Phys 5405

HW 3

2.3 (a) Using method of image, substitute the intersecting planes with a straight-line charge with constant linear charge density λ', located perpendicular to the $x - y$ plane at point (a, b). Since the system is translation invariant along z axis. We write the potential as a function of x and y. By symmetry, we can simply write down the answer,

$$\begin{aligned}\Phi(x,y) = &\frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{(x-x_0)^2+(y-y_0)^2} - \frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{(x+x_0)^2+(y-y_0)^2} \\ &- \frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{(x-x_0)^2+(y+y_0)^2} + \frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\ .\end{aligned}$$

Cancelling the constant terms, we obtain

$$\begin{aligned}\Phi(x,y) = &-\frac{\lambda}{4\pi\epsilon_0}\ln\left((x-x_0)^2+(y-y_0)^2\right) + \frac{\lambda}{4\pi\epsilon_0}\ln\left((x+x_0)^2+(y-y_0)^2\right) \\ &+ \frac{\lambda}{4\pi\epsilon_0}\ln\left((x-x_0)^2+(y+y_0)^2\right) - \frac{\lambda}{4\pi\epsilon_0}\ln\left((x+x_0)^2+(y+y_0)^2\right)\ .\end{aligned}$$

It is easily verified that the potential vanishes on the boundary. Now consider the tangential electric field. At $x = 0$, the tangential electric field should be along y axis.

$$\begin{aligned}E_y(x,y) = &-\partial_y\Phi(x,y) \\ = &\frac{\lambda}{4\pi\epsilon_0}\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{\lambda}{4\pi\epsilon_0}\frac{2(y-y_0)}{(x+x_0)^2+(y-y_0)^2} \\ &- \frac{\lambda}{4\pi\epsilon_0}\frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2} + \frac{\lambda}{4\pi\epsilon_0}\frac{2(y+y_0)}{(x+x_0)^2+(y+y_0)^2} \qquad (1)\end{aligned}$$

It is easily seen that at $x = 0$,

$$E_y(x=0,y) = 0\ . \qquad (2)$$

Similarly, at $y = 0$, the tangential electric field should be along x axis.

$$\begin{aligned}E_x(x,y) = &-\partial_x\Phi(x,y) \\ = &\frac{\lambda}{4\pi\epsilon_0}\frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{\lambda}{4\pi\epsilon_0}\frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2} \\ &- \frac{\lambda}{4\pi\epsilon_0}\frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2} + \frac{\lambda}{4\pi\epsilon_0}\frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2} \qquad (3)\end{aligned}$$

It is easily seen that at $y = 0$,

$$E_x(x, y = 0) = 0 \ . \tag{4}$$

2.3 (c) At $y = 0$, the surface charge density is

$$\sigma(x) = -\epsilon_0 \frac{\partial \Phi}{\partial y}\bigg|_{y=0} \tag{5}$$

Now using result in (1), we have

$$\sigma(x) = -\frac{\lambda}{\pi}\left(\frac{y_0}{(x-x_0)^2 + y_0^2} - \frac{y_0}{(x+x_0)^2 + y_0^2}\right) \ . \tag{6}$$

Then the total charge, per unit length in z on the plane $y = 0, x \geq 0$ is

$$\begin{aligned} Q_x &= \int_0^\infty dx\, \sigma(x) = -\frac{\lambda}{\pi}\int_0^\infty dx \left(\frac{y_0}{(x-x_0)^2 + y_0^2} - \frac{y_0}{(x+x_0)^2 + y_0^2}\right) \\ &= -\frac{\lambda}{\pi}\left(\tan^{-1}\left(\frac{x-x_0}{y_0}\right)\bigg|_0^\infty - \tan^{-1}\left(\frac{x+x_0}{y_0}\right)\bigg|_0^\infty\right) \\ &= -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right) \ . \end{aligned} \tag{7}$$

2.3 (d) At position far from the origin $\rho \gg \rho_0$, where

$$\rho = \sqrt{x^2 + y^2}, \quad \rho_0 = \sqrt{x_0^2 + y_0^2} \tag{8}$$

We can expand the potential

$$\begin{aligned}
\Phi(x,y) = & -\frac{\lambda}{4\pi\epsilon_0}\ln\left((x-x_0)^2+(y-y_0)^2\right)+\frac{\lambda}{4\pi\epsilon_0}\ln\left((x+x_0)^2+(y-y_0)^2\right) \\
& +\frac{\lambda}{4\pi\epsilon_0}\ln\left((x-x_0)^2+(y+y_0)^2\right)-\frac{\lambda}{4\pi\epsilon_0}\ln\left((x+x_0)^2+(y+y_0)^2\right) \ . \\
= & -\frac{\lambda}{4\pi\epsilon_0}\ln\left(1+\frac{\rho_0^2-2x_0x-2y_0y}{\rho^2}\right)+\frac{\lambda}{4\pi\epsilon_0}\ln\left(1+\frac{\rho_0^2+2x_0x-2y_0y}{\rho^2}\right) \\
& +\frac{\lambda}{4\pi\epsilon_0}\ln\left(1+\frac{\rho_0^2-2x_0x+2y_0y}{\rho^2}\right)-\frac{\lambda}{4\pi\epsilon_0}\ln\left(1+\frac{\rho_0^2+2x_0x+2y_0y}{\rho^2}\right) \ .
\end{aligned}$$

Now using $\ln(1+x) \approx x - \frac{x^2}{2}$ when x is very small, we have

$$\begin{aligned}
\Phi(x,y) \approx & -\frac{\lambda}{4\pi\epsilon_0}\left(\frac{\rho_0^2-2x_0x-2y_0y}{\rho^2}-\frac{\rho_0^2+2x_0x-2y_0y}{\rho^2}\right. \\
& \qquad \left.-\frac{\rho_0^2-2x_0x+2y_0y}{\rho^2}+\frac{\rho_0^2+2x_0x+2y_0y}{\rho^2}\right) \\
& +\frac{\lambda}{8\pi\epsilon_0}\left[\left(\frac{\rho_0^2-2x_0x-2y_0y}{\rho^2}\right)^2-\left(\frac{\rho_0^2+2x_0x-2y_0y}{\rho^2}\right)^2\right. \\
& \qquad \left.-\left(\frac{\rho_0^2-2x_0x+2y_0y}{\rho^2}\right)^2+\left(\frac{\rho_0^2+2x_0x+2y_0y}{\rho^2}\right)^2\right] \\
= & \frac{\lambda}{8\pi\epsilon_0\rho^4}\times 32x_0y_0xy = \frac{4\lambda}{\pi\epsilon_0}\frac{(x_0y_0)(xy)}{\rho^4} \ .
\end{aligned} \tag{9}$$

2.7 (a) The general Green function is of the form

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}') \ , \tag{10}$$

where $\nabla^2 F(\vec{x}, \vec{x}') = 0$. And the Dirichlet boundary condition implies that the Green function has to vanish on $z = 0$ surface. We can view the problem as a unit charge at x', with a conducting plane at $z = 0$. Suppose $\vec{x}' = (x', y', z')$, we can then put the image charge at $(x', y', -z')$ with negative unit charge. The potential is then given by

$$\Phi \tag{11}$$