Phys 5405

HW 6

3.9

3.9 It's similar to what we did in class but with different boundary conditions. Use separation of variables

$$\Phi = R(\rho)Q(\phi)Z(z) \ . \tag{1}$$

Laplacian in cylindrical coordinates is given by

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2} \ . \tag{2}$$

So from $\nabla^2\Phi = 0$, we can derive

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 Q}\frac{d^2Q}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} \ . \tag{3}$$

Since the left hand side only depends on ρ and ϕ, the right hand side depends only on z, then they must equal to a constant.

$$\frac{d^2Z}{dz^2} = -k^2 Z \ , \quad \frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = k^2\rho^2 \ . \tag{4}$$

From the first equation, we can derive that

$$Z(z) \propto \sin(kz), \cos(kz) \ . \tag{5}$$

From the second equation, we can derive

$$\frac{d^2Q}{d\phi^2} = -\nu^2 Q \ , \quad \frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \left(k^2 + \frac{\nu^2}{\rho^2}\right)R = 0 \ . \tag{6}$$

The solution to the first equation is

$$Q(\phi) \propto e^{\pm i\nu\phi} \ . \tag{7}$$

Since ϕ is a cyclic variable, we need ν to be an integer, and we will write m instead of ν. For the second derivative, we can see that $R(\rho)$ should be a linear combination of $J_m(ik\rho)$ and $N_m(ik\rho)$. For finiteness at $\rho = 0$, the coefficient of N_m should be zero. We can also use the modified Bessel function I_m, which is related to J_m as

$$J_m(ix) = e^{\frac{m\pi i}{2}} I_m(x). \tag{8}$$

Then

$$R(\rho) \propto I_m(k\rho)\ . \tag{9}$$

Now consider the boundary conditions. Since the potential on the end faces is zero. We have

$$Z(0) = Z(L) = 0\ , \tag{10}$$

from which we can derive

$$k = \frac{n\pi}{L}, \quad n \in \mathbb{Z}\ . \tag{11}$$

We can write down the general solution as

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\left[A_{mn}\sin(m\phi) + B_{mn}\cos(m\phi)\right]\ . \tag{12}$$

Now we require

$$\Phi(\rho = b, \phi, z) = V(\phi, z)\ . \tag{13}$$

Now the left hand side is a Fourier series in ϕ, also a Fourier series in z, we have (for $m \neq 0$)

$$A_{mn} I_m\left(\frac{n\pi b}{L}\right) = \frac{2}{\pi L}\int_0^{2\pi} d\phi \int_0^L dz\, V(\phi, z)\sin(m\phi)\sin\left(\frac{n\pi z}{L}\right)\ , \tag{14}$$

$$B_{mn} I_m\left(\frac{n\pi b}{L}\right) = \frac{2}{\pi L}\int_0^{2\pi} d\phi \int_0^L dz\, V(\phi, z)\cos(m\phi)\sin\left(\frac{n\pi z}{L}\right)\ . \tag{15}$$

Since for $m = 0$, $\sin(m\phi)$ vanishes, we only need to fix B_{0n},

$$B_{0n} I_0\left(\frac{n\pi b}{L}\right) = \frac{1}{\pi L}\int_0^{2\pi} d\phi \int_0^L dz\, V(\phi, z)\sin\left(\frac{n\pi z}{L}\right)\ . \tag{16}$$

Therefore, for $m \neq 0$,

$$A_{mn} = \left[I_m\left(\frac{n\pi b}{L}\right)\right]^{-1} \frac{2}{\pi L}\int_0^{2\pi} d\phi \int_0^L dz\, V(\phi, z)\sin(m\phi)\sin\left(\frac{n\pi z}{L}\right)\ , \tag{17}$$

$$B_{mn} = \left[I_m\left(\frac{n\pi b}{L}\right)\right]^{-1} \frac{2}{\pi L}\int_0^{2\pi} d\phi \int_0^L dz\, V(\phi, z)\cos(m\phi)\sin\left(\frac{n\pi z}{L}\right)\ , \tag{18}$$

and

$$B_{0n} = \left[I_0 \left(\frac{n\pi b}{L} \right) \right]^{-1} \frac{1}{\pi L} \int_0^{2\pi} d\phi \int_0^L dz \, V(\phi, z) \sin\left(\frac{n\pi z}{L} \right) \ . \tag{19}$$