Understanding Analysis Solutions

Michael You

Contents

1 The Real Numbers **7**
1.2 Some Preliminaries . . . 7
1.3 The Axiom of Completeness . . . 11
1.4 Consequences of Completeness . . . 13
1.5 Cantor's Theorem . . . 17

2 Sequences and Series **19**
2.2 The Limit of a Sequence . . . 19
2.3 The Algebraic and Order Limit Theorems . . . 21
2.4 The Monotone Convergence Theorem and a First Look at Infinite Series . . . 24
2.5 Subsequences and the Bolzano-Weierstrass Theorem . . . 27
2.6 The Cauchy Criterion . . . 29
2.7 Properties of Infinite Series . . . 31
2.8 Double Summations and Products of Infinite Series . . . 36

3 Basic Topology of $\mathbb{R}$ **41**
3.2 Open and Closed Sets . . . 41
3.3 Compact Sets . . . 46
3.4 Perfect Sets and Connected Sets . . . 49
3.5 Baire's Theorem . . . 51

4 Functional Limits and Continuity **53**
4.2 Functional Limits . . . 53
4.3 Combinations of Continuous Functions . . . 56
4.4 Continuous Functions on Compact Sets . . . 60
4.5 The Intermediate Value Theorem . . . 63
4.6 Sets of Discontinuity . . . 65

A Extras **67**
A.1 Useful Tools . . . 67
A.2 Cool Things . . . 67
A.3 Important Theorems . . . 67
A.3.1 5 Characterizations of Completeness . . . 67
A.3.2 Sequence Convergence . . . 68
A.3.3 Function Continuity . . . 68
A.4 Identities . . . 68

Acknowledgment

I would like to thank Varun Gudibanda and Trajan Hammonds for giving me the inspiration for writing these solutions by hyping up Real Analysis. They also contributed to some solutions in chapter 1 and 2.

I would also like to thank Stephen Abbott for writing a fantastic introductory text to Real Analysis. If he ever reads this, he's probably wondering why someone would write solutions to his text if it already exists, but I hope he's happy to know someone learned a lot while doing it.

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Proof AFSOC $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$ such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in lowest reduced terms. Then we can square both sides, yielding $3 = \left(\frac{m}{n}\right)^2 \implies 3n^2 = m^2$. Now, we know m^2 is a multiple of 3 and thus m must also. Then, we can write $m = 3k$, and derive

$$\begin{aligned}(\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2\end{aligned}$$

Similar to before, we come to the conclusion that n is a multiple of 3. However, this is a contradiction since m, n are both multiples of 3 and we assumed $\frac{m}{n}$ was in lowest terms. Thus, we conclude $\sqrt{3}$ is irrational.

The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ implies that m is a multiple of 4, since we have

$$4n^2 = m^2 \quad \Rightarrow \quad 2n = m,$$

so we cannot reach our contradiction that m/n is not in lowest terms.

Exercise 1.2.2

(a) False. Consider

$$A_n = \left[0, \frac{1}{n}\right).$$

Then

$$\bigcap_{n=1}^{\infty} A_n = \{0\}.$$

(b) True. Since $\forall i, A_i \subseteq A_1$, $\exists x$ such that $\forall i, x \in A_i$. Therefore, the intersection cannot be empty. Then, every set is finite, and the intersection of any number of finite sets will be finite.

(c) False. Consider $A = \{1, 2\}, B = \{1\}, C = \{2, 3\}$.

$$\{1,2\} \cap \big(\{1\} \cup \{2,3\}\big) = \{1,2\} \neq \big(\{1,2\} \cap \{1\}\big) \cup \{2,3\} = \{1,2,3\}$$

(d) True. Intersection is associative.

(e) True. Intersection is distributive over union.

PROOF We will prove

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.1}$$

by set inclusion.

- Suppose $x \in A \cap (B \cup C)$. By the definition of intersection, we know $x \in A$ and $x \in B \cup C$, the latter which means $x \in B$ or $x \in C$.

  We can consider 2 cases for x,

  1. $x \in B$. Then we know $x \in A$ and $x \in B$, so $x \in A \cap B$ and therefore $x \in (A \cap B) \cup (A \cap C)$
  2. $x \in C$. Symmetric to the case above.

  in all cases, we see $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$, so

  $$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

- Suppose $x \in (A \cap B) \cup (A \cap C)$. Then we have two cases

  1. $x \in A \cap B$. This means $x \in A$ and $x \in B$. If $x \in B$, then $x \in B \cup C$, since $B \subseteq B \cup C$. Putting these facts together, we see $x \in A \cap (B \cup C)$.
  2. $x \in A \cap C$. Symmetric to the case above.

  in all cases, we see $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$, so

  $$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

Exercise 1.2.3

(a) If $x \in (A \cap B)^c$, then we have cases

- $x \in B$ and $x \notin A$. Then $x \notin A$ implies $x \in A^c \Rightarrow x \in A^c \cup B^c$.
- $x \in A$. Symmetric to above.
- $x \notin A$ and $x \notin B$. Then $x \in A^c$ so $x \in A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$, then we have cases

- $x \in A^c$. Then $x \notin A$ so x cannot be in the intersection of A and B, so $x \in (A \cap B)^c$.
- $x \in B^c$. Symmetric to above.

(c) Proof for $(A \cup B)^c = A^c \cap B^c$ pretty similar to above.

Exercise 1.2.4

We are verifying the triangle inequality with a, b.

(a) If a, b have the same sign, then

$$\begin{aligned} |a+b| &= a+b \\ |a|+|b| &= a+b \\ \Rightarrow\ |a+b| &= |a|+|b| \\ \Rightarrow\ |a+b| &\le |a|+|b| \end{aligned}$$

(b)
- $a \ge 0, b < 0$.

$$\begin{aligned} |a+b| &\le |a| \\ &\le |a|+|b| \end{aligned}$$

- $a + b \geq 0$. At most one of a, b is negative. If they are both positive, then we have already shown this in part (a). Otherwise, WLOG a is negative. Then

$$\begin{aligned} |a+b| &\leq |b| \\ &\leq |a| + |b| \end{aligned}$$

Exercise 1.2.5

(a) Substitute in $b' = -b$ into the triangle inequality.

(b) Easy to prove directly without using triangle inequality. **TODO**.

A direct proof will look something like:

- If a, b are the same sign, then equality holds
- If a, b are different signs, then if b is negative, then $|a - b| = |a| + |b|$, and if a is negative, then $|a - b| = |a| + |b|$, both of which bound $\big||a| - |b|\big|$.

Exercise 1.2.6

(a) Yes, since $f(A \cap B) = [1, 4] = [0, 4] \cap [1, 16] = f(A) \cap f(B)$. This is by coincidence though, as we will later see. Yes, since $f(A \cup B) = [0, 16] = [0, 4] \cup [1, 16] = f(A) \cup f(B)$.

(b) Choose $A = [-2, 0], B = [0, 2]$

(c) Suppose $x \in g(A \cap B)$, then $\exists x' \in A \cap B$ such that $g(x') = x$. Since $x' \in A$ and $x' \in B$, we know $x = g(x') \in g(A), g(B)$, so we conclude $x \in g(A) \cap g(B)$.

(d) Equality. **TODO** too lazy to write out the proof. Similar to above.

Exercise 1.2.7

(a) **TODO** I don't think we want to include $x \in \mathbb{I}$...

$$f^{-1}(A) = [0, 2] \tag{1.2}$$

$$f^{-1}(B) = [0, 1] \tag{1.3}$$

We see $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ is also true.

(b) **TODO**

Exercise 1.2.8

Negating statements. Took some liberties. Also notice that these statements are not necessarily true.

(a) There exists a real number satisfying $a < b$, such that $\forall n \in \mathbb{N}$, $a + 1/n \geq b$.

(b) There exists two distinct real numbers such that there is not a rational number between them.

(c) There exists a natural number $n \in \mathbb{N}$ such that $\sqrt{n}$ is not a natural number nor an irrational number.

(d) There exists a real number $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $n \leq x$.

Exercise 1.2.9

We are given the sequence

$$x_1 = 1, x_{n+1} = \frac{1}{2}x_n + 1 \tag{1.4}$$

and want to show $\forall i \geq 1, x_i < 2$.

We can show this with a direct proof of summation.

An alternative that the book probably wants to see is using **induction**.

- Base Case: $x_1 = 1 < 2$
- Inductive case. Assume $\forall i < n + 1, x_i < 2$. Then $x_i/2 + 1 < 2$ since $x_i/2 < 1$.

- By induction our original claim is proved.

Exercise 1.2.10

(a) Similar to Exercise 1.2.9. $y_n < 4$ means $(3/4)y_n < 3$ so $(3/4)y_n + 1 < 4$

(b) In brief,

$$\begin{aligned} y_n &\leq \frac{3}{4}y_n + \frac{1}{4}y_n && (1.5) \\ &< \frac{3}{4}y_n + 1 && \text{(Using } y_n < 4\text{)} \\ &< y_n + 1 && \text{(Sequence definition)} \end{aligned}$$

Exercise 1.2.11

A combinatorial argument is that in order to construct a set, we have 2 choices for every element, to include it or not to. Therefore, we have

$$\prod_{i=1}^{n} 2 = 2^n$$

Exercise 1.2.12

(a) We know that $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$. So if we are trying to show $(A_1 \cup A_2 \cup A_3)^c = (A_1 \cup A_2)^c \cap A_3^c = A_1^c \cap A_2^c \cap A_3^c$. Induction lets us apply the property on smaller parts of our expression.

(b) Induction only proves the property for some $n \in \mathbb{N}$, i.e. some finite n. It is not shown for an infinite n.

(c) **TODO**. Sketch: If x is not in the union of all the A_n, then x cannot be part of any particular A_n either, or else it would be in the union.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) We compute the additive inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned}0+0&\equiv 0\\1+4&\equiv 0\\2+3&\equiv 0\\3+2&\equiv 0\\4+1&\equiv 0\end{aligned}$$

(b) We compute the multiplicative inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned}1\times 1&\equiv 1\\2\times 3&\equiv 1\\3\times 2&\equiv 1\\4\times 4&\equiv 1\end{aligned}$$

(c) $\mathbb{Z}_4$ is not a field because multiplicative inverses do not exist for every single element. For example, 2 multiplied with any number is even, which cannot $\equiv 1 \pmod 4$.

We conjecture that $\mathbb{Z}_n$ always has additive inverses and only has multiplicative inverses if n is prime.

Exercise 1.3.2

We are writing a formal definition for the *infimum* of a set.

(a) $s = \inf A$ means

i) s is a lower bound for A

ii) if b is any lower bound for A, then $b \leq s$

(b) If $s \in \mathbb{R}$ is a lower bound for $A \subseteq \mathbb{R}$, then $s = \inf A$ iff $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$.

Proof ($\Rightarrow$) If $s = \inf A$, then s is the greatest lower bound for A, meaning any $s+\epsilon$ for $\epsilon > 0$ will be greater than some element of A, otherwise $s + \epsilon$ is a greater lower bound and leads to a contradiction that$s \neq \inf A$.

($\Leftarrow$) If $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$, then since s is a lower bound, $\forall b > s$, b will not be a lower bound for A since if, $b > s$, then we can choose $\epsilon = b - s > 0$, and we know that $\exists a \in A$ where $a < s + \epsilon < b$, which means b is not a lower bound. Thus, all lower bounds b must be such that $b \leq s$, and we conclude $s = \inf A$.

Exercise 1.3.3

(a) Since $\inf A$ is a lower bound for A, we know $\inf A \in B$. Now, we need to show $\inf A$ is the supremum of B. $\inf A$ is the least upper bound for B, since if $\exists b \in B, b > \inf A$, then we know that this b is not a lower bound for A, so no such b exists.

(b) There might be a typo in this question. I think the question was meant to read "explain why there is no need to assert that the greatest *lower bound* in the Axiom of Completeness." In this case, the answer would be that the Axiom of Completeness already implies the greatest lower bound property, so there is no need to explicitly state it.

(c) We can take the negative of all elements in A, find $\sup A$, and then negate again to get $\inf A$.

Exercise 1.3.4

If $B \subseteq A$, then

$$\begin{aligned}\sup A = s &\geq a, \forall a \in A &&\\ s &\geq b, \forall b \in B && \text{(since } B \subseteq A)\\ \Rightarrow\ s &\geq \sup B. && \text{(since } s \text{ is an upper bound for } B)\end{aligned}$$

Exercise 1.3.5

(a)

$$
\begin{aligned}
& s = \sup(c+A) \\
\Rightarrow\ & s \text{ is the least upper bound for } c+A \\
\Rightarrow\ & s-c \text{ is the least upper bound for } A \\
\Rightarrow\ & s-c = \sup A \\
& s = c + \sup A
\end{aligned}
$$

(b)

$$
\begin{aligned}
& s = \sup(cA) \\
\Rightarrow\ & s \text{ is the least upper bound for } cA \\
\Rightarrow\ & \frac{s}{c} \text{ is the least upper bound for } A \\
\Rightarrow\ & \frac{s}{c} = \sup A \\
& s = c \sup A
\end{aligned}
$$

(c) If $c < 0$, $\sup(cA) = -c \inf(A)$.

Exercise 1.3.6

(a) $\sup : 3; \inf : 1$

(b) $\sup : 1; \inf : 0$

(c) $\sup : \frac{1}{2}; \inf : \frac{1}{3}$

(d) $\sup : 9; \inf : \frac{1}{9}$

Exercise 1.3.7

If $a \geq a', \forall a' \in A$, and $a \in A$, then

$$\forall \epsilon > 0, a - \epsilon < a, \tag{1.6}$$

so a is the least upper bound for A, and $a = \sup A$.

Exercise 1.3.8

Let

$$\epsilon = \sup B - \sup A > 0. \tag{1.7}$$

since $s_b = \sup B$, $\exists b \in B \mid b > s_b - \epsilon/2$. Since $s_b - \frac{\epsilon}{2} > \sup A$, then $b \geq \sup A$, so this $b \in B$ is an upper bound for A.

Exercise 1.3.9

(a) True, take the largest element in the set as the supremum.

(b) False, $\sup(0,2) = 2$, but $2 > a \in (0,2)$, but $\sup A = 2 \not< 2 = L$.

(c) False $A = (0,2), B = [2,3)$. We have that $\sup A = \inf B$

(d) True.

(e) False, take $A = B = (0,2)$.

1.4 Consequences of Completeness

Exercise 1.4.1
If $a < 0$, then we have two cases,

1. If $b > 0$, then $a < 0 < b$.
2. If $b = 0$, then we can take $-b, -a$, which satisfies $0 \leq -b < -a$, and apply Theorem 1.4.3.

Exercise 1.4.2
(a) If $a, b \in \mathbb{Q}$, then

$$\begin{aligned} a &= \frac{a_1}{a_2} \\ b &= \frac{b_1}{b_2} \\ \implies a + b &= \frac{a_1 b_2 + a_2 b_1}{a_2 b_2} \in \mathbb{Q} \end{aligned}$$

(b) We can use contradiction,

- AFSOC $a + t \in \mathbb{Q}$. Let $a + t = \frac{m}{n}$. We know $a = \frac{a_1}{a_2}$ since $a \in \mathbb{Q}$, so

$$\begin{aligned} a + t &= \frac{m}{n} \\ t &= \frac{m}{n} - \frac{a_1}{a_2} \in \mathbb{Q}, \end{aligned}$$

which is a contradiction since we are given $t \in \mathbb{Q}$. Therefore, we conclude $a + t \in \mathbb{I}$.

- AFSOC $at \in \mathbb{Q}$. Let $at = \frac{m}{n}$. We know $a = \frac{a_1}{a_2}$ since $a \in \mathbb{Q}$, so

$$\begin{aligned} at &= \frac{m}{n} \\ t &= \frac{m}{n} \cdot \frac{a_2}{a_1} \in \mathbb{Q}, \end{aligned}$$

which is a contradiction since we are given $t \in \mathbb{Q}$. Therefore, we conclude $at \in \mathbb{I}$

(c) $\mathbb{I}$ is not closed under addition or multiplication.

$$\begin{aligned} \left(3 - \sqrt{2}\right) + \left(3 + \sqrt{2}\right) &= 6 \notin \mathbb{I} \\ \left(3 - \sqrt{2}\right) \cdot \left(3 + \sqrt{2}\right) &= 5 \notin \mathbb{I} \end{aligned}$$

Exercise 1.4.3
We can apply Theorem 1.4.3, to find $a < q < b, q \in \mathbb{Q}$, and then subtract an irrational number such as $\sqrt{2}$ to end up at

$$a - \sqrt{2} < q - \sqrt{2} < b - \sqrt{2}, \tag{1.8}$$

where $q - \sqrt{2} \in \mathbb{I}$.

Exercise 1.4.4
Suppose $\exists b$ lower bound such that $b > 0$. Then by Archimedean Property of $\mathbb{R}$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b$, which means b is not a valid lower bound. Thus $b \leq 0$, and 0 is a valid lower bound so the inf is 0.

Exercise 1.4.5
AFSOC $\exists \alpha \in \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$. Then $\alpha > 0$, but by Archimedean property of reals, we have that $\exists n \in \mathbb{N} \mid \frac{1}{n} < \alpha$. Since $\alpha \notin \left(0, \frac{1}{n}\right)$ leads to $\alpha \notin \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$, a contradiction, we conclude the set is empty.

Exercise 1.4.6

(a) If $\alpha^2 > 2$, then

$$\begin{aligned}\left(a - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}\end{aligned}$$

choose $\frac{1}{n_0} < \frac{\alpha^2-2}{2\alpha}$. Then

$$\begin{aligned}\left(a - \frac{1}{n_0}\right)^2 &> \alpha^2 - \frac{2\alpha}{2\alpha}(\alpha^2 - 2) \\ &> 2\end{aligned}$$

but $\alpha - \frac{1}{n_0} < \alpha$, so α is not the least upper bound for the se, so $\alpha \neq \sup T$.

(b) Just replace $\sqrt{2}$ with $\sqrt{b}$ for the proof above.

Exercise 1.4.7

Once we have assigned $g(i) = f(n_i)$, remove $f(n_i)$ from A. Now, there is a new $n_{i+1} = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(1), f(2), \ldots, f(n_i)\}\}$. Assign $g(i+1) = f(n_{i+1})$, and repeat.

Exercise 1.4.8

(a) If both are finite, then their union is finite and trivially countable. If one is finite, then first enumerate elements of the finite set. Then map the rest of $\mathbb{N}$ to the countably infinite set. If both are countably infinite, map one set to odds and the other to evens.

(b) Induction only holds for finite integers, not infinity.

(c) We can arrange each A_n into row n of a $\mathbb{N} \times \mathbb{N}$ matrix. Then, we enumerate by diagonalization.

Exercise 1.4.9

(a) If $A \sim B$, then there is a 1-to-1 mapping. We can just take the inverse of the mapping to derive $B \sim A$.

(b) If we have $f : A \to B$, $g : B \to C$, then we can compose the functions so $g(f(x)) : A \to C$.

Exercise 1.4.10

The set of all finite subsets of $\mathbb{N}$ can be ordered in increasing order by the sum of each subset.

Exercise 1.4.11

(a) $f(x) = (x, 0.5) \in S$

(b) Interweave the decimal expansion of x, y, e.g.

$$f(x, y) = 0.x_1y_1x_2y_2x_3y_3\ldots \tag{1.9}$$

Exercise 1.4.12

(a)

$$\begin{aligned}\sqrt{2} &: x^2 - 2 = 0 \\ \sqrt[3]{2} &: x^3 - 2 = 0\end{aligned}$$

$\sqrt{3} + \sqrt{2}$ is not as trivial, so we will do it out in more steps.

There are two approaches to finding the integer coefficient polynomial. One is to take advantage of symmetry, and derive that

$$\prod (x - (\pm\sqrt{3} \pm \sqrt{2}) \tag{1.10}$$

will work (using loose notation of course). A more general technique is to notice that

$$\begin{aligned} x &= \sqrt{3}+\sqrt{2} \\ x^2 &= 5+2\sqrt{6} \\ (x^2-5)^2 &= 24 \\ x^4-10x^2+1 &= 0. \end{aligned}$$

Notice that this is actually the exact same answer we get in (1.10) if you work it out.

(b) Each element of A_n is a root of a n degree polynomial, which we can represent as an $(n+1)$-tuple of coefficients $\in \mathbb{Z}$. Therefore, $|A_n| = k\left|\mathbb{N}^{n+1}\right| = \left|\mathbb{N}^{n+1}\right|$, which we know is countable.

(c) We proved earlier in Theorem 1.4.13 that a countably infinite union of countable sets is countable. Since there are a countable number of algebraic numbers, and reals are uncountable, we conclude that transcendentals are also uncountable.

Exercise 1.4.13
We are proving the **Schroöder-Bernstein Theorem**, which states if there exist 1-to-1 functions $f : X \to Y$ and $g : Y \to X$, then there exists a 1-to-1, onto function $h : X \to Y$, which implies $X \sim Y$.

(a) By the definition of 1-to-1, there must be a unique $x \in X$ such that $f(x) = y$. A 1-to-1 function maps distinct elements from the domain to distinct elements of the range, so if we take the inverse f^{-1}, it will still be 1-to-1, this time from $Y \to X$.

(b) Possibilities:

- Zero: g^{-1} is not guaranteed to be onto, it may not have an inverse for x.
- Finite: $g^{-1}(x)$ could exist, and similarly for f^{-1}. Once the element doesn't exist in the inverse domain, the chain will stop.
- Infinite: x is in the range of g and the domain of f

(c) We have 2 cases

- The chains are disjoint. Nothing to prove here.
- The chains are not disjoint, i.e. they have one common element. Let us call this element x. We know to the right of x, all the elements in the two chains will be equal. From the left of x, the elements must be equal as well. This is because the inverse chain must be unique starting from x, see part (a). Since all the elements are the same, the chains must be the same as well.

(d) Since we know this chain started with $x \in X$, this y could not have been created from the RHS, otherwise this y would be in the range of f. Therefore, this chain either has infinite or a finite of elements to the right.

- Finite: the chain must start with an element $y \notin Y$ but not in f's range. This is because if we start with $x \in X$, then as mentioned before, all elements $y' \in Y$ will be in f's range. Therefore, if we start with $y \in Y$, it will match the form indicated.
- Infinite: The chain could not have an infinite elements to the left, because then every y must have come from an $f(x')$ for some $x' \in X$.

Therefore, these chains only can have a finite number of elements to the left, and it matches the form indicated.

(e) By the definition of C_x, all the elements of $y \in C_x$ that are $\in Y$ are mapped by f from $x \in X_1$. This means f maps X_1 onto Y_1. Similar logic can be used for g mapping Y_2 onto X_2.

Since we know f is a 1-to-1 function from X to Y, and we just showed it maps X_1 to Y_1. We can conclude that $X_1 \sim Y_1$, since f is a bijection between X_1 and Y_1. We can similarly conclude $X_2 \sim Y_2$

with g. Since X_1, X_2 are a partition of X, since all the chains are disjoint, and similarly for Y_1, Y_2 of Y, and there exists a bijection f, g for $X_1 \sim Y_1$ and $X_2 \sim Y_2$ respectively, we conclude there must be a bijection between X and Y. Therefore, we conclude $X \sim Y$.

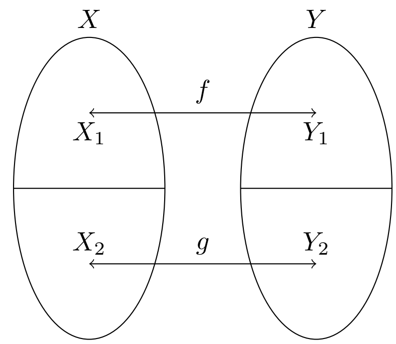

Figure 1.1: f and g mapping X and Y

1.5 Cantor's Theorem

Exercise 1.5.2

(a) Because b_1 differs from $f(1)$ in position 1

(b) b_i differs from $f(i)$ in position i.

(c) We reach a contradiction that we can enumerate all the elements of $(0, 1)$, since we found a real number that isn't enumerated, and thus $(0, 1)$ is uncountable.

Exercise 1.5.3

(a) $\frac{\sqrt{2}}{2} \in (0, 1)$ but is irrational

(b) We can just define our decimal representations to never have an infinite string of 9s.

Exercise 1.5.4

AFSOC S is countable. We will use a diagonalization proof. Then we can enumerate the elements of S using the natural numbers. Now, consider some $s = (s_1, s_2, \dots)$, where

$$s_i = \begin{cases} 0, \text{if } f(i), \text{position } i = 1 \\ 1, \text{otherwise} \end{cases} \tag{1.11}$$

Then since $s \neq f(i) \forall i$, we see $s \notin S$. But this is a contradiction since s only contains elements 0 or 1, and thus should be in S. Thus, we conclude that S is uncountable.

Exercise 1.5.5

(a)
$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \tag{1.12}$$

(b) Each element has two choices when constructing a subset of A. To be, or not to be [1], in the set.

Exercise 1.5.6

(a) Many different answers.
$$\begin{gathered} \{(a, \{a\}), (b, \{b\}), (c, \{c\})\} \\ \{(a, \emptyset), (b, \{b\}), (c, \{c\})\} \end{gathered} \tag{1.13}$$

(b)
$$\{(1, \{1\}), (2, \{2\}), (3, \{3\}), (4, \{4\})\}.$$

(c) Because in general, $\left|\mathcal{P}(A)\right| > |A|$ for any set $A \neq \emptyset$. The intuition is that the power set has strictly more elements than A, so A cannot map $\mathcal{P}(A)$ onto.

Exercise 1.5.7

Using the examples found in (1.13).

1. $B = \emptyset$
2. $B = \{a\}$

Exercise 1.5.8

(a) AFSOC $a' \in B$. Then that means $a \notin f(a')$ by the definition of B. But this is a contradiction since $a' \in B = f(a')$.

(b) AFSOC $a' \notin B = f(a')$. Then since $a' \notin f(a')$, by the construction of B, this implies $a' \in B$, but that is a contradiction from our original assumption.

Exercise 1.5.9

(a) This is the same as $\mathbb{N} \times \mathbb{N}$, which is countable.

---

[1] sorry, had to do it. *Addendum* For context, I took a Shakespeare class in college two semesters prior to when I first wrote this.

(b) Uncountable, since this is essentially constructing the power set of $\mathbb{N}$, and we know $\mathcal{P}(\mathbb{N})$ is uncountable.

(c) Is this question asking for the number of antichains or if there is an antichain with uncountable cardinality?

The latter is obvious, and *no* is the answer since any subset of $\mathbb{N}$ is countable.

If we want to count the number of antichains, we notice that an antichain is essentially a partition of some subset of B. We also notice that every element of $\mathcal{P}(B)$ is also technically a partition, just a partition of size one. This means that the cardinality of the set of antichains is at least the cardinality of $\mathcal{P}(B)$. If $B = \mathbb{N}$, then we know $\mathcal{P}(\mathbb{N})$ is already uncountable, so the set of antichains will also be uncountable.

Chapter 2

Sequences and Series

For the convergence proofs in this chapter, I will lean towards showing how to derive the N that works, rather than just going directly with the proof and supplying a magical N, since I think finding the N is the process that deserves more attention.

2.2 The Limit of a Sequence

Exercise 2.2.1
The proofs are essentially the same, so after the first proof, I'll just give the n that can be used to prove the convergence.

(a) Let $\epsilon > 0$ be arbitrary. Then choose $n \in \mathbb{N}$ such that $n > \frac{1}{\sqrt{6\epsilon}}$. Then

$$\begin{aligned}\left|\frac{1}{6n^2+1}\right| &< \left|\frac{1}{6\frac{1}{6\epsilon}+1}\right| \\ &< \left|\frac{1}{\frac{1}{\epsilon}+1}\right| \\ &< \frac{\epsilon}{\epsilon+1} \\ &< \epsilon\end{aligned}$$

as desired.

(b) Choose $n > \frac{13}{2\epsilon} - \frac{5}{2}$

(c) Choose $n > \frac{4}{\epsilon^2} - 3$

Exercise 2.2.2
Consider the sequence

$$x_n = (-1)^n, n \geq 1. \tag{2.1}$$

Then for $\epsilon > 2$, it is true that $|x_n - 0| < 2, \forall n \geq 1$.

The *vercongent* definition describes a sequence that can be finitely bounded past some n.

Exercise 2.2.3
(a) We have to find one school with a student shorter than 7 feet.

(b) We would have to find a college with a grade that is not A or B.

(c) We just have find a college where a student is shorter than 6 feet.

Exercise 2.2.4
For $\epsilon > \frac{1}{2}$, we can find a suitable N, since we can claim the sequence "converges" to $\frac{1}{2}$. For $\epsilon \leq \frac{1}{2}$, there is no suitable response.

Exercise 2.2.5

(a) $\lim a_n = 0$. Take $n > 1$. Then

$$\begin{aligned} \left|\left[\left[\frac{1}{n}\right]\right]\right| &\leq 0 \\ &< \epsilon. \end{aligned}$$

(b) $\lim a_n = 0$. Take $n > 10$. Then

$$\begin{aligned} \left|\left[\left[\frac{10+n}{2n}\right]\right]\right| &= \left|\left[\left[\frac{5}{n} + \frac{1}{2}\right]\right]\right| \\ &\leq 0 \\ &< \epsilon. \end{aligned}$$

Usually, the sequence converges to some value by getting closer and closer eventually. This means for a smaller ϵ-neighborhood, we have to enumerate more elements, so we need a larger N.

Sometimes, the sequence converges to the exact value very fast, which means for some n, we don't need to choose a larger n. E.g. if we had the sequence of all 0s, we can choose any n and claim the sequence converges to 0.

Exercise 2.2.6

(a) Any *larger* N will work, since succeeding elements should stay in the neighborhood.

(b) Any *larger* ϵ will work, since we already guaranteed succeeding elements will stay in the ϵ-neighborhood, so any $\epsilon' > \epsilon$ will also bound the rest of the sequence.

Exercise 2.2.7

(a) We say a sequence x_n *converges* to ∞ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$ we have that $|x_n| > \epsilon$

(b) With our definition, we say this sequence diverges, but does not converge to ∞.

Exercise 2.2.8

(a) Frequently, since -1 will leave the set $\{1\}$.

(b) Eventually is stronger, and implies frequently.

(c) We say that a sequence x_n converges to x if it eventually is in a neighborhood of radius ϵ of x for all $\epsilon > 0$.

(d) x_n is only necessarily frequently in $(1.9, 2.1)$, even if there are an infinite number of elements equal to 2, you could have something like $(-2)^n$, where it keeps on leaving the ϵ-neighborhood of 2.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1
Let $\epsilon > 0$. Consider $n \geq 1$, then

$$|a - a| = 0 < \epsilon.$$

Exercise 2.3.2

(a) We are given $(x_n) \to 0$, so we can make $|x_n - 0|$ as small as we want.

In particular, for some $\epsilon > 0$, we choose N such that $\forall n \geq N$,

$$|x_n| < \epsilon^2 \quad \Rightarrow \quad \left|\sqrt{x_n}\right| < \epsilon \tag{2.2}$$

The implication follows since we know $x_n \geq 0, \epsilon > 0$.

To see that this N works, observe that for all $n \geq N$,

$$\left|\sqrt{x_n} - 0\right| < \epsilon \qquad \text{(by (2.2))}$$

so we conclude $(\sqrt{x_n}) \to 0$.

(b) We have two cases. If the sequence converges to 0, then we just have part (a).

If $x \neq 0$, then notice

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{|x_n - x|}{\left|\sqrt{x_n} + \sqrt{x}\right|}$$

since we know $x_n \geq 0$ and $x \neq 0$. Now, this expression is hard to bound when the denominator is small, since that would make the overall expression big. Fortunately, we can put a bound on the denominator, namely, since we know $x \neq 0 \to x > 0$, the denominator is > 0. Let us call the denominator value d. Then the following N will work for the convergence proof,

$$N : \forall n \geq N \quad |x_n - x| < \epsilon \cdot d \tag{2.3}$$

Exercise 2.3.3
By the Order Limit Theorem, since

$$\forall n, x_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \geq \lim_{n\to\infty} x_n = l$$
$$\forall n, z_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \leq \lim_{n\to\infty} z_n = l$$

so $l \leq \lim_{n\to\infty} y_n \leq l \Rightarrow \lim_{n\to\infty} y_n = 1$.

Exercise 2.3.4
AFSOC $\lim a_n = l_1$ and l_2, for $l_1 \neq l_2$. Then we have that $\forall \epsilon > 0$, for sufficiently large n, that

$$|a_n - l_1| < \epsilon$$
$$e|a_n - l_2| < \epsilon$$

But this is a contradiction, since if we let $d = |l_1 - l_2|$, and $\epsilon = \frac{d}{2}$, then

$$|l_2 - l_1| \leq |a_n - l_1| + \left|-(a_n - l_2)\right| < 2\epsilon \qquad \text{(Triangle Inequality)}$$
$$d \leq |a_n - l_1| + \left|-(a_n - l_2)\right| < d,$$

which leads to $d < d$. Thus, we must conclude that $l_1 = l_2$, and limits are unique.

Exercise 2.3.5
$(\Rightarrow)$ If (z_n) is convergent to some l, then $\forall \epsilon > 0$, we have that $\exists N \in \mathbb{N}$ such that for $n \geq N$, that

$$|z_n - l| < \epsilon \Longrightarrow |x_n - l| < \epsilon, |y_n - l| < \epsilon, \tag{2.4}$$

because z_n appears before or at the same time as x_n and y_n in the sequence.

($\Leftarrow$) If $(x_n), (y_n)$ are both convergent to some limit l, then we have for any $\epsilon > 0$, $\exists N_x : n_x \geq N_x$ and $\exists N_y : n_y \geq N_y$, that

$$\begin{aligned} |x_{n_x} - l| &< \epsilon \\ \left|y_{n_y} - l\right| &< \epsilon, \end{aligned} \tag{2.5}$$

respectively.

Choose $N_z > 2 \cdot \max(N_x, N_y)$. Then for $n_z \geq N_z$, z_{n_z} is either equal to x_i for $i > N_x$ or y_j for $j > N_y$. Using (2.5), we can see that

$$|z_{n_z} - l| < \epsilon$$

so (z_n) is also convergent to l.

Exercise 2.3.6

(a) By triangle inequality, we have $\big||b_n| - |b|\big| \leq |b_n - b| < \epsilon$, so the N that proves convergence for (b_n) will also work for $(|b_n|)$.

(b) The converse is not true. Consider the sequence $a_n = (-1)^n$.

Exercise 2.3.7

(a) Since (a_n) is bounded, call M the upper bound of (a_n). Then since $|b_n|$ can get arbitrarily small, we choose $n \geq N$ such that $|b_n| < \frac{\epsilon}{M}$. Then we have

$$\begin{aligned} |a_n b_n| &\leq |a_n||b_n| \\ &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

We cannot use the Algebraic Limit Theorem because we are not given that (a_n) necessarily converges.

(b) No. For example, take $a_n = (-1)^n$, $b_n = 3$. This is because we can no longer make $|b_n|$ arbitrarily small.

(c) When $a = 0$, we have

$$|a_n b_n - ab| \leq |b_n||a_n - a|.$$

We can bound $|b_n| \leq M$, and then choose n such that $|a_n - a| < \frac{\epsilon}{M}$. Then,

$$\begin{aligned} |a_n b_n - ab| &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

Exercise 2.3.8

(a) $x_n = (-1)^n, y_n = (-1)^{n-1}$. Sum is just $\{0, 0, \dots\}$

(b) **Impossible**, since if $x_n + y_n$ converges and x_n also converges, we can show that y_n must converge, which is a contradiction.

(c) $b_n = \frac{1}{n}$

(d) **Impossible**, since if b_n converges to some b, for any $\epsilon > 0$, past some N, for $n \geq N$,

$$|b_n - b| < \epsilon.$$

Any a_n that is unbounded will grow in magnitude for larger n, so b_n cannot help bound a_n.

(e) $a_n = 0, b_n = n$

Exercise 2.3.9

Yes, the strict inequalities will provide an upper and lower bound still. Sort of like a sup, inf of the sequence.

Exercise 2.3.10
Since $|a_n|$ gets arbitrarily small, for any $\epsilon > 0$ we know $\exists N : n \geq N$ such that,

$$|b_n - b| \leq |a_n| < \epsilon. \tag{2.6}$$

Exercise 2.3.11
Let $\lim x_n = x$. Then, for any $\epsilon_x > 0$, $\exists N_x : n \geq N_x$, we have $|x_n - x| < \epsilon_x$.

Now, our goal is, given some $\epsilon_y > 0$, to find some $N_y : n \geq N_y$ so we can bound y_n. The intuition is, since we know (x_n) converges, after some point, x_i will be close to the limit x. Our goal is to choose some N_y large enough so the x_i' prior to these x_i are "averaged out" enough, so they are essentially gone, and that the weight on the x_i that are close to x is very high.

$$\begin{aligned}
|y_n - x| &= \left|\frac{1}{n}\left[\sum_{i=1}^{N_x}(x_i - x) + \sum_{i=N_x+1}^{N_y}(x_i - x)\right]\right| \\
&\leq \left|\frac{1}{n}\left[\sum_{i=1}^{N_x} M + \sum_{i=N_x+1}^{N_y}\epsilon_x\right]\right| && \text{(Let } M \text{ bound the difference from } x_i \text{ to } x.\text{)} \\
&\leq \left|\frac{1}{n}\Big[N_x M + \big(N_y - N_x\big)\epsilon_x\Big]\right| \\
&\leq \left|\frac{N_x}{n}M + \epsilon_x\right| < \epsilon_y
\end{aligned}$$

Now, we have quite a few choices for our N_y. One such solution, is

- Given some $\epsilon > 0$
- First choose N_x such $n \geq N_x \quad |x_n - x| < \epsilon/2$
- Then, choose $N_y > \frac{2N_xM}{\epsilon}$. This means for $n \geq N_y$,

$$\begin{aligned}
|y_n - x| &\leq \left|\frac{N_x}{n}M + \epsilon_x\right| \\
&< \left|\frac{N_x M}{\frac{2N_xM}{\epsilon}} + \epsilon/2\right| \leq \epsilon
\end{aligned}$$

Consider when $x_n = (-1)^n$. (x_n) does not converge but (y_n) does.

Exercise 2.3.12

(a) Intuitively, the limit should go to 1, since we have $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = 1$$
$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} = 0$$

(b) A sequence $(a_{m,n})$ converges to l if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$, we have that

$$\left|\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} - l\right| < \epsilon$$
$$\left|\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} - l\right| < \epsilon.$$

i.e. we approach the same limit no matter what permutation of the index variables we iterate through.

This definition is motivated by multivariable calculus, but unsure if this makes sense in the context of analysis.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1
Suppose $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Fix m, k so that $m \geq 2^{k+1} - 1$, then

$$\begin{aligned}\sum_{i=1}^{m} b_i &\geq \sum_{i=1}^{2^{k+1}-1} b_i \\ &= s_{2^{k+1}-1} \\ &= t_k\end{aligned}$$

Since t_k is a diverging sequence, then b_m will also diverge.

Exercise 2.4.2

(a) We can show by induction that the sequence is decreasing. Thus, because the sequence starts at 3, we know it is bounded below by 0. Thus, the sequence converges.

(b) If $\lim x_n$ exists, then $\lim x_{n+1}$ must be the same limit, because if the limit is a different value or doesn't exist, then (x_n) does not converge.

(c) Suppose $\lim x_n = \lim x_{n+1} = x$. Then

$$\begin{aligned}x &= \frac{1}{4-x} \\ x^2 - 4x + 1 &= 0 \\ &\implies x = 2 - \sqrt{3}\end{aligned}$$

The other root is too large and does not work with the initial conditions.

Exercise 2.4.3
We can use induction to show that (y_n) is increasing. Since the sequence is increasing and starts at 1, we know that (y_n) is bounded above by 4 and below by 0. Thus, by the Monotone Convergence Theorem, we conclude that (y_n) converges. Now, we find the limit of the recurrence by taking the limits of both sides of the equation,

$$\begin{aligned}y &= 4 - \frac{1}{y} \\ y^2 - 4y + 1 &= 0 \\ y &= 2 + \sqrt{3}\end{aligned}$$

Exercise 2.4.4
We can define the recurrence of this sequence as

$$a_{n+1} = \sqrt{2a_n}. \tag{2.7}$$

We can prove by induction that this sequence is increasing. We can also bound the sequence since this sequence can also be viewed as

$$2^{\frac{1}{2}}, 2^{\frac{1}{2}+\frac{1}{4}}, 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}, \cdots$$

You can take the infinite sum $\sum_{i=1}^{\infty} 2^{-i} = 1$ and get $2^1 = 2$ as your final answer.

The other way to solve this problem is to look at the limits of x_n, x_{n+1}, which must be equal. Let's say their limit is x, then

$$\begin{aligned}x_{n+1} &= \sqrt{2x_n}x & &= \sqrt{2x} \\ x^2 - 2x &= 0 \\ x &= 2. & &\text{(from } x_0 = 1\text{)}\end{aligned}$$

Exercise 2.4.5

(a) By induction, we have

Base Case: $x_1 = 2 \implies x_1^2 = 4 \geq 2$.

Inductive Hypothesis: Given that for some $x_n, x_n^2 \geq 2$.

Inductive Step: Consider

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) \\ &\geq \frac{1}{4}\big(2 + 4 + 4/2\big) = 2. \end{aligned}$$

Therefore, we conclude $\forall n, x_n \geq 2$.

Now we can show

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{\frac{1}{2}x_n^2 - 1}{x_n} \\ &\geq 0, \end{aligned}$$

which means the sequence is decreasing, so by the Monotone Convergence Theorem we know that (x_n) converges. We now take limits of x on both sides of the recurrence, yielding,

$$\begin{aligned} x &= \frac{1}{2}\left(x + \frac{2}{x}\right) \\ \frac{1}{2}x - \frac{1}{x} &= 0 \\ x^2 - 2 &= 0 \\ \implies x &= \sqrt{2}. \end{aligned}$$

(b) We can modify the sequence to converge to $\sqrt{c}, c \geq 0$ by setting $x_1 = c$, and

$$x_{n+1} = \frac{1}{c}\left((c-1)x_n + \frac{c}{x_n}\right) \tag{2.8}$$

Exercise 2.4.6

(a) Since we know that (a_n) is bounded, it must also be the case that $\sup(a_n)$ is bounded. Then, $\sup\{a_k\}$ is a decreasing sequence, so by the Monotone Convergence Theorem, we know that (y_n) converges.

(b) We can define

$$\liminf a_n = \lim z_n, \text{ where} \tag{2.9}$$
$$\lim z_n = \inf\{a_k : k \geq n\}. \tag{2.10}$$

Since $\inf\{a_k\}$ is a increasing sequence, and (a_n) is bounded, we know it converges.

(c) For any set A, $\inf A \leq \sup A$, so $\forall n, \inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$.

An example when the inequality is strict is

$$a_n = (-1)^n, \tag{2.11}$$

since $\liminf a_n = -1, \limsup a_n = 1$.

(d) ($\Rightarrow$) Suppose

$$\liminf a_n = \limsup a_n = L, \tag{2.12}$$

then given some $\epsilon > 0$, we know $\exists N : n \geq N$ so that, define $A_n = \{a_k : k \geq n\}$

$$|\inf A_n - L| < \epsilon$$
$$|\sup A_n - L| < \epsilon$$

since every element $k \geq n, \inf A_n \leq a_k \leq \sup A_n$, we conclude

$$k \geq n \geq N \quad |a_k - L| < \epsilon,$$

so $\lim a_n = L$.

($\Leftarrow$) Suppose

$$\lim a_n = L,$$

then given some $\epsilon > 0$, we know $\exists N : n \geq N$ so thats

$$|a_n - L| < \epsilon/2$$

This means every element after a_n lives in this $\epsilon/2$-neighborhood of L. Now, $\sup A_n$ must be arbitrarily close to the largest element of A_n, so we can make this distance $\epsilon/2$. That means

$$|\sup A_n - L| = \big|\max\{A_n\} + \epsilon/2 - L\big| < \epsilon,$$

which means $\limsup a_n = L$. This is similar for inf.

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Exercise 2.5.1

Suppose we have a convergent sequence with limit l. Then given any $\epsilon > 0$, we can always find $N : n \geq N$ such that $|a_n - l| < \epsilon$. For any subsequence of (a_n), (a'_m), any element of this subsequence, call it a'_k will be from some a_n in the original sequence, where $n \geq k$. So we can choose N from earlier, and for $m \geq N$ we will have $\left|a'_m - l\right| < \epsilon$.

Exercise 2.5.2

(a) Define

$$s_i = \sum_{j=1}^{i} a_j \tag{2.13}$$

$$b_i = \sum_{k=1}^{i} a_{n_k}, \tag{2.14}$$

where the series regrouping a_i is divided into groups of $n_1, n_2, \ldots$. Then b_i is a subsequence of s_n, which means they converge to the same limit, namely L in this case.

(b) Our proof does not apply to that example because that series did not converge in the first place.

Exercise 2.5.3

(a) Consider

$$a_n = \begin{cases} \sum_{i=1}^{n} \frac{1}{2^i}, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \tag{2.15}$$

Then we have that $b_n = a_{2n-1}$ converges to 1 and $c_n = a_{2n}$ converges to 0.

(b) A monotone sequence that diverges means that sequence is not bounded. Thus, every subsequence will also be unbounded and thus impossible to be convergent.

(c) Consider the sequence

$$\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \tag{2.16}$$

(d) Consider

$$a_n = \begin{cases} 2^i, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \tag{2.17}$$

(e) By Bolzano-Weierstrass, since we have a subsequence that is bounded, we know we can find a convergent subsequence within this subsequence that converges.

Exercise 2.5.4

AFSOC (a_n) converges to $b \neq a$. Then we have that $|a_n - b|$ can be arbitrarily small. But this implies that every subsequence will also converge to b, which is a contradiction.

AFSOC (a_n) does not converge. Then since (a_n) is bounded, we must have an infinite number of elements in two different ϵ-neighborhoods. But this would imply we have convergent subsequences to different limits, which contradicts the original problem statement.

Therefore, we conclude (a_n) converges to a.

Exercise 2.5.5

Consider $|b^n|$. Since $|b| < 1$, we have that $|b^n|$ is a decreasing sequence that is bounded below by 0, so we have

$$|b| > l \geq 0.$$

We notice that $\left|b^{2n}\right|$ is a subsequence that also converges to L, and since $\left|b^{2n}\right| = |b|^2$, by the Algebraic Limit Theorem, we have that $\left|b^{2n}\right| \to l^2 = l \implies l = 0$. Since $|b^n| \to 0$, we conclude $b^n \to 0$.

Exercise 2.5.6

We have $s = \sup S$, which means for any $\epsilon > 0$,

$$\exists x : s - \epsilon < x \in S < a'_n$$
$$\epsilon > \left|s - a'_n\right| = \left|a'_n - s\right|$$

where a'_n is an element of the infinite subsequence of $a_n : a_n > x \in S$.

2.6 The Cauchy Criterion

Exercise 2.6.1

(a) $a_n = 1 + \left(-\frac{1}{2}\right)^n$

(b) $a_n = n$

(c) Impossible, since a Cauchy sequence implies convergence, which means every subsequence will also converge.

(d) You can use Equation (2.17). Literally anything that diverges but has a convergent subsequence.

Exercise 2.6.2

If we have that $(x_n) \to x$, then we can make $|x_n - x|$ arbitrarily small. Consider

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| && \text{(Triangle Inequality)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Exercise 2.6.3

(a) The *pseudo-Cauchy* definition is different because it only looks at consecutive terms

(b) Consider the harmonic series, where $|s_{n+1} - s_n| = \frac{1}{n(n+1)}$.

Exercise 2.6.4

$$\begin{aligned} |c_{n+1} - c_n| &= \big||a_{n+1} - b_{n+1}| - |a_n - b_n|\big| \\ &\leq |a_{n+1} - a_n + b_{n+1} - b_n| \\ &\leq |a_{n+1} - a_n| + |b_{n+1} - b_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Exercise 2.6.5

(a) Let $a_n = x_n + y_n$, then

$$\begin{aligned} |a_{n+1} - a_n| &= |x_{n+1} - x_n + y_{n+1} - y_n| \\ &\leq |x_{n+1} - x_n| + |y_{n+1} - y_n| \\ &< \epsilon && \text{(for proper choice of } N\text{)} \end{aligned}$$

(b) Let $a_n = x_n y_n$, then

$$\begin{aligned} |a_{n+1} - a_n| &= |x_{n+1}y_{n+1} - x_n y_n| \\ &= |x_{n+1}y_{n+1} - x_n y_{n+1} - x_n y_n + x_n y_{n+1}| \\ &\leq \big|y_{n+1}(x_{n+1} - x_n)\big| + \big|x_n(y_{n+1} - y_n)\big| \end{aligned}$$

we have shown we can bound this before by

- $(x_n), (y_n)$ are convergent sequences, so they must be bounded. Call this bound M
- Now, $|x_{n+1} - x_n|$ can be made arbitrarily small. Given some ϵ, we can make it $< \frac{\epsilon}{2M}$.
- We do the same for $|y_{n+1} - y_n|$, and thus the overall bound is $< \epsilon$.

Exercise 2.6.6

I'm not going to write these down super rigorously, but will write down most of the ideas.

(a) Suppose we have some set of real numbers that is bounded above. We want to show that there exists a least upper bound, assuming the Nested Interval Property is true.

Let B be the set of upper bounds. Define $I_1 = B$, and for each subsequent I_i, define it as

$$I_{n+1} = \{b \in I_n : b < \ell_{n+1}\},$$

where $\ell_{n+1} \in I_n$ is arbitrarily chosen. The idea is that we are creating intervals that have a smaller and smaller maximum value.

Now we have 2 cases,

a $\exists n \in \mathbb{N} : I_n = \emptyset$. In this case, there must have been some ℓ_n where $\forall b \in I_n, b \neq \ell_n, \ell_n < b$. Since I_n is a subset of the smaller elements of B, we also have $\forall b \in B, b \neq \ell_n, \ell_n < b$, which means this ℓ_n is the least upper bound.

b None of the I_n are empty. Now, consider $\bigcap_{n=1}^{\infty} I_n$. By NIP, this intersection is nonempty. By our construction of the I_n, we know any element b in this intersection must be less than all the elements in the set preceding it. However, we reach a contradiction, because if b is in this infinite intersection, **TODO** this proof doesn't work... probably need to change the interval construction.

(b) Define i_n to be $\inf\left(\bigcup_{k=n}^{\infty} I_k\right)$, and s_n to be $\sup\left(\bigcup_{k=n}^{\infty} I_k\right)$. i_n is an increasing sequence, and s_n is a decreasing sequence. Since for any set, $\inf A \leq \sup A$, we know that $\forall n : i_n \leq s_n$. This means $\exists x : \lim i_n \leq x \leq \lim s_n$, which exists in every single interval I_k, so we know their infinite intersection is nonempty.

(c) We are using the BW Theorem to prove NIP. Let x_i be an arbitrary element of I_i. since I_1 is bounded, then all $I_k, k \geq 1$ are also bounded, so the sequence (x_n) is also bounded. By BW, we know that $\exists (s_n)$, a subsequence of (x_n) that converges. Suppose $\lim s_n = L$. We will show that L is in the infinite intersection of all the sets. For any I_k, the interval is (a_k, b_k). Call $\epsilon = |a_k - b_k|$, the size of the interval. Since (s_n) converges to L, we know $\exists N : n \geq N$,

$$|s_n - L| < \epsilon/2.$$

since $s_n \in I_n$ by definition, we see L also falls in this interval. Thus, we know $L \in I_n$ for $n \geq N$. Since I_n is also contained within all sets before as well, L also is contained in those sets. Therefore, we have shown that L is in every set, so the infinite intersection must contain at least L, so therefore it is nonempty.

(d) We are given a bounded sequence, and want to show that there exists a convergent subsequence.

We will construct a subsequence, and show that it is convergent.

Since we have a bounded sequence (x_n), we know $|x_n| \leq M$.

Let $I_1 = [-M, M]$. Now construct subsequent I_n from $I_{n-1} = [a, b]$ as

$$I_n = [a, (a+b)/2] \text{ or } [(a+b)/2, b],$$

depending on which half has an infinite number of elements.

Now, our subsequence is defined as $a_n : a_n \in I_n$. Given $\epsilon > 0$, since we can make the bound of I_n as arbitrarily small as possible, since we are halving the interval size every time, we know $\exists N : n \geq N \to |I_n| < \epsilon$. Now, since all future elements a_k will be chosen from this interval and its subsets, and the interval size is $< \epsilon$, we can conclude for $m, n \geq N$,

$$|a_n - a_m| < \epsilon.$$

This means (a_n) is a Cauchy Sequence, and by the Cauchy Criterion, we can conclude that (a_n) is convergent.

2.7 Properties of Infinite Series

Exercise 2.7.1

(a) For any $\epsilon > 0$, we know since $(a_n) \to 0$, $\exists N : n \geq N, |a_n| < \epsilon$. Now, let $n > m \geq N, n = m + 1$, $|s_n - s_m| = |a_n| < \epsilon$, which means (s_n) is a Cauchy sequence.

(b) Construct intervals I_k so that, initially, $I_1 = [-absa_1, |a_1|]$. For $n \geq 1$, if $I_n = [b_n, c_n]$,

$$I_{n+1} = (b_n, s_{n+1}) \text{ if } s_{n+1} > s_n$$
$$I_{n+1} = (s_{n+1}, c_n) \text{ if } s_{n+1} < s_n$$

Now, take any $L \in \bigcap_{k=1}^{\infty} I_k$, which we know exists by NIP since $I_{n+1} \subseteq I_n$. We can show s_n converges to L, since the size of any interval I_n is $|s_n - s_{n-1}| = |a_n|$, so for any $\epsilon > 0$, we can show that s_n past some N will be within an ϵ-neighborhood of L.

(c) (s_n) is bounded by $|a_1|$, so $(s_{2n}), (s_{2n+1})$ are both bounded. These sequences also happen to be monotonic, since one is increasing and the other is decreasing. Therefore, the two subsequences are convergent, and we can add them together to get another convergent sequence, which is (s_n).

Exercise 2.7.2

(a) The hints in the text are already a lot.

If the (b_n) series converges, then for any $\epsilon > 0$, we know $\exists N : n > m \geq N$, such that

$$\epsilon > \left| \sum_{i=m+1}^{n} b_i \right| > \left| \sum_{i=m+1}^{n} a_i \right|$$

so this N works for (a_n) series too.

If the (a_n) series diverges, then we can AFSOC (b_n) series converges, and use what we proved above to show by contradiction that (a_n) series converges.

(b) (a_n) series is increasing and bounded by (b_n) series, so it must converge. For (a_n) series diverging, We can do a similar AFSOC argument in part (a), where we can AFSOC (b_n) converges, which then we can show by contradiction that (a_n) series is converging.

Exercise 2.7.3

(a) If $\sum a_n$ diverges, then AFSOC $\sum p_n$ and $\sum q_n$ converge. Then $\sum p_n + \sum q_n = \sum a_n$ converges, but this is a contradiction.

(b) If $\sum a_n$ converges conditionally, WLOG AFSOC $\sum p_n$ converges. Then $\sum a_n - \sum p_n = \sum q_n$ must converge as well. $\left|\sum q_n\right|$ will also converge, since $\sum q_n$ does, and this equals $\sum |q_n|$. Then,

$$\sum |a_n| = \sum |p_n| + \sum |q_n|$$

which we know converges since $\sum |p_n| = \sum p_n$ and we just showed $\sum |q_n|$ converges. This is a contradiction since we assumed $\sum a_n$ converges conditionally.

Exercise 2.7.4

Define

$$x_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases} \qquad y_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Then $\sum x_n, \sum y_n$ both diverge, but $\sum x_n y_n = 0$.

Exercise 2.7.5

(a) If $\sum a_n$ converges absolutely, then $\sum |a_n|$ converges to some L, so

$$L^2 = \left(\sum |a_n|\right)^2 = \sum |a_n|^2 + S$$
$$L^2 \leq \sum |a_n|^2 = \sum a_n^2$$

Since $\sum_{n=1}^k a_n^2$ is an increasing sequence and is bounded, we conclude $\sum a_n^2 = \sum \left|a_n^2\right|$ converges.

This proposition does not hold without absolute convergence. Take $a_n = (-1)^n \frac{1}{\sqrt{n}}$, which converges by the alternating series test. Then $a_n^2 = \frac{1}{n}$, which is the harmonic series, and we know this does not converge.

(b) No, take $a_n = \frac{1}{n^2}$, which converges. Then $\sum \sqrt{a_n}$ is the harmonic series, which diverges.

Exercise 2.7.6

(a) Call M the bound of y_n. If $\sum x_n$ converges absolutely to L, then

$$\begin{aligned}\sum x_n y_n &\leq \sum |x_n y_n| \\ &\leq \sum |x_n||y_n| \\ &\leq \sum |x_n| M \\ &\leq LM\end{aligned}$$

$\sum |x_n y_n|$ converges by the Monotone Convergence Theorem because the partial sums are increasing and it is bounded above. Then, by the Absolute Convergence Test we can conclude $\sum x_n y_n$ also converges.

(b) Let $x_n = \frac{(-1)^n}{n}$ be the alternating harmonic series, and $y_n = (-1)^n$. Then $\sum x_n$ converges but $\sum x_n y_n$ is the harmonic series, which does not converge.

Exercise 2.7.7

We are going to bound our p-series with another series, and show that the other series converges.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \cdots \\ &\leq 1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{8^p} + \cdots \\ &= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots \\ &= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots \\ &= \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^p}{2^p - 2} && \text{(Only if } p > 1\text{)}\end{aligned}$$

By the Monotone Convergence Theorem, since the partial sums of the p-series is increasing, and there is an upper bound, we conclude that the p-series converges.

Notice that the convergence of p-series is often proved with calculus, but this is a nice alternative.

Exercise 2.7.8

Informally, you use the fact that both partial sums s_n^a, s_n^b will converge, then use the N_a, N_b and choose $N = \max(N_a, N_b)$, so that both partial sums are $\epsilon/2$ close to A, B. Then, triangle inequality to bound s_n^{a+b}, which will be $< 2 \cdot \epsilon/2 = \epsilon$.

Exercise 2.7.9

(a) This r' exists, since $a = \frac{r+1}{2}, r < a < 1$. We know

$$\begin{aligned} \left| \left| \frac{a_{n+1}}{a_n} \right| - r \right| &< \epsilon \\ \left| \frac{a_{n+1}}{a_n} \right| &< r + \epsilon && \text{(Also } > r - \epsilon\text{, but we don't need it)} \\ |a_{n+1}| &< |a_n|(r + \epsilon) \end{aligned}$$

We can choose N large enough so that $\epsilon < 1 - r$.

(b) Since $|r'| < 1$, we know $\sum r'^n$ converges, so $|a_N|$ times that also converges.

(c) We can bound the leading terms up to $n < N$ of $\sum |a_n|$ by some M. For the tail end, we can bound it from part (b). Since there exists an upper bound for this series, and its partial sums are increasing, we conclude that the partial sums converge and the overall sum does too.

Exercise 2.7.10

These are not proved very rigorously, but outline most of the ideas.

(a) If $\lim(na_n) = l$, then for any $\epsilon > 0, \exists N : n \geq N$ such that

$$\begin{aligned} |na_n - l| &< \epsilon \\ |a_n - l/n| &< \frac{\epsilon}{n} < \epsilon, \end{aligned}$$

so $\lim a_n = l/n$. Then since $l \neq 0$, $\sum a_n$ converges to a multiple of the harmonic series, which diverges, so $\sum a_n$ will too.

(b) Similar to the proof above in part (a), we can show $\lim a_n = l/n^2$. This converges to a multiple of $1/n^2$, which converges, so $\sum a_n$ also converges.

Exercise 2.7.11

An easy example,

$$\begin{aligned} (a_n) &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \\ (b_n) &= 2 - \frac{1}{2} + 2 - \frac{1}{4} + 2 - \frac{1}{6} + \cdots \end{aligned}$$

Then their $\sum \min\{a_n, b_n\}$ is the alternating harmonic series, and

- (a_n) diverges because it is the Harmonic series
- (b_n) diverges because every pair sums to > 1, so it sums an infinite number of numbers > 1.

For the challenge, an idea is to interweave a convergent and a divergent sequence together, so that (a_n) and (b_n) will both be divergent, since they both contain the divergent sequence, but the min only selects elements from the convergent sequence.

Exercise 2.7.12
Just verifying an identity,

$$\begin{aligned}
s_ny_{n+1} - s_my_{m+1} + \sum_{j=m+1}^{n} s_j(y_j - y_{j+1}) &= s_ny_{n+1} - s_my_{m+1} + (x_1 + \cdots x_{m+1})(y_{m+1} - y_{m+2}) + \\
&\quad (x_1 + \cdots x_{m+2})(y_{m+2} - y_{m+3}) + \cdots + (x_1 + \cdots x_n)(y_n - y_{n+1}) \\
&= s_ny_{n+1} - s_my_{m+1} + (x_1 + \cdots x_{m+1})y_{m+1} - \\
&\quad (x_1 + \cdots x_n)y_{n+1} + \sum_{j=m+2}^{n} x_jy_j \\
&= s_ny_{n+1} - s_ny_{n+1} - s_my_{m+1} + s_{m+1}y_{m+1} + \sum_{j=m+2}^{n} x_jy_j \\
&= x_{m+1}y_{m+1} + \sum_{j=m+2}^{n} x_jy_j \\
&= \sum_{j=m+1}^{n} x_jy_j
\end{aligned}$$

Exercise 2.7.13
(a) Using Exercise 2.7.12, we have

$$\begin{aligned}
\left|\sum_{j=m+1}^{n} x_jy_j\right| &= \left|s_ny_{n+1} - s_my_{m+1} + \sum_{j=m+1}^{n} s_j(y_j - y_{j+1})\right| \\
&= |s_ny_{n+1} - s_my_{m+1}| + \left|\sum_{j=m+1}^{n} s_j(y_j - y_{j+1})\right| && (\triangle \text{ inequality}) \\
&= \big|(s_n - s_m)y_{m+1}\big| + \left|\sum_{j=m+1}^{n} s_j(y_j - y_{j+1})\right| && (y_{m+1} > y_{n+1}) \\
&\le M|y_{m+1}| + M|y_{m+1} - y_{n+1}| \\
&\le 2M|y_{m+1}|
\end{aligned}$$

(b) For any $\epsilon > 0$, since (y_n) converges, make $|y_{m+1}| < \epsilon/(3M)$. Then

$$\begin{aligned}
\left|\sum_{j=m+1}^{\infty} x_j y_j\right| &\leq \left|\sum_{j=m+1}^{n} x_j y_j\right| + \left|\sum_{j=n+1}^{\infty} x_j y_j\right| \\
&\leq 2M|y_{m+1}| + \left|\sum_{j=n+1}^{\infty} x_j y_j\right| && \text{(From part (a))} \\
&\leq 2M|y_{m+1}| + \left|\sum_{j=n+1}^{\infty} x_j y_{m+1}\right| && \text{(Since for } n \geq m+1, y_{m+1} \geq y_n\text{)} \\
&\leq 2M|y_{m+1}| + |y_{m+1}|\left|\sum_{j=n+1}^{\infty} x_j\right| \\
&\leq 2M|y_{m+1}| + |y_{m+1}|M && \text{(Partial sums of } (x_n) \text{ bounded by } M\text{)} \\
&\leq 3M|y_{m+1}| \\
&\leq 3M\frac{\epsilon}{3M} = \epsilon
\end{aligned}$$

(c) We have $x_n = (-1)^{n+1}, y_n = a_n$.

Exercise 2.7.14

(a) Abel's test requires that $\sum x_n$ converges, which is stronger than the boundedness of the partial sums of (x_n). However, it only needs that (y_n) is non-negative and decreasing, which is weaker than Dirichlet, which in addition needs the limit to converge to 0.

(b) Using Exercise 2.7.13, part (a), we have

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leq 2A|b_1|$$

(c) We can define $a_n = x_{m+n}, b_n = y_{m+n}$, and bound

$$\left|\sum_{j=m+1}^{n} x_j y_j\right| = \left|\sum_{j=1}^{n} a_j b_j\right| \leq 2A|b_1|.$$

Now, we want to show we can make this bound arbitrarily small, since if we can make the tail end of this series arbitrarily small, then by the Cauchy Criterion for series we can conclude $\sum x_n y_n$ converges. We know that $\sum x_n$ converges, so by the Cauchy Criterion, for any $\epsilon > 0$, we can find some $N : \forall n' > m' \geq N$ such that

$$\left|\sum_{j=m'}^{n'} x_j\right| < \epsilon$$

Now, all we have to do is choose N so that $\left|\sum_{j=m'}^{n'} x_j\right| < \epsilon/(2b_1)$ then $A < \epsilon/(2b_1)$ and $\left|\sum_{j=m+1}^{n} x_j y_j\right| < 2|b_1| \cdot \frac{\epsilon}{2|b_1|}$, which means we have the Cauchy Criterion for $\sum x_n y_n$, and therefore it converges.

2.8 Double Summations and Products of Infinite Series

Exercise 2.8.1

$$\begin{aligned}\lim s_{nn} &= -1 + -\frac{1}{2} - \frac{1}{4} - \cdots \\ &= -\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i \\ &= -2.\end{aligned}$$

The value is equal to summing column-wise.

Exercise 2.8.2
By the Absolute Convergence test, since we know for fixed i that $\sum_{j=1}^{\infty}\left|a_{ij}\right|$ converges, then we know for fixed i that each $\sum_{j=1}^{\infty} a_{ij}$ converges to some c_i as well.

Then, since

$$\begin{aligned}\sum_{j=1}^{\infty}\left|a_{ij}\right| &\geq \left|\sum_{j=1}^{\infty} a_{ij}\right| \\ \Rightarrow b_i &\geq |c_i| \\ |b_i| &\geq |c_i|,\end{aligned}$$

and we know that $\sum_{i=1}^{\infty} b_i$ converges, we conclude that $\sum_{i=1}^{\infty} c_i$ must converge as well by the Absolute Convergence test, implying that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \tag{2.18}$$

converges as well.

Exercise 2.8.3

(a) Since $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right|$ converges, we have that

$$t_{mn} \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right| = L$$

Since t_{nn} is an increasing sequence, and is bounded above, by the Monotone Convergence Theorem, t_{nn} converges.

(b) For any $\epsilon > 0$, $\exists N : n > m \geq N$ such that $|t_{nn} - t_{mm}| < \epsilon$. Now, consider

$$\begin{aligned}\left|s_{n+1,n+1} - s_{nn}\right| &\leq \left|t_{n+1,n+1} - t_{nn}\right| \\ &< \epsilon.\end{aligned}$$

So (s_{nn}) is a Cauchy Sequence and converges.

Exercise 2.8.4

(a) Since we know there exists a $t_{m_0n_0}$ such that $t_{n_0n_0} > B - \frac{\epsilon}{2}$, and t_{nn} is increasing and that B is an upper bound, we can conclude that for $N_1 = \max\{m_0, n_0\} : m, n \geq N_1$,

$$B - \frac{\epsilon}{2} < t_{mn} \leq B. \tag{2.19}$$

(b) For any $\epsilon > 0$, since (t_{mn}) is bounded above by $A = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$, from part (a) we can choose $N : m, n \geq N$ such that

$$\begin{aligned} A + \frac{\epsilon}{2} &< t_{mn} < A + \epsilon \\ \Rightarrow \frac{\epsilon}{2} &< |t_{mn} - A| < \epsilon \\ &\left| t_{mn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| \right| < \epsilon. \end{aligned}$$

Then, we can see that this N also works to show

$$\begin{aligned} |s_{mn} - S| &= \left| s_{mn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \right| \\ &= \left| \sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} a_{ij} \right| \\ &< \left| \sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} |a_{ij}| \right| \\ &= \left| t_{mn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| \right| \\ &< \epsilon. \end{aligned}$$

Exercise 2.8.5
We know $\lim_{n\to\infty}\sum_{j=1}^{n} a_{ij} = r_i$, so for any $\epsilon > 0$, $\exists N : n \geq N$ such that

$$\left| \sum_{j=1}^{n} a_{ij} - r_i \right| < \frac{\epsilon}{m}.$$

if we fix $m \geq N$.

Then

$$\begin{aligned} \left|(r_1 + r_2 + \cdots + r_m) - S\right| &= \left| \sum_{i=1}^{m}\left(r_i - \sum_{j=1}^{n} a_{ij} \right) \right| \\ &\leq \sum_{i=1}^{m}\left| r_i - \sum_{j=1}^{n} a_{ij} \right| \\ &\leq m \cdot \frac{\epsilon}{m} = \epsilon \end{aligned}$$

Therefore, we conclude that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S.

TODO not sure where I have to use the Order Limit Theorem...

Exercise 2.8.6
For $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$, the proof is essentially the same as Exercise 2.8.5 to show it converges to S, except we fix n this time instead of m.

Exercise 2.8.7

(a) Define $t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|$. Also define $u_n = \sum_{k=2}^{n}|d_k|$ Then we know for $n \geq 2$,

$$u_n \leq t_{nn} = L \qquad (t_{nn} \text{ converges})$$

Since u_n is an increasing sequence and is bounded above, we conclude from the Monotone Convergence Theorem that $u_n = \sum_{k=2}^{n} |d_k|$ also converges. Then, $\sum_{k=2}^{n} d_k$ converges absolutely.

(b) We need to bound $|d_k - S|$ somehow. Consider the following diagram

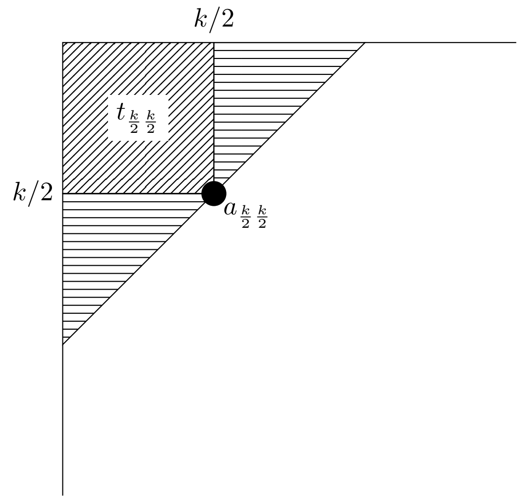

Figure 2.1: Demonstrating how we bound our sum

We know that for any $\epsilon > 0$, we can choose $N : n \geq N$ so that

$$|t_{nn} - S| < \epsilon \tag{2.20}$$

Now, choose $N_1 = 2N$. We can use Figure 2.1 to see that for $k \geq N_1$,

$$\begin{aligned} |d_{kk} - S| &\leq \left| t_{\frac{k}{2}\frac{k}{2}} - S \right| \\ &< \epsilon && \text{(From Equation (2.20))} \end{aligned}$$

Therefore, we can conclude $\sum_{k=2}^{\infty} d_k$ converges to S.

Exercise 2.8.8

(a) See that

$$\begin{aligned} AB &\geq \left(\sum_{i=1}^{\infty} |a_i| \right) \left(\sum_{j=1}^{\infty} |b_j| \right) \\ &= \sum_{i=1}^{\infty} \left(|a_i| \sum_{j=1}^{\infty} |b_j| \right) && \left(\textstyle\sum_{j=1}^{\infty} |b_j| \text{ is constant wrt } i\right) \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_i||b_j| \right) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j| \end{aligned}$$

Since $\sum_{i=1}^{m} \sum_{j=1}^{n} |a_i b_j|$ is bounded, and the partial sums $s'_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_i b_j|$ are increasing, we can conclude $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges by the Monotone Convergence Theorem.

(b) Let s_n^a, s_n^b be the partial sums of $(a_n), (b_n)$ respectively. Then,

$$\begin{aligned}
\lim_{n\to\infty} s_{nn} &= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \\
&= \lim_{n\to\infty} \sum_{i=1}^{n}\left(a_i \sum_{j=1}^{n} b_j\right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right) \\
&= \lim_{n\to\infty} s_n^a s_n^b \\
&= AB
\end{aligned}$$

Therefore, by Theorem 2.8.1, we can conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_i b_j = \sum_{i=2}^{\infty} d_k = AB$$

Chapter 3

Basic Topology of $\mathbb{R}$

3.2 Open and Closed Sets

Exercise 3.2.1

(a) We need a finite number of sets when we are choosing the minimum ϵ for our V_ϵ. If we had an infinite number of sets, this minimum may not exist.

(b) Let

$$O_n = \left(\sum_{i=1}^{n} \frac{1}{2^i}, 3 - \sum_{i=1}^{n} \frac{1}{2^i} \right)$$

Then $\bigcap_{n=1}^{\infty} O_n = [1, 2]$.

Exercise 3.2.2

(a) 1 and -1 are the only limit points of B. For any fixed element of B, the distance between it and its neighbors is $\geq \frac{n}{n+1} - \frac{n+2}{n+3} = \frac{n}{(n+1)(n+3)}$, so we can just choose ϵ smaller than this, and show that any element of B is isolated.

For 1, we can show that for any $\epsilon > 0$, we can choose $\frac{1}{N+1} < \epsilon \Rightarrow N > \frac{1}{\epsilon} - 1$, and we know $\frac{n}{n+1}$ for $n \geq N$ for even n is in the ϵ-neighborhood of 1. Doing a similar analysis for negative terms and -1 yields the same result.

(b) B does not contain its limit points, so it is not closed.

(c) B is not an open set, continuous ϵ neighborhoods are not subsets of B.

(d) All of B's elements are isolated

(e) $\overline{B} = B \cup \{-1, 1\}$.

Exercise 3.2.3

(a) $\mathbb{Q}$ is not open, because it doesn't have irrationals that can be in the ϵ-neighborhoods. It is not closed, because it contains irrational limit points. Therefore it is **neither**.

(b) $\mathbb{N}$ does not have any limit points, so it is **closed**.

(c) $\mathbb{R}^+$ cannot be closed, because 0 is a limit point and not contained. It is **open** because every element has an ϵ-neighborhood that is a subset.

(d) Not closed, doesn't contain 0, a limit point. Not open, since 1 has no ϵ-neighborhood subset. Therefore, **neither**.

(e) The sequence converges, but this limit point is not in the set. No ϵ-neighborhoods exist for certain ϵ, for certain elements, so not open. **Neither**.

Exercise 3.2.4
FOr any $\epsilon > 0$, we know $\exists N : n \geq N$ such that

$$|a_n - x| < \epsilon,$$

so every ϵ-neighborhood of x has points other than itself.

Exercise 3.2.5
If there exists exists such an ϵ-neighborhood, then by the definition of a limit point, since there are no other elements other than x itself, then this is not a limit point, so it is isolated.

Exercise 3.2.6
If a set $F \subseteq \mathbb{R}$ is closed, then it contains all its limit points. For any Cauchy Sequence in F, it is also convergent to some L, which we know is a limit point and thus must be in F.

If every Cauchy Sequence of an F has its limit as an element of F, then every limit point, which comes from the limit of some subsequence, which we know is a Cauchy Sequence. From our original assumption, this limit must be in F, so F contains all its limit points and is closed.

Exercise 3.2.7
AFSOC an infinite number of (x_n) terms not in O. Then since $(x_n) \to x$, $\epsilon =$ distance of x from O boundary, then $\forall N, : n \geq N$ we have that $\exists x_n : |x_n - x| \geq \epsilon$, since we can choose some x_n not in O. This means this sequence does not converge. This contradicts our original assumption.

Exercise 3.2.8

(a) We want to show that L, which contains all the limit points of of A, is closed. We can do this by showing all limit points of L are in L.

Suppose we have some limit point ℓ of L, then this means some subsequence of L,

$$(l_n) \to \ell$$

By the definition of convergence, for any $\epsilon > 0$, we can find $N : n \geq N$ such that

$$|l_n - \ell| < \frac{\epsilon}{2}$$

Now, since l_n are limit points of A, we know $\exists a \in A$ such that a is arbitrarily close to l_n. Define a subsequence in A

$$\left\{a_n \in A, |a_n - l_n| < \frac{\epsilon}{2}\right\}$$

Then for $n \geq N$,

$$|a_n - \ell| < |l_n - \ell| + \frac{\epsilon}{2} < 2 \cdot \frac{\epsilon}{2} = \epsilon,$$

which means (a_n) converges to this ℓ as well, so ℓ is a limit point of A and $\ell \in L$.

Therefore, we conclude L contains all of its limit points, and therefore it is closed.

(b) For any limit point ℓ of $A \cup L$, it must the limit of some convergent subsequence of $A \cup L$. This subsequence will contain elements from A and L. What we can do is for every element in L, use a similar technique we did in part (a) to replace all the $x \in L$ subsequence elements with elements in A instead, that are arbitrarily close enough. Then, we have constructed a subsequence that entirely lies in A, so this limit point must be of A. Therefore, all limit points of $A \cup L$ are limit points of A.

We can then conclude that $\overline{A} = A \cup L$ is a closed set, since all of its limit points are of A, and those limit points are contained in L, which means $\overline{A}$ contains all of its limit points and is closed.

Exercise 3.2.9

(a) Suppose y is a limit point of $A \cup B$, then there must exist a subsequence $(x_n), x_n \in A \cup B$ where $(x_n) \to y$. Now, this subsequence must contain either an infinite number of elements from A or B (or both).

WLOG, (x_n) contains an infinite number of elements from A, then we know $\exists$ subsequence $(x'_n) \to y$ where $x'_n \in A$. This means y is a limit point of A.

Therefore, we conclude if y is a limit point of $A \cup B$, y is either a limit point of A or B.

(b) Let L_S be the set of limit points for a set S.

$$\begin{aligned} \overline{A \cup B} &= A \cup B \cup L_{A \cup B} && (3.1)\\ &= A \cup B \cup (L_A \cup L_B) \qquad = (A \cup L_A) \cup (B \cup L_B) && (3.2)\\ &= \overline{A} \cup \overline{B} && (3.3) \end{aligned}$$

(c) We notice that in our proof, we were able to find a subsequence that was entirely in one set. Therefore, if it is possible to construct a subsequence that doesn't fit entirely in one set, then we can find a limit point that is not necessarily a limit point of an individual set.

With an infinite number of sets, we can take advantage of this property.

Suppose we have some $(a_n) \to L$, where $\forall n \, a_n \neq L$. Then construct sets the following way,

$$S_n = \{a_n\}$$

Now, S_n has no limit points, since it only has a single point which is isolated. Therefore, $\bigcup_{n=1}^{\infty} \overline{S_n} = \bigcup_{n=1}^{\infty} S_n$, but we have $\overline{\bigcup_{n=1}^{\infty} S_n} = \left(\bigcup_{n=1}^{\infty} S_n\right) \cup \{L\}$, and $L \notin \bigcup_{n=1}^{\infty} S_n$. So the property in part (b) does not apply for infinite sets.

Exercise 3.2.10

(a) A direct proof (double containment is another way to do it)

$$\begin{aligned} x \in \left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^c &\Leftrightarrow \forall \lambda, x \notin E_\lambda\\ &\Leftrightarrow \forall \lambda, x \in E_\lambda^c\\ &\Leftrightarrow x \in \bigcap_{\lambda \in \Lambda} E_\lambda^c \end{aligned}$$

$$\begin{aligned} x \in \left(\bigcap_{\lambda \in \Lambda} E_\lambda\right)^c &\Leftrightarrow \exists \lambda, x \notin E_\lambda\\ &\Leftrightarrow \exists \lambda, x \in E_\lambda^c\\ &\Leftrightarrow x \in \bigcup_{\lambda \in \Lambda} E_\lambda^c \end{aligned}$$

(b) We want to show that

(i) *The union of a finite collection of closed sets is closed.* Suppose we have a collection of closed sets $\{E_\lambda, \lambda \in \Lambda\}$, then, if we take the complement of the union of all these sets, by DeMorgan's, we get the intersection of the complements of all these sets. The complements of all these sets is open, and we know the intersection of a finite number of open sets is also open. Finally, taking the complement again, we must have a closed set, which is equal to our original union.

(ii) *The intersection of an arbitrary collection of closed sets is closed.* Take the intersection of these closed sets, and then take the complement. By DeMorgan's, we know have the union of the complement of these sets, which we know is open. We know that the union of an arbitrary number of open sets is also open. Finally, taking the complement of this entire expression again, we now have a closed set, which is equal to our original intersection.

Exercise 3.2.11

If $s = \sup A$ exists, we have 2 cases. Either $s \in A \Rightarrow s \in \overline{A}$ since $A \subseteq \overline{A}$, or, $s \notin A$. In the second case, since we know for any $\epsilon > 0$, $\exists a \in A$ such that $a > s - \epsilon \Rightarrow \epsilon > |s - a|$, we can construct a subsequence in A that converges to s. This means s is a limit point of A, and therefore $s \in \overline{A}$.

Exercise 3.2.12

(a) True. $\overline{A}$ is closed, so $\overline{A}^c$ must be open.

(b) True. There is no ϵ-neighborhood around this point that is contained in A.

(c) False. Take the harmonic sequence $\{1/n\}$.

(d) True. See Exercise 3.2.11

(e) True. A finite set only contains isolated points, so therefore it has no limit points, and vacuously contains all of its limit points and is closed.

(f) True. We know that around $q \in \mathbb{Q}$, exists some ϵ-neighborhood around it that is contained in the set. Suppose we have an arbitrary $r \in \mathbb{R}$, then we want to show it is contained in the ϵ-neighborhood of some $q \in \mathbb{Q}$. We can show this by contradiction. AFSOC r is not in any of these ϵ-neighborhoods. That means $\forall \epsilon > 0, |r - q| > \epsilon$, for all q. But for any ϵ, we can always find some rational number that is closer than ϵ to r, which means this statement is false. We have reached a contradiction, and must assume our original hypothesis was true.

Exercise 3.2.13

We can verify $\mathbb{R}$ is open because any ϵ-neighborhood only contains elements of $\mathbb{R}$, so therefore $\subseteq \mathbb{R}$. In addition, any limit point $\in \mathbb{R}$, so $\mathbb{R}$ also contains all of its limit points and is closed.

$\emptyset$ is closed and open by vacuity.

Now, we need to show that there are no other sets with this property. We know the complement of an open set is closed and vice versa, so AFSOC $\exists A \neq \mathbb{R}, \emptyset$, then we know $\exists x \in A^c, \notin A$. Now, x cannot be an isolated point, since then it would not have an ϵ-neighborhood around it that is contained in A^c. Therefore, we conclude x must be in some continuous set S, where it either

(i) Has a $\sup S$. In this case, either $\sup S \in S$, in which case there does not exist an ϵ-neighborhood around $\sup S$, which means S is not open, or $\sup S \notin S$, and then $\sup S$ is a limit point, but then S is not closed since it doesn't contain all of its limit points. This case is not possible.

(ii) Does not have an upper bound. Then look at the portion less than x and apply the argument in part (i) but with $\inf S$. It must have a lower bound, or else $A = \mathbb{R}$.

Therefore, we reach a contradiction in all cases, and therefore we conclude that it is not possible for this A to exist.

Exercise 3.2.14

(a) $[a, b] = \bigcap_{i=1}^{n} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$. Any $x \in [a, b]$ will be $< b + \frac{1}{n}$, and $> a - \frac{1}{n}$. Now, let us consider some $y < a$. $y \notin$ the set we created, bc suppose $|y - a| = \epsilon$. Then for $n' > \frac{1}{\epsilon}$, $a - \frac{1}{n'} > y$, so y is not in this set. The argument for an element larger than b is symmetric. Therefore, we conclude the set we constructed is equivalent to $\{a, b\}$, and is an intersection of a countable number of open sets.

(b) We can write

$$(a, b] = \bigcup_{i=1}^{n} \left[a + \frac{1}{n}, b\right] \tag{3.4}$$

$$(a, b] = \bigcap_{i=1}^{n} \left(a, b + \frac{1}{n}\right) \tag{3.5}$$

(c) We know $\mathbb{Q}$ is countable, so just union all the sets containing only one element of $\mathbb{Q}$ together. Since each set has one element which is an isolated point, each set is closed.

$$\bigcup_{q \in \mathbb{Q}} \{q\}$$

We know that $\mathbb{Q}^c = \mathbb{I}$, and by DeMorgan's law, we know

$$\left(\bigcup_{i=1}^{\infty} S\right)^c = \bigcap_{i=1}^{\infty} S^c$$

Since S are all closed, S^c are all open. We can use the infinitely countable union of the construction of $\mathbb{Q}$ and then take the complement to get $\mathbb{I}$, which by DeMorgan's is constructed as a countably infinite intersection of open sets.

3.3 Compact Sets

Exercise 3.3.1
Since we know K is compact, it must also be closed and bounded.

Since K is bounded, it must have a least upper and largest lower bound, by the Axiom of Completeness, which means $\sup K$ and $\inf K$ must exist. Now, we can construct subsequences of K that converge to $\sup K, \inf K$, since by the property of $\sup K$, for example, we can always find an element of K that is some ϵ close to it. So for example we can construct a subsequence where $a_n = k_n, k_n \in K$ such that $|\sup K - k| < \frac{1}{2^n}$. The same logic applies to $\inf K$, so they are limit points of K. Since K is closed, $\sup K, \inf K \in K$.

Exercise 3.3.2
Suppose we have some $K \subseteq \mathbb{R}$ that is closed and bounded. We want to show that it is compact.

We know that any sequence of K must be contained in K, which is bounded, so therefore by the Bolzano-Weierstrauss Theorem, we know that this sequence must have a convergent subsequence. Since K is also closed, this limit must be in K as well.

This shows that K is compact.

Exercise 3.3.3
We want to show the Cantor set is compact.

We know the Cantor set is $\subseteq [0, 1]$, so it is bounded.

Then, we know the complement of the Cantor set is

$$(-\infty, 0) \cup (1, \infty) \cup \left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \cdots \right]$$

which is the union of an arbitrary number of open sets, which we know is open. Therefore, the Cantor set is the complement of an open set, which is closed.

Therefore, since the Cantor set is bounded and closed, we conclude it is compact.

Exercise 3.3.4
We have that K is compact and F is closed. Since K is compact, it is also bounded and closed.

If we take $K \cap F$, we know that this must also be bounded, since $x \in K \cap F \Rightarrow x \in K$.

The intersection of two closed sets is also closed, so $K \cap F$ is closed.

Therefore, $K \cap F$ is bounded and closed, and thus is compact.

Exercise 3.3.5

(a) We can find a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$, but we know that $\sqrt{2} \notin \mathbb{Q}$, so $\mathbb{Q}$ is not compact.

(b) Again, similar to part (a), we can find a sequence that converges to $\sqrt{2}/2 \notin [0, 1] \cap \mathbb{Q}$.

(c) Take the sequence $a_n = n$. There is no limit, so this sequence does not have a subsequence that converges.

(d) $\mathbb{R} \cap [0, 1] = [0, 1]$ is closed and bounded, so it is compact.

(e) This sequence converges to 0, but does not contain 0, so it is not closed and thus not compact.

(f) Every subsequence of this set converges to 1, which is in the set, so therefore this set is compact.

Exercise 3.3.6

(a) We will prove this by induction.

- **Base Case:** $n = 1$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ We know a combination of two elements in $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ covers $\left[0, \frac{4}{3}\right]$. Then, combination of two elements in the latter set covers $\left[\frac{4}{3}, 2\right]$. Therefore, two elements $x, y \in C_1$ can add up to any element $\in [0, 2]$.
- **Inductive Hypothesis:** Suppose for $k \geq 1$, any two elements of C_k can add up to any element $\in [0, 2]$.

- **Inductive Step** We know $C_{k+1} = C_k/3 + \left\{\frac{2}{3} + C_k/3\right\}$, in other words, we are now missing the middle thirds of both the head and tail sets. What we want to show is that for the head set (and the same argument holds for the tail set), that with the middle third removed, we can still cover all of the original set, which means since the original sets are still covered, we can still cover $[0, 2]$. By the IH, any two elements of $C_k/3$ will cover $\left[0, \frac{2}{3}\right]$, and any two elements of $\left\{\frac{2}{3} + C_k/3\right\}$ covers $\left[\frac{4}{3}, 2\right]$. Choosing two elements from one of each set covers $\left[\frac{2}{3}, \frac{4}{3}\right]$. These three intervals cover $[0, 2]$.

(b) The reason (x_n), (y_n) may not converge is they can be picked out of sets, jumping across different subsets of C infinitely many times.

However, by the Bolzano-Weierstrauss Theorem, since (x_n) is contained entirely in the Cantor set, which is bounded, then (x_n) is also bounded. Therefore, it must contain a convergent subsequence. The same applies for (y_n), and we can take their limits l_x, l_y such that $l_x + l_y = s$.

Exercise 3.3.7

(a) True. The intersection will be bounded, since we can take any bound of a set, which will bound the intersection, and the intersection of an arbitrary number of closed sets is still closed. Therefore, this arbitrary intersection of compact sets is closed and bounded, which means it is also compact.

(b) False. Let $A = (0, 1), K = [0, 1]$, then $A \cap K = (0, 1)$, which is not closed so it is not compact.

(c) True. This is the Nested Interval Property.

(d) True. A finite set always closed, since there are no limit points, and bounded.

(e) False. Choose an unbounded countable set like $a_n = n$.

Exercise 3.3.8

(a) If they both have finite subcovers, then we can union those two finite subcovers to get a finite subcover for K, so we need at least one of them to not have a finite subcover.

(b) Create I_{n+1} by bisecting I_n, and taking a half that has no finite subcover. Such a half has to exist, because if both have finite subcovers, then the whole must have a finite subcover. The interval will half in size every iteration.

(c) By the Nested Interval Property, $\exists x \in K \forall x \in I_n$.

(d) Since the interval sizes get arbitrarily small, we can find some n_0 such that I_{n_0} fits entirely in O_{λ_0}. We have reached a contradiction, because we can finitely cover $I_n \cap K$ by taking

$$\left(\bigcup_{i=1}^{n_0-1} I_i\right) \cup O_{\lambda_0}$$

Exercise 3.3.9

(a) Open cover where you take some $\epsilon > 0$ around all rational numbers, and then union them together. No finite subcover exists, since that would bound $\mathbb{Q}$, which is not bounded.

(b) We can construct an open cover like

$$\left(\bigcup_{i=1}^{\infty}(-0.5, \frac{\sqrt{2}}{2} - \frac{1}{i})\right) \cup \left(\bigcup_{i=1}^{\infty}(\frac{\sqrt{2}}{2} + \frac{1}{i}, 1.1)\right)$$

This will contain all the rational numbers between $[0, 1]$, but needs an infinite number of sets to cover the entire set, because if we stop prematurely, we won't be able to capture the rationals that are very close to $\frac{\sqrt{2}}{2}$.

(e) An open cover for this set is

$$\bigcup_{i=1}^{\infty}\left(1.1, 1-\frac{i-1}{i}\right)$$

We need all of these open sets, because otherwise if for some N we stop, then we won't have the elements $< \frac{1}{N}$.

Exercise 3.3.10

For any closed set with an interval $[a, b]$, we can make intervals $I_n = \left[a, b - 1/n\right]$, and then union with $[b, b+1]$ to cover the set; with open intervals, we don't need the end. However, we will need all of the intervals, otherwise we won't include all the elements close to b. Therefore, any *clompact* subset must be a finite set of isolated points.

3.4 Perfect Sets and Connected Sets

Exercise 3.4.1
A perfect and a compact set are always closed, so their intersection is also closed. Since a compact set is bounded, the intersection of it and another set must also be bounded. Therefore, $P \cap K$ is a compact set.

We cannot guarantee that $P \cap K$ does not have isolated points, so for example, $[0,1] \cap \{1/2\} = \{1/2\}$ which is not perfect.

Exercise 3.4.2
A perfect set cannot only consist of rationals, because it would be nonempty and countable, since $\mathbb{Q}$ is countable, but this is impossible since any nonempty perfect set is uncountable.

Exercise 3.4.3

(a) $x_1 \in C_1$ implies that x_1 is in a closed interval of size $1/3$, so we can choose any element in this interval such that $x \neq x_1$, and we must have $|x - x_1| \leq 1/3$

(b) We can make the argument for any $x_n \in C_n$, since x_n must exist in a closed interval of size $\frac{1}{3}^n$, so we can any other element in this interval x, so that $|x - x_n| \leq \frac{1}{3}^n$. Now, we can construct a subsequence (x_n) such that $(x_n) \to x$, and this shows that there are no isolated points in C. We know the Cantor set is closed from earlier exercises.

Exercise 3.4.4

(a) This set is bounded, and is also closed since it is an intersection of an arbitrary collection of closed sets. Therefore, this construction is compact.

This set is also perfect, because we can use the same argument from Exercise (b) to show that there are no isolated points.

(b) We can compute the

- **Length**: We will compute the removed interval lengths,

$$\frac{1}{4} + 2 \cdot \frac{3}{32} + 4 \cdot \frac{27}{256} + \cdots = \frac{1/4}{1 - \frac{3}{2^2}} = \boxed{1}$$

So this Cantor-like set has length 0.

- **Dimension**: We have $3 - 3 \cdot \frac{1}{4} = \frac{9}{4}$, so solving

$$3^x = \frac{9}{4} \Rightarrow \boxed{0.738}$$

This is "larger" in dimension than the ternary Cantor set.

Exercise 3.4.5
If we have that $A \subseteq U, B \subseteq V$ such that U, V are disjoint open sets, then we know $A \cap B = \emptyset$. Therefore, if we want to show that they are separated, we just need to show that the limit points of A are disjoint from B, and vice versa.

AFSOC that a limit point of $A, \ell_A \in B$. Then this ℓ_A is also a limit point of U, since $A \subseteq U$. Now, this $\ell_A \in V$, since $B \subseteq V$. This means ℓ_A is ϵ far away from an element $\in U$, and since ℓ_A is also in the open set V, we must have that $\ell_A \in [v_1, v_2] \subseteq V$, where $v_1 < \ell_A < v_2$. Let $\epsilon = (\ell_A - v_1)/2$, then $\ell_A - \epsilon \in V$, and since ℓ_A is a limit point of U, we must also have that $\ell_A - \epsilon \in U$. However, this is a contradiction, because we just showed that $\ell_A - \epsilon \in U$ and $\in V$, which means $U \cap V \neq \emptyset$, and they are not disjoint. The same argument applies for the limit points of B not being in A. Therefore, we can conclude that $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, and that A, B are separated.

Exercise 3.4.6
($\Rightarrow$) Suppose $E \subseteq \mathbb{R}$ is connected. Then consider some sets A, B such that $A \cup B = E$, and A, B are nonempty and disjoint. AFSOC every convergent sequence $(x_n) \to x$ with (x_n) contained in A or B, $x \notin$ the other set. Then this must mean that every limit point of A or B is not in the other set, which means $\overline{A} \cap B = \overline{B} \cap A = \emptyset$,

and A, B are separated. This means $E = A \cup B$ for separated A, B, which is a contradiction since we assumed E was connected.

($\Leftarrow$) Suppose for all nonempty and disjoint A, B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A, or B, and x is an element of the other. Suppose (x_n) is contained within A. Then x must be a limit point of A, since $x \in B$, and A is disjoint of B so $x \notin A$. This means $\overline{A} \cap B = x \cup S \neq \emptyset$, which means A, B are not separated. Therefore, we cannot find separated sets A, B such that $E = A \cup B$, which means E is not disconnected, and therefore is connected.

Exercise 3.4.7

(a) Take $E = (-\infty, 1) \cup (1, \infty)$, then the closure is $\mathbb{R}$, which is closed, but this set is disconnected because $\big((-\infty, 1) \cup \{1\}\big) \cap (1, \infty) = \emptyset$ and $(-\infty, 1) \cap \big((1, \infty) \cup \{1\}\big) = \emptyset$ so these two sets are disconnected, and their union is equal to E.

(b) If A is connected, we can show that any limit points must already be in A, so $A = \overline{A}$ and $\overline{A}$ is still connected. If A is perfect, it is already closed, so it contains all of its limit points, and $\overline{A} = A$, and therefore $\overline{A}$ is still perfect.

Exercise 3.4.8

(a) Given any two rational x, y, WLOG $x < y$. Then $\exists r \in (x, y)$ such that $r \in \mathbb{I}$, i.e. it is not rational. Then we have $\mathbb{Q} = \big(\mathbb{Q} \cap (-\infty, r)\big) \cup \big(\mathbb{Q} \cap (r, \infty)\big)$, and these two sets are disconnected.

(b) Irrational numbers are also totally disconnected using the same argument in part ((a))

Exercise 3.4.9

(a) We know in C_n, it consists of intervals of size $\frac{1}{3}^n$. If the intervals are smaller than ϵ, i.e. $\frac{1}{3}^n < \epsilon$, then x, y must be in different intervals.

(b) If we know x, y are in different intervals, then between their intervals there must exist removed intervals in the construction of C. We can take z to be in one of these removed intervals. Given any $(a, b), a < b$, we have a few cases. If a or b is not in C, then (a, b) is not in C. If $a, b \in C$, then we can use the argument we just made to find some $z \in (a, b)$, such that $z \notin C$, which means $(a, b) \not\subseteq C$.

(c) For any $x, y \in C$, WLOG $x < y$, we can find $z \notin C, x < z < y$ such that $C = \big(C \cap (-\infty, z)\big) \cup \big(C \cap (z, \infty)\big)$. Therefore, C is totally disconnected.

Exercise 3.4.10

(a) O contains all the rational numbers (and some irrational numbers), so the complement O^c must only consist of irrational numbers.

(b) F only consists of closed intervals. F is totally disconnected, because for any $x, y \in \mathbb{I}, x < y$, we can find a rational number q where $q = r_n, \epsilon_n = 1/2^n < |x - y|/2$, so it fits in between x, y. Then we can make F with the union of two open sets intersected with F at that boundary.

(c) We know F is closed, since we are taking an arbitrary intersection of closed sets. F is not always perfect, since we can create isolated points, for example, by having open sets have end points that converge to some irrational number. The issue with our construction was that we would allow the ϵ neighborhoods to get arbitrarily close to irrational numbers, and sort of "squeeze" them into isolated points.

One trivial way to prevent this is to have some sort of minimum neighborhood size, but then $F = \emptyset$.

A less trivial, but vague way of construction, is to just get rid of the isolated irrational points. There can only be a countably infinite number of these, and there are uncountably many irrationals, so in this case $F \neq \emptyset$.

TODO Find a better construction for a perfect set of irrationals.

3.5 Baire's Theorem

Exercise 3.5.1
We can use DeMorgan's Law to show both directions, so that an countable union of closed sets becomes a countable intersection of open sets, and vice versa.

Exercise 3.5.2
(a) Countable, can use a similar proof to $\mathbb{N}^2$ countability to show it is still countable.

(b) Finite.

(c) Finite.

(d) Countable.

Exercise 3.5.3
Already done in Exercise 3.2.14.

Exercise 3.5.4
(a) Suppose $G_i = \left(g_i^1, g_i^2\right)$, then let $M = \left|g_i^2 - g_i^1\right|$, and define

$$I_i = \left[g_i^1 + M/4, g_i^2 - M/4\right]$$

Essentially we are making each G_i interval smaller so it can be closed, and we still have $I_i \subseteq G_i$.

(b) Now, we can use the Nested Interval Property to show that there exists an subsequence converging to some $x \in G_i \forall i$, so therefore this intersection is not empty.

Exercise 3.5.5
Suppose we could write $\mathbb{R}$ as a F_σ set, then we must have $\mathbb{R}^c = \bigcap_{n=1}^{\infty} G_n$, where G_n are dense, open sets. We just showed in Exercise 3.5.4 that this intersection is not empty, which is a contradiction, since $\mathbb{R}^c = \emptyset$.

Exercise 3.5.6
We know $\mathbb{Q}$ is an F_σ set, so if $\mathbb{I}$ were also an F_σ set, then that would imply $\mathbb{R}$ is also an F_σ set, which we proved in Exercise 3.5.5 is not.

Therefore, $\mathbb{I}$ is not an F_σ set, and its complement, $\mathbb{Q}$, cannot be a G_σ set.

Exercise 3.5.7
Take the construction of the Cantor set, except start with $(0, 1)$ and remove the closed 1/3 interval in the middle, i.e. $[1/3, 2/3]$ in the start, each time.

We can show there are an uncountable number of sets by using a diagonalization argument.

Exercise 3.5.8
($\Rightarrow$) If E is nowhere-dense then $\overline{E}$ contains no nonempty open intervals. This means $\overline{E}^c$ consists only of open intervals, and is $\mathbb{R}$ without $\overline{E}$. We also can see that $\overline{\overline{E}^c}$ will be $\mathbb{R}$, since the limit points of $\overline{E^c}$ are just $\overline{E}$. Therefore, $\overline{E}^c$ is dense.

($\Leftarrow$) If $\overline{E}^c$ is dense in $\mathbb{R}$, then $\overline{E}^c \cup L = \mathbb{R}$. Taking the complement and applying DeMorgan's, we get $\overline{E} \cap L^c = \emptyset$. If $\overline{E}$ is non-empty, then it consists of the limit points of $\overline{E}^c$, i.e. $\overline{E} \subseteq L$.

Now, $\overline{E}$ cannot consist of any open intervals, because otherwise, suppose some limit point of $\overline{E}^c$ is in some interval (a, b). Then there is some distance M from the limit point l to a or b. There is no sequence in $\overline{E}^c$ that converges to this point, since it is too far away from any point in $\overline{E}^c$; (a, b) is not in $\overline{E}^c$ at all.

Therefore, since $\overline{E}$ has no nonempty open intervals, we conclude E is nowhere-dense.

Exercise 3.5.9
(a) In between.

(b) Nowhere-dense.

(c) Dense in $\mathbb{R}$.

(d) In between.

Exercise 3.5.10

If we can write $\mathbb{R}$ as countable union of nowhere-dense sets, these nowhere-dense sets have no nonempty open sets, which means they are closed.

Therefore, we can write $\mathbb{R}$ as a F_σ set, which we showed in Exercise 3.5.5 was not possible.

Chapter 4

Functional Limits and Continuity

4.2 Functional Limits

Exercise 4.2.1

(a) Let $\epsilon > 0$, then notice

$$\begin{aligned}\big|f(x) - 8\big| &= |2x + 4 - 8| \\ &= |2x - 4| \\ &= 2|x - 2|\end{aligned}$$

So we can choose $\delta = \epsilon/2$, which will imply $2|x - 2| < 2 \cdot \epsilon/2 = \epsilon$.

(b) Choose $\delta = \sqrt[3]{\epsilon}$

(c) We can simplify

$$\begin{aligned}\big|f(x) - 8\big| &= \Big|x^3 - 8\Big| \\ &= |x - 2|\Big|x^2 + 2x + 4\Big|\end{aligned}$$

Now, we can say $\delta = \min\big\{1, \epsilon/7\big\}$, which means $\big|x^2 + 2x + 4\big| \leq 7$. Continuing, we get

$$|x - 2|\Big|x^2 + 2x + 4\Big| < \frac{\epsilon}{7} \cdot 7 = \epsilon$$

(d) Let $\epsilon > 0$. Since $\pi = 3.14\ldots$, if we choose $\delta < 0.14$, then $\big[[x]\big] = 3$, which means

$$\Big|\big[[x]\big] - 3\Big| = |3 - 3| = 0 < \epsilon$$

Exercise 4.2.2

If we have a δ that already works for an ϵ-challenge, then a smaller δ should also work.

Exercise 4.2.3

Remember that if two sequences converging to the same limit give different functional values, then the limit at that point does not exist.

(a) If we take $x_n^1 = 1/n$, then we get a limit of 1, but if we take $x_n^2 = -1/n$, we get a limit of -1, so therefore the limit does not exist.

(b) If we approach in the rationals space we get 1, but in the irrational space we get 0.

Exercise 4.2.4

(a) We can choose

- $x_n = \frac{n-1}{n}$
- $y_n = 1 + \frac{1}{n}$
- $z_n = \sqrt{\frac{n^2+1}{n^2}}$

(b) We can compute the limits with Thomae's function

- $\lim t(x_n) = 0$, since we get larger denominators
- $\lim t(y_n) = 0$, since we get larger denominators
- $\lim t(z_n) = 0$, since once $z_n > 1$, it is impossible for adjacent numbers to both be perfect squares, so this number is always irrational.

(c) We can conjecture $\lim_{x\to 1} t(x) = 0$. To prove this, informally, if we receive an ϵ-challenge, we can choose a δ-neighborhood small enough so that x is close enough to 1 so that it is either an irrational number, in which case $t(x) = 0$, or x is rational. If x is rational, since it is so close to 1, this small distance must be representable by a rational number, which only happens when the denominator is very large. This means as we get closer to 1, $t(x) \to \lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, we can find such a δ so that all elements in this neighborhood are such that $\left|t(x) - 0\right| < \epsilon$.

Exercise 4.2.5

(a) If we have $f(x_n) \to L$, then we essentially have a sequence that converges to L, which means we can use all the properties of the Algebraic Limit Theorem.

(b) For some $\epsilon > 0$, we can find δ_f, δ_g so that $\left|f(x_f) - L\right| < \epsilon/2, \left|g(x_g) - M\right| < \epsilon/2$, and then take $\delta = \min\{\delta_f, \delta_g\}$, so that for $0 < |x - c| < \delta$, we have $\left|f(x) + g(x) - (L + M)\right| < \epsilon/2 + \epsilon/2 = \epsilon$.

I'm too lazy to do the Algebraic Limit Theorem proof. But it's basically just using the fact that we have a convergent sequence, as mentioned in part (a).

For the proof without the Algebraic Limit Theorem, in shorthand, we need to do something like,

$$\begin{aligned}\left|f(x)g(x) - LM\right| &< \left|f(x)g(x) - g(x)L + g(x)L - LM\right| \\ &< \Big|g(x)\big(f(x) - L\big)\Big| + \Big|L\big(g(x) - M\big)\Big|\end{aligned}$$

Now, we can bound $g(x)$, since we know $g(x)$ is a sequence that converges to M. Then, just choose δ_f, δ_g, and take $\delta = \min\{\delta_f, \delta_g\}$ to find a δ-neighborhood that satisfies this inequality.

Exercise 4.2.6

For any $\epsilon > 0$, choose $\epsilon_g < \epsilon/M$, then we can find a δ_g, so that

$$\forall 0 < |x - c| < \delta_g, \left|g(x)f(x)\right| = \left|g(x)\right|\left|f(x)\right| < \frac{\epsilon}{M} \cdot M = \epsilon \tag{4.1}$$

Exercise 4.2.7

(a) We can say for any $M \in \mathbb{R}, \epsilon > 0, \exists\delta > 0$ such that

$$0 < |x| < \delta, \left|f(x) - M\right| > \epsilon \tag{4.2}$$

(b) We say for some $\epsilon > 0$, $\exists N$ such that for $x \geq N$, $\left|f(x) - L\right| < \epsilon$. We can show for any $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$.

(c) For any $M \in \mathbb{R}, \epsilon > 0$, $\exists N$ such that for $x \geq N$, we have $\left|f(x) - M\right| > \epsilon$. An example would be $f(x) = x$.

Exercise 4.2.8

If c is a limit point of A, then $\lim_{x\to c} f(x) = L, \lim_{x\to c} g(x) = M$. Now AFSOC $M > L$. Let $\epsilon = |M - L|$. Then we can show $\exists N, n \geq N, \left|g(x_n) - M\right| < \epsilon$, and $\exists N', n \geq N', \left|f(x_n) - L\right| < \epsilon/2$. But since $M > L$, this implies $\exists x_n', f(x_n') < g(x_n')$, which contracts that $\forall x, f(x) \geq g(x)$. Therefore, we reject our hypothesis and conclude that $L \geq M$.

Exercise 4.2.9
Let the limits of f, g, h be L_f, L_g, L_h respectively. Then using Exercise 4.2.8, we can show that

$$\begin{aligned} L_f \leq L_g &\Rightarrow L \leq L_g \\ L_h \geq L_g &\Rightarrow L \geq L_g \\ \Rightarrow L \leq L_g &\leq L \Rightarrow L_g = L, \end{aligned}$$

so we can conclude $\lim_{x \to c} g(x) = L_g = L$.

4.3 Combinations of Continuous Functions

Exercise 4.3.1

(a) For some $\epsilon > 0$, choose $\delta = \epsilon^3$, then

$$\left|f(x) - 0\right| = \left|\sqrt[3]{\epsilon^3}\right| < \epsilon$$

(b) If we are given some $\epsilon > 0$, $c \in \mathbb{R}$, choose $\delta = \epsilon\sqrt[3]{c^2}$, then

$$\begin{aligned}
\left|\left(\sqrt[3]{x} - \sqrt[3]{c}\right)\left(\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}\right)\right| &= |x - c| \\
\left|\sqrt[3]{x} - \sqrt[3]{c}\right| &= \frac{|x-c|}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}} \\
&< \frac{|x-c|}{\sqrt[3]{c^2}} \\
&= \frac{\epsilon\sqrt[3]{c^2}}{\sqrt[3]{c^2}} = \epsilon
\end{aligned}$$

Exercise 4.3.2

(a) For any $\epsilon > 0$, Since g is continuous at $f(c)$, choose δ_g such that if $f(x) \in V_{\delta_g}(f(c))$, then

$$\left|g(f(x)) - f(c)\right| < \epsilon.$$

Now, since $f(x)$ is continuous at c, there is a δ_f such that if $x \in V_{\delta_f}(c)$, then

$$\left|f(x) - f(c)\right| < \delta_g.$$

Choose this δ_f, then we will have for any $\epsilon > 0$,

$$x \in V_{\delta_f}(c) \Rightarrow \left|g(f(x)) - f(c)\right| < \epsilon$$

(b) Using sequential characterization, since $f(x)$ is continuous at c, we know that $(x_n) \to c$ implies $f(x_n) \to f(c)$.

Now, $g \circ f(x)$ is well-defined on A, so if we let $y_n = f(x_n)$, we know that since $g(y)$ is continuous at $f(c)$, $y_n \to f(c)$ means $g(y) \to g(f(c))$.

Putting these together, we just showed $(x_n) \to c$ implies $f(x_n) \to f(c)$, which finally implies $g(f(x_n)) \to g(f(c))$. By the sequential characterization, we can conclude that $g \circ f(x)$ is continuous at $x = c$.

Exercise 4.3.3

For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{a}$. Then

$$\left|f(x) - f(c)\right| = \left|a(x-c)\right| < a\frac{\epsilon}{a} < \epsilon$$

Exercise 4.3.4

(a) For $\epsilon > 0$, choose $\delta < 1$, then $x, c \in \mathbb{Z}$ and $0 < |x - c| < \delta$ implies that $x = c$, so $\left|f(x) - f(c)\right| = 0 < \epsilon$.

(b) For any isolated point, we can choose a δ-neighborhood small enough so that $0 < |x - c| < \delta$ implies $x = c$, since there are no other elements in the domain near c other than c itself. This will make $\left|f(x) - f(c)\right| = 0 < \epsilon$.

Exercise 4.3.5

We can find a δ_g so that $x \in V_{\delta_g}(c)$ means $\left|g(x) - g(c)\right| < \epsilon = \left|g(c)\right|$. This means $\forall x, g(x) \neq 0$, since it is too far away from 0, so therefore the denominator is always nonzero and $f(x)/g(x)$ is defined in this open interval.

Exercise 4.3.6

(a) Choose some $c \in \mathbb{R}$, we have 2 cases,

1. $f(c)$ is rational. We can choose an irrational sequence $(x_n) \to c$, so that $f(x_n) \to 0$, since $\forall x_n, f(x_n) = 0$, since x_n is irrational. However, we know that $f(c) = 1$, since c is rational.
2. $f(c)$ is irrational. We can choose an rational sequence $(x_n) \to c$, so that $f(x_n) \to 1$, since $\forall x_n, f(x_n) = 1$, since x_n is rational. However, we know that $f(c) = 0$, since c is irrational.

Therefore, we conclude the Dirichlet function is everywhere-discontinuous.

(b) Proving Thomae's function is discontinuous at every rational point is the same proof as showing Dirichlet's function is discontinuous at every rational point. See part (a).

(c) I'm extremely confused about this hint $t(x) \geq \epsilon$. All of these x that satisfy this must be rational numbers, and they must be isolated if we force a limit on their denominator, which is what $t(x) \geq \epsilon$ does. I think then the argument goes, take the complement of this set, so $\{x : t(x) < \epsilon\}$, then we know this is bounded somehow? And then we choose a δ so that our neighborhood only contains elements from this set? Maybe something about an open set existing in this set too, and we can choose that. Overall...unsure. **TODO**

Using Theorem 4.3.2 (iii), which is the ϵ, δ-neighborhood argument, What you can do is choose a δ so small so that to represent δ as a rational, you must use a denominator large enough such that $\frac{m}{n}$, $\frac{1}{n} < \epsilon$.

Exercise 4.3.7

To show K is a closed set, we need to show that K contains all of its limit points. Take any $(x_n) \to l$ from K. Since we know $h(x)$ is continuous, any $(x_n) \to l$ implies $h(x_n) \to h(l)$. Now, since all terms of $h(x_n)$ are 0, we must have $h(l) = 0$, otherwise, if $h(1) = y$, then choose $\epsilon < |y|$, and you cannot prove convergence. We conclude that $l \in K$, and therefore K contains all of its limit points and is closed.

Exercise 4.3.8

(a) AFSOC $\exists c \in \mathbb{R}, f(c) \neq 0$. Then $\exists (x_n) \to c, x_n \in \mathbb{Q}$. Since f is continuous on $\mathbb{R}$, we must have $(x_n) \to c$ implies $f(x) \to f(c)$. But $f(x_n) = 0$, and $f(c) \neq 0$, so this means f is not continuous at c. But this is a contradiction, since f is continuous on $\mathbb{R}$.

(b) There is no claim about continuity, so no. We can have $f(r) = g(r), r \in \mathbb{Q}$, but have $f(i) = -1, g(i) = 1, i \in \mathbb{I}$. Then f and g are not the same function.

Exercise 4.3.9

(a) We can do the ϵ-δ proof, but choose $\delta = \frac{\epsilon}{c}$, so for any $z \in \mathbb{R}$, we can show that for $0 < |x - z| < \delta$, that $\big|f(x) - f(z)\big| \leq c|x - z| < c\frac{\epsilon}{c} = \epsilon$.

(b) We notice that for every y_{n+1}, its distance from y_n is c times the distance of $|y_n - y_{n-1}|$. Since $c \in (0, 1)$, this distance will get smaller every single time. Namely,

$$|y_{n+1} - y_n| = \big|f(y_n) - f(y_{n-1})\big| \leq c|y_n - y_{n-1}| \leq c^{n-1}|y_2 - y_1|$$

So for any $\epsilon > 0$, we can choose N such that $c^{N-1} < \frac{\epsilon}{|y_2 - y_1|}$, and then for subsequent $n \geq N$, we have $|y_n - y| < \epsilon$. Since $\lim y_n = y$, this sequence converges to y, which is equivalent to showing it is a Cauchy sequence.

(c) We can choose the sequence $x_n = y_n$ where y_n was defined in part (b), and we see that $(x_n) \to y$, and since f is continuous on $\mathbb{R}$, we conclude that $f(x_n) \to f(y)$. Now, $f(x_n) = x_n + 1$, so $f(x_n) \to y$. This means $f(y) = y$. To prove uniqueness, we need to show no other x exists such that $f(x) = x$. AFSOC such an x exists, then we can do the following:

- We have $\big|f(x) - f(y)\big| \leq c|x - y|$.

- If we apply this property again with $f(f(x)), f(f(y))$, we get

$$\begin{aligned}\big|f(f(x)) - f(f(y))\big| &\le c\big|f(x) - f(y)\big| \\ \big|f(x) - f(y)\big| &\le c^2|x-y| \qquad \text{(Using } f(y)=y, f(x)=x \text{ property, and the prev result)}\end{aligned}$$

- We can continue this indefinitely, and show that $y = f(y) = f(x) = x$, which means y is unique.

(d) Let us say $(x_n) \to L$. Then we can find for any $\epsilon > 0$, a N such that for $n \ge N$, $|x_n - L| < \epsilon$.

Now, if we take $\big|f(x_n) - f(L)\big| = \big|x_n - f(L)\big| \le c|x_n - L| < \epsilon$, we see this sequence also converges to $f(L)$, which means $f(L) = L$. In the previous part, we showed y is the unique fixed point, so we conclude $L = y$, and thus sequence also converges to y.

Exercise 4.3.10

(a) We have two problems,

- $f(0+0) = f(0) = f(0) + f(0)$. If we have $x = 2x$, $x = 0$ is the only solution, so $f(0) = 0$.
- $f(x-x) = f(0) = f(x) + f(-x) \Rightarrow -f(x) = f(-x)$.

(b) For any $\epsilon > 0$, since $f(0)$ is continuous, $\exists\delta$ such that $\big|f(\delta) - f(0)\big| < \epsilon$. Now, if we have $c \in \mathbb{R}$, if we have some $(x_n) \to c$, we can choose N such that for $n \ge N$, $|x_n - c| < \delta$. Then,

$$\big|f(x_n) - f(c)\big| = \big|f(x_n - c)\big| = \big|f(\delta)\big| < \epsilon$$

(c) Just expand $f(1 + 1 + \cdots + 1)$, so e.g. $f(3) = f(1+1+1) = f(1) + f(1+1) = f(1) + f(1) + f(1) = 3f(1) = 3k$. Using $f(-x) = f(x)$, we can show $f(z) = kz, z \in \mathbb{Z}$. For rationals, let's say we have $\frac{m}{n} \in \mathbb{Q}$, where $m, n \in \mathbb{Z}$, then we can write $f(m)$ as $f(n \cdot \frac{m}{n})$, which means $f(m) = nf\big(\frac{m}{n}\big)$. Now, since $m \in \mathbb{Z}$, we know $f(m) = mk$, so $mk = nf\big(\frac{m}{n}\big) \Rightarrow f\big(\frac{m}{n}\big) = \frac{m}{n}k$.

(d) We only haven't shown the property $f(x) = kx$ is true for irrationals. We can do this by constructing a rational sequence $(x_n) \to c$ for any $c \in \mathbb{R}$. Now since f is continuous on $\mathbb{R}$, we conclude that $f(x_n) \to f(c)$. We also know for every $f(x_n) = kx_n$, so $f(c) = kc$.

Exercise 4.3.11

(a) Let

$$f(x) = \begin{cases} x, & x \notin \mathbb{Z} \\ x+1, & x \in \mathbb{Z} \end{cases}$$

(b) We basically define a function that is "normal", except inside $(0,1)$ we make it behave crazy like the Dirichlet function. We make sure that the surrounding area connects to this region smoothly, so that $0, 1$ are continuous.

$$f(x) = \begin{cases} 0, x \in \mathbb{Q} \cap (0,1) \\ 1, x \in \mathbb{I} \cap (0,1) \\ x, x \notin \mathbb{Z} \end{cases}$$

(c) Basically the same as part (b) except we make the surrounding area of the interval disconnected, so that $0, 1$ are discontinuous.

$$f(x) = \begin{cases} 0, x \in \mathbb{Q} \cap [0,1] \\ 1, x \in \mathbb{I} \cap [0,1] \\ x + 100, x \notin \mathbb{Z} \end{cases}$$

(d) We have to pay attention to 0, since there exists a subsequence $\in A$ that converges to 0. Since this subsequence will converge to 0, we just have to make sure that $f(x_n) \to f(0)$.

$$f(x) = \begin{cases} x, & x \in A \\ 0, & x \notin A \end{cases}$$

Exercise 4.3.12

(a) We can construct two subsequences that converge to c, one which is always in C, and the other that is not. Then $g(x_n)$ will converge to 1 for the first subsequence, but 0 for the second. Therefore every $c \in C$ is discontinuous.

(b) If $c \notin C$, then c is part of an open set, which means there exists an ϵ neighborhood around it, where all points x are such that $g(x) = 0$.

4.4 Continuous Functions on Compact Sets

Exercise 4.4.1

(a) For any $\epsilon > 0, c \in \mathbb{R}$, choose $\delta < \sqrt[3]{\epsilon}$.

(b) Take $x = n$, $y_n = n + 1/n$. Choose $\epsilon = 1$, then $x_n, y_n \to n, so|x_n - y_n| \to 0$, but

$$\left|f(x) - f(y)\right| = \left|n - (()n^3 + 3n + 3/n + 1/n^3)\right| = \left|3n + 3/n + 1/n^3\right| > 3 > \epsilon = 1$$

(c) For any bounded subset, let M be the largest absolute value of any element in the subset. Then choose $\delta < \frac{\epsilon}{3M^2}$. Then we have

$$\left|x^3 - y^3\right| = \left|(x-y)\left(x^2 + xy + y^2\right)\right| < \frac{\epsilon}{3M^2} \cdot \left(M^2 + M \cdot M + M^2\right) = \epsilon$$

Exercise 4.4.2

Given any ϵ, choose $\delta < \epsilon/2$. Then for $|x - y| < \delta$,

$$\begin{aligned}
\left|f(x) - f(y)\right| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| \\
&= \left|\frac{y^2 - x^2}{x^2y^2}\right| \\
&= \left|\frac{(y-x)(y+x)}{x^2y^2}\right| \\
&\leq |x-y|\left|\frac{x+y}{x^2y^2}\right| && \text{(Since } \tfrac{1}{x^2y^2} \leq 1\text{)} \\
&= |x-y|\left|\frac{1}{x^2y} + \frac{1}{y^2x}\right| \\
&< \frac{\epsilon}{2}(1+1) && (\tfrac{1}{x^2y} \leq 1\text{, since } x, y \geq 1) \\
&= \epsilon
\end{aligned}$$

Now, for $(0,1]$, choose $x_n = \frac{1}{n}, y_n = \frac{1}{n + \frac{1}{n}}$. Also choose $\epsilon_0 = 1/2$. Then $|x_n = y_n| = \left|\frac{1}{n^3+n}\right| \to 0$, but

$$\left|f(x_n) - f(y_n)\right| = \left|n^2 - \left(n + \frac{1}{n}\right)^2\right| = \left|-2 - \frac{1}{n^2}\right| \geq 1 > \epsilon_0$$

Exercise 4.4.3

Since K is compact, by the preservation of compact sets, since f is continuous on K, $f(K)$ is also compact. Now, we already showed in Exercise 3.3.1 that any compact set contains its maximum and minimum values, so therefore we can conclude that f will attain its maximum and minimum values of this range.

Exercise 4.4.4

We know $[a,b]$ is closed and bounded, so therefore it is compact. Thus, $f([a,b])$ is also compact, since f is continuous on this interval. Let M be the lower bound of $f([a,b])$, which we know exists since it is compact, then we can conclude $1/f$ is bounded by $1/M$ on $[a,b]$.

Exercise 4.4.5

Choose $\epsilon = \epsilon_0$. Now, we can show

$$\exists m, n \in \mathbb{N}, |x_m - y_m| < \delta_n = \frac{1}{n}$$

such that $\left|f(x_m) - f(y_m)\right| \geq \epsilon_0$. Since we can make δ_n arbitrarily small, any δ claim can be shown to be faulty, so therefore this function cannot be uniformly continuous.

Exercise 4.4.6

(a) We can take advantage of the fact that f is not necessarily continuous at $0, 1$. An example is $f(x) = \frac{1}{x}$, where if we take $x_n = \frac{1}{n}$, then $x_n \to 0$, but $f(x) = \frac{1}{1/n} = n$ does not converge.

(b) Impossible. Since $[0, 1]$ is closed, any $x_n \to x$ will have $x \in [0, 1]$. Since f is continuous over $[0, 1]$, it must also be continuous at x, so therefore $\lim f(x_n) = f(x)$, and therefore $f(x_n)$ is a Cauchy sequence.

(c) Impossible. Same reasoning as part , since $[0, \infty)$ is closed, any $x_n \to x$ will have its limit also in the set.

(d) An upside down parabola with a peak in $(0, 1)$ will work. A function like $\frac{1}{x(x-1)}$ is a bit more fun.

Exercise 4.4.7

Suppose we have some $\epsilon > 0$.

Now if we have two arbitrary $x, y \in (a, c)$, if x, y are both in the left set or the right set, then we can just use the fact that g is uniformly continuous on those sets to show it. The only other case we have is that they are in different sets, WLOG $x \in (a, b], y \in [b, c)$,

$$\left|f(x) - f(y)\right| = \left|f(x) - f(b) - (f(b) - f(y))\right| \leq \left|f(x) - f(b)\right| + \left|f(y) - f(b)\right|$$

Now, since b is in both sets, We know for $x_a \in (a, b], y \in [b, c)$, we can find δ_a, δ_b respectively such that $|x - b| < \delta_a \Rightarrow \left|f(x) - f(b)\right| < \epsilon/2$ and $|y - b| < \delta_b \Rightarrow \left|f(y) - f(b)\right| < \epsilon/2$.

So we can finish up our proof by choosing $\delta < \min(\delta_a, \delta_b)$, so that

$$|x - y| < \delta \Rightarrow \left|f(x) - f(y)\right| \leq \left|f(x) - f(b)\right| + \left|f(y) - f(b)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The last part is motivated by that if x, y are in separate sets, we try to make them as close to b as possible so we can use their uniformly continuous properties around b.

Exercise 4.4.8

(a) We can show f is uniformly continuous on $[0, b]$, since it is a compact set. Then, using the same argument as in Exercise 4.4.7, we can show that if f is uniformly continuous on $[0, b], [b, \infty)$, then f is also uniformly continuous on $[0, \infty)$.

(b) We can show $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$, because on this interval, any $x, y \in [1, \infty)$ means $\sqrt{x} + \sqrt{y} \geq 2$. Then choose $\delta < 2\epsilon$, then

$$\begin{aligned}
|x - y| < \delta \Rightarrow \left|\sqrt{x} - \sqrt{y}\right| &= \left|\frac{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} + \sqrt{y}\right)}\right| \\
&= \frac{|x - y|}{\left|\sqrt{x} + \sqrt{y}\right|} \\
&< 2\epsilon \cdot \frac{1}{2} = \epsilon
\end{aligned}$$

Now, $[0, 1]$ is closed and bounded, so it is compact, and $\sqrt{x}$ is continuous on it, so therefore $\sqrt{x}$ is uniformly continuous on $[0, 1]$.

Using what we just proved in part (a), we can combine these results to show that $\sqrt{x}$ is uniformly continuous on $[0, 1] \cup [1, \infty) = [0, \infty)$.

Exercise 4.4.9

(a) Choose $\delta < \frac{\epsilon}{M}$, then if we have $|x - y| < \delta = \frac{\epsilon}{M}$,

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M \Rightarrow \left|f(x) - f(y)\right| \leq |x - y|M < \frac{\epsilon}{M}M = \epsilon$$

(b) No, take $f(x) = \sqrt{x}$. We showed in Exercise 4.4.8, part (b) that this function is uniformly on $[0, 1]$, but its slope $\frac{1}{2\sqrt{x}}$ tends to ∞ as $x \to 0$. The issue stems from that we will have $M = \frac{\epsilon}{\delta}$, but this quantity may not be bounded, since ϵ, δ vary.

Exercise 4.4.10
Yes, uniform continuousness preserves boundedness over a set.

Choose some $\epsilon > 0$, say $\epsilon = \epsilon_0$. Now we know since f is uniformly continuous, $\exists\delta |x - y| < \delta \Rightarrow \big|f(x) - f(y)\big| < \epsilon_0$. Now, since A is a bounded set, we can find a finite open cover for it, let this cover be in the form

$$A = \bigcup_{i=1}^{N} V_\delta(x_i)$$

where $x_i \in A$. Now, take $M_1 = \min_{i=1}^{N}\big(f(x_i) - \epsilon_0\big), M_2 = \max_{i=1}^{N}\big(f(x_i) + \epsilon_0\big)$, and M_1 will be a lower bound for $f(A)$, and M_2 an upper bound.

Exercise 4.4.11
($\Rightarrow$) Let $O \in \mathbb{R}$ be any open set. Consider some $x \in g^{-1}(O)$, We know $g(x) = y \in O$. Now, since O is an open set, we know $\exists\epsilon$ such that $V_\epsilon(y) \subseteq O$. We also know that g is continuous, which means for any ϵ, we know $\exists V_\delta(x)$ such that $x' \in V_\delta(x)$ means $g(x') \in V_\epsilon(y)$. Since $V_\epsilon(y) \subseteq O$, we have $g(x') \in O$ for any of these x', which means by the definition of g^{-1}, all these $x' \in g^{-1}(O)$, and therefore $V_\delta(x) \subseteq g^{-1}(O)$, which means there exists an open neighborhood around any element in $g^{-1}(O)$, and therefore $g^{-1}(O)$ is an open set.

($\Leftarrow$) If we know $g^{-1}(O)$ is open for any open $O \subseteq \mathbb{R}$, suppose we get some $y \in O, \epsilon > 0$. Then consider the set $V_\epsilon(y) \in O$. Now, consider $g^{-1}(V_\epsilon(y))$, which we are guaranteed is open. Since $x = g^{-1}(y)$ is in $g^{-1}(V_\epsilon(y))$, because $y \in V_\epsilon(y)$, we know since g is open, that $\exists\delta > 0$ such that $V_\delta(x) \subseteq g^{-1}(V_\epsilon(y))$. Therefore, for any $V_\epsilon(y)$, we have found a corresponding $V_\delta(x)$ such that $x' \in V_\delta(x)$ implies $g(x') \in V_\epsilon(y)$, which means we conclude g is continuous.

Exercise 4.4.12
We want to prove that

A function that is continuous on a compact set K is uniformly continuous on K.

TODO

Exercise 4.4.13

(a) We know for every $\epsilon > 0$, $\exists\delta$ such that $x' \in V_\delta(x)$ implies $f(x') \in V_\epsilon(f(x'))$. Now, if we have some Cauchy Sequence x_n, we know $\exists N$ such that for $m, n \geq N$,

$$|x_m - x_n| < \delta,$$

which means $\big|f(x_m) - f(x_n)\big| < \epsilon$, since $x_m, x_n \in V_\delta(x)$. Therefore, we know for this N, $m, n \geq N$ means $\big|f(x_m) - f(x_n)\big| < \epsilon$, so therefore $f(x_n)$ is also a Cauchy Sequence.

(b) ($\Rightarrow$) We can come up with a sequence $a_n \to a, a_n \in (a, b)$. Now, since (a_n) is also a Cauchy Sequence, that means $g(a_n)$ is also a Cauchy sequence, since g is a uniformly continuous function. Therefore, define $g(a)$ as $g(a_n) \to g(a)$. We know this exists since $g(a_n)$ is Cauchy. Notice that any sequence $(a'_n) \to a$ is Cauchy, and therefore we must have $g(a_n) \to g(a')$. We want to show that $g(a') = g(a)$ for g to be continuous at a.

AFSOC $g(a') \neq g(a)$, then that means we have two sequences $a'_n, a_n \to a$, where these sequences are contained inside (a, b), but $\big|g(a') - g(a)\big| \geq \epsilon_0$, which means g is not uniformly continuous. This is a contradiction, so therefore we conclude that $g(a') = g(a)$, and therefore any $(a'_n) \to a$ implies $g(a'_n) \to g(a)$, which means g is continuous at a. we conclude that same for b, using the same argument, and therefore g is continuous over $[a, b]$.

($\Leftarrow$) $[a, b]$ is a compact set, and since g is continuous over this set, we have that g is also uniformly continuous over $[a, b]$, which means we also have that g is uniformly continuous over $(a, b) \subseteq [a, b]$.

4.5 The Intermediate Value Theorem

Exercise 4.5.1
If we have a continuous function f on $[a,b]$, and we know that $[a,b]$ is connected, then that means $f([a,b])$ is also connected.

By the property of connected sets, it follows directly that if we have $f(a) < L < f(b)$, we can always find this $L \in f([a,b])$. Then, this L must correspond to some $c \in (a,b)$, which proves the IVT.

Exercise 4.5.2

(a) No, you can imagine taking $(-1,1)$ to $[1,2)$ with $f(x) = x^2 + 1$. The idea for making this was that the bounds don't necessarily have to come from the open endpoints.

(b) Can use the same example from part (a)

(c) Yes. If we have bounded closed interval, then it is compact, so a continuous function will take it to another compact interval, which is bounded and closed. This function will also preserve connectedness.

Exercise 4.5.3
No, since $\mathbb{R}$ is connected, and if f is continuous, it will preserve connectedness. $\mathbb{Q}$ is not connected.

Exercise 4.5.4
Suppose we have some function f with the Intermediate value property and is also increasing. We want to show that f is continuous.

Suppose we have some $\epsilon > 0$ and a point $c \in [a,b]$. Let

$$L_1 = \max(f(a), f(c) - \epsilon), \min(f(b), L_2 = f(c) + \epsilon)$$

By the IVT property, we know $\exists x_1, x_2 \in [a,b]$ where $f(x_1) = L_1, f(x_2) = L_2$.

By the increasing property of f, we have that any $x \in [x_1, x_2]$ will satisfy

$$\left|f(x) - f(c)\right| < \epsilon$$

since $f(x_1) \leq f(c) \leq f(x_2)$.

Choose $\delta = \min(|c - x_1|, |c - x_2|)$, and this will work for any ϵ challenge.

Exercise 4.5.5
To complete the IVT proof with the Axiom of Completeness, we are trying to show $\exists c \in (a,b)$ such that $f(c) = 0$ if $f(a) < 0 < f(b)$, and f is continuous. With our definition of $K = \{x \in [a,b] : f(x) \leq 0\}$, since we know K is bounded above by b, and $a \in K$, K is not empty. By the Axiom of Completeness, we know $c = \sup K$ exists.

- We cannot have $f(c) > 0$ since all elements of K satisfy $f(x) \leq 0$.
- We cannot have $f(c) < 0$, because it is not an upper bound that includes 0.
- Therefore, we conclude $f(c) = 0$, which means we have found a $c \in (a,b)$ such that $f(a) < f(c) = 0 < f(b)$

To extend it for a general $f(a) < c < f(b)$, just consider the function $f'(x) = f(x) - c$, and then we can do the $f(a) < 0 < f(b)$ case, which we have already proved.

Exercise 4.5.6
With the binary set construction described in the text, once we take the $\bigcap_{n=1}^{\infty} I_n$, we must have an element in this intersection by NIP. Call this element x.

Now, our claim is that $f(x) = 0$. Notice that a_n and b_n converge to x, since $|a_n - b_n| < \frac{|a-b|}{2^n}$. Then, since f is continuous, $f(a_n)$ and $f(b_n)$ must converge to the same L. Since $f(a_n) < 0, f(b_n) \geq 0$, it must be the case that $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = 0$, since otherwise, for example if $L > 0$, then $\left|f(a_n) - L\right| \geq \epsilon_0$. Therefore, we have found $x \in (a,b)$ such that $f(x) = 0$.

Exercise 4.5.7

Construct the folowing sequence of intervals, and let $c_{n-1} = \frac{a_{n-1}+b_{n-1}}{2}$,

$$I_n = [a_n, b_n] = \begin{cases} [0,1], & n = 0 \\ [c_{n-1}, b_{n-1}], & \text{if } f(c_{n-1}) > c_{n-1} \\ [a_{n-1}, c_{n-1}], & \text{if } f(c_{n-1}) \leq c_{n-1} \end{cases}$$

Take $x \in \bigcap_{n=1}^{\infty} I_n$. By the NIP this x exists.

Now, our claim is that $f(x) = x$. This is similar to the proof in Exercise 4.5.6, where we use the continuity of f and that $(a_n) \to x, (b_n) \to x$, so that $\lim_{n\to\infty} f(a_n) = a_n$, and therefore we conclude $f(x) = x$.

There is probably a more straightforward proof with IVT.

Exercise 4.5.8

We cannot, since there are times that are ambiguous.

For example, consider the set of times from 1:00 - 2:00. Here, the hour hand goes from $30° - 60°$, while the minute hand goes from $0° - 360°$,

We can find a time that is isomorphic to this time range by considering that $30° - 60°$ is the 5-10 minute range, and if the minute hand is past some hour enough to the point where it crosses the same point where the hour point would be with a minute in the range of 5-10 minutes, then we would have an isomorphic time.

For example, consider when the minute hand goes past 2. The minute hand will go from 2 - 3, but in between there, at some point it can represent some 2-hour time where the minute is between 5-10 minutes.

There's probably a better setup and application of IVT, but this is enough to convince me that there are isomorphic times on the clock.

4.6 Sets of Discontinuity

Exercise 4.6.1

(a)

$$f(x) = \begin{cases} 1, & x \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$f(x) = \begin{cases} \text{Dirichlet's Function}, & x \in (0, 1] \\ 100, & \text{otherwise} \end{cases}$$

Exercise 4.6.2

(Left-hand Limit) Given a limit point of a set A and a function $f : A \to \mathbb{R}$, we write

$$\lim_{x \to c^-} f(x) = L$$

if $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$\left|f(x) - L\right| < \epsilon \text{ whenever } 0 < c - x < \delta$$

Equivalently, $\lim_{x \to c^-} f(x) = L$ if $\lim f(x_n) = L$ for all sequences (x_n) satisfying $x_n < c$ and $\lim(x_n) = c$.

Exercise 4.6.3

($\Rightarrow$) If $\lim_{x \to c} f(x) = L$, then all sequences that converge to c will have $\lim f(x_n) = L$, so we have $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$.

($\Leftarrow$) If we have $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$, we want to be able to show for any sequence $(x_n) \to c$, that we also have $\lim f(x_n) = L$. For any (x_n), we know there must be an infinite number of elements greater than c and/or less than c if $(x_n) \to c$, so WLOG take this subsequence that is all greater than c, which we know satisfies $\lim f(x_n) = L$.

Exercise 4.6.4

($x \to c^-$) For any $\epsilon > 0$, we can find $f(x') = f(c) - \epsilon$ by the IVT, and since f is monotone, we know $0 < c - x < \delta = \left|c - x'\right|$ will satisfy the ϵ challenge, since any x in this range will satisfy $f(x') \leq f(x) \leq f(c)$, since $x' \leq x \leq c$, and $\left|f(x) - f(x')\right| < \epsilon$. Similar logic applies to the other direction.

Since the limit exists at every point, but could potentially be different, a monotone function can only have jump discontinuity.

Exercise 4.6.5

For any monotone function f with a jump discontinuity at c, where $\lim_{x \to c^-} f(x) = L^-, \lim_{x \to c^+} f(x) = L^+$, it must be the case that $L^- < L^+$. In addition, because $\mathbb{Q}$ is dense, we know that $\exists r \in \mathbb{Q}$ such that $L^- < r < L^+$. Now, there is a slight concern that maybe this r could be repeated at other discontinuities, but since we know f is monotone, we know that for $c_1 < c_2$, we must have $L_1^- < L_2^-$ so therefore $r_1 < r_2$. Therefore, we can map all of D_f to unique $r \in \mathbb{Q}$, which means D_f is a subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, we conclude D_f is finite or countable.

Exercise 4.6.6

(a) $\mathbb{R}$ is closed

(b) $\mathbb{R} \setminus \{0\} = \bigcup_{n=1}^{\infty} \left(-\infty, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, \infty\right)$

(c) $\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} [r]$, $\mathbb{Q}$ is countable

(d) $\mathbb{Z} = \bigcup_{n \in \mathbb{Z}} [z]$, $\mathbb{Z}$ is countable

(e) $(0, 1] = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right]$

Exercise 4.6.7

The set that f is α continuous is some aribitrary union of open sets $(x - \delta, x + \delta)$, which means $\overline{D_\alpha}$ is open. Then, we conclude D_α is closed, since it is the complement of an open set.

Exercise 4.6.8
If we have $\alpha_1 < \alpha_2$, we know that the set of α_1-continuous points is a subset of the α_2-continuous points since if $\exists \delta$ such that $y, z \in (x-\delta, x+\delta)$ and $\left|f(y) - f(z)\right| < \alpha_1 < \alpha_2$. Therefore, $\overline{D_{\alpha_1}} \subseteq \overline{D_{\alpha_2}} \Rightarrow D_{\alpha_2} \subseteq D_{\alpha_1}$.

Exercise 4.6.9
If f is continuous at x, we know for any $V_\epsilon(f(c))$, we can find $\exists \delta, x \in V_\delta(c)$ such that $f(x) \in V_\epsilon(f(c))$. Choose $\epsilon = \alpha/2$, then we know that we have a delta that satisfies for $y, z \in V_\delta(x)$,

$$\begin{aligned}\left|f(y) - f(z)\right| &= \Big|f(y) - f(c) - \big[f(z) - f(c)\big]\Big| \\ &\leq \left|f(y) - f(c)\right| + \left|f(z) - f(c)\right| \\ &< \alpha/2 + \alpha/2 = \alpha\end{aligned}$$

Since f is α-continuous at every point it is continuous, we conclude that for points where it is not α-continuous, it must be not in the set of continuities of f, which is namely D_f. Therefore, $D_\alpha \subseteq D_f$.

Exercise 4.6.10
If f is not continuous at x, it must be the case that f is not α-continuous at x for some $\alpha > 0$, otherwise, we can show that f is continuous at this point by showing that $\exists \delta, \forall y \in V_\delta(x)$, we have $\left|f(y) - f(x)\right| < \alpha$.

To show that

$$D_f = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}},$$

for any $\alpha > 0$, $\exists m \in \mathbb{N}$ such that $0 < \frac{1}{m} < \alpha$. We showed $\frac{1}{m} < \alpha$ implies $D_\alpha \subseteq D_{\frac{1}{m}}$, so we can conclude for any discontinuity of f, it will be included in some $D_{\frac{1}{n}}$.

This union of a countable number of closed sets shows that the discontinuities of f is a F_σ set.

Appendix A

Extras

A.1 Useful Tools

Collection of useful tools and methods to solve problems.

Tip A.1.1
Template for a proof that $(x_n) \to x$:

- Let $\epsilon > 0$ be arbitrary
- Demonstrate a choice for $N \in \mathbb{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N works.
- Assume $n \geq N$
- With N well chosen, you should be able to show $|x_n - x| < \epsilon$.

A.2 Cool Things

- In Chapter 2, we learn that addition in infinite sums is not commutative.
- In Chapter 2, we learn that if $\sum_{n=1}^{\infty} a_n$ converges conditionally, then for any $r \in \mathbb{R}$, there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to r.
- In Chapter 3, we learn that $\mathbb{R}$ and $\emptyset$ are both open and closed, but they are the only subsets in $\mathbb{R}$ with this property.

A.3 Important Theorems

A.3.1 5 Characterizations of Completeness

Theorem 1 (Axiom of Completeness) *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Theorem 2 (Nested Interval Property) *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \cdots$$

has a nonempty intersection, that is $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem 3 (Monotone Convergence) *If a sequence is monotonic and bounded, then it converges.*

Theorem 4 (Bolzano-Weierstrass) *Every bounded sequence contains a convergent subsequence.*

Theorem 5 (Cauchy Criterion) *A sequence converges if and only if it is a Cauchy Sequence.*
A sequence (a_n) is called a Cauchy sequence if, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $m, n \geq N$, it follows that $|a_n - a_m| < \epsilon$.

A.3.2 Sequence Convergence

Theorem 6 (Convergence of a Sequence) *A sequence $\{a_n\}$ converges to $a \in \mathbb{R}$ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$, it follows that $|a_n - a| < \epsilon$.*

Theorem 7 (Cauchy Criterion) *A sequence converges if and only if it is a Cauchy sequence, which is defined as a sequence (a_n) that for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $m, n \geq N$, it follows that $|a_n - a_m| < \epsilon$.*

A.3.3 Function Continuity

Theorem 8 (Characterizations of Continuity) *Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. The function f is continuous at c if and only if any one of the following conditions is met:*

(i) *For all $\epsilon > 0$, $\exists \delta > 0$ such that for $x \in A, |x - c| < \delta$ implies $\big|f(x) - f(c)\big| < \epsilon$*

(ii) $\lim_{x \to c} f(x) = f(c)$

(iii) *For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in A, x \in V_\delta(c)$ implies $f(x) \in V_\epsilon(f(c))$*

(iv) *If $(x_n) \to c$, with $x_n \in A$, then $f(x_n) \to f(c)$*

A.4 Identities

Identity A.4.1
(Triangle Inequality) The triangle inequality states that for $x, y \in \mathbb{R}$,

$$|x| - |y| \leq |x + y| \leq |x| + |y| \tag{A.1}$$

Identity A.4.2
(Geometric Series)

$$\sum_{k=0}^{m} ar^k = \frac{a(1 - r^{m+1})}{(1 - r)} \tag{A.2}$$

and converges to

$$\lim_{m \to \infty} \sum_{k=0}^{m} ar^k = \frac{a}{1 - r}$$

iff $|r| < 1$.