Phys 5405

HW 6

3.13 3.14 3.16 b,c,d

3.13 The Green function for a spherical shell bounded by $r = a$ and $r = b$ is

$$G(\vec{x}, \vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) . \tag{1}$$

Since we have azimuthal symmetry, we only need to consider $m = 0$, therefore,

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) . \tag{2}$$

Now the potential is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}') G(\vec{x}, \vec{x}') d^3x' + \frac{1}{4\pi} \int_S \left[G(\vec{x}, \vec{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'}\right] da' . \tag{3}$$

In between the two spheres, the charge density is zero, and the Green function vanishes on the boundary. We have

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \int_S \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} da' . \tag{4}$$

Then we have to calculate the derivative of the Green function. For the boundary at radius a, we have $r > r'$,

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) . \tag{5}$$

Calculate its derivative and evaluate it at $r' = a$

$$\begin{aligned}
\left.\frac{\partial G(\vec{x}, \vec{x}')}{\partial n'}\right|_{r'=a} &= -\left.\frac{\partial G(\vec{x}, \vec{x}')}{\partial r'}\right|_{r'=a} \\
&= -\sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left(l r'^{l-1} + (l+1)\frac{a^{2l+1}}{r'^{l+2}}\right)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\Bigg|_{r'=a} \\
&= -\sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} a^{l-1}(2l+1)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) .
\end{aligned} \tag{6}$$

Then

$$\begin{aligned}
\int_{S_a} \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} da' &= \int_0^{\pi/2} a^2 \sin\theta' d\theta' \int_0^{2\pi} d\phi' \, V \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \\
&= -2\pi V \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{1 - (a/b)^{2l+1}} a^{l+1}(2l+1)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) \int_0^{\pi/2} P_l(\cos\theta') \sin\theta' d\theta'
\end{aligned} \tag{7}$$

Now evaluate for $l \neq 0$,

$$\int_0^{\pi/2} P_l(\cos\theta')\sin\theta' d\theta' = \int_0^1 P_l(x)dx = \frac{1}{2l+1}\left[P_{l-1}(0) - P_{l+1}(0)\right] \tag{8}$$

For $l = 0$, we simply have

$$\int_0^{\pi/2} P_l(\cos\theta')\sin\theta' d\theta' = \int_0^1 P_l(x)dx = 1 \ . \tag{9}$$

Therefore,

$$\begin{aligned}
&\int_{S_a} \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da' \\
&= -2\pi V\frac{a}{1-(a/b)}\left(\frac{1}{r}-\frac{1}{b}\right) - 2\pi V\sum_{l=1}^{\infty}\frac{P_l(\cos\theta)a^{l+1}}{1-(a/b)^{2l+1}}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\left[P_{l-1}(0)-P_{l+1}(0)\right] \ .
\end{aligned}$$

Similarly, for boundary at radius b, we have $r < r'$,

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\frac{P_l(\cos\theta')P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) \ . \tag{10}$$

Calculate its derivative and evaluate it at $r' = b$

$$\begin{aligned}
\left.\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right|_{r'=b} &= \left.\frac{\partial G(\vec{x},\vec{x}')}{\partial r'}\right|_{r'=b} \\
&= \left.\sum_{l=0}^{\infty}\frac{P_l(\cos\theta')P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left(-(l+1)\frac{1}{r'^{l+2}} - l\frac{r'^{l-1}}{b^{2l+1}}\right)\right|_{r'=b} \\
&= -\sum_{l=0}^{\infty}\frac{P_l(\cos\theta')P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)(2l+1)b^{-l-2} \ . \qquad (11)
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{S_b} \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da' = \int_{\pi/2}^{\pi} b^2\sin\theta' d\theta'\int_0^{2\pi}d\phi'\, V\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} \\
&\quad= -2\pi V\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)}{1-(a/b)^{2l+1}}b^{-l}(2l+1)\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\int_{\pi/2}^{\pi}P_l(\cos\theta')\sin\theta' d\theta' \\
&\quad= -2\pi V\frac{1}{1-a/b}(1-a/r) - 2\pi V\sum_{l=1}^{\infty}(-1)^l\frac{P_l(\cos\theta)b^{-l}}{1-(a/b)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left[P_{l-1}(0)-P_{l+1}(0)\right]
\end{aligned}$$

Then the potential between the two spheres are simply given by

$$\begin{aligned}\Phi(\vec{x}) = & -\frac{1}{4\pi}\int_{S_a}\Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da' - \frac{1}{4\pi}\int_{S_b}\Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da' \\ = & \frac{V}{2}\frac{1}{1-a/b}\left(\frac{a}{r}-\frac{a}{b}\right)+\frac{V}{2}\frac{1}{1-a/b}\left(1-\frac{a}{r}\right) \\ & +\frac{V}{2}\sum_{l=1}^{\infty}[P_{l-1}(0)-P_{l+1}(0)]P_l(\cos\theta) \\ & \times\left[\frac{a^{l+1}}{1-(a/b)^{2l+1}}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)+(-1)^l\frac{b^{-l}}{1-(a/b)^{2l+1}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\right] \\ = & \frac{V}{2}+\frac{V}{2}\sum_{l=1}^{\infty}(P_{l-1}(0)-P_{l+1}(0))P_l(\cos\theta)\left(\frac{(-1)^l b^{l+1}-a^{l+1}}{b^{2l+1}-a^{2l+1}}r^l+\frac{(-1)^l b^{-l}-a^{-l}}{b^{-(2l+1)}-a^{-(2l+1)}}r^{-(l+1)}\right)\ .\end{aligned}$$

This is exactly what I got in homework 5, Jackson problem 3.1.