Phys 5405
HW 4 2.12 2.13 2.17 2.24

2.12 Starting with

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n) \ . \quad (1)$$

For potential inside the cylinder, we have to set $b_n = 0$ for $n \geq 0$. Then,

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \left[\cos\alpha_n \sin(n\phi) + \sin\alpha_n \cos(n\phi)\right] \ . \quad (2)$$

Since we have specified the potential on the surface of the cylinder of radius b, we can set $\rho = b$, and obtain,

$$\Phi(b,\phi) = a_0 + \sum_{n=1}^{\infty} a_n b^n \left[\cos\alpha_n \sin(n\phi) + \sin\alpha_n \cos(n\phi)\right] \ . \quad (3)$$

Then for $n \geq 1$, evaluating,

$$\int_0^{2\pi} \Phi(b,\phi) \sin(n\phi) d\phi = \pi a_n b^n \cos\alpha_n \ , \quad (4)$$

$$\int_0^{2\pi} \Phi(b,\phi) \cos(n\phi) d\phi = \pi a_n b^n \sin\alpha_n \ . \quad (5)$$

Therefore, we obtain for $n \geq 1$,

$$a_n = \frac{b^{-n}}{\pi\cos\alpha_n} \int_0^{2\pi} \Phi(\rho,\phi) \sin(n\phi) d\phi = \frac{b^{-n}}{\pi\sin\alpha_n} \int_0^{2\pi} \Phi(\rho,\phi) \cos(n\phi) d\phi \ . \quad (6)$$

For a_0, we have

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \Phi(b,\phi) \cos(0\phi) d\phi = \frac{1}{2\pi} \int_0^{2\pi} \Phi(b,\phi) d\phi \ . \quad (7)$$

we obtain

$$
\begin{aligned}
\Phi(\rho,\phi) &= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')d\phi' \\
&\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}\int_0^{2\pi}\Phi(b,\phi')\left[\sin(n\phi')\sin(n\phi)+\cos(n\phi')\cos(n\phi)\right]d\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')d\phi' + \frac{1}{\pi}\int_0^{2\pi}d\phi'\Phi(b,\phi')\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}\cos[n(\phi-\phi')] \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')d\phi' + \frac{1}{\pi}\int_0^{2\pi}d\phi'\Phi(b,\phi')\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}e^{in(\phi-\phi')} \qquad (8)
\end{aligned}
$$

We can evaluate the summation (using $\theta=\phi-\phi'$ for short),

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{\rho}{b}e^{i(\phi-\phi')}\right)^n &= \frac{\rho e^{i(\phi-\phi')}}{b-\rho e^{i(\phi-\phi')}} = \frac{\rho\cos\theta+i\rho\sin\theta}{b-\rho\cos\theta-i\rho\sin\theta} \\
&= \frac{(\rho\cos\theta+i\rho\sin\theta)(b-\rho\cos\theta+i\rho\sin\theta)}{(b-\rho\cos\theta)^2+\rho^2\sin^2\theta} \qquad (9)
\end{aligned}
$$

Therefore its real part is given by

$$
\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{\rho}{b}e^{i(\phi-\phi')}\right)^n = \frac{b\rho\cos\theta-\rho^2}{b^2+\rho^2-2b\rho\cos\theta}\ . \qquad (10)
$$

Therefore, for potential inside the cylinder,

$$
\begin{aligned}
\Phi(\rho,\phi) &= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\left(1+\frac{2b\rho\cos\theta-2\rho^2}{b^2+\rho^2-2b\rho\cos\theta}\right)d\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')}d\phi'\ . \qquad (11)
\end{aligned}
$$

For potential outside the cylinder, we have to set $a_n=0, n\geq 1$ and $b_0=0$, we just need to swap b and ρ in the fraction in the above expression. Therefore, for potential outside the cylinder,

$$
\Phi(\rho,\phi) = -\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')}d\phi'\ . \qquad (12)
$$