

**Phys 5405**

HW 10

5.1, 5.3, 5.4(a), 5.7(a,b,c,d), 5.8, 5.10(a,b)

**5.1** We have the differential expression

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} . \quad (1)$$

Then

$$\mathbf{B} = \int d\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} . \quad (2)$$

It suffices to show that

$$\oint d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \nabla \Omega . \quad (3)$$

First, we can make use of the identity,

$$\boxed{\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \nabla' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \equiv \mathbf{V} .} \quad (4)$$

Then, consider each components

$$\boxed{\hat{\mathbf{x}}_i \cdot \oint d\mathbf{l}' \times \mathbf{V} = \oint d\mathbf{l}' \cdot (\mathbf{V} \times \hat{\mathbf{x}}_i) = \int_S d\mathbf{a}' \cdot (\nabla' \times (\mathbf{V} \times \hat{\mathbf{x}}_i)) .} \quad (5)$$

Now we evaluate  $\nabla' \times (\mathbf{V} \times \hat{\mathbf{x}}_i)$ , using the identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} , \quad (6)$$

so we have

$$\boxed{\nabla' \times (\mathbf{V} \times \hat{\mathbf{x}}_i) = -\hat{\mathbf{x}}_i(\nabla' \cdot \mathbf{V}) + (\hat{\mathbf{x}}_i \cdot \nabla')\mathbf{V} = (\hat{\mathbf{x}}_i \cdot \nabla')\mathbf{V} = \frac{\partial}{\partial x'_i} \mathbf{V} ,} \quad (7)$$

where we have used the fact that for  $\mathbf{x} \neq \mathbf{x}'$ ,

$$\nabla' \cdot \mathbf{V} = \nabla'^2 \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = 0 . \quad (8)$$

Finally, we have achieved that

$$\hat{\mathbf{x}}_i \cdot \oint d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \int_S d\mathbf{a}' \cdot \frac{\partial}{\partial x'_i} \left( \nabla' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) = -\frac{\partial}{\partial x_i} \int_S d\mathbf{a}' \cdot \nabla' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \quad (9)$$

From Jackson, the equation below (1.25), we have

$$d\mathbf{a}' \cdot \nabla' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -d\Omega . \quad (10)$$

Therefore,

$$\hat{\mathbf{x}}_i \cdot \oint d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \frac{\partial}{\partial x_i} \int_S d\Omega , \quad (11)$$

which implies that

$$\oint d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \nabla \Omega . \quad (12)$$