Understanding Analysis Solutions

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Chapter 1

The Real Numbers

1.2 Some Preliminaries

Exercise 1.2.1

(a) Proof AFSOC $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$ such that

$$\sqrt{3} = \frac{m}{n},$$

where $\frac{m}{n}$ is in lowest reduced terms. Then we can square both sides, yielding $3 = \left(\frac{m}{n}\right)^2 \implies 3n^2 = m^2$. Now, we know m^2 is a multiple of 3 and thus m must also. Then, we can write $m = 3k$, and derive

$$\begin{aligned}(\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2\end{aligned}$$

Similar to before, we come to the conclusion that n is a multiple of 3. However, this is a contradiction since m, n are both multiples of 3 and we assumed $\frac{m}{n}$ was in lowest terms. Thus, we conclude $\sqrt{3}$ is irrational.

The same proof for $\sqrt{3}$ works for $\sqrt{6}$ as well.

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ implies that m is a multiple of 4, since we have

$$4n^2 = m^2 \quad \Rightarrow \quad 2n = m,$$

so we cannot reach our contradiction that m/n is not in lowest terms.

Exercise 1.2.2

(a) False. Consider

$$A_n = \left[0, \frac{1}{n}\right).$$

Then

$$\bigcap_{n=1}^{\infty} A_n = \{0\}.$$

(b) True. Since $\forall i, A_i \subseteq A_1$, $\exists x$ such that $\forall i, x \in A_i$. Therefore, the intersection cannot be empty. Then, every set is finite, and the intersection of any number of finite sets will be finite.

(c) False. Consider $A = \{1, 2\}, B = \{1\}, C = \{2, 3\}$.

$$\{1,2\} \cap \big(\{1\} \cup \{2,3\}\big) = \{1,2\} \neq \big(\{1,2\} \cap \{1\}\big) \cup \{2,3\} = \{1,2,3\}$$

(d) True. Intersection is associative.

(e) True. Intersection is distributive over union.

PROOF We will prove

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.1}$$

by set inclusion.

- Suppose $x \in A \cap (B \cup C)$. By the definition of intersection, we know $x \in A$ and $x \in B \cup C$, the latter which means $x \in B$ or $x \in C$.

  We can consider 2 cases for x,

  1. $x \in B$. Then we know $x \in A$ and $x \in B$, so $x \in A \cap B$ and therefore $x \in (A \cap B) \cup (A \cap C)$
  2. $x \in C$. Symmetric to the case above.

  in all cases, we see $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$, so

  $$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

- Suppose $x \in (A \cap B) \cup (A \cap C)$. Then we have two cases

  1. $x \in A \cap B$. This means $x \in A$ and $x \in B$. If $x \in B$, then $x \in B \cup C$, since $B \subseteq B \cup C$. Putting these facts together, we see $x \in A \cap (B \cup C)$.
  2. $x \in A \cap C$. Symmetric to the case above.

  in all cases, we see $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$, so

  $$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

Exercise 1.2.3

(a) If $x \in (A \cap B)^c$, then we have cases

- $x \in B$ and $x \notin A$. Then $x \notin A$ implies $x \in A^c \Rightarrow x \in A^c \cup B^c$.
- $x \in A$. Symmetric to above.
- $x \notin A$ and $x \notin B$. Then $x \in A^c$ so $x \in A^c \cup B^c$.

(b) If $x \in A^c \cup B^c$, then we have cases

- $x \in A^c$. Then $x \notin A$ so x cannot be in the intersection of A and B, so $x \in (A \cap B)^c$.
- $x \in B^c$. Symmetric to above.

(c) Proof for $(A \cup B)^c = A^c \cap B^c$ pretty similar to above.

Exercise 1.2.4

We are verifying the triangle inequality with a, b.

(a) If a, b have the same sign, then

$$\begin{aligned}
|a + b| &= a + b \\
|a| + |b| &= a + b \\
\Rightarrow \ |a + b| &= |a| + |b| \\
\Rightarrow \ |a + b| &\leq |a| + |b|
\end{aligned}$$

(b) - $a \geq 0, b < 0$.

$$\begin{aligned}
|a + b| &\leq |a| \\
&\leq |a| + |b|
\end{aligned}$$

- $a+b \geq 0$. At most one of a, b is negative. If they are both positive, then we have already shown this in part (a). Otherwise, WLOG a is negative. Then

$$\begin{aligned} |a+b| &\leq |b| \\ &\leq |a| + |b| \end{aligned}$$

Exercise 1.2.5

(a) Substitute in $b' = -b$ into the triangle inequality.

(b) Easy to prove directly without using triangle inequality. **TODO**.

A direct proof will look something like:

- If a, b are the same sign, then equality holds
- If a, b are different signs, then if b is negative, then $|a-b| = |a| + |b|$, and if a is negative, then $|a-b| = |a| + |b|$, both of which bound $\big||a| - |b|\big|$.

Exercise 1.2.6

(a) Yes, since $f(A \cap B) = [1,4] = [0,4] \cap [1,16] = f(A) \cap f(B)$. This is by coincidence though, as we will later see. Yes, since $f(A \cup B) = [0,16] = [0,4] \cup [1,16] = f(A) \cup f(B)$.

(b) Choose $A = [-2,0], B = [0,2]$

(c) Suppose $x \in g(A \cap B)$, then $\exists x' \in A \cap B$ such that $g(x') = x$. Since $x' \in A$ and $x' \in B$, we know $x = g(x') \in g(A), g(B)$, so we conclude $x \in g(A) \cap g(B)$.

(d) Equality. **TODO** too lazy to write out the proof. Similar to above.

Exercise 1.2.7

(a) **TODO** I don't think we want to include $x \in \mathbb{I}$...

$$f^{-1}(A) = [0,2] \tag{1.2}$$

$$f^{-1}(B) = [0,1] \tag{1.3}$$

We see $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ is also true.

(b) **TODO**

Exercise 1.2.8

Negating statements. Took some liberties. Also notice that these statements are not necessarily true.

(a) There exists a real number satisfying $a < b$, such that $\forall n \in \mathbb{N}$, $a + 1/n \geq b$.

(b) There exists two distinct real numbers such that there is not a rational number between them.

(c) There exists a natural number $n \in \mathbb{N}$ such that $\sqrt{n}$ is not a natural number nor an irrational number.

(d) There exists a real number $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $n \leq x$.

Exercise 1.2.9

We are given the sequence

$$x_1 = 1, x_{n+1} = \frac{1}{2}x_n + 1 \tag{1.4}$$

and want to show $\forall i \geq 1, x_i < 2$.

We can show this with a direct proof of summation.

An alternative that the book probably wants to see is using **induction**.

- Base Case: $x_1 = 1 < 2$
- Inductive case. Assume $\forall i < n+1, x_i < 2$. Then $x_i/2 + 1 < 2$ since $x_i/2 < 1$.

- By induction our original claim is proved.

Exercise 1.2.10

(a) Similar to Exercise 1.2.9. $y_n < 4$ means $(3/4)y_n < 3$ so $(3/4)y_n + 1 < 4$

(b) In brief,

$$\begin{aligned} y_n &\leq \frac{3}{4}y_n + \frac{1}{4}y_n && (1.5) \\ &< \frac{3}{4}y_n + 1 && \text{(Using } y_n < 4\text{)} \\ &< y_n + 1 && \text{(Sequence definition)} \end{aligned}$$

Exercise 1.2.11

A combinatorial argument is that in order to construct a set, we have 2 choices for every element, to include it or not to. Therefore, we have

$$\prod_{i=1}^{n} 2 = 2^n$$

Exercise 1.2.12

(a) We know that $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$. So if we are trying to show $(A_1 \cup A_2 \cup A_3)^c = (A_1 \cup A_2)^c \cap A_3^c = A_1^c \cap A_2^c \cap A_3^c$. Induction lets us apply the property on smaller parts of our expression.

(b) Induction only proves the property for some $n \in \mathbb{N}$, i.e. some finite n. It is not shown for an infinite n.

(c) **TODO**. Sketch: If x is not in the union of all the A_n, then x cannot be part of any particular A_n either, or else it would be in the union.

1.3 The Axiom of Completeness

Exercise 1.3.1

(a) We compute the additive inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned} 0+0 &\equiv 0 \\ 1+4 &\equiv 0 \\ 2+3 &\equiv 0 \\ 3+2 &\equiv 0 \\ 4+1 &\equiv 0 \end{aligned}$$

(b) We compute the multiplicative inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned} 1\times 1 &\equiv 1 \\ 2\times 3 &\equiv 1 \\ 3\times 2 &\equiv 1 \\ 4\times 4 &\equiv 1 \end{aligned}$$

(c) $\mathbb{Z}_4$ is not a field because multiplicative inverses do not exist for every single element. For example, 2 multiplied with any number is even, which cannot $\equiv 1 \pmod 4$.

We conjecture that $\mathbb{Z}_n$ always has additive inverses and only has multiplicative inverses if n is prime.

Exercise 1.3.2

We are writing a formal definition for the *infimum* of a set.

(a) $s = \inf A$ means

i) s is a lower bound for A

ii) if b is any lower bound for A, then $b \leq s$

(b) If $s \in \mathbb{R}$ is a lower bound for $A \subseteq \mathbb{R}$, then $s = \inf A$ iff $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$.

Proof ($\Rightarrow$) If $s = \inf A$, then s is the greatest lower bound for A, meaning any $s + \epsilon$ for $\epsilon > 0$ will be greater than some element of A, otherwise $s + \epsilon$ is a greater lower bound and leads to a contradiction that$s \neq \inf A$.

($\Leftarrow$) If $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$, then since s is a lower bound, $\forall b > s$, b will not be a lower bound for A since if, $b > s$, then we can choose $\epsilon = b - s > 0$, and we know that $\exists a \in A$ where $a < s + \epsilon < b$, which means b is not a lower bound. Thus, all lower bounds b must be such that $b \leq s$, and we conclude $s = \inf A$.

Exercise 1.3.3

(a) Since $\inf A$ is a lower bound for A, we know $\inf A \in B$. Now, we need to show $\inf A$ is the supremum of B. $\inf A$ is the least upper bound for B, since if $\exists b \in B, b > \inf A$, then we know that this b is not a lower bound for A, so no such b exists.

(b) There might be a typo in this question. I think the question was meant to read "explain why there is no need to assert that the greatest *lower bound* in the Axiom of Completeness." In this case, the answer would be that the Axiom of Completeness already implies the greatest lower bound property, so there is no need to explicitly state it.

(c) We can take the negative of all elements in A, find $\sup A$, and then negate again to get $\inf A$.

Exercise 1.3.4

If $B \subseteq A$, then

$$\begin{aligned} \sup A = s &\geq a, \forall a \in A \\ s &\geq b, \forall b \in B && \text{(since } B \subseteq A) \\ \Rightarrow\ s &\geq \sup B. && \text{(since } s \text{ is an upper bound for } B) \end{aligned}$$

Exercise 1.3.5
(a)

$$
\begin{aligned}
&s = \sup(c + A) \\
\Rightarrow\, &s \text{ is the least upper bound for } c + A \\
\Rightarrow\, &s - c \text{ is the least upper bound for } A \\
\Rightarrow\, &s - c = \sup A \\
&s = c + \sup A
\end{aligned}
$$

(b)

$$
\begin{aligned}
&s = \sup(cA) \\
&\Rightarrow\ s \text{ is the least upper bound for } cA \\
&\Rightarrow\ \frac{s}{c} \text{ is the least upper bound for } A \\
&\Rightarrow\ \frac{s}{c} = \sup A \\
&s = c \sup A
\end{aligned}
$$

(c) If $c < 0$, $\sup(cA) = -c \inf(A)$.

Exercise 1.3.6
(a) $\sup : 3; \inf : 1$

(b) $\sup : 1; \inf : 0$

(c) $\sup : \frac{1}{2}; \inf : \frac{1}{3}$

(d) $\sup : 9; \inf : \frac{1}{9}$

Exercise 1.3.7
If $a \geq a', \forall a' \in A$, and $a \in A$, then

$$\forall \epsilon > 0, a - \epsilon < a, \tag{1.6}$$

so a is the least upper bound for A, and $a = \sup A$.

Exercise 1.3.8
Let

$$\epsilon = \sup B - \sup A > 0. \tag{1.7}$$

since $s_b = \sup B$, $\exists b \in B \mid b > s_b - \epsilon/2$. Since $s_b - \frac{\epsilon}{2} > \sup A$, then $b \geq \sup A$, so this $b \in B$ is an upper bound for A.

Exercise 1.3.9
(a) True, take the largest element in the set as the supremum.

(b) False, $\sup(0, 2) = 2$, but $2 > a \in (0, 2)$, but $\sup A = 2 \not< 2 = L$.

(c) False $A = (0, 2), B = [2, 3)$. We have that $\sup A = \inf B$

(d) True.

(e) False, take $A = B = (0, 2)$.

1.4 Consequences of Completeness

Exercise 1.4.1
If $a < 0$, then we have two cases,

1. If $b > 0$, then $a < 0 < b$.
2. If $b = 0$, then we can take $-b, -a$, which satisfies $0 \leq -b < -a$, and apply Theorem 1.4.3.

Exercise 1.4.2
(a) If $a, b \in \mathbb{Q}$, then

$$\begin{aligned} a &= \frac{a_1}{a_2} \\ b &= \frac{b_1}{b_2} \\ \implies a + b &= \frac{a_1 b_2 + a_2 b_1}{a_2 b_2} \in \mathbb{Q} \end{aligned}$$

(b) We can use contradiction,

- AFSOC $a + t \in \mathbb{Q}$. Let $a + t = \frac{m}{n}$. We know $a = \frac{a_1}{a_2}$ since $a \in \mathbb{Q}$, so

$$\begin{aligned} a + t &= \frac{m}{n} \\ t &= \frac{m}{n} - \frac{a_1}{a_2} \in \mathbb{Q}, \end{aligned}$$

  which is a contradiction since we are given $t \in \mathbb{Q}$. Therefore, we conclude $a + t \in \mathbb{I}$.

- AFSOC $at \in \mathbb{Q}$. Let $at = \frac{m}{n}$. We know $a = \frac{a_1}{a_2}$ since $a \in \mathbb{Q}$, so

$$\begin{aligned} at &= \frac{m}{n} \\ t &= \frac{m}{n} \cdot \frac{a_2}{a_1} \in \mathbb{Q}, \end{aligned}$$

  which is a contradiction since we are given $t \in \mathbb{Q}$. Therefore, we conclude $at \in \mathbb{I}$

(c) $\mathbb{I}$ is not closed under addition or multiplication.

$$\begin{aligned} \left(3 - \sqrt{2}\right) + \left(3 + \sqrt{2}\right) &= 6 \notin \mathbb{I} \\ \left(3 - \sqrt{2}\right) \cdot \left(3 + \sqrt{2}\right) &= 5 \notin \mathbb{I} \end{aligned}$$

Exercise 1.4.3
We can apply Theorem 1.4.3, to find $a < q < b, q \in \mathbb{Q}$, and then subtract an irrational number such as $\sqrt{2}$ to end up at

$$a - \sqrt{2} < q - \sqrt{2} < b - \sqrt{2}, \tag{1.8}$$

where $q - \sqrt{2} \in \mathbb{I}$.

Exercise 1.4.4
Suppose $\exists b$ lower bound such that $b > 0$. Then by Archimedean Property of $\mathbb{R}$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b$, which means b is not a valid lower bound. Thus $b \leq 0$, and 0 is a valid lower bound so the inf is 0.

Exercise 1.4.5
AFSOC $\exists \alpha \in \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$. Then $\alpha > 0$, but by Archimedean property of reals, we have that $\exists n \in \mathbb{N} \mid \frac{1}{n} < \alpha$. Since $\alpha \notin \left(0, \frac{1}{n}\right)$ leads to $\alpha \notin \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$, a contradiction, we conclude the set is empty.

Exercise 1.4.6

(a) If $\alpha^2 > 2$, then

$$\begin{aligned}\left(a - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}\end{aligned}$$

choose $\frac{1}{n_0} < \frac{\alpha^2-2}{2\alpha}$. Then

$$\begin{aligned}\left(a - \frac{1}{n_0}\right)^2 &> \alpha^2 - \frac{2\alpha}{2\alpha}(\alpha^2 - 2) \\ &> 2\end{aligned}$$

but $\alpha - \frac{1}{n_0} < \alpha$, so α is not the least upper bound for the se, so $\alpha \neq \sup T$.

(b) Just replace $\sqrt{2}$ with $\sqrt{b}$ for the proof above.

Exercise 1.4.7

Once we have assigned $g(i) = f(n_i)$, remove $f(n_i)$ from A. Now, there is a new $n_{i+1} = \min\{n \in \mathbb{N} : f(n) \in A \setminus \{f(1), f(2), \dots, f(n_i)\}\}$. Assign $g(i+1) = f(n_{i+1})$, and repeat.

Exercise 1.4.8

(a) If both are finite, then their union is finite and trivially countable. If one is finite, then first enumerate elements of the finite set. Then map the rest of $\mathbb{N}$ to the countably infinite set. If both are countably infinite, map one set to odds and the other to evens.

(b) Induction only holds for finite integers, not infinity.

(c) We can arrange each A_n into row n of a $\mathbb{N} \times \mathbb{N}$ matrix. Then, we enumerate by diagonalization.

Exercise 1.4.9

(a) If $A \sim B$, then there is a 1-to-1 mapping. We can just take the inverse of the mapping to derive $B \sim A$.

(b) If we have $f : A \to B$, $g : B \to C$, then we can compose the functions so $g(f(x)) : A \to C$.

Exercise 1.4.10

The set of all finite subsets of $\mathbb{N}$ can be ordered in increasing order by the sum of each subset.

Exercise 1.4.11

(a) $f(x) = (x, 0.5) \in S$

(b) Interweave the decimal expansion of x, y, e.g.

$$f(x, y) = 0.x_1y_1x_2y_2x_3y_3\dots \tag{1.9}$$

Exercise 1.4.12

(a)

$$\begin{aligned}\sqrt{2} &: x^2 - 2 = 0 \\ \sqrt[3]{2} &: x^3 - 2 = 0\end{aligned}$$

$\sqrt{3} + \sqrt{2}$ is not as trivial, so we will do it out in more steps.

There are two approaches to finding the integer coefficient polynomial. One is to take advantage of symmetry, and derive that

$$\prod(x - (\pm\sqrt{3} \pm \sqrt{2}) \tag{1.10}$$

will work (using loose notation of course). A more general technique is to notice that

$$\begin{aligned} x &= \sqrt{3}+\sqrt{2} \\ x^2 &= 5+2\sqrt{6} \\ (x^2-5)^2 &= 24 \\ x^4-10x^2+1 &= 0. \end{aligned}$$

Notice that this is actually the exact same answer we get in (1.10) if you work it out.

(b) Each element of A_n is a root of a n degree polynomial, which we can represent as an $(n+1)$-tuple of coefficients $\in \mathbb{Z}$. Therefore, $|A_n| = k\left|\mathbb{N}^{n+1}\right| = \left|\mathbb{N}^{n+1}\right|$, which we know is countable.

(c) We proved earlier in Theorem 1.4.13 that a countably infinite union of countable sets is countable. Since there are a countable number of algebraic numbers, and reals are uncountable, we conclude that transcendentals are also uncountable.

Exercise 1.4.13
We are proving the **Schroöder-Bernstein Theorem**, which states if there exist 1-to-1 functions $f : X \to Y$ and $g : Y \to X$, then there exists a 1-to-1, onto function $h : X \to Y$, which implies $X \sim Y$.

(a) By the definition of 1-to-1, there must be a unique $x \in X$ such that $f(x) = y$. A 1-to-1 function maps distinct elements from the domain to distinct elements of the range, so if we take the inverse f^{-1}, it will still be 1-to-1, this time from $Y \to X$.

(b) Possibilities:

- Zero: g^{-1} is not guaranteed to be onto, it may not have an inverse for x.
- Finite: $g^{-1}(x)$ could exist, and similarly for f^{-1}. Once the element doesn't exist in the inverse domain, the chain will stop.
- Infinite: x is in the range of g and the domain of f

(c) We have 2 cases

- The chains are disjoint. Nothing to prove here.
- The chains are not disjoint, i.e. they have one common element. Let us call this element x. We know to the right of x, all the elements in the two chains will be equal. From the left of x, the elements must be equal as well. This is because the inverse chain must be unique starting from x, see part (a). Since all the elements are the same, the chains must be the same as well.

(d) Since we know this chain started with $x \in X$, this y could not have been created from the RHS, otherwise this y would be in the range of f. Therefore, this chain either has infinite or a finite of elements to the right.

- Finite: the chain must start with an element $y \notin Y$ but not in f's range. This is because if we start with $x \in X$, then as mentioned before, all elements $y' \in Y$ will be in f's range. Therefore, if we start with $y \in Y$, it will match the form indicated.
- Infinite: The chain could not have an infinite elements to the left, because then every y must have come from an $f(x')$ for some $x' \in X$.

Therefore, these chains only can have a finite number of elements to the left, and it matches the form indicated.

(e) By the definition of C_x, all the elements of $y \in C_x$ that are $\in Y$ are mapped by f from $x \in X_1$. This means f maps X_1 onto Y_1. Similar logic can be used for g mapping Y_2 onto X_2.

Since we know f is a 1-to-1 function from X to Y, and we just showed it maps X_1 to Y_1. We can conclude that $X_1 \sim Y_1$, since f is a bijection between X_1 and Y_1. We can similarly conclude $X_2 \sim Y_2$

with g. Since X_1, X_2 are a partition of X, since all the chains are disjoint, and similarly for Y_1, Y_2 of Y, and there exists a bijection f, g for $X_1 \sim Y_1$ and $X_2 \sim Y_2$ respectively, we conclude there must be a bijection between X and Y. Therefore, we conclude $X \sim Y$.

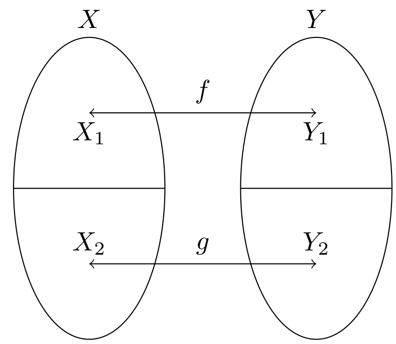


Figure 1.1: f and g mapping X and Y

1.5 Cantor's Theorem

Exercise 1.5.2

(a) Because b_1 differs from $f(1)$ in position 1

(b) b_i differs from $f(i)$ in position i.

(c) We reach a contradiction that we can enumerate all the elements of $(0, 1)$, since we found a real number that isn't enumerated, and thus $(0, 1)$ is uncountable.

Exercise 1.5.3

(a) $\frac{\sqrt{2}}{2} \in (0, 1)$ but is irrational

(b) We can just define our decimal representations to never have an infinite string of 9s.

Exercise 1.5.4

AFSOC S is countable. We will use a diagonalization proof. Then we can enumerate the elements of S using the natural numbers. Now, consider some $s = (s_1, s_2, \dots)$, where

$$s_i = \begin{cases} 0, \text{if } f(i), \text{position } i = 1 \\ 1, \text{otherwise} \end{cases} \tag{1.11}$$

Then since $s \neq f(i) \forall i$, we see $s \notin S$. But this is a contradiction since s only contains elements 0 or 1, and thus should be in S. Thus, we conclude that S is uncountable.

Exercise 1.5.5

(a)
$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \tag{1.12}$$

(b) Each element has two choices when constructing a subset of A. To be, or not to be [1], in the set.

Exercise 1.5.6

(a) Many different answers.
$$\begin{aligned} &\{(a, \{a\}), (b, \{b\}), (c, \{c\})\} \\ &\{(a, \emptyset), (b, \{b\}), (c, \{c\})\} \end{aligned} \tag{1.13}$$

(b)
$$\{(1, \{1\}), (2, \{2\}), (3, \{3\}), (4, \{4\})\}.$$

(c) Because in general, $\left|\mathcal{P}(A)\right| > |A|$ for any set $A \neq \emptyset$. The intuition is that the power set has strictly more elements than A, so A cannot map $\mathcal{P}(A)$ onto.

Exercise 1.5.7

Using the examples found in (1.13).

1. $B = \emptyset$
2. $B = \{a\}$

Exercise 1.5.8

(a) AFSOC $a' \in B$. Then that means $a \notin f(a')$ by the definition of B. But this is a contradiction since $a' \in B = f(a')$.

(b) AFSOC $a' \notin B = f(a')$. Then since $a' \notin f(a')$, by the construction of B, this implies $a' \in B$, but that is a contradiction from our original assumption.

Exercise 1.5.9

(a) This is the same as $\mathbb{N} \times \mathbb{N}$, which is countable.

[1] sorry, had to do it. *Addendum* For context, I took a Shakespeare class in college two semesters prior to when I first wrote this.

(b) Uncountable, since this is essentially constructing the power set of $\mathbb{N}$, and we know $\mathcal{P}(\mathbb{N})$ is uncountable.

(c) Is this question asking for the number of antichains or if there is an antichain with uncountable cardinality?

The latter is obvious, and *no* is the answer since any subset of $\mathbb{N}$ is countable.

If we want to count the number of antichains, we notice that an antichain is essentially a partition of some subset of B. We also notice that every element of $\mathcal{P}(B)$ is also technically a partition, just a partition of size one. This means that the cardinality of the set of antichains is at least the cardinality of $\mathcal{P}(B)$. If $B = \mathbb{N}$, then we know $\mathcal{P}(\mathbb{N})$ is already uncountable, so the set of antichains will also be uncountable.

Chapter 2

Sequences and Series

For the convergence proofs in this chapter, I will lean towards showing how to derive the N that works, rather than just going directly with the proof and supplying a magical N, since I think finding the N is the process that deserves more attention.

2.2 The Limit of a Sequence

Exercise 2.2.1
The proofs are essentially the same, so after the first proof, I'll just give the n that can be used to prove the convergence.

(a) Let $\epsilon > 0$ be arbitrary. Then choose $n \in \mathbb{N}$ such that $n > \frac{1}{\sqrt{6\epsilon}}$. Then

$$\begin{aligned}
\left|\frac{1}{6n^2+1}\right| &< \left|\frac{1}{6\frac{1}{6\epsilon}+1}\right| \\
&< \left|\frac{1}{\frac{1}{\epsilon}+1}\right| \\
&< \frac{\epsilon}{\epsilon+1} \\
&< \epsilon
\end{aligned}$$

as desired.

(b) Choose $n > \frac{13}{2\epsilon} - \frac{5}{2}$

(c) Choose $n > \frac{4}{\epsilon^2} - 3$

Exercise 2.2.2
Consider the sequence

$$x_n = (-1)^n, n \geq 1. \tag{2.1}$$

Then for $\epsilon > 2$, it is true that $|x_n - 0| < 2, \forall n \geq 1$.

The *vercongent* definition describes a sequence that can be finitely bounded past some n.

Exercise 2.2.3

(a) We have to find one school with a student shorter than 7 feet.

(b) We would have to find a college with a grade that is not A or B.

(c) We just have find a college where a student is shorter than 6 feet.

Exercise 2.2.4
For $\epsilon > \frac{1}{2}$, we can find a suitable N, since we can claim the sequence "converges" to $\frac{1}{2}$. For $\epsilon \leq \frac{1}{2}$, there is no suitable response.

Exercise 2.2.5
(a) $\lim a_n = 0$. Take $n > 1$. Then

$$\begin{aligned} \left| \left[\left[\frac{1}{n} \right]\right] \right| &\leq 0 \\ &< \epsilon. \end{aligned}$$

(b) $\lim a_n = 0$. Take $n > 10$. Then

$$\begin{aligned} \left| \left[\left[\frac{10+n}{2n} \right]\right] \right| &= \left| \left[\left[\frac{5}{n} + \frac{1}{2} \right]\right] \right| \\ &\leq 0 \\ &< \epsilon. \end{aligned}$$

Usually, the sequence converges to some value by getting closer and closer eventually. This means for a smaller ϵ-neighborhood, we have to enumerate more elements, so we need a larger N.

Sometimes, the sequence converges to the exact value very fast, which means for some n, we don't need to choose a larger n. E.g. if we had the sequence of all 0s, we can choose any n and claim the sequence converges to 0.

Exercise 2.2.6
(a) Any *larger* N will work, since succeeding elements should stay in the neighborhood.

(b) Any *larger* ϵ will work, since we already guaranteed succeeding elements will stay in the ϵ-neighborhood, so any $\epsilon' > \epsilon$ will also bound the rest of the sequence.

Exercise 2.2.7
(a) We say a sequence x_n *converges* to ∞ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$ we have that $|x_n| > \epsilon$

(b) With our definition, we say this sequence diverges, but does not converge to ∞.

Exercise 2.2.8
(a) Frequently, since -1 will leave the set $\{1\}$.

(b) Eventually is stronger, and implies frequently.

(c) We say that a sequence x_n converges to x if it eventually is in a neighborhood of radius ϵ of x for all $\epsilon > 0$.

(d) x_n is only necessarily frequently in $(1.9, 2.1)$, even if there are an infinite number of elements equal to 2, you could have something like $(-2)^n$, where it keeps on leaving the ϵ-neighborhood of 2.

2.3 The Algebraic and Order Limit Theorems

Exercise 2.3.1
Let $\epsilon > 0$. Consider $n \geq 1$, then

$$|a - a| = 0 < \epsilon.$$

Exercise 2.3.2

(a) We are given $(x_n) \to 0$, so we can make $|x_n - 0|$ as small as we want.

In particular, for some $\epsilon > 0$, we choose N such that $\forall n \geq N$,

$$|x_n| < \epsilon^2 \quad \Rightarrow \quad \left|\sqrt{x_n}\right| < \epsilon \tag{2.2}$$

The implication follows since we know $x_n \geq 0, \epsilon > 0$.

To see that this N works, observe that for all $n \geq N$,

$$\left|\sqrt{x_n} - 0\right| < \epsilon \qquad \text{(by (2.2))}$$

so we conclude $(\sqrt{x_n}) \to 0$.

(b) We have two cases. If the sequence converges to 0, then we just have part (a).

If $x \neq 0$, then notice

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{|x_n - x|}{\left|\sqrt{x_n} + \sqrt{x}\right|}$$

since we know $x_n \geq 0$ and $x \neq 0$. Now, this expression is hard to bound when the denominator is small, since that would make the overall expression big. Fortunately, we can put a bound on the denominator, namely, since we know $x \neq 0 \to x > 0$, the denominator is > 0. Let us call the denominator value d. Then the following N will work for the convergence proof,

$$N : \forall n \geq N \quad |x_n - x| < \epsilon \cdot d \tag{2.3}$$

Exercise 2.3.3
By the Order Limit Theorem, since

$$\forall n, x_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \geq \lim_{n\to\infty} x_n = l$$
$$\forall n, z_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \leq \lim_{n\to\infty} z_n = l$$

so $l \leq \lim_{n\to\infty} y_n \leq l \Rightarrow \lim_{n\to\infty} y_n = 1$.

Exercise 2.3.4
AFSOC $\lim a_n = l_1$ and l_2, for $l_1 \neq l_2$. Then we have that $\forall \epsilon > 0$, for sufficiently large n, that

$$|a_n - l_1| < \epsilon$$
$$e|a_n - l_2| < \epsilon$$

But this is a contradiction, since if we let $d = |l_1 - l_2|$, and $\epsilon = \frac{d}{2}$, then

$$|l_2 - l_1| \leq |a_n - l_1| + \left|-(a_n - l_2)\right| < 2\epsilon \qquad \text{(Triangle Inequality)}$$
$$d \leq |a_n - l_1| + \left|-(a_n - l_2)\right| < d,$$

which leads to $d < d$. Thus, we must conclude that $l_1 = l_2$, and limits are unique.

Exercise 2.3.5
($\Rightarrow$) If (z_n) is convergent to some l, then $\forall \epsilon > 0$, we have that $\exists N \in \mathbb{N}$ such that for $n \geq N$, that

$$|z_n - l| < \epsilon \Longrightarrow |x_n - l| < \epsilon, |y_n - l| < \epsilon, \tag{2.4}$$

because z_n appears before or at the same time as x_n and y_n in the sequence.

($\Leftarrow$) If $(x_n), (y_n)$ are both convergent to some limit l, then we have for any $\epsilon > 0$, $\exists N_x : n_x \geq N_x$ and $\exists N_y : n_y \geq N_y$, that

$$\begin{aligned} |x_{n_x} - l| &< \epsilon \\ \left|y_{n_y} - l\right| &< \epsilon, \end{aligned} \tag{2.5}$$

respectively.

Choose $N_z > 2 \cdot \max(N_x, N_y)$. Then for $n_z \geq N_z$, z_{n_z} is either equal to x_i for $i > N_x$ or y_j for $j > N_y$. Using (2.5), we can see that

$$|z_{n_z} - l| < \epsilon$$

so (z_n) is also convergent to l.

Exercise 2.3.6

(a) By triangle inequality, we have $\big||b_n| - |b|\big| \leq |b_n - b| < \epsilon$, so the N that proves convergence for (b_n) will also work for $(|b_n|)$.

(b) The converse is not true. Consider the sequence $a_n = (-1)^n$.

Exercise 2.3.7

(a) Since (a_n) is bounded, call M the upper bound of (a_n). Then since $|b_n|$ can get arbitrarily small, we choose $n \geq N$ such that $|b_n| < \frac{\epsilon}{M}$. Then we have

$$\begin{aligned} |a_n b_n| &\leq |a_n||b_n| \\ &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

We cannot use the Algebraic Limit Theorem because we are not given that (a_n) necessarily converges.

(b) No. For example, take $a_n = (-1)^n$, $b_n = 3$. This is because we can no longer make $|b_n|$ arbitrarily small.

(c) When $a = 0$, we have

$$|a_n b_n - ab| \leq |b_n||a_n - a|.$$

We can bound $|b_n| \leq M$, and then choose n such that $|a_n - a| < \frac{\epsilon}{M}$. Then,

$$\begin{aligned} |a_n b_n - ab| &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

Exercise 2.3.8

(a) $x_n = (-1)^n, y_n = (-1)^{n-1}$. Sum is just $\{0, 0, \dots\}$

(b) **Impossible**, since if $x_n + y_n$ converges and x_n also converges, we can show that y_n must converge, which is a contradiction.

(c) $b_n = \frac{1}{n}$

(d) **Impossible**, since if b_n converges to some b, for any $\epsilon > 0$, past some N, for $n \geq N$,

$$|b_n - b| < \epsilon.$$

Any a_n that is unbounded will grow in magnitude for larger n, so b_n cannot help bound a_n.

(e) $a_n = 0, b_n = n$

Exercise 2.3.9

Yes, the strict inequalities will provide an upper and lower bound still. Sort of like a sup, inf of the sequence.

Exercise 2.3.10
Since $|a_n|$ gets arbitrarily small, for any $\epsilon > 0$ we know $\exists N : n \geq N$ such that,

$$|b_n - b| \leq |a_n| < \epsilon. \tag{2.6}$$

Exercise 2.3.11
Let $\lim x_n = x$. Then, for any $\epsilon_x > 0$, $\exists N_x : n \geq N_x$, we have $|x_n - x| < \epsilon_x$.

Now, our goal is, given some $\epsilon_y > 0$, to find some $N_y : n \geq N_y$ so we can bound y_n. The intuition is, since we know (x_n) converges, after some point, x_i will be close to the limit x. Our goal is to choose some N_y large enough so the x_i' prior to these x_i are "averaged out" enough, so they are essentially gone, and that the weight on the x_i that are close to x is very high.

$$\begin{aligned}
|y_n - x| &= \left| \frac{1}{n} \left[\sum_{i=1}^{N_x} (x_i - x) + \sum_{i=N_x+1}^{N_y} (x_i - x) \right] \right| \\
&\leq \left| \frac{1}{n} \left[\sum_{i=1}^{N_x} M + \sum_{i=N_x+1}^{N_y} \epsilon_x \right] \right| && \text{(Let } M \text{ bound the difference from } x_i \text{ to } x.) \\
&\leq \left| \frac{1}{n} \Big[N_x M + \big(N_y - N_x\big)\epsilon_x \Big] \right| \\
&\leq \left| \frac{N_x}{n} M + \epsilon_x \right| < \epsilon_y
\end{aligned}$$

Now, we have quite a few choices for our N_y. One such solution, is

- Given some $\epsilon > 0$
- First choose N_x such $n \geq N_x \quad |x_n - x| < \epsilon/2$
- Then, choose $N_y > \frac{2N_x M}{\epsilon}$. This means for $n \geq N_y$,

$$\begin{aligned}
|y_n - x| &\leq \left| \frac{N_x}{n} M + \epsilon_x \right| \\
&< \left| \frac{N_x M}{\frac{2N_x M}{\epsilon}} + \epsilon/2 \right| \leq \epsilon
\end{aligned}$$

Consider when $x_n = (-1)^n$. (x_n) does not converge but (y_n) does.

Exercise 2.3.12

(a) Intuitively, the limit should go to 1, since we have $\frac{\infty}{\infty}$.

$$\begin{aligned}
\lim_{n\to\infty} \lim_{m\to\infty} a_{m,n} &= 1 \\
\lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} &= 0
\end{aligned}$$

(b) A sequence $(a_{m,n})$ converges to l if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$, we have that

$$\left| \lim_{n\to\infty} \lim_{m\to\infty} a_{m,n} - l \right| < \epsilon$$

$$\left| \lim_{m\to\infty} \lim_{n\to\infty} a_{m,n} - l \right| < \epsilon.$$

i.e. we approach the same limit no matter what permutation of the index variables we iterate through.

This definition is motivated by multivariable calculus, but unsure if this makes sense in the context of analysis.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Exercise 2.4.1
Suppose $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges. Fix m, k so that $m \geq 2^{k+1} - 1$, then

$$\begin{aligned} \sum_{i=1}^{m} b_i &\geq \sum_{i=1}^{2^{k+1}-1} b_i \\ &= s_{2^{k+1}-1} \\ &= t_k \end{aligned}$$

Since t_k is a diverging sequence, then b_m will also diverge.

Exercise 2.4.2

(a) We can show by induction that the sequence is decreasing. Thus, because the sequence starts at 3, we know it is bounded below by 0. Thus, the sequence converges.

(b) If $\lim x_n$ exists, then $\lim x_{n+1}$ must be the same limit, because if the limit is a different value or doesn't exist, then (x_n) does not converge.

(c) Suppose $\lim x_n = \lim x_{n+1} = x$. Then

$$\begin{aligned} x &= \frac{1}{4-x} \\ x^2 - 4x + 1 &= 0 \\ &\implies x = 2 - \sqrt{3} \end{aligned}$$

The other root is too large and does not work with the initial conditions.

Exercise 2.4.3
We can use induction to show that (y_n) is increasing. Since the sequence is increasing and starts at 1, we know that (y_n) is bounded above by 4 and below by 0. Thus, by the Monotone Convergence Theorem, we conclude that (y_n) converges. Now, we find the limit of the recurrence by taking the limits of both sides of the equation,

$$\begin{aligned} y &= 4 - \frac{1}{y} \\ y^2 - 4y + 1 &= 0 \\ y &= 2 + \sqrt{3} \end{aligned}$$

Exercise 2.4.4
We can define the recurrence of this sequence as

$$a_{n+1} = \sqrt{2a_n}. \tag{2.7}$$

We can prove by induction that this sequence is increasing. We can also bound the sequence since this sequence can also be viewed as

$$2^{\frac{1}{2}}, 2^{\frac{1}{2}+\frac{1}{4}}, 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}, \cdots$$

You can take the infinite sum $\sum_{i=1}^{\infty} 2^{-i} = 1$ and get $2^1 = 2$ as your final answer.

The other way to solve this problem is to look at the limits of x_n, x_{n+1}, which must be equal. Let's say their limit is x, then

$$\begin{aligned} x_{n+1} &= \sqrt{2x_n} x & &= \sqrt{2x} \\ x^2 - 2x &= 0 \\ x &= 2. & & \text{(from } x_0 = 1\text{)} \end{aligned}$$

Exercise 2.4.5

(a) By induction, we have

Base Case: $x_1 = 2 \implies x_1^2 = 4 \geq 2$.

Inductive Hypothesis: Given that for some $x_n, x_n^2 \geq 2$.

Inductive Step: Consider

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{4}\left(x_n^2 + 4 + \frac{4}{x_n^2}\right) \\ &\geq \frac{1}{4}\left(2 + 4 + 4/2\right) = 2. \end{aligned}$$

Therefore, we conclude $\forall n, x_n \geq 2$.

Now we can show

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &= \frac{\frac{1}{2}x_n^2 - 1}{x_n} \\ &\geq 0, \end{aligned}$$

which means the sequence is decreasing, so by the Monotone Convergence Theorem we know that (x_n) converges. We now take limits of x on both sides of the recurrence, yielding,

$$\begin{aligned} x &= \frac{1}{2}\left(x + \frac{2}{x}\right) \\ \frac{1}{2}x - \frac{1}{x} &= 0 \\ x^2 - 2 &= 0 \\ \implies x &= \sqrt{2}. \end{aligned}$$

(b) We can modify the sequence to converge to $\sqrt{c}, c \geq 0$ by setting $x_1 = c$, and

$$x_{n+1} = \frac{1}{c}\left((c-1)x_n + \frac{c}{x_n}\right) \tag{2.8}$$

Exercise 2.4.6

(a) Since we know that (a_n) is bounded, it must also be the case that $\sup(a_n)$ is bounded. Then, $\sup\{a_k\}$ is a decreasing sequence, so by the Monotone Convergence Theorem, we know that (y_n) converges.

(b) We can define

$$\liminf a_n = \lim z_n, \text{ where} \tag{2.9}$$
$$\lim z_n = \inf\{a_k : k \geq n\}. \tag{2.10}$$

Since $\inf\{a_k\}$ is a increasing sequence, and (a_n) is bounded, we know it converges.

(c) For any set A, $\inf A \leq \sup A$, so $\forall n, \inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$.

An example when the inequality is strict is

$$a_n = (-1)^n, \tag{2.11}$$

since $\liminf a_n = -1, \limsup a_n = 1$.

(d) ($\Rightarrow$) Suppose

$$\liminf a_n = \limsup a_n = L, \tag{2.12}$$

then given some $\epsilon > 0$, we know $\exists N : n \geq N$ so that, define $A_n = \{a_k : k \geq n\}$

$$|\inf A_n - L| < \epsilon$$
$$|\sup A_n - L| < \epsilon$$

since every element $k \geq n, \inf A_n \leq a_k \leq \sup A_n$, we conclude

$$k \geq n \geq N \quad |a_k - L| < \epsilon,$$

so $\lim a_n = L$.

($\Leftarrow$) Suppose

$$\lim a_n = L,$$

then given some $\epsilon > 0$, we know $\exists N : n \geq N$ so thats

$$|a_n - L| < \epsilon/2$$

This means every element after a_n lives in this $\epsilon/2$-neighborhood of L. Now, $\sup A_n$ must be arbitrarily close to the largest element of A_n, so we can make this distance $\epsilon/2$. That means

$$|\sup A_n - L| = \left|\max\{A_n\} + \epsilon/2 - L\right| < \epsilon,$$

which means $\limsup a_n = L$. This is similar for inf.

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Exercise 2.5.1
Suppose we have a convergent sequence with limit l. Then given any $\epsilon > 0$, we can always find $N : n \geq N$ such that $|a_n - l| < \epsilon$. For any subsequence of (a_n), (a'_m), any element of this subsequence, call it a'_k will be from some a_n in the original sequence, where $n \geq k$. So we can choose N from earlier, and for $m \geq N$ we will have $\left|a'_m - l\right| < \epsilon$.

Exercise 2.5.2

(a) Define

$$s_i = \sum_{j=1}^{i} a_j \tag{2.13}$$

$$b_i = \sum_{k=1}^{i} a_{n_k}, \tag{2.14}$$

where the series regrouping a_i is divided into groups of $n_1, n_2, \ldots$. Then b_i is a subsequence of s_n, which means they converge to the same limit, namely L in this case.

(b) Our proof does not apply to that example because that series did not converge in the first place.

Exercise 2.5.3

(a) Consider

$$a_n = \begin{cases} \sum_{i=1}^{n} \frac{1}{2^i}, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \tag{2.15}$$

Then we have that $b_n = a_{2n-1}$ converges to 1 and $c_n = a_{2n}$ converges to 0.

(b) A monotone sequence that diverges means that sequence is not bounded. Thus, every subsequence will also be unbounded and thus impossible to be convergent.

(c) Consider the sequence

$$\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \tag{2.16}$$

(d) Consider

$$a_n = \begin{cases} 2^i, & n \text{ odd} \\ \frac{1}{2^i}, & n \text{ even} \end{cases} \tag{2.17}$$

(e) By Bolzano-Weierstrass, since we have a subsequence that is bounded, we know we can find a convergent subsequence within this subsequence that converges.

Exercise 2.5.4
AFSOC (a_n) converges to $b \neq a$. Then we have that $|a_n - b|$ can be arbitrarily small. But this implies that every subsequence will also converge to b, which is a contradiction.

AFSOC (a_n) does not converge. Then since (a_n) is bounded, we must have an infinite number of elements in two different ϵ-neighborhoods. But this would imply we have convergent subsequences to different limits, which contradicts the original problem statement.

Therefore, we conclude (a_n) converges to a.

Exercise 2.5.5
Consider $|b^n|$. Since $|b| < 1$, we have that $|b^n|$ is a decreasing sequence that is bounded below by 0, so we have

$$|b| > l \geq 0.$$

We notice that $\left|b^{2n}\right|$ is a subsequence that also converges to L, and since $\left|b^{2n}\right| = |b|^2$, by the Algebraic Limit Theorem, we have that $\left|b^{2n}\right| \to l^2 = l \implies l = 0$. Since $|b^n| \to 0$, we conclude $b^n \to 0$.

Exercise 2.5.6

We have $s = \sup S$, which means for any $\epsilon > 0$,

$$\exists x : s - \epsilon < x \in S < a_n'$$
$$\epsilon > \left|s - a_n'\right| = \left|a_n' - s\right|$$

where a_n' is an element of the infinite subsequence of $a_n : a_n > x \in S$.

2.6 The Cauchy Criterion

Exercise 2.6.1

(a) $a_n = 1 + \left(-\frac{1}{2}\right)^n$

(b) $a_n = n$

(c) Impossible, since a Cauchy sequence implies convergence, which means every subsequence will also converge.

(d) You can use Equation (2.17). Literally anything that diverges but has a convergent subsequence.

Exercise 2.6.2

If we have that $(x_n) \to x$, then we can make $|x_n - x|$ arbitrarily small. Consider

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &\leq |x_n - x| + |x_m - x| && \text{(Triangle Inequality)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Exercise 2.6.3

(a) The *pseudo-Cauchy* definition is different because it only looks at consecutive terms

(b) Consider the harmonic series, where $|s_{n+1} - s_n| = \frac{1}{n(n+1)}$.

Exercise 2.6.4

$$\begin{aligned} |c_{n+1} - c_n| &= \big||a_{n+1} - b_{n+1}| - |a_n - b_n|\big| \\ &\leq |a_{n+1} - a_n + b_{n+1} - b_n| \\ &\leq |a_{n+1} - a_n| + |b_{n+1} - b_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Exercise 2.6.5

(a) Let $a_n = x_n + y_n$, then

$$\begin{aligned} |a_{n+1} - a_n| &= |x_{n+1} - x_n + y_{n+1} - y_n| \\ &\leq |x_{n+1} - x_n| + |y_{n+1} - y_n| \\ &< \epsilon && \text{(for proper choice of } N\text{)} \end{aligned}$$

(b) Let $a_n = x_n y_n$, then

$$\begin{aligned} |a_{n+1} - a_n| &= |x_{n+1}y_{n+1} - x_n y_n| \\ &= |x_{n+1}y_{n+1} - x_n y_{n+1} - x_n y_n + x_n y_{n+1}| \\ &\leq \big|y_{n+1}(x_{n+1} - x_n)\big| + \big|x_n(y_{n+1} - y_n)\big| \end{aligned}$$

we have shown we can bound this before by

- $(x_n), (y_n)$ are convergent sequences, so they must be bounded. Call this bound M
- Now, $|x_{n+1} - x_n|$ can be made arbitrarily small. Given some ϵ, we can make it $< \frac{\epsilon}{2M}$.
- We do the same for $|y_{n+1} - y_n|$, and thus the overall bound is $< \epsilon$.

Exercise 2.6.6

I'm not going to write these down super rigorously, but will write down most of the ideas.

(a) Suppose we have some set of real numbers that is bounded above. We want to show that there exists a least upper bound, assuming the Nested Interval Property is true.

Let B be the set of upper bounds. Define $I_1 = B$, and for each subsequent I_i, define it as

$$I_{n+1} = \{b \in I_n : b < \ell_{n+1}\},$$

where $\ell_{n+1} \in I_n$ is arbitrarily chosen. The idea is that we are creating intervals that have a smaller and smaller maximum value.

Now we have 2 cases,

a $\exists n \in \mathbb{N} : I_n = \emptyset$. In this case, there must have been some ℓ_n where $\forall b \in I_n, b \neq \ell_n, \ell_n < b$. Since I_n is a subset of the smaller elements of B, we also have $\forall b \in B, b \neq \ell_n, \ell_n < b$, which means this ℓ_n is the least upper bound.

b None of the I_n are empty. Now, consider $\bigcap_{n=1}^{\infty} I_n$. By NIP, this intersection is nonempty. By our construction of the I_n, we know any element b in this intersection must be less than all the elements in the set preceding it. However, we reach a contradiction, because if b is in this infinite intersection, **TODO** this proof doesn't work... probably need to change the interval construction.

(b) Define i_n to be $\inf\left(\bigcup_{k=n}^{\infty} I_k\right)$, and s_n to be $\sup\left(\bigcup_{k=n}^{\infty} I_k\right)$. i_n is an increasing sequence, and s_n is a decreasing sequence. Since for any set, $\inf A \leq \sup A$, we know that $\forall n : i_n \leq s_n$. This means $\exists x : \lim i_n \leq x \leq \lim s_n$, which exists in every single interval I_k, so we know their infinite intersection is nonempty.

(c) We are using the BW Theorem to prove NIP. Let x_i be an arbitrary element of I_i. since I_1 is bounded, then all $I_k, k \geq 1$ are also bounded, so the sequence (x_n) is also bounded. By BW, we know that $\exists (s_n)$, a subsequence of (x_n) that converges. Suppose $\lim s_n = L$. We will show that L is in the infinite intersection of all the sets. For any I_k, the interval is (a_k, b_k). Call $\epsilon = |a_k - b_k|$, the size of the interval. Since (s_n) converges to L, we know $\exists N : n \geq N$,

$$|s_n - L| < \epsilon/2.$$

since $s_n \in I_n$ by definition, we see L also falls in this interval. Thus, we know $L \in I_n$ for $n \geq N$. Since I_n is also contained within all sets before as well, L also is contained in those sets. Therefore, we have shown that L is in every set, so the infinite intersection must contain at least L, so therefore it is nonempty.

(d) We are given a bounded sequence, and want to show that there exists a convergent subsequence.

We will construct a subsequence, and show that it is convergent.

Since we have a bounded sequence (x_n), we know $|x_n| \leq M$.

Let $I_1 = [-M, M]$. Now construct subsequent I_n from $I_{n-1} = [a, b]$ as

$$I_n = [a, (a+b)/2] \text{ or } [(a+b)/2, b],$$

depending on which half has an infinite number of elements.

Now, our subsequence is defined as $a_n : a_n \in I_n$. Given $\epsilon > 0$, since we can make the bound of I_n as arbitrarily small as possible, since we are halving the interval size every time, we know $\exists N : n \geq N \rightarrow |I_n| < \epsilon$. Now, since all future elements a_k will be chosen from this interval and its subsets, and the interval size is $< \epsilon$, we can conclude for $m, n \geq N$,

$$|a_n - a_m| < \epsilon.$$

This means (a_n) is a Cauchy Sequence, and by the Cauchy Criterion, we can conclude that (a_n) is convergent.

2.7 Properties of Infinite Series

Exercise 2.7.1

(a) For any $\epsilon > 0$, we know since $(a_n) \to 0$, $\exists N : n \geq N, |a_n| < \epsilon$. Now, let $n > m \geq N, n = m+1$, $|s_n - s_m| = |a_n| < \epsilon$, which means (s_n) is a Cauchy sequence.

(b) Construct intervals I_k so that, initially, $I_1 = [-abs a_1, |a_1|]$. For $n \geq 1$, if $I_n = [b_n, c_n]$,

$$I_{n+1} = (b_n, s_{n+1}) \text{ if } s_{n+1} > s_n$$
$$I_{n+1} = (s_{n+1}, c_n) \text{ if } s_{n+1} < s_n$$

Now, take any $L \in \bigcap_{k=1}^{\infty} I_k$, which we know exists by NIP since $I_{n+1} \subseteq I_n$. We can show s_n converges to L, since the size of any interval I_n is $|s_n - s_{n-1}| = |a_n|$, so for any $\epsilon > 0$, we can show that s_n past some N will be within an ϵ-neighborhood of L.

(c) (s_n) is bounded by $|a_1|$, so $(s_{2n}), (s_{2n+1})$ are both bounded. These sequences also happen to be monotonic, since one is increasing and the other is decreasing. Therefore, the two subsequences are convergent, and we can add them together to get another convergent sequence, which is (s_n).

Exercise 2.7.2

(a) The hints in the text are already a lot.

If the (b_n) series converges, then for any $\epsilon > 0$, we know $\exists N : n > m \geq N$, such that

$$\epsilon > \left| \sum_{i=m+1}^{n} b_i \right| > \left| \sum_{i=m+1}^{n} a_i \right|$$

so this N works for (a_n) series too.

If the (a_n) series diverges, then we can AFSOC (b_n) series converges, and use what we proved above to show by contradiction that (a_n) series converges.

(b) (a_n) series is increasing and bounded by (b_n) series, so it must converge. For (a_n) series diverging, We can do a similar AFSOC argument in part (a), where we can AFSOC (b_n) converges, which then we can show by contradiction that (a_n) series is converging.

Exercise 2.7.3

(a) If $\sum a_n$ diverges, then AFSOC $\sum p_n$ and $\sum q_n$ converge. Then $\sum p_n + \sum q_n = \sum a_n$ converges, but this is a contradiction.

(b) If $\sum a_n$ converges conditionally, WLOG AFSOC $\sum p_n$ converges. Then $\sum a_n - \sum p_n = \sum q_n$ must converge as well. $\left|\sum q_n\right|$ will also converge, since $\sum q_n$ does, and this equals $\sum |q_n|$. Then,

$$\sum |a_n| = \sum |p_n| + \sum |q_n|$$

which we know converges since $\sum |p_n| = \sum p_n$ and we just showed $\sum |q_n|$ converges. This is a contradiction since we assumed $\sum a_n$ converges conditionally.

Exercise 2.7.4

Define

$$x_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases} \qquad y_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Then $\sum x_n, \sum y_n$ both diverge, but $\sum x_n y_n = 0$.

Exercise 2.7.5

(a) If $\sum a_n$ converges absolutely, then $\sum |a_n|$ converges to some L, so

$$L^2 = \left(\sum |a_n|\right)^2 = \sum |a_n|^2 + S$$
$$L^2 \leq \sum |a_n|^2 = \sum a_n^2$$

Since $\sum_{n=1}^k a_n^2$ is an increasing sequence and is bounded, we conclude $\sum a_n^2 = \sum \left|a_n^2\right|$ converges.

This proposition does not hold without absolute convergence. Take $a_n = (-1)^n \frac{1}{\sqrt{n}}$, which converges by the alternating series test. Then $a_n^2 = \frac{1}{n}$, which is the harmonic series, and we know this does not converge.

(b) No, take $a_n = \frac{1}{n^2}$, which converges. Then $\sum \sqrt{a_n}$ is the harmonic series, which diverges.

Exercise 2.7.6

(a) Call M the bound of y_n. If $\sum x_n$ converges absolutely to L, then

$$\begin{aligned}\sum x_n y_n &\leq \sum |x_n y_n| \\ &\leq \sum |x_n||y_n| \\ &\leq \sum |x_n| M \\ &\leq LM\end{aligned}$$

$\sum |x_n y_n|$ converges by the Monotone Convergence Theorem because the partial sums are increasing and it is bounded above. Then, by the Absolute Convergence Test we can conclude $\sum x_n y_n$ also converges.

(b) Let $x_n = \frac{(-1)^n}{n}$ be the alternating harmonic series, and $y_n = (-1)^n$. Then $\sum x_n$ converges but $\sum x_n y_n$ is the harmonic series, which does not converge.

Exercise 2.7.7

We are going to bound our p-series with another series, and show that the other series converges.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \cdots \\ &\leq 1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{8^p} + \cdots \\ &= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots \\ &= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots \\ &= \frac{1}{1 - \frac{1}{2^{p-1}}} = \frac{2^p}{2^p - 2} \qquad \text{(Only if } p > 1\text{)}\end{aligned}$$

By the Monotone Convergence Theorem, since the partial sums of the p-series is increasing, and there is an upper bound, we conclude that the p-series converges.

Notice that the convergence of p-series is often proved with calculus, but this is a nice alternative.

Exercise 2.7.8

Informally, you use the fact that both partial sums s_n^a, s_n^b will converge, then use the N_a, N_b and choose $N = \max(N_a, N_b)$, so that both partial sums are $\epsilon/2$ close to A, B. Then, triangle inequality to bound s_n^{a+b}, which will be $< 2 \cdot \epsilon/2 = \epsilon$.

Exercise 2.7.9

(a) This r' exists, since $a = \frac{r+1}{2}, r < a < 1$. We know

$$\begin{aligned} \left| \left| \frac{a_{n+1}}{a_n} \right| - r \right| &< \epsilon \\ \left| \frac{a_{n+1}}{a_n} \right| &< r + \epsilon && \text{(Also } > r - \epsilon\text{, but we don't need it)} \\ |a_{n+1}| &< |a_n|(r + \epsilon) \end{aligned}$$

We can choose N large enough so that $\epsilon < 1 - r$.

(b) Since $|r'| < 1$, we know $\sum r'^n$ converges, so $|a_N|$ times that also converges.

(c) We can bound the leading terms up to $n < N$ of $\sum |a_n|$ by some M. For the tail end, we can bound it from part (b). Since there exists an upper bound for this series, and its partial sums are increasing, we conclude that the partial sums converge and the overall sum does too.

Exercise 2.7.10

These are not proved very rigorously, but outline most of the ideas.

(a) If $\lim(na_n) = l$, then for any $\epsilon > 0, \exists N : n \geq N$ such that

$$\begin{aligned} |na_n - l| &< \epsilon \\ |a_n - l/n| &< \frac{\epsilon}{n} < \epsilon, \end{aligned}$$

so $\lim a_n = l/n$. Then since $l \neq 0$, $\sum a_n$ converges to a multiple of the harmonic series, which diverges, so $\sum a_n$ will too.

(b) Similar to the proof above in part (a), we can show $\lim a_n = l/n^2$. This converges to a multiple of $1/n^2$, which converges, so $\sum a_n$ also converges.

Exercise 2.7.11

An easy example,

$$\begin{aligned} (a_n) &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \\ (b_n) &= 2 - \frac{1}{2} + 2 - \frac{1}{4} + 2 - \frac{1}{6} + \cdots \end{aligned}$$

Then their $\sum \min\{a_n, b_n\}$ is the alternating harmonic series, and

- (a_n) diverges because it is the Harmonic series
- (b_n) diverges because every pair sums to > 1, so it sums an infinite number of numbers > 1.

For the challenge, an idea is to interweave a convergent and a divergent sequence together, so that (a_n) and (b_n) will both be divergent, since they both contain the divergent sequence, but the min only selects elements from the convergent sequence.

Exercise 2.7.12
Just verifying an identity,

$$
\begin{aligned}
s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j(y_j - y_{j+1}) &= s_n y_{n+1} - s_m y_{m+1} + (x_1 + \cdots x_{m+1})(y_{m+1} - y_{m+2}) + \\
&\quad (x_1 + \cdots x_{m+2})(y_{m+2} - y_{m+3}) + \cdots + (x_1 + \cdots x_n)(y_n - y_{n+1}) \\
&= s_n y_{n+1} - s_m y_{m+1} + (x_1 + \cdots x_{m+1}) y_{m+1} - \\
&\quad (x_1 + \cdots x_n) y_{n+1} + \sum_{j=m+2}^{n} x_j y_j \\
&= s_n y_{n+1} - s_n y_{n+1} - s_m y_{m+1} + s_{m+1} y_{m+1} + \sum_{j=m+2}^{n} x_j y_j \\
&= x_{m+1} y_{m+1} + \sum_{j=m+2}^{n} x_j y_j \\
&= \sum_{j=m+1}^{n} x_j y_j
\end{aligned}
$$

Exercise 2.7.13
(a) Using Exercise 2.7.12, we have

$$
\begin{aligned}
\left| \sum_{j=m+1}^{n} x_j y_j \right| &= \left| s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \right| \\
&= |s_n y_{n+1} - s_m y_{m+1}| + \left| \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \right| && (\triangle \text{ inequality}) \\
&= \big|(s_n - s_m) y_{m+1}\big| + \left| \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}) \right| && (y_{m+1} > y_{n+1}) \\
&\le M|y_{m+1}| + M|y_{m+1} - y_{n+1}| \\
&\le 2M|y_{m+1}|
\end{aligned}
$$

(b) For any $\epsilon > 0$, since (y_n) converges, make $|y_{m+1}| < \epsilon/(3M)$. Then

$$\begin{aligned}
\left|\sum_{j=m+1}^{\infty} x_j y_j\right| &\le \left|\sum_{j=m+1}^{n} x_j y_j\right| + \left|\sum_{j=n+1}^{\infty} x_j y_j\right| \\
&\le 2M|y_{m+1}| + \left|\sum_{j=n+1}^{\infty} x_j y_j\right| && \text{(From part (a))} \\
&\le 2M|y_{m+1}| + \left|\sum_{j=n+1}^{\infty} x_j y_{m+1}\right| && \text{(Since for } n \ge m+1, y_{m+1} \ge y_n) \\
&\le 2M|y_{m+1}| + |y_{m+1}|\left|\sum_{j=n+1}^{\infty} x_j\right| \\
&\le 2M|y_{m+1}| + |y_{m+1}|M && \text{(Partial sums of } (x_n) \text{ bounded by } M) \\
&\le 3M|y_{m+1}| \\
&\le 3M\frac{\epsilon}{3M} = \epsilon
\end{aligned}$$

(c) We have $x_n = (-1)^{n+1}, y_n = a_n$.

Exercise 2.7.14

(a) Abel's test requires that $\sum x_n$ converges, which is stronger than the boundedness of the partial sums of (x_n). However, it only needs that (y_n) is non-negative and decreasing, which is weaker than Dirichlet, which in addition needs the limit to converge to 0.

(b) Using Exercise 2.7.13, part (a), we have

$$\left|\sum_{j=1}^{n} a_j b_j\right| \le 2A|b_1|$$

(c) We can define $a_n = x_{m+n}, b_n = y_{m+n}$, and bound

$$\left|\sum_{j=m+1}^{n} x_j y_j\right| = \left|\sum_{j=1}^{n} a_j b_j\right| \le 2A|b_1|.$$

Now, we want to show we can make this bound arbitrarily small, since if we can make the tail end of this series arbitrarily small, then by the Cauchy Criterion for series we can conclude $\sum x_n y_n$ converges. We know that $\sum x_n$ converges, so by the Cauchy Criterion, for any $\epsilon > 0$, we can find some $N : \forall n' > m' \ge N$ such that

$$\left|\sum_{j=m'}^{n'} x_j\right| < \epsilon$$

Now, all we have to do is choose N so that $\left|\sum_{j=m'}^{n'} x_j\right| < \epsilon/(2b_1)$ then $A < \epsilon/(2b_1)$ and $\left|\sum_{j=m+1}^{n} x_j y_j\right| < 2|b_1| \cdot \frac{\epsilon}{2|b_1|}$, which means we have the Cauchy Criterion for $\sum x_n y_n$, and therefore it converges.

2.8 Double Summations and Products of Infinite Series

Exercise 2.8.1

$$\begin{aligned}\lim s_{nn} &= -1 + -\frac{1}{2} - \frac{1}{4} - \cdots \\ &= -\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i \\ &= -2.\end{aligned}$$

The value is equal to summing column-wise.

Exercise 2.8.2
By the Absolute Convergence test, since we know for fixed i that $\sum_{j=1}^{\infty}\left|a_{ij}\right|$ converges, then we know for fixed i that each $\sum_{j=1}^{\infty} a_{ij}$ converges to some c_i as well.

Then, since

$$\begin{aligned}\sum_{j=1}^{\infty}\left|a_{ij}\right| &\geq \left|\sum_{j=1}^{\infty} a_{ij}\right| \\ \Rightarrow b_i &\geq |c_i| \\ |b_i| &\geq |c_i|,\end{aligned}$$

and we know that $\sum_{i=1}^{\infty} b_i$ converges, we conclude that $\sum_{i=1}^{\infty} c_i$ must converge as well by the Absolute Convergence test, implying that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \tag{2.18}$$

converges as well.

Exercise 2.8.3

(a) Since $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right|$ converges, we have that

$$t_{mn} \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right| = L$$

Since t_{nn} is an increasing sequence, and is bounded above, by the Monotone Convergence Theorem, t_{nn} converges.

(b) For any $\epsilon > 0$, $\exists N : n > m \geq N$ such that $|t_{nn} - t_{mm}| < \epsilon$. Now, consider

$$\begin{aligned}\left|s_{n+1,n+1} - s_{nn}\right| &\leq \left|t_{n+1,n+1} - t_{nn}\right| \\ &< \epsilon.\end{aligned}$$

So (s_{nn}) is a Cauchy Sequence and converges.

Exercise 2.8.4

(a) Since we know there exists a $t_{m_0 n_0}$ such that $t_{n_0 n_0} > B - \frac{\epsilon}{2}$, and t_{nn} is increasing and that B is an upper bound, we can conclude that for $N_1 = \max\{m_0, n_0\} : m, n \geq N_1$,

$$B - \frac{\epsilon}{2} < t_{mn} \leq B. \tag{2.19}$$

(b) For any $\epsilon > 0$, since (t_{mn}) is bounded above by $A = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right|$, from part (a) we can choose $N : m, n \geq N$ such that

$$\begin{aligned} A + \frac{\epsilon}{2} &< t_{mn} < A + \epsilon \\ \Rightarrow \frac{\epsilon}{2} &< |t_{mn} - A| < \epsilon \\ &\left|t_{mn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right|\right| < \epsilon. \end{aligned}$$

Then, we can see that this N also works to show

$$\begin{aligned} |s_{mn} - S| &= \left|s_{mn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}\right| \\ &= \left|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} a_{ij}\right| \\ &< \left|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} \left|a_{ij}\right|\right| \\ &= \left|t_{mn} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|a_{ij}\right|\right| \\ &< \epsilon. \end{aligned}$$

Exercise 2.8.5
We know $\lim_{n\to\infty}\sum_{j=1}^{n} a_{ij} = r_i$, so for any $\epsilon > 0$, $\exists N : n \geq N$ such that

$$\left|\sum_{j=1}^{n} a_{ij} - r_i\right| < \frac{\epsilon}{m}.$$

if we fix $m \geq N$.
Then

$$\begin{aligned} \left|(r_1 + r_2 + \cdots + r_m) - S\right| &= \left|\sum_{i=1}^{m}\left(r_i - \sum_{j=1}^{n} a_{ij}\right)\right| \\ &\leq \sum_{i=1}^{m}\left|r_i - \sum_{j=1}^{n} a_{ij}\right| \\ &\leq m \cdot \frac{\epsilon}{m} = \epsilon \end{aligned}$$

Therefore, we conclude that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ converges to S.
TODO not sure where I have to use the Order Limit Theorem...

Exercise 2.8.6
For $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$, the proof is essentially the same as Exercise 2.8.5 to show it converges to S, except we fix n this time instead of m.

Exercise 2.8.7
(a) Define $t_{nn} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|$. Also define $u_n = \sum_{k=2}^{n}|d_k|$ Then we know for $n \geq 2$,

$$u_n \leq t_{nn} = L \qquad (t_{nn} \text{ converges})$$

Since u_n is an increasing sequence and is bounded above, we conclude from the Monotone Convergence Theorem that $u_n = \sum_{k=2}^{n} |d_k|$ also converges. Then, $\sum_{k=2}^{n} d_k$ converges absolutely.

(b) We need to bound $|d_k - S|$ somehow. Consider the following diagram

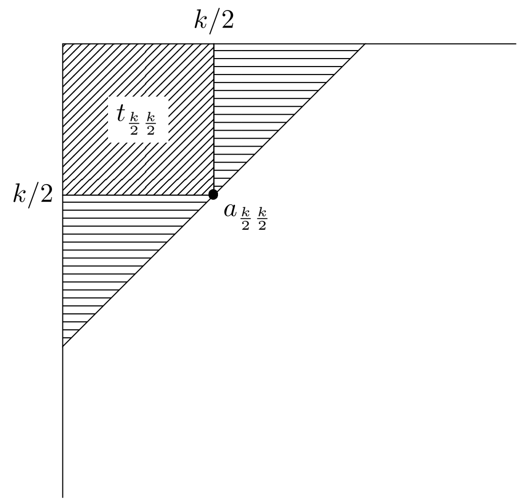


Figure 2.1: Demonstrating how we bound our sum

We know that for any $\epsilon > 0$, we can choose $N : n \geq N$ so that

$$|t_{nn} - S| < \epsilon \tag{2.20}$$

Now, choose $N_1 = 2N$. We can use Figure 2.1 to see that for $k \geq N_1$,

$$\begin{aligned} |d_{kk} - S| &\leq \left| t_{\frac{k}{2}\frac{k}{2}} - S \right| \\ &< \epsilon && \text{(From Equation (2.20))} \end{aligned}$$

Therefore, we can conclude $\sum_{k=2}^{\infty} d_k$ converges to S.

Exercise 2.8.8

(a) See that

$$\begin{aligned} AB &\geq \left(\sum_{i=1}^{\infty} |a_i| \right) \left(\sum_{j=1}^{\infty} |b_j| \right) \\ &= \sum_{i=1}^{\infty} \left(|a_i| \sum_{j=1}^{\infty} |b_j| \right) && \left(\textstyle\sum_{j=1}^{\infty} |b_j| \text{ is constant wrt } i\right) \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_i||b_j| \right) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j| \end{aligned}$$

Since $\sum_{i=1}^{m} \sum_{j=1}^{n} |a_i b_j|$ is bounded, and the partial sums $s'_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_i b_j|$ are increasing, we can conclude $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_i b_j|$ converges by the Monotone Convergence Theorem.

(b) Let s_n^a, s_n^b be the partial sums of $(a_n), (b_n)$ respectively. Then,

$$\begin{aligned}
\lim_{n\to\infty} s_{nn} &= \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a_i \sum_{j=1}^{n} b_j \right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i \right) \left(\sum_{j=1}^{n} b_j \right) \\
&= \lim_{n\to\infty} s_n^a s_n^b \\
&= AB
\end{aligned}$$

Therefore, by Theorem 2.8.1, we can conclude that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_i b_j = \sum_{i=2}^{\infty} d_k = AB$$

Chapter 3

Basic Topology of $\mathbb{R}$

3.2 Open and Closed Sets

Exercise 3.2.1

(a) We need a finite number of sets when we are choosing the minimum ϵ for our V_ϵ. If we had an infinite number of sets, this minimum may not exist.

(b) Let

$$O_n = \left(\sum_{i=1}^{n} \frac{1}{2^i}, 3 - \sum_{i=1}^{n} \frac{1}{2^i} \right)$$

Then $\bigcap_{n=1}^{\infty} O_n = [1, 2]$.

Exercise 3.2.2

(a) 1 and -1 are the only limit points of B. For any fixed element of B, the distance between it and its neighbors is $\geq \frac{n}{n+1} - \frac{n+2}{n+3} = \frac{n}{(n+1)(n+3)}$, so we can just choose ϵ smaller than this, and show that any element of B is isolated.

For 1, we can show that for any $\epsilon > 0$, we can choose $\frac{1}{N+1} < \epsilon \Rightarrow N > \frac{1}{\epsilon} - 1$, and we know $\frac{n}{n+1}$ for $n \geq N$ for even n is in the ϵ-neighborhood of 1. Doing a similar analysis for negative terms and -1 yields the same result.

(b) B does not contain its limit points, so it is not closed.

(c) B is not an open set, continuous ϵ neighborhoods are not subsets of B.

(d) All of B's elements are isolated

(e) $\overline{B} = B \cup \{-1, 1\}$.

Exercise 3.2.3

(a) $\mathbb{Q}$ is not open, because it doesn't have irrationals that can be in the ϵ-neighborhoods. It is not closed, because it contains irrational limit points. Therefore it is **neither**.

(b) $\mathbb{N}$ does not have any limit points, so it is **closed**.

(c) $\mathbb{R}^+$ cannot be closed, because 0 is a limit point and not contained. It is **open** because every element has an ϵ-neighborhood that is a subset.

(d) Not closed, doesn't contain 0, a limit point. Not open, since 1 has no ϵ-neighborhood subset. Therefore, **neither**.

(e) The sequence converges, but this limit point is not in the set. No ϵ-neighborhoods exist for certain ϵ, for certain elements, so not open. **Neither**.

Exercise 3.2.4
FOr any $\epsilon > 0$, we know $\exists N : n \geq N$ such that

$$|a_n - x| < \epsilon,$$

so every ϵ-neighborhood of x has points other than itself.

Exercise 3.2.5
If there exists exists such an ϵ-neighborhood, then by the definition of a limit point, since there are no other elements other than x itself, then this is not a limit point, so it is isolated.

Exercise 3.2.6
If a set $F \subseteq \mathbb{R}$ is closed, then it contains all its limit points. For any Cauchy Sequence in F, it is also convergent to some L, which we know is a limit point and thus must be in F.

If every Cauchy Sequence of an F has its limit as an element of F, then every limit point, which comes from the limit of some subsequence, which we know is a Cauchy Sequence. From our original assumption, this limit must be in F, so F contains all its limit points and is closed.

Exercise 3.2.7
AFSOC an infinite number of (x_n) terms not in O. Then since $(x_n) \to x$, $\epsilon =$ distance of x from O boundary, then $\forall N, : n \geq N$ we have that $\exists x_n : |x_n - x| \geq \epsilon$, since we can choose some x_n not in O. This means this sequence does not converge. This contradicts our original assumption.

Exercise 3.2.8

(a) We want to show that L, which contains all the limit points of of A, is closed. We can do this by showing all limit points of L are in L.

Suppose we have some limit point ℓ of L, then this means some subsequence of L,

$$(l_n) \to \ell$$

By the definition of convergence, for any $\epsilon > 0$, we can find $N : n \geq N$ such that

$$|l_n - \ell| < \frac{\epsilon}{2}$$

Now, since l_n are limit points of A, we know $\exists a \in A$ such that a is arbitrarily close to l_n. Define a subsequence in A

$$\left\{ a_n \in A, |a_n - l_n| < \frac{\epsilon}{2} \right\}$$

Then for $n \geq N$,

$$|a_n - \ell| < |l_n - \ell| + \frac{\epsilon}{2} < 2 \cdot \frac{\epsilon}{2} = \epsilon,$$

which means (a_n) converges to this ℓ as well, so ℓ is a limit point of A and $\ell \in L$.

Therefore, we conclude L contains all of its limit points, and therefore it is closed.

(b) For any limit point ℓ of $A \cup L$, it must the limit of some convergent subsequence of $A \cup L$. This subsequence will contain elements from A and L. What we can do is for every element in L, use a similar technique we did in part (a) to replace all the $x \in L$ subsequence elements with elements in A instead, that are arbitrarily close enough. Then, we have constructed a subsequence that entirely lies in A, so this limit point must be of A. Therefore, all limit points of $A \cup L$ are limit points of A.

We can then conclude that $\overline{A} = A \cup L$ is a closed set, since all of its limit points are of A, and those limit points are contained in L, which means $\overline{A}$ contains all of its limit points and is closed.

Exercise 3.2.9

(a) Suppose y is a limit point of $A \cup B$, then there must exist a subsequence $(x_n), x_n \in A \cup B$ where $(x_n) \to y$. Now, this subsequence must contain either an infinite number of elements from A or B (or both).

WLOG, (x_n) contains an infinite number of elements from A, then we know $\exists$ subsequence $(x'_n) \to y$ where $x'_n \in A$. This means y is a limit point of A.

Therefore, we conclude if y is a limit point of $A \cup B$, y is either a limit point of A or B.

(b) Let L_S be the set of limit points for a set S.

$$\begin{aligned}\overline{A \cup B} &= A \cup B \cup L_{A\cup B} && && (3.1)\\ &= A \cup B \cup (L_A \cup L_B) && = (A \cup L_A) \cup (B \cup L_B) && (3.2)\\ &= \overline{A} \cup \overline{B} && && (3.3)\end{aligned}$$

(c) We notice that in our proof, we were able to find a subsequence that was entirely in one set. Therefore, if it is possible to construct a subsequence that doesn't fit entirely in one set, then we can find a limit point that is not necessarily a limit point of an individual set.

With an infinite number of sets, we can take advantage of this property.

Suppose we have some $(a_n) \to L$, where $\forall n\, a_n \neq L$. Then construct sets the following way,

$$S_n = \{a_n\}$$

Now, S_n has no limit points, since it only has a single point which is isolated. Therefore, $\bigcup_{n=1}^{\infty} \overline{S_n} = \bigcup_{n=1}^{\infty} S_n$, but we have $\overline{\bigcup_{n=1}^{\infty} S_n} = \left(\bigcup_{n=1}^{\infty} S_n\right) \cup \{L\}$, and $L \notin \bigcup_{n=1}^{\infty} S_n$. So the property in part (b) does not apply for infinite sets.

Exercise 3.2.10

(a) A direct proof (double containment is another way to do it)

$$\begin{aligned} x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c &\Leftrightarrow \forall \lambda, x \notin E_\lambda \\ &\Leftrightarrow \forall \lambda, x \in E_\lambda^c \\ &\Leftrightarrow x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c \end{aligned}$$

$$\begin{aligned} x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c &\Leftrightarrow \exists \lambda, x \notin E_\lambda \\ &\Leftrightarrow \exists \lambda, x \in E_\lambda^c \\ &\Leftrightarrow x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c \end{aligned}$$

(b) We want to show that

(i) *The union of a finite collection of closed sets is closed.* Suppose we have a collection of closed sets $\{E_\lambda, \lambda \in \Lambda\}$, then, if we take the complement of the union of all these sets, by DeMorgan's, we get the intersection of the complements of all these sets. The complements of all these sets is open, and we know the intersection of a finite number of open sets is also open. Finally, taking the complement again, we must have a closed set, which is equal to our original union.

(ii) *The intersection of an arbitrary collection of closed sets is closed.* Take the intersection of these closed sets, and then take the complement. By DeMorgan's, we know have the union of the complement of these sets, which we know is open. We know that the union of an arbitrary number of open sets is also open. Finally, taking the complement of this entire expression again, we now have a closed set, which is equal to our original intersection.

Exercise 3.2.11

If $s = \sup A$ exists, we have 2 cases. Either $s \in A \Rightarrow s \in \overline{A}$ since $A \subseteq \overline{A}$, or, $s \notin A$. In the second case, since we know for any $\epsilon > 0$, $\exists a \in A$ such that $a > s - \epsilon \Rightarrow \epsilon > |s - a|$, we can construct a subsequence in A that converges to s. This means s is a limit point of A, and therefore $s \in \overline{A}$.

Exercise 3.2.12

(a) True. $\overline{A}$ is closed, so $\overline{A}^c$ must be open.

(b) True. There is no ϵ-neighborhood around this point that is contained in A.

(c) False. Take the harmonic sequence $\{1/n\}$.

(d) True. See Exercise 3.2.11

(e) True. A finite set only contains isolated points, so therefore it has no limit points, and vacuously contains all of its limit points and is closed.

(f) True. We know that around $q \in \mathbb{Q}$, exists some ϵ-neighborhood around it that is contained in the set. Suppose we have an arbitrary $r \in \mathbb{R}$, then we want to show it is contained in the ϵ-neighborhood of some $q \in \mathbb{Q}$. We can show this by contradiction. AFSOC r is not in any of these ϵ-neighborhoods. That means $\forall \epsilon > 0, |r - q| > \epsilon$, for all q. But for any ϵ, we can always find some rational number that is closer than ϵ to r, which means this statement is false. We have reached a contradiction, and must assume our original hypothesis was true.

Exercise 3.2.13

We can verify $\mathbb{R}$ is open because any ϵ-neighborhood only contains elements of $\mathbb{R}$, so therefore $\subseteq \mathbb{R}$. In addition, any limit point $\in \mathbb{R}$, so $\mathbb{R}$ also contains all of its limit points and is closed.

$\emptyset$ is closed and open by vacuity.

Now, we need to show that there are no other sets with this property. We know the complement of an open set is closed and vice versa, so AFSOC $\exists A \neq \mathbb{R}, \emptyset$, then we know $\exists x \in A^c, \notin A$. Now, x cannot be an isolated point, since then it would not have an ϵ-neighborhood around it that is contained in A^c. Therefore, we conclude x must be in some continuous set S, where it either

(i) Has a $\sup S$. In this case, either $\sup S \in S$, in which case there does not exist an ϵ-neighborhood around $\sup S$, which means S is not open, or $\sup S \notin S$, and then $\sup S$ is a limit point, but then S is not closed since it doesn't contain all of its limit points. This case is not possible.

(ii) Does not have an upper bound. Then look at the portion less than x and apply the argument in part (i) but with $\inf S$. It must have a lower bound, or else $A = \mathbb{R}$.

Therefore, we reach a contradiction in all cases, and therefore we conclude that it is not possible for this A to exist.

Exercise 3.2.14

(a) $[a, b] = \bigcap_{i=1}^{n} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$. Any $x \in [a, b]$ will be $< b + \frac{1}{n}$, and $> a - \frac{1}{n}$. Now, let us consider some $y < a$. $y \notin$ the set we created, bc suppose $|y - a| = \epsilon$. Then for $n' > \frac{1}{\epsilon}$, $a - \frac{1}{n'} > y$, so y is not in this set. The argument for an element larger than b is symmetric. Therefore, we conclude the set we constructed is equivalent to $\{a, b\}$, and is an intersection of a countable number of open sets.

(b) We can write

$$(a, b] = \bigcup_{i=1}^{n} \left[a + \frac{1}{n}, b\right] \tag{3.4}$$

$$(a, b] = \bigcap_{i=1}^{n} \left(a, b + \frac{1}{n}\right) \tag{3.5}$$

(c) We know $\mathbb{Q}$ is countable, so just union all the sets containing only one element of $\mathbb{Q}$ together. Since each set has one element which is an isolated point, each set is closed.

$$\bigcup_{q \in \mathbb{Q}} \{q\}$$

We know that $\mathbb{Q}^c = \mathbb{I}$, and by DeMorgan's law, we know

$$\left(\bigcup_{i=1}^{\infty} S\right)^c = \bigcap_{i=1}^{\infty} S^c$$

Since S are all closed, S^c are all open. We can use the infinitely countable union of the construction of $\mathbb{Q}$ and then take the complement to get $\mathbb{I}$, which by DeMorgan's is constructed as a countably infinite intersection of open sets.

3.3 Compact Sets

Exercise 3.3.1
Since we know K is compact, it must also be closed and bounded.

Since K is bounded, it must have a least upper and largest lower bound, by the Axiom of Completeness, which means $\sup K$ and $\inf K$ must exist. Now, we can construct subsequences of K that converge to $\sup K, \inf K$, since by the property of $\sup K$, for example, we can always find an element of K that is some ϵ close to it. So for example we can construct a subsequence where $a_n = k_n, k_n \in K$ such that $|\sup K - k| < \frac{1}{2^n}$. The same logic applies to $\inf K$, so they are limit points of K. Since K is closed, $\sup K, \inf K \in K$.

Exercise 3.3.2
Suppose we have some $K \subseteq \mathbb{R}$ that is closed and bounded. We want to show that it is compact.

We know that any sequence of K must be contained in K, which is bounded, so therefore by the Bolzano-Weierstrauss Theorem, we know that this sequence must have a convergent subsequence. Since K is also closed, this limit must be in K as well.

This shows that K is compact.

Exercise 3.3.3
We want to show the Cantor set is compact.

We know the Cantor set is $\subseteq [0,1]$, so it is bounded.

Then, we know the complement of the Cantor set is

$$(-\infty, 0) \cup (1, \infty) \cup \left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \cdots\right]$$

which is the union of an arbitrary number of open sets, which we know is open. Therefore, the Cantor set is the complement of an open set, which is closed.

Therefore, since the Cantor set is bounded and closed, we conclude it is compact.

Exercise 3.3.4
We have that K is compact and F is closed. Since K is compact, it is also bounded and closed.

If we take $K \cap F$, we know that this must also be bounded, since $x \in K \cap F \Rightarrow x \in K$.

The intersection of two closed sets is also closed, so $K \cap F$ is closed.

Therefore, $K \cap F$ is bounded and closed, and thus is compact.

Exercise 3.3.5

(a) We can find a sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$, but we know that $\sqrt{2} \notin \mathbb{Q}$, so $\mathbb{Q}$ is not compact.

(b) Again, similar to part (a), we can find a sequence that converges to $\sqrt{2}/2 \notin [0,1] \cap \mathbb{Q}$.

(c) Take the sequence $a_n = n$. There is no limit, so this sequence does not have a subsequence that converges.

(d) $\mathbb{R} \cap [0,1] = [0,1]$ is closed and bounded, so it is compact.

(e) This sequence converges to 0, but does not contain 0, so it is not closed and thus not compact.

(f) Every subsequence of this set converges to 1, which is in the set, so therefore this set is compact.

Exercise 3.3.6

(a) We will prove this by induction.

- **Base Case:** $n = 1$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ We know a combination of two elements in $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ covers $\left[0, \frac{4}{3}\right]$. Then, combination of two elements in the latter set covers $\left[\frac{4}{3}, 2\right]$. Therefore, two elements $x, y \in C_1$ can add up to any element $\in [0,2]$.
- **Inductive Hypothesis:** Suppose for $k \geq 1$, any two elements of C_k can add up to any element $\in [0,2]$.

- **Inductive Step** We know $C_{k+1} = C_k/3 + \left\{\frac{2}{3} + C_k/3\right\}$, in other words, we are now missing the middle thirds of both the head and tail sets. What we want to show is that for the head set (and the same argument holds for the tail set), that with the middle third removed, we can still cover all of the original set, which means since the original sets are still covered, we can still cover $[0, 2]$. By the IH, any two elements of $C_k/3$ will cover $\left[0, \frac{2}{3}\right]$, and any two elements of $\left\{\frac{2}{3} + C_k/3\right\}$ covers $\left[\frac{4}{3}, 2\right]$. Choosing two elements from one of each set covers $\left[\frac{2}{3}, \frac{4}{3}\right]$. These three intervals cover $[0, 2]$.

(b) The reason (x_n), (y_n) may not converge is they can be picked out of sets, jumping across different subsets of C infinitely many times.

However, by the Bolzano-Weierstrauss Theorem, since (x_n) is contained entirely in the Cantor set, which is bounded, then (x_n) is also bounded. Therefore, it must contain a convergent subsequence. The same applies for (y_n), and we can take their limits l_x, l_y such that $l_x + l_y = s$.

Exercise 3.3.7

(a) True. The intersection will be bounded, since we can take any bound of a set, which will bound the intersection, and the intersection of an arbitrary number of closed sets is still closed. Therefore, this arbitrary intersection of compact sets is closed and bounded, which means it is also compact.

(b) False. Let $A = (0, 1), K = [0, 1]$, then $A \cap K = (0, 1)$, which is not closed so it is not compact.

(c) True. This is the Nested Interval Property.

(d) True. A finite set always closed, since there are no limit points, and bounded.

(e) False. Choose an unbounded countable set like $a_n = n$.

Exercise 3.3.8

(a) If they both have finite subcovers, then we can union those two finite subcovers to get a finite subcover for K, so we need at least one of them to not have a finite subcover.

(b) Create I_{n+1} by bisecting I_n, and taking a half that has no finite subcover. Such a half has to exist, because if both have finite subcovers, then the whole must have a finite subcover. The interval will half in size every iteration.

(c) By the Nested Interval Property, $\exists x \in K \forall x \in I_n$.

(d) Since the interval sizes get arbitrarily small, we can find some n_0 such that I_{n_0} fits entirely in O_{λ_0}. We have reached a contradiction, because we can finitely cover $I_{n_0} \cap K$ by using O_{λ_0}.

Exercise 3.3.9

(a) Open cover where you take some $\epsilon > 0$ around all rational numbers, and then union them together. No finite subcover exists, since that would bound $\mathbb{Q}$, which is not bounded.

(b) We can construct an open cover like

$$\left(\bigcup_{i=1}^{\infty}(-0.5, \frac{\sqrt{2}}{2} - \frac{1}{i})\right) \cup \left(\bigcup_{i=1}^{\infty}(\frac{\sqrt{2}}{2} + \frac{1}{i}, 1.1)\right)$$

This will contain all the rational numbers between $[0, 1]$, but needs an infinite number of sets to cover the entire set, because if we stop prematurely, we won't be able to capture the rationals that are very close to $\frac{\sqrt{2}}{2}$.

(e) An open cover for this set is

$$\bigcup_{i=1}^{\infty}\left(1.1, 1 - \frac{i-1}{i}\right)$$

We need all of these open sets, because otherwise if for some N we stop, then we won't have the elements $< \frac{1}{N}$.

Exercise 3.3.10

For any closed set with an interval $[a, b]$, we can make intervals $I_n = \left[a, b - 1/n\right]$, and then union with $[b, b+1]$ to cover the set; with open intervals, we don't need the end. However, we will need all of the intervals, otherwise we won't include all the elements close to b. Therefore, any *clompact* subset must be a finite set of isolated points.

3.4 Perfect Sets and Connected Sets

Exercise 3.4.1

A perfect and a compact set are always closed, so their intersection is also closed. Since a compact set is bounded, the intersection of it and another set must also be bounded. Therefore, $P \cap K$ is a compact set.

We cannot guarantee that $P \cap K$ does not have isolated points, so for example, $[0, 1] \cap \{1/2\} = \{1/2\}$ which is not perfect.

Exercise 3.4.2

A perfect set cannot only consist of rationals, because it would be nonempty and countable, since $\mathbb{Q}$ is countable, but this is impossible since any nonempty perfect set is uncountable.

Exercise 3.4.3

(a) $x_1 \in C_1$ implies that x_1 is in a closed interval of size $1/3$, so we can choose any element in this interval such that $x \neq x_1$, and we must have $|x - x_1| \leq 1/3$

(b) We can make the argument for any $x_n \in C_n$, since x_n must exist in a closed interval of size $\frac{1}{3}^n$, so we can any other element in this interval x, so that $|x - x_n| \leq \frac{1}{3}^n$. Now, we can construct a subsequence (x_n) such that $(x_n) \to x$, and this shows that there are no isolated points in C. We know the Cantor set is closed from earlier exercises.

Exercise 3.4.4

(a) This set is bounded, and is also closed since it is an intersection of an arbitrary collection of closed sets. Therefore, this construction is compact.

This set is also perfect, because we can use the same argument from Exercise (b) to show that there are no isolated points.

(b) We can compute the

- **Length**: We will compute the removed interval lengths,

$$\frac{1}{4} + 2 \cdot \frac{3}{32} + 4 \cdot \frac{27}{256} + \cdots = \frac{1/4}{1 - \frac{3}{2^2}} = \boxed{1}$$

So this Cantor-like set has length 0.

- **Dimension**: We have $3 - 3 \cdot \frac{1}{4} = \frac{9}{4}$, so solving

$$3^x = \frac{9}{4} \Rightarrow \boxed{0.738}$$

This is "larger" in dimension than the ternary Cantor set.

Exercise 3.4.5

If we have that $A \subseteq U, B \subseteq V$ such that U, V are disjoint open sets, then we know $A \cap B = \emptyset$. Therefore, if we want to show that they are separated, we just need to show that the limit points of A are disjoint from B, and vice versa.

AFSOC that a limit point of $A, \ell_A \in B$. Then this ℓ_A is also a limit point of U, since $A \subseteq U$. Now, this $\ell_A \in V$, since $B \subseteq V$. This means ℓ_A is ϵ far away from an element $\in U$, and since ℓ_A is also in the open set V, we must have that $\ell_A \in [v_1, v_2] \subseteq V$, where $v_1 < \ell_A < v_2$. Let $\epsilon = (\ell_A - v_1)/2$, then $\ell_A - \epsilon \in V$, and since ℓ_A is a limit point of U, we must also have that $\ell_A - \epsilon \in U$. However, this is a contradiction, because we just showed that $\ell_A - \epsilon \in U$ and $\in V$, which means $U \cap V \neq \emptyset$, and they are not disjoint. The same argument applies for the limit points of B not being in A. Therefore, we can conclude that $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, and that A, B are separated.

Exercise 3.4.6

($\Rightarrow$) Suppose $E \subseteq \mathbb{R}$ is connected. Then consider some sets A, B such that $A \cup B = E$, and A, B are nonempty and disjoint. AFSOC every convergent sequence $(x_n) \to x$ with (x_n) contained in A or B, $x \notin$ the other set. Then this must mean that every limit point of A or B is not in the other set, which means $\overline{A} \cap B = \overline{B} \cap A = \emptyset$, and A, B are separated. This means $E = A \cup B$ for separated A, B, which is a contradiction since we assumed E was connected.

($\Leftarrow$) Suppose for all nonempty and disjoint A, B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A, or B, and x is an element of the other. Suppose (x_n) is contained within A. Then x must be a limit point of A, since $x \in B$, and A is disjoint of B so $x \notin A$. This means $\overline{A} \cap B = x \cup S \neq \emptyset$, which means A, B are not separated. Therefore, we cannot find separated sets A, B such that $E = A \cup B$, which means E is not disconnected, and therefore is connected.

Exercise 3.4.7

(a) Take $E = (-\infty, 1) \cup (1, \infty)$, then the closure is $\mathbb{R}$, which is closed, but this set is disconnected because $\big((-\infty, 1) \cup \{1\}\big) \cap (1, \infty) = \emptyset$ and $(-\infty, 1) \cap \big((1, \infty) \cup \{1\}\big) = \emptyset$ so these two sets are disconnected, and their union is equal to E.

(b) If A is connected, we can show that any limit points must already be in A, so $A = \overline{A}$ and $\overline{A}$ is still connected. If A is perfect, it is already closed, so it contains all of its limit points, and $\overline{A} = A$, and therefore $\overline{A}$ is still perfect.

Exercise 3.4.8

(a) Given any two rational x, y, WLOG $x < y$. Then $\exists r \in (x, y)$ such that $r \in \mathbb{I}$, i.e. it is not rational. Then we have $\mathbb{Q} = \big(\mathbb{Q} \cap (-\infty, r)\big) \cup \big(\mathbb{Q} \cap (r, \infty)\big)$, and these two sets are disconnected.

(b) Irrational numbers are also totally disconnected using the same argument in part ((a))

Exercise 3.4.9

(a) We know in C_n, it consists of intervals of size $\frac{1}{3}^n$. If the intervals are smaller than ϵ, i.e. $\frac{1}{3}^n < \epsilon$, then x, y must be in different intervals.

(b) If we know x, y are in different intervals, then between their intervals there must exist removed intervals in the construction of C. We can take z to be in one of these removed intervals. Given any $(a, b), a < b$, we have a few cases. If a or b is not in C, then (a, b) is not in C. If $a, b \in C$, then we can use the argument we just made to find some $z \in (a, b)$, such that $z \notin C$, which means $(a, b) \not\subseteq C$.

(c) For any $x, y \in C$, WLOG $x < y$, we can find $z \notin C, x < z < y$ such that $C = \big(C \cap (-\infty, z)\big) \cup \big(C \cap (z, \infty)\big)$. Therefore, C is totally disconnected.

Exercise 3.4.10

(a) O contains all the rational numbers (and some irrational numbers), so the complement O^c must only consist of irrational numbers.

(b) F only consists of closed intervals. F is totally disconnected, because for any $x, y \in \mathbb{I}, x < y$, we can find a rational number q where $q = r_n, \epsilon_n = 1/2^n < |x - y|/2$, so it fits in between x, y. Then we can make F with the union of two open sets intersected with F at that boundary.

(c) We know F is closed, since we are taking an arbitrary intersection of closed sets. F is not always perfect, since we can create isolated points, for example, by having open sets have end points that converge to some irrational number. The issue with our construction was that we would allow the ϵ neighborhoods to get arbitrarily close to irrational numbers, and sort of “squeeze” them into isolated points.

One trivial way to prevent this is to have some sort of minimum neighborhood size, but then $F = \emptyset$.

A less trivial, but vague way of construction, is to just get rid of the isolated irrational points. There can only be a countably infinite number of these, and there are uncountably many irrationals, so in this case $F \neq \emptyset$.

TODO Find a better construction for a perfect set of irrationals.

3.5 Baire's Theorem

Exercise 3.5.1
We can use DeMorgan's Law to show both directions, so that an countable union of closed sets becomes a countable intersection of open sets, and vice versa.

Exercise 3.5.2
(a) Countable, can use a similar proof to $\mathbb{N}^2$ countability to show it is still countable.

(b) Finite.

(c) Finite.

(d) Countable.

Exercise 3.5.3
Already done in Exercise 3.2.14.

Exercise 3.5.4
(a) Suppose $G_i = \left(g_i^1, g_i^2\right)$, then let $M = \left|g_i^2 - g_i^1\right|$, and define

$$I_i = \left[g_i^1 + M/4, g_i^2 - M/4\right]$$

Essentially we are making each G_i interval smaller so it can be closed, and we still have $I_i \subseteq G_i$.

(b) Now, we can use the Nested Interval Property to show that there exists an subsequence converging to some $x \in G_i \forall i$, so therefore this intersection is not empty.

Exercise 3.5.5
Suppose we could write $\mathbb{R}$ as a F_σ set, then we must have $\mathbb{R}^c = \bigcap_{n=1}^{\infty} G_n$, where G_n are dense, open sets. We just showed in Exercise 3.5.4 that this intersection is not empty, which is a contradiction, since $\mathbb{R}^c = \emptyset$.

Exercise 3.5.6
We know $\mathbb{Q}$ is an F_σ set, so if $\mathbb{I}$ were also an F_σ set, then that would imply $\mathbb{R}$ is also an F_σ set, which we proved in Exercise 3.5.5 is not.

Therefore, $\mathbb{I}$ is not an F_σ set, and its complement, $\mathbb{Q}$, cannot be a G_σ set.

Exercise 3.5.7
Take the construction of the Cantor set, except start with $(0, 1)$ and remove the closed $1/3$ interval in the middle, i.e. $[1/3, 2/3]$ in the start, each time.

We can show there are an uncountable number of sets by using a diagonalization argument.

Exercise 3.5.8
($\Rightarrow$) If E is nowhere-dense then $\overline{E}$ contains no nonempty open intervals. This means $\overline{E}^c$ consists only of open intervals, and is $\mathbb{R}$ without $\overline{E}$. We also can see that $\overline{\overline{E}^c}$ will be $\mathbb{R}$, since the limit points of $\overline{E^c}$ are just $\overline{E}$. Therefore, $\overline{E}^c$ is dense.

($\Leftarrow$) If $\overline{E}^c$ is dense in $\mathbb{R}$, then $\overline{E}^c \cup L = \mathbb{R}$. Taking the complement and applying DeMorgan's, we get $\overline{E} \cap L^c = \emptyset$. If $\overline{E}$ is non-empty, then it consists of the limit points of $\overline{E}^c$, i.e. $\overline{E} \subseteq L$.

Now, $\overline{E}$ cannot consist of any open intervals, because otherwise, suppose some limit point of $\overline{E}^c$ is in some interval (a, b). Then there is some distance M from the limit point l to a or b. There is no sequence in $\overline{E}^c$ that converges to this point, since it is too far away from any point in $\overline{E}^c$; (a, b) is not in $\overline{E}^c$ at all.

Therefore, since $\overline{E}$ has no nonempty open intervals, we conclude E is nowhere-dense.

Exercise 3.5.9
(a) In between.

(b) Nowhere-dense.

(c) Dense in $\mathbb{R}$.

(d) In between.

Exercise 3.5.10
If we can write $\mathbb{R}$ as countable union of nowhere-dense sets, these nowhere-dense sets have no nonempty open sets, which means they are closed.

Therefore, we can write $\mathbb{R}$ as a F_σ set, which we showed in Exercise 3.5.5 was not possible.

Chapter 4

Functional Limits and Continuity

4.2 Functional Limits

Exercise 4.2.1

(a) Let $\epsilon > 0$, then notice

$$\begin{aligned}\big|f(x) - 8\big| &= |2x + 4 - 8| \\ &= |2x - 4| \\ &= 2|x - 2|\end{aligned}$$

So we can choose $\delta = \epsilon/2$, which will imply $2|x - 2| < 2 \cdot \epsilon/2 = \epsilon$.

(b) Choose $\delta = \sqrt[3]{\epsilon}$

(c) We can simplify

$$\begin{aligned}\big|f(x) - 8\big| &= \Big|x^3 - 8\Big| \\ &= |x - 2|\Big|x^2 + 2x + 4\Big|\end{aligned}$$

Now, we can say $\delta = \min\big\{1, \epsilon/7\big\}$, which means $\big|x^2 + 2x + 4\big| \leq 7$. Continuing, we get

$$|x - 2|\Big|x^2 + 2x + 4\Big| < \frac{\epsilon}{7} \cdot 7 = \epsilon$$

(d) Let $\epsilon > 0$. Since $\pi = 3.14\ldots$, if we choose $\delta < 0.14$, then $\big[[x]\big] = 3$, which means

$$\Big|\big[[x]\big] - 3\Big| = |3 - 3| = 0 < \epsilon$$

Exercise 4.2.2

If we have a δ that already works for an ϵ-challenge, then a smaller δ should also work.

Exercise 4.2.3

Remember that if two sequences converging to the same limit give different functional values, then the limit at that point does not exist.

(a) If we take $x_n^1 = 1/n$, then we get a limit of 1, but if we take $x_n^2 = -1/n$, we get a limit of -1, so therefore the limit does not exist.

(b) If we approach in the rationals space we get 1, but in the irrational space we get 0.

Exercise 4.2.4

(a) We can choose

- $x_n = \frac{n-1}{n}$
- $y_n = 1 + \frac{1}{n}$
- $z_n = \sqrt{\frac{n^2+1}{n^2}}$

(b) We can compute the limits with Thomae's function

- $\lim t(x_n) = 0$, since we get larger denominators
- $\lim t(y_n) = 0$, since we get larger denominators
- $\lim t(z_n) = 0$, since once $z_n > 1$, it is impossible for adjacent numbers to both be perfect squares, so this number is always irrational.

(c) We can conjecture $\lim_{x\to 1} t(x) = 0$. To prove this, informally, if we receive an ϵ-challenge, we can choose a δ-neighborhood small enough so that x is close enough to 1 so that it is either an irrational number, in which case $t(x) = 0$, or x is rational. If x is rational, since it is so close to 1, this small distance must be representable by a rational number, which only happens when the denominator is very large. This means as we get closer to 1, $t(x) \to \lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, we can find such a δ so that all elements in this neighborhood are such that $\left|t(x) - 0\right| < \epsilon$.

Exercise 4.2.5

(a) If we have $f(x_n) \to L$, then we essentially have a sequence that converges to L, which means we can use all the properties of the Algebraic Limit Theorem.

(b) For some $\epsilon > 0$, we can find δ_f, δ_g so that $\left|f(x_f) - L\right| < \epsilon/2, \left|g(x_g) - M\right| < \epsilon/2$, and then take $\delta = \min\{\delta_f, \delta_g\}$, so that for $0 < |x - c| < \delta$, we have $\left|f(x) + g(x) - (L + M)\right| < \epsilon/2 + \epsilon/2 = \epsilon$.

I'm too lazy to do the Algebraic Limit Theorem proof. But it's basically just using the fact that we have a convergent sequence, as mentioned in part (a).

For the proof without the Algebraic Limit Theorem, in shorthand, we need to do something like,

$$\begin{aligned}\left|f(x)g(x) - LM\right| &< \left|f(x)g(x) - g(x)L + g(x)L - LM\right| \\ &< \Big|g(x)\big(f(x) - L\big)\Big| + \Big|L\big(g(x) - M\big)\Big|\end{aligned}$$

Now, we can bound $g(x)$, since we know $g(x)$ is a sequence that converges to M. Then, just choose δ_f, δ_g, and take $\delta = \min\{\delta_f, \delta_g\}$ to find a δ-neighborhood that satisfies this inequality.

Exercise 4.2.6

For any $\epsilon > 0$, choose $\epsilon_g < \epsilon/M$, then we can find a δ_g, so that

$$\forall 0 < |x - c| < \delta_g, \left|g(x)f(x)\right| = \left|g(x)\right|\left|f(x)\right| < \frac{\epsilon}{M} \cdot M = \epsilon \tag{4.1}$$

Exercise 4.2.7

(a) We can say for any $M \in \mathbb{R}, \epsilon > 0, \exists \delta > 0$ such that

$$0 < |x| < \delta, \left|f(x) - M\right| > \epsilon \tag{4.2}$$

(b) We say for some $\epsilon > 0$, $\exists N$ such that for $x \geq N$, $\left|f(x) - L\right| < \epsilon$. We can show for any $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$.

(c) For any $M \in \mathbb{R}, \epsilon > 0$, $\exists N$ such that for $x \geq N$, we have $\left|f(x) - M\right| > \epsilon$. An example would be $f(x) = x$.

Exercise 4.2.8

If c is a limit point of A, then $\lim_{x\to c} f(x) = L, \lim_{x\to c} g(x) = M$. Now AFSOC $M > L$. Let $\epsilon = |M - L|$. Then we can show $\exists N, n \geq N, \left|g(x_n) - M\right| < \epsilon$, and $\exists N', n \geq N', \left|f(x_n) - L\right| < \epsilon/2$. But since $M > L$, this implies $\exists x_n', f(x_n') < g(x_n')$, which contracts that $\forall x, f(x) \geq g(x)$. Therefore, we reject our hypothesis and conclude that $L \geq M$.

Exercise 4.2.9
Let the limits of f, g, h be L_f, L_g, L_h respectively. Then using Exercise 4.2.8, we can show that

$$\begin{gathered}L_f \leq L_g \Rightarrow L \leq L_g \\ L_h \geq L_g \Rightarrow L \geq L_g \\ \Rightarrow L \leq L_g \leq L \Rightarrow L_g = L,\end{gathered}$$

so we can conclude $\lim_{x\to c} g(x) = L_g = L$.

4.3 Combinations of Continuous Functions

Exercise 4.3.1

(a) For some $\epsilon > 0$, choose $\delta = \epsilon^3$, then

$$\left|f(x) - 0\right| = \left|\sqrt[3]{\epsilon^3}\right| < \epsilon$$

(b) If we are given some $\epsilon > 0$, $c \in \mathbb{R}$, choose $\delta = \epsilon\sqrt[3]{c^2}$, then

$$\begin{aligned}\left|\left(\sqrt[3]{x} - \sqrt[3]{c}\right)\left(\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}\right)\right| &= |x - c| \\ \left|\sqrt[3]{x} - \sqrt[3]{c}\right| &= \frac{|x-c|}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{c^2}} \\ &< \frac{|x-c|}{\sqrt[3]{c^2}} \\ &= \frac{\epsilon\sqrt[3]{c^2}}{\sqrt[3]{c^2}} = \epsilon\end{aligned}$$

Exercise 4.3.2

(a) For any $\epsilon > 0$, Since g is continuous at $f(c)$, choose δ_g such that if $f(x) \in V_{\delta_g}(f(c))$, then

$$\left|g(f(x)) - f(c)\right| < \epsilon.$$

Now, since $f(x)$ is continuous at c, there is a δ_f such that if $x \in V_{\delta_f}(c)$, then

$$\left|f(x) - f(c)\right| < \delta_g.$$

Choose this δ_f, then we will have for any $\epsilon > 0$,

$$x \in V_{\delta_f}(c) \Rightarrow \left|g(f(x)) - f(c)\right| < \epsilon$$

(b) Using sequential characterization, since $f(x)$ is continuous at c, we know that $(x_n) \to c$ implies $f(x_n) \to f(c)$.

Now, $g \circ f(x)$ is well-defined on A, so if we let $y_n = f(x_n)$, we know that since $g(y)$ is continuous at $f(c)$, $y_n \to f(c)$ means $g(y) \to g(f(c))$.

Putting these together, we just showed $(x_n) \to c$ implies $f(x_n) \to f(c)$, which finally implies $g(f(x_n)) \to g(f(c))$. By the sequential characterization, we can conclude that $g \circ f(x)$ is continuous at $x = c$.

Exercise 4.3.3

For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{a}$. Then

$$\left|f(x) - f(c)\right| = \left|a(x - c)\right| < a\frac{\epsilon}{a} < \epsilon$$

Exercise 4.3.4

(a) For $\epsilon > 0$, choose $\delta < 1$, then $x, c \in \mathbb{Z}$ and $0 < |x - c| < \delta$ implies that $x = c$, so $\left|f(x) - f(c)\right| = 0 < \epsilon$.

(b) For any isolated point, we can choose a δ-neighborhood small enough so that $0 < |x - c| < \delta$ implies $x = c$, since there are no other elements in the domain near c other than c itself. This will make $\left|f(x) - f(c)\right| = 0 < \epsilon$.

Exercise 4.3.5

We can find a δ_g so that $x \in V_{\delta_g}(c)$ means $\left|g(x) - g(c)\right| < \epsilon = \left|g(c)\right|$. This means $\forall x, g(x) \neq 0$, since it is too far away from 0, so therefore the denominator is always nonzero and $f(x)/g(x)$ is defined in this open interval.

Exercise 4.3.6

(a) Choose some $c \in \mathbb{R}$, we have 2 cases,

1. $f(c)$ is rational. We can choose an irrational sequence $(x_n) \to c$, so that $f(x_n) \to 0$, since $\forall x_n, f(x_n) = 0$, since x_n is irrational. However, we know that $f(c) = 1$, since c is rational.
2. $f(c)$ is irrational. We can choose an rational sequence $(x_n) \to c$, so that $f(x_n) \to 1$, since $\forall x_n, f(x_n) = 1$, since x_n is rational. However, we know that $f(c) = 0$, since c is irrational.

Therefore, we conclude the Dirichlet function is everywhere-discontinuous.

(b) Proving Thomae's function is discontinuous at every rational point is the same proof as showing Dirichlet's function is discontinuous at every rational point. See part (a).

(c) I'm extremely confused about this hint $t(x) \geq \epsilon$. All of these x that satisfy this must be rational numbers, and they must be isolated if we force a limit on their denominator, which is what $t(x) \geq \epsilon$ does. I think then the argument goes, take the complement of this set, so $\{x : t(x) < \epsilon\}$, then we know this is bounded somehow? And then we choose a δ so that our neighborhood only contains elements from this set? Maybe something about an open set existing in this set too, and we can choose that. Overall...unsure. **TODO**

Using Theorem 4.3.2 (iii), which is the ϵ, δ-neighborhood argument, What you can do is choose a δ so small so that to represent δ as a rational, you must use a denominator large enough such that $\frac{m}{n}$, $\frac{1}{n} < \epsilon$.

Exercise 4.3.7

To show K is a closed set, we need to show that K contains all of its limit points. Take any $(x_n) \to l$ from K. Since we know $h(x)$ is continuous, any $(x_n) \to l$ implies $h(x_n) \to h(l)$. Now, since all terms of $h(x_n)$ are 0, we must have $h(l) = 0$, otherwise, if $h(1) = y$, then choose $\epsilon < |y|$, and you cannot prove convergence. We conclude that $l \in K$, and therefore K contains all of its limit points and is closed.

Exercise 4.3.8

(a) AFSOC $\exists c \in \mathbb{R}, f(c) \neq 0$. Then $\exists (x_n) \to c, x_n \in \mathbb{Q}$. Since f is continuous on $\mathbb{R}$, we must have $(x_n) \to c$ implies $f(x) \to f(c)$. But $f(x_n) = 0$, and $f(c) \neq 0$, so this means f is not continuous at c. But this is a contradiction, since f is continuous on $\mathbb{R}$.

(b) There is no claim about continuity, so no. We can have $f(r) = g(r), r \in \mathbb{Q}$, but have $f(i) = -1, g(i) = 1, i \in \mathbb{I}$. Then f and g are not the same function.

Exercise 4.3.9

(a) We can do the ϵ-δ proof, but choose $\delta = \frac{\epsilon}{c}$, so for any $z \in \mathbb{R}$, we can show that for $0 < |x - z| < \delta$, that $\big|f(x) - f(z)\big| \leq c|x - z| < c\frac{\epsilon}{c} = \epsilon$.

(b) We notice that for every y_{n+1}, its distance from y_n is c times the distance of $|y_n - y_{n-1}|$. Since $c \in (0, 1)$, this distance will get smaller every single time. Namely,

$$|y_{n+1} - y_n| = \big|f(y_n) - f(y_{n-1})\big| \leq c|y_n - y_{n-1}| \leq c^{n-1}|y_2 - y_1|$$

So for any $\epsilon > 0$, we can choose N such that $c^{N-1} < \frac{\epsilon}{|y_2 - y_1|}$, and then for subsequent $n \geq N$, we have $|y_n - y| < \epsilon$. Since $\lim y_n = y$, this sequence converges to y, which is equivalent to showing it is a Cauchy sequence.

(c) We can choose the sequence $x_n = y_n$ where y_n was defined in part (b), and we see that $(x_n) \to y$, and since f is continuous on $\mathbb{R}$, we conclude that $f(x_n) \to f(y)$. Now, $f(x_n) = x_n + 1$, so $f(x_n) \to y$. This means $f(y) = y$. To prove uniqueness, we need to show no other x exists such that $f(x) = x$. AFSOC such an x exists, then we can do the following:

- We have $\big|f(x) - f(y)\big| \leq c|x - y|$.

- If we apply this property again with $f(f(x)), f(f(y))$, we get

$$\begin{aligned}\big|f(f(x)) - f(f(y))\big| &\le c\big|f(x) - f(y)\big| \\ \big|f(x) - f(y)\big| &\le c^2|x - y| \qquad \text{(Using } f(y) = y, f(x) = x \text{ property, and the prev result)}\end{aligned}$$

- We can continue this indefinitely, and show that $y = f(y) = f(x) = x$, which means y is unique.

(d) Let us say $(x_n) \to L$. Then we can find for any $\epsilon > 0$, a N such that for $n \ge N$, $|x_n - L| < \epsilon$.

Now, if we take $\big|f(x_n) - f(L)\big| = \big|x_n - f(L)\big| \le c|x_n - L| < \epsilon$, we see this sequence also converges to $f(L)$, which means $f(L) = L$. In the previous part, we showed y is the unique fixed point, so we conclude $L = y$, and thus sequence also converges to y.

Exercise 4.3.10

(a) We have two problems,

- $f(0 + 0) = f(0) = f(0) + f(0)$. If we have $x = 2x$, $x = 0$ is the only solution, so $f(0) = 0$.
- $f(x - x) = f(0) = f(x) + f(-x) \Rightarrow -f(x) = f(-x)$.

(b) For any $\epsilon > 0$, since $f(0)$ is continuous, $\exists\delta$ such that $\big|f(\delta) - f(0)\big| < \epsilon$. Now, if we have $c \in \mathbb{R}$, if we have some $(x_n) \to c$, we can choose N such that for $n \ge N$, $|x_n - c| < \delta$. Then,

$$\big|f(x_n) - f(c)\big| = \big|f(x_n - c)\big| = \big|f(\delta)\big| < \epsilon$$

(c) Just expand $f(1 + 1 + \cdots + 1)$, so e.g. $f(3) = f(1 + 1 + 1) = f(1) + f(1 + 1) = f(1) + f(1) + f(1) = 3f(1) = 3k$. Using $f(-x) = f(x)$, we can show $f(z) = kz, z \in \mathbb{Z}$. For rationals, let's say we have $\frac{m}{n} \in \mathbb{Q}$, where $m, n \in \mathbb{Z}$, then we can write $f(m)$ as $f(n \cdot \frac{m}{n})$, which means $f(m) = nf\big(\frac{m}{n}\big)$. Now, since $m \in \mathbb{Z}$, we know $f(m) = mk$, so $mk = nf\big(\frac{m}{n}\big) \Rightarrow f\big(\frac{m}{n}\big) = \frac{m}{n}k$.

(d) We only haven't shown the property $f(x) = kx$ is true for irrationals. We can do this by constructing a rational sequence $(x_n) \to c$ for any $c \in \mathbb{R}$. Now since f is continuous on $\mathbb{R}$, we conclude that $f(x_n) \to f(c)$. We also know for every $f(x_n) = kx_n$, so $f(c) = kc$.

Exercise 4.3.11

(a) Let

$$f(x) = \begin{cases} x, & x \notin \mathbb{Z} \\ x + 1, & x \in \mathbb{Z} \end{cases}$$

(b) We basically define a function that is "normal", except inside $(0, 1)$ we make it behave crazy like the Dirichlet function. We make sure that the surrounding area connects to this region smoothly, so that $0, 1$ are continuous.

$$f(x) = \begin{cases} 0, x \in \mathbb{Q} \cap (0, 1) \\ 1, x \in \mathbb{I} \cap (0, 1) \\ x, x \notin \mathbb{Z} \end{cases}$$

(c) Basically the same as part (b) except we make the surrounding area of the interval disconnected, so that $0, 1$ are discontinuous.

$$f(x) = \begin{cases} 0, x \in \mathbb{Q} \cap [0, 1] \\ 1, x \in \mathbb{I} \cap [0, 1] \\ x + 100, x \notin \mathbb{Z} \end{cases}$$

(d) We have to pay attention to 0, since there exists a subsequence $\in A$ that converges to 0. Since this subsequence will converge to 0, we just have to make sure that $f(x_n) \to f(0)$.

$$f(x) = \begin{cases} x, & x \in A \\ 0, & x \notin A \end{cases}$$

Exercise 4.3.12

(a) We can construct two subsequences that converge to c, one which is always in C, and the other that is not. Then $g(x_n)$ will converge to 1 for the first subsequence, but 0 for the second. Therefore every $c \in C$ is discontinuous.

(b) If $c \notin C$, then c is part of an open set, which means there exists an ϵ neighborhood around it, where all points x are such that $g(x) = 0$.

4.4 Continuous Functions on Compact Sets

Exercise 4.4.1

(a) For any $\epsilon > 0, c \in \mathbb{R}$, choose $\delta < \sqrt[3]{\epsilon}$.

(b) Take $x = n$, $y_n = n + 1/n$. Choose $\epsilon = 1$, then $x_n, y_n \to n, so|x_n - y_n| \to 0$, but

$$\left|f(x) - f(y)\right| = \left|n - (()n^3 + 3n + 3/n + 1/n^3)\right| = \left|3n + 3/n + 1/n^3\right| > 3 > \epsilon = 1$$

(c) For any bounded subset, let M be the largest absolute value of any element in the subset. Then choose $\delta < \frac{\epsilon}{3M^2}$. Then we have

$$\left|x^3 - y^3\right| = \left|(x-y)\left(x^2 + xy + y^2\right)\right| < \frac{\epsilon}{3M^2} \cdot \left(M^2 + M \cdot M + M^2\right) = \epsilon$$

Exercise 4.4.2

Given any ϵ, choose $\delta < \epsilon/2$. Then for $|x - y| < \delta$,

$$\begin{aligned}
\left|f(x) - f(y)\right| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| \\
&= \left|\frac{y^2 - x^2}{x^2y^2}\right| \\
&= \left|\frac{(y-x)(y+x)}{x^2y^2}\right| \\
&\leq |x - y|\left|\frac{x+y}{x^2y^2}\right| && (\text{Since } \tfrac{1}{x^2y^2} \leq 1) \\
&= |x - y|\left|\frac{1}{x^2y} + \frac{1}{y^2x}\right| \\
&< \frac{\epsilon}{2}(1+1) && (\tfrac{1}{x^2y} \leq 1, \text{ since } x, y \geq 1) \\
&= \epsilon
\end{aligned}$$

Now, for $(0, 1]$, choose $x_n = \frac{1}{n}, y_n = \frac{1}{n + \frac{1}{n}}$. Also choose $\epsilon_0 = 1/2$. Then $|x_n = y_n| = \left|\frac{1}{n^3+n}\right| \to 0$, but

$$\left|f(x_n) - f(y_n)\right| = \left|n^2 - \left(n + \frac{1}{n}\right)^2\right| = \left|-2 - \frac{1}{n^2}\right| \geq 1 > \epsilon_0$$

Exercise 4.4.3

Since K is compact, by the preservation of compact sets, since f is continuous on K, $f(K)$ is also compact. Now, we already showed in Exercise 3.3.1 that any compact set contains its maximum and minimum values, so therefore we can conclude that f will attain its maximum and minimum values of this range.

Exercise 4.4.4

We know $[a, b]$ is closed and bounded, so therefore it is compact. Thus, $f([a, b])$ is also compact, since f is continuous on this interval. Let M be the lower bound of $f([a, b])$, which we know exists since it is compact, then we can conclude $1/f$ is bounded by $1/M$ on $[a, b]$.

Exercise 4.4.5

Choose $\epsilon = \epsilon_0$. Now, we can show

$$\exists m, n \in \mathbb{N}, |x_m - y_m| < \delta_n = \frac{1}{n}$$

such that $\left|f(x_m) - f(y_m)\right| \geq \epsilon_0$. Since we can make δ_n arbitrarily small, any δ claim can be shown to be faulty, so therefore this function cannot be uniformly continuous.

Exercise 4.4.6

(a) We can take advantage of the fact that f is not necessarily continuous at $0, 1$. An example is $f(x) = \frac{1}{x}$, where if we take $x_n = \frac{1}{n}$, then $x_n \to 0$, but $f(x) = \frac{1}{1/n} = n$ does not converge.

(b) Impossible. Since $[0, 1]$ is closed, any $x_n \to x$ will have $x \in [0, 1]$. Since f is continuous over $[0, 1]$, it must also be continuous at x, so therefore $\lim f(x_n) = f(x)$, and therefore $f(x_n)$ is a Cauchy sequence.

(c) Impossible. Same reasoning as part (b), since $[0, \infty)$ is closed, any $x_n \to x$ will have its limit also in the set.

(d) An upside down parabola with a peak in $(0, 1)$ will work. A function like $\frac{1}{x(x-1)}$ is a bit more fun.

Exercise 4.4.7

Suppose we have some $\epsilon > 0$.

Now if we have two arbitrary $x, y \in (a, c)$, if x, y are both in the left set or the right set, then we can just use the fact that g is uniformly continuous on those sets to show it. The only other case we have is that they are in different sets, WLOG $x \in (a, b], y \in [b, c)$,

$$\left|f(x) - f(y)\right| = \left|f(x) - f(b) - (f(b) - f(y))\right| \leq \left|f(x) - f(b)\right| + \left|f(y) - f(b)\right|$$

Now, since b is in both sets, We know for $x_a \in (a, b], y \in [b, c)$, we can find δ_a, δ_b respectively such that $|x - b| < \delta_a \Rightarrow \left|f(x) - f(b)\right| < \epsilon/2$ and $|y - b| < \delta_b \Rightarrow \left|f(y) - f(b)\right| < \epsilon/2$.

So we can finish up our proof by choosing $\delta < \min(\delta_a, \delta_b)$, so that

$$|x - y| < \delta \Rightarrow \left|f(x) - f(y)\right| \leq \left|f(x) - f(b)\right| + \left|f(y) - f(b)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The last part is motivated by that if x, y are in separate sets, we try to make them as close to b as possible so we can use their uniformly continuous properties around b.

Exercise 4.4.8

(a) We can show f is uniformly continuous on $[0, b]$, since it is a compact set. Then, using the same argument as in Exercise 4.4.7, we can show that if f is uniformly continuous on $[0, b], [b, \infty)$, then f is also uniformly continuous on $[0, \infty)$.

(b) We can show $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$, because on this interval, any $x, y \in [1, \infty)$ means $\sqrt{x} + \sqrt{y} \geq 2$. Then choose $\delta < 2\epsilon$, then

$$\begin{aligned} |x - y| < \delta \Rightarrow \left|\sqrt{x} - \sqrt{y}\right| &= \left|\frac{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}{\left(\sqrt{x} + \sqrt{y}\right)}\right| \\ &= \frac{|x - y|}{\left|\sqrt{x} + \sqrt{y}\right|} \\ &< 2\epsilon \cdot \frac{1}{2} = \epsilon \end{aligned}$$

Now, $[0, 1]$ is closed and bounded, so it is compact, and $\sqrt{x}$ is continuous on it, so therefore $\sqrt{x}$ is uniformly continuous on $[0, 1]$.

Using what we just proved in part (a), we can combine these results to show that $\sqrt{x}$ is uniformly continuous on $[0, 1] \cup [1, \infty) = [0, \infty)$.

Exercise 4.4.9

(a) Choose $\delta < \frac{\epsilon}{M}$, then if we have $|x - y| < \delta = \frac{\epsilon}{M}$,

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M \Rightarrow \left|f(x) - f(y)\right| \leq |x - y|M < \frac{\epsilon}{M}M = \epsilon$$

(b) No, take $f(x) = \sqrt{x}$. We showed in Exercise 4.4.8, part (b) that this function is uniformly on $[0, 1]$, but its slope $\frac{1}{2\sqrt{x}}$ tends to ∞ as $x \to 0$. The issue stems from that we will have $M = \frac{\epsilon}{\delta}$, but this quantity may not be bounded, since ϵ, δ vary.

Exercise 4.4.10
Yes, uniform continuousness preserves boundedness over a set.

Choose some $\epsilon > 0$, say $\epsilon = \epsilon_0$. Now we know since f is uniformly continuous, $\exists\delta |x - y| < \delta \Rightarrow \big|f(x) - f(y)\big| < \epsilon_0$. Now, since A is a bounded set, we can find a finite open cover for it, let this cover be in the form

$$A = \bigcup_{i=1}^{N} V_\delta(x_i)$$

where $x_i \in A$. Now, take $M_1 = \min_{i=1}^{N}\big(f(x_i) - \epsilon_0\big), M_2 = \max_{i=1}^{N}\big(f(x_i) + \epsilon_0\big)$, and M_1 will be a lower bound for $f(A)$, and M_2 an upper bound.

Exercise 4.4.11
($\Rightarrow$) Let $O \in \mathbb{R}$ be any open set. Consider some $x \in g^{-1}(O)$, We know $g(x) = y \in O$. Now, since O is an open set, we know $\exists\epsilon$ such that $V_\epsilon(y) \subseteq O$. We also know that g is continuous, which means for any ϵ, we know $\exists V_\delta(x)$ such that $x' \in V_\delta(x)$ means $g(x') \in V_\epsilon(y)$. Since $V_\epsilon(y) \subseteq O$, we have $g(x') \in O$ for any of these x', which means by the definition of g^{-1}, all these $x' \in g^{-1}(O)$, and therefore $V_\delta(x) \subseteq g^{-1}(O)$, which means there exists an open neighborhood around any element in $g^{-1}(O)$, and therefore $g^{-1}(O)$ is an open set.

($\Leftarrow$) If we know $g^{-1}(O)$ is open for any open $O \subseteq \mathbb{R}$, suppose we get some $y \in O, \epsilon > 0$. Then consider the set $V_\epsilon(y) \in O$. Now, consider $g^{-1}(V_\epsilon(y))$, which we are guaranteed is open. Since $x = g^{-1}(y)$ is in $g^{-1}(V_\epsilon(y))$, because $y \in V_\epsilon(y)$, we know since g is open, that $\exists\delta > 0$ such that $V_\delta(x) \subseteq g^{-1}(V_\epsilon(y))$. Therefore, for any $V_\epsilon(y)$, we have found a corresponding $V_\delta(x)$ such that $x' \in V_\delta(x)$ implies $g(x') \in V_\epsilon(y)$, which means we conclude g is continuous.

Exercise 4.4.12
We want to prove that

A function that is continuous on a compact set K is uniformly continuous on K.

TODO

Exercise 4.4.13

(a) We know for every $\epsilon > 0$, $\exists\delta$ such that $x' \in V_\delta(x)$ implies $f(x') \in V_\epsilon(f(x'))$. Now, if we have some Cauchy Sequence x_n, we know $\exists N$ such that for $m, n \geq N$,

$$|x_m - x_n| < \delta,$$

which means $\big|f(x_m) - f(x_n)\big| < \epsilon$, since $x_m, x_n \in V_\delta(x)$. Therefore, we know for this N, $m, n \geq N$ means $\big|f(x_m) - f(x_n)\big| < \epsilon$, so therefore $f(x_n)$ is also a Cauchy Sequence.

(b) ($\Rightarrow$) We can come up with a sequence $a_n \to a, a_n \in (a, b)$. Now, since (a_n) is also a Cauchy Sequence, that means $g(a_n)$ is also a Cauchy sequence, since g is a uniformly continuous function. Therefore, define $g(a)$ as $g(a_n) \to g(a)$. We know this exists since $g(a_n)$ is Cauchy. Notice that any sequence $(a'_n) \to a$ is Cauchy, and therefore we must have $g(a_n) \to g(a')$. We want to show that $g(a') = g(a)$ for g to be continuous at a.

AFSOC $g(a') \neq g(a)$, then that means we have two sequences $a'_n, a_n \to a$, where these sequences are contained inside (a, b), but $\big|g(a') - g(a)\big| \geq \epsilon_0$, which means g is not uniformly continuous. This is a contradiction, so therefore we conclude that $g(a') = g(a)$, and therefore any $(a'_n) \to a$ implies $g(a'_n) \to g(a)$, which means g is continuous at a. we conclude that same for b, using the same argument, and therefore g is continuous over $[a, b]$.

($\Leftarrow$) $[a, b]$ is a compact set, and since g is continuous over this set, we have that g is also uniformly continuous over $[a, b]$, which means we also have that g is uniformly continuous over $(a, b) \subseteq [a, b]$.

4.5 The Intermediate Value Theorem

Exercise 4.5.1
If we have a continuous function f on $[a, b]$, and we know that $[a, b]$ is connected, then that means $f([a, b])$ is also connected.

By the property of connected sets, it follows directly that if we have $f(a) < L < f(b)$, we can always find this $L \in f([a, b])$. Then, this L must correspond to some $c \in (a, b)$, which proves the IVT.

Exercise 4.5.2

(a) No, you can imagine taking $(-1, 1)$ to $[1, 2)$ with $f(x) = x^2 + 1$. The idea for making this was that the bounds don't necessarily have to come from the open endpoints.

(b) Can use the same example from part (a)

(c) Yes. If we have bounded closed interval, then it is compact, so a continuous function will take it to another compact interval, which is bounded and closed. This function will also preserve connectedness.

Exercise 4.5.3
No, since $\mathbb{R}$ is connected, and if f is continuous, it will preserve connectedness. $\mathbb{Q}$ is not connected.

Exercise 4.5.4
Suppose we have some function f with the Intermediate value property and is also increasing. We want to show that f is continuous.

Suppose we have some $\epsilon > 0$ and a point $c \in [a, b]$. Let

$$L_1 = \max(f(a), f(c) - \epsilon), \min(f(b), L_2 = f(c) + \epsilon)$$

By the IVT property, we know $\exists x_1, x_2 \in [a, b]$ where $f(x_1) = L_1, f(x_2) = L_2$.

By the increasing property of f, we have that any $x \in [x_1, x_2]$ will satisfy

$$\left|f(x) - f(c)\right| < \epsilon$$

since $f(x_1) \leq f(c) \leq f(x_2)$.

Choose $\delta = \min(|c - x_1|, |c - x_2|)$, and this will work for any ϵ challenge.

Exercise 4.5.5
To complete the IVT proof with the Axiom of Completeness, we are trying to show $\exists c \in (a, b)$ such that $f(c) = 0$ if $f(a) < 0 < f(b)$, and f is continuous. With our definition of $K = \left\{x \in [a, b] : f(x) \leq 0\right\}$, since we know K is bounded above by b, and $a \in K$, K is not empty. By the Axiom of Completeness, we know $c = \sup K$ exists.

- We cannot have $f(c) > 0$ since all elements of K satisfy $f(x) \leq 0$.
- We cannot have $f(c) < 0$, because it is not an upper bound that includes 0.
- Therefore, we conclude $f(c) = 0$, which means we have found a $c \in (a, b)$ such that $f(a) < f(c) = 0 < f(b)$

To extend it for a general $f(a) < c < f(b)$, just consider the function $f'(x) = f(x) - c$, and then we can do the $f(a) < 0 < f(b)$ case, which we have already proved.

Exercise 4.5.6
With the binary set construction described in the text, once we take the $\bigcap_{n=1}^{\infty} I_n$, we must have an element in this intersection by NIP. Call this element x.

Now, our claim is that $f(x) = 0$. Notice that a_n and b_n converge to x, since $|a_n - b_n| < \frac{|a-b|}{2^n}$. Then, since f is continuous, $f(a_n)$ and $f(b_n)$ must converge to the same L. Since $f(a_n) < 0, f(b_n) \geq 0$, it must be the case that $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = 0$, since otherwise, for example if $L > 0$, then $\left|f(a_n) - L\right| \geq \epsilon_0$. Therefore, we have found $x \in (a, b)$ such that $f(x) = 0$.

Exercise 4.5.7
Construct the folowing sequence of intervals, and let $c_{n-1} = \frac{a_{n-1}+b_{n-1}}{2}$,

$$I_n = [a_n, b_n] = \begin{cases} [0,1], & n = 0 \\ [c_{n-1}, b_{n-1}], & \text{if } f(c_{n-1}) > c_{n-1} \\ [a_{n-1}, c_{n-1}], & \text{if } f(c_{n-1}) \leq c_{n-1} \end{cases}$$

Take $x \in \bigcap_{n=1}^{\infty} I_n$. By the NIP this x exists.

Now, our claim is that $f(x) = x$. This is similar to the proof in Exercise 4.5.6, where we use the continuity of f and that $(a_n) \to x, (b_n) \to x$, so that $\lim_{n\to\infty} f(a_n) = a_n$, and therefore we conclude $f(x) = x$.

A more straightforward proof with IVT is to define $g(x) = f(x) - x$. Then we have $g(0) = f(0) - 0 = f(0) \in [0,1] \Rightarrow g(0) \geq 0$, and also $g(1) = f(1) - 1 \Rightarrow f(1) - 1 \in [-1, 0] \Rightarrow g(1) \leq 0$. Putting these two together, we have either $g(0) = 0, g(1) = 0$, or neither are equal to 1. If either are equal to one, then we have the trivial case where $g(c) = 0 \Rightarrow f(c) = c$ for $c = 0, 1$.

Otherwise, if $g(0) \neq 0, g(1) \neq 0$, then we have $g(1) < L < g(0)$, and then we can use the IVT to say $\exists c$ such that $g(1) < g(c) < g(0)$ and $c \in (0,1)$. This implies that for this c, that $f(c) = c$.

Therefore, we have shown in all cases that we have a fixed point in this range.

Exercise 4.5.8
We cannot, since there are times that are ambiguous.

For example, consider the set of times from 1:00 - 2:00. Here, the hour hand goes from $30° - 60°$, while the minute hand goes from $0° - 360°$,

We can find a time that is isomorphic to this time range by considering that $30° - 60°$ is the 5-10 minute range, and if the minute hand is past some hour enough to the point where it crosses the same point where the hour point would be with a minute in the range of 5-10 minutes, then we would have an isomorphic time.

For example, consider when the minute hand goes past 2. The minute hand will go from 2 - 3, but in between there, at some point it can represent some 2-hour time where the minute is between 5-10 minutes.

There's probably a better setup and application of IVT, but this is enough to convince me that there are isomorphic times on the clock.

4.6 Sets of Discontinuity

Exercise 4.6.1

(a)

$$f(x) = \begin{cases} 1, & x \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$f(x) = \begin{cases} \text{Dirichlet's Function}, & x \in (0,1] \\ 100, & \text{otherwise} \end{cases}$$

Exercise 4.6.2

(Left-hand Limit) Given a limit point of a set A and a function $f : A \to \mathbb{R}$, we write

$$\lim_{x \to c^-} f(x) = L$$

if $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$\left|f(x) - L\right| < \epsilon \text{ whenever } 0 < c - x < \delta$$

Equivalently, $\lim_{x \to c^-} f(x) = L$ if $\lim f(x_n) = L$ for all sequences (x_n) satisfying $x_n < c$ and $\lim(x_n) = c$.

Exercise 4.6.3

($\Rightarrow$) If $\lim_{x \to c} f(x) = L$, then all sequences that converge to c will have $\lim f(x_n) = L$, so we have $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$.

($\Leftarrow$) If we have $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$, we want to be able to show for any sequence $(x_n) \to c$, that we also have $\lim f(x_n) = L$. For any (x_n), we know there must be an infinite number of elements greater than c and/or less than c if $(x_n) \to c$, so WLOG take this subsequence that is all greater than c, which we know satisfies $\lim f(x_n) = L$.

Exercise 4.6.4

($x \to c^-$) For any $\epsilon > 0$, we can find $f(x') = f(c) - \epsilon$ by the IVT, and since f is monotone, we know $0 < c - x < \delta = \left|c - x'\right|$ will satisfy the ϵ challenge, since any x in this range will satisfy $f(x') \leq f(x) \leq f(c)$, since $x' \leq x \leq c$, and $\left|f(x) - f(x')\right| < \epsilon$. Similar logic applies to the other direction.

Since the limit exists at every point, but could potentially be different, a monotone function can only have jump discontinuity.

Exercise 4.6.5

For any monotone function f with a jump discontinuity at c, where $\lim_{x \to c^-} f(x) = L^-, \lim_{x \to c^+} f(x) = L^+$, it must be the case that $L^- < L^+$. In addition, because $\mathbb{Q}$ is dense, we know that $\exists r \in \mathbb{Q}$ such that $L^- < r < L^+$. Now, there is a slight concern that maybe this r could be repeated at other discontinuities, but since we know f is monotone, we know that for $c_1 < c_2$, we must have $L_1^- < L_2^-$ so therefore $r_1 < r_2$. Therefore, we can map all of D_f to unique $r \in \mathbb{Q}$, which means D_f is a subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, we conclude D_f is finite or countable.

Exercise 4.6.6

(a) $\mathbb{R}$ is closed

(b) $\mathbb{R} \setminus \{0\} = \bigcup_{n=1}^{\infty} \left(-\infty, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, \infty\right)$

(c) $\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} [r]$, $\mathbb{Q}$ is countable

(d) $\mathbb{Z} = \bigcup_{n \in \mathbb{Z}} [z]$, $\mathbb{Z}$ is countable

(e) $(0, 1] = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right]$

Exercise 4.6.7

The set that f is α continuous is some aribitrary union of open sets $(x - \delta, x + \delta)$, which means $\overline{D_\alpha}$ is open. Then, we conclude D_α is closed, since it is the complement of an open set.

Exercise 4.6.8
If we have $\alpha_1 < \alpha_2$, we know that the set of α_1-continuous points is a subset of the α_2-continuous points since if $\exists\delta$ such that $y, z \in (x-\delta, x+\delta)$ and $\left|f(y) - f(z)\right| < \alpha_1 < \alpha_2$. Therefore, $\overline{D_{\alpha_1}} \subseteq \overline{D_{\alpha_2}} \Rightarrow D_{\alpha_2} \subseteq D_{\alpha_1}$.

Exercise 4.6.9
If f is continuous at x, we know for any $V_\epsilon(f(c))$, we can find $\exists\delta, x \in V_\delta(c)$ such that $f(x) \in V_\epsilon(f(c))$. Choose $\epsilon = \alpha/2$, then we know that we have a delta that satisfies for $y, z \in V_\delta(x)$,

$$\begin{aligned}\left|f(y) - f(z)\right| &= \Big|f(y) - f(c) - \big[f(z) - f(c)\big]\Big| \\ &\leq \left|f(y) - f(c)\right| + \left|f(z) - f(c)\right| \\ &< \alpha/2 + \alpha/2 = \alpha\end{aligned}$$

Since f is α-continuous at every point it is continuous, we conclude that for points where it is not α-continuous, it must be not in the set of continuities of f, which is namely D_f. Therefore, $D_\alpha \subseteq D_f$.

Exercise 4.6.10
If f is not continuous at x, it must be the case that f is not α-continuous at x for some $\alpha > 0$, otherwise, we can show that f is continuous at this point by showing that $\exists\delta, \forall y \in V_\delta(x)$, we have $\left|f(y) - f(x)\right| < \alpha$.

To show that

$$D_f = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}},$$

for any $\alpha > 0$, $\exists m \in \mathbb{N}$ such that $0 < \frac{1}{m} < \alpha$. We showed $\frac{1}{m} < \alpha$ implies $D_\alpha \subseteq D_{\frac{1}{m}}$, so we can conclude for any discontinuity of f, it will be included in some $D_{\frac{1}{n}}$.

This union of a countable number of closed sets shows that the discontinuities of f is a F_σ set.

Chapter 5

The Derivative

5.2 Derivatives and the Intermediate Value Property

Exercise 5.2.1

(i) We can show that

$$\begin{aligned}
(f+g)'(c) &= \lim_{x\to c} \frac{f(x)+g(x)-\big(f(c)+g(c)\big)}{x-c} \\
&= \lim_{x\to c} \frac{\big[f(x)-f(c)\big]+\big(g(x)-g(c)\big)}{x-c} \\
&= \lim_{x\to c} \left[\frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}\right] \\
&= f'(c)+g'(c)
\end{aligned}$$

(ii) Straightforward factoring

$$\begin{aligned}
(kf)'(c) &= \lim_{x\to c} \frac{kf(x)-kf(c)}{x-c} \\
&= k\cdot \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= kf'(c)
\end{aligned}$$

Exercise 5.2.2

(a) Computing the derivative of $1/x$

$$\begin{aligned}
\big(1/x\big)'_{x=c} &= \lim_{x\to c} \frac{\frac{1}{x}-\frac{1}{c}}{x-c} \\
&= \lim_{x\to c} \frac{\frac{c-x}{xc}}{x-c} \\
&= \lim_{x\to c} \frac{1}{xc} \\
&= -\frac{1}{c^2}
\end{aligned}$$

(b) We can instead prove the quotient rule with the $(1/x)'$ and the chain rule,

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \frac{f'(x)}{g(x)} + f(x)\frac{-1}{g(x)^2}g'(x) \\
&= \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}
\end{aligned}$$

(c) Another proof for the quotient rule

$$\begin{aligned}\left(\frac{f(x)}{g(x)}\right)' &= \lim_{x\to c}\frac{\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}}{x-c}\\ &= \lim_{x\to c}\frac{\frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)}}{x-c}\\ &= \lim_{x\to c}\frac{\frac{f(x)g(c)-f(c)g(c)-\left[f(c)g(x)-f(c)g(c)\right]}{g(x)g(c)}}{x-c}\\ &= \lim_{x\to c}\frac{g(c)\big(f(x)-f(c)\big)}{g(x)g(c)(x-c)}-\frac{f(c)\big(g(x)-g(c)\big)}{g(x)g(c)(x-c)}\\ &= \frac{f'(c)g(c)-f(c)g'(c)}{g(c)^2}\end{aligned}$$

Exercise 5.2.3
Choose the function

$$f(x)=\begin{cases}x^3, & x\in\mathbb{Q}\\ 0, & \text{otherwise}\end{cases}$$

This function is not continuous for any $x\neq 0$, since we can find sequences approaching both $x^3\neq 0$ and 0, if we choose sequences in $\mathbb{Q}$ and not in $\mathbb{Q}$ respectively. Since any function is continuous at its differentiable points, we know these points can't be differentiable.

Now, we want to show that this function is differentiable at $x=0$. In order to do this, we need to show that

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}$$

exists. We can show this by saying that for any $\epsilon>0$, choose $\delta=\sqrt{\epsilon}$. Then we have

$$\left|\frac{f(x)}{x}\right|\leq \max(0/x, x^3/x)\leq x^2<\left(\sqrt{\epsilon}\right)^2=\epsilon \qquad \text{(Since } |x-0|<\sqrt{\epsilon}\text{)}$$

So we conclude this limit exists and is 0.

This $f(x)$ we defined is differentiable only at $x=0$.

Exercise 5.2.4
(a) $a>0$, $f(x)$ will be continuous at 0

(b) Differentiable at 0 for $a>1$, since otherwise, $f'(x)=ax^{a-1}$ will be dividing by 0 if $a<1$

(c) $f''(x)=a(a-1)x^{a-2}$, so $a>2$

Exercise 5.2.5
First, we compute the derivative in general, which is

$$g_a'(x)=ax^{a-1}\sin\left(\frac{1}{x}\right)-x^{a-2}\cos\left(\frac{1}{x}\right)$$

and at 0,

$$g_a'(0)=\lim_{x\to 0}\frac{x^a\sin\big(1/x\big)-0}{x-0}=\lim_{x\to 0}x^{a-1}\sin\big(1/x\big),$$

which is defined for $a>1$, since that will help bound the oscillations near 0.

(a) Choose $a=1.5$, then the derivative exists everywhere, but $x^{a-2}\cos\big(1/x\big)$ could be unbounded, since for $x=1/(2\pi n)$, this expression is $(2\pi n)^\alpha$, for $\alpha=0.5$.

(b) I just want to note that the wording for this question is confusing, because I originally interpreted it as find a such that g_a' is continuous at 0, but g_a is not differentiable at 0. The question is asking for g_a' is continuous at 0, and g_a' is not differentiable at 0, which is a much easier task.

First, $g_a'(x)$ will be continuous if $g_a'(0) = 0$, which we know is true if $a > 2$.

Next, we should find $g_a''(0)$, to determine when it is differentiable

$$g_a''(0) = \lim_{x\to 0} \frac{g_a'(x) - 0}{x - 0} = ax^{a-2} \sin\left(\frac{1}{x}\right) - x^{a-3} \cos\left(\frac{1}{x}\right)$$

This is not differentiable when $a \leq 3$, since the cos limit will not exist so we can choose something like $a = 2.5 \in (2, 3]$.

(c) Let us first find $g_a''(x)$,

$$g_a''(x) = \sin\left(\frac{1}{x}\right)\left[a(a-1)x^{a-2} - x^{a-4}\right] - \cos\left(\frac{1}{x}\right)\left[x^{a-3}(2a-2)\right].$$

We see that for $a \leq 4$, $\lim_{x\to 0} g_a''(x)$ will not exist, so therefore $g_a''(x)$ will not be continuous at 0 if $a \leq 4$. From part (b), we saw that $g_a'(x)$ is differentiable when $a > 3$, so we can choose something like $a = 3.5 \in (3, 4]$.

Exercise 5.2.6

(a) Since $g'(a) < 0$, if we AFSOC that there are no x such that $g(x) < g(a), x \in (a, b)$, then if we choose $(x_n) \to a$ where $x_n \in (a, b)$, we get $g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a} > 0$, which contradicts that $g'(a) < 0$.

We can similarly apply this logic to $g'(b) > 0$ to show that $\exists y \in (a, b)$ where $g(y) < g(b)$

(b) From part (a), we can finish up Darboux's Theorem proof by showing that $\exists c \in (a, b)$ such that $g'(c) = 0$. If the derivative is 0, we know this implies that $g(c)$ is either a maximum or minimum, so we'd like to show that the max or min of the interval $[a, b]$ is in (a, b).

Namely, consider the minimum of $[a, b]$. Since we found $g(x) < g(a), g(y) < g(b)$, that means the minimum of the interval $[a, b]$ must be in (a, b), which means there must exist some $c \in (a, b)$ where $g'(c) = 0$. This completes the proof.

Exercise 5.2.7

(a) We can say $f : A \to \mathbb{R}$ is uniformly differentiable on A if for any $\epsilon > 0$, $\exists \delta > 0$ such that $|x - c| < \delta$ implies

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \epsilon$$

(b) Kind of a trivial example, but choose $f(x) = x$, then

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| = \left|\frac{x - c}{x - c} - 1\right| = |0| < \epsilon$$

(c) We found earlier in Exercise 5.2.5 that for $a \in (1, 2]$, $g_a(x)$ is differentiable on $[0, 1]$ but g_a' unbounded on $[0, 1]$. If g_a' is unbounded on this interval, it will not be uniformly differentiable, because if we observe the difference near the unbounded part, we will see that it can get very large.

Consider $x = 0, t_n = \frac{1}{2\pi n}$ and $g_2(x)$. Then we have

$$\left|\frac{g_2(0) - g_2(t_n)}{0 - t_n} - g_2'(t_n)\right| = \left|t_n \sin\big(1/t_n\big) + \cos\big(1/t_n\big) - 2t_n \sin\big(1/t_n\big)\right| \qquad = |0 + 1 - 0| = 1,$$

which does not satisfy all $\epsilon > 0$ challenges.

So we see that for these $g_2'(t_n)$, we are showing a smaller $\delta = 1/(2\pi n)$ doesn't work for $x = 0, t_n = 1/(2\pi n)$, which means we cannot possibly find a fixed δ that makes g_2 uniformly continuous.

Exercise 5.2.8

(a) Even if we have discontinuities in the function, Darboux's Theorem tells us we can always find some c such that $f'(a) < f'(c) < f'(b)$, and since irrationals are dense, the derivative will be irrational for some point.

(b) We can do $f(x) = x/2 + x^2 \sin\big(1/x\big)$, which has a derivative of $1/2$ at 0, and outside, will have a derivative of $f'(x) = 1/2 + 2x \sin\big(1/x\big) - \cos\big(1/x\big)$, which can be potentially negative for any δ neighborhood around 0, since we can consider $x = \frac{1}{2\pi n}$, where $\cos\big(1/x\big) = 1$. Therefore, such a neighborhood where $f'(x) > 0$ does not always exist.

(c) AFSOC $L \neq f'(0)$, then we know we can find ϵ_0 such that $0 < \epsilon_0 < \big|L - f'(0)\big|$. Now WLOG $f'(0) < L$ (the $f'(0) > L$ case is symmetric) Since ϵ_0 is less than the distance between $f'(0)$ and L, we know $\exists \alpha, f'(0) < \alpha < L - \epsilon_0$, i.e. that it fits between $f'(0)$ but is outside $V_{\epsilon_0}(L)$.

Now, we know $\exists \delta$ such that $\exists x_2 \in V_\delta(0)$ and $f'(x_2) \in V_{\epsilon_0}(L)$, since $\lim_{x \to 0} f'(x) = L$. We also know that $x_1 = 0$ means $f'(x_1) = f'(0)$. Since we have $f'(x_1) = f'(0) < \alpha < L - \epsilon = f'(x_2)$, by Darboux's Theorem, we know $\exists c \in (x_1, x_2) \in V_\delta(0)$ such that $f'(c) = \alpha$.

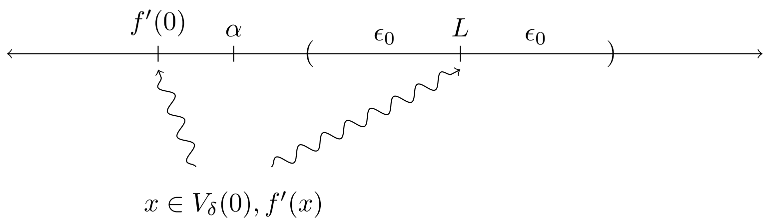


Figure 5.1: Diagram indicating the setup for our contradiction proof.

However, we reach a contradiction here, since we originally chose $\alpha \notin V_{\epsilon_0}(L)$, but if $f'(c) = \alpha$ and $c \in V_\delta(0)$, we must have $f'(c) \in V_{\epsilon_0}(L)$. Therefore, we conclude that $f'(0) = 0$.

(d) **TODO**. The official solutions says this is true, but can't you choose

$$f(x) = \begin{cases} x, & x \neq 0 \\ -1, & x = 0 \end{cases}$$

and then we have $f'(x) = 1$ for $x \neq 0$, and any sequence of $x_n \to 0$ will have $f'(x_n) \to 1$ since $f'(x_n) = 1$ for any $x_n \neq 0$.

However, we have that $f'(0)$ does not exist.

5.3 The Mean Value Theorem

Exercise 5.3.1
We know that by the Mean Value Theorem, $\exists c \in (a,b)$ such that $f'(c) = \frac{f(x)-f(y)}{x-y}$.

Now, since f' is continuous on $[a,b]$, which is a compact set, from the Extreme Value Theorem, we know that f' will achieve some maximum and minimum value on this interval, which means $f'(c) = \frac{f(x)-f(y)}{x-y}$ is bounded, and we can conclude f is Lipschitz on $[a,b]$.

Exercise 5.3.2
By the Mean Value Theorem, we have for any x, y,

$$\left|\frac{f(x)-f(y)}{x-y}\right| = \big|f'(c)\big| < 1$$

for some c in the same closed interval that x, y is on.

This implies that $\big|f(x) - f(y)\big| < s|x-y|$ for $0 < s < 1$.

Exercise 5.3.3

(a) Since h is differentiable on $[0,3]$, it must also be continuous on this set. If we have $h(1) = 2, h(3) = 2$, then by the IVT we must have some c such that $h(c) = c$. (See Exercise 4.5.7)

(b) Applying the Mean Value Theorem directly,

$$\frac{h(3)-h(0)}{3-0} = \frac{2-1}{3} = \frac{1}{3} = h'(c)$$

for some $c \in [0,3]$.

(c) We can apply the MVT to $h(1), h(3)$ to get that $\exists c_2 \in [1,3]$ such that $h'(c_2) = 0$.

Now, Using Darboux's Theorem, since we have $c_1, c_2 \in [0,3]$ such that

$$h'(c_2) = 0 < \frac{1}{4} < \frac{1}{3} = h'(c_1),$$

we conclude $\exists c_3 \in [0,3]$ such that $h'(c_3) = \frac{1}{4}$.

Exercise 5.3.4

(a) Consider the function

$$h(x) = \big[f(b) - f(a)\big]g(x) - \big[g(b) - g(a)\big]f(x).$$

Now, consider

$$\begin{aligned} h(a) &= \big[f(b) - f(a)\big]g(a) - \big[g(b) - g(a)\big]f(a) = f(b)g(a) - g(b)f(a) \\ h(b) &= \big[f(b) - f(a)\big]g(b) - \big[g(b) - g(a)\big]f(b) = g(a)f(b) - f(a)g(b) \\ \Rightarrow \frac{h(a)-h(b)}{a-b} &= \frac{\big[f(b)g(a) - f(a)g(b)\big] - \big[f(b)g(a) - f(a)g(b)\big]}{a-b} \\ &= 0 = h'(c) \end{aligned}$$

this c exists by the MVT. Now, we can look at

$$\begin{aligned} h'(c) &= \big[f(b) - f(a)\big]g'(c) - \big[g(b) - g(a)\big]f'(c) \\ \Rightarrow \big[f(b) - f(a)\big]g'(c) &= \big[g(b) - g(a)\big]f'(c). \end{aligned}$$

(b) I think there are 2 interpretations. The author wants the interpretation where if you set parametric equations

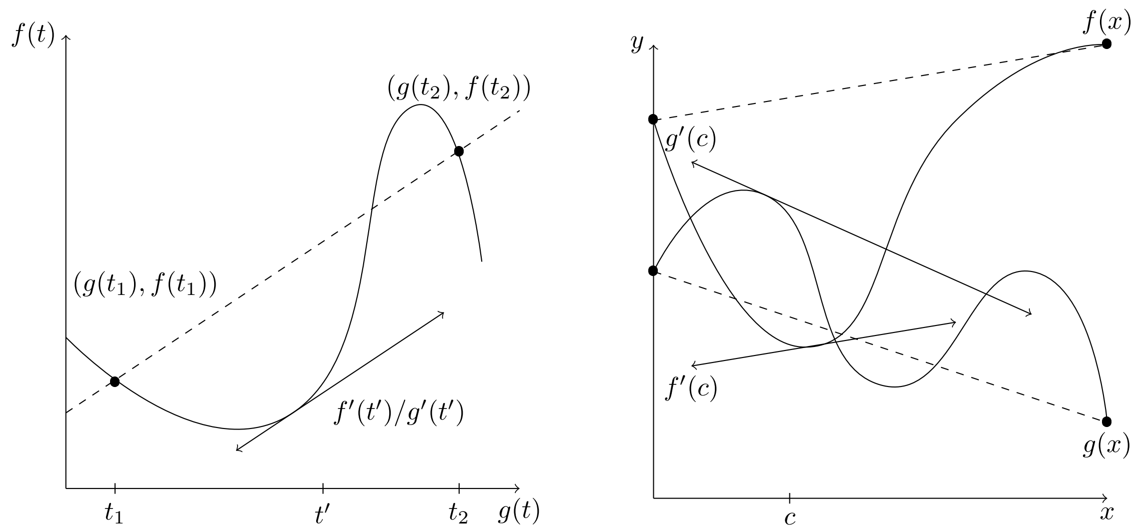
$$\begin{aligned} x(t) &= g(t) \\ y(t) &= f(t) \end{aligned}$$

and consider the $x(t) - y(t)$ plane, then it's basically just the MVT interpretations – the average slope of a curve over an interval exists as a slope somewhere on this interval – but with a different plane.

On the non-parametric plane, we can imagine

$$\frac{f(a) - f(b)}{g(a) - g(b)}$$

as some ratio of the average slope of f over g on the interval $[a, b]$, and the generalized MVT tells us that there is a single point where the slope of f and g has this ratio. Notice that this is stronger than saying there is some $c_1, c_2 \in [a, b]$ such that $f'(c_1)/g'(c_2)$ is equal to this average slope ratio.



(a) Parametric Interpretation (b) Cartesian Interpretation

Figure 5.2: Geometric Interpretations of the Generalized MVT

Exercise 5.3.5

Suppose f has two fixed points, at x_1, x_2. Then we can apply the MVT to show that there exists c such that

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{x_1 - x_2}{x_1 - x_2} = 1$$

which cannot be true, since we assumed $f'(x) \neq 1$. Therefore, we conclude f can have at most one fixed point.

Exercise 5.3.6

We are given that

- $g''(x) > 0$
- $g(0) > 0, g(1) = 1$

($\Rightarrow$) We wanna show $g(d) \to g'(1) > 1$.
We can use the MVT to show that $\exists c \in (0, 1)$ such that

$$g'(c) = \frac{g(1) - g(d)}{1 - d} = \frac{1 - d}{1 - d} = 1.$$

Now, we can again apply MVT to $g'(x)$, to see that $\exists c_2 \in (0, 1)$ such that

$$\frac{g'(1) - g'(c)}{1 - c} + g''(c_2) > 0,$$

where the last part follows because we are given $g''(x) > 0$ for all x. Since $g'(c) = 1$, and $1 - c > 0$, we conclude $g'(1) > g'(c) = 1$.

($\Leftarrow$) Since $g'(1) > 1$, we know $\exists x'$ such that $g(x') < x'$. Now, since $g(0) > 0$, we know $g(0)$ is above the $f(x) = x$ line, while $g(x')$ is below the line. We must have an intersection between the curve $g(x)$ and $f(x) = x$ in $[0, x']$, and this intersection represents a point c when $g(c) = c$. The way to do this with IVT is to define $h(x) = g(x) - x$, and show that $h(c) = 0$ must exist, since $h(0) = g(0) - 0 > 0$, and $h(x') = g(x') - x' < 0$.

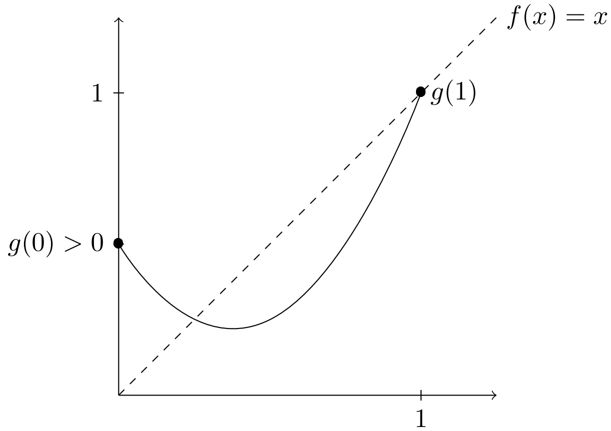


Figure 5.3: Showing how this concave-up curve will intersect $f(x) = x$

As a sidenote, it seems like we can make the same conjecture except without $g''(x) > 0$, and that

Exercise 5.3.7

(a) ($\Rightarrow$) If we know that f is increasing, then consider some $(x_n) \to c$ such that $x_n \geq c$, then since we know f is differentiable on (a, b),

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x_n} \frac{f(x_n) - f(c)}{x_n - c} \geq 0 \qquad (x_n > c)$$

therefore $f'(c) \geq 0$ for any $c \in (a, b)$.

($\Leftarrow$) If we have $\forall c \in (a, b), f'(c) \geq 0$, then we know for any $x, y \in (a, b), x < y$, if we apply the MVT, we get

$$\frac{f(y) - f(x)}{y - x} = f'(b) \geq 0$$

for some $b \in (a, b)$. Now, since $y > x$, it must be the case that $f(y) \geq f(x)$. Since x, y were arbitrary, this shows that f is increasing on this interval.

(b) We have

$$g'(x) = \begin{cases} 1/2 + 2x\sin\big(1/x\big) + -cos(1/x), & x \neq 0 \\ 1/2, & x = 0 \end{cases}$$

We have for $x = 1/(2\pi n)$, that

$$g'(x) = 1/2 + \frac{2}{2\pi n} \cdot 0 - 1 = -1/2$$

Any open interval S containing 0 will have some $V_\delta(0) \in S$, and we can always find some $x_n = \frac{1}{2\pi n} \in V_\delta(0)$, where $g'(x_n) = -1/2 < 0$, which means $g'(x)$ cannot be increasing on any interval containing zero.

Exercise 5.3.8

I think there is a typo in this question, because $c \in V_\delta(c)$ and we certainly have $g(x = c) = g(c)$. But we can find another open set.

Consider $(x_n) \to c$, where $x_n > c$. Then since g is differentiable at c,

$$\lim_{x_n} \frac{g(x_n) - g(c)}{x_n - c} = \lim x \to c \frac{g(x) - g(c)}{x - c} = g'(c) \neq 0 \quad \Rightarrow \quad g(x_n) - g(c) = (x_n - c)g'(c)$$

Since we know $x_n \to c$, we can find some δ such that $|x_n - c| < \delta$ after some $n \geq N$.

Then we will choose our neighborhood as

$$V_{\delta/2}(c + \delta/2)$$

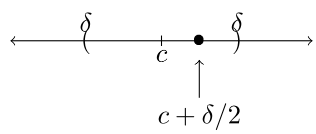


Figure 5.4: Choosing a δ-neighborhood

Exercise 5.3.9

For any $M > 0$, first let $\epsilon_1 = |L|/2$, this way $L \pm \epsilon_1 \neq 0$. We know we can find δ_1 such that $x \in V_{\delta_1}(c)$ implies $\left|f(x) - L\right| < \epsilon_1$.

Now let $\epsilon_2 = \frac{\left||L| - |\epsilon_1|\right|}{M}$. Since $\lim x \to cg(x) = 0$, we can choose δ_2 such that $|x - c| < \delta_2$ implies $\left|g(x) - 0\right| < \epsilon_2$¿

Now, choose $\delta = \min(\delta_1, \delta_2)$, then for any $x \in V_\delta(c)$

$$\frac{f(x)}{g(x)} \geq \frac{\left||L| - |\epsilon_1|\right|}{\frac{\left||L| - |\epsilon_1|\right|}{M}} = M$$

Exercise 5.3.10

If f is a bounded function, let $\ell = \min_x \left|f(x)\right|$, then we can do Exercise 5.3.9 except we choose $\epsilon = \frac{\ell}{M}$, and then the δ that comes along with that.

Exercise 5.3.11

Since we know

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

that means we can find a δ-neighborhood around a such that if $x \in V_\delta(a)$ implies this limit is ϵ close to L.

Now, if we apply the GMVT to some points $x, a \in V_\delta(a)$, WLOG $x > a$, then we have

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

for some $c \in (a, x) \subseteq V_\delta(a)$. Since this c is also in $V_\delta(a)$, we conclude that

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \epsilon$$

which proves that $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L$.

Exercise 5.3.12
If we know that f, g are differentiable at a and f', g' are continuous at a, then we can use properties of the derivative,

$$\begin{aligned}
L &= \lim_{x \to a} \frac{f'(x)}{g'(x)} \\
&= \frac{\lim_{x \to a} f'(x)}{\lim_{x \to a} g'(x)} \\
&= \frac{f'(a)}{g'(a)} && \text{(Since } f', g' \text{ cont. at } a) \\
&= \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} && \text{(Def. of derivative)} \\
&= \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} && \text{(Limit properties)} \\
&= \lim_{x \to a} \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} \\
L &= \lim_{x \to a} \frac{f(x)}{g(x)}
\end{aligned}$$

Exercise 5.3.13
With this slightly weaker hypothesis, an idea is to bound the difference

$$\left| \frac{f(x)}{g(x)} - \frac{f(x) - f(a)}{g(x) - g(a)} \right| < \epsilon$$

and then we can do the proof as we did before in Exercise 5.3.11.

5.4 A Continuous Nowhere-Differentiable Function

Exercise 5.4.1
Sketch of $(1/2)h(2x)$ on $x \in [-2, 3]$.

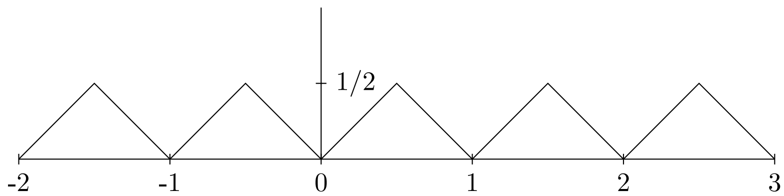


Figure 5.5: $(1/2)h(x)$

Notice that this is just the original $h(x)$ plot but y-scaled down by half and the x-scale is squeezed towards the origin by half.

If we consider $h_n(x) = \frac{1}{2^n}h(2^n x)$ in general, it will be y-scaled by $1/2^n$ and x-squeezed by 2^n.

Exercise 5.4.2
We can use the fact that $h(x) \leq 1$, to show that

$$\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x) \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

Since $g(x)$ is defined at every $x \in \mathbb{R}$, it is properly defined.

Exercise 5.4.3
Since we know $f(x) + g(x)$ is continuous for f, g continuous, and that $kf(x)$ is continuous if f is continuous, we see that $g_m(x)$ is a sum of constant factored continuous functions $h(x)$, and therefore is continuous itself.

This is using the Algebraic Continuity Theorem from Chapter 4.

Exercise 5.4.4
First, it is useful to show that

$$n \in \mathbb{Z}, h(2n) = 0$$

We know this is true because $h(x)$ is periodic every width 2, and $h(0) = 0$.

$$\begin{aligned}
g'(0) &= \lim_{x_m} \frac{g(x_m) - g(0)}{x_m - 0} \\
&= \frac{\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n 2^{-m}) - \sum_{n=0}^{\infty} \frac{1}{2^n} h(0)}{2^{-m}} \\
&= 2^m \left[\sum_{n=0}^{m} 2^{-n} h(2^{n-m}) + \sum_{n=m+1}^{\infty} 2^{-n} h(2^{n-m}) \right] \\
&= 2^m \left[\sum_{n=0}^{m} 2^{-n} 2^{n-m} + \sum_{n=m+1}^{\infty} 2^{-n} \cdot 0 \right] && \text{(We know that } h(2n) = 0\text{)} \\
&= \sum_{n=0}^{m} 1 \\
&= m + 1
\end{aligned}$$

Since $m + 1$ does not converge, we conclude that $g'(0)$ does not exist.

Exercise 5.4.5

(a) We can choose $x_m = 1 - \frac{1}{2^m}$, which means $x_m \to 1$ and use the same argument as Exercise 5.4.4 to show that $g'(1)$ does not exist. The only differences are that first, we have to argue that

$$h\left(2^n\left(1 - \frac{1}{2^m}\right)\right) = \begin{cases} 1 - \frac{1}{2^m}, & n = 0 \\ 2^{n-m}, & 1 \le n \le m \\ 0, & n > m \end{cases} \tag{5.1}$$

This is the case, because when $n \le m$, we have that $0 \le 2^{n-m} \le 1$, and we know that $h(x) = x$ when $x \in [0, 1]$. When we have $n > m$, $2^{n-m} = 2k$, so we know $h(2k) = 0$.

Second, our denominator is $x_m - 1$, which will actually help us simplify this expression, see below...

$$\begin{aligned}
\frac{\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x_m) - \sum_{n=0}^{\infty} \frac{1}{2^n} h(1)}{\left(1 - \frac{1}{2^m}\right) - 1} &= \frac{\left(1 - \frac{1}{2^m} + \sum_{n=1}^{m} 2^{-n} \cdot 2^{n-m} + \sum_{n=m+1}^{\infty} 2^{-n} \cdot 0\right) - 1}{-\frac{1}{2^m}} \\
&\qquad\qquad \text{(Using (5.1))} \\
&= -2^m \left[-\frac{1}{2^m} + \sum_{n=1}^{m} 2^{-m} \right] \\
&= 1 - \sum_{n=1}^{m} 1 \\
&= -m + 1,
\end{aligned}$$

which does not converge.

For $g'(1/2)$, we can first compute that $g(1/2) = h\left(\frac{1}{2}\right) + \frac{1}{2} h(1) = 1$. Then, we choose $x_m = \frac{1}{2} - \frac{1}{2^m}$, and compute as usual,

$$\begin{aligned}
\frac{\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x_m) - \sum_{n=0}^{\infty} \frac{1}{2^n} h(1/2)}{\left(1/2 - \frac{1}{2^m}\right) - 1/2} &= \frac{\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^{n-1} - 2^{n-m}) - 1}{-\frac{1}{2^m}} \\
&= \frac{\left(h\left(\frac{1}{2} - \frac{1}{2^m}\right) + h\left(1 - \frac{2}{2^m}\right) + \sum_{n=2}^{m} 2^{-n}\left(2^{n-1} - 2^{n-m}\right)\right) - 1}{-\frac{1}{2^m}} \\
&= -2^m \left[\frac{3}{2} - \frac{3}{2^m} \sum_{n=2}^{m} 2^{-m} - 1 \right] \\
&= 3 \cdot 2^{m-1} - \sum_{n=0}^{m} -2^{m-1} + 3 \\
&= 3 \cdot 2^{m-1} - m + 4.
\end{aligned}$$

(b) We can define for any $x = p/2^k$ the sequence

$$x_m = \frac{p}{2^k} - \frac{1}{2^m}$$

and do what we've been doing in part (a).

Exercise 5.4.6

(a) h_m is not differentiable at the multiples of $\frac{1}{2^m}$, but differentiable everywhere else. Since x is between a multiple of any power of 2, we conclude that g_m is differentiable at x.

We know that

$$\left|g'_{m+1}(x) - g'_m(x)\right| = \left|\sum_{i=0}^{m+1} h'_i - \sum_{j=0}^{m} h'_j\right| = \left|h'_{m+1}\right| = 1$$

since the derivative of $h'_k(x)$ at anywhere defined will be either 1 or -1.

(b) The key observation for this problem is that

$$g(x_m) = g_m(x_m) + \sum_{n=m+1}^{\infty} 2^{-n} h(p2^{n-m}) = g_m(x_m).$$

And same for $g(y_m) = g_m(y_m)$. In addition, notice that on the interval $[x_m, y_m]$, it must be a straight line, since for g_m, the granularity of the sawtooths is at $1/2^m$. Therefore, we can conclude

$$g'_m(x) = \frac{g(y_m) - g(x_m)}{y_m - x_m}$$

since x sits between x_m, y_m, adjacent dyadic numbers.

We also need to see that $g(x) > g_m(x_m)$, since $g(x) = g(x_m) + \text{stuff with } h(x)$, and $h(x) \geq 0$.

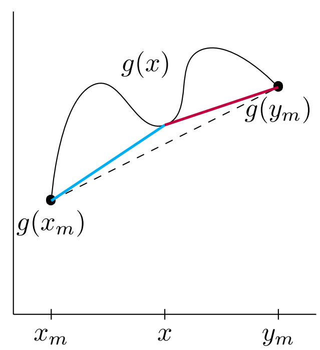


Figure 5.6: $g'_m(x)$ is dashed, $g(x)$ is solid.

Now the rest of the problem is easy, since we can use our above graph to see geometrically,

$$\frac{g(y_m) - g(x)}{y_m - x} < g'_m(x) \qquad \text{(Purple line in Figure 5.6)}$$
$$\frac{g(x_m) - g(x)}{x_m - x} > g'_m(x) \qquad \text{(Cyan line in Figure 5.6)}$$

(c) I don't think we even need part (b).

We know from part (a) that $g'_m(x)$ is not a Cauchy sequence, so therefore it does not converge, and we can conclude tht $g'(x)$ does not exist.

Exercise 5.4.7
We review our Cauchy Sequence argument for $g'_m(x)$, which was

$$\left|g'_{m+1}(x) - g'_m(x)\right| = \left|h'_{m+1}(x)\right|$$

With $\sum_{n=0}^{\infty}(1/2^n)h(3^n x)$, we get that

$$\left|g'_{m+1}(x) - g'_m(x)\right| = \frac{1}{2^m+1} \cdot 3^{m+1} = \left(\frac{3}{2}\right)^{m+1}$$

which means that the sequence diverges, and we can still conclude $g'(x)$ does not exist.

On the other hand, with $\sum_{n=0}^{\infty}(1/3^n)h(2^n x)$, we get that

$$\left|g'_{m+1}(x) - g'_m(x)\right| = \frac{1}{3^m + 1} \cdot 2^{m+1} = \left(\frac{2}{3}\right)^{m+1},$$

which actually does converge, so we cannot use this argument.

Chapter 6

Sequences and Series of Functions

6.2 Uniform Convergence of a Sequence of Functions

Exercise 6.2.1

(a) We take $\lim n \to \infty$,

$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = \frac{x}{x^2} = \frac{1}{x}$$

(b) We can see for

$$\left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| = \frac{1}{x+nx^3}$$

which requires us to choose N as

$$N > \frac{1-\epsilon x}{\epsilon x^3}.$$

this quantity grows as $x \to 0$, so this is not converging uniformly.

(c) No, since x can tend towards 0, this will still not converge.

(d) Yes, since we can bound $x > 1$, and choose

$$N > \max\left(\frac{1}{\epsilon} - 1, 1\right) > \frac{1-\epsilon x}{\epsilon x^3}.$$

Exercise 6.2.2

Pointwise limit is $x/2$, since $\left|\sin(nx)\right| \leq 1$.

It is uniformly convergent on all of $\mathbb{R}$, since

$$\left|\frac{nx+\sin(nx)}{2n} - \frac{x}{2}\right| = \left|\frac{\sin(nx)}{2n}\right| \leq \frac{1}{2n}$$

so we can choose $N > 1/(2\epsilon)$.

Exercise 6.2.3

(a) Pointwise limit should be split into a few cases. The idea is that $x^n \to 0$ if $x \in [0,1)$, but $x^n \to \infty$ if $x \in (1,\infty)$. So we have

$$\lim_{n\to\infty} h_n(x) = \begin{cases} x, & x < 1 \\ 1/2, & x = 1 \\ 0, & x > 1 \end{cases}$$

(b) The limit is discontinuous, at $x = 1$, $x \to 1/2 \to 0$.

(c) We can show that this sequence converges on $(1,2)$,

$$
\begin{aligned}
\left|\frac{x}{1+x^n} - 0\right| &< \epsilon \\
x &< \epsilon + x^n\epsilon \\
x - \epsilon &< x^n\epsilon \\
\ln(x-\epsilon) &< n\ln(x) + \ln(\epsilon) \\
\ln\left(\frac{x}{\epsilon} - 1\right) &< n\ln(x) \\
\frac{\ln\left(\frac{x}{\epsilon}-1\right)}{\ln(x)} \leq \frac{\ln\left(\frac{2}{\epsilon}-1\right)}{\ln(2)} &< n
\end{aligned}
$$

Exercise 6.2.4
$f_n(x)$ achieves optima points at

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} = 0$$

so when $x = \pm\sqrt{\frac{1}{n}}$. At these points, $f_n(x) = \pm\frac{1}{2\sqrt{n}}$.

The limit function is 0, since the denominator goes to ∞. We can then prove convergence by saying $\left|f_n(x) - 0\right| \leq \frac{1}{2\sqrt{n}}$, and then just choose $n > \frac{1}{4\epsilon^2}$.

Exercise 6.2.5
(a)

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1, & |x| \geq 0 \\ 0, & |x| < 0 \end{cases}$$

This is not uniform convergence since at $x = 0$, we have a discontinuity.

(b) The idea is that our function from part (a) is continuous everywhere, but has freedom to change the convergence point at $x = 1/n$. If we change this point to converge to n instead of 1, so make

$$f_n(x) = \begin{cases} 1/x, & |x| \geq 1/n \\ n^2|x|, & |x| < 1/n \end{cases}$$

we have that the limit function goes to infinite at $x = 1/n$.

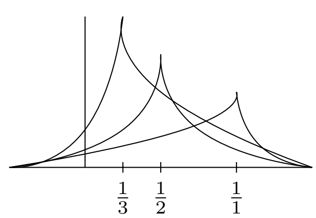


Figure 6.1: The convergence at $x = 1/n$ rises

Exercise 6.2.6
($\Rightarrow$) If we know (f_n) converges uniformly, choose N such that $\left|f_n(x) - f(x)\right| < \epsilon/2$, then we have for this N that if we choose $n, m > N$,

$$\left|f_n(x) - f_m(x)\right| = \left|f_n(x) - f(x) - (f_m(x) - f(x))\right| \leq \left|f_n(x) - f(x)\right| + \left|f_m(x) - f(x)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

($\Leftarrow$) We know for some N that for $m, n \geq N$, that

$$\left|f_n(x) - f_m(x)\right| < \epsilon \Rightarrow -\epsilon < f_n(x) - f_m(x) < \epsilon$$

Now, we are going to claim that $f(x) = \lim_{n\to\infty} f_n(x)$.

Using the Algebraic Limit Theorem, we can say

$$\lim_{n\to\infty} -\epsilon < \lim_{n\to\infty} \big(f_n(x) - f_m(x)\big) < \lim_{n\to\infty} \epsilon$$
$$-\epsilon < f(x) - f_m(x) < \epsilon$$

so we can conclude

$$\big|f_m(x) - f(x)\big| < \epsilon$$

for this N we chose in the beginning, and we have shown uniform convergence.

Exercise 6.2.7

We have that

$$\begin{aligned}\big|f(x) - f(y)\big| &= \big|f(x) - f_n(x) + f_n(x) - f_n(y) - (f(y) - f_n(y))\big| \\ &\le \big|f(x) - f_n(x)\big| + \big|f_n(x) - f_n(y)\big| + \big|f(y) - f_n(y)\big| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

We know from the uniform convergence $f_n \to f$ that we can find N_x, N_y such that for $n_x \ge N_x, n_y \ge N_y$, that

$$\big|f_{n_x}(x) - f(x)\big| < \epsilon/3, \big|f_{n_y}(y) - f(y)\big| < \epsilon/3$$

Choose $N = \max\big(N_x, N_y\big)$. Then, from the uniform convergence of every f_n for fixed $n \ge N$, with $\exists\delta, |x - y| < \delta$, that

$$\big|f_n(x) - f_n(y)\big| < \epsilon/3$$

Choose this δ, after finding N, and our proof is complete.

Exercise 6.2.8

(a) We saw that

$$f_n(x) = \frac{x^2 + nx}{n}$$

converges pointwise on $\mathbb{R}$, a compact set, to x, but is not uniformly continuous, since larger x values require larger N's to bound.

(b) Yes, we just need to account for the bounded value of g.

(c) We could consider $f_n(x) = nx^2$ over the interval $[-1, 1]$. Each $\big|f_n(x)\big| \le n$, but as $n \to \infty$, this is no longer bounded. However, this example doesn't work since nx^2 does not converge to a limit function.

To show that f is bounded if it converges uniformly, since we can pick some fixed $\epsilon > 0$, and find N such that

$$\big|f_N(x) - f(x)\big| \le \epsilon$$

Now, we know that $f_N(x) \le M$ for all x if N is fixed, so therefore we can conclude that

$$\big|f(x)\big| \le \epsilon + M,$$

for all x, and therefore f is bounded.

(d) Yes, we can choose the larger of N_a, N_b to be our N, and it will satisfy the uniform convergence criteria.

(e) We know that for $y > x$

$$\forall n, f_n(y) > f_n(x)$$

now, we can use the Order Limit Theorem, to get

$$\lim_{n\to\infty} f_n(y) > \lim_{n\to\infty} f_n(x)$$
$$f(y) > f(x) \qquad \text{(Since } f_n \to f\text{)}$$

(f) Our proof in part (e) did not require uniform convergence, just convergence, so we can use the same proof.

Exercise 6.2.9
Since K is compact, g is bounded over K. Now,

$$\left|\frac{f_n(x)}{g(x)} - \frac{f(x)}{g(x)}\right| = \left|1/g(x)\right|\left|f_n(x) - f(x)\right| < \epsilon$$

If we have $g(x)$ bounded, we can say $1/g(x) \leq M'$ then we can choose N such that for $n \geq N$,

$$\left|f_n(x) - f(x)\right| < \epsilon/M'$$

Exercise 6.2.10
For any $\epsilon > 0$, we know $\exists\delta$ such that for $|x - y| < \delta$, $\left|f(x) - f(y)\right| < \epsilon$. We can choose $1/n < \delta \Rightarrow n > 1/\delta$, to show that

$$\left|f_n(x) - f(x)\right| = \left|f(x + 1/n) - f(x)\right| < \epsilon,$$

since $\left|\left(x + \frac{1}{n}\right) - x\right| = \left|\frac{1}{n}\right| < \delta$.

Now, with just normal continuity, we can choose a function like $f(x) = x^2$, which will give us

$$\left|f(x + 1/n) - f(x)\right| = \left|2x/n + 1/n^2\right|$$

which grows bigger as x gets bigger.

Exercise 6.2.11

(a) This is a straightforward application of the triangle inequality,

$$\left|\big(f_n(x) + g_n(x)\big) - \big(f(x) + g(x)\big)\right| \leq \left|f_n(x) - f(x)\right| + \left|g_n(x) - g(x)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(b) Try $f_n(x), g_n(x) = \frac{1}{x} + \frac{1}{n}$, then we run into issues with bounding because their limit function is $f(x)g(x) = \frac{1}{x^2}$, but we have troubles with

$$\left|\left(\frac{1}{x^2} + \frac{2}{xn} + \frac{1}{n^2}\right) - \frac{1}{x^2}\right| = \left|\frac{2}{xn} + \frac{1}{n^2}\right|,$$

which grows as $x \to 0$.

(c) This is the trick where you add and subtract $f_n(x)g(x)$ inside the absolute value, take the triangle inequality, and bound. The only tricky part is that we need to bound $\left|g(x)\right|\left|f_n(x) - f(x)\right|$, but we can do this because g_n is bounded for all n, and we know that g_n is uniformly convergent.

Exercise 6.2.12

(a) Fix some $x \in K$. Since f_n is increasing in n, $g_n(x) = f(x) - f_n(x)$ is decreasing in n. We also have that $g_n(x)$ is continuous, since it is the subtraction of two continuous functions, f, f_n. In addition, we have that $g_n(x) \to 0$, since $f_n \to f$.

(b) This means for any $x \in K_j$, since we have $g_j(x) \geq \epsilon$, it must be the case that for $i < j, g_i(x) \geq \epsilon$ as well, since $g_i(x) \geq g_j(x)$. Therefore, $x \in K_i$.

This shows that

$$K_i \supseteq K_j \Rightarrow K_1 \supseteq K_2 \supseteq K_3 \supseteqq \cdots \tag{6.1}$$

We also want to have that every K_n is compact, so we can use the Nested Interval Property. For any sequence $x_m \to x$, since g is continuous, we must have

$$\lim_{x_m \to x} g_n(x_m) = g_n(x)$$

By the properties of K_n, since $g_n(x_m) \geq \epsilon$, we know by the Order Limit Theorem that $g_n(x) \geq \epsilon$ as well. This means that $x \in K_n$, by definition of K_n. Therefore, since K_n has all of its limit points, it is a closed set. Since $K_n \subseteq K_{n-1}$, which is compact (induction...), and therefore bounded, we conclude that K_n is closed and bounded, i.e. compact.

Now, from Equation 6.1, if we AFSOC that every $K_n \neq \emptyset$, we know that from the NIP that $\exists x \in K_n$ for all n, which means $g_n(x) \to \epsilon_0 \geq \epsilon > 0$. But this is a contradiction, since $g_n(x) \to 0$.

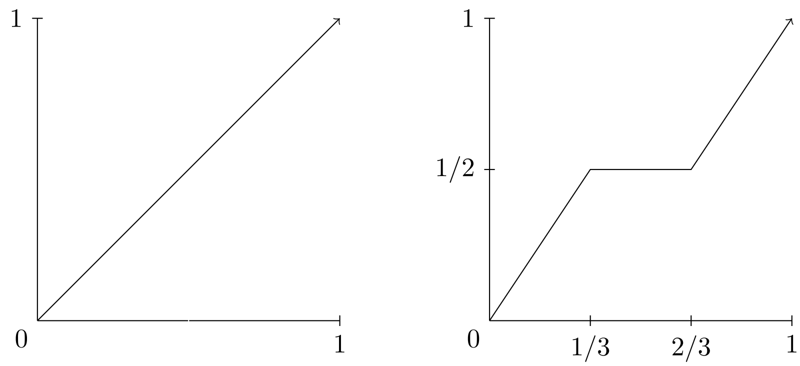
So therefore we must have that $\exists K_n = \emptyset$, which means

$$\exists n, g_n(x) < \epsilon \Rightarrow \left|f(x) - f_n(x)\right| < \epsilon$$

for all x, which means that $f_n \to f$ uniformly.

Exercise 6.2.13

(a) Sketches



(a) $f_0(x)$ (b) $f_1(x)$

Figure 6.2: Sketches for $f_n(x), n = 0, 1$

(b) Sketching $f_2(x)$,

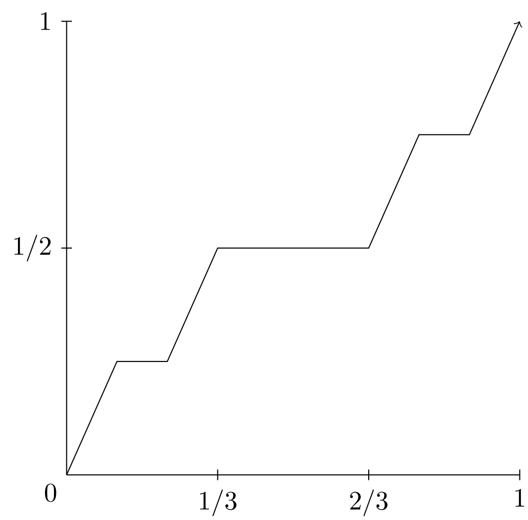


Figure 6.3: Sketch for $f_2(x)$

(c) For any $m < n$, the difference between $f_m(x)$ and $f_n(x)$ will be at the first time f_m is a upward sloping line, while f_n potentially has more "kinks" in it.

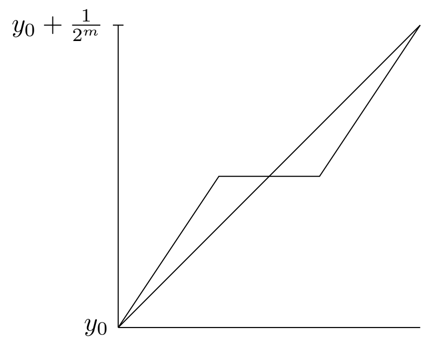


Figure 6.4: Difference between f_m, f_n at kinks

But what we notice is despite the kinks, since they happen between these upward lines of f_m, which are at most height $\frac{1}{2^m}$, we can say that

$$\left|f_m(x) - f_n(x)\right| \leq \frac{1}{2^m}. \tag{6.2}$$

This means for any $\epsilon > 0$, we can find $\frac{1}{2^m} < \epsilon$, and show the Cauchy Criterion for Uniform Convergence for function sequences, which implies uniform convergence for f_n.

(d) Every f_n is continuous and increasing, and we showed that $f_n \to f$ uniformly, so therefore f is also continuous and increasing.

We have $f'(x) = 0 \ \forall x \in [0,1] \setminus C$, since these are the portions where the function is flat.

Exercise 6.2.14

(a) $f_n(x_1)$ contains a convergent subsequence because $f_n(x)$ is bounded, and this is a direct application of the Bolzano-Weierstrass Theorem.

(b) Same reason as above, we stil have a bounded sequence on reals, so we can still apply Bolzano-Weierstrass.

(c) We can construct f_n by the following

$$f_n(x) = \text{First element of } f_{n,k}$$

since every $f_{n,k} \in f_{m,k}$ for $m < n$, we conclude that for any $x_m \in A$, $f_n(x_m) \in f_{m,k}$ after $n > m$, and thus this $f_n(x_m)$ for $n > m$ is a convergent subsequence, and therefore f_n converges at every point of A.

Exercise 6.2.15

(a) The difference is that we need a single δ that works for $|x - y| < \delta$, whereas if we say for each f_n is uniformly continuous, we can choose a different δ for each f_n. This equicontinuous definition is stronger.

(b) We saw that as $n \to \infty$, x_n has a jump at 1, where for $x < 1$, $x \to 0$, and $x = 1$ if $x = 1$. This means for any ϵ, δ, we can always choose a large enough n so that the δ-neighborhood is not small enough.

Each g_n is uniformly continuous, since we can say

$$\begin{aligned}
\left|g_n(x) - g_n(y)\right| &= |x^n - y^n| \\
&= \left|(x-y)\left(\sum_{i=1}^{n} x^{n-i}y^i\right)\right| \\
&\leq |x-y|n && \text{(Since } x, y \in [0,1], x^k y^k \leq 1) \\
\Rightarrow \delta &< \frac{\epsilon}{n} && \text{(Will win the } \epsilon\text{-challenge)}
\end{aligned}$$

Exercise 6.2.16

(a) $\mathbb{Q}$ is countable, so we can find this $g_k = f_{n_k}$ that converges at every $q \in \mathbb{Q}$ on $[0,1]$.

(b) We know for every r_i, $\exists N_i, s, t \geq N_i$ such that

$$\left|g_s(r_i) - g_t(r_i)\right| < \frac{\epsilon}{3},$$

since we know g_k converges so all $r \in \mathbb{Q}$ on $[0,1]$, so we can take $N = \max_{i=1}^{m}(N_i)$.

We need $\{r_i\}_{i=1}^m$ to be finite since we wanted to take the max N from $\{N_i\}_{i=1}^m$, which may not exist if the set is not finite.

(c) We can show for any $x \in [0,1]$, choose $r \in \mathbb{Q}$ such that $|x - r| < \delta$, then

$$\begin{aligned}
\left|g_s(x) - g_t(x)\right| &= \left|\big(g_s(x) - g_s(r)\big) + \big(g_s(r) - g_t(r)\big) - \big(g_t(x) - g_t(r)\big)\right| \\
&\leq \left|g_s(x) - g_s(r)\right| + \left|g_s(r) - g_t(r)\right| + \left|g_t(x) - g_t(r)\right| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

6.3 Uniform Convergence and Differentiation

Exercise 6.3.1

(a)
$$\left|\frac{\sin(nx)}{n}\right| \leq \frac{1}{n} \qquad (\left|\sin(y)\right| < 1)$$

We first find
$$h_n'(x) = \frac{n\cos(nx)}{n} = \cos(nx)$$
this sequence has issues converging when $n \to \infty$ for $\cos(nx)$, but does converge when $x = 2\pi k$, since $\cos(2\pi kn) = 1$. The official solutions says $x = \pi/2 + \pi k$, which works but I think misses on the $x = 2\pi k$ solution. We can verify $x = \pi/2$ for example,
$$\cos\big(\pi/2\big), \cos\big(3\pi/2\big), \cos\big(5\pi/2\big), \cos\big(7\pi/2\big), \dots \Rightarrow 0, 0, 0, 0, \dots$$

(b) Use $h_n(x) = \frac{\sin(n^2 x)}{n}$, then
$$h_n'(x) = n\cos\Big(n^2 x\Big)$$

Exercise 6.3.2

(a) We see that $\lim g_n = 0$ since $x^n \leq 1$, while the denominator grows with n linearly. Then,
$$\left|\frac{x^n}{n} - 0\right| \leq \frac{1}{n},$$
which shows uniform convergence.

$g(x) = 0$, so $g'(x) = 0$.

(b) We have
$$g_n'(x) = \frac{nx^{n-1}}{n} = x^{n-1}$$
which exists $\forall x \in [0, 1]$.

We can show that $g_n'(x)$ converges to 0 for $x < 1$, and 1 if $x = 1$, so we run into problems for convergence if $x = 1$.

We have
$$h = \lim g_n' = \begin{cases} 0, & x \in [0, 1) \\ 1, & x = 1 \end{cases}$$
which is $\neq g'(x) = 0$.

This tells us in general,
$$\lim g_n' = h \neq g'$$

Exercise 6.3.3

We have
$$f = \lim f_n = 0$$
and
$$f_n'(x) = \frac{1 + nx^2 - x(2nx)}{\left(1 + nx^2\right)^3} = \frac{1 - nx^2}{\left(1 + nx^2\right)^2}$$
We know $f(x) = 0$, so $f'(x) = 0$. We see that for $x = 0$,
$$f_n'(0) = \frac{1 - 0}{\left(1 + 0\right)^2} = 1 \neq 0 = f'(x),$$
but for all other values of x, we end up with
$$\frac{\text{stuff}}{\text{less dominant terms} + n^2x^4} \to 0 = f'(x)$$

Exercise 6.3.4

(a) If we algebraically calculate the limit,

$$g(x) = \lim_{n\to\infty} \frac{nx + x^2}{2n} = \frac{x}{2}$$

Then

$$g'(x) = \frac{1}{2}$$

(b) We can find that

$$g_n'(x) = \frac{n + 2x}{2n} = \frac{1}{2} + \frac{x}{n}$$

On any interval $x \in [-M, M]$, we have

$$\left|g_n'(x) - g(x)\right| = \left|\frac{1}{2} + \frac{x}{n} - \frac{1}{2}\right| = \left|\frac{x}{n}\right| \leq \frac{M}{n}$$

so a choice of $n > \frac{M}{\epsilon}$ will suffice.

To show that $g'(x) = \lim g_n'(x)$, we need

- g_n be a sequence of differentiable functions, yes we have this.
- g_n' converge uniformly to g, yes.
- $x_0 \in [M, -M]$ such that $g_n(x_0)$ is convergent. We can choose $x = 0$, then $g_n(0) \to 0 = g(0)$.

(c) algebraically calculating the limit,

$$f(x) = \lim_{n\to\infty} \frac{nx^2 + 1}{2n + x} = \frac{x^2}{2}$$

So therefore

$$f'(x) = 2x/2 = x$$

Now,

$$f_n'(x) = \frac{(2nx)(2n + x) - (nx^2 + 1)(1)}{(2n + x)^2} = \frac{4n^2x + nx^2 - 1}{4n^2 + 4nx + x^2}$$

To show uniform convergence,

$$\begin{aligned}
\left|\frac{4n^2x + nx^2 - 1}{4n^2 + 4nx + x^2} - x\right| &= \left|\frac{4n^2x + nx^2 - 1}{4n^2 + 4nx + x^2} - \frac{4n^2x + 4nx^2 + x^3}{4n^2 + 4nx + x^2}\right| \\
&= \left|\frac{-3nx^2 - x^3 - 1}{4n^2 + 4nx + x^2}\right| \\
&\leq \frac{\left|3nx^2\right| + \left|x^3\right| + |1|}{|4n^2 + 4nx + x^2|} && \text{(Triangle Inequality)} \\
&\leq \frac{\left|3nx^2\right| + \left|x^3\right| + |1|}{|4n^2 - 4nM|} && \text{(Making the denominator smaller)} \\
&\leq \frac{3nM^2 + M^3 + 1}{|4n^2 - 4nM|}
\end{aligned}$$

Since $n \to \infty$ makes this expression $\to 1/n$, we can make this quantity as small as we want, to be smaller than any $\epsilon > 0$.

To summarize, we have

- f_n all differentiable

- $f_n' \to x = f'$ uniformly
- f_n is convergent for any x.

Exercise 6.3.5
We want to show for any $x \in [a,b]$ that

$$\left|f_n(x) - f(x)\right| < \epsilon.$$

We can try to prove the Cauchy Criterion instead, so

$$\begin{aligned}
\left|f_n(x) - f_m(x)\right| &= \Big|f_n(x) - f_m - \big(f_n(x_0) - f_m(x_0)\big) + \big(f_n(x_0) - f_m(x_0)\big)\Big| \\
&= \Big|f_n(x) - f_m - \big(f_n(x_0) - f_m(x_0)\big)\Big| + \left|f_n(x_0) - f_m(x_0)\right| \\
&= \Big|\big(f_n(x) - f_n(x_0)\big) - \big(f_m(x) - f_m(x_0)\big)\Big| + \left|f_n(x_0) - f_m(x_0)\right|
\end{aligned}$$

Now, we first choose N_1 such that for $m, n \geq N_1$,

$$\left|f_n(x_0) - f_m(x_0)\right| < \frac{\epsilon}{2}$$

Now, we apply MVT to

$$\begin{aligned}
\Big|\big(f_n(x) - f_n(x_0)\big) - \big(f_m(x) - f_m(x_0)\big)\Big| &= |x - x_0|\left|\left(\frac{f_n(x) - f_n(x_0)}{x - x_0}\right) - \left(\frac{f_m(x) - f_m(x_0)}{x - x_0}\right)\right| \\
&= M\left|f_n'(c_1) - f_n'(c_2)\right| \qquad (|x - x_0| \text{ bounded since } x, x_0 \in [a,b])
\end{aligned}$$

By MVT, $c_1, c_2 \in [a,b]$. Since we know that f_n' converges uniformly on $[a,b]$, we know we can find N_2 such that

$$\left|f_n'(c_1) - f_n'(c_2)\right| < \frac{\epsilon}{2M}$$

Putting everything together, if we choose $N = \max(N_1, N_2)$, we can show that for $m, n \geq N$, that

$$\left|f_n(x) - f_m(x)\right| = \Big|\big(f_n(x) - f_n(x_0)\big) - \big(f_m(x) - f_m(x_0)\big)\Big| + \left|f_n(x_0) - f_m(x_0)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

6.4 Series of Functions

Exercise 6.4.1
AFSOC $g_n \to \epsilon_0 \neq 0$. Then we can find some N where $|g_n - \epsilon_0| < \epsilon_0/2$. WLOG $\epsilon_0 > 0$, then we will be summing an infinite number of positive numbers, which diverges to $+\infty$.

Exercise 6.4.2
WLOG $k_1 < k_2$,

$$\begin{aligned} |s_{k_1} - s_{k_2}| &\leq \left| \sum_{n=k_1+1}^{k_2} f_n(x) \right| \\ &\leq \sum_{n=k_1+1}^{k_2} \left| f_n(x) \right| \\ &\leq \sum_{n=k_1+1}^{k_2} M_n && (\left| f_n(x) \right| < M_n) \\ &\leq \left| \sum_{n=k_1+1}^{k_2} M_n \right| < \epsilon \end{aligned}$$

The last part follows, since we know $\sum_{n=1}^{\infty} M_n$ converges, so we can find this N such that for $k_1, k_2 \geq$, we have convergence.

Exercise 6.4.3
(a) WLOG $k_1 < k_2$, then we have

$$\begin{aligned} \left| g_{k_1}(x) - g_{k_2}(x) \right| &= \left| \sum_{n=1}^{k_1} \frac{\cos(2^n x)}{2^n} - \sum_{n=1}^{k_2} \frac{\cos(2^n x)}{2^n} \right| \\ &= \left| \sum_{n=k_1+2}^{k_2} \frac{\cos(2^n x)}{2^n} \right| \\ &\leq \left| \sum_{n=k_1+2}^{k_2} \frac{1}{2^n} \right| \\ &\leq \left| \sum_{n=k_1+2}^{\infty} \frac{1}{2^n} \right| \\ &\leq \left| \frac{1}{2^{k_1-1}} \right| \end{aligned}$$

So we just need to choose $N > \log_2\left(\frac{2}{\epsilon}\right)$ Since $g(x)$ converges uniformly, and each

$$f_n = \frac{\cos(2^n x)}{2^n}$$

is continuous, since we can find a derivative that is defined everywhere, we conclude that $g(x)$ is continuous.A

Ok so apparently I did way more work than you had to, you can use

$$M_n = \frac{1}{2^n} \geq \left| \cos(2^n x)/2^n \right|$$

and the Weierstrass M-Test to prove uniform convergence too...

(b) Each

$$f_n = x^n/n^2 \Rightarrow f_n' = x^{n-1}/n$$

has a derivative that exists, so we conclude each f_n is continuous. We just need to show that $h(x)$ converges uniformly.

Define

$$M_n = \frac{1}{n^2} \geq \left|\frac{x^n}{n^2}\right|$$

This is true since $|x^n| \leq 1$ since $x \in [-1, 1]$. Now, since $\sum_{n=1}^{\infty} 1/n^2$ converges, we can use the Weierstrass M-Test to conclude that $h(x)$ converges uniformly.

Exercise 6.4.4
We can show that

$$M_n = \frac{1}{2^n} \geq \left|\frac{h(2^n x)}{2^n}\right|$$

We know that $\sum_{n=1}^{\infty} 1/2^n$ converges. Therefore, $g(x)$ converges uniformly.

Finally, each $h(2^n x)/2^n$ is continuous, so we can conclude that $g(x)$ is therefore continuous as well.

Exercise 6.4.5
(a) We have

$$f_n'(x) = \frac{\cos(kx)}{k^2}$$

We can show this converges uniformly to 0,

$$\left|\frac{\cos(kx)}{k^2}\right| \leq \frac{1}{k^2}$$

We can show that $f(x) \to 0$ for $x = 2\pi$, since $\sin(k2\pi) = 0$. Therefore, we can conclude, since $\sum f_n'$ converges uniformly, that $f(x)$ is differentiable.

Since each f_n' is continuous, and $\sum f_n' \to f'$ uniformly, we can conclude that f' is continuous.

(b) No, since we can only do

$$\frac{1}{n} \geq \left|-\sin(nx)/n\right|,$$

but we know the Harmonic series does not converge.

Exercise 6.4.6
For any $x_0 \in (0, 1)$ fixed, choose $y \in (x_0, 1)$. Then we can show

$$\begin{aligned} 1 &\leq y^0 \\ x &\leq y \\ x^2 \leq y^2 &\Rightarrow \frac{x^2}{2} \leq y^2 \\ x^3 \leq y^3 &\Rightarrow \frac{x^3}{3} \leq y^3 \end{aligned}$$

which means for $M_n = y^n$, we can show $\sum M_n \to \frac{1}{1-y}$, which exists when $y \in (0, 1)$, and therefore $f(x)$ converges uniformly.

We know each $\left(\frac{x^n}{n}\right)' = x^{n-1}$, so each f_n is continuous.

Therefore, we conclude $f(x)$ is continuous at x_0.

Exercise 6.4.7

(a) We have

$$M_n = 1/n^2 \geq h_n(x),$$

so $h(x)$ converges uniformly. Each

$$h_n'(x) = \frac{-2x}{(x^2+n^2)^2}$$

exists for all x, so they are all continuous. Therefore, we conclude h is continuous on all of $\mathbb{R}$.

(b) We can show that for $x \in [-M, M]$, that

$$\left|\frac{-2x}{(x^2+n^2)^2}\right| \leq \frac{2M}{n^4}$$

Since $\sum 2M/n^4$ converges, we can conclude with the Weierstrass M-Test that h_n' converges uniformly. Therefore, h is differentiable, and also h' is continuous from uniform convergence.

Since we can choose any M for our argument, we conclude that h is differentiable and continuous on all of $\mathbb{R}$.

Exercise 6.4.8

For $x \notin \mathbb{Q}$, we can show $u_n(x)$ is continuous, since for $x < r_n$, we can choose a small enough δ such that $u_n(y) = 0$ for $y \in V_\delta(x)$. Similar reasoning applies to $x > r_n$. We can then show $h(x)$ converges uniformly, by the Weierstrass M-Test since

$$M_n = \frac{1}{2^n} \geq u_n(x)$$

Therefore, since u_n are all continuous, and h converges uniformly, we conclude that h is continuous.

h is monotone, since every $u_n(x)$ is increasing, so for $x < y$,

$$\begin{aligned}
\forall n u_n(x) &\leq u_n(y) \\
\sum_{n=1}^{k} u_n(x) &\leq \sum_{n=1}^{k} u_n(y) \\
\lim_k \sum_{n=1}^{k} u_n(x) &\leq \lim_k \sum_{n=1}^{k} u_n(y) \\
h(x) &\leq h(y) && (h \text{ converges uniformly})
\end{aligned}$$

6.5 Power Series

Exercise 6.5.1

(a) g is defined on $(-1, 1)$, since we can bound

$$\left|\frac{x^n}{n}\right| \leq x^n \leq y^n$$

for y that is in between ± 1 and x.

g converges uniformly on this set, and every g_n is continuous, so g is continuous.

g is still defined on $(-1, 1]$, since at $x = 1$, we have the alternating harmonic series, which converges.

For $x = -1$, we have

$$g(-1) = -1 - 1/2 - 1/3 - 1/4 - \cdots$$

which diverges.

For $|x| > 1$, g will diverge, since x^n/n is increasing.

$g(x)$ cannot converge for other x, since we know that if g did for $|x| > 1$, that would imply $x = -1$ has to converge, which we know is not the case.

(b) g' is defined for $x \in (-1, 1)$.

$$g'(x) = 1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1+x}$$

we can use the geometric series formula since $|x| < 1$.

Exercise 6.5.2

(a) $a_n = 1/n^2$.

(b) $a_0 = 0$, for $n \geq 1, a_n = \frac{1}{n}$. This will create the alternating harmonic series for $x = -1$, but the harmonic series for $x = 1$.

(c) You can define

$$a_n = \begin{cases} 0, & n = 0, n \text{ odd} \\ \frac{-1^{n/2}}{n/2}, & n \geq 2, n \text{ even} \end{cases}$$

The idea is that $x = -1$ will alternate between $-1, 1$ for x^n, but will always be 1 if x^{2k}, so we can easily make a conditionally convergent sequence for the even terms, e.g.

$$x = -1 : a_0(1) + a_1(-1) + a_2(1) + a_3(-1) + a_4(1) + \dots$$
$$x = 1 : a_0(1) + a_1(-1) + a_2(1) + a_3(-1) + a_4(1) + \dots$$

(d) No, because if the power series converges absolutely at $x = 1$, we have

$$\left|a_n(-1)^n\right| = \left|a_n(1)^n\right|$$

which means the $x = -1$ series will also converge absolutely.

Exercise 6.5.3

AFSOC we have conditional convergence at 3 points, x_1, x_2, x_3. Since we have 3 different points, if we look at their absolute values, it must be the case that at least one of the points has a different absolute value than the others. Let this point be x_3 WLOG.

We have 2 cases,

- $|x_3| < |x_1|$. Since the series converges at $|x_1|$, it must converge on $S = \big[-|x_1|, |x_1|\big]$, which means since $x_3 \in S$, that x_3 must converge absolutely.

- $|x_3| > |x_1|$. Since the series converges at $|x_3|$, it must converge on $S = \big[-|x_3|, |x_3|\big]$, which means since $x_1 \in S$, that x_1 must converge absolutely.

In all cases, we show that some point out of the 3 converges absolutely, so therefore at most two points can converge conditionally.

We have shown in Exercise 6.5.2 an example where we have conditional convergence at two points x_1, x_2.

Exercise 6.5.4

(a) For any $x \in (-R, R)$, choose $c \in (|x|, R)$. Then since the power series converges at c, it must converge absolutely at x, since $|x| < c$, from Theorem 6.5.1. Theorem 6.5.2 then tells us that on $[-|x|, |x|]$, the power series will converge uniformly. From Theorem 6.4.2, if $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on A, then f is continuous on $A = \big[-|x|, |x|\big]$, and we know $x \in A$.

(b) By Theorem 6.5.1, we know that the power series will converge at any $|x| < R$, which is the set $(-R, R)$, so our convergence is

$$(-R, R) \cup \{R\} = (-R, R]$$

Exercise 6.5.5

If we have absolute convergence at a point x_0, then we know $\sum |a_n x_0^n|$ converges.

Then for any $x \in \big[-|x_0|, |x_0|\big]$, we have

$$M_n = |a_n x_0^n| \geq |a_n x^n|$$

so therefore by the Weierstrass M-Test, we can conclude that this $\sum a_n x^n$ converges.

Exercise 6.5.6

Take some compact set $K \subseteq A$. We know that K must be bounded, since it is compact, so say $K \subseteq [-M, M] \subseteq A$.

We know the power series converges at M, since $M \in A$. Now, by Abel's Theorem, we have that since the power series converges at M, we conclude the series converges uniformly on $[0, M]$. Similarly, it converges at $-M$, so the power series converges uniformly on $[-M, 0]$ as well.

Since the power series converges uniformly on

$$[-M, 0] \cup [0, M] \supseteq K,$$

we conclude the series converges uniformly on K as well.

Exercise 6.5.7

(a) We have

$$\lim_n \left| \frac{(n+1)s^n}{ns^{n-1}} \right| = \lim_n \left| \frac{n+1}{n} s \right| = |s| < 1$$

(b) If we have $\sum_{n=0}^{\infty} a_n x^n$ converging on $x \in (-R, R)$, we can use Theorem 6.5.5 from Exercise 6.5.6 to show that this series also converges uniformly on compact sets contained in $(-R, R)$.

Now for the differentiated series, choose a $t \in (|x|, R)$, we can write

$$\sum \left| n a_n x^{n-1} \right| = \sum \frac{1}{t} \left(n \left| \frac{x}{t} \right|^{n-1} \right) |a_n t^n|$$

Now, since we know $\sum n \big|x/t\big|^{n-1} = L$ because it converges, from part (a), we can now write

$$\sum \left| n a_n x^{n-1} \right| \leq \sum \frac{1}{t}(L)|a_n t^n| = \frac{L}{t} \sum |a_n t^n| = \frac{LL'}{t}$$

we know $\sum |a_n t^n|$ converges since we know the power series converges at $t' \in (t, R)$.

Therefore, the differentiated series also converges. Using Theorem 6.5.5 again, we can conclude the differentiated series also converges uniformly on compact sets in $(-R, R)$.

Exercise 6.5.8

(a) We can use the ratio test,

$$\lim_n \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = L|x| < 1.$$

which shows that the series converges.

(b) We can use the ratio test as in part (a).

(c) If $L' = \sup\Big\{ \big|a_{k+1}/a_k\big| \Big\}$, then $L \leq L'$, so for $x \in (-1/L', 1/L')$, we apply the ratio test again,

$$\lim_n \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = L|x| \leq L'|x| < 1.$$

If $L' = 0$, this is not possible since that would imply $a_k = 0$ for all k, but we know $a_k \neq 0$.

(d) When $x = 0$, this series $= 0$, so it converges trivially. When $x \neq 0$, this proof is more involved. If we have $|x| \geq 1$, this proof is not too hard. We can show $\left|\sum a_n x^n\right|$ does not converge, because we can choose some $M > 1$, and then we can find n such that $\big|a_{n+1}/a_n\big| \geq M$. This means

$$\left|a_{n+1}x^{n+1} - a_n x^n\right| \geq \left|a_{n+1}x^{n+1}\right| - |a_n x^n| = (M-1)|a_{n-1}|\big(|x|-1\big)\left|x^{n-1}\right| > \epsilon_1 |a_{n-1}|\delta_1 \cdot 1 > 0$$

which means after this n, the sum keeps on oscillating with an amplitude that is always greater than 0, which means it cannot possibly converge.

For $x \in (-1, 1)$, what we need to show is somehow that $\big|a_{n+1}/a_n\big|$ "grows faster" than x^n decreases. Otherwise, it is possible that x^n can diminish the unboundedness of the coefficients. The author probably did not intend for this proof, so I will not prove it. It might be not too hard to show, but I don't feel like doing it...

The solution says that the condition should be $\big|a_{k+1}/a_k\big| \geq 1$ after some $k \geq N$, but I don't think this is necessarily true, because if you take some $|x| < 1$, and $a_n = 1$, then the series will still converge.

Exercise 6.5.9

We can show that

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} b_n x^n = a_0 - b_0 = 0$$

so therefore $a_0 = b_0$. We can inductively continue by considering for $x \neq 0$,

$$\frac{1}{x}\left(\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} b_n x^n \right) = a_1 - b_1 = 0 \Rightarrow a_1 = b_1$$

So therefore $\forall a_n = b_n$, and we conclude the power series is unique.

Exercise 6.5.10

We are given that

$$\sum a_n, \sum b_n, \sum d_n$$

all converge. Notice that

$$f(1) = \sum a_n 1^n = \sum a_n,$$

and that similar observations can be made about g, h.

Then, the assumptions mean that $f(1), g(1), h(1)$ all converge, and by Abel's Theorem, this means that f, g, h converge uniformly on $[0, 1]$, and thus are also continuous on this set.

Now, we also know that since f, g, h converge at 1, that they converge absolutely for any $x \in [0, 1)$, which means in this range, we have that

$$h(x) = f(x)g(x)$$

Therefore, since f, g, h are continuous on $[0, 1)$, and they also converge at 1, we can conclude that at $x = 1$, we also have

$$h(1) = f(1)g(1) \Rightarrow \sum d_n = \left(\sum a_n\right)\left(\sum b_n\right) = AB$$

Exercise 6.5.11

(a) Any series $\sum a_n = L$ means that

$$f(x = 1) = \sum a_n 1^n = L$$

so $f(x)$ converges uniformly on $[0, 1]$ by Abel's Theorem. Since all $a_n x^n$ are continuous on $[0, 1]$, $f(x)$ is also continuous over this set, so we can conclude $\lim_{x\to 1^-} f(x) = f(1) = L$.

(b) This is just the geometric series with $b_0 = 1$, $r = -1 \cdot x$, so we have

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1 - (-x)} = \frac{1}{1 + x}$$

If we take this $\lim_{x\to 1^-}$, we get a limit of $L = 1/2$.

Exercise 6.5.12

The key observation here is that $G(0) = d_1$, and G is strictly increasing.

This means that $d_{r+1} = G(d_r) \le G(d_0)$, so by the Order Limit Theorem, we conclude $G(d_r) \to G(d_0) = d_0$. So $\lim d_r = d_0$.

6.6 Taylor Series

Exercise 6.6.1
If we know that $\arctan(x)$ is continuous, we just need to show the series is also continuous over the interval $[-1, 1]$. We know that the series converges at 1, which by Abel's Theorem gives us uniform convergence over $[-1, 1]$. Every $a_n x^n$ is also continuous, which means that the series is continuous over this interval as well. Since the series and $\arctan(x)$ have the same value over $[0, 1)$, we can conclude they must have the same value at 1 as well.

We get the property that

$$\arctan(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1}$$

Exercise 6.6.2
We know

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(1+x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

So if we take the integral,

$$\begin{aligned} \ln(1+x) &= \int_0^x \left(1 - x + x^2 - x^3 + \cdots\right) \mathrm{d}x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \end{aligned}$$

This is value for $|x| < 1$, if we want to use the geometric series identity, and also converges for $x = 1$ since it is the alternating harmonic series, so converges over $x \in (-1, 1]$.

Exercise 6.6.3

(a) Taking derivatives

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

so we see $f'(x) = a_1$.

(b) We can show that after $f^{(n)}(x)$, the coefficient of x^n is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(a_n x^n) = n! \cdot a_n,$$

by the power rule of differentiation.

Then we have

$$f^{(n)}(x) = n! \cdot a_n + x(\text{stuff}) \Rightarrow f^n(0) = n! \cdot a_n$$

so therefore

$$a_n = \frac{f^{(n)}(0)}{n!}$$

We can use Theorem 6.3.3 to justify our actions, since we know all the terms in the series are differentiable, since they are in the form $a_n x^n$, and also we know the point $x_0 = 0$ has a convergent point, so therefore $\left(f^{(n)}\right)' = \sum (b_n)'$ where b_n are the terms of $f^{(n)}$.

Exercise 6.6.4
We can try a few terms,

$$
\begin{aligned}
a_0 &= \sin(0) = 0 \\
a_1 &= \frac{\cos(0)}{1!} = 1 \\
a_2 &= \frac{-\sin(0)}{2!} = 0 \\
a_3 &= \frac{-\cos(0)}{3!} = -\frac{1}{3!} \\
a_4 &= \frac{\sin(0)}{4!} = 0 \\
a_5 &= \frac{\cos(0)}{5!} = \frac{1}{5!} \\
&\dots
\end{aligned}
$$

and we can verify the pattern for the Taylor series for $\sin(x)$.

Exercise 6.6.5
We know by Lagrange's remainder theorem that $\exists c$ such that

$$E_N(x) = \sin(x) - S_N(x) \leq \left|\frac{\cos(c)}{(N+1)!}\right| 2^{N+1} \leq \frac{2^{N+1}}{(N+1)!}$$

and this quantity $\to 0$ as $N \to \infty$ since factorial grows faster than exponential. Therefore, $E_N(x) \to 0$, uniformly. Our argument works for any interval $[-R, R]$.

Exercise 6.6.6
(a) We can find

$$
\begin{aligned}
a_0 &= e^0 = 1 \\
a_1 &= \frac{e^0}{1!} = 1 \\
a_2 &= \frac{e^0}{2!} = \frac{1}{2!} \\
a_3 &= \frac{e^0}{3!} = \frac{1}{3!} \\
a_4 &= \frac{e^0}{4!} = \frac{1}{4!} \\
a_5 &= \frac{e^0}{5!} = \frac{1}{5!} \\
&\dots
\end{aligned}
$$

So therefore

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

On any interval $[-R, R]$, we can bound $e^c \leq M$ for $c \in [-R, R]$. Now,

$$\left|E_N(x)\right| = \left|\frac{e^c}{(N+1)!} x^{N+1}\right| \leq \frac{M \cdot R^{N+1}}{(N+1)!}$$

this shows $\left|E_N(x)\right| \to 0$, so it converges uniformly to 0.

(b) We can verify that

$$\begin{aligned}
(e^x)' &= (1)' + \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} \\
&= 0 + \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{n'=0}^{\infty} \frac{x^{n'}}{(n')!} && (n' = n-1)
\end{aligned}$$

(c) We can make

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}.$$

and find that

$$\begin{aligned}
e^x \cdot e^{-x} &= 1 + x(1-1) + x^2\left(2\frac{1}{2!} - \left(\frac{1}{1!}\right)^2\right) + x^3\left(\frac{1}{3!}(1-1) + \frac{1}{2!}(1-1)\right) + x^4\left(\frac{2}{4!} - \frac{2}{3!} + \left(\frac{1}{2!}\right)^2\right) + \cdots \\
&= 1 + 0 + 0 + \cdots \\
&= 1,
\end{aligned}$$

which is expected since $e^x e^{-x} = e^0 = 1$.

For a more rigorous justification, it is not hard to show for x^k k odd, all the terms will cancel each other out. For even powers, the task is more difficult.

The easiest way to show they cancel out is to notice that the coefficients of x^n after the multiplication is just

$$\frac{1}{n!}(-1+1)^n = 0$$

Exercise 6.6.7

Not sure if this problem has a typo...but I think the author meant

$$E_N(x) = f(x) - S_N(x)$$

and not $f_N(x)$...whatever that is supposed to mean.

$$\begin{aligned}
E_N^{(n)}(0) &= f^{(n)}(0) - S_N^{(n)}(0) \\
&= f^{(n)}(0) - a_N \cdot N! \\
&= f^{(n)}(0) - \frac{f^{(n)}(0)}{N!} \cdot N! \\
&= 0
\end{aligned}$$

Exercise 6.6.8
We have

$$\begin{aligned}
E_N(x) &= \frac{x^{N+1}}{N+1}\frac{E_N'(x_1)}{x_1^N} \\
&= \frac{x^{N+1}}{N+1}\cdot\frac{1}{N}\cdot\frac{E_N^{(2)}(x_2)}{x_2^{N-1}} \\
&= \frac{x^{N+1}}{N+1}\cdot\frac{1}{N}\cdot\frac{1}{N-1}\cdot\frac{E_N^{(3)}(x_3)}{x_3^{N-2}} \\
&= \frac{x^{N+1}}{(N+1)!}E^{(N+1)}(x_{N+1}) \\
&= \frac{x^{N+1}}{(N+1)!}\left[f^{(N+1)}(x_{N+1}) - S_N^{(N+1)}(x_{N+1})\right] \\
&= \frac{f^{(N+1)}(x_{N+1})}{(N+1)!}x^{N+1}
\end{aligned}$$

For the last step, S_N highest degree term is x^N, so after $N+1$ derivatives, $S_N^{(N+1)}(x) = 0$. We see the overall strategy is just repeatedly apply the GMVT between $x = x_k, 0$, and we can find

$$x_2 \in (0, x_1), x_3 \in (0, x_2), \ldots, x_k \in (0, x_{k-1}),$$

So we set $c = x_{N+1}$ to be used in the theorem.

The negative direction is basically the same proof, just we choose $x_k \in (x_{k-1}, 0)$.

Exercise 6.6.9
Computing

$$\begin{aligned}
g'(0) &= \lim_{x\to 0}\frac{e^{-1/x^2} - 0}{x} \\
&= \lim_{x\to 0}\frac{x^{-1}}{e^{1/x^2}} \\
&= \lim_{x\to 0}\frac{-x^{-2}}{e^{-1/x^2}(2x^{-3})} && \text{(L'Hopital's)} \\
&= \lim_{x\to 0}\frac{x}{2e^{1/x^2}} \\
&= 0
\end{aligned}$$

Exercise 6.6.10
For $x \neq 0$,

$$\begin{aligned}
g'(x) &= \frac{2x^{-3}}{e^{1/x^2}} \\
g''(x) &= \frac{4x^{-6} - 6x^{-4}}{e^{1/x^2}} \\
g'''(x) &= \frac{8x^{-9} - 36x^{-7} + 24x^{-5}}{e^{1/x^2}}
\end{aligned}$$

We can use the quotient rule to show that for

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f_k}{e^{1/x^2}} = \frac{f_k'e^{1/x^2} - e^{1/x^2}(-2x^{-3})f_k}{\left(e^{1/x^2}\right)^2} = \frac{f_k' + 2x^{-3}f_k}{e^{1/x^2}}$$

So we can make a sequence-like definition for $g'(x)$.

Exercise 6.6.11
Computing

$$\begin{aligned}
\lim_{x\to 0} \frac{\frac{2x^{-3}}{e^{1/x^2}} - 0}{x - 0} &= \lim_{x\to 0} \frac{2x^{-4}}{e^{-1/x^2}} \\
&= \lim_{x\to 0} \frac{-8x^{-5}}{e^{1/x^2}(-2x^{-3})} \\
&= \lim_{x\to 0} \frac{4x^{-2}}{e^{1/x^2}} \\
&= \lim_{x\to 0} \frac{-8x^{-3}}{e^{1/x^2}(-2x^{-3})} \\
&= \lim_{x\to 0} \frac{4}{e^{1/x^2}} \\
&= 0
\end{aligned}$$

We can see in general, the L'Hopital's will produce enough $-2x^{-3}$ to cancel out any factors in the numerator, until the numerator has a non-negative degree polynomial in x, which means

$$\lim_{x\to 0} \frac{f(x)}{e^{1/x^2}} = 0/\infty = 0 = g^{(n)}(0)$$

Exercise 6.6.12
g is infinitely differentiable, since we found a general formula for its n^{th} derivative.

We found that $g^{(k)}(0) = 0$, which means the Taylor Series will be $a_n = 0$. Of course, since this is $f(x) = 0$, this converges everywhere. However, we clearly see that $g(x) \neq 0$, so the Taylor series does not work for this example.

Chapter 7

The Riemann Integral

I'm not sure why, but there a quite of lot typos in this chapter. I did not point out all of the typos, only the ones I felt like could potentially interfere with understanding the content.

Some of the problems also don't give you all the assumptions you need to solve them, which is pretty frustrating if you've spent a good amount of time trying to figure out a problem that is much harder than it needs to be.

7.2 The Definition of the Riemann Integral

The partition theorems may be a little confusing at first to understand. The basic idea is that if we have a refinement of some partition P, then we have all the original points of P, with additional points in between.

Now, if we consider the $L(f, P)$, we see that adding more points in some interval $[x_k, x_{k+1}]$ will only yield estimates m_k that are larger than the $\inf(f(x), x \in [x_k, x_{k+1}])$, since in this interval, any $f(x)$ will be larger than the inf. Therefore, we can conclude

$$L(f, P) \leq L(f, \text{refinement})$$

On the other hand, if we look at $U(f, P)$, we get the opposite relation. The refinement produces a U estimate that is less than the original, since if you have some maximum over some interval, adding more points gives you chance to take estimates that are less than the maximum. Therefore, we conclude

$$U(f, P) \geq U(f, \text{refinement})$$

There are some typos in this section:

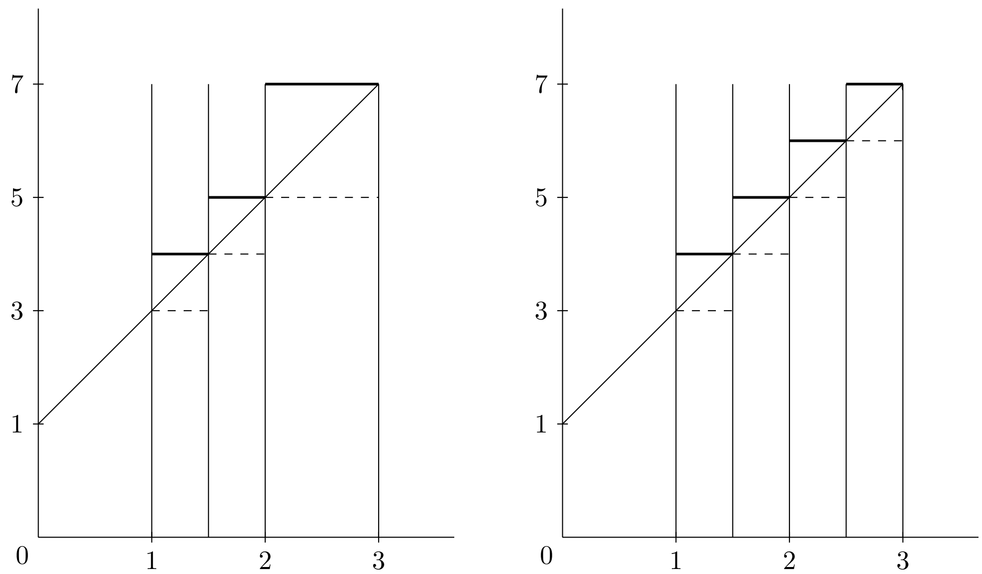
- In Lemma 7.2.4, the author meant to say if $Q = P_1 \cup P_2$, then $P_1 \subseteq Q, P_2, \subseteq Q$.
- In Definition 7.2.5, where the *Lower Integral* of f should be defined as

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}$$

Exercise 7.2.1
For any partition P, we have $U(f) \geq L(f, P)$ since over any interval, the estimate for f will be $\sup f(x)$ over this interval, and for $L(f, P)$, it must be the $\inf f(x)$ over an interval that contains part of this interval, which must be less than $\sup f(x)$.

Then, $U(f, P) \geq L(f)$ since we chose an arbitrary P.

Exercise 7.2.2
We are considering $f(x) = 2x + 1$ over $x \in [1, 3]$.



(a) $P = \{1, 3/2, 2, 3\}$ (b) $P = \{1, 3/2, 2, 5/2, 3\}$

Figure 7.1: Riemann Sum estimates for $f(x) = 2x + 1$. Dashed lines indicate U, thick lines indicate L.

(a) We can compute

$$\begin{aligned} L(f, P) &= \frac{1}{2}(3 + 4) + 1 \cdot 5 = 8.5 \\ U(f, P) &= \frac{1}{2}(4 + 5) + 1 \cdot 7 = 11.5 \\ U(f, P) - L(f, P) &= 3 \end{aligned}$$

(b) With an extra point 5/2 in the partition,

$$\begin{aligned} L(f, P) &= \frac{1}{2}(3 + 4 + 5 + 6) = 9 \\ U(f, P) &= \frac{1}{2}(4 + 5 + 6 + 7) = 11 \\ U(f, P) - L(f, P) &= 2 \end{aligned}$$

So $U - L$ decreases.

(c) If we add midpoints between all the current intervals, then we wil have

$$U(f, P') - L(f, P') = \frac{7 - 3}{4} = 1 < 2$$

Exercise 7.2.3
The constant function satisfies $U(f) = L(f)$, since the $\sup, \inf$ over any interval is the same.

$$\int_a^b f(x)\,\mathrm{d}x = b - a$$

Exercise 7.2.4
(a) AFSOC $U(f) \neq L(f)$. Then we must have $U(f) > L(f)$. Let $\epsilon = U(f) - L(f)$. We can find P_n such that

$$\left|U(f, P_n) - L(f, P_n)\right| < \epsilon,$$

which means either $U(f, P_n) < U(f)$ or $L(f, P_n) > L(f)$, both of which are contradictions, so therefore we conclude $U(f) = L(f)$.

(b) We can find

$$L(f, P_n) = \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right) = \frac{n(n-1)}{2n^2} = \frac{n-1}{2n}$$
$$U(f, P_n) = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n} + \frac{n}{n}\right) = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}$$

(c) We take the difference

$$U(f, P_n) - L(f, P_n) = \frac{n+1-(n-1)}{2n} = \frac{1}{n}$$

so the difference $\to 0$, and therefore $f(x) = x$ is integrable on $[0, 1]$.

Exercise 7.2.5
Because $f_n \to f$ uniformly, hoose n_1 such that

$$\left|f_{n_1}(x) - f(x)\right| < \frac{\epsilon}{3 \cdot (b-a)}$$

and since each f_n is integrable, choose n_2 such that

$$\left|U(f_{n_1}, P_{n_2}) - L(f_{n_1}, P_{n_2})\right| < \frac{\epsilon}{3}.$$

Now, choose $n = \max(n_1, n_2)$,

Notice that

$$\begin{aligned}
\left|U(f, P_n) - U(f_n, P_n)\right| &\leq \sum_{x_k} \left|f(x_k) - f_n(x_k)\right| \Delta x_k \\
&< \sum_{x_k} \frac{\epsilon}{3(b-a)} \Delta x_k \\
&= \frac{\epsilon}{3(b-a)} \sum_{x_k} \Delta x_k \\
&= \frac{\epsilon}{3(b-a)} (b-a) \\
&= \frac{\epsilon}{3}
\end{aligned}$$

The key observation here is that over $[x_k, x_{k+1}]$, $\left|\sup f(x) - \sup f_n(x)\right| \leq \left|f_n(x) - f(x)\right|$ since every point of f_n is close to f.

A similar results holds for

$$\left|L(f, P_{n_2}) - L(f_n, P_{n_2})\right| < \frac{\epsilon}{3}$$

So therefore, we can say

$$\begin{aligned}
\left|U(f, P_n) - L(f, P_n)\right| &\leq \left|U(f, P_n) - U(f_n, P_n) + U(f_n, P_n) - L(f_n, P_n) - \big(L(f, P_n) - L(f_n, P_n)\big)\right| \\
&\leq \left|U(f, P_n) - U(f_n, P_n)\right| + \left|U(f_n, P_n) - L(f_n, P_n)\right| + \left|L(f, P_n) - L(f_n, P_n)\right| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

Exercise 7.2.6
The key observation here is that for any interval $[x_k, x_{k+1}]$, the sup of this set is at x_{k+1}, since that is the largest element of this interval, and f is increasing. Likewise, $\inf f(x) = f(x_k)$.

Take P_n to be a partition with all the subintervals equal length $\Delta x = \frac{b-a}{n}$. We can compute

$$\begin{aligned}
U(f) - L(f) &= \sum_{k=0}^{n-1} \big[f(x_{k+1}) - f(x_k)\big](x_{k+1} - x_k) \\
&= \sum_{k=0}^{n-1} \big[f(x_{k+1}) - f(x_k)\big]\Delta x \\
&= \Delta x\Big[\big(f(x_n) - f(x_{n-1})\big) + \big(f(x_{n-1}) - f(x_{n-2})\big) + \cdots + \big(f(x_1) - f(x_0)\big)\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Telescoping series)} \\
&= \frac{b-a}{n}\big[f(b) - f(a)\big]
\end{aligned}$$

Now, let M be $\big|f(b) - f(a)\big|$. Then choose

$$n > \frac{(b-a)\big[f(b) - f(a)\big]}{\epsilon}$$

to satisfy $U(f, P_n) - L(f, P_n) < \epsilon$.

The official solutions for this problem is done wrong...the author got $\Delta x(b-a)$ at the end, but I think he meant $\Delta x\big[f(b) - f(a)\big]$. Of course, the idea is the same, but it's just confusing to see an error like that.

7.3 Integrating Functions with Discontinuities

Exercise 7.3.1

(a) Every partition contains some $x \in [x_k, x_{k+1}]$, where $x_k < x_{k=1}$.

This means every interval contains some element $y \neq 1$, which will mean $\exists f(y) = 1$ in every interval, and thus

$$L(f, P) = 1$$

(b) Just make a little interval around $x = 1$, so for example

$$P = \left\{0, 1 - \frac{1}{15}, 1\right\}$$

Then the difference is the interval $\left[1 - \frac{1}{15}, 2\right]$ where we have $\Delta x = \frac{1}{15}$ and thus $U(f, P) = 1 + \frac{1}{15}$.

(c) Construct

$$P_\epsilon = \left\{0, 1 - \frac{\epsilon}{2}, 1\right\}$$

Then

$$U(f, P_\epsilon) = 1 + \frac{\epsilon}{2}$$

Exercise 7.3.2

Fix some enumeration of the rationals. Let $S_n = \{r_1, r_2, \ldots, r_n\}$, or the first n rationals on this list.

Define

$$g_n(x) = \begin{cases} 1, & x \in S_n \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Then $g_n \to g$ since for any $x \in \mathbb{R}$, if $x \notin \mathbb{Q}$, then $g_n(x) = 0 = g(x)$. Otherwise, we know $x = r_k$ for some k, and we can just choose $n = k$, and then $g_n(x) = 1 = g(x)$.

Exercise 7.3.3

We can make an ϵ-neighborhood around each discontinuity and remove them from $[a, b]$.

The remaining $[a, b] \setminus O$ must be continuous for f since we just removed all the discontinuities.

Now, $[a, b] \setminus O$ is a compact set, becauase it is still bounded, and is the union of a finite number of closed sets. Therefore, f is uniformly continuous on $[a, b] \setminus O$.

Since f is uniformly continuous, we have

$$\left|f(x) - f(y)\right| < \epsilon$$

whenever $|x - y| < \delta$.

This means for any ϵ challenge, we can define a partition such that the subintervals are small enough around the discontinuities, and also so that $|x - y| < \delta$ is true for the δ we need.

Then we can bound $U - L$ at the discontinuities and the area that is continuous however we'd like.

Exercise 7.3.4

(a) Let the new value of this point be M. We can do the putting a small $\epsilon'/2$-neighborhood around the new value We can then show that

$$U(f) - U(f, P'_\epsilon) \leq |M|\epsilon' < \epsilon/3, L(f) - L(f, P'_\epsilon) \leq |M|\epsilon' < \epsilon/3 \tag{7.1}$$

for appropriate values of ϵ'. Then, we can bound

$$\left|U(f, P'_\epsilon) - L(f, P'_\epsilon)\right| \leq \left|U(f, P'_\epsilon) - U(f)\right| + \left|+U(f) - L(f)\right| + \left|L(f, P'_\epsilon) - L(f)\right| < \epsilon$$

(b) For a finite number of f changed, we can choose an even smaller ϵ' around each change, so something like if M bounds the changes, and there are N changes, then choose

$$N \cdot (2\epsilon')M < \epsilon/3$$

(c) We can just use Dirichlet's function idea, and change the value of f at every rational $x \in [a, b]$.

Exercise 7.3.5
We can see that any interval $[x, y], x < y$ containing $1/n$ means $\exists z \neq 1/n$ so therefore the minimum $f(x)$ value on any interval $= 0$, so we conclude

$$L(f) = 0$$

Define P to be the partition so that for P_N, we have

$$P_N = \left\{0, \frac{1}{N}, 1\right\} \cup S_r$$

where S_r contains the points to add little intervals of radius $r = \frac{1}{N(N+1)}$ around each $1/k \in [0, 1]$ for $k < N$.

Then we have

$$U(f, P_N) - L(f, P_N) = r + 2r(N-2) + \frac{1}{N} = \frac{1}{N(N+1)} + \frac{2(N-2)}{N(N+1)} + \frac{1}{N}$$

which $\to 0$ as $N \to \infty$.

We showed earlier that $L(f) = 0$, so

$$\int_0^1 f(x)\,\mathrm{d}x = L(f) = 0$$

Exercise 7.3.6

(a) Let the bound of f on $[a, b]$ be M. Choose a content zero set for the discontinuities so that their length of intervals is $< \epsilon/(4M)$ Also let P_n be some partition so that the continuous part of f, which is integrable, satisfies

$$U(f^{\text{cont}}, P_n) - L(f^{\text{cont}}, P_n) < \epsilon/2$$

and define P_N be the union of P_n with points to give us the content zero intervals.

Then we can show

$$\begin{aligned} U(f, P_N) - L(f, P_N) &< \epsilon/2 + 2M \cdot \frac{\epsilon}{4M} \\ &= \epsilon \end{aligned}$$

(b) For any finite set of N elements,

$$S_N = \{a_1, a_2, \ldots, a_N\},$$

define

$$O_i = \left(a_i - \frac{\epsilon}{2N}, a_i + \frac{\epsilon}{2N}\right)$$

Then

$$\sum_{k=1}^{N} |O_k| = N \cdot 2 \cdot \frac{\epsilon}{2N} = \epsilon$$

(c) We know for any $m < n$, that $C_m \supseteq C_n$, since C_n is C_m but with more open intervals removed. In addition, we know that C_m consists of 2^m closed intervals each with length $\frac{1}{3^m}$, so the total length of C_m is $\left(\frac{2}{3}\right)^m$.

To get a finite zero set, we need open intervals. We can do this by adding a small ϵ' radius open set around each endpoint of the 2^m sets. Since this means adding length of

$$2\epsilon' \cdot 2 \cdot 2^m,$$

since each interval is $2\epsilon'$ wide, and there are 2 endpoints per interval. Now, choose m such that

$$\left(\frac{2}{3}\right)^m < \epsilon$$

and let $\delta = \epsilon - \left(\frac{2}{3}\right)^m$, and then we can choose

$$2\epsilon' \cdot 2 \cdot 2^m < \delta \Rightarrow \epsilon' < \frac{\delta}{2^{m+2}}$$

So the length of these open intervals will be less than ϵ, and therefore we can conclude that C is a content zero set.

(d) $h(x)$ is discontinuous at the $x \in C$ points, but we showed this has content zero, so we can conclude $h(x)$ is integrable.

We can find $x \notin C$ for any open interval of C, so therefore we can conclude

$$\int_0^1 h(x)\,\mathrm{d}x = 0$$

7.4 Properties of the Integral

Exercise 7.4.1

(a) If both $\sup, \inf$ are negative or both nonnegative (i.e. same sign), then

$$M - m = M' - m'$$

If sup is positive and inf is negative, then $M' = M$, but $m' = \inf \left|f(x)\right| > \inf f(x) = m$, so

$$M' - m' < M - m$$

(b) Using our work from part (a), we can see that we can choose a partition P so that

$$\epsilon > U(f, P) - L(f, P) \geq U(|f|, P) - L(|f|, P)$$

(c) It is always the case that $U(|f|) \geq U(f)$, so we can conclude that

$$\begin{aligned} U(|f|) &\geq U(f) \\ \int_a^b |f| &\geq \int_a^b f \\ \Rightarrow \left| \int_a^b |f| \right| &\geq \left| \int_a^b f \right| \\ \int_a^b |f| &\geq \left| \int_a^b f \right| \qquad (|f| \geq 0) \end{aligned}$$

Exercise 7.4.2

For $c \leq a \leq b$, we can use the property

$$\begin{aligned} \int_c^b f &= \int_c^a f + \int_a^b f \\ \int_c^b f - \int_c^a f &= \int_a^b f \\ \int_c^b f + \int_a^c f &= \int_a^b f \end{aligned}$$

Exercise 7.4.3

From Exercise 7.2.5, we saw that f was integrable if we had $f_n \to f$ uniformly, and each f_n was integrable.

We need to still show that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

We can do this by first using the uniform convergence of f_n, which gives us

$$\begin{aligned} &|f_n - f| < \epsilon \\ &\Rightarrow f - \epsilon < f_n < f + \epsilon \\ &\int_a^b (f - \epsilon) < \int_a^b f_n < \int_a^b (f + \epsilon) \\ \Rightarrow &\left| \int_a^b f_n - \int_a^b f \right| < \epsilon(b - a) \end{aligned}$$

so we can make their difference arbitrarily small.

Exercise 7.4.4

(a) False. Use Dirichlet's function, except define -1 for irrationals,

$$g(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & \text{otherwise} \end{cases}$$

Then we have $\left|g(x)\right| = 1$, which is integrable, but we have $U(g) = 1, L(g) = -1$, which is not integrable.

(b) False, consider $f(x) = 1, x = 1/n, 0$ otherwise we saw in Exercise 7.3.5, where the integral is still 0.

(c) True. By the continuity of g, we can find a δ neighborhood such that $x \in V_\delta(x_0) \Rightarrow \left|g(x) - g(x_0)\right| < g(x_0)$, which means $g(x) > 0$ on this interval. Choose a partition that is entirely contained in this interval, then we have $L(g, P) > 0$, since that interval will yield a positive contribution and the others are ≥ 0 since $g \geq$. Therefore,

$$\int_a^b g = L(g) \geq L(g, P) > 0$$

(d) We know $L(f) = \int_a^b f > 0$, which means there must have been some closed interval $[c, d]$ where $\inf f(x) > 0$. So we can choose this $\delta \in (\inf f(x), 0)$.

Exercise 7.4.5

(a) At any point $x \in [y_k, y_{k+1}]$, we have

$$(f + g)(x) \leq \sup f(x) + \sup g(x) \Rightarrow \sup(f + g)(x) \leq \sup f(x) + \sup g(x)$$

so we can conclude

$$U(f + g, P) = \sum_{x \in [y_k, y_{k+1}]} \sup(f + g)(x) \leq \sum_{x \in [y_k, y_{k+1}]} \sup f(x) + g(x) = U(f, P) + U(g, P)$$

We have equality when the supremums are at the same x in f, g. So a trivial example is $f(x) = x, g(x) = x$.

The lower sums will abide by

$$L(f + g, P) \geq L(f, P) + L(g, P)$$

(b) First, from our results in part (a), we have

$$\begin{aligned} U(f + g, P) &\leq U(f, P) + U(g, P) \\ L(f + g, P) &\geq L(f, P) + L(g, P) \end{aligned}$$

We can show $f + g$ is integrable by showing that

$$\begin{aligned} U(f + g, P_m) - L(f + g, P_m) &\leq \big[U(f, P_m) - L(f, P_m)\big] + \big[U(g, P_m) - L(g, P_m)\big] \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

with the appropriate P_m that bounds f, g.

Now, we can also conclude

$$\begin{aligned} U(f + g) &\leq U(f) + U(g) \\ L(f + g) &\geq L(f) + L(g) \end{aligned}$$

which helps us show that

$$\begin{aligned}\int_a^b f + \int_a^b g &= L(f) + L(g)\\ &\le L(f+g)\\ &\le U(f+g)\\ &\le U(f) + U(g)\\ &= \int_a^b f + \int_a^b g\end{aligned}$$

so we can conclude

$$\int_a^b f + g = U(f+g) = \int_a^b f + \int_a^b g$$

Exercise 7.4.6

(a) We can define

$$f_n(x) = \begin{cases} n^2, & x \in (0, 1/n) \\ 0, & \text{otherwise} \end{cases}$$

We see $f_n \to 0$, since we can always find $1/n < x$, which means $f_n(x) = 0$ for that particular n. Then $\int_0^1 f_n(x) = n$, so the limit does not exist.

(b) We can use the same example in part (a), since the integral gives the sequence $i_n = n$.

(c) We can modify our example in part (a) so that instead of a jump from 0 to n^2, we define a line from $(1/2n, n^2) \to (1/n, 0)$.

Then this function will be continuous, but have $\int_0^1 f = \frac{3}{4}n$.

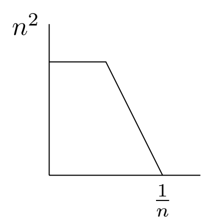


Figure 7.2: The modified continuous $f_n(x)$

(d) No, since if f_n is bounded, we can just bound the integral by M.

Exercise 7.4.7

Let the bound of g, g_n be M.

We want to bound

$$\begin{aligned}\left|\int_0^1 g_n - \int_0^1 g\right| &= \left|\int_0^1 (g_n - g)\right| && (g, g_n \text{ integrable})\\ &\le \int_0^1 |g_n - g|\\ &\le \int_0^\delta |g_n - g| + \int_\delta^1 |g_n - g|\end{aligned}$$

We know $|g_n - g| \le 2M$, from their bounded property.

$$\int_0^\delta |g_n - g| + \int_\delta^1 |g_n - g| \le \int_0^\delta 2M + \int_\delta^1 |g_n - g|$$

and we can choose δ first, and then an n such that

$$\delta < \frac{\epsilon}{2M}, |g_n - g| < \frac{\epsilon}{2(1-\delta)}$$

and we can conclude

$$\begin{aligned}
\int_0^\delta 2M + \int_\delta^1 |g_n - g| &< \int_0^\delta 2M + \int_\delta^1 \frac{\epsilon}{2(1-\delta)} \\
&= \frac{\epsilon}{2M} \cdot 2M + (1-\delta)\frac{\epsilon}{2(1-\delta)} \\
&= \epsilon
\end{aligned}$$

7.5 The Fundamental Theorem of Calculus

Exercise 7.5.1
If f is continuous, then we know from Theorem 7.2.9 that is is integrable.

Then, from the Fundamental Theorem of Calculus, that we can define

$$F(x) = \int_a^x f,$$

where $F'(c) = f(c)$ at all the continuous points of f. This means f is the derivative of F.

Exercise 7.5.2

(a) For $x \in [-1, 0]$, we just have

$$F(x) = \frac{(-x+1)(x+1)}{2} = \frac{1-x^2}{2}$$

For $x \in (-\infty, 1)$,

$$F(x) = \int_{-1}^x f(x) = -\int_x^{-1} f(x) = -\frac{(1-x)(-1-x)}{2} = \frac{1-x^2}{2}$$

and for $x \in (1, \infty)$, we have

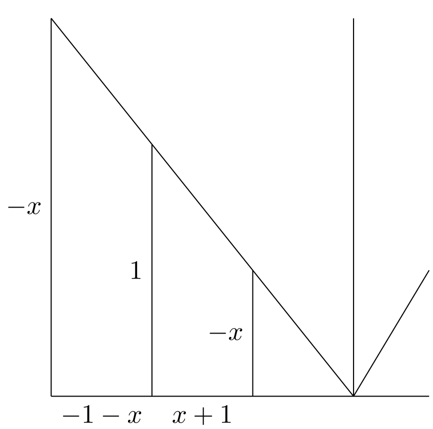
$$F(x) = \frac{1}{2} + \frac{x^2}{2}$$

Putting these together gives

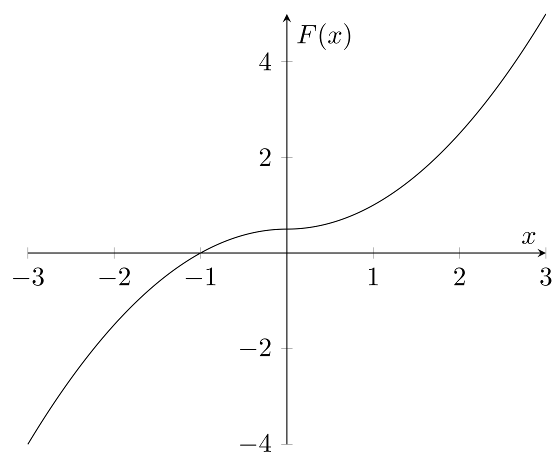
$$F(x) = \begin{cases} \frac{1-x^2}{2}, & x \leq 0 \\ \frac{1+x^2}{2}, & x \geq 0 \end{cases}$$

See Figure 7.3b for the plot of $F(X)$.

I computed these by geometrically looking at the area of a trapezoid. See Figure 7.3a for more details.



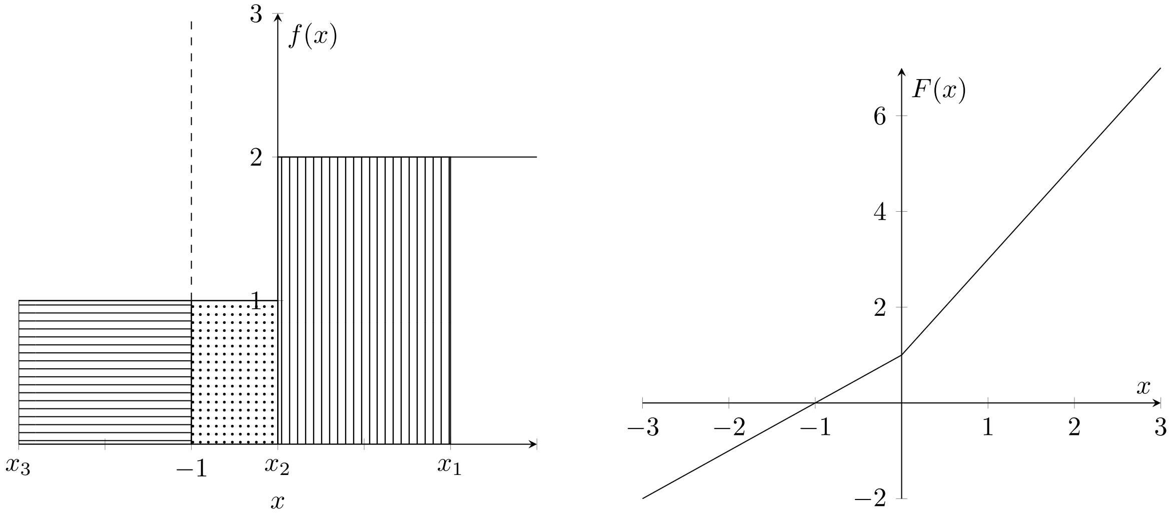
(a) Finding $F(x)$



(b) $F(x)$

Figure 7.3

F is continuous everywhere, and differentiable everywhere. Therefore, $F'(x) = f(x)$ everywhere as well.



(a) Finding $F(x)$

(b) $F(x)$

Figure 7.4

(b) We can see from Figure 7.4a that we have 3 cases, $x_3 \in (-\infty, -1), x_2 \in [-1, 0], x_1 \in (0, \infty)$,

$$F(x) = \begin{cases} -(-1-x)\cdot 1, & x \leq -1 \\ (x-(-1))\cdot 1, & x \in [-1, 0] \\ 1 + x \cdot 2, & x \geq 0 \end{cases} \Rightarrow F(x) = \begin{cases} x+1, & x \leq 0 \\ 2x+1, & x \geq 0 \end{cases}$$

From Figure 7.4b we see that F is continuous everywhere, and bur not differentiable at $x = 0$, since there is a kink. However, we still have $F'(x) = f(x)$ everywhere, thanks to the Fundamental Theorem of Calculus.

Exercise 7.5.3

We only need $F'(x) = f(x)$ for x over (a, b), since we are applyin the MVT.

Exercise 7.5.4

(a) We can compute

$$H(1) = \int_1^1 \frac{1}{t}\,\mathrm{d}t = 0$$

We know $1/t$ is differentiable over $t > 0$, so by the FTC

$$H'(x) = \frac{1}{x}$$

(b) We have for $0 < x < y$

$$H(y) - H(x) = \int_1^y \frac{1}{t}\,\mathrm{d}t - \int_1^x \frac{1}{t}\,\mathrm{d}t = \int_x^y \frac{1}{t}\,\mathrm{d}t > 0$$

Since $1/t > 0$ for $t > 0$, and we are finding its area over $[x, y]$.

(c) We observe that

$$H'(cx) = c \cdot \frac{1}{cx} = \frac{1}{x} = H'(x) = H'(x) + H'(c) \qquad (H'(c) = 0 \text{ since } c \text{ constant})$$

$$\int_1^x H'(ct)\,\mathrm{d}t = \int_1^x H'(t)\,\mathrm{d}t + \int_1^x H'(c)\,\mathrm{d}t$$

$$H(cx) = H(x) + H(c)$$

I think the point of this exercise is to point out that with the properties shown, ln satisfies all of them, so we can conclude that $H(x) = \ln(x)$.

Exercise 7.5.5
We can use Theorem 7.4.4 to show that since $f_n' \to g$ uniformly, and each f_n' is integrable, so we get

$$\begin{aligned} \lim_{n\to\infty} \int_a^x f_n' &= \int_a^x g \\ \lim_{n\to\infty} f_n(x) - f_n(a) &= \int_a^x g \\ f(x) - f(a) &= \int_a^x g && (f_n \to f) \end{aligned}$$

Now, we know g is continuous on $[a, b]$, since f_n' are all continuous, so we can use the FTC to conclude that

$$f'(x) = g(x)$$

Exercise 7.5.6
A missing assumption is that f is continuous. This is important, because if we now define

$$G(x) = \int_a^x f,$$

we now know that G is differentiable, so $G'(x) = f(x)$. We are given that $F(x)$ satisfies $F'(x) = f(x)$, so combining,

$$G'(x) = f(x) = F'(x)$$

and we have that $G'(x) = F'(x)$. Now, this is useful, since we know that this implies

$$G(x) = F(x) + k.$$

Finally, notice that $G(a) = 0 = F(a) + k$, so $k = -F(a)$.

We can now compute

$$\int_a^b f = G(b) = F(b) - F(a)$$

Exercise 7.5.7
We can use FTC to show $\exists G, G'(x) = g(x)$, and

$$\frac{1}{b-1} \int_a^b g = \frac{1}{b-a} \big[G(b) - G(a)\big]$$

Now by the MVT, we can conclude

$$\exists c \in (a, b), G'(c) = \frac{1}{b-a} \big[G(b) - G(a)\big].$$

Finally, $G'(c) = g(c)$, so

$$g(c) = G'(c) = \frac{1}{b-a} \big[G(b) - G(a)\big] = \frac{1}{b-1} \int_a^b g$$

Exercise 7.5.8

(a) We can write

$$\begin{aligned}
V f &= \sup\left\{\sum_{k=1}^{n} \left|f(x_k) - f(x_{k-1})\right|\right\} \\
&= \sup\left\{\sum_{k=1}^{n} \left|\int_{x_{k-1}}^{x_k} f'\right|\right\} \\
&= \left\{\left|\int_a^b f'\right|\right\} && \text{(Endpoints are } a, b\text{)} \\
&\leq \int_a^b \left|f'\right|
\end{aligned}$$

(b) With the MVT, we can show

$$\begin{aligned}
V f &= \sup\left\{\sum_{k=1}^{n} \left|f(x_k) - f(x_{k-1})\right|\right\} \\
&= \sup\left\{\sum_{k=1}^{n} \left|\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}(x_k - x_{k-1})\right|\right\} \\
&= \sup\left\{\sum_{k=1}^{n} \left|f(c_k')\right|\Delta x_k\right\} \\
&= L(\left|f'\right|, P_{\sup}) && \left(\text{Since } \left|f(c_k') \geq \inf\left|f(x)\right|\right|\right) \\
&\leq L(\left|f'\right|, P) = \int_a^b \left|f'\right|
\end{aligned}$$

We could have alternatively said we know it is $= U(\left|f'\right|, P)$, for some P, which must be $\geq U(\left|f'\right|) = \int_a^b \left|f'\right|$.

Exercise 7.5.9

We see $U(h) = L(h) = b$ for $x \in [0, b]$, so h is integrable.

We also can see that $H(x) = x$, so H is differentiable at $x = 1$.

Exercise 7.5.10

We can show $F(x)$ is not differentiable at $x = c$ by showing that $F'(c)$ equals different values depending on how we take the limit.

We'll show $F'(c) \to L_1 = \lim_{x\to c^-} f(x)$, and a similar argument can be made for $F'(c) \to L_2 = \lim_{x\to c^+} f(x)$.

$$\begin{aligned}
\left|F'(c) - L_1\right| &= \left|\lim_{x\to c^-} \frac{F(x) - F(c)}{x - c} - L_1\right| \\
&= \left|\frac{1}{x - c}\int_c^x \left[f(t) - L_1\right] \mathrm{d}t\right|
\end{aligned}$$

Now, we can find

$$t', \left|f(t') - L_1\right| < \frac{\epsilon}{2}$$

We can also find

$$\delta, |x-c| \Rightarrow \left|f(x)-f(t')\right| < \frac{\epsilon}{2}$$

Then,

$$\begin{aligned}
\left|\frac{1}{x-c}\int_c^x \left[f(t)-L_1\right] \mathrm{d}t\right| &= \frac{1}{x-c}\int_c^x \left|f(t)-f(t')\right| + \left|f(t')-L_1\right| \mathrm{d}t \\
&\le \frac{1}{x-c}\int_c^x \left[\frac{\epsilon}{2}+\frac{\epsilon}{2}\right] \mathrm{d}t \\
&= \frac{1}{x-c}(x-c)\epsilon = \epsilon
\end{aligned}$$

A similar argument occurs for $\lim_{x\to c^+} F'(c) = L_2$ And since $L_1 \neq L_2$, we conclude $F'(c)$ does not exist.

Exercise 7.5.11

If we take $h(x)$ from Exercise 6.4.8, we see there is a discontinuity at every rational r_n, since $u_n(x)$ contributes a jump at r_n of $1/2^n$.

Now define $H(x) = \int_a^x h(x)$ for some $[a, b]$. From Exercise 7.5.10, we know that since $h(x)$ has a jump discontinuity at every r_n, we conclude $H(x)$ is not differentiable at any r_n, which is $\mathbb{Q}$ and dense.

We still need to show that $H(x)$ is continuous and monotonic.

$H(x)$ is monotonic since $h(x) > 0$, over any nonzero interval, so any $y > x$,

$$H(y) - H(x) = \int_x^y h(t)\,\mathrm{d}t > 0 \tag{7.2}$$

We can show $H(x) - H(c)$ is continuous, because given any $\epsilon > 0$, choose n such that

$$2 \cdot \frac{1}{2^n} < \epsilon$$

The reason we have this condition is that in the worst case, we have an infinite number of r_n near $H(c)$, which means the difference of

$$\left|H(x)-H(c)\right| \le \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots \le 2 \cdot \frac{1}{2^n}$$

Then choose a radius δ around c such that there only exists r_m with $m > n$. This is always possible, since there are a finite number of r_m such that $r_m \ge n$.

Then

$$\left|H(x)-H(c)\right| \le 2 \cdot \frac{1}{2^n} < \epsilon$$

7.6 Lebesgue's Criterion for Riemann Integrability

Exercise 7.6.1

(a) $L(t, P) = 0$ since every interval will have an irrational number, so the inf $= 0$ in all intervals.

(b) The set of points $\geq \epsilon/2$ are

$$\begin{aligned} x &= 0 \\ x &= \frac{1}{1} \\ x &= \frac{1}{2} \\ x &= \frac{1}{3} \\ &\cdots \\ x &= \frac{1}{\lfloor 2/\epsilon \rfloor} \end{aligned}$$

So we have

$$\left\lfloor \frac{2}{\epsilon} \right\rfloor + 1$$

numbers on this list.

(c) Choose a partition that is

$$\left\{0, \frac{1}{\lfloor 2/\epsilon \rfloor}\right\} \cup \left\{V_{\epsilon^2/9}(x)\right\}$$

Then we have

$$\begin{aligned} U(t, P_\epsilon) &= \frac{\epsilon}{2} \cdot 1 + \left[\left(\left\lfloor \frac{2}{\epsilon} \right\rfloor + 1\right) \cdot \frac{\epsilon^2}{9}\right] \\ &= \frac{\epsilon}{2} + \frac{\epsilon^2}{3} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{3} && (\text{for } \epsilon < 1) \\ &< \epsilon \end{aligned}$$

Note that $\epsilon \geq 1$ is trivial, since $\sup U(t, P) = 1$, so any partition will work for $\epsilon \geq 1$.

Exercise 7.6.2

We can compute that $L(g) = 0$, since every interval must contain some $x \notin C$, so the inf $= 0$ for any interval.

We know C_n has closed sets with length $(2/3)^n$, so we can define

$$P_n = \text{finite union of closed intervals that make up } C_n \cup \text{the rest}$$

The only worry is that we can still include the endpoints of the closed intervals in "the rest". So what we do is we make the finite union of closed intervals slightly wider, so that the length is still small. We can add L length extensions at the endpoints, where

$$2 \cdot 2^m \cdot L < \epsilon/2$$

Then if we choose $\left(\frac{2}{3}\right)^m < \epsilon/2$, and then choose L as indicated above,

$$\begin{aligned} U(g, P_m) &= 2^m \left(\frac{1}{3^m} + 2L\right) + 0 \\ &= \left(\frac{2}{3}\right)^m + 2 \cdot 2^m \cdot L \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Exercise 7.6.3
For any countable set, we have some enumeration, call it

$$S = \{a_1, a_2, a_3, \ldots\}$$

We can then construct the interval

$$O_n = \left(a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}\right)$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} |O_n| &= \sum_{n=1}^{\infty} 2 \cdot \frac{\epsilon}{2^{n+1}} \\ &= \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \epsilon\end{aligned}$$

Exercise 7.6.4
We can do the same strategy we did in Exericse 7.6.2, where we choose a C_n with really small intervals, and in order to include everything in open intervals, we extend each closed interval by a small amount. Then,

$$\begin{aligned}\sum |O_n| &= 2^n \left(\frac{1}{3^n} + 2L\right) && (7.3) \\ &< \epsilon && \text{(See Exercise 7.6.2)}\end{aligned}$$

Exercise 7.6.5
If A and B are measure zero, then we can find open covers O_n^1, O_n^2 such that

$$\begin{aligned}\sum \left|O_n^1\right| < \frac{\epsilon}{2} \\ \sum \left|O_n^2\right| < \frac{\epsilon}{2}\end{aligned}$$

So then $A \cup B \subseteq \bigcup_n O_n^1 \cup O_n^2$, and

$$\sum \left|O_n^1 \cup O_n^2\right| \leq \sum \left|O_n^1\right| + \sum \left|O_n^2\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} - \epsilon$$

For a countable union, I feel like we can assign

$$\sum_n \left|O_n^k\right| < \frac{\epsilon}{2^k}$$

for each set A_k.

Then if we sum these together we get $< \epsilon$.

The official solutions seems to need some reordering of a summation...I'm not sure why we need that, since we can say the open cover is

$$\mathcal{O} = \bigcup_k \bigcup_n O_n^k$$

So if we want to figure out

$$\sum |\mathcal{O}| = \sum_k \sum_n \left|O_n^k\right| = \sum_k \epsilon/2^k = \epsilon$$

I think the more rigorous proof also shows the other sum also equals $< \epsilon$.

Apparently the next exercises were already done in Section 4.6, but I'm going to do them again for review.

Exercise 7.6.6
Let $x \in D_{\alpha_2}$, then $\forall\delta$,

$$\left|f(y) - f(z)\right| \geq \alpha_2 > \alpha_1$$

So therefore f is also not α_1-continuous, and therefore $x \in D_{\alpha_1}$.

Exercise 7.6.7

(a) If we know that f is continuous at x, then we can always find a δ such that for $y, z \in V_\delta(x)$, that

$$\left|f(y) - f(z)\right| < \epsilon < \alpha \tag{7.4}$$

as long as we choose $\epsilon < \alpha$.

If f is not α-continuous at x, then f is discontinuous at that point, which means

$$D_\alpha \subseteq D$$

(b) This follows directly from the definition of continuity, since if we take the negation of the definition of continuity, we get that $\exists\epsilon_0 > 0$ such that $\left|f(y) - f(z)\right| \geq \epsilon_0$ for any $y, z \in V_\delta(x)$.

We can write

$$D = \bigcup_{n=1}^{\infty} D_{1/n}$$

since for any discontinuity at x, we can find this ϵ_0, which because of $\exists n, 1/n < \epsilon_0$, which shows that $x \in D_{1/n}$.

Exercise 7.6.8
Consider some limit point of D_α. $\exists(x_n) \to x, x_n \in D_\alpha$.

Now AFSOC f is α-continuous at x. Then we know $\exists\delta, y, z \in V_\delta(x)$ implies $\left|f(y) - f(z)\right| < \alpha$.

We know since $(x_n) \to x$, we can find x_k, x_j such that $|x_k - x|, \left|x_j - x\right| < \delta$. In this case, we have

$$\left|f(x_k) - f(x_j)\right| < \alpha$$

But this is a contradiction, since $x_k, x_j \in D_\alpha$, which means for any δ, we should be able to show that $\left|f(y) - f(z)\right| \geq \alpha$. But with our assumption, no matter what δ we choose, we can always produce some x_k close enough to an element of D_α such that $\left|f(x_k) - f(x)\right|$ is $< \alpha$.

Therefore, we must conclude that f is α-discontinuous at x, and therefore $x \in D_\alpha$.

Since D_α contains all of its limit points, it must be closed.

Exercise 7.6.9
AFSOC f is not uniformly α-continuous on K, then we know

$$\exists x_n, y_n, \lim |x_n - y_n| = 0 \text{ while } \left|f(x_n) - f(y_n)\right| \geq \alpha$$

Since K is compact, we know x_n, y_n both have convergent subsequences $(x_{n_k}), (y_{n_k})$ with limit $L_x = \lim x_{n_k}, L_y = \lim y_{n_k}$. By the Algebraic Limit Theorem, we have

$$\lim y_{n_k} = \lim\Big(\big(y_{n_k} - x_{n_k}\big) + x_{n_k}\Big) = 0 + L_x$$

Therefore, we have $L_y = L_x$, so the two subsequences converge to the same limit.

Now, since f is α-continuous at every point x, there $\exists\delta$ such that $y, z \in V_\delta(x)$ implies $\left|f(y) - f(z)\right| < \alpha$. Since $(x_{n_k}), (y_{n_k}) \to L$, we can find elements in both sequences that are $\delta/2$ away from L, call them x', y', which means $\left|x' - y'\right| < \delta$, and therefore

$$\left|f(x') - f(y')\right| < \alpha$$

But this is a contradiction, since we assumed this quantity $\geq \alpha$.

Therefore, we conclude that no such sequence exists, and therefore $|x - y| < \delta$ implies $\left|f(x) - f(y)\right| < \alpha$.

Exercise 7.6.10

(a) Since we are assuming none of $I_n = \emptyset$, we can use the NIP to show that there must be an x in an infinite intersection of nonempty closed sets.

(b) Since $x \in K$, $\exists G_{\alpha_0}$ that contains x. Let $S = (x-\epsilon, x+\epsilon) \subseteq G_{\alpha_0}$. We know such an ϵ exists since G_{α_0} is open.

We also know $\lim_n |I_n| = 0$, and since K is compact and therefore bounded, this means we can choose n such that

$$|K \cap I_n| < \epsilon$$

Now, we must have

$$K \cap I_n \subseteq (x-\epsilon, x+\epsilon) \subseteq G_{\alpha_0}$$

But this is a contradiction since we assumed $K \cap I_n$ had no finite subcover.

Exercise 7.6.11

From the definition of measure zero, we know there exists some set of open intervals G_k such that D_α is contained within it, and that

$$\sum |G_k| < \epsilon$$

Now, since D_α is bounded by $[a, b]$, and is closed, we know that D_α is compact, which means it has a finite subcover from G_k.

The last thing we have to do is make these open sets disjoint, which we can do by merging any G_k together that have any overlap, and we know their union is still an open set.

Finally, the measure of these open sets is still arbitrarily small, since first, we took a subset of the already measure zero set, then we took unions of sets, which satisfy

$$|G_1 \cup G_2| \leq |G_1| + |G_2|$$

Exercise 7.6.12

We know f is α-continuous on K, since D_α is removed. After removing a finite set of disjoint open intervals on $[a, b]$, we must have K equal a union of a finite number of closed sets, which is still closed. Since K is bounded, we conclude K is compact. We showed in Exercise 7.6.9 that if f is α-continuous over a compact set, it must also be uniformly α-continuous over it as well.

Exercise 7.6.13

We can choose the parition so that, for δ that makes f uniformly α-continuous on $[a, b]$,

$$P_\epsilon = \{\text{Even intervals of width } < \delta\} \cup (\text{endpoints of } G_i)$$

Now, if we calculate

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) &\leq \frac{\epsilon}{4M} \cdot 2M + (b-a)\frac{\epsilon}{2(b-a)} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

For some explaining here, when we're inside the G_i regions, the maximum difference of $U - L$ is $2M$, since M is the bound of f, and we could potentially have $M - (-M) = 2M$. The length of these intervals is conveniently $\epsilon/(4M)$ that we chose from earlier.

For the intervals outside of the G_i, they are at most width δ, so by α-continuity, any x, y in this interval will satisfy

$$\big|f(x) - f(y)\big| < \alpha$$

As an upper bound, we can say $< \alpha$, and then say the sum of the length of these intervals is $(b-a)$.

Therefore, in this direction, we can conclude that f is integrable.

Exercise 7.6.14

(a) If we have $U(f, P_\epsilon) - L(f, P_\epsilon) < \alpha\epsilon$, then let G_k be the intervals that contain D_α. Notice that because it is α-discontinuous on these intervals, any difference of $\left|f(x) - f(y)\right| \geq \alpha$, so

If we now compute

$$\begin{aligned}\sum_k \alpha|G_k| &\leq \sum_k (M_k - m_k)|G_k| \\ &\leq \sum_j (M_j - m_j)\Delta x_j \\ &= U(f, P_\epsilon) - L(f, P_\epsilon) \\ &< \alpha\epsilon\end{aligned}$$

So we have

$$\sum_k \alpha|G_k| < \alpha\epsilon \Rightarrow |D_\alpha| \leq \sum_k |G_k| < \epsilon$$

(b) Since we know each D_α as measure zero, we can define

$$D = \bigcup_{n=1}^{\infty} D_{1/n}$$

Now, this has measure zero since every $\left|D_{1/n}\right|$ can be bounded by $\epsilon/2^n$, and thus $|D| < \epsilon$.

Exercise 7.6.15

(a)

$$\begin{aligned}g'(0) &= \lim_{x\to 0} \frac{g(x) - 0}{x - 0} \\ &= \lim_{x\to 0} \frac{x^2 \sin\left(1/x\right)}{x} \\ &= \lim_{x\to 0} x \sin\left(1/x\right) \\ &= 0\end{aligned}$$

(b) We can compute

$$g'(x) = 2x\sin\left(1/x\right) - \cos\left(1/x\right)$$

(c) For any $\frac{1}{2\pi n}, \frac{1}{\pi + 2\pi n} \in (-\delta, \delta)$,

$$\begin{aligned}g'\left(\frac{1}{2\pi n}\right) &= -1 \\ g'\left(\frac{1}{\pi + 2\pi n}\right) &= 1\end{aligned}$$

Since $g'(x)$ is continuous between

$$x \in \left[\frac{1}{\pi + 2\pi n}, \frac{1}{2\pi n}\right],$$

we see that $g'(x)$ attains every value between $-1, 1$ over $V_\delta(0)$.

We can conclude g' is not continuous at $x = 0$, because no matter how small of a $V_\delta(0)$, we choose, we can always fine

$$\left|g'(x) - g'(0)\right| = 1$$

Exercise 7.6.16

(a) Eventually $f_n(c) = 0$, since $x \in C_n$ for some n. So $\lim_{n\to\infty} f_n(c) = 0$.

(b) If $x \notin C$, then it will be part of some fragment of $g(x)$, which is constant after it is created. This fragment is also continuous.

Exercise 7.6.17

(a) $f'(x)$ is the differentiable $g(x)$ fragment we showed earlier was differentiable.

(b) We can imagine $\left|f(x)\right|$ as a sine wave that oscillates under the parabola x^2 centered around some endpoint $\in C$.

Then, $(x-c)^2$ is a parabola centered one of these endpoints, which means it is the upper envelope for one of these sine waves. See Figure 7.5 for how this envelope bounds the sine-parabola.

Now, this property implies that

$$\left|f(c+\delta)\right| \leq (c+\delta-c)^2 = \delta^2$$

so we can find

$$f'(c) = \frac{f(c+\delta)-f(c)}{c+\delta-c} = \frac{\delta^2}{\delta} = \delta$$

which we can make arbitrarily small, so $f'(c) = 0$.

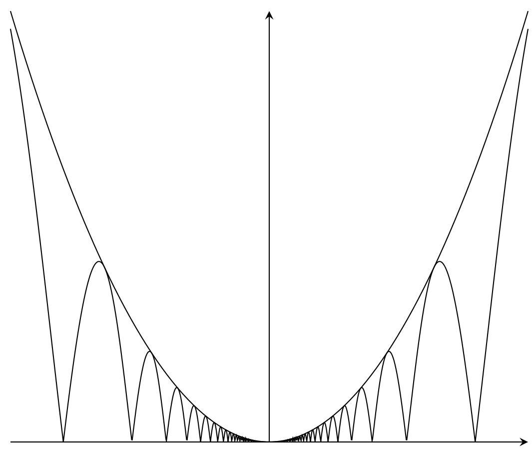


Figure 7.5: Showing how $\left|f(x)\right| \leq (x-c)^2$

(c) We have $f'(c) = 0$ for every $c \in C$. Now, in any δ-neighborhood of c, we can always find some $c' \notin C$, which because is on an open set of a fragment of g, $\exists c''$ where $f'(c'') \neq 0$.

This means for any δ, $\exists c'' \in V_\delta(c)$ such that

$$\left|f'(c'') - f'(c)\right| = \left|f'(c'')\right| \neq 0$$

So therefore, $f'(x)$ is not continuous.

Exercise 7.6.18

By Lebesgue's Theorem, we know that f' is bounded (≤ 1), and f' is not continuous at every $x \in C$, and we know C is a measure zero set, so therefore f is Riemann-integrable.

Exercise 7.6.19

Now, the sum of the lengths of C are the complement of the sum of the segments which we remove, which is

$$\frac{1}{9} + 2 \cdot \frac{1}{27} + \cdots = \frac{1/9}{1-2/3} = \frac{1}{3}$$

So we can conclude $\lim_{n\to\infty} |C_n| = \frac{2}{3}$.

I also wanted to do this directly...but it is more difficult. The sequence for $|C_k|$ looks like

$$|C_0| = 1, |C_k| = |C_{k-1}| - \frac{2^{k-1}}{3^{k+1}}.$$

Now, I couldn't find a smart way to figure out this convergence, other than noticing that

$$|C_k| = |C_0| - \left(\frac{1}{9} + \frac{2}{27} + \cdots + \frac{2^{k-1}}{3^{k+1}}\right),$$

which just leads to the result we found earlier.

Chapter 8

Additional Topics

8.1 The Generalized Riemann Integral

Exercise 8.1.1

(a) The Riemann sum $R(f,P)$ chooses tags that are in between the inf, sup of every interval, so it must be between L, U.

We know $\int_a^b f = U = L$, so it is $\geq L(f,P)$ and $\leq U(f,P)$.

(b) We chose P_ϵ such that $U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon/3$, and P' contains more points than P_ϵ, so the difference can only be smaller.

Exercise 8.1.2

$P \subseteq P'$, so as we saw in Chapter 7, adding more points to the partition gives us more opportunities to have a smaller sup in every interval, so we conclude

$$U(f,P) - U(f,P') \geq 0$$

Exercise 8.1.3

(a) The intervals that will be different will be when P_ϵ contributes a point between an interval of P, where it will break into two segments that are different from the original. This produces a total of 3 segments that are potentially different. Since P_ϵ has $n+1$ points, 2 of the points are for the endpoints, so we can have $n-1$ of these 3 segment differences, i.e. a total of $3(n-1)$ differences

(b) We have $3(n-1)$ segment differences, and each segment is bounded by contribution

$$|M \cdot \delta|$$

since we chose δ-fine segments.

Thus, we can show

$$U(f,P) - U(f,P') \leq 3(n-1) \cdot \delta \cdot M < \frac{\epsilon}{9nM} \cdot 3(n-1) \cdot M < \epsilon/3$$

The proof finishes by showing a similar result for the other side, and then we can conclude that $R(f,P), \int_a^b f$ are sandwiched between $L(f,P), U(f,P)$, which in turn are sandwiched between $L(f,P') - \epsilon/3, U(f,P') + \epsilon/3$, and $L(f,P'), U(f,P')$ themselves are $< \epsilon/3$ apart, so in all, $L(f,P), U(f,P)$ are sandwiched in an interval of length ϵ, so we conclude $R(f,P), \int_a^b f$ are also $< \epsilon$ apart.

Exercise 8.1.4

(a) We want to say we can just pick $c_k = \sup_x[x_k, x_{k+1}]$, but the issue is sup might not exist in this interval; imagine approaching $f(x) = 3$ infinitely closely, but never getting there. Then we know $\sup f = 3$, but $\nexists x' f(x') = 3$ in this interval.

If we have f continuous, then sup (and inf) exists, so we can choose these c_k to make $R(f,P) = U(f,P)$.

(b) If f is not continuous, we still know f is bounded, so no matter what $\sup f$ is on an interval, we can always get arbitrarily close to it.

Then, all we have to do is make sure that

$$\left|\sup f_k - f(c_k)\right| < \epsilon_1$$

is small enough such that when we add all the intervals together, it is still $< \epsilon$.

We can choose $\epsilon_1 < \frac{\epsilon}{b-a}$ to do that.

Exercise 8.1.5

We showed in Exercise 8.1.4 that

$$\begin{aligned} U(f, P) - R(f, P^1) &< \epsilon \\ R(f, P^2) - L(f, P) &< \epsilon \end{aligned}$$

Where P^1, P^2 are different taggings of P.

Now, if we add these two questions together, we get

$$U(f, P) - L(f, P) < 2\epsilon + \left[R(f, P^1) - R(f, P^2)\right]$$

So if we could bound the difference between the tagged partitions, we can finish our proof.

Fortunately, in this direction, we know that for some δ-fine P, it is possible to show that for *any* tagging of P, we have

$$\left|R(f, P) - A\right| < \epsilon \Rightarrow \left|R(f, P^1) - R(f, P^2)\right| < \epsilon$$

Therefore, we can do the following,

1. Choose δ so that
$$\left|R(f, P) - A\right| < \frac{\epsilon}{4} \Rightarrow \left|R(f, P^j) - R(f, P^k)\right| < \frac{\epsilon}{4}$$

2. Choose partitions P^1, P^2 so that we have
$$\begin{aligned} U(f, P) - R(f, P^1) &< \frac{\epsilon}{4} \\ R(f, P^2) - L(f, P) &< \frac{\epsilon}{4} \end{aligned}$$
so we can say
$$U(f, P) - L(f, P) < \frac{\epsilon}{2} + \left[R(f, P^1) - R(f, P^2)\right] < \epsilon$$

Therefore, we can conclude f is integrable.

To show that $\int_a^b f = A$, see that $R(f, P^1)$ is $\epsilon/4$ away from A, and $U(f, P)$ is $\epsilon/4$ away from $R(f, P^1)$, so therefore $U(f, P)$ is $\epsilon/2$ away from A. We also see that $L(f, P)$ is $\epsilon/2$ away from A as well, so therefore we know $U(f) = \int_a^b f$, is close to A

$$\left|\int_a^b f - A\right| < \epsilon$$

Exercise 8.1.6

(a) We just need $x_k - x_{k-1} < \delta(c_k) = 1/9$, so any partition with intervals width $< 1/9$ will do.

(b) We can choose the partition

$$P = \left\{0, 1/4, 1/4 + 1/12, 1/4 + 2/12, \ldots, 1/4 + 8/12, 1\right\}$$

and choose tags as the left endpoint of the first interval, and the right endpoint of the rest.

The reason this works is in the initial interval has tag 1/4 and size 1/4. For every subsequent interval, the minimum tag has $> 1/12$, so as long as we make the interval smaller than that, we are fine.

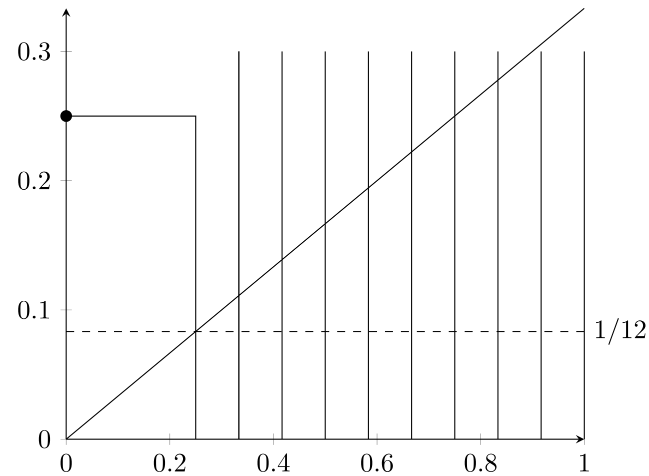


Figure 8.1: Choosing a $\delta(x)$-fine partition

Exercise 8.1.7
The intuition here is that eventually, the partitions are so small that it must be $< \delta(x)$ for some x in the interval.

AFSOC this bisecting never ends. Then by the NIP, we can find an x that exists in the never-ending bisected intervals. From our hypothesis, we know that for this x, every

$$|I_n| > \delta(x)$$

However, since $\delta(x)$ is fixed here, and we can find an arbitrarily small interval size, this implies $\delta(x) = 0$, but this is a contradiction since $\delta(x) > 0$.

Therefore, we conclude that there must be a finite number of bisections, and when the bisecting stops, $\exists x$ such that $\delta(x) > x_k - x_{k-1}$.

Exercise 8.1.8
Define

$$\delta(x) = \min(\delta_1(x), \delta_2(x)),$$

then we can find some tagged partition $(P, \{c_k\}_{k=1}^n)$ that is $\delta(x)$-fine.

Now, this tagged partition is also δ_1, δ_2-fine since in every interval,

$$x_k - x_{k-1} < \delta(c_k) = \min(\delta_1(c_k), \delta_2(c_k)),$$

and therefore $x_k - x_{k-1}$ is smaller than both δ_1, δ_2.

Therefore A_1, A_2 are $\epsilon/2$ away from a common element, which means we can show they are

$$|A_1 - A_2| < \epsilon$$

Exercise 8.1.9
If a function is Riemann Integrable, we know that it has a δ_0 such that for any δ_0-fine tagged partition we have

$$\left|R(f, P) - A\right| < \epsilon$$

We can choose $\delta(x) = \delta_0$ to satisfy the above property, and therefore we have found a gauge that works and shows f is Generalized Riemann Integrable.

Exercise 8.1.10
If we have a tagged partition that is $\delta(x)$-fine by our definition, we know that every interval containing a rational number r_k is width at most $\epsilon/2^{k+1}$. Now, this r_k could also be on the endpoint of an interval, in which case it appears twice.

So in the worst case, we can calculate

$$R(f,P) < \sum_{k=1}^{\infty} 2 \cdot \frac{\epsilon}{2^{k+1}} \cdot \big(\sup g(c_k)\big) = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} \cdot 1 = \epsilon$$

Exercise 8.1.11
This is just a telescoping series, with the endpoints a, b.

Exercise 8.1.12
We know $F'(c) = f(c)$, so we must have some $V_\delta(c)$ such that for any x in this neighborhood, the derivative is ϵ close to $f(c)$.

Then, we can define the gauage to be this δ for every c.

Exercise 8.1.13

(a) Both of these results follow from Exercise 8.1.12, since our P is $\delta(c)$-fine, so any c chosen in the $[x_{k-1}, x_k]$ interval is $\delta(c)$ close to either endpoint.

(b) We can add up

$$\begin{aligned}
-\epsilon(x_k - c_k) < F(x_k) - F(c_k) - f(c_k)(x_k - c_k) < \epsilon(x_k - c_k)\\
-\epsilon(c_k - x_{k-1}) < F(c_k) - F(x_{k-1}) - f(c_k)(c_k - x_{k-1}) < \epsilon(c_k - x_{k-1})\\
-\epsilon(x_k - x_{k-1}) < F(x_k) - F(x_{k-1}) - f(c_k)(x_k - x_{k-1}) < \epsilon(x_k - x_{k-1})\\
\Rightarrow \big|F(x_k) - F(x_{k-1}) - f(c_k)(x_k - x_{k-1})\big| < \epsilon(x_k - x_{k-1})
\end{aligned}$$

From here, we know that our difference

$$\big|F(b) - F(a) - R(f,P)\big| < \epsilon'$$

Exercise 8.1.14

(a) We just need to show that

$$\big(F(g(x))\big)' = F'(g(x)) \cdot g'(x)$$

exists, but this is true since we said g is differentiable on $[a,b]$ and F is differentiable on the set $g([a,b])$.

(b) We can show

$$\begin{aligned}
F(g(b)) - F(g(a)) &= \int_a^b \big(F(g(x))\big)' \,\mathrm{d}x\\
&= \int_a^b F'(g(x)) \cdot g'(x) \,\mathrm{d}x\\
&= \int_a^b f(g(x)) \cdot g'(x) \,\mathrm{d}x
\end{aligned}$$

(c) Since F is differentiable at each point $g([a,b])$, we can directly apply Theorem 8.1.9 and conclude that

$$\int_{g(a))}^{g(b)} f = F(g(b)) - F(g(a))$$

8.2 Metric Spaces and the Baire Category Theorem

8.3 Fourier Series

8.4 A Construction of $\mathbb{R}$ From $\mathbb{Q}$

Appendix A

Extras

A.1 Useful Tools

Collection of useful tools and methods to solve problems.

Tip A.1.1
Template for a proof that $(x_n) \to x$:

- Let $\epsilon > 0$ be arbitrary
- Demonstrate a choice for $N \in \mathbb{N}$. This step usually requires the most work, almost all of which is done prior to actually writing the formal proof.
- Now, show that N works.
- Assume $n \geq N$
- With N well chosen, you should be able to show $|x_n - x| < \epsilon$.

A.2 Cool Things

- In Chapter 2, we learn that addition in infinite sums is not commutative.
- In Chapter 2, we learn that if $\sum_{n=1}^{\infty} a_n$ converges conditionally, then for any $r \in \mathbb{R}$, there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to r.
- In Chapter 3, we learn that $\mathbb{R}$ and $\emptyset$ are both open and closed, but they are the only subsets in $\mathbb{R}$ with this property.
- In chapter 7, we talk about the Lebesgue Integral as a generalization of the Riemann Integral, which allows us to integrate more functions. This is cool, because it uses an intuitive idea about partitioning the x axis as partitions of measures, instead of relying on a defined partition like the Riemann Integral does.

A.3 Important Theorems

A.3.1 5 Characterizations of Completeness

Theorem 1 (Axiom of Completeness) *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Theorem 2 (Nested Interval Property) *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \cdots$$

has a nonempty intersection, that is $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem 3 (Monotone Convergence) *If a sequence is monotonic and bounded, then it converges.*

Theorem 4 (Bolzano-Weierstrass) *Every bounded sequence contains a convergent subsequence.*

Theorem 5 (Cauchy Criterion) *A sequence converges if and only if it is a Cauchy Sequence.*

A sequence (a_n) is called a Cauchy sequence if, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $m, n \geq N$, it follows that $|a_n - a_m| < \epsilon$.

A.3.2 Sequence Convergence

Theorem 6 (Convergence of a Sequence) *A sequence $\{a_n\}$ converges to $a \in \mathbb{R}$ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$, it follows that $|a_n - a| < \epsilon$.*

Theorem 7 (Cauchy Criterion) *A sequence converges if and only if it is a Cauchy sequence, which is defined as a sequence (a_n) that for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $m, n \geq N$, it follows that $|a_n - a_m| < \epsilon$.*

A.3.3 Function Continuity

Theorem 8 (Characterizations of Continuity) *Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. The function f is continuous at c if and only if any one of the following conditions is met:*

(i) *For all $\epsilon > 0$, $\exists \delta > 0$ such that for $x \in A, |x - c| < \delta$ implies $\big|f(x) - f(c)\big| < \epsilon$*

(ii) $\lim_{x \to c} f(x) = f(c)$

(iii) *For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in A, x \in V_\delta(c)$ implies $f(x) \in V_\epsilon(f(c))$*

(iv) *If $(x_n) \to c$, with $x_n \in A$, then $f(x_n) \to f(c)$*

Theorem 9 (Uniform Continuity) *A function $f : A \to \mathbb{R}$ is uniformly continuous on A if for every $\epsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta$ implies $\big|f(x) - f(y)\big| < \epsilon$.*

A.3.4 Function Convergence

Definition 1 (Uniform Convergence) Let f_n be a sequence of functions defined on a set $A \subseteq \mathbb{R}$. Then, (f_n) converges uniformly on A to a limit function f defined on A if, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\big|f_n(x) - f(x)\big| < \epsilon$ whenever $n \geq N$ and $x \in A$.

A.4 Identities

Identity A.4.1
(Triangle Inequality) The triangle inequality states that for $x, y \in \mathbb{R}$,

$$|x| - |y| \leq |x + y| \leq |x| + |y| \tag{A.1}$$

Identity A.4.2
(Geometric Series)

$$\sum_{k=0}^{m} ar^k = \frac{a(1 - r^{m+1})}{(1 - r)} \tag{A.2}$$

and converges to

$$\lim_{m\to\infty}\sum_{k=0}^{m} ar^k = \frac{a}{1-r}$$

iff $|r| < 1$.