Statistical Mechanics is a useful study of the thermodynamics related to a system of any arrangement and number of particles. By employing statistical mechanics, the behaviors of a series of particles can be treated as a range of probabilities and related to quantities like energy and entropy. This can directly be applied to magnetism.

Magnetism can be described as the cumulative behavior and alignment of the individual magnetic moments of electrons. Magnetic moments can be further described as the normal vector of a magnetic dipoles. The magnetic moment of a dipole tends to align with an external magnetic field, with varying levels of alignment across a material's volume, based on the materials property of Magnetic Susceptibility (X_M). The application of statistical mechanics allows for a prediction of the dipole alignment. This can be used to further derive a material's magnetization.

All Figures and Graphs in this paper were produced using Python accessible here sss

1 Related Concepts and Equations

1.1 Magnetism

Magnetic Dipoles and Magnetism

A Magnetic dipole can be described as a current loop, with the magnetic moment being perpendicular to the plane of the loop (Eq. 1) . As an example, electrons orbiting a nucleus can be treated as a current loop.

A magnetic dipole will experience a torque (Eq. 3) cause by the Lorentz Force when in the presence of an external magnetic field (Eq. 2). A series of dipoles will experience the same torque, and consequently the magnetic moments will align. The cumulation of aligned magnetic moments contributes to the magnetic field generated by a material, and an overall current loop forms around the region of dipole alignments.

$$\vec{\mu} = I \cdot \vec{A} \tag{1}$$

$$\vec{F} = q(\vec{v} \times \vec{B}) \tag{2}$$

$$\vec{\tau} = \vec{r} \times \vec{F} = \mu \times \vec{B} \tag{3}$$

Magnetism in Media

Magnetism in media is expressed differently to account for a material's specific properties of permeability (μ) and susceptibility (X_M). The former scales the ability for magnetic fields to travel (eq. 4), while the latter scales the effects/ magnitude of magnetism (Eq. 5). The auxiliary magnetic field $\vec{H}$ can be used to express the external magnetic field instead of the altered expression of $\vec{B}$. Expressions for bound surface (Eq. 7) and volume density currents (Eq. 8) are also affected.

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1 + X_M)\vec{H} = \mu\vec{H} \tag{4}$$

$$M = X_m\vec{H} \tag{5}$$

$$\mu = \mu_0(1 + X_M) \tag{6}$$

$$K_B = \vec{M} \times \hat{n} \tag{7}$$

$$J_B = \nabla \times \vec{M} \tag{8}$$

1.2 Statistical Mechanics

Multiplicity and Entropy

Multiplicity is useful for determining the probability of a macrostate to exist, taking into account the number of microstates found within it (Eq. 9). In doing so, multiplicity becomes a description of a the varying probability of all states of existence, which can be related to entropy (Eq. 11). A common trend in multiplicity is that a system's value will usually peak where a macrostate is likely to have the most number of microstates, at equilibrium. This also happens to be where the system's entropy peaks.

$$\text{for a two state system: } \Omega(N, N_\uparrow) = \frac{N!}{N_\uparrow!((N-N_\uparrow)!} \approx 2^N \tag{9}$$

$$N_{microstates} = 2^{N_{particle}} \tag{10}$$

$$S = k_b ln\Omega \tag{11}$$

Return to Magnetism

With the directional component of the magnetic moment μ, a spin up state will be assigned to be in the positive y direction. Spin down will be a negative y. Accordingly, a spin up state would be represented $+\mu$ and spin down $-\mu$. For a moment along the x direction, this can be considered as a 0 energy situation for simplicity. Then the energy required to flip the direction from one state to another would be twice the energy needed to reach one of the spin states.

A thermodynamic equation for total energy can be derived by considering the microstate for a given macrostate (Eq 12). The term B is considering the applied magnetic field, which can be thought of as H. Using the identity from Eq. 4, Magnetization can be derived via statistical mechanics (Eq. 13) as an expression of the cumulative spin state of the system at a given macrostate. The energy and magnetization dependence of temperature proves to be useful in thermodynamics, and will be included in this standard.

$$U = \mu B(N_\downarrow - N_\uparrow) = \mu B(N - 2N_\uparrow) \tag{12}$$

$$M = \mu(N_\uparrow - N_\downarrow) = -\frac{U}{B} \tag{13}$$

Distribution Functions

Because magnetism happens to be related to temperature and energy, the Boltzmann factor is applicable. The Boltzmann factor allows the use of a weighted probability of a system's energy state, or net dipole alignment. A certain macrostate will consist of the numerous microstates possible underneath it, and can be expressed using the partition function. Further connection between two state systems of metals can be used to produce Distribution functions of such states. These equations may not be used depending on how much time I have to work on this, but I'd like to at least mention this extent of statistical mechanics application for characterizing conductance and magnetization at low temperatures.

2 Two State System

A magnetic moment can represent the spin of a particle. Because the spin position is quantized, the spin direction can be labeled as either spin up or spin down, each representing one of two states. It is the alignment of moments that contributes to magnetism. As seen in the table below, the magnetization aligns with the direction of spin.

For a system of 100 particles, the multiplicity function following Eq. 9, 11, 12, 13, and a thermodynamic identity for temperature shown in Table 1. At the point when $N_\uparrow = 50 = N_\downarrow$, multiplicity is at a peak point along with entropy. This defines the equilibrium point, which houses to most number of possible spin orientations. Should an external magnetic field begin to influence the orientation of spin to point upwards, values for multiplicity and entropy will decrease, while magnetization aligns with the cumulation of spin alignment. Its then seen that multiplicity and entropy reach a minimum at the extremities when $N_\uparrow = 100$ or $N_\downarrow = 100$. At these points, the possible energy of the system has already been distributed such that this microstate will certainly consist of a single microstate of dipole alignment, with magnetization seemingly in agreement.

Up	Down	Multiplicity	Entropy (S / kB)	Energy (U/μ B)	Mag.(M/μB)	kT / μB
100	0	1	0	-100	1	0.434294482
99	1	100	4.605170186	-98	0.98	0.470193115
98	2	4950	8.507142856	-96	0.96	0.541394491
97	3	161700	11.99349805	-94	0.94	0.599271423
96	4	3921225	15.18191466	-92	0.92	0.651112999
95	5	75287520	18.13682494	-90	0.9	0.699664268
...	...	...	...	...	...	...
51	49	9.89E+28	66.76403902	-2	0.02	50.48844556
50	50	1.01E+29	66.78384165	0	0	inf
49	51	9.89E+28	66.76403902	2	-0.02	-50.48844556
...	...	...	...	...	...	...
5	95	75287520	18.13682494	90	-0.9	-0.699664268
4	96	3921225	15.18191466	92	-0.92	-0.651112999
3	97	161700	11.99349805	94	-0.94	-0.599271423
2	98	4950	8.507142856	96	-0.96	-0.541394491
1	99	100	4.605170186	98	-0.98	-0.470193115
0	100	1	0	100	-1	-0.434294482

Table 1: Mutliplicity table for N=100 of a two state system. Schroeder [2]

The figures below are visual representations of a two state system, displaying trends in entropy and magnetism. The use of entropy is crucial for associating quantities like temperature. Finding trends of magnetism related to temperature is also helpful in understanding the relation of thermal energy to a dipole's willingness to align with the external magnetic field. The relationships of magnetism, entropy, and temperature are crucial to understanding the dipole behaviors of a material. Sub Figure b shows that magnetization is most prominent at lower temperatures.

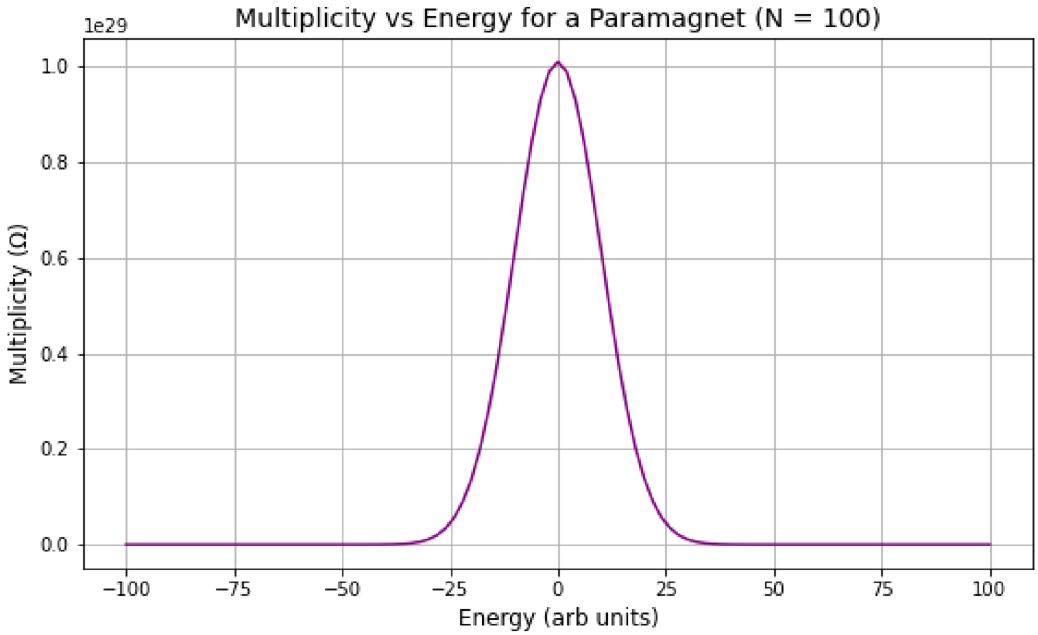
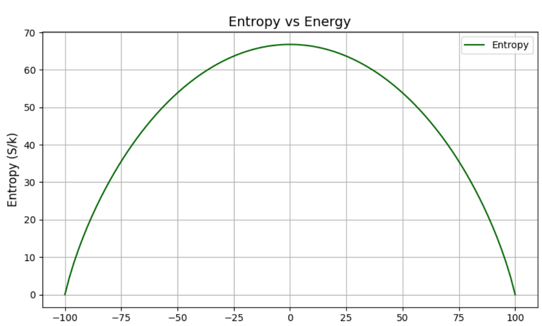
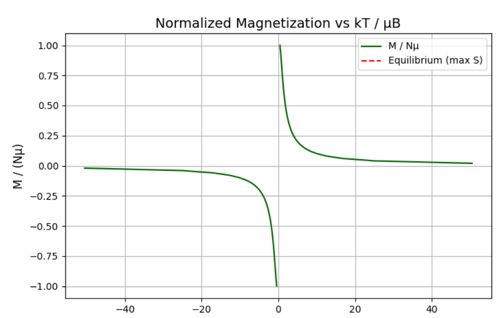


Figure 1: Multiplicity Distribution



(a) Entropy across the range of energies



(b) Magnetization related to temperature

Eq. 13 for magnetism can be expressed differently when considering a two-state system along with the implications when using the thermodynamic identities involved. A combination of sterling approximations used for entropy along with shuffling identities allows a derivation of the Brillouin function (Eq. 14) as an expression for magnetism in terms of temperature, stemming from the observed relationships from Eq. 13. The Brillouin function may also be derived by considering the average magnetism under a statistical approach. Another important use of temperature is deriving an expression of the magnetic susceptibility of a material (X_M) via Curie's Law (Eq. 15).

$$M = N\mu tanh(\frac{\mu B}{kT}) \tag{14}$$

$$X_M = (\frac{\partial M}{\partial B})_{B\to 0} = \frac{N\mu^2}{kT} \tag{15}$$

From the equation of magnetism in matter (Eq. 5), it understood that χ_m of a material is important in determining the scale by which a material may magnetize. The value of χ_m can be expressed with Curie's law, and a numeric expression is shown by using the derivative of relevant values from Eq. 15 in Figure 3. The trend of χ_m vs Temperature is similar to that of magnetism, which should be the case. In showing that χ_M is highest at low temperatures, it would be understood magnetism would also proporitonaly scale.

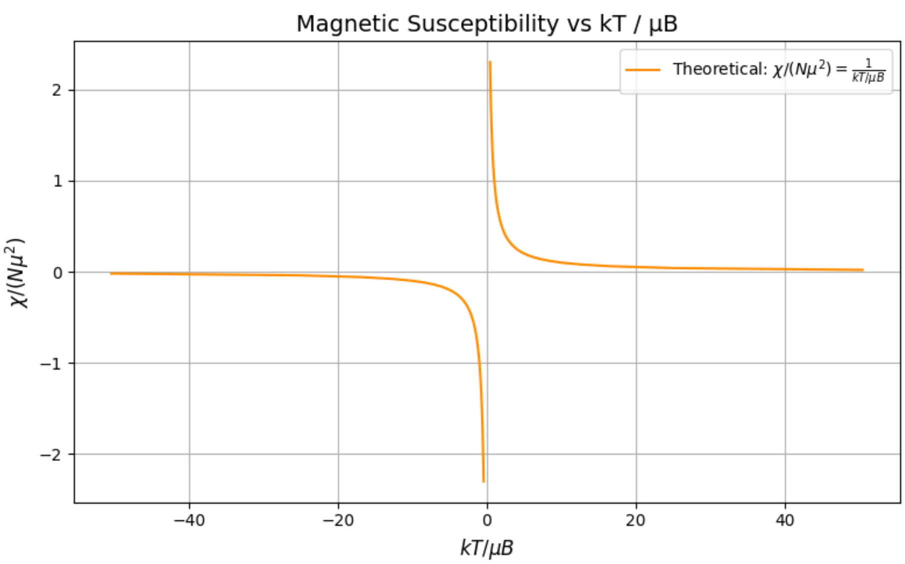


Figure 3: Magnetic Susceptibility vs Temperature.

Partition function...

$$M = -\frac{\partial F}{\partial B} \tag{16}$$

$$\chi = \frac{\partial M}{\partial B} = -\frac{\partial F^2}{\partial B^2} \tag{17}$$

3 Identifying an Unknown Metal

To demonstrate the application of statistical mechanics in magnetism, I will simulate an experiment to measure the magnetic responses of an unknown metal and then perform an analysis to determine the identity of this metal purely based on statistical mechanics. The simulation is performed computationally using Python generated by Claude AI. In doing so, the specific properties of the material were necessary to accurately model the behaviors so that the analytical findings would reflect a physical experiment. (Eq. 5)

3.1 Additional Concepts

When generating the code for this simulation, the AI chose to use some higher level concepts that weren't previously mentioned. I dont think that they are necessary for the scope, but I decided to keep them to further emphasize the utilization of statistical mechanics in Electro-Magnetism Theory.

Pauli's Exclusion Principal and Fermi Energy [1]

The benefit to modeling metals in this fashion is the ability to utilize Pauli's exclusion principal, which states that no two electrons may share the same quantum numbers. Electrons can be viewed as Fermions in a two-state system of spin up and spin down Duzer and Turner [1]. This construction relating to energy provides another explanation as to why the multiplicity function decreases as the net dipole allignment of a magnet begins to favor a direction.

Free Electron Theory

The valence electrons of a metal maintain mobility, as described by the surface bound charge density of a magnetized material. Building from this concept, the electrons can be considered as cloud of delocalized particles with free mobility. The free electron theory works well with Pauli's Principal and the Fermi-Dirac model Duzer and Turner [1].

$$\chi = \mu_0 \mu_e^2 n / (2 k_B T) \tag{18}$$

3.2 Simulation and Analysis

The experiment simulates the magnetization of a metal in response to an increasing external magnetic field. A line of best fit was applied with the slope being associated to the χ_M of the material. The overall analysis doesn't describe an actual experiment, as the material's properties are needed to properly simulate it.

The Magnetization was calculated with Eq 13 using the Pauli Susceptibility (Eq. 18). A value for N was calculated by using the material properties of Aluminum. The magnetization was then calculated across a varying magnetic field under a constant temperature (T=300k).

Because Eq. 18 does not represent susceptibility in terms of the changing magnetic quantities, a fit line was still used to determine χ_m being measured.

As added verification, further analysis was performed by modeling the relationship between values of χ_m vs Temperature. A fit curve was used, which matches the expected Curie relationship.

[1] I was delighted but not surprised to see these concepts here. They came up in my capstone proposal when I was investigating the theory behind superconductivity.

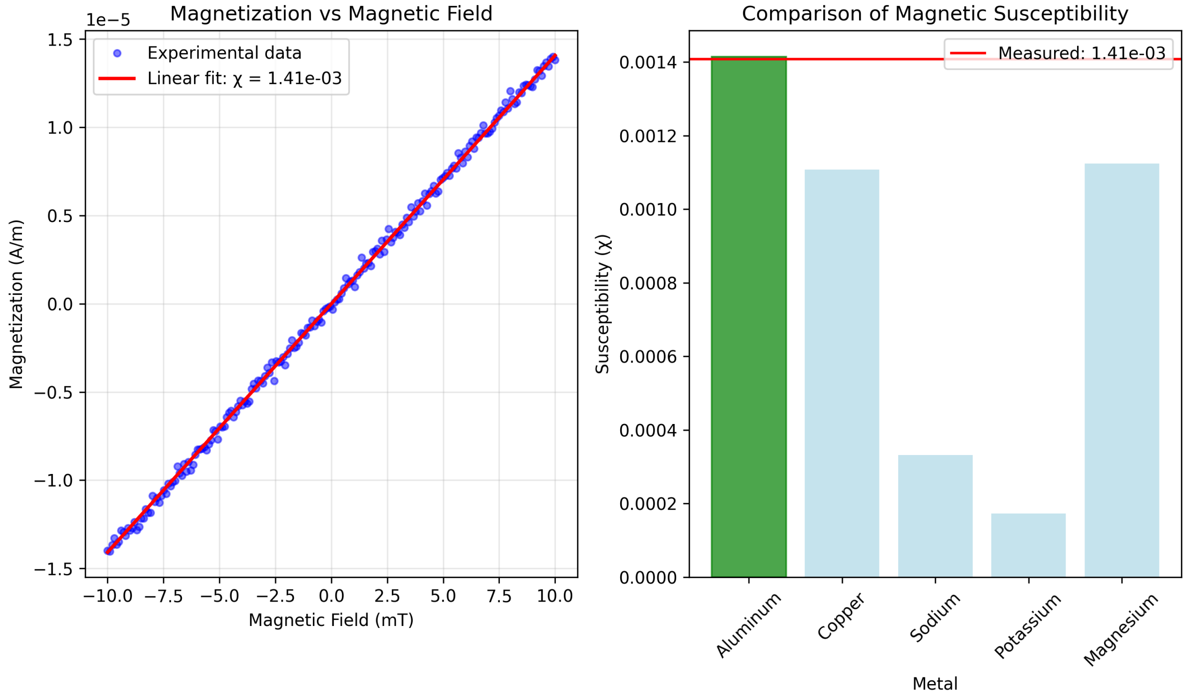


Figure 4: Magnetization trend with a varying magnetic field used to determine χ_m of the material.

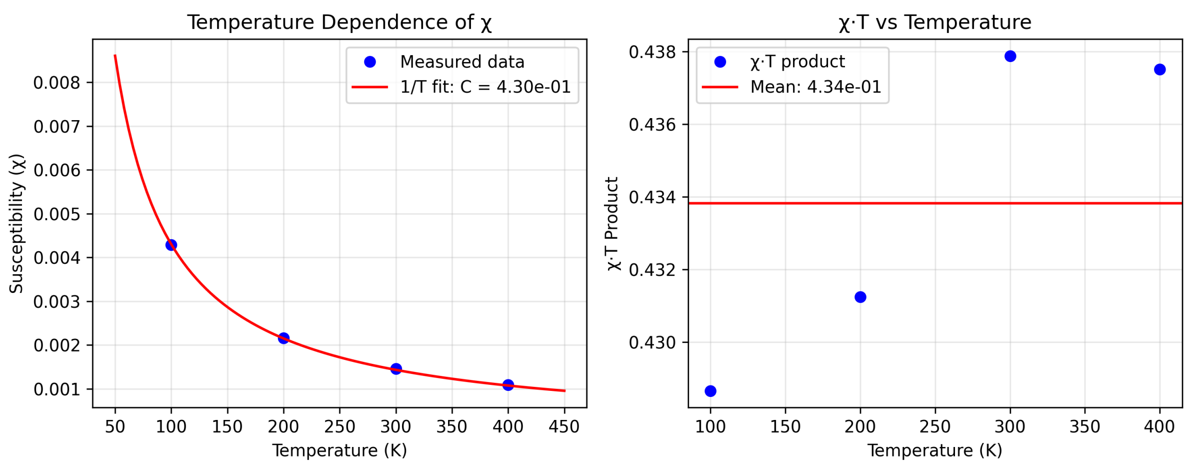


Figure 5: Caption

3.3 Boundaries

Another script was Generated using Claude AI to describe the boundary conditions along with display the surface charge loop. The model isn't perfect, but there isn't much time to polish it up. The internal magnetic field is calculated using the expression for

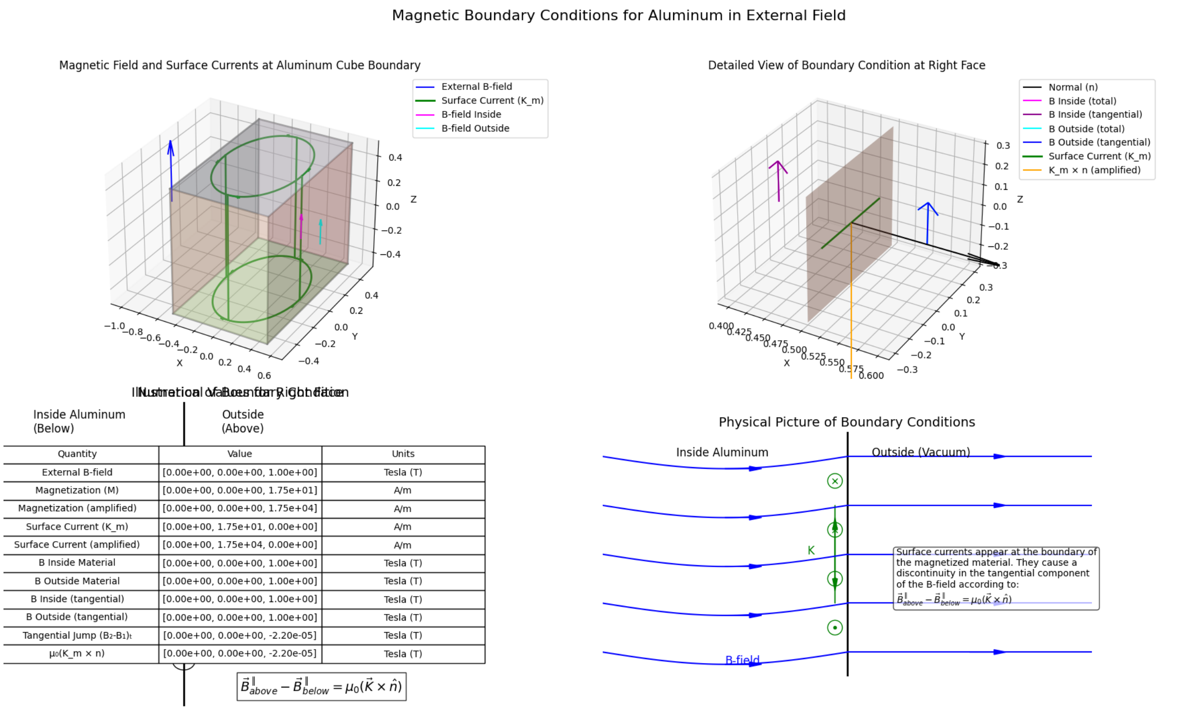


Figure 6: 3D model of Boundary Conditions Using Aluminum

4 Analysis of Magnetism and Low Temperatures

As seen in Figure 3, changes in magnetization and susceptibility are most prominent at lower temperatures. A deeper analysis within this temperature range provides a comprehensive insight between theoretical valuations using the Brillouin function (Eq. 14) and numerical valuations using the quantitative description of the changes in each microstate.

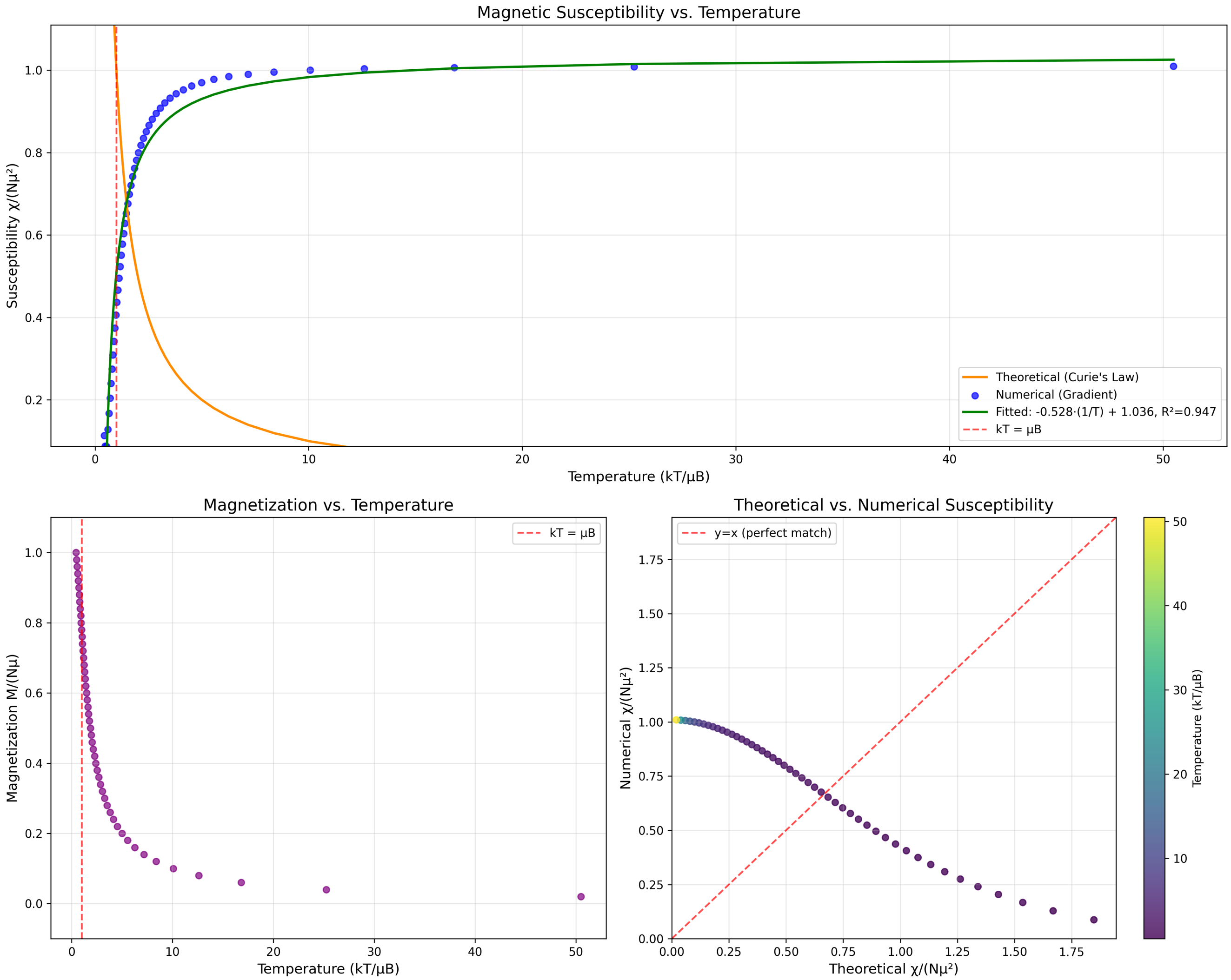


Figure 7: Caption

References

[1] Theoder Van Duzer and Charles W. Turner. *Principles of superconductive devices and circuits 2nd ed.* Prentice Hall PTR, Jan 1999.

[2] Daniel V. Schroeder. *Introduction to Thermal Physics.* Oxford University Press, 2021.