Phys 5405
HW 11
5.13, 5.17, 5.19(a), 5.20, 5.21, 5.26, 5.27

5.13 We can write down the current density,

$$\mathbf{J}(\mathbf{x}) = \sigma\omega a \sin\theta\delta(r-a)\hat{\phi} \ . \tag{1}$$

Then the vector potential is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{\mu_0\sigma}{4\pi}\omega a \int r^2 dr \sin\theta' d\theta' d\phi' \frac{\sin\theta'\delta(r'-a)\hat{\phi}'}{|\mathbf{x}-\mathbf{x}'|} \ . \tag{2}$$

We can expand the Green function in spherical coordinates

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \ , \tag{3}$$

where $r_< = \min(r, r')$ and $r_> = \max(r, r')$. Therefore, we get

$$\mathbf{A}(\mathbf{x}) = \mu_0\sigma\omega a^3 \sum_{l,m} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) \int d\Omega' Y_{lm}^*(\theta',\phi') \sin\theta' \hat{\phi}' \ , \tag{4}$$

where $r_< = \min(r, a)$ and $r_> = \max(r, a)$. Now we decompose $\hat{\phi}'$ in Cartesian coordinates,

$$\hat{\phi}' = \cos\phi' \hat{i} + \sin\phi' \hat{j} \ . \tag{5}$$

We can evaluate the integrals,

$$\begin{aligned} \int d\Omega' Y_{lm}^*(\theta',\phi')\sin\theta'\cos\phi' &= \sqrt{\frac{2\pi}{3}}\int d\Omega' Y_{lm}^*(\theta',\phi')(-Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')) \\ &= \sqrt{\frac{2\pi}{3}}(-\delta_{l1}\delta_{m1} + \delta_{l1}\delta_{m,-1}) \ . \end{aligned} \tag{6}$$

$$\begin{aligned} \int d\Omega' Y_{lm}^*(\theta',\phi')\sin\theta'\sin\phi' &= \sqrt{\frac{2\pi}{3}} i \int d\Omega' Y_{lm}^*(\theta',\phi')(Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')) \\ &= \sqrt{\frac{2\pi}{3}} i(\delta_{l1}\delta_{m1} + \delta_{l1}\delta_{m,-1}) \ . \end{aligned} \tag{7}$$

Therefore, we have

$$\mathbf{A}(\mathbf{x}) = \frac{1}{3}\mu_0\sigma\omega a^3 \frac{r_<}{r_>^2}\sin\theta(\cos\phi\hat{i} + \sin\phi\hat{j}) = \frac{1}{3}\mu_0\sigma\omega a^3\frac{r_<}{r_>^2}\sin\theta\hat{\phi} \ . \tag{8}$$

Then, inside the sphere $r_< = r$ and $r_> = a$,

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{1}{3}\mu_0\sigma\omega a r\sin\theta\hat{\phi}} \tag{9}$$

Outside the sphere, $r_< = a$ and $r_> = r$,

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{1}{3}\mu_0\sigma\omega\frac{a^4}{r^2}\sin\theta\hat{\phi}} \tag{10}$$

The magnetic flux density is given by

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta}\ . \tag{11}$$

So inside the sphere, we have

$$\boxed{\mathbf{B} = \frac{2}{3}\mu_0\sigma\omega a(\cos\theta\hat{r} - \sin\theta\hat{\theta})} \tag{12}$$

Outside the sphere, we have

$$\boxed{\mathbf{B} = \frac{1}{3}\mu_0\sigma\omega\frac{a^4}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})} \tag{13}$$

5.17 For $z > 0$, the magnetic induction is generated by the current $\mathbf{J}$ and the image current $\mathbf{J}^*$,

$$\mathbf{B}^+(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{(\mathbf{J}(\mathbf{x}') + \mathbf{J}^*(\mathbf{x}')) \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \ . \tag{14}$$

It is an integration over the whole region. For $z < 0$, the magnetic induction is generated by the current $k\mathbf{J}$, where k is a scaling constant because of different permeability,

$$\mathbf{B}^-(\mathbf{x}) = \frac{\mu_0\mu_r k}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \ . \tag{15}$$

It is an integration over the region where $z' > 0$. Now we want to transform the first integral such that these two integrals have the same integration domain. Suppose the component of $\mathbf{x}'$ is (x', y', z'), then we define $\mathbf{x}'' = (x', y', -z')$ and we can write

$$\mathbf{B}^+(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} + \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}^*(\mathbf{x}'') \times (\mathbf{x}-\mathbf{x}'')}{|\mathbf{x}-\mathbf{x}''|^3} \ . \tag{16}$$

Now this integral is defined in the region where $z' > 0$. And especially, when $z = 0$, there is no difference between $|\mathbf{x}-\mathbf{x}'|$ and $|\mathbf{x}-\mathbf{x}''|$.

Now the boundary conditions are given by

$$\mathbf{B}_z^+(z=0) = \mathbf{B}_z^-(z=0) \ , \quad \mathbf{B}_{x,y}^+(z=0) = \frac{1}{\mu_r}\mathbf{B}_{x,y}^-(z=0) \ . \tag{17}$$

For the first equation, we can equate the numerator in the integrand. When $z = 0$,

$$\hat{z}\cdot[\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')] + \hat{z}\cdot[\mathbf{J}^*(\mathbf{x}'') \times (\mathbf{x}-\mathbf{x}'')] = \mu_r k\hat{z}\cdot(\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')) \ , \tag{18}$$

from which we can get that at $z = 0$,

$$(\mathbf{x}-\mathbf{x}')\cdot[\hat{z}\times\mathbf{J}(\mathbf{x}')] + (\mathbf{x}-\mathbf{x}'')\cdot[\hat{z}\times\mathbf{J}^*(\mathbf{x}'')] = \mu_r k(\mathbf{x}-\mathbf{x}')\cdot(\hat{z}\times\mathbf{J}(\mathbf{x}')) \ , \tag{19}$$

and expanding it in components we can get

$$\boxed{J_y(\mathbf{x}') + J_y^*(\mathbf{x}'') = \mu_r k J_y(\mathbf{x}'), \quad J_x(\mathbf{x}') + J_x^*(\mathbf{x}'') = \mu_r k J_x(\mathbf{x}')} \tag{20}$$

Another equation gives that at $z = 0$,

$$\hat{z}\times[\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')] + \hat{z}\times[\mathbf{J}^*(\mathbf{x}'') \times (\mathbf{x}-\mathbf{x}'')] = k\hat{z}\times[\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')] \ . \tag{21}$$

Now using,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \ , \tag{22}$$

we can further simplify the above equation to,

$$J_z^*(\mathbf{x}'')(\mathbf{x}-\mathbf{x}'') - z'\mathbf{J}^*(\mathbf{x}'') = (k-1)J_z(\mathbf{x}')(\mathbf{x}-\mathbf{x}') + (k-1)z'\mathbf{J}(\mathbf{x}') \ . \tag{23}$$

Expanding it into components and using the fact that the equation holds for arbitrary $\mathbf{x}$, we have,

$$\boxed{J_x^*(\mathbf{x}'') = (1-k)J_x(\mathbf{x}'), \quad J_y^*(\mathbf{x}'') = (1-k)J_y(\mathbf{x}'), \quad J_z^*(\mathbf{x}'') = (k-1)J_z(\mathbf{x}')} \tag{24}$$

From (20) and (24), we can solve for $k = 2/(1+\mu_r)$. Plug it back into (24), we have the image current distribution $\mathbf{J}^*$, with components,

$$\boxed{\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_x(x, y, -z), \quad \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_y(x, y, -z), \quad -\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_z(x, y, -z)}$$

Since $k = 2/(1+\mu_r)$, we have stated that for $z < 0$, the magnetic induction is due to a current distribution $k\mathbf{J}$ in a medium of relative permeability μ_r. We can also consider it due to a current distribution

$$\boxed{k\mu_r\mathbf{J} = \frac{2\mu_r}{1+\mu_r}\mathbf{J}} \tag{25}$$

in a medium of unit relative permeability.

5.19 (a) Since $\mathbf{J} = 0$, we can use the magnetic scalar potential Φ_M. Since the magnetization is uniform, we have

$$\Phi_M(\mathbf{x}) = \frac{1}{4\pi} \oint_S \frac{\mathbf{n}' \cdot \hat{z}\, M_0\, da'}{|\mathbf{x} - \mathbf{x}'|} \tag{26}$$

Since the magnetization points along the z-direction, we only need to consider the top boundary (say, at $z = L$) and the bottom boundary (at $z = 0$) and let the axis of the cylinder lying in the z-axis. Then when $\mathbf{x}$ is on the axis, at top, $|\mathbf{x} - \mathbf{x}'| = \sqrt{x'^2 + y'^2 + (z - L)^2}$ and at bottom, $|\mathbf{x} - \mathbf{x}'| = \sqrt{x'^2 + y'^2 + z^2}$. For the two dimensional surface integral, we can also use polar coordinates. Then, we have,

$$\begin{aligned}
\Phi_M(\mathbf{x}) &= \frac{M_0}{4\pi} \int_0^a \rho' d\rho' \int_0^{2\pi} d\phi \left(\frac{1}{\sqrt{\rho'^2 + (z - L)^2}} - \frac{1}{\sqrt{\rho'^2 + z^2}} \right) \\
&= \frac{M_0}{2} \int_0^a d\rho' \left(\frac{\rho'}{\sqrt{\rho'^2 + (z - L)^2}} - \frac{\rho'}{\sqrt{\rho'^2 + z^2}} \right) \\
&= \frac{M_0}{2} \left(\sqrt{\rho'^2 + (z - L)^2} - \sqrt{\rho'^2 + z^2} \right) \Big|_{\rho'=0}^{\rho'=a} \\
&= \frac{M_0}{2} \left(\sqrt{a^2 + (z - L)^2} - |z - L| - \sqrt{a^2 + z^2} + |z| \right) .
\end{aligned} \tag{27}$$

Therefore,

$$\Phi_M(z) = \left\{ \frac{M_0}{2} \left(\sqrt{} \right) \right. \tag{28}$$

5.20

5.21

5.26

5.27