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Perturbation theory applied to a variational analogy between reactor theory and classical mechanics

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PERTURBATION THEORY APPLIED TO A VARIATIONAL ANALOGY
BETWEEN
REACTOR THEORY AND CLASSICAL MECHANICS

by

Richard Eyring Turley

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
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INTRODUCTION

It has been said by those much more knowledgeable than the author of this thesis that variational methods do not break ground for a new advance but do help in unifying and consolidating a subject (27). It was with this thought in mind that an investigation was begun on the theoretical subject of perturbation theory and its application to a variational analogy between reactor theory and classical mechanics.

Perturbation methods have been used extensively in the solution of physicomathematical problems. The general procedure in solving a problem by the method of perturbations is to associate with the problem in question a similar problem which has an exact solution. The solution to the problem which apparently does not have an exact solution is then treated as the sum of the simpler exact solution and a power series expansion of that solution. The desired solution is thus obtained in open form and truncated usually so that only the first one or two terms of the power expansion are used. The solution is then called a first-order or second-order approximation accordingly.

Ordinarily the variational method and the perturbation method are two distinct approaches to the same problem. In the solution of scalar field problems when perturbation methods become unwieldy and difficult to interpret, or solve, one then tries a variational method which in many cases provides a simpler and perhaps more meaningful result.

In the work which follows, the variational calculus is used in setting up the analogy between the Hamiltonian in classical mechanics and an analogous Hamiltonian density in neutron diffusion theory. Once this analogy is made, then the perturbation method is developed around the Hamiltonian density.

Two examples are selected to illustrate the use of this perturbation method in reactor theory. The first example treats the source of a neutron diffusion problem as a perturbation. The second example illustrates how the method can be used in determining not only the magnitude but the shape of the perturbed flux. This second illustration alone justifies the development of this method in reactor theory inasmuch as currently used perturbation methods yield only the shape of the perturbed flux and not the magnitude.

REVIEW OF PERTURBATION METHODS APPLIED TO REACTOR THEORY

In reviewing the literature relating perturbation theory to reactor calculations it is found that Wigner was the originator of its use (8). As early as May 1945, a progress report was issued by Nordheim (28) wherein he states that effects of small inhomogeneous disturbances in "piles" are generally computed by a weight formula in which reactivity changes produced by a disturbance are proportional to the square of the neutron density. Further in his report he recognizes Wigner as making suggestions on the use of adjoint operators. Approximately one month later Wigner released a report describing his work as an endeavor to go beyond Fermi's simple theory which applied only to a uniform one-group thermal bare reactor (35).

Wigner's statement causes one to wonder if perhaps Fermi had not already utilized perturbation theory in a more elementary form and that Wigner's contribution was to extend and formalize it. Both men were familiar with perturbation methods. Fermi in his lectures on the "Quantum Theory of Radiation" at the Symposium for Theoretical Physics at the University of Michigan in 1930 included perturbation theory as a part of his treatment of Dirac's theory of radiation (5). Also Wigner as early as 1935 had used the Rayleigh-Schrodinger theory (17).

Wigner in his report mentioned above sets up the neutron diffusion equation as an eigenvalue problem. He then solves it as he would have solved a problem in quantum mechanics by using a modified version of the Rayleigh-Schrodinger perturbation theory, i.e., with suitable modi-

fications to take into account operator differences. This method is now published in a number of references (8,23,33).

The Eigenvalue Problem

Glasstone and Edlund in their work on Reactor Theory (8) assume that the ith group of a multi-group treatment of a reactor may be written as

$$\sum_{j=1}^m M_{ij} \phi_j = \frac{\partial \phi_i}{\partial t} \quad (1)$$

where M_{ij} is an operator and ϕ_i is the neutron scalar flux in the ith group. If the neutron spectrum can be suitably partitioned into m groups, the resulting m equations in the form of Eq. 1 may be represented in matrix form as

$$M\phi = \frac{\partial \phi}{\partial t} \quad (2)$$

where M is a matrix operator which suitably combines the M_{ij} operators, and ϕ is a vector whose m elements are the ϕ_i .

They next consider solutions of the form

$$\phi_i = \phi_{i0} e^{\omega_i t} \quad (3)$$

which yield the set of equations

$$\sum_{j=1}^m M_{ij} \phi_j = \omega_i \phi_i \quad i=1,2,\dots,m \quad (4)$$

This form of solution is quite commonly used in the literature.

The assumption in arriving at this form is that higher harmonics in the

reactor kinetics equations have decayed out and the i th scalar flux is either statically or transiently stable with period ω_i^{-1} . Since there is dependence among the groups all the way from fission energy down to the lowest energy group, all the ω_i are equal. Thus the set of equations represented by Eq. 4 may be expressed as an eigenvalue problem in matrix form as

$$M\phi = \omega\phi \quad (5)$$

where ω is now a scalar eigenvalue and ϕ an eigenfunction column matrix.

Perturbation theory enters the picture when it is assumed that the reactor has been perturbed. Equation 5 changes to

$$(M + \lambda P)\phi' = \omega'\phi' \quad (6)$$

where λP represents the operator matrix which describes the change that has occurred, ϕ' the new value of the scalar flux matrix, ω' the new eigenvalue, and λ is some parameter which may arbitrarily vary between zero and one. If λ is zero, Eq. 6 reverts to Eq. 5 and the state of the system is unperturbed with reference to its initial state as described by Eq. 5. The parameter λ is customarily absorbed into the perturbation matrix by assigning to it a value of one.

In the event that the eigenfunctions of Eq. 5 do not form an orthogonal set it is necessary to define an adjoint operator M^\dagger which is the "tranjugate" of M , i.e., the transposed complex conjugate of M .

Associated with the adjoint operator M^+ is the adjoint equation

$$M^+ \psi = \omega^* \psi \quad (7)$$

where M^+ is the adjoint matrix operator, ψ the adjoint flux, and ω^* the complex conjugate of ω . The equations associated with this phase of reactor theory are all real. The adjoint operator is therefore the transpose of M and $\omega = \omega^*$ since the eigenvalues are all real.

The function ψ is called the adjoint flux by virtue of the fact that it satisfies the associated adjoint equation. The physical meaning associated with the adjoint flux has become a study in itself. Lewins (20) and Selengut (30) have both studied the adjoint flux for linear reactor problems. Becker (1) has done work of a more general nature in studying the adjoint function for a broader class of problems, e.g., heat conduction, thermal stress, and certain nonlinear problems. More will be mentioned later about the function ψ .

Along with the above remarks about the adjoint operator, it is required that the following equality hold:

$$\int_V \psi M \phi \, dV = \int_V \phi M^+ \psi \, dV \quad (8)$$

This requirement, or definition as some prefer to call it, imposes certain boundary conditions on the problem. For example, in the case of the bare reactor, it is found that

$$\psi M \phi - \phi M^+ \psi = \bar{\nabla} \cdot \psi D \bar{\nabla} \phi \quad (9)$$

where D is a diffusion coefficient, or a diagonal matrix of coefficients. For Eq. 8 to hold, Eq. 9 requires that

$$\int_V \vec{\nabla} \cdot \psi D \vec{\nabla} \phi dV = 0 \quad (10)$$

From Gauss's divergence theorem this implies that

$$\oint_S \psi D \vec{\nabla} \phi \cdot d\vec{\sigma} = 0 \quad (11)$$

i.e., the vector quantity $\psi D \vec{\nabla} \phi$ must either be zero on the extrapolated surface or else tangent to the surface. The zero condition is the only physically possible condition since by definition the extrapolated boundary corresponds to the zero isotimic surface of the flux ϕ . Thus the requirement that Eq. 8 hold imposes the boundary condition that the adjoint function ψ be zero on the boundary or else the gradient of ϕ must be zero there. The vector quantity $\psi D \vec{\nabla} \phi$ corresponds to Morse and Feshbach's bilinear concomitant in their generalization of Green's theorem (27).

If Eq. 6, with the parameter λ set at unity, is multiplied on the left by the adjoint flux ψ , and Eq. 7 is multiplied by the perturbed flux ϕ' on the left, and their difference taken, the following equation results:

$$\psi M \phi' - \phi' M^+ \psi + \psi P \phi' = (\omega' - \omega) \psi \phi' \quad (12)$$

After integrating Eq. 12 over the volume and noting that ϕ' satisfies the same extrapolated boundary condition as ϕ , the following pertur-

bation equation results:

$$(\omega' - \omega) = \frac{\int \psi \rho \phi' dV}{\int \psi \phi' dV} \quad (13)$$

It is then customary to assume that the perturbation is first order so that ϕ' may be approximated by ϕ , thus

$$(\omega' - \omega) = \frac{\int \psi \rho \phi dV}{\int \psi \phi dV} \quad (14)$$

And if $M = M^\dagger$, i.e., the operator is self-adjoint, then

$$(\omega' - \omega) = \frac{\int \phi \rho \phi dV}{\int \phi^2 dV} \quad (15)$$

This equation corresponds to Glasstone and Edlund's Eq. 13.10.2 (8), Weinberg and Wigner's Eq. 16.48 (33), and Isbin's Eq. 12-44 (12).

There are other forms of Eq. 15 depending upon what parameter one wishes to analyze, e.g., it is customary and desirable to express Eq. 15 in terms of a change in reactivity. For stable periods using a one-group model it can be assumed that

$$\omega = \frac{\Delta K_{eff}}{l} = K \Sigma_a v \frac{\Delta K_{eff}}{K_{eff}} \quad (16)$$

where l is the neutron lifetime, K the infinite multiplication constant, Σ_a the total absorption cross-section, v the one-group neutron velocity, and K_{eff} the effective multiplication constant.

If the case where $\omega = 0$ is considered, the following equation is obtained from Eq. 12 by substituting Eq. 16 as ω' .

$$\frac{\Delta K_{eff}}{K_{eff}} = \frac{\int \phi P \phi dV}{\int K \Sigma_a v \phi^2 dV} \quad (17)$$

This equation corresponds to Glasstone and Edlund's Eq. 13.15.3 (8), and Meghreglian and Holmes' Eq. 13.58 (23).

Similar equations may be obtained by using a variational approach to the eigenvalue problem as done by Meneghetti (24). The variational method is generally suggested by Margenau and Murphy (21), Morse and Feshbach (27), Korn and Korn (16), Hildebrand (11), and others.

In practice the criticality determination, and thus Eq. 17, is of vital use in reactor core design. Numerical machine codes such as the MACH1 code presently being prepared by Argonne (25) incorporate a perturbation code based on an equation similar to Eq. 17. The MACH1 code uses the DEL perturbation code (18) to determine the fractional change in the multiplication constant due to a perturbation in the macroscopic cross-section of a particular nuclide or of a material consisting of several nuclides, or the change in the reactor buckling.

Perturbation of the neutron scalar flux

In the present work the author is concerned with the perturbation of the scalar flux, i.e., what shape does ϕ' assume after the perturbation? This is a problem neglected by almost all the before-mentioned authors except Weinberg and Wigner (33). Had the others fully utilized

the Schrodinger perturbation theory they would have obtained an equation for the flux. Weinberg and Wigner expand the perturbed flux ϕ' in terms of the eigenfunctions associated with the unperturbed equation, Eq. 5. In their treatment they consider only perturbations of such a nature that the perturbed reactor is again critical, i.e., $\Delta\omega=0$.

Thus if $\omega=0$, then $\omega'=0$, and Eq. 6 therefore becomes

$$(M + P)\phi' = 0 \quad (18)$$

with the parameter λ set at unity. If Eq. 18 is multiplied on the left by ψ , and integrated, then

$$\int \psi, M\phi' dV + \int \psi, P\phi' dV = 0 \quad (19)$$

If the integral identity, Eq. 8, is utilized, Eq. 19 becomes

$$\int \phi' M^+ \psi, dV + \int \psi, P\phi' dV = 0 \quad (20)$$

But since $M^+ \psi = \omega, \psi$, and $\omega=0$,

$$\int \psi, P\phi' dV = 0 \quad (21)$$

becomes a condition necessary for the stable criticality of the reactor.

To the first order, Eq. 21 becomes

$$\int \psi, P\phi, dV = 0 \quad (22)$$

since the statically stable condition defines the flux ϕ as ϕ .

The difference between the perturbed flux and the steady state

flux is expressed in terms of the eigenfunctions of the unperturbed system, and yields the expression

$$\phi' - \phi_1 = \sum_n A_n \phi_n \quad (23)$$

If Eq. 23 is substituted into Eq. 18 the following equations result.

$$(M + P)(\phi_1 + \sum_n A_n \phi_n) = 0 \quad (24)$$

$$M\phi_1 + \sum_n A_n M\phi_n + P\phi_1 + \sum_n A_n P\phi_n = 0 \quad (25)$$

Since the terms in the last sum are of second order they are neglected, and since the first term is zero by virtue of the fact that $\omega = 0$, Eq. 25 becomes

$$\sum_n A_n M\phi_n + P\phi_1 = 0 \quad (26)$$

Since $M\phi_n = \omega_n \phi_n$ (27)

$$\sum_n A_n \omega_n \phi_n + P\phi_1 = 0 \quad (28)$$

Equation 26 is now multiplied by ψ_m and then integrated over the volume to determine the A_n , thus

$$\sum_n A_n \omega_n \int \psi_m \phi_n dV = - \int \psi_m P\phi_1 dV \quad (29)$$

or $A_m \omega_m = - \int \psi_m P\phi_1 dV$ (30)

if it is assumed that ψ_m and ϕ_n are chosen to be orthonormal over the reactor.

For the case where $m = 1$, since $\omega_1 = 0$, it is not possible to find A_1 by Eq. 29. The perturbed flux thus becomes

$$\phi' = (1 + A_1) \phi_1 - \sum_{n=2} \left(\frac{1}{\omega_n} \int \psi_n P \phi_1 dV \right) \phi_n \quad (31)$$

This equation corresponds to Weinberg and Wigner's Eq. 16.53 except that A_1 is included here. They indicate that A_1 remains indeterminate and that if one desires he may include it as being "of the order of magnitude of the increments" or else leave it out. They indicate that in a first order approximation, the only effect this term has on ϕ' is to change its normalization. Equation 31 thus becomes a "shape" equation with a magnitude not necessarily indicative of the real situation. Such an equation has some application but would prove more useful if A_1 were known.

Bellman (2) solves a problem similar to the problem of Weinberg and Wigner with the added generality that the perturbing function be expanded in an orthogonal series.

The object of this author's work is to approach the problem using another method which will yield a more exact expression for ϕ' .

VARIATIONAL PRINCIPLES APPLIED TO REACTOR THEORY

In order to arrive at the perturbation method used in this work it is necessary first to develop a certain functional which has its derivation within the realm of the calculus of variations. By way of an analogy, classical mechanics plays a major role in the discussion which follows.

It is quite well known that generalized variational principles have been developed with discrete particle systems as well as with the field theories of classical and quantum mechanics. Morse and Feshbach (27) have collected and summarized the variational equations for the following scalar and vector field problems: the flexible string or membrane, compressible nonviscous fluid flow, the diffusion equation, the Schrodinger wave equation, the Klein-Gordon equation, the elastic wave equation, the electromagnetic equations, and the Dirac equation of Quantum Mechanics. Wentzel (34) has also developed a generalized canonical formalism associated with field equations as an introduction to his work on quantum theory of fields. Recently Kaplan (14) has summarized the variational field equations of neutron diffusion theory and demonstrated, up to a point, the analogy with classical mechanics.

The general approach to developing the variational principle is to begin with the differential equations which form a model of the physical system.

One-group bare reactor

For the case of a one-group bare reactor, the equation which de-

scribes the steady state scalar flux ϕ is

$$M\phi = 0 \quad (32)$$

where from Eq. 2 the partial of the flux with respect to time has been set equal to zero. The operator in this case is

$$M = \bar{\nabla} D \bar{\nabla} + B_0^2 D \quad (33)$$

and Eq. 32 can be written in differential form as

$$\bar{\nabla} \cdot D \bar{\nabla} \phi + B_0^2 D \phi = 0 \quad (34)$$

Corresponding to Eq. 7 the equation for the adjoint function ψ may be similarly obtained, i.e.,

$$\bar{\nabla} \cdot D \bar{\nabla} \psi + B_0^2 D \psi = 0 \quad (35)$$

The next step is to multiply Eq. 34 by a variation in the adjoint flux $\delta\psi$ and Eq. 35 by a variation in the flux $\delta\phi$. The resulting equations are then added and subsequently integrated over the volume. The following integral results.

$$\int [(\bar{\nabla} \cdot D \bar{\nabla} \phi) \delta\psi + (\bar{\nabla} \cdot D \bar{\nabla} \psi) \delta\phi + B_0^2 \delta(\phi\psi)] dV = 0 \quad (36)$$

There are certain operations which are permissible with the δ variational operator. These are discussed by Hildebrand (11). Equation 36 when integrated by parts gives a Lagrangian functional and certain conditions on the boundary. The resulting integral is

as follows:

$$\int \bar{\nabla} \cdot [(\mathcal{D} \bar{\nabla} \phi) \delta \psi + (\mathcal{D} \bar{\nabla} \psi) \delta \phi] dV - \delta \int [\mathcal{D} \bar{\nabla} \phi \cdot \bar{\nabla} \psi - \mathcal{B}_0^2 \mathcal{D} \phi \psi] dV = 0 \quad (37)$$

The first integral of Eq. 37 may be transformed to a surface integral by using Gauss's divergence theorem. Therefore,

$$\oint [(\mathcal{D} \bar{\nabla} \phi) \delta \psi + (\mathcal{D} \bar{\nabla} \psi) \delta \phi] \cdot d\vec{\sigma} - \delta \int [\mathcal{D} \bar{\nabla} \phi \cdot \bar{\nabla} \psi - \mathcal{B}_0^2 \mathcal{D} \phi \psi] dV = 0 \quad (38)$$

Since the flux and adjoint flux are zero at the extrapolated boundary and therefore fixed from a variational standpoint, the variations $\delta \phi$ and $\delta \psi$ go to zero there, and the desired functional equation is

$$\delta \int [\mathcal{D} \bar{\nabla} \phi \cdot \bar{\nabla} \psi - \mathcal{B}_0^2 \mathcal{D} \phi \psi] dV = 0 \quad (39)$$

The bracketed term within the integral of Eq. 39 is referred to as the Lagrange density. The integral itself is called the Lagrangian integral, and Eq. 39 implies that this integral is stationary, i.e., any small variations of ϕ or ψ will not greatly change its value. This stationary property is the property which allows the use of approximation methods, e.g., the methods of Rayleigh-Ritz, and Kantorovich (3). Wachspress and Becker (32) have recently utilized this stationary property in their work with discontinuous trial functions. Selengut (30) has worked on a method to simplify the preceding step-ladder approach used in obtaining the functional, Eq. 39. This appears to be especially important when many independent variables occur in the problem. Recently Pomraning (29) has shown that it is possible to derive several variational principles of a general nature, and that many of the familiar

principles, in particular those due to Roussopoulos and Schwinger, correspond to special cases of these.

Hamilton's principle

If Eq. 39 is written in the following form

$$\delta \int \mathcal{L}[\phi, \psi, \bar{\nabla}\phi, \bar{\nabla}\psi] dV = 0 \quad (40)$$

where \mathcal{L} represents the Lagrangian density, immediately one recognizes this as being analogous to Hamilton's principle from classical dynamics (9,19). The integrand which has been called the Lagrangian density is analogous to the Lagrangian and in an analogous manner Lagrange's equations yield the equations of the system. Kaplan (14) in his work writes Lagrange's equations as follows:

$$\bar{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \bar{\nabla}\psi} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad (41a)$$

$$\bar{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial \bar{\nabla}\phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (41b)$$

He then substitutes the Lagrangian density \mathcal{L} into Eqs. 41 and obtains the following equations:

$$\bar{\nabla} \cdot D \bar{\nabla}\phi + B_0^2 D\phi = 0 \quad (42a)$$

$$\bar{\nabla} \cdot D \bar{\nabla}\psi + B_0^2 D\psi = 0 \quad (42b)$$

It is not clear to the author of this work how Kaplan justifies differentiating a scalar with respect to a vector and obtaining a vector. It seems preferable to write the Lagrangian density explicitly

and then add each component to obtain the vector form. For example, the Lagrangian can be written as

$$L = D \left[\frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial z} \frac{\partial \psi}{\partial z} \right] - B_0^2 \phi \psi \quad (43)$$

Since the problem is linear and isotropy is assumed, the flux ϕ and the adjoint flux ψ may be split into equal scalar components as follows:

$$\phi = \phi_x = \phi_y = \phi_z \quad (44a)$$

$$\psi = \psi_x = \psi_y = \psi_z \quad (44b)$$

therefore

$$\phi = \frac{\phi_x + \phi_y + \phi_z}{3} \quad (45a)$$

$$\psi = \frac{\psi_x + \psi_y + \psi_z}{3} \quad (45b)$$

Lagrange's equations thus become

$$\frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial \left(\frac{\partial \phi}{\partial x_i} \right)} \right) - \frac{\partial L}{\partial \phi_{x_i}} = 0 \quad (46a)$$

$$\frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial \left(\frac{\partial \psi}{\partial x_i} \right)} \right) - \frac{\partial L}{\partial \psi_{x_i}} = 0 \quad (46b)$$

The Lagrangian may be substituted into Eqs. 46b to give the component equations of the system.

The component equations are as follows:

$$\frac{\partial}{\partial x} D \frac{\partial \phi}{\partial x} + \frac{1}{3} B_o^2 D \phi = 0 \quad (47a)$$

$$\frac{\partial}{\partial y} D \frac{\partial \phi}{\partial y} + \frac{1}{3} B_o^2 D \phi = 0 \quad (47b)$$

$$\frac{\partial}{\partial z} D \frac{\partial \phi}{\partial z} + \frac{1}{3} B_o^2 D \phi = 0 \quad (47c)$$

The sum of Eqs. 47 becomes

$$\frac{\partial}{\partial x} D \frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y} D \frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z} D \frac{\partial \phi}{\partial z} + B_o^2 D \phi = 0 \quad (48)$$

or in vector form

$$\bar{\nabla} \cdot D \bar{\nabla} \phi + B_o^2 D \phi = 0 \quad (49)$$

If the Lagrangian is substituted into Eq. 46a, a similar equation may be obtained for the adjoint flux ψ . These results may be compared with Eqs. 34 and 35.

In addition, it is possible by means of a Legendre transformation to change the Lagrangian density to what is called a Hamiltonian density. This is done as follows:

$$H[\phi, \psi, D\bar{\nabla}\phi, D\bar{\nabla}\psi] = 2 D\bar{\nabla}\phi \cdot \bar{\nabla}\psi - \mathcal{L}[\phi, \psi, \bar{\nabla}\phi, \bar{\nabla}\psi] \quad (50)$$

If the bracketed term of Eq. 39 is substituted into Eq. 50 for the Lagrangian density \mathcal{L} , the result becomes

$$H[\phi, \psi, D\bar{\nabla}\phi, D\bar{\nabla}\psi] = \frac{D\bar{\nabla}\phi \cdot D\bar{\nabla}\psi}{D} + B_o^2 D\phi\psi \quad (51)$$

where $H[\phi, \psi, D\bar{\nabla}\phi, D\bar{\nabla}\psi]$ represents the Hamiltonian density and should in

an analogous manner yield Hamilton's so-called canonical equations.

It is customary and convenient to let

$$D\bar{\nabla}\phi = \vec{\mathcal{P}}_\phi \quad (52)$$

and
$$D\bar{\nabla}\psi = \vec{\mathcal{P}}_\psi \quad (53)$$

This again is analogous to regular Hamiltonian notation where \mathcal{P} would correspond to the momentum. Equation 51 may then be written as

$$H[\phi, \psi, \vec{\mathcal{P}}_\phi, \vec{\mathcal{P}}_\psi] = \frac{\vec{\mathcal{P}}_\phi \cdot \vec{\mathcal{P}}_\psi}{D} + B_0^2 D\phi\psi \quad (54)$$

Kaplan (14) derives the canonical equations from the Hamiltonian as follows:

$$\frac{\partial H}{\partial \vec{\mathcal{P}}_\psi} = \bar{\nabla}\phi = \frac{\vec{\mathcal{P}}_\phi}{D} \quad (55)$$

$$\frac{\partial H}{\partial \vec{\mathcal{P}}_\phi} = \bar{\nabla}\psi = \frac{\vec{\mathcal{P}}_\psi}{D} \quad (56)$$

$$\frac{\partial H}{\partial \psi} = -\bar{\nabla} \cdot \vec{\mathcal{P}}_\phi = B_0^2 D\phi \quad (57)$$

$$\frac{\partial H}{\partial \phi} = -\bar{\nabla} \cdot \vec{\mathcal{P}}_\psi = B_0^2 D\psi \quad (58)$$

Here again he is not explicit as to how he performs these operations.

This author proposes the following operator which can be used in deriving Hamilton's canonical equations:

$$\text{Let } \vec{\nabla}_{\mathcal{P}} = \frac{\partial}{\partial \mathcal{P}_x} \hat{x} + \frac{\partial}{\partial \mathcal{P}_y} \hat{y} + \frac{\partial}{\partial \mathcal{P}_z} \hat{z} \quad (59)$$

with properties, some of which are as follows:

$$\vec{\nabla}_p \cdot \vec{p} = 3 \quad (60)$$

$$\vec{\nabla}_p \times \vec{p} = 0 \quad (61)$$

$$\vec{\nabla}_p (\vec{A} \cdot \vec{p}) = \vec{A} \quad (62)$$

$$\vec{\nabla}_p p = \hat{e}_p \quad (63)$$

where \hat{e}_p is a unit vector in the \vec{p} direction, and \vec{A} is a constant vector. One may note the similarity with the gradient vector operator

$\vec{\nabla}$, thus the $\vec{\nabla}_p$ operator will be called the p-gradient operator.

If this p-gradient operator operates on the Hamiltonian the following is the result.

$$\vec{\nabla}_p H = \frac{\partial H}{\partial p_x} \hat{i} + \frac{\partial H}{\partial p_y} \hat{j} + \frac{\partial H}{\partial p_z} \hat{k} \quad (64)$$

$$= \frac{\partial \psi}{\partial x} \hat{i} + \frac{\partial \psi}{\partial y} \hat{j} + \frac{\partial \psi}{\partial z} \hat{k} \quad (65)$$

$$= \vec{\nabla} \psi = \frac{\vec{p}_\psi}{D} \quad (66)$$

which is the desired result. In a similar manner,

$$\vec{\nabla}_{p_\psi} H = \vec{\nabla} \phi = \frac{\vec{p}_\phi}{D} \quad (67)$$

The remaining canonical equations are:

$$\frac{\partial H}{\partial \phi} = B_0^2 D \psi = - \vec{\nabla} \cdot \vec{p}_\psi \quad (68)$$

and

$$\frac{\partial H}{\partial \psi} = B_0^2 D \phi = -\vec{\nabla} \cdot \vec{\rho} \phi \quad (69)$$

One-group case In a one-group case, where ψ is proportional to ϕ , the ψ is replaced by $\frac{\phi}{2}$ in both the Lagrangian and the Hamiltonian. The Hamiltonian thus becomes

$$H[\phi, \vec{\rho} \phi] = \frac{1}{2} \frac{\vec{\rho} \phi^2}{D} + \frac{1}{2} B_0^2 D \phi^2 \quad (70)$$

and only one set of canonical equations result. These equations are as follows:

$$\vec{\nabla}_{\rho} H = \frac{\vec{\rho} \phi}{D} = \vec{\nabla} \phi \quad (71)$$

$$\frac{\partial H}{\partial \phi} = B_0^2 D \phi = -\vec{\nabla} \cdot D \vec{\nabla} \phi \quad (72)$$

In a dynamics problem the phase space (q,p,t) is used, i.e., displacement, momentum, and time. In the neutron diffusion problem an analogous space becomes the $(\phi, \vec{\rho}, r)$ space as already indicated, i.e., flux, neutron current, and position.

The Hamiltonian

For a better understanding of the Hamiltonian it is necessary to see how it changes with position. This is accomplished as follows:

$$\begin{aligned} \vec{\nabla} H[\phi, \psi, \vec{\rho} \phi, \vec{\rho} \psi] &= \frac{\partial H}{\partial \phi} \vec{\nabla} \phi + \frac{\partial H}{\partial \psi} \vec{\nabla} \psi \\ &+ (\vec{\nabla}_{\rho} H) \vec{\nabla} \cdot \vec{\rho} \phi + (\vec{\nabla}_{\rho} H) \vec{\nabla} \cdot \vec{\rho} \psi \quad (73) \end{aligned}$$

Use is now made of Hamilton's canonical equations. Equations 66 to 69 are substituted into Eq. 73 above and the result is as follows:

$$\begin{aligned} \vec{\nabla} H = & (-\vec{\nabla} \cdot \vec{p}_\psi) \vec{\nabla} \phi + (-\vec{\nabla} \cdot \vec{p}_\phi) \vec{\nabla} \psi \\ & + (\vec{\nabla} \psi) \vec{\nabla} \cdot \vec{p}_\phi + (\vec{\nabla} \phi) \vec{\nabla} \cdot \vec{p}_\psi \end{aligned} \quad (74)$$

$$\vec{\nabla} H = 0 \quad (75)$$

It is seen above that unless the Hamiltonian is an explicit function of position its gradient is zero, that is, it is a constant over the reactor. This is an interesting and important property of the Hamiltonian. This property corresponds to the Hamiltonian being a constant of the motion in a dynamic conservative system, i.e., when $H(q,p,t)$ is not an explicit function of the time. Kaplan did not show this, however, his needs were different than the author's. In the perturbation method which follows, the Hamiltonian density plays a central role in the theory.

The Hamiltonian belonging to classical dynamics represents the total energy of the system and when not an explicit function of time is a constant of the motion. To gain physical insight into the significance of the Hamiltonian density it would be more meaningful perhaps if physical significance were first given to Eq. 39. In dynamics this equation would correspond to the integrated difference between the kinetic and the potential energy. In Eq. 39 the integrated difference between a term proportional to the gradient of the flux squared and a term proportional to the flux squared must be made

stationary. This Lagrangian integral may be interpreted as a net reactivity "importance" integral wherein the integrated difference between the weighted neutron losses and gains is kept constant if the reactor is to be maintained critical.

Equation 39 can be rewritten as

$$\delta \int [(K \epsilon_a \phi) \psi - (\epsilon_a \phi) \psi - (D \nabla \phi) \cdot (\nabla \psi)] dV = 0 \quad (76)$$

where the ψ may be interpreted as an "importance" function. The term in brackets is a density function and therefore pertains to a volume element at the point (x,y,z) in the reactor. Each term in parentheses is a reaction rate per unit volume, or a flow rate per unit area. The adjoint function ψ may be interpreted as the importance of a neutron to the future progeny (i.e., neutrons) of the system. For the one-group model, ψ is proportional to ϕ . At the boundary the neutrons will have zero importance, and where ϕ reaches a maximum they will have maximum importance.

The production and loss densities when multiplied by the importance function ψ are therefore weighted according to how important the gain or loss of neutrons is at the point (x,y,z) . As Weinberg and Wigner (33) point out, the neutron current term when multiplied by the gradient of ψ is a measure of whether those neutrons leaving or entering the volume element are moving into an area of greater or lesser importance. From one point of view, as discussed in Appendix A, the first and second terms of Eq. 76 may be called the Potential importance rate density, and the term involving current may be called the Kinetic importance rate

density. From this point of view, the integrated difference between the kinetic and potential importance rate densities must remain constant for the integral to be stationary, and therefore for the original differential equations, Eqs. 34 and 35, to describe the behavior of the system.

The Hamiltonian density, Eq. 51, may be written as

$$H[\phi, \psi, D\bar{\phi}, D\bar{\psi}] = \frac{D\bar{\phi} \cdot D\bar{\psi}}{D} + K\varepsilon_a \phi \psi - \varepsilon_a \phi \psi \quad (77)$$

The Hamiltonian density when interpreted according to the foregoing discussion becomes the sum of the kinetic and potential importance rate densities at position (x,y,z). If the Hamiltonian density does not depend explicitly upon position, then it is a constant at each point in the reactor, and therefore the sum of the kinetic and potential importance rate densities must be constant over the reactor in this case.

As an example of the above case, consider a one-group infinite slab reactor described by the following Hamiltonian density.

$$H[\phi, \varphi] = \frac{1}{2} \frac{\varphi^2}{D} + \frac{1}{2} B_0^2 D \phi^2 \quad (78)$$

The solution to the reactor system described by Eq. 78 is obtained by using the canonical equations. The results are as follows:

$$\phi = \phi_0 \sin B_0 x \quad (79)$$

$$\varphi = D \phi_0 B_0 \cos B_0 x \quad (80)$$

The resulting Hamiltonian density is as follows:

$$H = \frac{D\phi_0^2 B_0^2}{2} (\cos^2 B_0 x + \sin^2 B_0 x) \quad (81)$$

$$H = \frac{D\phi_0^2 B_0^2}{2} \quad (82)$$

Equation 85 shows that H is a constant of the system in question when not an explicit function of position.

PERTURBATION THEORY USING THE HAMILTONIAN DENSITY

The Hamiltonian principles developed in the previous section make it possible to develop another approach to the solution of a perturbed reactor. The following method is an operator method wherein Hamilton's canonical equations play a major part. In order to simplify this initial use of the method in reactor theory, the author will treat only the one dimensional reactor.

This operator method is similar to that used by Heisenberg (10) in his development of the physical principles of quantum mechanics. It is also similar to the hybrid Schrodinger-Heisenberg representation described by McConnell (22) in his work on quantum particle dynamics and by Kramers (15) in his treatment of time dependent perturbations. The particular method as used by this author was applied to classical mechanics by Garrido (6) and later formalized by Garrido and Gascon (7). The operator symbolism used by Garrido and Gascon will be used in the work which follows.

Liouville's operator

The perturbation method which follows is basically an operator method. Fundamental to its development is Liouville's differential operator which appears in Liouville's theorem in Statistical Mechanics (4). The operator is as follows:

$$\Omega = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial \phi_i} - \frac{\partial H}{\partial \phi_i} \frac{\partial}{\partial p_i} \right) \quad (83)$$

where H is the Hamiltonian density developed in the previous section, and ϕ_x and φ_x are the canonically conjugate variables associated with the Hamiltonian density.

Suppose that there exists a functional $F[\phi(x), \varphi(x)]$ which does not depend explicitly upon position. If the Hamiltonian density is substituted into Liouville's differential operator, and Hamilton's canonical equations utilized, the total derivative of F with respect to x takes the operational form

$$\frac{dF}{dx} = [\Omega, F] \quad (84)$$

where the brackets in this case denote the commutator of Ω and F , namely

$$[\Omega, F] = \Omega F - F \Omega \quad (85a)$$

$$[\Omega, F] = \frac{\partial H}{\partial \varphi} \frac{\partial F}{\partial \phi} - \frac{\partial H}{\partial \phi} \frac{\partial F}{\partial \varphi} - F \frac{\partial H}{\partial \varphi} \frac{\partial}{\partial \phi} + F \frac{\partial H}{\partial \phi} \frac{\partial}{\partial \varphi} \quad (85b)$$

When the commutator operates on the unit scalar quantity one, the last two terms become zero. When the commutator operates on the scalar one it becomes equivalent to the Poisson bracket (26), for H and F , that is,

$$(H, F) = \frac{\partial H}{\partial \varphi} \frac{\partial F}{\partial \phi} - \frac{\partial H}{\partial \phi} \frac{\partial F}{\partial \varphi} \quad (86a)$$

$$= \frac{d\phi}{dx} \frac{\partial F}{\partial \phi} + \frac{d\varphi}{dx} \frac{\partial F}{\partial \varphi} \quad (86b)$$

$$= \frac{dF}{dx} \quad (86c)$$

Similarly it can be shown that

$$\frac{d^n F}{dx^n} \circ 1 = \left[\Omega, \dots \left[\Omega, \left[\Omega, F \right]_1 \right]_2 \dots \right]_n \circ 1 \quad (87)$$

where again the brackets denote the commutator operation.

The operator $S(x)$

Let $S(x)$ be an operator which has the following operational properties.

$$F[\phi(x), p(x)] = S(x) F[\phi_0, p_0] S^{-1}(x) \quad (88)$$

$$\Omega[\phi(x), p(x), x] = S(x) \Omega[\phi_0, p_0, x] S^{-1}(x) \quad (89)$$

where the brackets denote functional dependence upon the canonically conjugate variables $\phi(x)$ and $p(x)$. In this instance ϕ_0 and p_0 are the values of $\phi(x)$ and $p(x)$ at the position $x = 0$. The operator $S(x)$ thus evolves or expands the functional about the point $x = 0$ much like a Taylor's expansion operator. In fact it will be shown that the two are equivalent under certain conditions.

The operator $S^{-1}(x)$ is assumed to exist and represents the inverse operator so that

$$S(x) S^{-1}(x) = 1 \quad (90)$$

The derivative of Eq. 90 is as follows:

$$\frac{d}{dx} (S S^{-1}) = S \frac{dS^{-1}}{dx} + \frac{dS}{dx} S^{-1} = 0 \quad (91)$$

If Eq. 91 is multiplied on the left by the inverse operator S^{-1} the

following equation results.

$$\frac{d\bar{S}^{-1}}{dx} = -\bar{S}^{-1} \frac{dS}{dx} \bar{S}^{-1} \quad (92)$$

In order to simplify the notation which follows, let the functionals be abbreviated so that $F[x] = F[\phi(x), p(x)]$ and $F[0] = F[\phi_0, p_0]$. Equation 88 may then be written as

$$F[x] = S F[0] S^{-1} \quad (93)$$

The derivative of $F[x]$ will be

$$\frac{dF[x]}{dx} = \frac{dS}{dx} F[0] S^{-1} + S F[0] \frac{d\bar{S}^{-1}}{dx} \quad (94a)$$

$$= \frac{dS}{dx} \bar{S}^{-1} S F[0] S^{-1} + S F[0] \left[-\bar{S}^{-1} \frac{dS}{dx} \bar{S}^{-1} \right] \quad (94b)$$

$$= \frac{dS}{dx} \bar{S}^{-1} F[0] - F[x] \frac{dS}{dx} S^{-1} \quad (94c)$$

$$= \left[\frac{dS}{dx} \bar{S}^{-1}, F[x] \right] \quad (94d)$$

where in Eq. 94d the brackets indicate the commutator. When this latter equation is compared with Eq. 84, it is seen that

$$\frac{dS}{dx} \bar{S}^{-1} = \Omega[x] \quad (95)$$

Since Eq. 95 and Eq. 89 are equal,

$$\frac{dS}{dx} \bar{S}^{-1} = S \Omega[\phi_0, p_0, x] S^{-1} \quad (96)$$

and when Eq. 96 is multiplied on the right by the operator $S(x)$,

$$\frac{dS}{dx} = S \Omega[\phi_0, p_0, x] \quad (97)$$

In the special case where Ω does not depend upon position, this equation may be integrated to give

$$S(x) = e^{x\Omega} \quad (98)$$

since $S(0) = 1$ by definition, i.e.,

$$S(0)F[0]S^{-1}(0) = F[0] \quad (99)$$

To illustrate the operational significance of Eq. 98, let the exponential be expanded about zero and then operate on $F[0]$. This is shown as follows:

$$e^{x\Omega} = 1 + x\Omega + \frac{(x\Omega)^2}{2!} + \dots \quad (100a)$$

$$e^{-x\Omega} = 1 - x\Omega + \frac{(x\Omega)^2}{2!} - \dots \quad (100b)$$

Thus from Eq. 93,

$$F[x] = e^{x\Omega} F[0] e^{-x\Omega} \quad (101a)$$

$$F[x] = \left(1 + x\Omega + \frac{(x\Omega)^2}{2!} + \dots\right) F[0] \left(1 - x\Omega + \frac{(x\Omega)^2}{2!} - \dots\right) \quad (101b)$$

and when multiplied by the scalar one on the right

$$F[x] = \left(1 + x\Omega + \frac{(x\Omega)^2}{2!} + \dots\right) F[0] \quad (102a)$$

$$F[x] = F[0] + x\Omega F[0] + \frac{(x\Omega)^2}{2} F[0] + \dots \quad (102b)$$

which is recognized as the Taylor's series about the point $x = 0$, or the Maclaurin's series.

Derivation of the $S_1(x)$ operator

In the development which follows let it be assumed that the perturbed reactor remains statically stable, i.e., the reactivity is kept zero by whatever adjustments are necessary. Delayed neutrons are neglected, although this simplification should cause no particular effect on the problem to be considered. The important point is that the reactor system be time independent.

In the last section, the Hamiltonian density was derived. In the perturbed reactor it will be assumed that the perturbed Hamiltonian density can be divided into two parts. Let H_0 represent the unperturbed part and H_1 the part which is due to the perturbation, therefore,

$$H = H_0 + H_1 \quad (103)$$

If it is assumed that the equations of the system in the unperturbed state are solved, then the solution to the unperturbed system may be written in the form

$$F[\phi_0(x), \rho_0(x)] = S_0(x) F[\phi_0, \rho_0] S_0^{-1}(x) \quad (104)$$

Also, Eq. 97 for the unperturbed system may be written as

$$\frac{dS_0}{dx} = S_0 \Omega_0[\phi_0, \rho_0, x] \quad (105)$$

where Ω_0 is Liouville's operator corresponding to the unperturbed Hamiltonian density H_0 .

Let $S(x)$ be the operator whose associated Hamiltonian density is

H . Then

$$F[x] = S(x) F[\phi_0, \rho_0] S^{-1}(x) \quad (106)$$

will be the expression for $F[x]$ when the system is perturbed. If Eq. 104 is multiplied on the left by $S_0^{-1}(x)$ and on the right by $S_0(x)$, an expression for $F[\phi_0, \rho_0]$ is obtained, i.e.,

$$F[\phi_0, \rho_0] = S_0^{-1}(x) F[\phi_0(x), \rho_0(x)] S_0(x) \quad (107)$$

Equation 107 amounts to an inverse operation. This expression when substituted into Eq. 106 yields an expression for $F[x]$ based on knowledge of the unperturbed system at some position x . This is an important point. The perturbed system thus becomes

$$F[x] = S(x) S_0^{-1}(x) F[\phi_0(x), \rho_0(x)] S_0(x) S^{-1}(x) \quad (108)$$

$$\text{and if} \quad S_1(x) = S(x) S_0^{-1}(x) \quad (109)$$

$$F[x] = S_1(x) F[\phi_0(x), \rho_0(x)] S_1^{-1}(x) \quad (110)$$

The operator $S(x)$ in Eq. 106 expresses the perturbed functional in terms of the initial value $F[\phi_0, \rho_0] = F[\phi_0(0), \rho_0(0)]$. The operator $S_1(x)$ in Eq. 110 expresses the perturbed functional in terms of the unperturbed system $F[\phi_0(x), \rho_0(x)]$ at the location x , not at the initial point as before.

From the foregoing discussion it becomes apparent that the operator $S_1(x)$ is the principal operator. Once $S_1(x)$ is known and the unperturbed equations for the system are known, the problem is solved by

multiplying Eq. 110 on the right by the scalar one, i.e.,

$$F[x] \circ 1 = S_1(x) F[\phi_0(x), \varphi_0(x)] \quad (111)$$

The operator $S_1(x)$ is determined in the following manner. From Eq. 109 it is seen that

$$\frac{dS_1}{dx} = \frac{dS}{dx} S_0^{-1} + S \frac{dS_0^{-1}}{dx} \quad (112a)$$

$$= \frac{dS}{dx} S_0^{-1} + S (-S_0^{-1} \frac{dS_0}{dx} S_0^{-1}) \quad (112b)$$

After substituting Eqs. 97 and 105,

$$\frac{dS_1}{dx} = S (\Omega - \Omega_0) S_0^{-1} \quad (113)$$

The difference between the Ω operators may be determined as follows:

$$\Omega - \Omega_0 = \left(\frac{\partial H}{\partial \varphi} \frac{\partial}{\partial \phi} - \frac{\partial H}{\partial \phi} \frac{\partial}{\partial \varphi} \right) - \left(\frac{\partial H_0}{\partial \varphi} \frac{\partial}{\partial \phi} - \frac{\partial H_0}{\partial \phi} \frac{\partial}{\partial \varphi} \right) \quad (114)$$

$$= \frac{\partial H_1}{\partial \varphi} \frac{\partial}{\partial \phi} - \frac{\partial H_1}{\partial \phi} \frac{\partial}{\partial \varphi} \quad (115)$$

$$\Omega - \Omega_0 = \Omega_1 \quad (116)$$

where $H = H_0 + H_1$, and $\Omega_1[\phi_0(x), \varphi_0(x), x]$ represents the operator associated with the perturbed part of the Hamiltonian density. Therefore Eq. 113 becomes,

$$\frac{dS_1}{dx} = S \Omega_1 S_0^{-1} \quad (117)$$

and utilizing the identity of Eq. 109,

$$\frac{dS_1}{dx} = S_1 S_0 \Omega_1[\phi_0, \varphi_0, x] S_0^{-1} \quad (118)$$

and therefore

$$\frac{dS_1}{dx} = S_1 \Omega, [\phi_0(x), \varphi_0(x), x] \quad (119)$$

Equation 119 is now integrated from the extrapolated boundary at $x = 0$ where the perturbed and unperturbed systems share a common boundary condition. The operator $S_1(x)$ therefore becomes

$$S_1(x) = 1 + \int_0^x S_1(x_1) \Omega, [\phi_0(x_1), \varphi_0(x_1), x_1] dx_1, \quad (120)$$

By the method of successive approximations an expression for the nth approximation may be written as

$$\begin{aligned} S_1^{(n)}(x) = & 1 + \int_0^x \Omega, [\phi_0(x_1), \varphi_0(x_1), x_1] dx_1 \\ & + \int_0^x \int_0^{x_1} \Omega, [\dots x_2] \Omega, [\dots x_1] dx_2 dx_1 \\ & + \dots \dots \dots + \\ & + \int_0^x \int_0^{x_1} \dots \int_0^{x_n} \Omega, [\dots x_n] \dots \Omega, [\dots x_1] dx_n \dots dx_1, \end{aligned} \quad (121)$$

Equation 121 when used in Eq. 111 as an approximation for the exact operator $S_1(x)$ yields a truncated solution for $F[x]$, the solution to the perturbed system, i.e.,

$$F[x] \simeq F^{(n)}[x] = S_1^{(n)}(x) F[\phi_0(x), \varphi_0(x)] \quad (122)$$

As noted in Eq. 121, it is necessary to express the conjugate variables $\phi_0(x)$ and $\varphi_0(x)$ as functions of initial values at position x_v as follows:

$$\phi_0(x) = \phi_0[\phi_0(x_v), \varphi_0(x_v), x - x_v] \quad (123a)$$

$$\varphi_0(x) = \varphi_0[\phi_0(x_v), \varphi_0(x_v), x - x_v] \quad (123b)$$

EXAMPLES IN REACTOR THEORY

Two examples will be presented to illustrate the use of the perturbation method applied to reactor theory. The first example illustrates that an exact solution can be obtained when the canonically conjugate variables ϕ and φ occur in a linear manner in the perturbed part of the Hamiltonian density. The second example shows how the method can be applied when the perturbed variables appear in quadratic form.

Infinite plane source as a perturbation

An interesting example using the perturbation method derived in the previous section can be shown for the case of monoenergetic neutrons diffusing through a nonmultiplying semi-infinite slab region from an infinite plane source.

The neutron diffusion equation for the unperturbed system is as follows:

$$D \frac{d^2 \phi}{dx^2} - \Sigma_a \phi = 0 \quad (124)$$

where ϕ goes to zero at $x = a$. The Lagrangian density becomes

$$\mathcal{L} = \frac{D}{2} \left(\frac{d\phi}{dx} \right)^2 + \frac{\Sigma_a}{2} \phi^2 \quad (125)$$

and the resulting Hamiltonian density for the unperturbed system is

$$H_0 = \frac{1}{2} \frac{\varphi^2}{D} - \frac{\Sigma_a}{2} \phi^2 \quad (126)$$

The differential equation for the region in question has as its

general solution the following equation.

$$\phi_0(x) = A e^{-Kx} + C e^{Kx} \quad (127)$$

where $K = \sqrt{\frac{\epsilon_a}{D}}$. One boundary condition already stated is that the flux be zero at the extrapolated boundary $x = a$. This boundary condition eliminates one of the arbitrary constants, thus

$$\phi_0(x) = A [e^{-Kx} - e^{-K(2a-x)}] \quad (128)$$

The other arbitrary constant is eliminated by assuming that the unperturbed flux is known at some point x_v . Equation 128 thus becomes

$$\phi_0(x) = \phi_0(x_v) \frac{[e^{-Kx} - e^{-K(2a-x)}]}{[e^{-Kx_v} - e^{-K(2a-x_v)}]} \quad (129)$$

where $\phi_0(x)$ is shown to be a function of x and of the value of the flux at the point x_v . The flux at x_v may be shown to be dependent upon the current at that point, i.e.,

$$\phi_0(x_v) = - \frac{P_0(x_v) [e^{-Kx_v} - e^{-K(2a-x_v)}]}{DK [e^{-Kx_v} + e^{-K(2a-x_v)}]} \quad (130)$$

therefore Eq. 129 has as a companion equation, the expression

$$\phi_0(x) = - \frac{P_0(x_v) [e^{-Kx} - e^{-K(2a-x)}]}{DK [e^{-Kx_v} + e^{-K(2a-x_v)}]} \quad (131)$$

Let the infinite plane source perturb the system and be represented

by a Dirac delta function δ_D such that the new Hamiltonian density becomes

$$H = \frac{1}{2} \frac{\varphi^2}{D} - \frac{\Sigma a}{2} \phi^2 + \delta_D(x) \phi \quad (132)$$

If Eq. 126 is subtracted from Eq. 132, it is seen that

$$H_1 = \delta_D(x) \phi(x) \quad (133)$$

Liouville's operator can thus be written as

$$\Omega_1 = \frac{\partial H_1}{\partial \varphi} \frac{\partial}{\partial \phi} - \frac{\partial H_1}{\partial \phi} \frac{\partial}{\partial \varphi} \quad (134)$$

and after substituting Eq. 133 for H_1 ,

$$\Omega_1 = -\delta_D(x) \frac{\partial}{\partial \varphi(x)} \quad (135)$$

Equation 121 for the operator $S_1(x)$ is now allowed to operate on Eq. 131 which is an expression for the unperturbed flux in terms of the current $\varphi_0(x)$. Therefore, the perturbed flux becomes,

$$\begin{aligned} \phi(x) = S_1(x) \phi_0(x) &= \phi_0(x) + \int_0^x \Omega_1[\dots x_1] \phi_0(x) d\varphi_1 \\ &+ \int_0^x \int_0^{x_1} \Omega_1[\dots x_2] \Omega_1[\dots x_1] \phi_0(x) d\varphi_2 d\varphi_1 \\ &+ \dots \dots \dots \end{aligned} \quad (136a)$$

$$\phi(x) = \phi_0(x) + \int_0^x (-1) \delta_D(x_1) \frac{\partial}{\partial \varphi(x_1)} \left\{ \frac{-\varphi_0(x_1) [e^{-Kx} - e^{-K(2a-x)}]}{DK [e^{-Kx_1} + e^{-K(2a-x_1)}]} \right\} d\varphi_1 \quad (136b)$$

$$= \phi_0(x) + \int_0^x \delta_D(x_1) \frac{1}{DK} \frac{[e^{-Kx} - e^{-K(2a-x)}]}{[e^{-Kx_1} + e^{-K(2a-x_1)}]} d\varphi_1 \quad (136c)$$

By using the integrating properties of the Dirac delta function Eq. 136c becomes

$$\phi(k) = \phi_0(k) + \frac{1}{2DK} \left[\frac{e^{-kx} - e^{-k(2a-x)}}{1 + e^{-2ak}} \right] \quad (137)$$

This equation may be compared to Glasstone and Edlund's Eq. 5.55.1 (8) where they assume that $\phi_0(k) = 0$.

It should be noted in this example that the answer is exact since all other terms are zero. This happens because of the linear manner in which the Hamiltonian was perturbed.

Fission product poisoning as a perturbation

The previously mentioned authors Weinberg and Wigner (33, p.564) use their modified version of the Rayleigh-Schrodinger perturbation method to solve a one-group semi-infinite bare reactor problem. The perturbation is caused by a sinusoidal fission product poisoning. This author will use this example since it offers an excellent opportunity to compare results. Their method was described in the first section of this work.

Equation 31 represents their general equation for the perturbed flux based on a perturbation operator P . They consider a reactor which has been poisoned by fission products and at the same time fuel has been added to keep the reactor critical. The fuel is added in such a manner that the fission cross-section remains independent of position throughout the reactor.

The unperturbed flux and current

The differential equation

describing the unperturbed reactor is

$$D \frac{d^2 \phi_0}{dx^2} + (\nu \Sigma_f - \Sigma_a) \phi_0 = 0 \quad (138)$$

where ν is the number of neutrons produced per fission, Σ_f the macroscopic fission cross-section, Σ_a the total absorption cross-section, and D the diffusion coefficient of the medium.

For the geometry in question, the solution to the unperturbed system is as follows:

$$\text{Let} \quad B_0^2 = \frac{\nu \Sigma_f - \Sigma_a}{D} \quad (139a)$$

$$\text{then since} \quad \phi_0(0) = \phi_0(a) = 0 \quad (139b)$$

$$\text{and } \phi_0 \text{ is symmetric about the point } x = a/2, \quad (139c)$$

$$\phi_0(x) = \sin B_0 x = \sin \frac{\pi x}{a} \quad (140)$$

where the unperturbed flux has been normalized to unity at $x = a/2$, and only the fundamental eigenvalue $B_0 = \pi/a$ has been retained. The neutron current can thus be expressed as

$$J_0(x) = D \frac{d\phi_0}{dx} = D \frac{\pi}{a} \cos \frac{\pi x}{a} \quad (141)$$

In order to use Eqs. 140 and 141 it is necessary to express them in terms of some position x_y in the reactor. When this is done the

resulting equations are

$$\phi(x) = \phi_0(x_v) \cos \frac{\pi}{a}(x-x_v) + \frac{\mathcal{P}_0(x_v)}{\delta\pi/a} \sin \frac{\pi}{a}(x-x_v) \quad (142)$$

$$\mathcal{P}_0(x) = \mathcal{P}_0(x_v) \cos \frac{\pi}{a}(x-x_v) - \frac{\phi_0(x_v)}{a/\pi} \sin \frac{\pi}{a}(x-x_v) \quad (143)$$

The perturbed equation

The differential equation for the per-

turbed flux will be

$$D \frac{d^2 \phi}{dx^2} + (\nu \Sigma_f' - \Sigma_a') \phi = 0 \quad (144)$$

where primed quantities refer to the perturbed values. As stipulated, the fission cross-section Σ_f' is independent of position, but the absorption cross-section changes to

$$\Sigma_a' = \Sigma_a + (1+\alpha)(\Sigma_f' - \Sigma_f) + c \sin \frac{\pi}{a} x \quad (145)$$

where α is the capture to fission ratio of the fuel.

The Hamiltonian density

The Hamiltonian density for the per-

turbed system becomes

$$H = \frac{1}{2} \frac{\mathcal{P}^2}{D} + \frac{1}{2} \left[\nu \Sigma_f' - \Sigma_a - (1+\alpha)(\Sigma_f' - \Sigma_f) - c \sin \frac{\pi}{a} x \right] \phi^2 \quad (146)$$

whereas for the unperturbed system

$$H_0 = \frac{1}{2} \frac{\mathcal{P}^2}{D} + \frac{1}{2} (\nu \Sigma_f - \Sigma_a) \phi^2 \quad (147)$$

The difference between the two expressions above gives the change in the Hamiltonian due to the perturbation. This change in the Hamiltonian density due to the perturbation is thus

$$H_1 = \frac{1}{2} \left[(\Sigma_f' - \Sigma_f)(\nu - 1 - \alpha) - c \sin \frac{\pi}{a} x \right] \phi^2 \quad (148)$$

Stationary property of the Lagrangian

The stationary property

of the Lagrangian as shown in Eq. 39 can be utilized to show that the following equation holds true:

$$\int_0^a H_1 d\chi = 0 \quad (149)$$

This equation adds a condition from which the perturbing parameters can be related. That is, if Eq. 148 is substituted into Eq. 149 the result becomes

$$(\Sigma_f' - \Sigma_f) = \frac{c}{(\nu - 1 - \alpha)} \frac{\int_0^a (\sin \frac{\pi}{a} x) \phi^2 dx}{\int_0^a \phi^2 dx} \quad (150)$$

In as much as the square of the perturbed flux occurs in both the numerator and the denominator, a good first approximation is that

$$(\Sigma_f' - \Sigma_f) \approx \frac{c}{(\nu - 1 - \alpha)} \frac{\int_0^a \sin^3 \frac{\pi}{a} x dx}{\int_0^a \sin^2 \frac{\pi}{a} x dx} \quad (151)$$

where the unperturbed flux of Eq. 140 has been substituted in lieu of the perturbed flux. Equation 151 may be integrated to yield the relation

$$(\Sigma_f' - \Sigma_f) \approx \frac{8c}{3\pi(\nu - 1 - \alpha)} \quad (152)$$

As mentioned previously, this problem has been solved by Weinberg and Wigner (33, p.565) and Eq. 151 corresponds exactly with their Eq. 16.56a which was obtained by an entirely different approach using the method described in the first section of this work. Equation 21

of that section was the condition necessary for the stable criticality of the reactor and would be equivalent to Eq. 150.

The Liouville operator The Liouville operator for this problem may be determined by substituting the expression for H_1 from Eq. 148 into the expression for Ω , which is Eq. 115. The result is as follows:

$$\Omega_1 = \frac{\partial H_1}{\partial \varphi} \frac{\partial}{\partial \phi} - \frac{\partial H_1}{\partial \phi} \frac{\partial}{\partial \varphi} \quad (153)$$

$$\Omega_1 = -[(\varepsilon'_f - \varepsilon_f)(\nu - 1 - \alpha) - C \sin \frac{\pi}{a} x] \phi \frac{\partial}{\partial \varphi} \quad (154)$$

where it may be observed that since the perturbed Hamiltonian density was not expressed as a function of the current φ , the first term drops out. Use may now be made of Eq. 152, which when substituted into Eq. 154 yields the Liouville operator for the problem.

$$\Omega_1 = C \left(\sin \frac{\pi}{a} x - \frac{\theta}{3\pi} \right) \phi \frac{\partial}{\partial \varphi} \quad (155)$$

The perturbed flux All the operator elements necessary to use Eq. 121 in determining an expression for the perturbed flux have been obtained. In Eq. 122, if $F[x]$ is replaced by $\phi(x)$, that equation becomes the following:

$$\phi(x) \simeq S_1^{(n)}(x) \phi_0(x) \quad (156)$$

where $\phi_0(x)$ is the unperturbed flux. Substituting for $S_1^{(n)}(x)$ from Eq. 121 and from Eq. 142 for $\phi_0(x)$, the expression to be evaluated for $\phi(x)$ becomes

$$\begin{aligned} \phi(x) \simeq & \phi_0(x) + C \int_0^x \left(\sin \frac{\pi}{a} x_1 - \frac{\theta}{3\pi} \right) \phi(x_1) \frac{\partial}{\partial \varphi(x_1)} \phi_0[\phi_0(x_1), \varphi_0(x_1)] dx_1 \\ & + C^2 \int_0^x \int_0^{x_1} \left(\sin \frac{\pi}{a} x_2 - \frac{\theta}{3\pi} \right) \phi(x_2) \frac{\partial}{\partial \varphi(x_2)} \left(\sin \frac{\pi}{a} x_1 - \frac{\theta}{3\pi} \right) \\ & \cdot \phi(x_1) \frac{\partial}{\partial \varphi(x_1)} \phi_0[\phi_0(x_1), \varphi_0(x_1)] dx_1 dx_2 + \dots \end{aligned} \quad (157)$$

If all terms of Eq. 157 beyond the first integral term are dropped, the equation becomes,

$$\phi^{(1)}(x) = \phi_0(x) + \frac{c a}{2D\pi} \int_0^x \left(\sin \frac{\pi}{a} x_1 - \frac{B}{3\pi} \right) \sin \frac{\pi}{a} x_1 \sin \frac{\pi}{a} (x - x_1) dx_1 \quad (158)$$

This equation integrates to the following expression

$$\begin{aligned} \phi^{(1)}(x) = \phi_0(x) + \frac{c a^2}{2D\pi^2} \left\{ \left(1 - \frac{4}{3} \cos \frac{\pi}{a} x + \frac{1}{3} \cos \frac{\pi}{a} (2x) \right) \right. \\ \left. - \frac{B}{3} \left(\sin \frac{\pi}{a} x - \frac{x}{a} \pi \cos \frac{\pi}{a} x \right) \right\} \quad (159) \end{aligned}$$

Values chosen for this particular problem are as follows:

$$\text{Let} \quad C = \frac{\pi \epsilon_a}{22.2} \quad (160a)$$

$$a = 200 \quad (160b)$$

$$L = 0.1a = \sqrt{\frac{D}{\epsilon_a}} \quad (160c)$$

$$\text{so that} \quad \frac{c a^2}{2D\pi^2} = 0.716 \quad (160d)$$

and Eq. 159 thus becomes

$$\begin{aligned} \phi^{(1)}(x) = \phi_0(x) + 0.716 \left\{ \left(1 - \frac{4}{3} \cos \frac{\pi}{a} x + \frac{1}{3} \cos \frac{\pi}{a} (2x) \right) \right. \\ \left. - 0.849 \left(\sin \frac{\pi}{a} x - \frac{x}{a} \pi \cos \frac{\pi}{a} x \right) \right\} \quad (161) \end{aligned}$$

If the boundary conditions are imposed, it is seen that

$$\phi^{(1)}(0) = \phi_0(0) + 0.716 \left\{ \left[1 - \frac{4}{3} + \frac{1}{3} \right] - 0.849[0] \right\} \quad (162a)$$

$$\phi^{(1)}(0) = 0 \quad (162b)$$

$$\phi^{(1)}(a) = \phi_0(a) + 0.716 \left\{ \left[1 + \frac{4}{3} + \frac{1}{3} \right] - .849[\pi] \right\} \quad (163a)$$

$$= 0 + 0.716 \left\{ \frac{8}{3} - .849\pi \right\} \quad (163b)$$

$$\phi^{(1)}(a) = 0 \quad (163c)$$

so that this approximation satisfies the boundary conditions. It may also be shown that the current in this first approximation is zero at the mid-point $x = a/2$ which is a condition imposed by the symmetry of the problem.

Figure 1 compares the unperturbed flux with the perturbed flux. The flux at the center has decreased by 13%. Two additional terms were calculated for Eq. 159 but were insignificant. The convergence was rapid with the second integral contributing only 0.2% at the point $x = a/2$. Another check was made to see how good the approximation would be if Eq. 159 was used as the expression for the perturbed flux. Equation 159 was substituted into Eq. 150 and the result was as follows:

$$\left[\frac{(\xi'_f - \xi_f)(\nu - 1 - \alpha)}{c} \right] = 0.837 \quad (164a)$$

As a comparison, Eq. 152 when put into this form becomes

$$\left[\frac{(\xi'_f - \xi_f)(\nu - 1 - \alpha)}{c} \right] = 0.849 \quad (164b)$$

so it is seen that since these two results differ by only 1.4% that the method is quite good and the truncation of terms is satisfactory

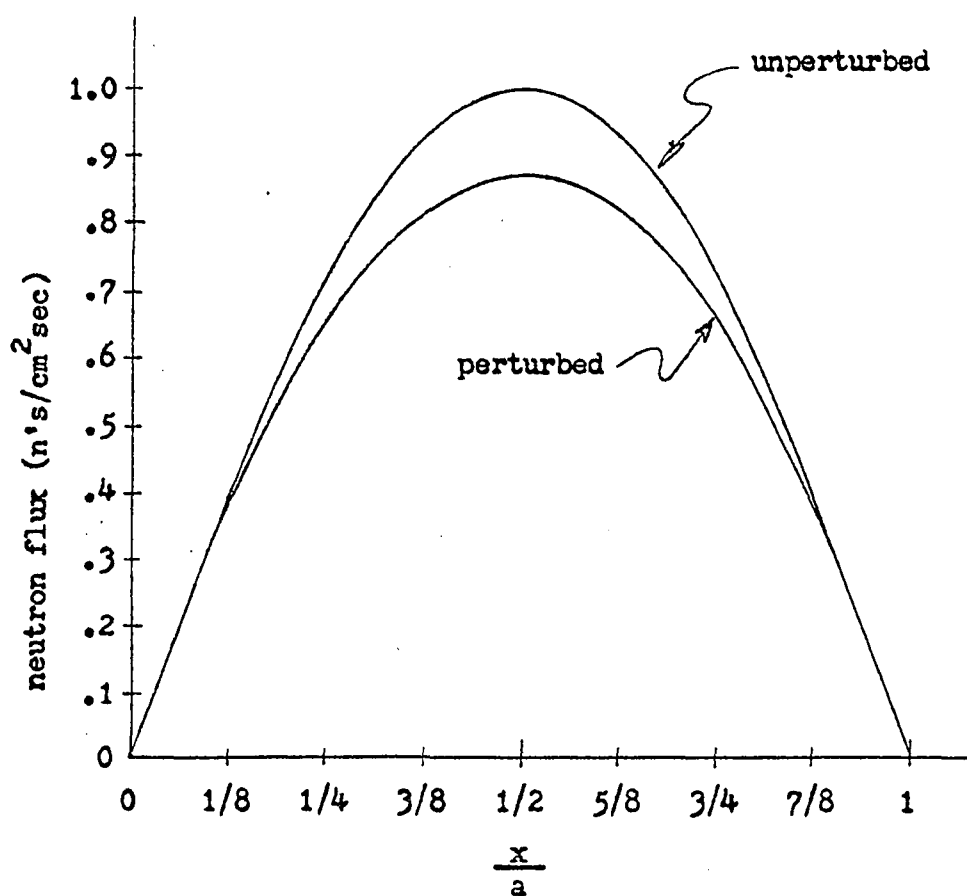


Fig. 1. Perturbation of the neutron flux by an absorber of magnitude proportional to the unperturbed flux.

for a reasonably reliable answer in this case. The determination was done with the thought that perhaps one or two iterations on the value of the relation in Eq. 164a would yield a better answer. This was not done because the two only differed by such a slight amount.

Figure 2 compares the two flux profiles where the perturbed flux has normalized at the center. A flattening of the flux is evident. This flattening was expected from the fission product poisoning coupling with the addition of a flat fissile component. In order to compare these graphs with Weinberg and Wigner's it would be necessary to normalize the curves at the point $x = a/3$. This would be necessary since the second and last term ($\sin 3x/a$) in their truncated expansion has a zero at this point and therefore does not add to the unperturbed first term.

Analog solution as a check An analog computer was used to check the solution of the problem. The results are shown in Table 1 along with the results from Eq. 161. The analog solution was interesting since the approximate value for the perturbing parameters as obtained by Eq. 152 influences the period of the perturbed solution. The periods represent the conditions on the extrapolated boundaries and therefore must be the same. This analog solution has been included as Appendix B.

The correspondence between the analog method and the perturbation method illustrates the potential of the perturbation method used herein. In contrast to the Rayleigh-Schrodinger method, the method used herein, provides a method of determining the magnitude as well as the shape of the perturbed flux.

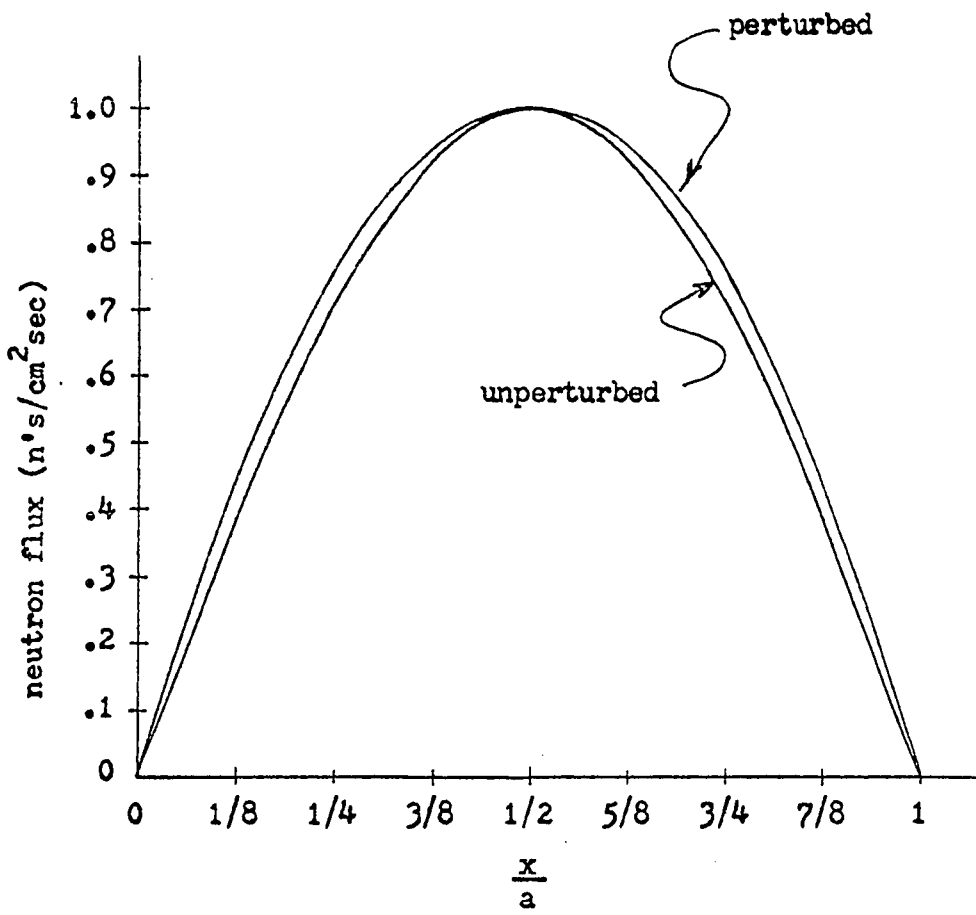


Fig. 2. Normalized flux profiles

Table 1. Comparison between the perturbed and the unperturbed flux as computed by the perturbation method and substantiated by an analog computer solution.

$\frac{x}{a}$	Perturbed flux			Unperturbed flux
	analog solution	Eq. 161	Eq. 161 normalized	$\sin x$
0	0	0	0	0
1/8	.38	.375	.431	.383
1/4	.66	.656	.754	.707
3/8	.82	.818	.940	.924
1/2	.87	.870	1.00	1.00
5/8	.82	.819	.941	.924
3/4	.66	.654	.752	.707
7/8	.38	.373	.429	.383
1	0	0	0	0

DISCUSSION AND SUMMARY

The perturbation method as outlined in the text of this work lends itself quite readily to the solution of certain problems in neutron diffusion theory. The formulation of the problem from a variational principle is both utile as well as aesthetic. There is always something fascinating about a variational formulation. The Hamiltonian density and its integral over the reactor volume which resulted from this formulation illustrates this appreciation by providing a physically significant result which would have otherwise been difficult to obtain.

This operator method provides a general way of expressing a perturbed solution in terms of an unperturbed solution by using a canonical Liouville operator. This method has conceptual significance which equals or perhaps exceeds that of Taylor's theorem in mathematics.

When the foregoing methods are put together they form a practical and interesting method of treating certain types of problems in neutron diffusion theory, not to mention all the other areas where by analogy the method can be applied.

Two examples were treated by this perturbation method. The first example illustrated the method wherein the Hamiltonian density was perturbed by a term linear in the neutron flux ϕ . The result of this problem was exact. The second example illustrated the method when the perturbed Hamiltonian density involves a term quadratic in flux. The solution to this problem was not expressible in closed form, however convergence occurred quite rapidly. This second example suggests that

this method could possibly be used to solve nonlinear problems in neutron diffusion theory. To treat a more general class of problems, it will be necessary to study the problem of convergence.

One of the outstanding results of the method used in the second example was that it provides the magnitude as well as the shape of the solution. The ordinarily used modified Rayleigh-Schrodinger method does not.

The method can appropriately be applied to the multi-group problem. The Liouville operator Ω described in Eq. 83 provides for any number of conjugate ϕ_i and p_i . The operator $S_i(x)$, which includes the Ω operator, may be applied in the same manner as herein described except that it would now operate on a column matrix composed of the group fluxes.

It has been pointed out herein that when the Hamiltonian density is not explicitly a function of position it becomes a constant over the reactor volume. This result has greater potential use than has been indicated especially since reactor calculations are predominantly based on homogenized partitioning of the reactor.

SUGGESTIONS FOR FUTURE WORK

It was stated previously that this perturbation method could be applied to the multi-group diffusion problem. The operators which are used in this method allow for any number of canonically conjugate variables. The multi-group perturbation problem should be investigated.

In reactor calculations it is customary to partition the reactor into regions. The multi-region perturbation problem should be studied. One of the difficulties will be with boundary conditions since one of the conditions utilized herein was that both the perturbed and unperturbed fluxes had a common initial boundary condition at $x = 0$.

The question of convergence should be investigated. The method of successive approximations was used. This method is also used in the same manner to obtain iterative type solutions to Volterra integral equations of the second kind. In the problem at hand the integral equations which need to be solved have differential operators rather than algebraic kernels. The method of proof, however, might proceed along similar lines as indicated by Hildebrand (11, p.260) in his work with linear integral equations.

The Hamiltonian densities occurring in the illustrated examples were derived from linear differential equations. A logical extension of the method would be to study the nonlinear diffusion equation as a perturbation of a linear equation. This work should be done. The perturbed portion of the Hamiltonian, H_1 , which occurs in the Liouville operator would contain that part due to the nonlinear com-

ponent of the problem. Here also since the problem would be in open form, convergence of the resultant infinite series must be studied.

This work has completely excluded the case of the time dependent reactor. The author believes there are distinct possibilities here also for future work. When the reactor is on a stable positive period the Lagrangian should again be stationary. If the reactor period was not stable, the Lagrangian would be unstable and therefore not stationary. There are many possibilities to investigate here.

In developing this operator method it has been assumed that all operations or transformations were valid. A study should be made to mathematically formalize this operator method. An approach to the problem might begin by consulting Von Neumann's work (31) on the mathematical foundations of quantum mechanics wherein he considers the mathematics of transformation theory. Jammer (13) has also recently written a text which also covers transformation theory and should be helpful in initiating work in this area.

Last but not least perhaps, it would be well to see if computer techniques could be used in solving problems by this method. The author did not use or investigate the use of digital computers for the method.

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Last but not least the author is indebted to He who governs the infinite source of all knowledge and wisdom from whence this fragment has been taken.

APPENDIX A

Interpretation for the Lagrangian integral

There are at least two points of view possible for interpreting the stationary Lagrangian integral in Eq. 40. From a reaction rate point of view the integrand represents the difference between weighted reaction rates. For the integral to be stationary the integrated difference between the weighted reaction rates must be stationary.

The other point of view is developed in the following discussion. The importance function ψ is dimensionally equivalent to the importance per neutron. The importance function as normally defined, therefore, is a unit-importance function, i.e.,

$$\psi = \frac{\text{importance}}{\text{neutron}} \quad (\text{A-1})$$

The unit-importance function when multiplied by the number of neutrons present should give the importance of the neutrons in question. This is shown as follows:

$$1. \quad \begin{array}{l} \text{Production} \\ \text{Reaction Rate} \end{array} \quad K \Sigma_a \phi = \frac{\text{neutrons produced}}{\text{cm}^3 - \text{sec}} \quad (\text{A-2})$$

$$\begin{array}{l} \text{Production} \\ \text{Importance Rate} \end{array} \quad K \Sigma_a \phi \psi = \frac{\text{importance produced}}{\text{cm}^3 - \text{sec}} \quad (\text{A-3})$$

$$2. \quad \begin{array}{l} \text{Absorption} \\ \text{Reaction Rate} \end{array} \quad \Sigma_a \phi = \frac{\text{neutrons absorbed}}{\text{cm}^3 - \text{sec}} \quad (\text{A-4})$$

$$\begin{array}{l} \text{Absorption} \\ \text{Importance Rate} \end{array} \quad \Sigma_a \phi \psi = \frac{\text{importance lost}}{\text{cm}^3 - \text{sec}} \quad (\text{A-5})$$

$$\begin{array}{l} \text{3. Migration} \\ \text{Rate} \end{array} \quad D \bar{\nabla} \phi = \frac{\text{neutrons migrating(in,out)}}{\text{cm}^2 - \text{sec}} \quad (\text{A-6})$$

$$\begin{array}{l} \text{Importance} \\ \text{Migration} \\ \text{Rate} \end{array} \quad D \bar{\nabla} \phi \cdot \bar{\nabla} \psi = \frac{\text{importance migrating(in,out)}}{\text{cm}^3 - \text{sec}} \quad (\text{A-7})$$

From a dimensional standpoint the following equivalence relations hold true:

For variables of displacement, momentum, and time as used in dynamics, i.e., (L,P,T),

$$\text{Energy} \doteq \frac{PL}{T} \quad (\text{A-8})$$

For variables of flux, current, and position as used in one-group reactor theory, i.e., (ϕ , p, r),

$$\begin{array}{l} \text{Importance} \\ \text{Rate} \end{array} = \frac{p \phi}{r} \quad (\text{A-9})$$

Importance rate therefore plays a role analogous to energy. The flux gradient term by way of analogy is associated with the time derivative term. Therefore, kinetic importance rate would be the analog to kinetic energy. Similarly one may associate by analogy potential importance rate with the potential energy. This correlation in nomenclature may be demonstrated by comparing the following equations:

One-group reactor
Semi-infinite
slab geometry

Differential equation:

$$D \frac{d^2 \phi}{dx^2} + B_0^2 D \phi = 0 \quad (\text{A-10})$$

Hamilton's principle:

$$\delta \int \left[\frac{1}{2} \frac{P^2}{D} - \frac{1}{2} B_0^2 D \phi^2 \right] dx = 0 \quad (\text{A-11})$$

Linear harmonic oscillator

Differential equation:

$$m \frac{d^2 x}{dt^2} + Kx = 0 \quad (A-12)$$

Hamilton's principle:

$$\delta \int \left[\frac{1}{2} \frac{p^2}{m} - \frac{1}{2} Kx^2 \right] dt = 0 \quad (A-13)$$

From the above equations it is possible to make the following comparison:

$$\text{Kinetic importance rate} = \frac{p^2}{2D} \quad (A-14)$$

$$\text{Potential importance rate} = \frac{1}{2} B_0^2 D \phi^2 \quad (A-15)$$

$$\text{Kinetic energy} = \frac{p^2}{2m} \quad (A-16)$$

$$\text{Potential energy} = \frac{1}{2} Kx^2 \quad (A-17)$$

APPENDIX B

Analog solution of the problem

An EAI-Pace TR-10 analog computer was used to check the solution to the problem described by Eqs. 140, 144, and 145. The circuit for the analog solution is shown in Fig. 3.

The lower branch of the circuit generates the term $\frac{c}{D} \sin \frac{\pi}{a} x$ which describes the fission product poisoning. This term is then multiplied by the perturbed flux term exhibited on the outputs of amplifiers 2 and 4. This result is then added to the other terms which make up Eq. 144, in the following form:

$$D \frac{d^2 \phi}{dx^2} = -\nu \Sigma_f' \phi + \Sigma_a' \phi \quad (B-1)$$

$$\text{or} \quad \frac{d^2 \phi}{dx^2} = -\left[\frac{(\Sigma_f' - \Sigma_f)(\nu - 1 - \alpha)}{D} + \left(\frac{\pi}{a}\right)^2 \right] \phi(x) + \left[\frac{c}{D} \sin \frac{\pi}{a} x \right] \phi(x) \quad (B-2)$$

When the values of the problem worked in the text of the thesis are substituted into Eq. B-2 the following equation results:

$$\frac{d^2 \phi(x)}{dx^2} = -.000547 \phi(x) + .000353 \phi(x) \sin \frac{\pi}{a} x \quad (B-3)$$

Since the coefficients of Eq. B-3 are so small it was necessary to set up a time scale on the analog computer as follows:

$$\text{Let} \quad x = (10)^{1.5} t_m \quad (B-4)$$

$$\text{then} \quad \frac{d^2 \phi(x)}{dx^2} = 10^{-3} \frac{d^2 \phi(t_m)}{dt_m^2} \quad (B-5)$$

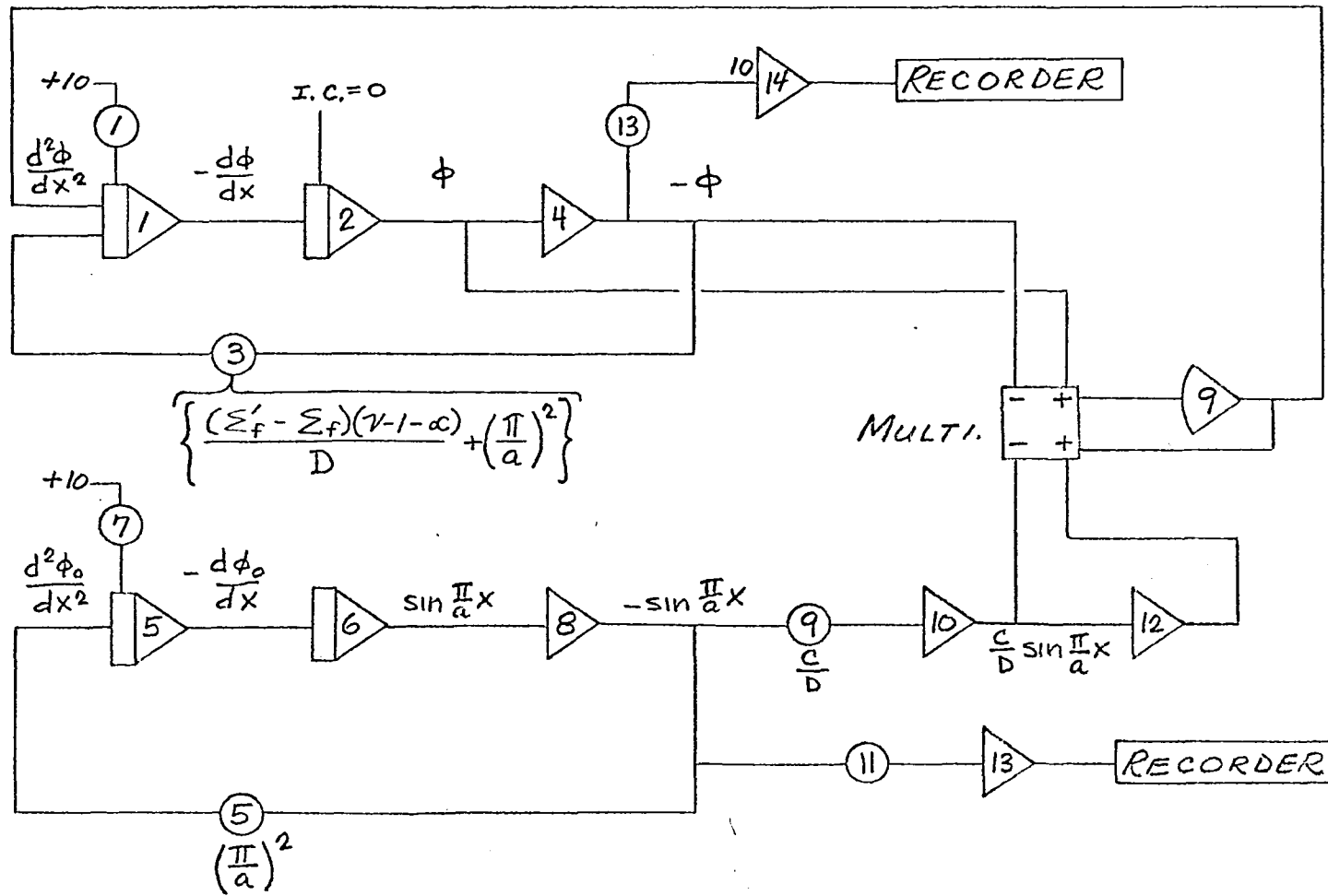


Fig. 3. Circuit for an analog computer solution to a problem where the perturbation of the neutron flux is by an absorber which is proportional to the unperturbed flux.

Substitution of Eq. B-5 into Eq. B-3 yields the following equation:

$$\frac{d^2 \phi(t_m)}{dt_m^2} = -0.547 \phi(t_m) + 0.353 \phi(t_m) \sin(497 t_m) \quad (B-6)$$

with the boundary conditions that $\phi(0) = \phi(a) = 0$, where the point a in this case symbolizes the end of a half-period. The potentiometer settings which correspond to the solution of Eq. B-6 were as shown in the following table:

Table 2. Potentiometer settings corresponding to the solution of Eq. B-6 and the circuit of Fig. 3.

Potentiometer settings						
1	3	5	7	9	11	13
.200	.555	.233	.483	.353	.500	.125

In solving the problem, two boundary conditions were used. The initial condition that $\phi = 0$ at $x = 0$ and the condition that $\phi = 0$ at the point $x = a$. The initial conditions on amplifiers 2 and 6 were set at zero. The procedure for setting the end conditions was by no means exact but worked quite well. First the upper circuit was disconnected from the rest of the problem and set up so that it generated a simple unperturbed flux. Then the potentiometers of the lower circuit were set to give a magnitude of unity and a period which corresponded to

the period of the upper circuit.

If the system had been perfect, potentiometer settings for Pots. 5 and 7 would have been 0.247 and 0.497 respectively. However, in balancing the two unperturbed circuits Pot. 3 was allowed to control since its value would be changed in the perturbed problem. When the two circuits were set to have the same period, Pot. 3 had a setting of 0.247 and Pot. 5 had a setting of 0.233. Normalization of the lower circuit was then set at 0.483 which is the square root of 0.233. The upper circuit was then normalized for an unperturbed magnitude of two by setting the initial conditions on amplifier 1. Because of the nature of the problem the initial conditions on amplifiers 1 and 5 should be proportional to their respective unperturbed flux amplitudes as x approaches zero. The ratio in this case was 4:1 which explains the 4:1 ratio between the output Pots. 11 and 13.

The circuit was then completed and a value of 0.547 selected for Pot. 3. With this calculated value, the perturbed solution did not have the same period as the unperturbed solution and thus a discrepancy existed. To remove the discrepancy, Pot. 3 was adjusted until the two periods were in agreement. This disagreement arose because Eq. 152 was only an approximation and according to Eqs. 164a and 164b it was off by 1.4%. Likewise the potentiometer had to be changed from 0.547 to 0.555 which is a difference of 1.45%. When this change was made the periods were in agreement.

The output of amplifier 13 is the unperturbed flux amplified so

that when outputed on the recorder its amplitude is normalized to one. The output of amplifier 14 is the perturbed flux. When this flux is displayed on the same calibrated channel as the previously displayed unperturbed flux, the perturbed flux is then compared to the normalization of the unperturbed flux.

The results coincide exactly with those found by the perturbation method. Rather than plot the graph, values at quarter half periods were included in Table 1 of the main discussion.