to the exact value very fast, which means for some n, we don't need to choose a larger n. E.g. if we had the sequence of all 0s, we can choose any n and claim the sequence converges to 0.

Exercise 2.2.6

(a) Any *larger* N will work, since succeeding elements should stay in the neighborhood.

(b) Any *larger* ϵ will work, since we already guaranteed succeeding elements will stay in the ϵ-neighborhood, so any $\epsilon' > \epsilon$ will also bound the rest of the sequence.

Exercise 2.2.7

(a) We say a sequence x_n *converges* to ∞ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$ we have that $|x_n| > \epsilon$

(b) With our definition, we say this sequence diverges, but does not converge to ∞.

Exercise 2.2.8

(a) Frequently, since -1 will leave the set $\{1\}$.

(b) Eventually is stronger, and implies frequently.

(c) We say that a sequence x_n converges to x if it eventually is in a neighborhood of radius ϵ of x for all $\epsilon > 0$.

(d) x_n is only necessarily frequently in $(1.9, 2.1)$, even if there are an infinite number of elements equal to 2, you could have something like $(-2)^n$, where it keeps on leaving the ϵ-neighborhood of 2.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1
Let $\epsilon > 0$. Consider $n \geq 1$, then

$$|a - a| = 0 < \epsilon.$$

Exercise 2.3.2

(a) We are given $(x_n) \to 0$, so we can make $|x_n - 0|$ as small as we want.

In particular, for some $\epsilon > 0$, we choose N such that $\forall n \geq N$,

$$|x_n| < \epsilon^2 \quad \Rightarrow \quad \left|\sqrt{x_n}\right| < \epsilon \tag{2.2}$$

The implication follows since we know $x_n \geq 0, \epsilon > 0$.

To see that this N works, observe that for all $n \geq N$,

$$\left|\sqrt{x_n} - 0\right| < \epsilon \qquad \text{(by (2.2))}$$

so we conclude $(\sqrt{x_n}) \to 0$.

(b) We have two cases. If the sequence converges to 0, then we just have part ((a)).

If $x \neq 0$, then notice

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{|x_n - x|}{\left|\sqrt{x_n} + \sqrt{x}\right|}$$

since we know $x_n \geq 0$ and $x \neq 0$. Now, this expression is hard to bound when the denominator is small, since that would make the overall expression big. Fortunately, we can put a bound on the denominator, namely, since we know $x \neq 0 \to x > 0$, the denominator is > 0. Let us call the denominator value d. Then the following N will work for the convergence proof,

$$N : \forall n \geq N \quad |x_n - x| < \epsilon \cdot d \tag{2.3}$$

Exercise 2.3.3
By the Order Limit Theorem, since

$$\forall n, x_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \geq \lim_{n\to\infty} x_n = l$$
$$\forall n, z_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \leq \lim_{n\to\infty} z_n = l$$

so $l \leq \lim_{n\to\infty} y_n \leq l \Rightarrow \lim_{n\to\infty} y_n = 1$.

Exercise 2.3.4
AFSOC $\lim a_n = l_1$ and l_2, for $l_1 \neq l_2$. Then we have that $\forall \epsilon > 0$, for sufficiently large n, that

$$|a_n - l_1| < \epsilon$$
$$e|a_n - l_2| < \epsilon$$

But this is a contradiction, since if we let $d = |l_1 - l_2|$, and $\epsilon = \frac{d}{2}$, then

$$|l_2 - l_1| \leq |a_n - l_1| + \left|-(a_n - l_2)\right| < 2\epsilon \qquad \text{(Triangle Inequality)}$$
$$d \leq |a_n - l_1| + \left|-(a_n - l_2)\right| < d,$$

which leads to $d < d$. Thus, we must conclude that $l_1 = l_2$, and limits are unique.

Exercise 2.3.5
($\Rightarrow$) If (z_n) is convergent to some l, then $\forall \epsilon > 0$, we have that $\exists N \in \mathbb{N}$ such that for $n \geq N$, that

$$|z_n - l| < \epsilon \Longrightarrow |x_n - l| < \epsilon, |y_n - l| < \epsilon, \tag{2.4}$$

because z_n appears before or at the same time as x_n and y_n in the sequence.

($\Leftarrow$) If $(x_n), (y_n)$ are both convergent to some limit l, then we have for any $\epsilon > 0$, $\exists N_x : n_x \geq N_x$ and $\exists N_y : n_y \geq N_y$, that

$$\begin{aligned} |x_{n_x} - l| &< \epsilon \\ \left|y_{n_y} - l\right| &< \epsilon, \end{aligned} \tag{2.5}$$

respectively.

Choose $N_z > 2 \cdot \max(N_x, N_y)$. Then for $n_z \geq N_z$, z_{n_z} is either equal to x_i for $i > N_x$ or y_j for $j > N_y$. Using (2.5), we can see that

$$|z_{n_z} - l| < \epsilon$$

so (z_n) is also convergent to l.

Exercise 2.3.6

(a) By triangle inequality, we have $\big||b_n| - |b|\big| \leq |b_n - b| < \epsilon$, so the N that proves convergence for (b_n) will also work for $(|b_n|)$.

(b) The converse is not true. Consider the sequence $a_n = (-1)^n$.

Exercise 2.3.7

(a) Since (a_n) is bounded, call M the upper bound of (a_n). Then since $|b_n|$ can get arbitrarily small, we choose $n \geq N$ such that $|b_n| < \frac{\epsilon}{M}$. Then we have

$$\begin{aligned} |a_n b_n| &\leq |a_n||b_n| \\ &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

We cannot use the Algebraic Limit Theorem because we are not given that (a_n) necessarily converges.

(b) No. For example, take $a_n = (-1)^n$, $b_n = 3$. This is because we can no longer make $|b_n|$ arbitrarily small.

(c) When $a = 0$, we have

$$|a_n b_n - ab| \leq |b_n||a_n - a|.$$

We can bound $|b_n| \leq M$, and then choose n such that $|a_n - a| < \frac{\epsilon}{M}$. Then,

$$\begin{aligned} |a_n b_n - ab| &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

Exercise 2.3.8

(a) $x_n = (-1)^n, y_n = (-1)^{n-1}$. Sum is just $\{0, 0, \dots\}$

(b) **Impossible**, since if $x_n + y_n$ converges and x_n also converges, we can show that y_n must converge, which is a contradiction.

(c) $b_n = \frac{1}{n}$

(d) **Impossible**, since if b_n converges to some b, for any $\epsilon > 0$, past some N, for $n \geq N$,

$$|b_n - b| < \epsilon.$$

Any a_n that is unbounded will grow in magnitude for larger n, so b_n cannot help bound a_n.

(e) $a_n = 0, b_n = n$

Exercise 2.3.9

Yes, the strict inequalities will provide an upper and lower bound still. Sort of like a sup, inf of the sequence.

Exercise 2.3.10
Since $|a_n|$ gets arbitrarily small, for any $\epsilon > 0$ we know $\exists N : n \geq N$ such that,

$$|b_n - b| \leq |a_n| < \epsilon. \tag{2.6}$$

Exercise 2.3.11
Let $\lim x_n = x$. Then, for any $\epsilon_x > 0$, $\exists N_x : n \geq N_x$, we have $|x_n - x| < \epsilon_x$.

Now, our goal is, given some $\epsilon_y > 0$, to find some $N_y : n \geq N_y$ so we can bound y_n. The intuition is, since we know (x_n) converges, after some point, x_i will be close to the limit x. Our goal is to choose some N_y large enough so the x_i' prior to these x_i are "averaged out" enough, so they are essentially gone, and that the weight on the x_i that are close to x is very high.

$$\begin{aligned}
|y_n - x| &= \left| \frac{1}{n} \left[\sum_{i=1}^{N_x} (x_i - x) + \sum_{i=N_x+1}^{N_y} (x_i - x) \right] \right| \\
&\leq \left| \frac{1}{n} \left[\sum_{i=1}^{N_x} M + \sum_{i=N_x+1}^{N_y} \epsilon_x \right] \right| && \text{(Let } M \text{ bound the difference from } x_i \text{ to } x.) \\
&\leq \left| \frac{1}{n} \Big[N_x M + \big(N_y - N_x\big)\epsilon_x \Big] \right| \\
&\leq \left| \frac{N_x}{n} M + \epsilon_x \right| < \epsilon_y
\end{aligned}$$

Now, we have quite a few choices for our N_y. One such solution, is

- Given some $\epsilon > 0$
- First choose N_x such $n \geq N_x \quad |x_n - x| < \epsilon/2$
- Then, choose $N_y > \frac{2N_x M}{\epsilon}$. This means for $n \geq N_y$,

$$\begin{aligned}
|y_n - x| &\leq \left| \frac{N_x}{n} M + \epsilon_x \right| \\
&< \left| \frac{N_x M}{\frac{2N_x M}{\epsilon}} + \epsilon/2 \right| \leq \epsilon
\end{aligned}$$

Consider when $x_n = (-1)^n$. (x_n) does not converge but (y_n) does.

Exercise 2.3.12

(a) Intuitively, the limit should go to 1, since we have $\frac{\infty}{\infty}$.

$$\begin{aligned}
\lim_{n\to\infty} \lim_{m\to\infty} a_{m,n} &= 1 \\
\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} &= 0
\end{aligned}$$

(b) A sequence $(a_{m,n})$ converges to l if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$, we have that

$$\left| \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n} - l \right| < \epsilon$$
$$\left| \lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} - l \right| < \epsilon.$$

i.e. we approach the same limit no matter what permutation of the index variables we iterate through.

This definition is motivated by multivariable calculus, but unsure if this makes sense in the context of analysis.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1
Suppose $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Fix m, k so that $m \geq 2^{k+1} - 1$, then

$$\begin{aligned}\sum_{i=1}^{m} b_i &\geq \sum_{i=1}^{2^{k+1}-1} b_i \\ &= s_{2^{k+1}-1} \\ &= t_k\end{aligned}$$

Since t_k is a diverging sequence, then b_m will also diverge.

Exercise 2.4.2

(a) We can show by induction that the sequence is decreasing. Thus, because the sequence starts at 3, we know it is bounded below by 0. Thus, the sequence converges.

(b) If $\lim x_n$ exists, then $\lim x_{n+1}$ must be the same limit, because if the limit is a different value or doesn't exist, then (x_n) does not converge.

(c) Suppose $\lim x_n = \lim x_{n+1} = x$. Then

$$\begin{aligned}x &= \frac{1}{4-x} \\ x^2 - 4x + 1 &= 0 \\ &\implies x = 2 - \sqrt{3}\end{aligned}$$

The other root is too large and does not work with the initial conditions.

Exercise 2.4.3
We can use induction to show that (y_n) is increasing. Since the sequence is increasing and starts at 1, we know that (y_n) is bounded above by 4 and below by 0. Thus, by the Monotone Convergence Theorem, we conclude that (y_n) converges. Now, we find the limit of the recurrence by taking the limits of both sides of the equation,

$$\begin{aligned}y &= 4 - \frac{1}{y} \\ y^2 - 4y + 1 &= 0 \\ y &= 2 + \sqrt{3}\end{aligned}$$

Exercise 2.4.4
We can define the recurrence of this sequence as

$$a_{n+1} = \sqrt{2a_n}. \tag{2.7}$$

We can prove by induction that this sequence is increasing. We can also bound the sequence since this sequence can also be viewed as

$$2^{\frac{1}{2}}, 2^{\frac{1}{2}+\frac{1}{4}}, 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}, \cdots$$

You can take the infinite sum $\sum_{i=1}^{\infty} 2^{-i} = 1$ and get $2^1 = 2$ as your final answer.

The other way to solve this problem is to look at the limits of x_n, x_{n+1}, which must be equal. Let's say their limit is x, then

$$\begin{aligned}x_{n+1} = \sqrt{2x_n}x & &= \sqrt{2x} \\ x^2 - 2x &= 0 \\ x &= 2. & &\text{(from } x_0 = 1\text{)}\end{aligned}$$

Exercise 2.4.5

(a) By induction, we have

Base Case: $x_1 = 2 \implies x_1^2 = 4 \geq 2$.

Inductive Hypothesis: Given that for some $x_n, x_n^2 \geq 2$.

Inductive Step: Consider

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) \\ &\geq \frac{1}{4}\left(2 + 4 + 4/2\right) = 2. \end{aligned}$$

Therefore, we conclude $\forall n, x_n \geq 2$.

Now we can show

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{\frac{1}{2}x_n^2 - 1}{x_n} \\ &\geq 0, \end{aligned}$$

which means the sequence is decreasing, so by the Monotone Convergence Theorem we know that (x_n) converges. We now take limits of x on both sides of the recurrence, yielding,

$$\begin{aligned} x &= \frac{1}{2}\left(x + \frac{2}{x}\right) \\ \frac{1}{2}x - \frac{1}{x} &= 0 \\ x^2 - 2 &= 0 \\ \implies x &= \sqrt{2}. \end{aligned}$$

(b) We can modify the sequence to converge to $\sqrt{c}, c \geq 0$ by setting $x_1 = c$, and

$$x_{n+1} = \frac{1}{c}\left((c-1)x_n + \frac{c}{x_n}\right) \tag{2.8}$$

Exercise 2.4.6

(a) Since we know that (a_n) is bounded, it must also be the case that $\sup(a_n)$ is bounded. Then, $\sup\{a_k\}$ is a decreasing sequence, so by the Monotone Convergence Theorem, we know that (y_n) converges.

(b) We can define

$$\liminf a_n = \lim z_n, \text{ where} \tag{2.9}$$
$$\lim z_n = \inf\{a_k : k \geq n\}. \tag{2.10}$$

Since $\inf\{a_k\}$ is a increasing sequence, and (a_n) is bounded, we know it converges.

(c) For any set A, $\inf A \leq \sup A$, so $\forall n, \inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$.

An example when the inequality is strict is

$$a_n = (-1)^n, \tag{2.11}$$

since $\liminf a_n = -1, \limsup a_n = 1$.

(d) ($\Rightarrow$) Suppose

$$\liminf a_n = \limsup a_n = L, \tag{2.12}$$

then given some $\epsilon > 0$, we know $\exists N : n \geq N$ so that, define $A_n = \{a_k : k \geq n\}$

$$|\inf A_n - L| < \epsilon$$
$$|\sup A_n - L| < \epsilon$$

since every element $k \geq n, \inf A_n \leq a_k \leq \sup A_n$, we conclude

$$k \geq n \geq N \quad |a_k - L| < \epsilon,$$

so $\lim a_n = L$.

($\Leftarrow$) Suppose

$$\lim a_n = L,$$

then given some $\epsilon > 0$, we know $\exists N : n \geq N$ so thats

$$|a_n - L| < \epsilon/2$$

This means every element after a_n lives in this $\epsilon/2$-neighborhood of L. Now, $\sup A_n$ must be arbitrarily close to the largest element of A_n, so we can make this distance $\epsilon/2$. That means

$$|\sup A_n - L| = \left|\max\{A_n\} + \epsilon/2 - L\right| < \epsilon,$$

which means $\limsup a_n = L$. This is similar for inf.

Exercise 2.4.1
Suppose we have a convergent sequence. Then given any ϵ, we can always find for $n \geq N \in \mathbb{N}$ that $|a_n - l| < \epsilon$. For any subsequence of (a_n), $a'_m = a_n$ will be such that $m \geq n$, so we can choose $m \geq N$ from earlier and conclude that $\left|a'_m - l\right| < \epsilon$.

Exercise 2.4.2
(a) Define

$$s_i = \sum_{j=1}^{i} a_j \tag{2.13}$$

$$b_i = \sum_{k=1}^{i} a_{n_k}, \tag{2.14}$$

where the series regrouping a_i is divided into groups of $n_1, n_2, \ldots$. Then b_i is a subsequence of s_n, which means they converge to the same limit, namely L in this case.

(b) Our proof does not apply to that example because that series did not converge in the first place.

Exercise 2.4.3
(a) Consider

$$a_n = \begin{cases} \sum_{i=1}^{n} \frac{1}{2^i}, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \tag{2.15}$$

Then we have that $b_n = a_{2n-1}$ converges to 1 and $c_n = a_{2n}$ converges to 0.

(b) A monotone sequence that diverges means that sequence is not bounded. Thus, every subsequence will also be unbounded and thus impossible to be convergent.

(c) Consider the sequence

$$\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \tag{2.16}$$

(d) Consider

$$a_n = \begin{cases} 2^i, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \tag{2.17}$$

(e) By Bolzano-Weierstrass, since we have a subsequence that is bounded, we know we can find a convergent subsequence within this subsequence that converges.

Exercise 2.4.4
AFSOC (a_n) converges to $b \neq a$. Then we have that $|a_n - b|$ can be arbitrarily small. But this implies that every subsequence will also converge to b, which is a contradiction.

AFSOC (a_n) does not converge. Then since (a_n) is bounded, and every convergent subsequence converges to ??? i give up :(

Exercise 2.4.5
Consider $|b^n|$. Since $|b| < 1$, we have that $|b^n|$ is a decreasing sequence that is bounded below by 0, so we have

$$|b| > l \geq 0.$$

We notice that $\left|b^{2n}\right|$ is a subsequence that also converges to L, and since $\left|b^{2n}\right| = |b|^2$, by the Algebraic Limit Theorem, we have that $\left|b^{2n}\right| \to l^2 = l \implies l = 0$. Since $|b^n| \to 0$, we conclude $b^n \to 0$.

Exercise 2.4.6
We have $s = \sup S$, which means for any $\epsilon > 0$,

$$s - \epsilon < x \in S < a'_n$$
$$\implies \epsilon > \left|s - a'_n\right| = \left|a'_n - s\right|$$

where a'_n is a subsequence containing part of the infinite $a_n > x \in S$.

Exercise 2.4.1

(a) $a_n = 1 + \left(-\frac{1}{2}\right)^n$

(b) $a_n = n$

(c) Impossible, since a Cauchy sequence implies convergence, which means every subsequence will also converge.

(d) You can use Equation (2.17). Literally anything that diverges but has a convergent subsequence.

Exercise 2.4.2

If we have that $(x_n) \to x$, then we can make $|x_n - x|$ arbitrarily small. Consider

$$\begin{aligned} |x_n - x_m| = |x_n - x + x - x_m| & \\ \leq |x_n - x| + |x_m - x| & \qquad \text{(Triangle Inequality)} \\ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon & \end{aligned}$$

Exercise 2.4.3

(a) The *pseudo-Cauchy* definition is different because it only looks at consecutive terms

(b) Consider the harmonic series, where $s_{n+1} - s_n = \frac{1}{n}$.

Exercise 2.4.4

$$\begin{aligned} |c_{n+1} - c_n| &= \big||a_{n+1} - b_{n+1}| - |a_n - b_n|\big| \\ &\leq |a_{n+1} - a_n + b_{n+1} - b_n| \\ &\leq |a_{n+1} - a_n| + |b_{n+1} - b_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Exercise 2.4.5

(a) Let $a_n = x_n + y_n$, then

$$\begin{aligned} |a_{n+1} - a_n| &= |x_{n+1} - x_n + y_{n+1} - y_n| \\ &\leq |x_{n+1} - x_n| + |y_{n+1} - y_n| \\ &< \epsilon \end{aligned}$$

(b)

Exercise 2.4.6

(a)

Exercise 2.4.1

$$\begin{aligned}\lim s_{nn} &= \sum_{i=0}^{\infty} -\left(\frac{1}{2}\right)^i \\ &= -2.\end{aligned}$$

The value is equal to summing the columns first.

Exercise 2.4.2
By the Absolute Convergence test, since we know for fixed i that $\sum_{j=1}^{\infty} |a_{ij}|$ converges, then we know for fixed i that each $\sum_{j=1}^{\infty} a_{ij}$ converges to some c_i as well.

Then, since

$$\begin{aligned}\sum_{j=1}^{\infty} |a_{ij}| &\geq \sum_{j=1}^{\infty} a_{ij} \\ \implies b_i &\geq c_i,\end{aligned}$$

and we know that $\sum_{i=1}^{\infty} b_i$ converges, we conclude that $\sum_{i=1}^{\infty} c_i$ must converge as well, implying that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \tag{2.18}$$

converges as well.

Exercise 2.4.3

(a) Since $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$ converges, we know that we can find $m, n \geq N$ such that

$$|t_{mn} - L| < 0.9,$$

and then choose our upper bound as

$$M = \max\{l\} \cup \{a_{mn} \mid m, n < N\} \tag{2.19}$$

Since t_{nn} is an increasing sequence, and is bounded above, by the Monotone Convergence Theorem, t_{nn} converges.

(b) Consider

$$\begin{aligned}|s_{n+1,n+1} - s_{nn}| &= |a_{n+1,n+1}| \\ &= |t_{n+1,n+1} - t_{nn}| \\ &< \epsilon.\end{aligned}$$

So (s_{nn}) is a Cauchy Sequence and converges.

Exercise 2.4.4

(a) Since (t_{nn}) is an increasing sequence and is bounded above, we know there exists a $t_{n_0 n_0}$ such that

$$B - \frac{\epsilon}{2} < t_{n_0 n_0} \leq B. \tag{2.20}$$

For $N_1 = n_0$, since for $m, n \geq N_1$, $t_{mn} > t_{n_0 n_0}$, and $t_{mn} \leq B$ (upper bound B), we will have that

$$B - \frac{\epsilon}{2} < t_{m,n} \leq B. \tag{2.21}$$

(b) Since $(t_{nn}0$ converges, we can find $n \geq N$ such that

$$\begin{aligned}
|s_{nn} - S| &= \left| s_{nn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \right| \\
&= \left| \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} a_{ij} \right| \\
&< \left| \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} |a_{ij}| \right| \\
&= \left| t_{nn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}| \right| \\
&< \epsilon.
\end{aligned}$$

So for $m, n \geq N$,

$$|s_{mn} - S| \leq \tag{2.22}$$

hm?????????/

Appendix A

Extras

A.1 Useful Tools

Collection of useful

Tip A.1.1
Template for a proof that $(x_n) \to x$:

- Let $\epsilon > 0$ be arbitrary
- Demonstrate a choice for $N \in \mathbb{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N works.
- Assume $n \geq N$
- With N well chosen, you should be able to show $|x_n - x| < \epsilon$.

A.2 Cool Things

- In Chapter 2, we learn that addition in infinite sums is not commutative.