Understanding Analysis Solutions

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Acknowledgment

I would like to thank Varun Gudibanda and Trajan Hammonds for giving me the inspiration for writing these solutions by hyping up Real Analysis. They also contributed to some solutions in chapter 1 and 2.

I would also like to thank Stephen Abbott for writing a fantastic introductory text to Real Analysis. If he ever reads this, he's probably wondering why someone would write solutions to his text if it already exists, but I hope he's happy to know someone learned a lot while doing it.

Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) PROOF AFSOC $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$ such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in lowest reduced terms. Then we can square both sides, yielding $3 = \left(\frac{m}{n}\right)^2 \Longrightarrow 3n^2 = m^2$. Now, we know m^2 is a multiple of 3 and thus m must also. Then, we can write $m = 3k$, and derive

$$\begin{aligned}(\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2\end{aligned}$$

Similar to before, we come to the conclusion that n is a multiple of 3. However, this is a contradiction since m, n are both multiples of 3 and we assumed $\frac{m}{n}$ was in lowest terms. Thus, we conclude $\sqrt{3}$ is irrational.

The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ implies that m is a multiple of 4, since we have

$$4n^2 = m^2 \quad \Rightarrow \quad 2n = m,$$

so we cannot reach our contradiction that m/n is not in lowest terms.

Exercise 1.2.2

(a) False. Consider

$$A_n = \left[0, \frac{1}{n}\right).$$

Then

$$\bigcap_{n=1}^{\infty} A_n = \{0\}.$$

(b) True. Since $\forall i, A_i \subseteq A_1$, $\exists x$ such that $\forall i, x \in A_i$. Therefore, the intersection cannot be empty. Then, every set is finite, and the intersection of any number of finite sets will be finite.

(c) False. Consider $A = \{1, 2\}, B = \{1\}, C = \{2, 3\}$.

$$\{1,2\} \cap \big(\{1\} \cup \{2,3\}\big) = \{1,2\} \neq \big(\{1,2\} \cap \{1\}\big) \cup \{2,3\} = \{1,2,3\}$$

(d) True. Intersection is associative.

(e) True. Intersection is distributive over union.

PROOF We will prove

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.1}$$

by set inclusion.

- Suppose $x \in A \cap (B \cup C)$. By the definition of intersection, we know $x \in A$ and $x \in B \cup C$, the latter which means $x \in B$ or $x \in C$.
  We can consider 2 cases for x,
  1. $x \in B$. Then we know $x \in A$ and $x \in B$, so $x \in A \cap B$ and therefore $x \in (A \cap B) \cup (A \cap C)$
  2. $x \in C$. Symmetric to the case above.

  in all cases, we see $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$, so
  $$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$
- Suppose $x \in (A \cap B) \cup (A \cap C)$. Then we have two cases
  1. $x \in A \cap B$. This means $x \in A$ and $x \in B$. If $x \in B$, then $x \in B \cup C$, since $B \subseteq B \cup C$. Putting these facts together, we see $x \in A \cap (B \cup C)$.
  2. $x \in A \cap C$. Symmetric to the case above.

  in all cases, we see $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$, so
  $$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

Exercise 1.2.3

(a) If $x \in (A \cap B)^c$, then we have cases

- $x \in B$ and $x \notin A$. Then $x \notin A$ implies $x \in A^c \Rightarrow x \in A^c \cup B^c$.
- $x \in A$. Symmetric to above.
- $x \notin A$ and $x \notin B$. Then $x \in A^c$ so $x \in A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$, then we have cases

- $x \in A^c$. Then $x \notin A$ so x cannot be in the intersection of A and B, so $x \in (A \cap B)^c$.
- $x \in B^c$. Symmetric to above.

(c) Proof for $(A \cup B)^c = A^c \cap B^c$ pretty similar to above.

Exercise 1.2.4

We are verifying the triangle inequality with a, b.

(a) If a, b have the same sign, then

$$\begin{aligned} |a+b| &= a+b \\ |a|+|b| &= a+b \\ \Rightarrow |a+b| &= |a|+|b| \\ \Rightarrow |a+b| &\leq |a|+|b| \end{aligned}$$

(b)
- $a \geq 0, b < 0$.

$$\begin{aligned} |a+b| &\leq |a| \\ &\leq |a|+|b| \end{aligned}$$

- $a + b \geq 0$. At most one of a, b is negative. If they are both positive, then we have already shown this in part (a). Otherwise, WLOG a is negative. Then

$$\begin{aligned} |a+b| &\leq |b| \\ &\leq |a| + |b| \end{aligned}$$

Exercise 1.2.5

(a) Substitute in $b' = -b$ into the triangle inequality.

(b) Easy to prove directly without using triangle inequality. **TODO**.

A direct proof will look something like:

- If a, b are the same sign, then equality holds
- If a, b are different signs, then if b is negative, then $|a - b| = |a| + |b|$, and if a is negative, then $|a - b| = |a| + |b|$, both of which bound $\big||a| - |b|\big|$.

Exercise 1.2.6

(a) Yes, since $f(A \cap B) = [1, 4] = [0, 4] \cap [1, 16] = f(A) \cap f(B)$. This is by coincidence though, as we will later see. Yes, since $f(A \cup B) = [0, 16] = [0, 4] \cup [1, 16] = f(A) \cup f(B)$.

(b) Choose $A = [-2, 0], B = [0, 2]$

(c) Suppose $x \in g(A \cap B)$, then $\exists x' \in A \cap B$ such that $g(x') = x$. Since $x' \in A$ and $x' \in B$, we know $x = g(x') \in g(A), g(B)$, so we conclude $x \in g(A) \cap g(B)$.

(d) Equality. **TODO** too lazy to write out the proof. Similar to above.

Exercise 1.2.7

(a) **TODO** I don't think we want to include $x \in \mathbb{I}$...

$$f^{-1}(A) = [0, 2] \tag{1.2}$$

$$f^{-1}(B) = [0, 1] \tag{1.3}$$

We see $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ is also true.

(b) **TODO**

Exercise 1.2.8

Negating statements. Took some liberties. Also notice that these statements are not necessarily true.

(a) There exists a real number satisfying $a < b$, such that $\forall n \in \mathbb{N}$, $a + 1/n \geq b$.

(b) There exists two distinct real numbers such that there is not a rational number between them.

(c) There exists a natural number $n \in \mathbb{N}$ such that $\sqrt{n}$ is not a natural number nor an irrational number.

(d) There exists a real number $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $n \leq x$.

Exercise 1.2.9

We are given the sequence

$$x_1 = 1, x_{n+1} = \frac{1}{2}x_n + 1 \tag{1.4}$$

and want to show $\forall i \geq 1, x_i < 2$.

We can show this with a direct proof of summation.

An alternative that the book probably wants to see is using **induction**.

- Base Case: $x_1 = 1 < 2$
- Inductive case. Assume $\forall i < n + 1, x_i < 2$. Then $x_i/2 + 1 < 2$ since $x_i/2 < 1$.

- By induction our original claim is proved.

Exercise 1.2.10

(a) Similar to Exercise 1.2.9. $y_n < 4$ means $(3/4)y_n < 3$ so $(3/4)y_n + 1 < 4$

(b) In brief,

$$\begin{aligned} y_n &\leq \frac{3}{4}y_n + \frac{1}{4}y_n && (1.5) \\ &< \frac{3}{4}y_n + 1 && \text{(Using } y_n < 4\text{)} \\ &< y_n + 1 && \text{(Sequence definition)} \end{aligned}$$

Exercise 1.2.11

A combinatorial argument is that in order to construct a set, we have 2 choices for every element, to include it or not to. Therefore, we have

$$\prod_{i=1}^{n} 2 = 2^n$$

Exercise 1.2.12

(a) We know that $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$. So if we are trying to show $(A_1 \cup A_2 \cup A_3)^c = (A_1 \cup A_2)^c \cap A_3^c = A_1^c \cap A_2^c \cap A_3^c$. Induction lets us apply the property on smaller parts of our expression.

(b) Induction only proves the property for some $n \in \mathbb{N}$, i.e. some finite n. It is not shown for an infinite n.

(c) **TODO**. Sketch: If x is not in the union of all the A_n, then x cannot be part of any particular A_n either, or else it would be in the union.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) We compute the additive inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned} 0+0 &\equiv 0 \\ 1+4 &\equiv 0 \\ 2+3 &\equiv 0 \\ 3+2 &\equiv 0 \\ 4+1 &\equiv 0 \end{aligned}$$

(b) We compute the multiplicative inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned} 1\times 1 &\equiv 1 \\ 2\times 3 &\equiv 1 \\ 3\times 2 &\equiv 1 \\ 4\times 4 &\equiv 1 \end{aligned}$$

(c) $\mathbb{Z}_4$ is not a field because multiplicative inverses do not exist for every single element. For example, 2 multiplied with any number is even, which cannot $\equiv 1 \pmod 4$.

We conjecture that $\mathbb{Z}_n$ always has additive inverses and only has multiplicative inverses if n is prime.

Exercise 1.3.2

We are writing a formal definition for the *infimum* of a set.

(a) $s = \inf A$ means

i) s is a lower bound for A

ii) if b is any lower bound for A, then $b \le s$

(b) If $s \in \mathbb{R}$ is a lower bound for $A \subseteq \mathbb{R}$, then $s = \inf A$ iff $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$.

Proof ($\Rightarrow$) If $s = \inf A$, then s is the greatest lower bound for A, meaning any $s+\epsilon$ for $\epsilon > 0$ will be greater than some element of A, otherwise $s + \epsilon$ is a greater lower bound and leads to a contradiction that$s \ne \inf A$.

($\Leftarrow$) If $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$, then since s is a lower bound, $\forall b > s$, b will not be a lower bound for A since if, $b > s$, then we can choose $\epsilon = b - s > 0$, and we know that $\exists a \in A$ where $a < s + \epsilon < b$, which means b is not a lower bound. Thus, all lower bounds b must be such that $b \le s$, and we conclude $s = \inf A$.

Exercise 1.3.3

(a) Since $\inf A$ is a lower bound for A, we know $\inf A \in B$. Now, we need to show $\inf A$ is the supremum of B. $\inf A$ is the least upper bound for B, since if $\exists b \in B, b > \inf A$, then we know that this b is not a lower bound for A, so no such b exists.

(b) There might be a typo in this question. I think the question was meant to read “explain why there is no need to assert that the greatest *lower bound* in the Axiom of Completeness.” In this case, the answer would be that the Axiom of Completeness already implies the greatest lower bound property, so there is no need to explicitly state it.

(c) We can take the negative of all elements in A, find $\sup A$, and then negate again to get $\inf A$.

Exercise 1.3.4

If $B \subseteq A$, then

$$\begin{aligned} \sup A = s &\ge a, \forall a \in A \\ s &\ge b, \forall b \in B && \text{(since } B \subseteq A) \\ \Rightarrow\ s &\ge \sup B. && \text{(since } s \text{ is an upper bound for } B) \end{aligned}$$

Exercise 1.3.5

(a)

$$
\begin{aligned}
&s = \sup(c + A) \\
\Rightarrow\ & s \text{ is the least upper bound for } c + A \\
\Rightarrow\ & s - c \text{ is the least upper bound for } A \\
\Rightarrow\ & s - c = \sup A \\
& s = c + \sup A
\end{aligned}
$$

(b)

$$
\begin{aligned}
&s = \sup(cA) \\
\Rightarrow\ & s \text{ is the least upper bound for } cA \\
\Rightarrow\ & \frac{s}{c} \text{ is the least upper bound for } A \\
\Rightarrow\ & \frac{s}{c} = \sup A \\
&s = c \sup A
\end{aligned}
$$

(c) If $c < 0$, $\sup(cA) = -c \inf(A)$.

Exercise 1.3.6

(a) $\sup: 3; \inf: 1$

(b) $\sup: 1; \inf: 0$

(c) $\sup: \frac{1}{2}; \inf: \frac{1}{3}$

(d) $\sup: 9; \inf: \frac{1}{9}$

Exercise 1.3.7

If $a \geq a', \forall a' \in A$, and $a \in A$, then

$$\forall \epsilon > 0, a - \epsilon < a, \tag{1.6}$$

so a is the least upper bound for A, and $a = \sup A$.

Exercise 1.3.8

Let

$$\epsilon = \sup B - \sup A > 0. \tag{1.7}$$

since $s_b = \sup B$, $\exists b \in B \mid b > s_b - \epsilon/2$. Since $s_b - \frac{\epsilon}{2} > \sup A$, then $b \geq \sup A$, so this $b \in B$ is an upper bound for A.

Exercise 1.3.9

(a) True, take the largest element in the set as the supremum.

(b) False, $\sup(0, 2) = 2$, but $2 > a \in (0, 2)$, but $\sup A = 2 \not< 2 = L$.

(c) False $A = (0, 2), B = [2, 3)$. We have that $\sup A = \inf B$

(d) True.

(e) False, take $A = B = (0, 2)$.

1.4 Consequences of Completeness

Exercise 1.4.1
If $a < 0$, then we have two cases,

1. If $b > 0$, then $a < 0 < b$.
2. If $b = 0$, then we can take $-b, -a$, which satisfies $0 \leq -b < -a$, and apply Theorem 1.4.3.

Exercise 1.4.2
(a) If $a, b \in \mathbb{Q}$, then

$$\begin{aligned} a &= \frac{a_1}{a_2} \\ b &= \frac{b_1}{b_2} \\ \implies a + b &= \frac{a_1 b_2 + a_2 b_1}{a_2 b_2} \in \mathbb{Q} \end{aligned}$$

(b) We can use contradiction,

- AFSOC $a + t \in \mathbb{Q}$. Let $a + t = \frac{m}{n}$. We know $a = \frac{a_1}{a_2}$ since $a \in \mathbb{Q}$, so

$$\begin{aligned} a + t &= \frac{m}{n} \\ t &= \frac{m}{n} - \frac{a_1}{a_2} \in \mathbb{Q}, \end{aligned}$$

  which is a contradiction since we are given $t \in \mathbb{Q}$. Therefore, we conclude $a + t \in \mathbb{I}$.
- AFSOC $at \in \mathbb{Q}$. Let $at = \frac{m}{n}$. We know $a = \frac{a_1}{a_2}$ since $a \in \mathbb{Q}$, so

$$\begin{aligned} at &= \frac{m}{n} \\ t &= \frac{m}{n} \cdot \frac{a_2}{a_1} \in \mathbb{Q}, \end{aligned}$$

  which is a contradiction since we are given $t \in \mathbb{Q}$. Therefore, we conclude $at \in \mathbb{I}$

(c) $\mathbb{I}$ is not closed under addition or multiplication.

$$\begin{aligned} \left(3 - \sqrt{2}\right) + \left(3 + \sqrt{2}\right) &= 6 \notin \mathbb{I} \\ \left(3 - \sqrt{2}\right) \cdot \left(3 + \sqrt{2}\right) &= 5 \notin \mathbb{I} \end{aligned}$$

Exercise 1.4.3
We can apply Theorem 1.4.3, to find $a < q < b, q \in \mathbb{Q}$, and then subtract an irrational number such as $\sqrt{2}$ to end up at

$$a - \sqrt{2} < q - \sqrt{2} < b - \sqrt{2}, \tag{1.8}$$

where $q - \sqrt{2} \in \mathbb{I}$.

Exercise 1.4.4
Suppose $\exists b$ lower bound such that $b > 0$. Then by Archimedean Property of $\mathbb{R}$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b$, which means b is not a valid lower bound. Thus $b \leq 0$, and 0 is a valid lower bound so the inf is 0.

Exercise 1.4.5
AFSOC $\exists \alpha \in \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$. Then $\alpha > 0$, but by Archimedean property of reals, we have that $\exists n \in \mathbb{N} \mid \frac{1}{n} < \alpha$. Since $\alpha \notin \left(0, \frac{1}{n}\right)$ leads to $\alpha \notin \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$, a contradiction, we conclude the set is empty.

Exercise 1.4.6

(a) If $\alpha^2 > 2$, then

$$\begin{aligned}\left(a - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}\end{aligned}$$

choose $\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}$. Then

$$\begin{aligned}\left(a - \frac{1}{n_0}\right)^2 &> \alpha^2 - \frac{2\alpha}{2\alpha}(\alpha^2 - 2) \\ &> 2\end{aligned}$$

but $\alpha - \frac{1}{n_0} < \alpha$, so α is not the least upper bound for the se, so $\alpha \neq \sup T$.

(b) Just replace $\sqrt{2}$ with $\sqrt{b}$ for the proof above.

Exercise 1.4.7

Once we have assigned $g(i) = f(n_i)$, remove $f(n_i)$ from A. Now, there is a new $n_{i+1} = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(1), f(2), \ldots, f(n_i)\}\}$. Assign $g(i+1) = f(n_{i+1})$, and repeat.

Exercise 1.4.8

(a) If both are finite, then their union is finite and trivially countable. If one is finite, then first enumerate elements of the finite set. Then map the rest of $\mathbb{N}$ to the countably infinite set. If both are countably infinite, map one set to odds and the other to evens.

(b) Induction only holds for finite integers, not infinity.

(c) We can arrange each A_n into row n of a $\mathbb{N} \times \mathbb{N}$ matrix. Then, we enumerate by diagonalization.

Exercise 1.4.9

(a) If $A \sim B$, then there is a 1-to-1 mapping. We can just take the inverse of the mapping to derive $B \sim A$.

(b) If we have $f : A \to B$, $g : B \to C$, then we can compose the functions so $g(f(x)) : A \to C$.

Exercise 1.4.10

The set of all finite subsets of $\mathbb{N}$ can be ordered in increasing order by the sum of each subset.

Exercise 1.4.11

(a) $f(x) = (x, 0.5) \in S$

(b) Interweave the decimal expansion of x, y, e.g.

$$f(x, y) = 0.x_1y_1x_2y_2x_3y_3\ldots \tag{1.9}$$

Exercise 1.4.12

(a)

$$\begin{aligned}\sqrt{2} &: x^2 - 2 = 0 \\ \sqrt[3]{2} &: x^3 - 2 = 0\end{aligned}$$

$\sqrt{3} + \sqrt{2}$ is not as trivial, so we will do it out in more steps.

There are two approaches to finding the integer coefficient polynomial. One is to take advantage of symmetry, and derive that

$$\prod(x - (\pm\sqrt{3} \pm \sqrt{2}) \tag{1.10}$$

will work (using loose notation of course). A more general technique is to notice that

$$\begin{aligned} x &= \sqrt{3}+\sqrt{2} \\ x^2 &= 5+2\sqrt{6} \\ (x^2-5)^2 &= 24 \\ x^4-10x^2+1 &= 0. \end{aligned}$$

Notice that this is actually the exact same answer we get in (1.10) if you work it out.

(b) Each element of A_n is a root of a n degree polynomial, which we can represent as an $(n+1)$-tuple of coefficients $\in \mathbb{Z}$. Therefore, $|A_n| = k\left|\mathbb{N}^{n+1}\right| = \left|\mathbb{N}^{n+1}\right|$, which we know is countable.

(c) We proved earlier in Theorem 1.4.13 that a countably infinite union of countable sets is countable. Since there are a countable number of algebraic numbers, and reals are uncountable, we conclude that transcendentals are also uncountable.

Exercise 1.4.13
We are proving the **Schroöder-Bernstein Theorem**, which states if there exist 1-to-1 functions $f : X \to Y$ and $g : Y \to X$, then there exists a 1-to-1, onto function $h : X \to Y$, which implies $X \sim Y$.

(a) By the definition of 1-to-1, there must be a unique $x \in X$ such that $f(x) = y$. A 1-to-1 function maps distinct elements from the domain to distinct elements of the range, so if we take the inverse f^{-1}, it will still be 1-to-1, this time from $Y \to X$.

(b) Possibilities:

- Zero: g^{-1} is not guaranteed to be onto, it may not have an inverse for x.
- Finite: $g^{-1}(x)$ could exist, and similarly for f^{-1}. Once the element doesn't exist in the inverse domain, the chain will stop.
- Infinite: x is in the range of g and the domain of f

(c) We have 2 cases

- The chains are disjoint. Nothing to prove here.
- The chains are not disjoint, i.e. they have one common element. Let us call this element x. We know to the right of x, all the elements in the two chains will be equal. From the left of x, the elements must be equal as well. This is because the inverse chain must be unique starting from x, see part (a). Since all the elements are the same, the chains must be the same as well.

(d) Since we know this chain started with $x \in X$, this y could not have been created from the RHS, otherwise this y would be in the range of f. Therefore, this chain either has infinite or a finite of elements to the right.

- Finite: the chain must start with an element $y \notin Y$ but not in f's range. This is because if we start with $x \in X$, then as mentioned before, all elements $y' \in Y$ will be in f's range. Therefore, if we start with $y \in Y$, it will match the form indicated.
- Infinite: The chain could not have an infinite elements to the left, because then every y must have come from an $f(x')$ for some $x' \in X$.

Therefore, these chains only can have a finite number of elements to the left, and it matches the form indicated.

(e) By the definition of C_x, all the elements of $y \in C_x$ that are $\in Y$ are mapped by f from $x \in X_1$. This means f maps X_1 onto Y_1. Similar logic can be used for g mapping Y_2 onto X_2.

Since we know f is a 1-to-1 function from X to Y, and we just showed it maps X_1 to Y_1. We can conclude that $X_1 \sim Y_1$, since f is a bijection between X_1 and Y_1. We can similarly conclude $X_2 \sim Y_2$

with g. Since X_1, X_2 are a partition of X, since all the chains are disjoint, and similarly for Y_1, Y_2 of Y, and there exists a bijection f, g for $X_1 \sim Y_1$ and $X_2 \sim Y_2$ respectively, we conclude there must be a bijection between X and Y. Therefore, we conclude $X \sim Y$.

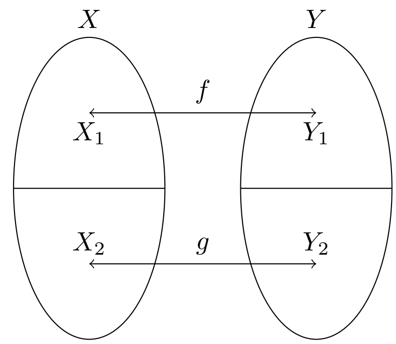

Figure 1.1: f and g mapping X and Y

1.5 Cantor's Theorem

Exercise 1.5.2

(a) Because b_1 differs from $f(1)$ in position 1

(b) b_i differs from $f(i)$ in position i.

(c) We reach a contradiction that we can enumerate all the elements of $(0,1)$, since we found a real number that isn't enumerated, and thus $(0,1)$ is uncountable.

Exercise 1.5.3

(a) $\frac{\sqrt{2}}{2} \in (0,1)$ but is irrational

(b) We can just define our decimal representations to never have an infinite string of 9s.

Exercise 1.5.4

AFSOC S is countable. We will use a diagonalization proof. Then we can enumerate the elements of S using the natural numbers. Now, consider some $s = (s_1, s_2, \dots)$, where

$$s_i = \begin{cases} 0, \text{if } f(i), \text{position } i = 1 \\ 1, \text{otherwise} \end{cases} \tag{1.11}$$

Then since $s \neq f(i) \forall i$, we see $s \notin S$. But this is a contradiction since s only contains elements 0 or 1, and thus should be in S. Thus, we conclude that S is uncountable.

Exercise 1.5.5

(a)

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\} \tag{1.12}$$

(b) Each element has two choices when constructing a subset of A. To be, or not to be [1], in the set.

Exercise 1.5.6

(a) Many different answers.

$$\begin{gathered} \{(a,\{a\}),(b,\{b\}),(c,\{c\})\} \\ \{(a,\emptyset),(b,\{b\}),(c,\{c\})\} \end{gathered} \tag{1.13}$$

(b)

$$\{(1,\{1\}),(2,\{2\}),(3,\{3\}),(4,\{4\})\}.$$

(c) Because in general, $\left|\mathcal{P}(A)\right| > |A|$ for any set $A \neq \emptyset$. The intuition is that the power set has strictly more elements than A, so A cannot map $\mathcal{P}(A)$ onto.

Exercise 1.5.7

Using the examples found in (1.13).

1. $B = \emptyset$
2. $B = \{a\}$

Exercise 1.5.8

(a) AFSOC $a' \in B$. Then that means $a \notin f(a')$ by the definition of B. But this is a contradiction since $a' \in B = f(a')$.

(b) AFSOC $a' \notin B = f(a')$. Then since $a' \notin f(a')$, by the construction of B, this implies $a' \in B$, but that is a contradiction from our original assumption.

Exercise 1.5.9

(a) This is the same as $\mathbb{N} \times \mathbb{N}$, which is countable.

[1] sorry, had to do it. *Addendum* For context, I took a Shakespeare class in college two semesters prior to when I first wrote this.

(b) Uncountable, since this is essentially constructing the power set of $\mathbb{N}$, and we know $\mathcal{P}(\mathbb{N})$ is uncountable.

(c) Is this question asking for the number of antichains or if there is an antichain with uncountable cardinality?

The latter is obvious, and *no* is the answer since any subset of $\mathbb{N}$ is countable.

If we want to count the number of antichains, we notice that an antichain is essentially a partition of some subset of B. We also notice that every element of $\mathcal{P}(B)$ is also technically a partition, just a partition of size one. This means that the cardinality of the set of antichains is at least the cardinality of $\mathcal{P}(B)$. If $B = \mathbb{N}$, then we know $\mathcal{P}(\mathbb{N})$ is already uncountable, so the set of antichains will also be uncountable.

Chapter 2

Sequences and Series

For the convergence proofs in this chapter, I will lean towards showing how to derive the N that works, rather than just going directly with the proof and supplying a magical N, since I think finding the N is the process that deserves more attention.

2.2 The Limit of a Sequence

Exercise 2.2.1
The proofs are essentially the same, so after the first proof, I'll just give the n that can be used to prove the convergence.

(a) Let $\epsilon > 0$ be arbitrary. Then choose $n \in \mathbb{N}$ such that $n > \frac{1}{\sqrt{6\epsilon}}$. Then

$$\begin{aligned}
\left|\frac{1}{6n^2+1}\right| &< \left|\frac{1}{6\frac{1}{6\epsilon}+1}\right| \\
&< \left|\frac{1}{\frac{1}{\epsilon}+1}\right| \\
&< \frac{\epsilon}{\epsilon+1} \\
&< \epsilon
\end{aligned}$$

as desired.

(b) Choose $n > \frac{13}{2\epsilon} - \frac{5}{2}$

(c) Choose $n > \frac{4}{\epsilon^2} - 3$

Exercise 2.2.2
Consider the sequence

$$x_n = (-1)^n, n \geq 1. \tag{2.1}$$

Then for $\epsilon > 2$, it is true that $|x_n - 0| < 2, \forall n \geq 1$.

The *vercongent* definition describes a sequence that can be finitely bounded past some n.

Exercise 2.2.3

(a) We have to find one school with a student shorter than 7 feet.

(b) We would have to find a college with a grade that is not A or B.

(c) We just have find a college where a student is shorter than 6 feet.

Exercise 2.2.4
For $\epsilon > \frac{1}{2}$, we can find a suitable N, since we can claim the sequence "converges" to $\frac{1}{2}$. For $\epsilon \leq \frac{1}{2}$, there is no suitable response.

Exercise 2.2.5

(a) $\lim a_n = 0$. Take $n > 1$. Then

$$\begin{aligned} \left|\left[\left[\frac{1}{n}\right]\right]\right| &\leq 0 \\ &< \epsilon. \end{aligned}$$

(b) $\lim a_n = 0$. Take $n > 10$. Then

$$\begin{aligned} \left|\left[\left[\frac{10+n}{2n}\right]\right]\right| &= \left|\left[\left[\frac{5}{n} + \frac{1}{2}\right]\right]\right| \\ &\leq 0 \\ &< \epsilon. \end{aligned}$$

Usually, the sequence converges to some value by getting closer and closer eventually. This means for a smaller ϵ-neighborhood, we have to enumerate more elements, so we need a larger N.

Sometimes, the sequence converges to the exact value very fast, which means for some n, we don't need to choose a larger n. E.g. if we had the sequence of all 0s, we can choose any n and claim the sequence converges to 0.

Exercise 2.2.6

(a) Any *larger* N will work, since succeeding elements should stay in the neighborhood.

(b) Any *larger* ϵ will work, since we already guaranteed succeeding elements will stay in the ϵ-neighborhood, so any $\epsilon' > \epsilon$ will also bound the rest of the sequence.

Exercise 2.2.7

(a) We say a sequence x_n *converges* to ∞ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$ we have that $|x_n| > \epsilon$

(b) With our definition, we say this sequence diverges, but does not converge to ∞.

Exercise 2.2.8

(a) Frequently, since -1 will leave the set $\{1\}$.

(b) Eventually is stronger, and implies frequently.

(c) We say that a sequence x_n converges to x if it eventually is in a neighborhood of radius ϵ of x for all $\epsilon > 0$.

(d) x_n is only necessarily frequently in $(1.9, 2.1)$, even if there are an infinite number of elements equal to 2, you could have something like $(-2)^n$, where it keeps on leaving the ϵ-neighborhood of 2.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1
Let $\epsilon > 0$. Consider $n \geq 1$, then

$$|a - a| = 0 < \epsilon.$$

Exercise 2.3.2

(a) We are given $(x_n) \to 0$, so we can make $|x_n - 0|$ as small as we want.

In particular, for some $\epsilon > 0$, we choose N such that $\forall n \geq N$,

$$|x_n| < \epsilon^2 \quad \Rightarrow \quad \left|\sqrt{x_n}\right| < \epsilon \tag{2.2}$$

The implication follows since we know $x_n \geq 0, \epsilon > 0$.

To see that this N works, observe that for all $n \geq N$,

$$\left|\sqrt{x_n} - 0\right| < \epsilon \qquad \text{(by (2.2))}$$

so we conclude $(\sqrt{x_n}) \to 0$.

(b) We have two cases. If the sequence converges to 0, then we just have part ((a)).

If $x \neq 0$, then notice

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{|x_n - x|}{\left|\sqrt{x_n} + \sqrt{x}\right|}$$

since we know $x_n \geq 0$ and $x \neq 0$. Now, this expression is hard to bound when the denominator is small, since that would make the overall expression big. Fortunately, we can put a bound on the denominator, namely, since we know $x \neq 0 \to x > 0$, the denominator is > 0. Let us call the denominator value d. Then the following N will work for the convergence proof,

$$N : \forall n \geq N \quad |x_n - x| < \epsilon \cdot d \tag{2.3}$$

Exercise 2.3.3
By the Order Limit Theorem, since

$$\forall n, x_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \geq \lim_{n\to\infty} x_n = l$$
$$\forall n, z_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \leq \lim_{n\to\infty} z_n = l$$

so $l \leq \lim_{n\to\infty} y_n \leq l \Rightarrow \lim_{n\to\infty} y_n = 1$.

Exercise 2.3.4
AFSOC $\lim a_n = l_1$ and l_2, for $l_1 \neq l_2$. Then we have that $\forall \epsilon > 0$, for sufficiently large n, that

$$|a_n - l_1| < \epsilon$$
$$e|a_n - l_2| < \epsilon$$

But this is a contradiction, since if we let $d = |l_1 - l_2|$, and $\epsilon = \frac{d}{2}$, then

$$|l_2 - l_1| \leq |a_n - l_1| + \left|-(a_n - l_2)\right| < 2\epsilon \qquad \text{(Triangle Inequality)}$$
$$d \leq |a_n - l_1| + \left|-(a_n - l_2)\right| < d,$$

which leads to $d < d$. Thus, we must conclude that $l_1 = l_2$, and limits are unique.

Exercise 2.3.5
($\Rightarrow$) If (z_n) is convergent to some l, then $\forall \epsilon > 0$, we have that $\exists N \in \mathbb{N}$ such that for $n \geq N$, that

$$|z_n - l| < \epsilon \Longrightarrow |x_n - l| < \epsilon, |y_n - l| < \epsilon, \tag{2.4}$$

because z_n appears before or at the same time as x_n and y_n in the sequence.

($\Leftarrow$) If $(x_n), (y_n)$ are both convergent to some limit l, then we have for any $\epsilon > 0$, $\exists N_x : n_x \geq N_x$ and $\exists N_y : n_y \geq N_y$, that

$$\begin{aligned} |x_{n_x} - l| &< \epsilon \\ \left|y_{n_y} - l\right| &< \epsilon, \end{aligned} \tag{2.5}$$

respectively.

Choose $N_z > 2 \cdot \max(N_x, N_y)$. Then for $n_z \geq N_z$, z_{n_z} is either equal to x_i for $i > N_x$ or y_j for $j > N_y$. Using (2.5), we can see that

$$|z_{n_z} - l| < \epsilon$$

so (z_n) is also convergent to l.

Exercise 2.3.6

(a) By triangle inequality, we have $\big||b_n| - |b|\big| \leq |b_n - b| < \epsilon$, so the N that proves convergence for (b_n) will also work for $(|b_n|)$.

(b) The converse is not true. Consider the sequence $a_n = (-1)^n$.

Exercise 2.3.7

(a) Since (a_n) is bounded, call M the upper bound of (a_n). Then since $|b_n|$ can get arbitrarily small, we choose $n \geq N$ such that $|b_n| < \frac{\epsilon}{M}$. Then we have

$$\begin{aligned} |a_n b_n| &\leq |a_n||b_n| \\ &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

We cannot use the Algebraic Limit Theorem because we are not given that (a_n) necessarily converges.

(b) No. For example, take $a_n = (-1)^n$, $b_n = 3$. This is because we can no longer make $|b_n|$ arbitrarily small.

(c) When $a = 0$, we have

$$|a_n b_n - ab| \leq |b_n||a_n - a|.$$

We can bound $|b_n| \leq M$, and then choose n such that $|a_n - a| < \frac{\epsilon}{M}$. Then,

$$\begin{aligned} |a_n b_n - ab| &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

Exercise 2.3.8

(a) $x_n = (-1)^n, y_n = (-1)^{n-1}$. Sum is just $\{0, 0, \dots\}$

(b) **Impossible**, since if $x_n + y_n$ converges and x_n also converges, we can show that y_n must converge, which is a contradiction.

(c) $b_n = \frac{1}{n}$

(d) **Impossible**, since if b_n converges to some b, for any $\epsilon > 0$, past some N, for $n \geq N$,

$$|b_n - b| < \epsilon.$$

Any a_n that is unbounded will grow in magnitude for larger n, so b_n cannot help bound a_n.

(e) $a_n = 0, b_n = n$

Exercise 2.3.9

Yes, the strict inequalities will provide an upper and lower bound still. Sort of like a $\sup, \inf$ of the sequence.

Exercise 2.3.10

Since $|a_n|$ gets arbitrarily small, for any $\epsilon > 0$ we know $\exists N : n \geq N$ such that,

$$|b_n - b| \leq |a_n| < \epsilon. \tag{2.6}$$

Exercise 2.3.11

Let $\lim x_n = x$. Then, for any $\epsilon_x > 0$, $\exists N_x : n \geq N_x$, we have $|x_n - x| < \epsilon_x$.

Now, our goal is, given some $\epsilon_y > 0$, to find some $N_y : n \geq N_y$ so we can bound y_n. The intuition is, since we know (x_n) converges, after some point, x_i will be close to the limit x. Our goal is to choose some N_y large enough so the x_i' prior to these x_i are "averaged out" enough, so they are essentially gone, and that the weight on the x_i that are close to x is very high.

$$\begin{aligned}
|y_n - x| &= \left|\frac{1}{n}\left[\sum_{i=1}^{N_x}(x_i - x) + \sum_{i=N_x+1}^{N_y}(x_i - x)\right]\right| \\
&\leq \left|\frac{1}{n}\left[\sum_{i=1}^{N_x} M + \sum_{i=N_x+1}^{N_y} \epsilon_x\right]\right| && \text{(Let } M \text{ bound the difference from } x_i \text{ to } x.) \\
&\leq \left|\frac{1}{n}\Big[N_x M + \big(N_y - N_x\big)\epsilon_x\Big]\right| \\
&\leq \left|\frac{N_x}{n} M + \epsilon_x\right| < \epsilon_y
\end{aligned}$$

Now, we have quite a few choices for our N_y. One such solution, is

- Given some $\epsilon > 0$
- First choose N_x such $n \geq N_x \quad |x_n - x| < \epsilon/2$
- Then, choose $N_y > \frac{2N_x M}{\epsilon}$. This means for $n \geq N_y$,

$$\begin{aligned}
|y_n - x| &\leq \left|\frac{N_x}{n} M + \epsilon_x\right| \\
&< \left|\frac{N_x M}{\frac{2N_x M}{\epsilon}} + \epsilon/2\right| \leq \epsilon
\end{aligned}$$

Consider when $x_n = (-1)^n$. (x_n) does not converge but (y_n) does.

Exercise 2.3.12

(a) Intuitively, the limit should go to 1, since we have $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = 1$$
$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} = 0$$

(b) A sequence $(a_{m,n})$ converges to l if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$, we have that

$$\left|\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} - l\right| < \epsilon$$
$$\left|\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} - l\right| < \epsilon.$$

i.e. we approach the same limit no matter what permutation of the index variables we iterate through.

This definition is motivated by multivariable calculus, but unsure if this makes sense in the context of analysis.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1
Suppose $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Fix m, k so that $m \geq 2^{k+1}-1$, then

$$\begin{aligned}\sum_{i=1}^{m} b_i &\geq \sum_{i=1}^{2^{k+1}-1} b_i \\ &= s_{2^{k+1}-1} \\ &= t_k\end{aligned}$$

Since t_k is a diverging sequence, then b_m will also diverge.

Exercise 2.4.2

(a) We can show by induction that the sequence is decreasing. Thus, because the sequence starts at 3, we know it is bounded below by 0. Thus, the sequence converges.

(b) If $\lim x_n$ exists, then $\lim x_{n+1}$ must be the same limit, because if the limit is a different value or doesn't exist, then (x_n) does not converge.

(c) Suppose $\lim x_n = \lim x_{n+1} = x$. Then

$$\begin{aligned}x &= \frac{1}{4-x} \\ x^2 - 4x + 1 &= 0 \\ &\implies x = 2 - \sqrt{3}\end{aligned}$$

The other root is too large and does not work with the initial conditions.

Exercise 2.4.3
We can use induction to show that (y_n) is increasing. Since the sequence is increasing and starts at 1, we know that (y_n) is bounded above by 4 and below by 0. Thus, by the Monotone Convergence Theorem, we conclude that (y_n) converges. Now, we find the limit of the recurrence by taking the limits of both sides of the equation,

$$\begin{aligned}y &= 4 - \frac{1}{y} \\ y^2 - 4y + 1 &= 0 \\ y &= 2 + \sqrt{3}\end{aligned}$$

Exercise 2.4.4
We can define the recurrence of this sequence as

$$a_{n+1} = \sqrt{2a_n}. \tag{2.7}$$

We can prove by induction that this sequence is increasing. We can also bound the sequence since this sequence can also be viewed as

$$2^{\frac{1}{2}}, 2^{\frac{1}{2}+\frac{1}{4}}, 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}, \cdots$$

You can take the infinite sum $\sum_{i=1}^{\infty} 2^{-i} = 1$ and get $2^1 = 2$ as your final answer.

The other way to solve this problem is to look at the limits of x_n, x_{n+1}, which must be equal. Let's say their limit is x, then

$$\begin{aligned}x_{n+1} = \sqrt{2x_n}x && = \sqrt{2x} \\ x^2 - 2x &= 0 \\ x &= 2. && \text{(from } x_0 = 1)\end{aligned}$$

Exercise 2.4.5

(a) By induction, we have

Base Case: $x_1 = 2 \implies x_1^2 = 4 \geq 2$.

Inductive Hypothesis: Given that for some $x_n, x_n^2 \geq 2$.

Inductive Step: Consider

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) \\ &\geq \frac{1}{4}\big(2 + 4 + 4/2\big) = 2. \end{aligned}$$

Therefore, we conclude $\forall n, x_n \geq 2$.

Now we can show

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{\frac{1}{2}x_n^2 - 1}{x_n} \\ &\geq 0, \end{aligned}$$

which means the sequence is decreasing, so by the Monotone Convergence Theorem we know that (x_n) converges. We now take limits of x on both sides of the recurrence, yielding,

$$\begin{aligned} x &= \frac{1}{2}\left(x + \frac{2}{x}\right) \\ \frac{1}{2}x - \frac{1}{x} &= 0 \\ x^2 - 2 &= 0 \\ \implies x &= \sqrt{2}. \end{aligned}$$

(b) We can modify the sequence to converge to $\sqrt{c}, c \geq 0$ by setting $x_1 = c$, and

$$x_{n+1} = \frac{1}{c}\left((c-1)x_n + \frac{c}{x_n}\right) \tag{2.8}$$

Exercise 2.4.6

(a) Since we know that (a_n) is bounded, it must also be the case that $\sup(a_n)$ is bounded. Then, $\sup\{a_k\}$ is a decreasing sequence, so by the Monotone Convergence Theorem, we know that (y_n) converges.

(b) We can define

$$\liminf a_n = \lim z_n, \text{ where} \tag{2.9}$$
$$\lim z_n = \inf\{a_k \,:\, k \geq n\}. \tag{2.10}$$

Since $\inf\{a_k\}$ is a increasing sequence, and (a_n) is bounded, we know it converges.

(c) For any set A, $\inf A \leq \sup A$, so $\forall n, \inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$.

An example when the inequality is strict is

$$a_n = (-1)^n, \tag{2.11}$$

since $\liminf a_n = -1, \limsup a_n = 1$.

(d) ($\Rightarrow$) Suppose

$$\liminf a_n = \limsup a_n = L, \tag{2.12}$$

then given some $\epsilon > 0$, we know $\exists N : n \geq N$ so that, define $A_n = \{a_k : k \geq n\}$

$$|\inf A_n - L| < \epsilon$$
$$|\sup A_n - L| < \epsilon$$

since every element $k \geq n, \inf A_n \leq a_k \leq \sup A_n$, we conclude

$$k \geq n \geq N \quad |a_k - L| < \epsilon,$$

so $\lim a_n = L$.

($\Leftarrow$) Suppose

$$\lim a_n = L,$$

then given some $\epsilon > 0$, we know $\exists N : n \geq N$ so thats

$$|a_n - L| < \epsilon/2$$

This means every element after a_n lives in this $\epsilon/2$-neighborhood of L. Now, $\sup A_n$ must be arbitrarily close to the largest element of A_n, so we can make this distance $\epsilon/2$. That means

$$|\sup A_n - L| = \left|\max\{A_n\} + \epsilon/2 - L\right| < \epsilon,$$

which means $\limsup a_n = L$. This is similar for inf.

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Exercise 2.5.1
Suppose we have a convergent sequence with limit l. Then given any $\epsilon > 0$, we can always find $N : n \geq N$ such that $|a_n - l| < \epsilon$. For any subsequence of (a_n), (a'_m), any element of this subsequence, call it a'_k will be from some a_n in the original sequence, where $n \geq k$. So we can choose N from earlier, and for $m \geq N$ we will have $\left|a'_m - l\right| < \epsilon$.

Exercise 2.5.2

(a) Define

$$s_i = \sum_{j=1}^{i} a_j \tag{2.13}$$

$$b_i = \sum_{k=1}^{i} a_{n_k}, \tag{2.14}$$

where the series regrouping a_i is divided into groups of $n_1, n_2, \ldots$. Then b_i is a subsequence of s_n, which means they converge to the same limit, namely L in this case.

(b) Our proof does not apply to that example because that series did not converge in the first place.

Exercise 2.5.3

(a) Consider

$$a_n = \begin{cases} \sum_{i=1}^{n} \frac{1}{2^i}, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \tag{2.15}$$

Then we have that $b_n = a_{2n-1}$ converges to 1 and $c_n = a_{2n}$ converges to 0.

(b) A monotone sequence that diverges means that sequence is not bounded. Thus, every subsequence will also be unbounded and thus impossible to be convergent.

(c) Consider the sequence

$$\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \tag{2.16}$$

(d) Consider

$$a_n = \begin{cases} 2^i, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \tag{2.17}$$

(e) By Bolzano-Weierstrass, since we have a subsequence that is bounded, we know we can find a convergent subsequence within this subsequence that converges.

Exercise 2.5.4
AFSOC (a_n) converges to $b \neq a$. Then we have that $|a_n - b|$ can be arbitrarily small. But this implies that every subsequence will also converge to b, which is a contradiction.

AFSOC (a_n) does not converge. Then since (a_n) is bounded, we must have an infinite number of elements in two different ϵ-neighborhoods. But this would imply we have convergent subsequences to different limits, which contradicts the original problem statement.

Therefore, we conclude (a_n) converges to a.

Exercise 2.5.5
Consider $|b^n|$. Since $|b| < 1$, we have that $|b^n|$ is a decreasing sequence that is bounded below by 0, so we have

$$|b| > l \geq 0.$$

We notice that $\left|b^{2n}\right|$ is a subsequence that also converges to L, and since $\left|b^{2n}\right| = |b|^2$, by the Algebraic Limit Theorem, we have that $\left|b^{2n}\right| \to l^2 = l \implies l = 0$. Since $|b^n| \to 0$, we conclude $b^n \to 0$.

Exercise 2.5.6

We have $s = \sup S$, which means for any $\epsilon > 0$,

$$\exists x : s - \epsilon < x \in S < a'_n$$
$$\epsilon > \left|s - a'_n\right| = \left|a'_n - s\right|$$

where a'_n is an element of the infinite subsequence of $a_n : a_n > x \in S$.

2.6 The Cauchy Criterion

Exercise 2.6.1

(a) $a_n = 1 + \left(-\frac{1}{2}\right)^n$

(b) $a_n = n$

(c) Impossible, since a Cauchy sequence implies convergence, which means every subsequence will also converge.

(d) You can use Equation (2.17). Literally anything that diverges but has a convergent subsequence.

Exercise 2.6.2

If we have that $(x_n) \to x$, then we can make $|x_n - x|$ arbitrarily small. Consider

$$\begin{aligned}
|x_n - x_m| &= |x_n - x + x - x_m| \\
&\leq |x_n - x| + |x_m - x| && \text{(Triangle Inequality)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Exercise 2.6.3

(a) The *pseudo-Cauchy* definition is different because it only looks at consecutive terms

(b) Consider the harmonic series, where $|s_{n+1} - s_n| = \frac{1}{n(n+1)}$.

Exercise 2.6.4

$$\begin{aligned}
|c_{n+1} - c_n| &= \big||a_{n+1} - b_{n+1}| - |a_n - b_n|\big| \\
&\leq |a_{n+1} - a_n + b_{n+1} - b_n| \\
&\leq |a_{n+1} - a_n| + |b_{n+1} - b_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Exercise 2.6.5

(a) Let $a_n = x_n + y_n$, then

$$\begin{aligned}
|a_{n+1} - a_n| &= |x_{n+1} - x_n + y_{n+1} - y_n| \\
&\leq |x_{n+1} - x_n| + |y_{n+1} - y_n| \\
&< \epsilon && \text{(for proper choice of } N\text{)}
\end{aligned}$$

(b) Let $a_n = x_n y_n$, then

$$\begin{aligned}
|a_{n+1} - a_n| &= |x_{n+1}y_{n+1} - x_n y_n| \\
&= |x_{n+1}y_{n+1} - x_n y_{n+1} - x_n y_n + x_n y_{n+1}| \\
&\leq \big|y_{n+1}(x_{n+1} - x_n)\big| + \big|x_n(y_{n+1} - y_n)\big|
\end{aligned}$$

we have shown we can bound this before by

- $(x_n), (y_n)$ are convergent sequences, so they must be bounded. Call this bound M
- Now, $|x_{n+1} - x_n|$ can be made arbitrarily small. Given some ϵ, we can make it $< \frac{\epsilon}{2M}$.
- We do the same for $|y_{n+1} - y_n|$, and thus the overall bound is $< \epsilon$.

Exercise 2.6.6

I'm not going to write these down super rigorously, but will write down most of the ideas.

(a) Suppose we have some set of real numbers that is bounded above. We want to show that there exists a least upper bound, assuming the Nested Interval Property is true.

Let B be the set of upper bounds. Define $I_1 = B$, and for each subsequent I_i, define it as

$$I_{n+1} = \{b \in I_n : b < \ell_{n+1}\},$$

where $\ell_{n+1} \in I_n$ is arbitrarily chosen. The idea is that we are creating intervals that have a smaller and smaller maximum value.

Now we have 2 cases,

a $\exists n \in \mathbb{N} : I_n = \emptyset$. In this case, there must have been some ℓ_n where $\forall b \in I_n, b \neq \ell_n, \ell_n < b$. Since I_n is a subset of the smaller elements of B, we also have $\forall b \in B, b \neq \ell_n, \ell_n < b$, which means this ℓ_n is the least upper bound.

b None of the I_n are empty. Now, consider $\bigcap_{n=1}^{\infty} I_n$. By NIP, this intersection is nonempty. By our construction of the I_n, we know any element b in this intersection must be less than all the elements in the set preceding it. However, we reach a contradiction, because if b is in this infinite intersection, **TODO** this proof doesn't work... probably need to change the interval construction.

(b) Define i_n to be $\inf\big(\bigcup_{k=n}^{\infty} I_k\big)$, and s_n to be $\sup\big(\bigcup_{k=n}^{\infty} I_k\big)$. i_n is an increasing sequence, and s_n is a decreasing sequence. Since for any set, $\inf A \leq \sup A$, we know that $\forall n : i_n \leq s_n$. This means $\exists x : \lim i_n \leq x \leq \lim s_n$, which exists in every single interval I_k, so we know their infinite intersection is nonempty.

(c) We are using the BW Theorem to prove NIP. Let x_i be an arbitrary element of I_i. since I_1 is bounded, then all $I_k, k \geq 1$ are also bounded, so the sequence (x_n) is also bounded. By BW, we know that $\exists (s_n)$, a subsequence of (x_n) that converges. Suppose $\lim s_n = L$. We will show that L is in the infinite intersection of all the sets. For any I_k, the interval is (a_k, b_k). Call $\epsilon = |a_k - b_k|$, the size of the interval. Since (s_n) converges to L, we know $\exists N : n \geq N$,

$$|s_n - L| < \epsilon/2.$$

since $s_n \in I_n$ by definition, we see L also falls in this interval. Thus, we know $L \in I_n$ for $n \geq N$. Since I_n is also contained within all sets before as well, L also is contained in those sets. Therefore, we have shown that L is in every set, so the infinite intersection must contain at least L, so therefore it is nonempty.

(d) We are given a bounded sequence, and want to show that there exists a convergent subsequence.

We will construct a subsequence, and show that it is convergent.

Since we have a bounded sequence (x_n), we know $|x_n| \leq M$.

Let $I_1 = [-M, M]$. Now construct subsequent I_n from $I_{n-1} = [a, b]$ as

$$I_n = [a, (a+b)/2] \text{ or } [(a+b)/2, b],$$

depending on which half has an infinite number of elements.

Now, our subsequence is defined as $a_n : a_n \in I_n$. Given $\epsilon > 0$, since we can make the bound of I_n as arbitrarily small as possible, since we are halving the interval size every time, we know $\exists N : n \geq N \to |I_n| < \epsilon$. Now, since all future elements a_k will be chosen from this interval and its subsets, and the interval size is $< \epsilon$, we can conclude for $m, n \geq N$,

$$|a_n - a_m| < \epsilon.$$

This means (a_n) is a Cauchy Sequence, and by the Cauchy Criterion, we can conclude that (a_n) is convergent.

2.7 Properties of Infinite Series

Exercise 2.7.1

(a) For any $\epsilon > 0$, we know since $(a_n) \to 0$, $\exists N : n \geq N, |a_n| < \epsilon$. Now, let $n > m \geq N, n = m + 1$, $|s_n - s_m| = |a_n| < \epsilon$, which means (s_n) is a Cauchy sequence.

(b) Construct intervals I_k so that, initially, $I_1 = [-absa_1, |a_1|]$. For $n \geq 1$, if $I_n = [b_n, c_n]$,

$$I_{n+1} = (b_n, s_{n+1}) \text{ if } s_{n+1} > s_n$$
$$I_{n+1} = (s_{n+1}, c_n) \text{ if } s_{n+1} < s_n$$

Now, take any $L \in \bigcap_{k=1}^{\infty} I_k$, which we know exists by NIP since $I_{n+1} \subseteq I_n$. We can show s_n converges to L, since the size of any interval I_n is $|s_n - s_{n-1}| = |a_n|$, so for any $\epsilon > 0$, we can show that s_n past some N will be within an ϵ-neighborhood of L.

(c) (s_n) is bounded by $|a_1|$, so $(s_{2n}), (s_{2n+1})$ are both bounded. These sequences also happen to be monotonic, since one is increasing and the other is decreasing. Therefore, the two subsequences are convergent, and we can add them together to get another convergent sequence, which is (s_n).

Exercise 2.7.2

(a) The hints in the text are already a lot.

If the (b_n) series converges, then for any $\epsilon > 0$, we know $\exists N : n > m \geq N$, such that

$$\epsilon > \left| \sum_{i=m+1}^{n} b_i \right| > \left| \sum_{i=m+1}^{n} a_i \right|$$

so this N works for (a_n) series too.

If the (a_n) series diverges, then we can AFSOC (b_n) series converges, and use what we proved above to show by contradiction that (a_n) series converges.

(b) (a_n) series is increasing and bounded by (b_n) series, so it must converge. For (a_n) series diverging, We can do a similar AFSOC argument in part ((a)), where we can AFSOC (b_n) converges, which then we can show by contradiction that (a_n) series is converging.

Exercise 2.7.3

(a) If $\sum a_n$ diverges, then AFSOC $\sum p_n$ and $\sum q_n$ converge. Then $\sum p_n + \sum q_n = \sum a_n$ converges, but this is a contradiction.

(b) If $\sum a_n$ converges conditionally, WLOG AFSOC $\sum p_n$ converges. Then $\sum a_n - \sum p_n = \sum q_n$ must converge as well. $\left|\sum q_n\right|$ will also converge, since $\sum q_n$ does, and this equals $\sum |q_n|$. Then,

$$\sum |a_n| = \sum |p_n| + \sum |q_n|$$

which we know converges since $\sum |p_n| = \sum p_n$ and we just showed $\sum |q_n|$ converges. This is a contradiction since we assumed $\sum a_n$ converges conditionally.

Exercise 2.7.4

Define

$$x_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases} \qquad y_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Then $\sum x_n, \sum y_n$ both diverge, but $\sum x_n y_n = 0$.

Exercise 2.7.5

(a) If $\sum a_n$ converges absolutely, then $\sum |a_n|$ converges to some L, so

$$L^2 = \left(\sum |a_n|\right)^2 = \sum |a_n|^2 + S$$
$$L^2 \leq \sum |a_n|^2 = \sum a_n^2$$

Since $\sum_{n=1}^{k} a_n^2$ is an increasing sequence and is bounded, we conclude $\sum a_n^2 = \sum \left|a_n^2\right|$ converges.

This proposition does not hold without absolute convergence. Take $a_n = (-1)^n \frac{1}{\sqrt{n}}$, which converges by the alternating series test. Then $a_n^2 = \frac{1}{n}$, which is the harmonic series, and we know this does not converge.

(b) No, take $a_n = \frac{1}{n^2}$, which converges. Then $\sum \sqrt{a_n}$ is the harmonic series, which diverges.

Exercise 2.7.6

(a) Call M the bound of y_n. If $\sum x_n$ converges absolutely to L, then

$$\begin{aligned}\sum x_n y_n &\leq \sum |x_n y_n| \\ &\leq \sum |x_n||y_n| \\ &\leq \sum |x_n| M \\ &\leq LM\end{aligned}$$

$\sum |x_n y_n|$ converges by the Monotone Convergence Theorem because the partial sums are increasing and it is bounded above. Then, by the Absolute Convergence Test we can conclude $\sum x_n y_n$ also converges.

(b) Let $x_n = \frac{(-1)^n}{n}$ be the alternating harmonic series, and $y_n = (-1)^n$. Then $\sum x_n$ converges but $\sum x_n y_n$ is the harmonic series, which does not converge.

Exercise 2.7.7

We are going to bound our p-series with another series, and show that the other series converges.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \cdots \\ &\leq 1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{8^p} + \cdots \\ &= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots \\ &= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots \\ &= \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^p}{2^p - 2} && \text{(Only if } p > 1\text{)}\end{aligned}$$

By the Monotone Convergence Theorem, since the partial sums of the p-series is increasing, and there is an upper bound, we conclude that the p-series converges.

Notice that the convergence of p-series is often proved with calculus, but this is a nice alternative.

Exercise 2.7.8

Informally, you use the fact that both partial sums s_n^a, s_n^b will converge, then use the N_a, N_b and choose $N = \max(N_a, N_b)$, so that both partial sums are $\epsilon/2$ close to A, B. Then, triangle inequality to bound s_n^{a+b}, which will be $< 2 \cdot \epsilon/2 = \epsilon$.

Exercise 2.7.9

(a) This r' exists, since $a = \frac{r+1}{2}, r < a < 1$. We know

$$\begin{aligned} \left| \left| \frac{a_{n+1}}{a_n} \right| - r \right| &< \epsilon \\ \left| \frac{a_{n+1}}{a_n} \right| &< r + \epsilon && \text{(Also } > r - \epsilon\text{, but we don't need it)} \\ |a_{n+1}| &< |a_n|(r + \epsilon) \end{aligned}$$

We can choose N large enough so that $\epsilon < 1 - r$.

(b) Since $|r'| < 1$, we know $\sum r'^n$ converges, so $|a_N|$ times that also converges.

(c) We can bound the leading terms up to $n < N$ of $\sum |a_n|$ by some M. For the tail end, we can bound it from part ((b)). Since there exists an upper bound for this series, and its partial sums are increasing, we conclude that the partial sums converge and the overall sum does too.

Exercise 2.7.10

These are not proved very rigorously, but outline most of the ideas.

(a) If $\lim(na_n) = l$, then for any $\epsilon > 0, \exists N : n \geq N$ such that

$$\begin{aligned} |na_n - l| &< \epsilon \\ |a_n - l/n| &< \frac{\epsilon}{n} < \epsilon, \end{aligned}$$

so $\lim a_n = l/n$. Then since $l \neq 0$, $\sum a_n$ converges to a multiple of the harmonic series, which diverges, so $\sum a_n$ will too.

(b) Similar to the proof above in part ((a)), we can show $\lim a_n = l/n^2$. This converges to a multiple of $1/n^2$, which converges, so $\sum a_n$ also converges.

Exercise 2.7.11

An easy example,

$$\begin{aligned} (a_n) &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \\ (b_n) &= 2 - \frac{1}{2} + 2 - \frac{1}{4} + 2 - \frac{1}{6} + \cdots \end{aligned}$$

Then their $\sum \min\{a_n, b_n\}$ is the alternating harmonic series, and

- (a_n) diverges because it is the Harmonic series
- (b_n) diverges because every pair sums to > 1, so it sums an infinite number of numbers > 1.

For the challenge, an idea is to interweave a convergent and a divergent sequence together, so that (a_n) and (b_n) will both be divergent, since they both contain the divergent sequence, but the min only selects elements from the convergent sequence.

Exercise 2.7.12
Just verifying an identity,

$$\begin{aligned}
s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j(y_j - y_{j+1}) &= s_n y_{n+1} - s_m y_{m+1} + (x_1 + \cdots x_{m+1})(y_{m+1} - y_{m+2}) + \\
&\quad (x_1 + \cdots x_{m+2})(y_{m+2} - y_{m+3}) + \cdots + (x_1 + \cdots x_n)(y_n - y_{n+1}) \\
&= s_n y_{n+1} - s_m y_{m+1} + (x_1 + \cdots x_{m+1}) y_{m+1} - \\
&\quad (x_1 + \cdots x_n) y_{n+1} + \sum_{j=m+2}^{n} x_j y_j \\
&= s_n y_{n+1} - s_n y_{n+1} - s_m y_{m+1} + s_{m+1} y_{m+1} + \sum_{j=m+2}^{n} x_j y_j \\
&= x_{m+1} y_{m+1} + \sum_{j=m+2}^{n} x_j y_j \\
&= \sum_{j=m+1}^{n} x_j y_j
\end{aligned}$$

Exercise 2.7.13
(a) Using Exercise 2.7.12, we have

$$\begin{aligned}
\left| \sum_{j=m+1}^{n} x_j y_j \right| &= \left| s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \right| \\
&= |s_n y_{n+1} - s_m y_{m+1}| + \left| \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \right| && (\triangle \text{ inequality}) \\
&= \big|(s_n - s_m) y_{m+1}\big| + \left| \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \right| && (y_{m+1} > y_{n+1}) \\
&\leq M|y_{m+1}| + M|y_{m+1} - y_{n+1}| \\
&\leq 2M|y_{m+1}|
\end{aligned}$$

(b) For any $\epsilon > 0$, since (y_n) converges, make $|y_{m+1}| < \epsilon/(3M)$. Then

$$\begin{aligned}
\left|\sum_{j=m+1}^{\infty} x_j y_j\right| &\leq \left|\sum_{j=m+1}^{n} x_j y_j\right| + \left|\sum_{j=n+1}^{\infty} x_j y_j\right| \\
&\leq 2M|y_{m+1}| + \left|\sum_{j=n+1}^{\infty} x_j y_j\right| && \text{(From part ((a)))} \\
&\leq 2M|y_{m+1}| + \left|\sum_{j=n+1}^{\infty} x_j y_{m+1}\right| && \text{(Since for } n \geq m+1, y_{m+1} \geq y_n\text{)} \\
&\leq 2M|y_{m+1}| + |y_{m+1}|\left|\sum_{j=n+1}^{\infty} x_j\right| \\
&\leq 2M|y_{m+1}| + |y_{m+1}|M && \text{(Partial sums of } (x_n) \text{ bounded by } M\text{)} \\
&\leq 3M|y_{m+1}| \\
&\leq 3M\frac{\epsilon}{3M} = \epsilon
\end{aligned}$$

(c) We have $x_n = (-1)^{n+1}, y_n = a_n$.

Exercise 2.7.14

(a) Abel's test requires that $\sum x_n$ converges, which is stronger than the boundedness of the partial sums of (x_n). However, it only needs that (y_n) is non-negative and decreasing, which is weaker than Dirichlet, which in addition needs the limit to converge to 0.

(b) Using Exercise 2.7.13, part ((a)), we have

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leq 2A|b_1|$$

(c) We can define $a_n = x_{m+n}, b_n = y_{m+n}$, and bound

$$\left|\sum_{j=m+1}^{n} x_j y_j\right| = \left|\sum_{j=1}^{n} a_j b_j\right| \leq 2A|b_1|.$$

Now, we want to show we can make this bound arbitrarily small, since if we can make the tail end of this series arbitrarily small, then by the Cauchy Criterion for series we can conclude $\sum x_n y_n$ converges. We know that $\sum x_n$ converges, so by the Cauchy Criterion, for any $\epsilon > 0$, we can find some $N : \forall n' > m' \geq N$ such that

$$\left|\sum_{j=m'}^{n'} x_j\right| < \epsilon$$

Now, all we have to do is choose N so that $\left|\sum_{j=m'}^{n'} x_j\right| < \epsilon/(2b_1)$ then $A < \epsilon/(2b_1)$ and $\left|\sum_{j=m+1}^{n} x_j y_j\right| < 2|b_1| \cdot \frac{\epsilon}{2|b_1|}$, which means we have the Cauchy Criterion for $\sum x_n y_n$, and therefore it converges.

2.8 Double Summations and Products of Infinite Series

Exercise 2.8.1

$$\begin{aligned}\lim s_{nn} &= -1 + -\frac{1}{2} - \frac{1}{4} - \cdots \\ &= -\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i \\ &= -2.\end{aligned}$$

The value is equal to summing column-wise.

Exercise 2.8.2
By the Absolute Convergence test, since we know for fixed i that $\sum_{j=1}^{\infty}\left|a_{ij}\right|$ converges, then we know for fixed i that each $\sum_{j=1}^{\infty} a_{ij}$ converges to some c_i as well.

Then, since

$$\begin{aligned}\sum_{j=1}^{\infty}\left|a_{ij}\right| &\geq \left|\sum_{j=1}^{\infty} a_{ij}\right| \\ \Rightarrow b_i &\geq |c_i| \\ |b_i| &\geq |c_i|,\end{aligned}$$

and we know that $\sum_{i=1}^{\infty} b_i$ converges, we conclude that $\sum_{i=1}^{\infty} c_i$ must converge as well by the Absolute Convergence test, implying that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \tag{2.18}$$

converges as well.

Exercise 2.8.3

(a) Since $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right|$ converges, we have that

$$t_{mn} \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right| = L$$

Since t_{nn} is an increasing sequence, and is bounded above, by the Monotone Convergence Theorem, t_{nn} converges.

(b) For any $\epsilon > 0$, $\exists N : n > m \geq N$ such that $|t_{nn} - t_{mm}| < \epsilon$. Now, consider

$$\begin{aligned}\left|s_{n+1,n+1} - s_{nn}\right| &\leq \left|t_{n+1,n+1} - t_{nn}\right| \\ &< \epsilon.\end{aligned}$$

So (s_{nn}) is a Cauchy Sequence and converges.

Exercise 2.8.4

(a) Since we know there exists a $t_{m_0 n_0}$ such that $t_{n_0 n_0} > B - \frac{\epsilon}{2}$, and t_{nn} is increasing and that B is an upper bound, we can conclude that for $N_1 = \max\{m_0, n_0\} : m, n \geq N_1$,

$$B - \frac{\epsilon}{2} < t_{mn} \leq B. \tag{2.19}$$

(b) For any $\epsilon > 0$, since (t_{mn}) is bounded above by $A = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$, from part ((a)) we can choose $N : m, n \geq N$ such that

$$\begin{aligned} A + \frac{\epsilon}{2} &< t_{mn} < A + \epsilon \\ \Rightarrow \frac{\epsilon}{2} &< |t_{mn} - A| < \epsilon \\ &\left|t_{mn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|\right| < \epsilon. \end{aligned}$$

Then, we can see that this N also works to show

$$\begin{aligned} |s_{mn} - S| &= \left|s_{mn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\right| \\ &= \left|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} a_{ij}\right| \\ &< \left|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} |a_{ij}|\right| \\ &= \left|t_{mn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|\right| \\ &< \epsilon. \end{aligned}$$

Exercise 2.8.5

We know $\lim_{n\to\infty}\sum_{j=1}^{n} a_{ij} = r_i$, so for any $\epsilon > 0$, $\exists N : n \geq N$ such that

$$\left|\sum_{j=1}^{n} a_{ij} - r_i\right| < \frac{\epsilon}{m}.$$

if we fix $m \geq N$.

Then

$$\begin{aligned} \left|(r_1 + r_2 + \cdots + r_m) - S\right| &= \left|\sum_{i=1}^{m}\left(r_i - \sum_{j=1}^{n} a_{ij}\right)\right| \\ &\leq \sum_{i=1}^{m}\left|r_i - \sum_{j=1}^{n} a_{ij}\right| \\ &\leq m \cdot \frac{\epsilon}{m} = \epsilon \end{aligned}$$

Therefore, we conclude that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S.

TODO not sure where I have to use the Order Limit Theorem...

Exercise 2.8.6

For $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$, the proof is essentially the same as Exercise 2.8.5 to show it converges to S, except we fix n this time instead of m.

Exercise 2.8.7

(a) Define $t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|$. Also define $u_n = \sum_{k=2}^{n}|d_k|$ Then we know for $n \geq 2$,

$$u_n \leq t_{nn} = L \qquad (t_{nn} \text{ converges})$$

Since u_n is an increasing sequence and is bounded above, we conclude from the Monotone Convergence Theorem that $u_n = \sum_{k=2}^{n} |d_k|$ also converges. Then, $\sum_{k=2}^{n} d_k$ converges absolutely.

(b) We need to bound $|d_k - S|$ somehow. Consider the following diagram

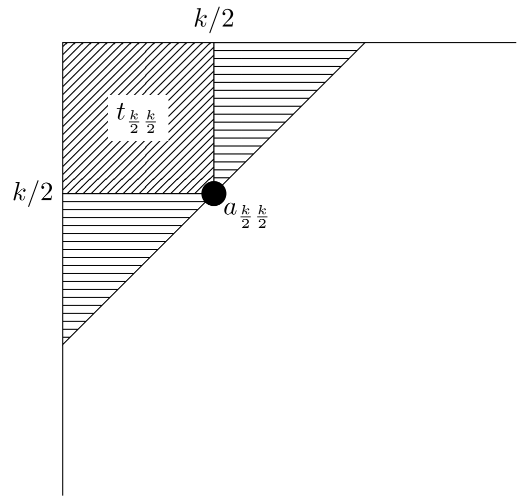

Figure 2.1: Demonstrating how we bound our sum

We know that for any $\epsilon > 0$, we can choose $N : n \geq N$ so that

$$|t_{nn} - S| < \epsilon \tag{2.20}$$

Now, choose $N_1 = 2N$. We can use Figure 2.1 to see that for $k \geq N_1$,

$$\begin{aligned} |d_{kk} - S| &\leq \left| t_{\frac{k}{2}\frac{k}{2}} - S \right| \\ &< \epsilon && \text{(From Equation (2.20))} \end{aligned}$$

Therefore, we can conclude $\sum_{k=2}^{\infty} d_k$ converges to S.

Exercise 2.8.8

(a) See that

$$\begin{aligned} AB &\geq \left(\sum_{i=1}^{\infty} |a_i| \right) \left(\sum_{j=1}^{\infty} |b_j| \right) \\ &= \sum_{i=1}^{\infty} \left(|a_i| \sum_{j=1}^{\infty} |b_j| \right) && (\textstyle\sum_{j=1}^{\infty} |b_j| \text{ is constant wrt } i) \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_i||b_j| \right) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j| \end{aligned}$$

Since $\sum_{i=1}^{m} \sum_{j=1}^{n} |a_i b_j|$ is bounded, and the partial sums $s'_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_i b_j|$ are increasing, we can conclude $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges by the Monotone Convergence Theorem.

(b) Let s_n^a, s_n^b be the partial sums of $(a_n), (b_n)$ respectively. Then,

$$\begin{aligned}
\lim_{n\to\infty} s_{nn} &= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a_i \sum_{j=1}^{n} b_j \right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{j=1}^{n} b_j \right) \\
&= \lim_{n\to\infty} s_n^a s_n^b \\
&= AB
\end{aligned}$$

Therefore, by Theorem 2.8.1, we can conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_i b_j = \sum_{i=2}^{\infty} d_k = AB$$

Chapter 3

Basic Topology of $\mathbb{R}$

3.2 Open and Closed Sets

Exercise 3.2.1

(a) We need a finite number of sets when we are choosing the minimum ϵ for our V_ϵ. If we had an infinite number of sets, this minimum may not exist.

(b) Let

$$O_n = \left(\sum_{i=1}^{n} \frac{1}{2^i}, 3 - \sum_{i=1}^{n} \frac{1}{2^i} \right)$$

Then $\bigcap_{n=1}^{\infty} O_n = [1, 2]$.

Exercise 3.2.2

(a) 1 and -1 are the only limit points of B. For any fixed element of B, the distance between it and its neighbors is $\geq \frac{n}{n+1} - \frac{n+2}{n+3} = \frac{n}{(n+1)(n+3)}$, so we can just choose ϵ smaller than this, and show that any element of B is isolated.

For 1, we can show that for any $\epsilon > 0$, we can choose $\frac{1}{N+1} < \epsilon \Rightarrow N > \frac{1}{\epsilon} - 1$, and we know $\frac{n}{n+1}$ for $n \geq N$ for even n is in the ϵ-neighborhood of 1. Doing a similar analysis for negative terms and -1 yields the same result.

(b) B does not contain its limit points, so it is not closed.

(c) B is not an open set, continuous ϵ neighborhoods are not subsets of B.

(d) All of B's elements are isolated

(e) $\overline{B} = B \cup \{-1, 1\}$.

Exercise 3.2.3

(a) $\mathbb{Q}$ is not open, because it doesn't have irrationals that can be in the ϵ-neighborhoods. It is not closed, because it contains irrational limit points. Therefore it is **neither**.

(b) $\mathbb{N}$ does not have any limit points, so it is **closed**.

(c) $\mathbb{R}^+$ cannot be closed, because 0 is a limit point and not contained. It is **open** because every element has an ϵ-neighborhood that is a subset.

(d) Not closed, doesn't contain 0, a limit point. Not open, since 1 has no ϵ-neighborhood subset. Therefore, **neither**.

(e) The sequence converges, but this limit point is not in the set. No ϵ-neighborhoods exist for certain ϵ, for certain elements, so not open. **Neither**.

Exercise 3.2.4
FOr any $\epsilon > 0$, we know $\exists N : n \geq N$ such that

$$|a_n - x| < \epsilon,$$

so every ϵ-neighborhood of x has points other than itself.

Exercise 3.2.5
If there exists exists such an ϵ-neighborhood, then by the definition of a limit point, since there are no other elements other than x itself, then this is not a limit point, so it is isolated.

Exercise 3.2.6
If a set $F \subseteq \mathbb{R}$ is closed, then it contains all its limit points. For any Cauchy Sequence in F, it is also convergent to some L, which we know is a limit point and thus must be in F.

If every Cauchy Sequence of an F has its limit as an element of F, then every limit point, which comes from the limit of some subsequence, which we know is a Cauchy Sequence. From our original assumption, this limit must be in F, so F contains all its limit points and is closed.

Exercise 3.2.7
AFSOC an infinite number of (x_n) terms not in O. Then since $(x_n) \to x$, $\epsilon =$ distance of x from O boundary, then $\forall N, : n \geq N$ we have that $\exists x_n : |x_n - x| \geq \epsilon$, since we can choose some x_n not in O. This means this sequence does not converge. This contradicts our original assumption.

Exercise 3.2.8

(a) We want to show that L, which contains all the limit points of of A, is closed. We can do this by showing all limit points of L are in L.

Suppose we have some limit point ℓ of L, then this means some subsequence of L,

$$(l_n) \to \ell$$

By the definition of convergence, for any $\epsilon > 0$, we can find $N : n \geq N$ such that

$$|l_n - \ell| < \frac{\epsilon}{2}$$

Now, since l_n are limit points of A, we know $\exists a \in A$ such that a is arbitrarily close to l_n. Define a subsequence in A

$$\left\{ a_n \in A, |a_n - l_n| < \frac{\epsilon}{2} \right\}$$

Then for $n \geq N$,

$$|a_n - \ell| < |l_n - \ell| + \frac{\epsilon}{2} < 2 \cdot \frac{\epsilon}{2} = \epsilon,$$

which means (a_n) converges to this ℓ as well, so ℓ is a limit point of A and $\ell \in L$.

Therefore, we conclude L contains all of its limit points, and therefore it is closed.

(b) For any limit point ℓ of $A \cup L$, it must the limit of some convergent subsequence of $A \cup L$. This subsequence will contain elements from A and L. What we can do is for every element in L, use a similar technique we did in part ((a)) to replace all the $x \in L$ subsequence elements with elements in A instead, that are arbitrarily close enough. Then, we have constructed a subsequence that entirely lies in A, so this limit point must be of A. Therefore, all limit points of $A \cup L$ are limit points of A.

We can then conclude that $\overline{A} = A \cup L$ is a closed set, since all of its limit points are of A, and those limit points are contained in L, which means $\overline{A}$ contains all of its limit points and is closed.

Exercise 3.2.9

(a) Suppose y is a limit point of $A \cup B$, then there must exist a subsequence $(x_n), x_n \in A \cup B$ where $(x_n) \to y$. Now, this subsequence must contain either an infinite number of elements from A or B (or both).

WLOG, (x_n) contains an infinite number of elements from A, then we know $\exists$ subsequence $(x'_n) \to y$ where $x'_n \in A$. This means y is a limit point of A.

Therefore, we conclude if y is a limit point of $A \cup B$, y is either a limit point of A or B.

(b) Let L_S be the set of limit points for a set S.

$$\overline{A \cup B} = A \cup B \cup L_{A\cup B} \tag{3.1}$$
$$= A \cup B \cup (L_A \cup L_B) \qquad = (A \cup L_A) \cup (B \cup L_B) \tag{3.2}$$
$$= \overline{A} \cup \overline{B} \tag{3.3}$$

(c) We notice that in our proof, we were able to find a subsequence that was entirely in one set. Therefore, if it is possible to construct a subsequence that doesn't fit entirely in one set, then we can find a limit point that is not necessarily a limit point of an individual set.

With an infinite number of sets, we can take advantage of this property.

Suppose we have some $(a_n) \to L$, where $\forall n\, a_n \neq L$. Then construct sets the following way,

$$S_n = \{a_n\}$$

Now, S_n has no limit points, since it only has a single point which is isolated. Therefore, $\bigcup_{n=1}^{\infty} \overline{S_n} = \bigcup_{n=1}^{\infty} S_n$, but we have $\overline{\bigcup_{n=1}^{\infty} S_n} = \left(\bigcup_{n=1}^{\infty} S_n\right) \cup \{L\}$, and $L \notin \bigcup_{n=1}^{\infty} S_n$. So the property in part ((b)) does not apply for infinite sets.

Exercise 3.2.10

(a) A direct proof (double containment is another way to do it)

$$\begin{aligned} x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c &\Leftrightarrow \forall\lambda, x \notin E_\lambda \\ &\Leftrightarrow \forall\lambda, x \in E_\lambda^c \\ &\Leftrightarrow x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c \end{aligned}$$

$$\begin{aligned} x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c &\Leftrightarrow \exists\lambda, x \notin E_\lambda \\ &\Leftrightarrow \exists\lambda, x \in E_\lambda^c \\ &\Leftrightarrow x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c \end{aligned}$$

(b) We want to show that

(i) *The union of a finite collection of closed sets is closed.* Suppose we have a collection of closed sets $\{E_\lambda, \lambda \in \Lambda\}$, then, if we take the complement of the union of all these sets, by DeMorgan's, we get the intersection of the complements of all these sets. The complements of all these sets is open, and we know the intersection of a finite number of open sets is also open. Finally, taking the complement again, we must have a closed set, which is equal to our original union.

(ii) *The intersection of an arbitrary collection of closed sets is closed.* Take the intersection of these closed sets, and then take the complement. By DeMorgan's, we know have the union of the complement of these sets, which we know is open. We know that the union of an arbitrary number of open sets is also open. Finally, taking the complement of this entire expression again, we now have a closed set, which is equal to our original intersection.

Exercise 3.2.11

If $s = \sup A$ exists, we have 2 cases. Either $s \in A \Rightarrow s \in \overline{A}$ since $A \subseteq \overline{A}$, or, $s \notin A$. In the second case, since we know for any $\epsilon > 0$, $\exists a \in A$ such that $a > s - \epsilon \Rightarrow \epsilon > |s - a|$, we can construct a subsequence in A that converges to s. This means s is a limit point of A, and therefore $s \in \overline{A}$.

Exercise 3.2.12

(a) True. $\overline{A}$ is closed, so $\overline{A}^c$ must be open.

(b) True. There is no ϵ-neighborhood around this point that is contained in A.

(c) False. Take the harmonic sequence $\{1/n\}$.

(d) True. See Exercise 3.2.11

(e) True. A finite set only contains isolated points, so therefore it has no limit points, and vacuously contains all of its limit points and is closed.

(f) True. We know that around $q \in \mathbb{Q}$, exists some ϵ-neighborhood around it that is contained in the set. Suppose we have an arbitrary $r \in \mathbb{R}$, then we want to show it is contained in the ϵ-neighborhood of some $q \in \mathbb{Q}$. We can show this by contradiction. AFSOC r is not in any of these ϵ-neighborhoods. That means $\forall \epsilon > 0, |r - q| > \epsilon$, for all q. But for any ϵ, we can always find some rational number that is closer than ϵ to r, which means this statement is false. We have reached a contradiction, and must assume our original hypothesis was true.

Exercise 3.2.13

We can verify $\mathbb{R}$ is open because any ϵ-neighborhood only contains elements of $\mathbb{R}$, so therefore $\subseteq \mathbb{R}$. In addition, any limit point $\in \mathbb{R}$, so $\mathbb{R}$ also contains all of its limit points and is closed.

$\emptyset$ is closed and open by vacuity.

Now, we need to show that there are no other sets with this property. We know the complement of an open set is closed and vice versa, so AFSOC $\exists A \neq \mathbb{R}, \emptyset$, then we know $\exists x \in A^c, \notin A$. Now, x cannot be an isolated point, since then it would not have an ϵ-neighborhood around it that is contained in A^c. Therefore, we conclude x must be in some continuous set S, where it either

(i) Has a $\sup S$. In this case, either $\sup S \in S$, in which case there does not exist an ϵ-neighborhood around $\sup S$, which means S is not open, or $\sup S \notin S$, and then $\sup S$ is a limit point, but then S is not closed since it doesn't contain all of its limit points. This case is not possible.

(ii) Does not have an upper bound. Then look at the portion less than x and apply the argument in part ((i)) but with $\inf S$. It must have a lower bound, or else $A = \mathbb{R}$.

Therefore, we reach a contradiction in all cases, and therefore we conclude that it is not possible for this A to exist.

Exercise 3.2.14

(a) $[a, b] = \bigcap_{i=1}^{n} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$. Any $x \in [a, b]$ will be $< b + \frac{1}{n}$, and $> a - \frac{1}{n}$. Now, let us consider some $y < a$. $y \notin$ the set we created, bc suppose $|y - a| = \epsilon$. Then for $n' > \frac{1}{\epsilon}$, $a - \frac{1}{n'} > y$, so y is not in this set. The argument for an element larger than b is symmetric. Therefore, we conclude the set we constructed is equivalent to $\{a, b\}$, and is an intersection of a countable number of open sets.

(b) We can write

$$(a, b] = \bigcup_{i=1}^{n} \left[a + \frac{1}{n}, b\right] \tag{3.4}$$

$$(a, b] = \bigcap_{i=1}^{n} \left(a, b + \frac{1}{n}\right) \tag{3.5}$$

(c) We know $\mathbb{Q}$ is countable, so just union all the sets containing only one element of $\mathbb{Q}$ together. Since each set has one element which is an isolated point, each set is closed.

$$\bigcup_{q \in \mathbb{Q}} \{q\}$$

We know that $\mathbb{Q}^c = \mathbb{I}$, and by DeMorgan's law, we know

$$\left(\bigcup_{i=1}^{\infty} S\right)^c = \bigcap_{i=1}^{\infty} S^c$$

Since S are all closed, S^c are all open. We can use the infinitely countable union of the construction of $\mathbb{Q}$ and then take the complement to get $\mathbb{I}$, which by DeMorgan's is constructed as a countably infinite intersection of open sets.

3.3 Compact Sets

Exercise 3.3.1
Since we know K is compact, it must also be closed and bounded.

Since K is bounded, it must have an upper and lower bound, which means $\sup K$ and $\inf K$ must exist.

Exercise 3.3.2
Suppose we have some $K \subseteq \mathbb{R}$ that is closed and bounded. We want to show that it is compact.

We know that any sequence of K must be contained in K, which is bounded, so therefore by the Bolzano-Weierstrauss Theorem, we know that this sequence must have a convergent subsequence. Since K is also closed, this limit must be in K as well.

This shows that K is compact.

Exercise 3.3.3
We want to show the Cantor set is compact.

We know the Cantor set is $\subseteq [0,1]$, so it is bounded.

Then, we know the complement of the Cantor set is

$$(-\infty,0)\cup(1,\infty)\cup\left[\left(\frac{1}{3},\frac{2}{3}\right)\cup\left(\frac{1}{9},\frac{2}{9}\right)\cup\left(\frac{1}{9},\frac{2}{9}\right)\cup\cdots\right]$$

which is the union of an arbitrary number of open sets, which we know is open. Therefore, the Cantor set is the complement of an open set, which is closed.

Therefore, since the Cantor set is bounded and closed, we conclude it is compact.

Exercise 3.3.4
We have that K is compact and F is closed. Since K is compact, it is also bounded and closed.

If we take $K \cap F$, we know that this must also be bounded, since $x \in K \cap F \Rightarrow x \in K$.

The intersection of two closed sets is also closed, so $K \cap F$ is closed.

Therefore, $K \cap F$ is bounded and closed, and thus is compact.

Exercise 3.3.5

(a) We can find a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$, but we know that $\sqrt{2} \notin \mathbb{Q}$, so $\mathbb{Q}$ is not compact.

(b) Again, similar to part ((a)), we can find a sequence that converges to $\sqrt{2}/2 \notin [0,1]\cap\mathbb{Q}$.

(c) Take the sequence $a_n = n$. There is no limit, so this sequence does not have a subsequence that converges.

(d) $\mathbb{R}\cap[0,1] = [0,1]$ is closed and bounded, so it is compact.

(e) This sequence converges to 0, but does not contain 0, so it is not closed and thus not compact.

(f) Every subsequence of this set converges to 1, which is in the set, so therefore this set is compact.

Exercise 3.3.6

(a) We will prove this by induction.

- **Base Case:** $n = 1$, $C_1 = \left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$ We know a combination of two elements in $\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$ covers $\left[0,\frac{4}{3}\right]$. Then, combination of two elements in the latter set covers $\left[\frac{4}{3},2\right]$. Therefore, two elements $x, y \in C_1$ can add up to any element $\in [0,2]$.
- **Inductive Hypothesis:** Suppose for $k \geq 1$, any two elements of C_k can add up to any element $\in [0,2]$.
- **Inductive Step** We know $C_{k+1} = C_k/3 + \left\{\frac{2}{3} + C_k/3\right\}$, in other words, we are now missing the middle thirds of both the head and tail sets. What we want to show is that for the head set (and the same argument holds for the tail set), that with the middle third removed, we can still cover all of the original set, which means since the original sets are still covered, we can still cover $[0,2]$. By the IH, any two elements of $C_k/3$ will cover $\left[0,\frac{2}{3}\right]$, and any two elements of $\left\{\frac{2}{3}+C_k/3\right\}$ covers $\left[\frac{4}{3},2\right]$. Choosing two elements from one of each set covers $\left[\frac{2}{3},\frac{4}{3}\right]$. These three intervals cover $[0,2]$.

(b) The reason (x_n), (y_n) may not converge is they can be picked out of sets, jumping across different subsets of C infinitely many times.

However, by the Bolzano-Weierstrauss Theorem, since (x_n) is contained entirely in the Cantor set, which is bounded, then (x_n) is also bounded. Therefore, it must contain a convergent subsequence. The same applies for (y_n), and we can take their limits l_x, l_y such that $l_x + l_y = s$.

Exercise 3.3.7

(a) True. The intersection will be bounded, since we can take any bound of a set, which will bound the intersection, and the intersection of an arbitrary number of closed sets is still closed. Therefore, this arbitrary intersection of compact sets is closed and bounded, which means it is also compact.

(b) False. Let $A = (0, 1), K = [0, 1]$, then $A \cap K = (0, 1)$, which is not closed so it is not compact.

(c) True. This is the Nested Interval Property.

(d) True. A finite set always closed, since there are no limit points, and bounded.

(e) False. Choose an unbounded countable set like $a_n = n$.

Exercise 3.3.8

(a) If they both have finite subcovers, then we can union those two finite subcovers to get a finite subcover for K, so we need at least one of them to not have a finite subcover.

(b) Create I_{n+1} by bisecting I_n, and taking a half that has no finite subcover. Such a half has to exist, because if both have finite subcovers, then the whole must have a finite subcover. The interval will half in size every iteration.

(c) By the Nested Interval Property, $\exists x \in K \forall x \in I_n$.

(d) Since the interval sizes get arbitrarily small, we can find some n_0 such that I_{n_0} fits entirely in O_{λ_0}. We have reached a contradiction, because we can finitely cover $I_n \cap K$ by taking

$$\left(\bigcup_{i=1}^{n_0-1} I_i\right) \cup O_{\lambda_0}$$

Exercise 3.3.9

(a) Open cover where you take some $\epsilon > 0$ around all rational numbers, and then union them together. No finite subcover exists, since that would bound $\mathbb{Q}$, which is not bounded.

(b) We can construct an open cover like

$$\left(\bigcup_{i=1}^{\infty}(-0.5, \frac{\sqrt{2}}{2} - \frac{1}{i})\right) \cup \left(\bigcup_{i=1}^{\infty}(\frac{\sqrt{2}}{2} + \frac{1}{i}, 1.1)\right)$$

This will contain all the rational numbers between $[0, 1]$, but needs an infinite number of sets to cover the entire set, because if we stop prematurely, we won't be able to capture the rationals that are very close to $\frac{\sqrt{2}}{2}$.

(e) An open cover for this set is

$$\bigcup_{i=1}^{\infty}\left(1.1, 1 - \frac{i-1}{i}\right)$$

We need all of these open sets, because otherwise if for some N we stop, then we won't have the elements $< \frac{1}{N}$.

Exercise 3.3.10

For any closed set with an interval $[a, b]$, we can make intervals $I_n = \left[a, b - 1/n\right]$, and then union with $[b, b + 1]$ to cover the set; with open intervals, we don't need the end. However, we will need all of the intervals, otherwise we won't include all the elements close to b. Therefore, any *clompact* subset must be a finite set of isolated points.

3.4 Perfect Sets and Connected Sets

Exercise 3.4.1

A perfect and a compact set are always closed, so their intersection is also closed. Since a compact set is bounded, the intersection of it and another set must also be bounded. Therefore, $P \cap K$ is a compact set.

We cannot guarantee that $P \cap K$ does not have isolated points, so for example, $[0,1] \cap \{1/2\} = \{1/2\}$ which is not perfect.

Exercise 3.4.2

A perfect set cannot only consist of rationals, because it would be nonempty and countable, since $\mathbb{Q}$ is countable, but this is impossible since any nonempty perfect set is uncountable.

Exercise 3.4.3

(a) $x_1 \in C_1$ implies that x_1 is in a closed interval of size $1/3$, so we can choose any element in this interval such that $x \neq x_1$, and we must have $|x - x_1| \leq 1/3$

(b) We can make the argument for any $x_n \in C_n$, since x_n must exist in a closed interval of size $\frac{1}{3}^n$, so we can any other element in this interval x, so that $|x - x_n| \leq \frac{1}{3}^n$. Now, we can construct a subsequence (x_n) such that $(x_n) \to x$, and this shows that there are no isolated points in C. We know the Cantor set is closed from earlier exercises.

Exercise 3.4.4

(a) This set is bounded, and is also closed since it is an intersection of an arbitrary collection of closed sets. Therefore, this construction is compact.

This set is also perfect, because we can use the same argument from Exercise (b) to show that there are no isolated points.

(b) We can compute the

- **Length**: We will compute the removed interval lengths,

$$\frac{1}{4} + 2 \cdot \frac{3}{32} + 4 \cdot \frac{27}{256} + \cdots = \frac{1/4}{1 - \frac{3}{2^2}} = \boxed{1}$$

  So this Cantor-like set has length 0.

- **Dimension**: We have $3 - 3 \cdot \frac{1}{4} = \frac{9}{4}$, so solving

$$3^x = \frac{9}{4} \Rightarrow \boxed{0.738}$$

  This is "larger" in dimension than the ternary Cantor set.

Exercise 3.4.5

If we have that $A \subseteq U, B \subseteq V$ such that U, V are disjoint open sets, then we know $A \cap B = \emptyset$. Therefore, if we want to show that they are separated, we just need to show that the limit points of A are disjoint from B, and vice versa.

AFSOC that a limit point of $A, \ell_A \in B$. Then this ℓ_A is also a limit point of U, since $A \subseteq U$. Now, this $\ell_A \in V$, since $B \subseteq V$. This means ℓ_A is ϵ far away from an element $\in U$, and since ℓ_A is also in the open set V, we must have that $\ell_A \in [v_1, v_2] \subseteq V$, where $v_1 < \ell_A < v_2$. Let $\epsilon = (\ell_A - v_1)/2$, then $\ell_A - \epsilon \in V$, and since ℓ_A is a limit point of U, we must also have that $\ell_A - \epsilon \in U$. However, this is a contradiction, because we just showed that $\ell_A - \epsilon \in U$ and $\in V$, which means $U \cap V \neq \emptyset$, and they are not disjoint. The same argument applies for the limit points of B not being in A. Therefore, we can conclude that $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, and that A, B are separated.

Exercise 3.4.6

($\Rightarrow$) Suppose $E \subseteq \mathbb{R}$ is connected. Then consider some sets A, B such that $A \cup B = E$, and A, B are nonempty and disjoint. AFSOC every convergent sequence $(x_n) \to x$ with (x_n) contained in A or B, $x \notin$ the other set. Then this must mean that every limit point of A or B is not in the other set, which means $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, and A, B are separated. This means $E = A \cup B$ for separated A, B, which is a contradiction since we assumed E was connected.

($\Leftarrow$) Suppose for all nonempty and disjoint A, B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A, or B, and x is an element of the other. Suppose (x_n) is contained within A. Then x must be a limit point of A, since $x \in B$, and A is disjoint of B so $x \notin A$. This means $\overline{A} \cap B = x \cup S \neq \emptyset$, which means A, B are not separated. Therefore, we cannot find separated sets A, B such that $E = A \cup B$, which means E is not disconnected, and therefore is connected.

Exercise 3.4.7

(a) Take $E = (-\infty, 1) \cup (1, \infty)$, then the closure is $\mathbb{R}$, which is closed, but this set is disconnected because $\big((-\infty, 1) \cup \{1\}\big) \cap (1, \infty) = \emptyset$ and $(-\infty, 1) \cap \big((1, \infty) \cup \{1\}\big) = \emptyset$ so these two sets are disconnected, and their union is equal to E.

(b) If A is connected, we can show that any limit points must already be in A, so $A = \overline{A}$ and $\overline{A}$ is still connected. If A is perfect, it is already closed, so it contains all of its limit points, and $\overline{A} = A$, and therefore $\overline{A}$ is still perfect.

Exercise 3.4.8

(a) Given any two rational x, y, WLOG $x < y$. Then $\exists r \in (x, y)$ such that $r \in \mathbb{I}$, i.e. it is not rational. Then we have $\mathbb{Q} = \big(\mathbb{Q} \cap (-\infty, r)\big) \cup \big(\mathbb{Q} \cap (r, \infty)\big)$, and these two sets are disconnected.

(b) Irrational numbers are also totally disconnected using the same argument in part ((a))

Exercise 3.4.9

(a) We know in C_n, it consists of intervals of size $\frac{1}{3}^n$. If the intervals are smaller than ϵ, i.e. $\frac{1}{3}^n < \epsilon$, then x, y must be in different intervals.

(b) If we know x, y are in different intervals, then between their intervals there must exist removed intervals in the construction of C. We can take z to be in one of these removed intervals. Given any $(a, b), a < b$, we have a few cases. If a or b is not in C, then (a, b) is not in C. If $a, b \in C$, then we can use the argument we just made to find some $z \in (a, b)$, such that $z \notin C$, which means $(a, b) \not\subseteq C$.

(c) For any $x, y \in C$, WLOG $x < y$, we can find $z \notin C, x < z < y$ such that $C = \big(C \cap (-\infty, z)\big) \cup \big(C \cap (z, \infty)\big)$. Therefore, C is totally disconnected.

Exercise 3.4.10

(a) O contains all the rational numbers (and some irrational numbers), so the complement O^c must only consist of irrational numbers.

(b) F only consists of closed intervals. F is totally disconnected, because for any $x, y \in \mathbb{I}, x < y$, we can find a rational number q where $q = r_n, \epsilon_n = 1/2^n < |x - y|/2$, so it fits in between x, y. Then we can make F with the union of two open sets intersected with F at that boundary.

(c) We know F is closed, since we are taking an arbitrary intersection of closed sets. F is not always perfect, since we can create isolated points, for example, by having open sets have end points that converge to some irrational number. The issue with our construction was that we would allow the ϵ neighborhoods to get arbitrarily close to irrational numbers, and sort of "squeeze" them into isolated points.

One trivial way to prevent this is to have some sort of minimum neighborhood size, but then $F = \emptyset$.

A less trivial, but vague way of construction, is to just get rid of the isolated irrational points. There can only be a countably infinite number of these, and there are uncountably many irrationals, so in this case $F \neq \emptyset$.

TODO Find a better construction for a perfect set of irrationals.

3.5 Baire's Theorem

Appendix A

Extras

A.1 Useful Tools

Collection of useful tools and methods to solve problems.

Tip A.1.1
Template for a proof that $(x_n) \to x$:

- Let $\epsilon > 0$ be arbitrary
- Demonstrate a choice for $N \in \mathbb{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N works.
- Assume $n \geq N$
- With N well chosen, you should be able to show $|x_n - x| < \epsilon$.

A.2 Cool Things

- In Chapter 2, we learn that addition in infinite sums is not commutative.
- In Chapter 2, we learn that if $\sum_{n=1}^{\infty} a_n$ converges conditionally, then for any $r \in \mathbb{R}$, there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to r.
- In Chapter 3, we learn that $\mathbb{R}$ and $\emptyset$ are both open and closed, but they are the only subsets in $\mathbb{R}$ with this property.

A.3 Important Theorems

A.3.1 5 Characterizations of Completeness

Theorem 1 (Axiom of Completeness) *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Theorem 2 (Nested Interval Property) *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \cdots$$

has a nonempty intersection, that is $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem 3 (Monotone Convergence) *If a sequence is monotonic and bounded, then it converges.*

Theorem 4 (Bolzano-Weierstrass) *Every bounded sequence contains a convergent subsequence.*

Theorem 5 (Cauchy Criterion) *A sequence converges if and only if it is a Cauchy Sequence.*
A sequence (a_n) *is called a Cauchy sequence if, for every* $\epsilon > 0$, $\exists N \in \mathbb{N}$ *such that whenever* $m, n \geq N$, *it follows that* $|a_n - a_m| < \epsilon$.

A.4 Identities

Identity A.4.1
(Geometric Series)

$$\sum_{k=0}^{m} ar^k = \frac{a(1-r^{m+1})}{(1-r)} \tag{A.1}$$

and converges to

$$\lim_{m\to\infty} \sum_{k=0}^{m} ar^k = \frac{a}{1-r}$$

iff $|r| < 1$.