

Understanding Analysis Solutions

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Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) PROOF AFSOC $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$ such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in lowest reduced terms. Then we can square both sides, yielding $3 = \left(\frac{m}{n}\right)^2 \implies 3n^2 = m^2$. Now, we know m^2 is a multiple of 3 and thus m must also. Then, we can write $m = 3k$, and derive

$$\begin{aligned}(\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2\end{aligned}$$

Similar to before, we come to the conclusion that n is a multiple of 3. However, this is a contradiction since m, n are both multiples of 3 and we assumed $\frac{m}{n}$ was in lowest terms. Thus, we conclude $\sqrt{3}$ is irrational.

The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ implies that m is a multiple of 4, since we have

$$4n^2 = m^2 \quad \Rightarrow \quad 2n = m,$$

so we cannot reach our contradiction that m/n is not in lowest terms.

Exercise 1.2.2

(a) False. Consider

$$A_n = \left[0, \frac{1}{n}\right).$$

Then

$$\bigcap_{n=1}^{\infty} A_n = \{0\}.$$

(b) True. Since $\forall i, A_i \subseteq A_1$, $\exists x$ such that $\forall i, x \in A_i$. Therefore, the intersection cannot be empty. Then, every set is finite, and the intersection of any number of finite sets will be finite.

(c) False. Consider $A = \{1, 2\}, B = \{1\}, C = \{2, 3\}$.

$$\{1, 2\} \cap (\{1\} \cup \{2, 3\}) = \{1, 2\} \neq (\{1, 2\} \cap \{1\}) \cup \{2, 3\} = \{1, 2, 3\}$$

- (d) True. Intersection is associative.
 (e) True. Intersection is distributive over union.

PROOF We will prove

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad (1.1)$$

by set inclusion.

- Suppose $x \in A \cap (B \cup C)$. By the definition of intersection, we know $x \in A$ and $x \in B \cup C$, the latter which means $x \in B$ or $x \in C$.

We can consider 2 cases for x ,

1. $x \in B$. Then we know $x \in A$ and $x \in B$, so $x \in A \cap B$ and therefore $x \in (A \cap B) \cup (A \cap C)$
2. $x \in C$. Symmetric to the case above.

in all cases, we see $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$, so

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

- Suppose $x \in (A \cap B) \cup (A \cap C)$. Then we have two cases
 1. $x \in A \cap B$. This means $x \in A$ and $x \in B$. If $x \in B$, then $x \in B \cup C$, since $B \subseteq B \cup C$. Putting these facts together, we see $x \in A \cap (B \cup C)$.
 2. $x \in A \cap C$. Symmetric to the case above.

in all cases, we see $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$, so

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

Exercise 1.2.3

(a) If $x \in (A \cap B)^c$, then we have cases

- $x \in B$ and $x \notin A$. Then $x \notin A$ implies $x \in A^c \Rightarrow x \in A^c \cup B^c$.
- $x \in A$. Symmetric to above.
- $x \notin A$ and $x \notin B$. Then $x \in A^c$ so $x \in A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$, then we have cases

- $x \in A^c$. Then $x \notin A$ so x cannot be in the intersection of A and B , so $x \in (A \cap B)^c$.
- $x \in B^c$. Symmetric to above.

(c) Proof for $(A \cup B)^c = A^c \cap B^c$ pretty similar to above.

Exercise 1.2.4

We are verifying the triangle inequality with a, b .

(a) If a, b have the same sign, then

$$\begin{aligned} |a + b| &= a + b \\ |a| + |b| &= a + b \\ \Rightarrow |a + b| &= |a| + |b| \\ \Rightarrow |a + b| &\leq |a| + |b| \end{aligned}$$

(b) • $a \geq 0, b < 0$.

$$\begin{aligned} |a + b| &\leq |a| \\ &\leq |a| + |b| \end{aligned}$$

- $a + b \geq 0$. At most one of a, b is negative. If they are both positive, then we have already shown this in part (a). Otherwise, WLOG a is negative. Then

$$\begin{aligned} |a + b| &\leq |b| \\ &\leq |a| + |b| \end{aligned}$$

Exercise 1.2.5

- (a) Substitute in $b' = -b$ into the triangle inequality.
- (b) Easy to prove directly without using triangle inequality. **TODO**.
A direct proof will look something like:

- If a, b are the same sign, then equality holds
- If a, b are different signs, then if b is negative, then $|a - b| = |a| + |b|$, and if a is negative, then $|a - b| = |a| + |b|$, both of which bound $||a| - |b||$.

Exercise 1.2.6

- (a) Yes, since $f(A \cap B) = [1, 4] = [0, 4] \cap [1, 16] = f(A) \cap f(B)$. This is by coincidence though, as we will later see. Yes, since $f(A \cup B) = [0, 16] = [0, 4] \cup [1, 16] = f(A) \cup f(B)$.
- (b) Choose $A = [-2, 0], B = [0, 2]$
- (c) Suppose $x \in g(A \cap B)$, then $\exists x' \in A \cap B$ such that $g(x') = x$. Since $x' \in A$ and $x' \in B$, we know $x = g(x') \in g(A), g(B)$, so we conclude $x \in g(A) \cap g(B)$.
- (d) Equality. **TODO** too lazy to write out the proof. Similar to above.

Exercise 1.2.7

- (a) **TODO** I don't think we want to include $x \in \mathbb{I} \dots$

$$f^{-1}(A) = [0, 2] \tag{1.2}$$

$$f^{-1}(B) = [0, 1] \tag{1.3}$$

We see $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ is also true.

- (b) **TODO**

Exercise 1.2.8

Negating statements. Took some liberties. Also notice that these statements are not necessarily true.

- (a) There exists a real number satisfying $a < b$, such that $\forall n \in \mathbb{N}, a + 1/n \geq b$.
- (b) There exists two distinct real numbers such that there is not a rational number between them.
- (c) There exists a natural number $n \in \mathbb{N}$ such that \sqrt{n} is not a natural number nor an irrational number.
- (d) There exists a real number $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, n \leq x$.

Exercise 1.2.9

We are given the sequence

$$x_1 = 1, x_{n+1} = \frac{1}{2}x_n + 1 \tag{1.4}$$

and want to show $\forall i \geq 1, x_i < 2$.

We can show this with a direct proof of summation.

An alternative that the book probably wants to see is using **induction**.

- Base Case: $x_1 = 1 < 2$
- Inductive case. Assume $\forall i < n + 1, x_i < 2$. Then $x_i/2 + 1 < 2$ since $x_i/2 < 1$.

- By induction our original claim is proved.

Exercise 1.2.10

- (a) Similar to Exercise 1.2.9. $y_n < 4$ means $(3/4)y_n < 3$ so $(3/4)y_n + 1 < 4$
- (b) In brief,

$$\begin{aligned} y_n &\leq \frac{3}{4}y_n + \frac{1}{4}y_n & (1.5) \\ &< \frac{3}{4}y_n + 1 & \text{(Using } y_n < 4\text{)} \\ &< y_n + 1 & \text{(Sequence definition)} \end{aligned}$$

Exercise 1.2.11

A combinatorial argument is that in order to construct a set, we have 2 choices for every element, to include it or not to. Therefore, we have

$$\prod_{i=1}^n 2 = 2^n$$

Exercise 1.2.12

- (a) We know that $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$. So if we are trying to show $(A_1 \cup A_2 \cup A_3)^c = (A_1 \cup A_2)^c \cap A_3^c = A_1^c \cap A_2^c \cap A_3^c$. Induction lets us apply the property on smaller parts of our expression.
- (b) Induction only proves the property for some $n \in \mathbb{N}$, i.e. some finite n . It is not shown for an infinite n .
- (c) **TODO**. Sketch: If x is not in the union of all the A_n , then x cannot be part of any particular A_n either, or else it would be in the union.

1.3 The Axiom of Completeness

Exercise 1.3.1

- (a) We compute the additive inverse for each element in \mathbb{Z}_5 .

$$\begin{aligned} 0 + 0 &\equiv 0 \\ 1 + 4 &\equiv 0 \\ 2 + 3 &\equiv 0 \\ 3 + 2 &\equiv 0 \\ 4 + 1 &\equiv 0 \end{aligned}$$

- (b) We compute the multiplicative inverse for each element in \mathbb{Z}_5 .

$$\begin{aligned} 1 \times 1 &\equiv 1 \\ 2 \times 3 &\equiv 1 \\ 3 \times 2 &\equiv 1 \\ 4 \times 4 &\equiv 1 \end{aligned}$$

- (c) \mathbb{Z}_4 is not a field because multiplicative inverses do not exist for every single element. For example, 2 multiplied with any number is even, which cannot $\equiv 1 \pmod{4}$.

We conjecture that \mathbb{Z}_n always has additive inverses and only has multiplicative inverses if n is prime.

Exercise 1.3.2

We are writing a formal definition for the *infimum* of a set.

- (a) $s = \inf A$ means

- i) s is a lower bound for A
 - ii) if b is any lower bound for A , then $b \leq s$
- (b) If $s \in \mathbb{R}$ is a lower bound for $A \subseteq \mathbb{R}$, then $s = \inf A$ iff $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$.

PROOF (\Rightarrow) If $s = \inf A$, then s is the greatest lower bound for A , meaning any $s + \epsilon$ for $\epsilon > 0$ will be greater than some element of A , otherwise $s + \epsilon$ is a greater lower bound and leads to a contradiction that $s \neq \inf A$.

(\Leftarrow) If $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$, then since s is a lower bound, $\forall b > s$, b will not be a lower bound for A since if, $b > s$, then we can choose $\epsilon = b - s > 0$, and we know that $\exists a \in A$ where $a < s + \epsilon < b$, which means b is not a lower bound. Thus, all lower bounds b must be such that $b \leq s$, and we conclude $s = \inf A$.

Exercise 1.3.3

- (a) Since $\inf A$ is a lower bound for A , we know $\inf A \in B$. Now, we need to show $\inf A$ is the supremum of B . $\inf A$ is the least upper bound for B , since if $\exists b \in B, b > \inf A$, then we know that this b is not a lower bound for A , so no such b exists.
- (b) There might be a typo in this question. I think the question was meant to read “explain why there is no need to assert that the greatest *lower bound* in the Axiom of Completeness.” In this case, the answer would be that the Axiom of Completeness already implies the greatest lower bound property, so there is no need to explicitly state it.
- (c) We can take the negative of all elements in A , find $\sup A$, and then negate again to get $\inf A$.

Exercise 1.3.4

If $B \subseteq A$, then

$$\begin{aligned} \sup A = s &\geq a, \forall a \in A \\ s &\geq b, \forall b \in B && (\text{since } B \subseteq A) \\ \Rightarrow s &\geq \sup B. && (\text{since } s \text{ is an upper bound for } B) \end{aligned}$$

Exercise 1.3.5

(a)

$$\begin{aligned} s &= \sup(c + A) \\ \Rightarrow s &\text{ is the least upper bound for } c + A \\ \Rightarrow s - c &\text{ is the least upper bound for } A \\ \Rightarrow s - c &= \sup A \\ s &= c + \sup A \end{aligned}$$

(b)

$$\begin{aligned} s &= \sup(cA) \\ \Rightarrow s &\text{ is the least upper bound for } cA \\ \Rightarrow \frac{s}{c} &\text{ is the least upper bound for } A \\ \Rightarrow \frac{s}{c} &= \sup A \\ s &= c \sup A \end{aligned}$$

(c) If $c < 0$, $\sup(cA) = -c \inf(A)$.

Exercise 1.3.6

(a) $\sup : 3; \inf : 1$

(b) $\sup : 1; \inf : 0$

(c) $\sup : \frac{1}{2}; \inf : \frac{1}{3}$

(d) $\sup : 9; \inf : \frac{1}{9}$

Exercise 1.3.7

If $a \geq a', \forall a' \in A$, and $a \in A$, then

$$\forall \epsilon > 0, a - \epsilon < a, \quad (1.6)$$

so a is the least upper bound for A , and $a = \sup A$.

Exercise 1.3.8

Let

$$\epsilon = \sup B - \sup A > 0. \quad (1.7)$$

since $s_b = \sup B$, $\exists b \in B \mid b > s_b - \epsilon/2$. Since $s_b - \frac{\epsilon}{2} > \sup A$, then $b \geq \sup A$, so this $b \in B$ is an upper bound for A .

Exercise 1.3.9

- (a) True, take the largest element in the set as the supremum.
- (b) False, $\sup(0, 2) = 2$, but $2 > a \in (0, 2)$, but $\sup A = 2 \not< 2 = L$.
- (c) False $A = (0, 2), B = [2, 3]$. We have that $\sup A = \inf B$
- (d) True.
- (e) False, take $A = B = (0, 2)$.

1.4 Consequences of Completeness

Exercise 1.4.1

If $a < 0$, then we have two cases,

- 1. If $b > 0$, then $a < 0 < b$.
- 2. If $b = 0$, then we can take $-b, -a$, which satisfies $0 \leq -b < -a$, and apply Theorem 1.4.3.

Exercise 1.4.2

- (a) If $a, b \in \mathbb{Q}$, then

$$\begin{aligned} a &= \frac{a_1}{a_2} \\ b &= \frac{b_1}{b_2} \\ \implies a + b &= \frac{a_1 b_2 + a_2 b_1}{a_2 b_2} \in \mathbb{Q} \end{aligned}$$

- (b) AFSOC $at \in \mathbb{Q}$. But if $a = \frac{a_1}{a_2}$, this implies

$$t = \frac{a_2}{a_1} \frac{m}{n} \in \mathbb{Q},$$

which is a contradiction, so we must have that $at \in \mathbb{I}$

- (c) \mathbb{I} is not closed under addition or multiplication.

Exercise 1.4.3

We can apply Theorem 1.4.3, to find $a < q < b, q \in \mathbb{Q}$, and then subtract an irrational number such as $\sqrt{2}$ to end up at

$$a - \sqrt{2} < q - \sqrt{2} < b - \sqrt{2}, \quad (1.8)$$

where $q - \sqrt{2} \in \mathbb{I}$.

Exercise 1.4.4

Suppose $\exists b$ lower bound such that $b > 0$. Then by Archimedean Property of \mathbb{R} , $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b$, which means b is not a valid lower bound. Thus $b \leq 0$, and 0 is a valid lower bound so the inf is 0.

Exercise 1.4.5

AFSOC $\exists \alpha \in \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$. Then $\alpha > 0$, but by Archimedean property of reals, we have that $\exists n \in \mathbb{N} \mid \frac{1}{n} < \alpha$. Since $\alpha \notin (0, \frac{1}{n})$, $\alpha \notin \bigcap_{n=1}^{\infty} (0, \frac{1}{n})$, a contradiction. Thus we conclude the set is empty.

Exercise 1.4.6

(a) If $\alpha^2 > 2$, then

$$\begin{aligned} \left(a - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n} \end{aligned}$$

choose $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$. Then

$$\begin{aligned} \left(a - \frac{1}{n_0}\right)^2 &> \alpha^2 - \frac{2\alpha}{2\alpha}(\alpha^2 - 2) \\ &> 2 \end{aligned}$$

but $\alpha - \frac{1}{n_0} < \alpha$, so α is not the least upper bound for the set.

(b) Just replace $\sqrt{2}$ with \sqrt{b}

Exercise 1.4.7

Take the minimum from the set. Assign as i , then remove the minimum from the set. Repeat for $i + 1$ and so on.

Exercise 1.4.8

- (a) If both are finite, then their union is finite and trivially countable. If one is finite, then first enumerate elements of the finite set. Then map the rest of \mathbb{N} to the countably infinite set. If both are countably infinite, map one set to odds and the other to evens.
- (b) Induction only holds for finite integers, not infinity.
- (c) We can arrange each A_n into row n of a $\mathbb{N} \times \mathbb{N}$ matrix. Then, we enumerate by diagonalization.

Exercise 1.4.9

- (a) If $A \sim B$, then there is a 1-to-1 mapping. We can just take the inverse of the mapping to derive $B \sim A$.
- (b) If we have $f : A \rightarrow B, g : B \rightarrow C$, then we can compose the functions so $g(f(x)) : A \rightarrow C$.

Exercise 1.4.10

The set of all finite subsets of \mathbb{N} can be ordered in increasing order by the sum of each subset.

Exercise 1.4.11

- (a) $f(x) = x$
- (b) Interweave the decimal expansion of x, y , e.g.

$$f(x, y) = 0.x_1y_1x_2y_2x_3y_3 \dots \quad (1.9)$$

Exercise 1.4.12

(a)

$$\sqrt{2} : x^2 - 22 = 0$$

$$\sqrt[3]{2} : x^3 - 2 = 0$$

$\sqrt{3} + \sqrt{2}$ is not as trivial, so we will do it out in more steps.

There are two approaches to finding the integer coefficient polynomial. One is to take advantage of symmetry, and derive that

$$\prod (x - (\pm\sqrt{3} \pm \sqrt{2})) \quad (1.10)$$

will work (using loose notation of course). A more general technique is to notice that

$$x = \sqrt{3} + \sqrt{2}$$

$$x^2 = 5 + 2\sqrt{6}$$

$$(x^2 - 5)^2 = 24$$

$$x^4 - 10x^2 + 1 = 0.$$

Notice that this is actually the exact same answer we get in (1.10) if you work it out.

(b) Each $|A_n| = |\mathbb{N}^n|$, which is countable

(c) We proved earlier in Theorem 1.4.13 that a countably infinite union of countable sets is countable. Since there are a countable number of algebraic numbers, and reals are uncountable, we conclude that transcendentals are also uncountable.

Exercise 1.4.13

(a) INCOMPLETE

1.5 Cantor's Theorem**Exercise 1.5.2**

(a) Because b_1 differs from $f(1)$ in position 1

(b) b_i differs from $f(i)$ in position i .

(c) We reach a contradiction that we can enumerate the elements of $(0, 1)$, and thus $(0, 1)$ is uncountable.

Exercise 1.5.3

(a) $\frac{\sqrt{2}}{2} \in (0, 1)$ but is irrational

(b) We can just define our decimal representations to never have an infinite string of 9s

Exercise 1.5.4

Suppose S is countable. Then we can enumerate the elements of S using the natural numbers. Now, consider some $s = (s_1, s_2, \dots)$, where

$$s_i = \begin{cases} 0, & \text{if } f(i), \text{ position } i = 11, \\ \text{otherwise} \end{cases} \quad (1.11)$$

Then since $s \neq f(i) \forall i$, $s \notin S$. But this is a contradiction since s only contains elements 0 or 1, and thus should be in S . Thus, we conclude that S is uncountable.

Exercise 1.5.5

(a)

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \quad (1.12)$$

- (b) Each element has two choices when constructing a subset of A . To be, or not to be¹, in the set.

Exercise 1.5.6

- (a) Many different answers.

$$\{(a, \{a\}), (b, \{b\}), (c, \{c\})\}$$

$$\{(a, \emptyset), (b, \{b\}), (c, \{c\})\}$$

- (b)

$$\{(1, \{1\}), (2, \{2\}), (3, \{3\}), (4, \{4\})\}.$$

- (c) Because $|\mathcal{P}(A)| > |A|$ and same for B .

Exercise 1.5.7

1. $B = \emptyset$
2. $B = \{a, d\}$

Exercise 1.5.8

- (a) AFSOC $a' \in B$. Then that means $a \notin f(a')$. But this is a contradiction since $a' \in B = f(a')$.
- (b) AFSOC $a' \notin B = f(a')$. Then since $a' \notin f(a')$, $a' \in B$, but that is a contradiction.

Exercise 1.5.9

- (a) This is the same as $\mathbb{N} \times \mathbb{N}$, which is countable.
- (b) Uncountable, since $\mathcal{P}(\mathbb{N})$ is uncountable.
- (c) Is this question asking for the number of antichains or if there is an antichain with uncountable cardinality? The latter is obvious, and *no* is the answer since any subset of \mathbb{N} is countable. The first case probably uncountable???

¹sorry, had to do it

Chapter 2

Sequences and Series

Exercise 2.0.1

(a) Let $\epsilon > 0$ be arbitrary. Then choose $n \in \mathbb{N}$ such that $n > \frac{1}{\sqrt{6\epsilon}}$. Then

$$\begin{aligned} \left| \frac{1}{6n^2 + 1} \right| &< \left| \frac{1}{6\frac{1}{6\epsilon} + 1} \right| \\ &< \left| \frac{1}{\frac{1}{\epsilon} + 1} \right| \\ &< \frac{\epsilon}{\epsilon + 1} \\ &< \epsilon \end{aligned}$$

as desired.

(b) Choose $n > \frac{13}{2\epsilon} - \frac{5}{2}$

(c) Choose $n > \frac{4}{\epsilon^2} - 3$

Exercise 2.0.2

Consider the sequence

$$x_n = (-1)^n, n \geq 1. \tag{2.1}$$

Then for $\epsilon > 2$, it is true that $|x_n - 0| < 2, \forall n \geq 1$.

The *vercongent* definition describes a sequence that can be finitely bounded past some n .

Exercise 2.0.3

(a) We have to find one school with a student shorter than 7 feet.

(b) We would have to find a college with a grade lower than B.

(c) We just have to check every college for a student who is shorter than 6 feet.

Exercise 2.0.4

For $\epsilon > \frac{1}{2}$, we can find a suitable N , since we can claim the sequence “converges” to $\frac{1}{2}$. For $\epsilon \leq \frac{1}{2}$, there is no suitable response.

Exercise 2.0.5

(a) $\lim a_n = 0$. Take $n > 1$. Then

$$\begin{aligned} \left| \left[\left[\frac{1}{n} \right] \right] \right| &\leq 0 \\ &< \epsilon. \end{aligned}$$

(b) $\lim a_n = 0$. Take $n > 10$. Then

$$\begin{aligned} \left| \left[\left[\frac{10+n}{2n} \right] \right] \right| &= \left| \left[\left[\frac{5}{n} + \frac{1}{2} \right] \right] \right| \\ &\leq 0 \\ &< \epsilon. \end{aligned}$$

Usually, the sequence converges to some value by getting closer and closer eventually. Sometimes, the sequence converges to the exact value very fast, which means for some n , we don't need to choose a larger n . E.g. if we had the sequence of all 0s, we can choose any n and claim the sequence converges to 0.

Exercise 2.0.6

(a) Larger

(b) Larger

Exercise 2.0.7

(a) We say a sequence x_n *converges* to ∞ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$ we have that $|x_n| > \epsilon$

(b) With our definition, we say this sequence diverges, but does not converge to ∞ .

Exercise 2.0.8

(a) Frequently.

(b) Eventually is stronger, and implies frequently.

(c) We say that a sequence x_n converges to x if it eventually is in a neighborhood of radius ϵ of x for all $\epsilon > 0$.

(d) x_n is only necessarily frequently in (1.9, 2.1).

Exercise 2.0.1

Let $\epsilon > 0$. Consider $n \geq 1$, then

$$|a - a| = 0 < \epsilon.$$

Exercise 2.0.2

(a) We are given $(x_n) \rightarrow 0$, so we can make $|x_n - 0|$ as small as we want. In particular, we choose N such that $|x_n| < \epsilon|\sqrt{x_n}|$, whenever $n \geq N$. To see that this N indeed works, observe that for all $n \geq N$,

$$|\sqrt{x_n}| = \frac{|x_n|}{|\sqrt{x_n}|} < \frac{1}{|\sqrt{x_n}|} \epsilon |\sqrt{x_n}| = \epsilon$$

so $(\sqrt{x_n}) \rightarrow 0$.

(b) We are given $(x_n) \rightarrow x$, so we can make $|x_n - x|$ as small as we want. We choose N such that

$$|x_n - x| < \epsilon|\sqrt{x_n} + \sqrt{x}|$$

whenever $n \geq N$. To see that this N works, notice that for all $n \geq N$,

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|} < \frac{1}{|\sqrt{x_n} + \sqrt{x}|} \epsilon |\sqrt{x_n} + \sqrt{x}| = \epsilon$$

Therefore, $(\sqrt{x_n}) \rightarrow \sqrt{x}$.

Exercise 2.0.3

By the Order Limit Theorem, since

$$\begin{aligned}\forall n, x_n \leq y_n &\Rightarrow \lim_{n \rightarrow \infty} y_n \geq \lim_{n \rightarrow \infty} x_n = l \\ \forall n, z_n \leq y_n &\Rightarrow \lim_{n \rightarrow \infty} y_n \leq \lim_{n \rightarrow \infty} z_n = l\end{aligned}$$

so $l \leq \lim_{n \rightarrow \infty} y_n \leq l \Rightarrow \lim_{n \rightarrow \infty} y_n = l$.

Exercise 2.0.4

AFSOC $\lim a_n = l_1$ and l_2 , for $l_1 \neq l_2$. Then we have that $\forall \epsilon > 0$, for sufficiently large n , that

$$\begin{aligned}|a_n - l_1| &< \epsilon \\ |a_n - l_2| &< \epsilon\end{aligned}$$

But this is a contradiction, since if we let $d = |l_1 - l_2|$, and $\epsilon = \frac{d}{2}$, then

$$\begin{aligned}|l_2 - l_1| &\leq |a_n - l_1| + |-(a_n - l_2)| < 2\epsilon \\ d &\leq |a_n - l_1| + |-(a_n - l_2)| < d,\end{aligned}$$

which leads to $d < d$. Thus, we must conclude that $l_1 = l_2$, and limits are unique.

Exercise 2.0.5

(*Rightarrow*) If (z_n) is convergent to some l , then $\forall \epsilon > 0$, we have that $\exists N \in \mathbb{N}$ such that for $n \geq N$, that

$$|z_n - l| < \epsilon \implies |x_n - l| < \epsilon, |y_n - l| < \epsilon, \quad (2.2)$$

because z_n appears before or at the same time as x_n and y_n in the sequence.

(\Leftarrow) If $(x_n), (y_n)$ are both convergent to some limit l , then we have for some $n \geq N \in \mathbb{N}$, that

$$\begin{aligned}|x_n - l| &< \epsilon \\ |y_n - l| &< \epsilon.\end{aligned}$$

Then choose $n' \geq 2N$, then we have two cases,

- If n' odd, then $z_{n'} = x_{(n'+1)/2}$. Since $\frac{n'+1}{2} \geq N$, $|x_{(n'+1)/2} - l| < \epsilon$.
- If n' even, then $z_{n'} = y_{n'/2}$. Since $\frac{n'}{2} \geq N$, $|y_{n'/2} - l| < \epsilon$.

In both cases we have that for every $\epsilon > 0$, we can find $n' \geq N' = 2N \in \mathbb{N}$ such that

$$|z_{n'} - l| < \epsilon, \quad (2.3)$$

so (z_n) is also convergent to l .

Exercise 2.0.6

(a) By triangle inequality, we have $||b_n| - |b|| \leq |b_n - b| < \epsilon$

(b) The converse is not true. Consider the sequence $a_n = (-1)^n$.

Exercise 2.0.7

(a) Since (a_n) is bounded, call M the upper bound of (a_n) . Then since $|b_n|$ can get arbitrarily small, we choose $n \geq N$ such that $|b_n| < \frac{\epsilon}{M}$. Then we have

$$\begin{aligned}|a_n b_n| &\leq |a_n| |b_n| \\ &< M \frac{\epsilon}{M} \\ &< \epsilon.\end{aligned}$$

We cannot use the Algebraic Limit Theorem because we are not given that (a_n) necessarily converges.

(b) No. For example, take $a_n = (-1)^n$, $b_n = 3$.

(c) When $a = 0$, we have

$$|a_n b_n - ab| \leq |b_n| |a_n - a|.$$

We can bound $|b_n| \leq M$, and then choose n such that $|a_n - a| < \frac{\epsilon}{M}$. Then,

$$\begin{aligned} |a_n b_n - ab| &< M \frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

Exercise 2.0.8

(a) $x_n = (-1)^n$, $y_n = (-1)^{n-1}$

(b) Impossible by theorem ???

(c) $b_n = \frac{1}{n}$

(d) Impossible by theorem ???

(e) $a_n = 0$, $b_n = n$

Exercise 2.0.9

No. Consider $a_n = \frac{1}{n}$, $a_n > 0$. $\lim a_n = 0$, but $0 \not\geq 0$.

Exercise 2.0.10

Since $|a_n|$ gets arbitrarily small, we know for $n \geq N$,

$$|b_n - b| \leq |a_n| < \epsilon. \quad (2.4)$$

Exercise 2.0.11

Let $\lim x_n = x$. Then, for some $n_\epsilon \geq N$, we have $|x_n - x| < \epsilon/2$. Now,

$$\begin{aligned} |y_n - x| &= \frac{1}{n} \left[\left| \sum_{i=1}^{n_\epsilon} (x_i - x) \right| + \left| \sum_{i=n_\epsilon}^n (x_i - x) \right| \right] \\ &= \frac{n_\epsilon}{n} \max_{i \in [1, n_\epsilon]} (x_i - x) + \frac{n - n_\epsilon}{n} \max_{i \in [n_\epsilon, n]} (x_i - x) \\ &= \frac{n_\epsilon}{n} \max_{i \in [1, n_\epsilon]} (x_i - x) + \frac{\epsilon}{2} \end{aligned}$$

now if we choose $n > \frac{n_\epsilon \max_{i \in [1, n_\epsilon]} (x_i - x)}{\epsilon/2}$, then we can bound the RHS by ϵ .

Consider when $x_n = (-1)^n$. (x_n) does not converge but (y_n) does.

Exercise 2.0.12

(a) Intuitively, the limit should go to 1, since we have $\frac{\infty}{\infty}$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} a_{m,n} &= 1 \\ \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a_{m,n} &= 0 \end{aligned}$$

(b) A sequence $(a_{m,n})$ converges to l if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$, we have that

$$\begin{aligned} \left| \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} a_{m,n} - l \right| &< \epsilon \\ \left| \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a_{m,n} - l \right| &< \epsilon. \end{aligned}$$

i.e. we approach the same limit no matter what permutation of the index variables we iterate through.

Exercise 2.0.1

Suppose $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Then consider

$$\begin{aligned} s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + \dots \\ &\geq b_2 + (b_4 + b_4) + \dots \\ &= (t_k - b_1)/2 \end{aligned}$$

Exercise 2.0.2

- (a) We can show by induction that the sequence is decreasing. Thus, because the sequence starts at 3, we know it is bounded above by 1 and below by 0. Thus, the sequence converges.
- (b) If $\lim x_n$ exists, then $\lim x_{n+1}$ must be the same limit, because if the limit is a different value or doesn't exist, then (x_n) does not converge.
- (c) Suppose $\lim x_n = \lim x_{n+1} = x$. Then

$$\begin{aligned} x &= \frac{1}{4-x} \\ x^2 - 4x + 1 &= 0 \\ \implies x &= 2 - \sqrt{3} \end{aligned}$$

Exercise 2.0.3

We can use induction to show that (y_n) is increasing. Since the sequence is increasing and starts at 1, we know that (y_n) is bounded above by 4 and below by 0. Thus, by the Monotone Convergence Theorem, we conclude that (y_n) converges. Now, we find the limit of the recurrence by taking the limits of both sides of the equation,

$$\begin{aligned} y &= 4 - \frac{1}{y} \\ y^2 - 4y + 1 &= 0 \\ y &= 2 + \sqrt{3} \end{aligned}$$

Exercise 2.0.4

We can define the recurrence of this sequence as

$$a_{n+1} = \sqrt{2a_n}. \quad (2.5)$$

We can prove by induction that this sequence is increasing. We can also bound the sequence since ???.

Taking the limits of both sides,

$$\begin{aligned} a &= \sqrt{2a} \\ a^2 - 2a &= 0 \\ a &= 2. \end{aligned} \quad (\text{from i.c. } a_0 = 1)$$

Exercise 2.0.5

- (a) By induction, we have

Base Case: $x_1 = 2 \implies x_1^2 = 4 \geq 2$.

Inductive Hypothesis: Given that for some $x_n, x_n^2 \geq 2$.

Inductive Step: Consider

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{4} \left(x_n^2 + 4 + \frac{4}{x_n^2} \right) \\ &\geq 1 + 1 = 2. \end{aligned}$$

Now consider

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2} \left(x_n + \frac{2}{x_n} \right) \\ &= \frac{\frac{1}{2}x_n^2 - 1}{x_n} \\ &\geq 0. \end{aligned}$$

Thus by the Monotone Convergence Theorem we know that (x_n) converges. We now take limits of x on both sides of the recurrence, yielding,

$$\begin{aligned} x &= \frac{1}{2} \left(x + \frac{2}{x} \right) \\ \frac{1}{2}x - \frac{1}{x} &= 0 \\ x^2 - 2 &= 0 \\ \implies x &= \sqrt{2}. \end{aligned}$$

(b) We can modify the sequence to converge to $\sqrt{c}, c \geq 0$ by setting $x_1 = c$, and

$$x_{n+1} = \frac{1}{c} \left((c-1)x_n + \frac{c}{x_n} \right) \quad (2.6)$$

Exercise 2.0.6

- (a) Since we know that (a_n) is bounded, it must also be the case that $\sup(a_n)$ is bounded. Then, $\sup\{a_k\}$ is a decreasing sequence, so by the Monotone Convergence Theorem, we know that (y_n) converges.
- (b) We can define

$$\liminf a_n = \lim z_n, \text{ where} \quad (2.7)$$

$$\lim z_n = \inf\{a_k : k \geq n\}. \quad (2.8)$$

Since $\inf\{a_k\}$ is an increasing sequence, and (a_n) is bounded, we know it converges.

(c) ??? x(An example when the inequality is strict is

$$a_n = 0 \quad (2.9)$$

(d) (*Rightarrow*) Suppose

$$\liminf a_n = \limsup a_n, \quad (2.10)$$

then

Exercise 2.0.1

Suppose we have a convergent sequence. Then given any ϵ , we can always find for $n \geq N \in \mathbb{N}$ that $|a_n - l| < \epsilon$. For any subsequence of (a_n) , $a'_m = a_n$ will be such that $m \geq n$, so we can choose $m \geq N$ from earlier and conclude that $|a'_m - l| < \epsilon$.

Exercise 2.0.2

(a) Define

$$s_i = \sum_{j=1}^i a_j \quad (2.11)$$

$$b_i = \sum_{k=1}^i a_{n_k}, \quad (2.12)$$

where the series regrouping a_i is divided into groups of n_1, n_2, \dots . Then b_i is a subsequence of s_n , which means they converge to the same limit, namely L in this case.

(b) Our proof does not apply to that example because that series did not converge in the first place.

Exercise 2.0.3

(a) Consider

$$a_n = \begin{cases} \sum_{i=1}^n \frac{1}{2^i}, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \quad (2.13)$$

Then we have that $b_n = a_{2n-1}$ converges to 1 and $c_n = a_{2n}$ converges to 0.

(b) A monotone sequence that diverges means that sequence is not bounded. Thus, every subsequence will also be unbounded and thus impossible to be convergent.

(c) Consider the sequence

$$\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \quad (2.14)$$

(d) Consider

$$a_n = \begin{cases} 2^i, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \quad (2.15)$$

(e) By Bolzano-Weierstrass, since we have a subsequence that is bounded, we know we can find a convergent subsequence within this subsequence that converges.

Exercise 2.0.4

AFSOC (a_n) converges to $b \neq a$. Then we have that $|a_n - b|$ can be arbitrarily small. But this implies that every subsequence will also converge to b , which is a contradiction.

AFSOC (a_n) does not converge. Then since (a_n) is bounded, and every convergent subsequence converges to ??? i give up :(

Exercise 2.0.5

Consider $|b^n|$. Since $|b| < 1$, we have that $|b^n|$ is a decreasing sequence that is bounded below by 0, so we have

$$|b| > l \geq 0.$$

We notice that $|b^{2n}|$ is a subsequence that also converges to L , and since $|b^{2n}| = |b|^2$, by the Algebraic Limit Theorem, we have that $|b^{2n}| \rightarrow l^2 = l \implies l = 0$. Since $|b^n| \rightarrow 0$, we conclude $b^n \rightarrow 0$.

Exercise 2.0.6

We have $s = \sup S$, which means for any $\epsilon > 0$,

$$\begin{aligned} s - \epsilon < x \in S < a'_n \\ \implies \epsilon > |s - a'_n| = |a'_n - s| \end{aligned}$$

where a'_n is a subsequence containing part of the infinite $a_n > x \in S$.

Exercise 2.0.1

(a) $a_n = 1 + \left(-\frac{1}{2}\right)^n$

(b) $a_n = n$

(c) Impossible, since a Cauchy sequence implies convergence, which means every subsequence will also converge.

(d) You can use Equation (2.15). Literally anything that diverges but has a convergent subsequence.

Exercise 2.0.2

If we have that $(x_n) \rightarrow x$, then we can make $|x_n - x|$ arbitrarily small. Consider

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| && \text{(Triangle Inequality)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Exercise 2.0.3

- (a) The *pseudo-Cauchy* definition is different because it only looks at consecutive terms
- (b) Consider the harmonic series, where $s_{n+1} - s_n = \frac{1}{n}$.

Exercise 2.0.4

$$\begin{aligned}
 |c_{n+1} - c_n| &= ||a_{n+1} - b_{n+1}| - |a_n - b_n|| \\
 &\leq |a_{n+1} - a_n + b_{n+1} - b_n| \\
 &\leq |a_{n+1} - a_n| + |b_{n+1} - b_n| \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
 \end{aligned}$$

Exercise 2.0.5

- (a) Let $a_n = x_n + y_n$, then

$$\begin{aligned}
 |a_{n+1} - a_n| &= |x_{n+1} - x_n + y_{n+1} - y_n| \\
 &\leq |x_{n+1} - x_n| + |y_{n+1} - y_n| \\
 &< \epsilon
 \end{aligned}$$

- (b)

Exercise 2.0.6

- (a)

Exercise 2.0.1

$$\begin{aligned}
 \lim s_{nn} &= \sum_{i=0}^{\infty} -\left(\frac{1}{2}\right)^i \\
 &= -2.
 \end{aligned}$$

The value is equal to summing the columns first.

Exercise 2.0.2

By the Absolute Convergence test, since we know for fixed i that $\sum_{j=1}^{\infty} |a_{ij}|$ converges, then we know for fixed i that each $\sum_{j=1}^{\infty} a_{ij}$ converges to some c_i as well.

Then, since

$$\begin{aligned}
 \sum_{j=1}^{\infty} |a_{ij}| &\geq \sum_{j=1}^{\infty} a_{ij} \\
 \implies b_i &\geq c_i,
 \end{aligned}$$

and we know that $\sum_{i=1}^{\infty} b_i$ converges, we conclude that $\sum_{i=1}^{\infty} c_i$ must converge as well, implying that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \tag{2.16}$$

converges as well.

Exercise 2.0.3

- (a) Since $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$ converges, we know that we can find $m, n \geq N$ such that

$$|t_{mn} - L| < 0.9,$$

and then choose our upper bound as

$$M = \max\{l\} \cup \{a_{mn} \mid m, n < N\} \quad (2.17)$$

Since t_{nn} is an increasing sequence, and is bounded above, by the Monotone Convergence Theorem, t_{nn} converges.

(b) Consider

$$\begin{aligned} |s_{n+1,n+1} - s_{nn}| &= |a_{n+1,n+1}| \\ &= |t_{n+1,n+1} - t_{nn}| \\ &< \epsilon. \end{aligned}$$

So (s_{nn}) is a Cauchy Sequence and converges.

Exercise 2.0.4

(a) Since (t_{nn}) is an increasing sequence and is bounded above, we know there exists a $t_{n_0 n_0}$ such that

$$B - \frac{\epsilon}{2} < t_{n_0 n_0} \leq B. \quad (2.18)$$

For $N_1 = n_0$, since for $m, n \geq N_1$, $t_{mn} > t_{n_0 n_0}$, and $t_{mn} \leq B$ (upper bound B), we will have that

$$B - \frac{\epsilon}{2} < t_{m,n} \leq B. \quad (2.19)$$

(b) Since (t_{nn}) converges, we can find $n \geq N$ such that

$$\begin{aligned} |s_{nn} - S| &= \left| s_{nn} - \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \right| \\ &= \left| \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij} \right| \\ &< \left| \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} |a_{ij}| \right| \\ &= \left| t_{nn} - \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| \right| \\ &< \epsilon. \end{aligned}$$

So for $m, n \geq N$,

$$|s_{mn} - S| \leq \quad (2.20)$$

hm????????/