**Phys 5405**
HW 7
3.13 3.14 3.16 b,c,d

**3.13** The Green function for a spherical shell bounded by $r = a$ and $r = b$ is

$$G(\vec{x}, \vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]} \left( r^l_< - \frac{a^{2l+1}}{r^{l+1}_<} \right) \left( \frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}} \right) . \tag{1}$$

Since we have azimuthal symmetry, we only need to consider $m = 0$, therefore,

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left( r^l_< - \frac{a^{2l+1}}{r^{l+1}_<} \right) \left( \frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}} \right) . \tag{2}$$

Now the potential is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}') G(\vec{x}, \vec{x}') d^3x' + \frac{1}{4\pi} \int_S \left[ G(\vec{x}, \vec{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \right] da' . \tag{3}$$

In between the two spheres, the charge density is zero, and the Green function vanishes on the boundary. We have

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \int_S \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} da' . \tag{4}$$

Then we have to calculate the derivative of the Green function. For the boundary at radius $a$, we have $r > r'$,

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left( r'^l - \frac{a^{2l+1}}{r'^{l+1}} \right) \left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) . \tag{5}$$

Calculate its derivative and evaluate it at $r' = a$

$$\begin{aligned} \left.\frac{\partial G(\vec{x}, \vec{x}')}{\partial n'}\right|_{r'=a} &= -\left.\frac{\partial G(\vec{x}, \vec{x}')}{\partial r'}\right|_{r'=a} \\ &= -\sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left( l r'^{l-1} + (l+1) \frac{a^{2l+1}}{r'^{l+2}} \right) \left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \Bigg|_{r'=a} \\ &= -\sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} a^{l-1} (2l+1) \left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) . \end{aligned} \tag{6}$$

Then

$$\begin{aligned} \int_{S_a} \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} da' &= \int_0^{\pi/2} a^2 \sin\theta' d\theta' \int_0^{2\pi} d\phi' \, V \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \\ &= -2\pi V \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{1 - (a/b)^{2l+1}} a^{l+1} (2l+1) \left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \int_0^{\pi/2} P_l(\cos\theta') \sin\theta' d\theta' \end{aligned} \tag{7}$$

Now evaluate for $l \neq 0$,

$$\int_0^{\pi/2} P_l(\cos\theta')\sin\theta' d\theta' = \int_0^1 P_l(x)dx = \frac{1}{2l+1}\left[P_{l-1}(0) - P_{l+1}(0)\right] \tag{8}$$

For $l = 0$, we simply have

$$\int_0^{\pi/2} P_l(\cos\theta')\sin\theta' d\theta' = \int_0^1 P_l(x)dx = 1 \ . \tag{9}$$

Therefore,

$$\int_{S_a} \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da'$$
$$= -2\pi V\frac{a}{1-(a/b)}\left(\frac{1}{r}-\frac{1}{b}\right) - 2\pi V\sum_{l=1}^{\infty}\frac{P_l(\cos\theta)a^{l+1}}{1-(a/b)^{2l+1}}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\left[P_{l-1}(0)-P_{l+1}(0)\right] \ .$$

Similarly, for boundary at radius $b$, we have $r < r'$,

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\frac{P_l(\cos\theta')P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) \ . \tag{10}$$

Calculate its derivative and evaluate it at $r' = b$

$$\left.\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right|_{r'=b} = \left.\frac{\partial G(\vec{x},\vec{x}')}{\partial r'}\right|_{r'=b}$$
$$= \sum_{l=0}^{\infty}\frac{P_l(\cos\theta')P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left.\left(-(l+1)\frac{1}{r'^{l+2}} - l\frac{r'^{l-1}}{b^{2l+1}}\right)\right|_{r'=b}$$
$$= -\sum_{l=0}^{\infty}\frac{P_l(\cos\theta')P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)(2l+1)b^{-l-2} \ . \tag{11}$$

Then

$$\int_{S_b}\Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da' = \int_{\pi/2}^{\pi} b^2\sin\theta' d\theta'\int_0^{2\pi}d\phi'\, V\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}$$
$$= -2\pi V\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)}{1-(a/b)^{2l+1}}b^{-l}(2l+1)\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\int_{\pi/2}^{\pi}P_l(\cos\theta')\sin\theta' d\theta'$$
$$= -2\pi V\frac{1}{1-a/b}(1-a/r) - 2\pi V\sum_{l=1}^{\infty}(-1)^l\frac{P_l(\cos\theta)b^{-l}}{1-(a/b)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left[P_{l-1}(0)-P_{l+1}(0)\right]$$

Then the potential between the two spheres are simply given by

$$
\begin{aligned}
\Phi(\vec{x}) = & -\frac{1}{4\pi}\int_{S_a}\Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da' - \frac{1}{4\pi}\int_{S_b}\Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da' \\
= & \frac{V}{2}\frac{1}{1-a/b}\left(\frac{a}{r}-\frac{a}{b}\right) + \frac{V}{2}\frac{1}{1-a/b}\left(1-\frac{a}{r}\right) \\
& + \frac{V}{2}\sum_{l=1}^{\infty}[P_{l-1}(0)-P_{l+1}(0)]P_l(\cos\theta) \\
& \times \left[\frac{a^{l+1}}{1-(a/b)^{2l+1}}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right) + (-1)^l\frac{b^{-l}}{1-(a/b)^{2l+1}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\right] \\
= & \frac{V}{2}+\frac{V}{2}\sum_{l=1}^{\infty}(P_{l-1}(0)-P_{l+1}(0))P_l(\cos\theta)\left(\frac{(-1)^l b^{l+1}-a^{l+1}}{b^{2l+1}-a^{2l+1}}r^l + \frac{(-1)^l b^{-l}-a^{-l}}{b^{-(2l+1)}-a^{-(2l+1)}}r^{-(l+1)}\right) \ .
\end{aligned}
$$

This is exactly what I got in homework 5, Jackson problem 3.1.

**3.14 (a)** Now we write the charge density as a function of the position vector, in terms of delta functions, for $|\vec{x}'| \leq d$,

$$\rho(\vec{x}') = \frac{3Q}{4d^3}(d^2 - r'^2)\frac{1}{2\pi r'^2}[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)] \ . \tag{12}$$

Otherwise, for $|\vec{x}| > d$, the charge density is zero. The Green function inside a sphere of radius $b$ with azimuthal symmetry is given by,

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \ . \tag{13}$$

Since on the boundary, the potential vanishes, we have

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x}, \vec{x}')d^3x' \ . \tag{14}$$

For $r > d$, we have $r_< = r'$ and $r_> = r$. Then the potential is

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{1}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) \\
&\quad \int_0^d dr' \int_{-1}^{1} d(\cos\theta')(d^2 - r'^2)r'^l P_l(\cos\theta')[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)] \\
&= \frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^d dr'(d^2 - r'^2)r'^l \\
&= \frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\frac{2d^{l+3}}{(l+1)(l+3)} \\
&= \frac{3Q}{4\pi\epsilon_0}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\frac{d^l}{(l+1)(l+3)} \ .
\end{aligned} \tag{15}$$

For $r < d$, we have two regimes, $r' < r$ and $r < r' < d$. Then in the first regime, its contribution to the potential is

$$\begin{aligned}
\Phi_1(\vec{x}) &= \frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^r dr'(d^2 - r'^2)r'^l \\
&= \frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\left(\frac{d^2r^{l+1}}{l+1} - \frac{r^{l+3}}{l+3}\right) \ .
\end{aligned} \tag{16}$$

In the second regime, its contribution to the potential is

$$\Phi_2(\vec{x}) = \frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)r^l\int_r^d dr'(d^2 - r'^2)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) \ . \tag{17}$$

Denote the integral by

$$\mathcal{I}_l = \int_r^d dr'(d^2 - r'^2)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) \ . \tag{18}$$

When $l = 0$, it's

$$\mathcal{I}_0 = d^2 \ln \frac{d}{r} - \frac{d^2}{b}(d - r) - \frac{1}{2}(d^2 - r^2) + \frac{1}{3b}(d^3 - r^3) \ . \tag{19}$$

When $l = 2$, it's

$$\mathcal{I}_2 = -\ln \frac{d}{r} - \frac{d^2}{2}(d^{-2} - r^{-2}) - \frac{d^2}{3b^5}(d^3 - r^3) + \frac{1}{5b^5}(d^5 - r^5) \ . \tag{20}$$

When $l \geq 4$, it'

$$\mathcal{I}_l = -\frac{d^2}{l}(d^{-l} - r^{-l}) - \frac{d^2}{(l+1)b^{2l+1}}(d^{l+1} - r^{l+1}) + \frac{1}{l-2}(d^{2-l} - r^{2-l}) + \frac{1}{(l+3)b^{2l+1}}(d^{l+3} - r^{l+3}) \ . \tag{21}$$

Therefore, for $r < d$

$$\Phi(\vec{x}) = \Phi_1(\vec{x}) + \Phi_2(\vec{x}) \ . \tag{22}$$

**3.14 (b)** For potential near the boundary, we have $r > d$ and

$$\Phi(\vec{x}) = \frac{3Q}{4\pi\epsilon_0} \sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta) \left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \frac{d^l}{(l+1)(l+3)} \ . \tag{23}$$

Then, the surface charge density is given by

$$\sigma = \epsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=b} = -\frac{3Q}{4\pi} \sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta) \frac{(2l+1)}{(l+1)(l+3)} \frac{d^l}{b^{l+2}} \ . \tag{24}$$

**3.14 (c)** In the limit that $d \ll b$, then also $d << r$. The potential in (a) is

$$\Phi(\vec{x}) = \frac{3Q}{4\pi\epsilon_0} \sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta) \left( \frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \frac{d^l}{(l+1)(l+3)} \ . \tag{25}$$

Taking the limit, only the $l = 0$ term matters,

$$\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0} \left( \frac{1}{r} - \frac{1}{b} \right) \ . \tag{26}$$

It consists the point charge potential and the induced surface charge potential. Similarly, the surface charge density is

$$\sigma = -\frac{Q}{4\pi b^2}, \tag{27}$$

from which we can see that the charge is uniformly distributed on the sphere with a total charge $Q$.

**3.16 (b)** Obtain the following expansion,

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k(z_>-z_<)} \tag{28}$$

The Green function satisfy

$$\nabla^2 G(\vec{x},\vec{x}') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') \ . \tag{29}$$

Now use the result from part (a), the delta functions can be written as

$$\frac{1}{\rho}\delta(\rho-\rho') = \int_0^\infty kJ_m(k\rho)J_m(k\rho')dk \tag{30}$$

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \tag{31}$$

Then we expand the Green function in a similar fashion,

$$G(\vec{x},\vec{x}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')} kJ_m(k\rho)J_m(k\rho')g(k,z,z') \ . \tag{32}$$

Then substitution into (29) leads to

$$\begin{aligned}\nabla^2 G(\vec{x},\vec{x}') = {} & \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\phi-\phi')} kJ_m(k\rho)J_m(k\rho')g(k,z,z')\left(\frac{m^2}{\rho^2}-k^2-\frac{m^2}{\rho^2}\right) \\ & + \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')} kJ_m(k\rho)J_m(k\rho')\partial_z^2 g(k,z,z') \ ,\end{aligned} \tag{33}$$

which leads to

$$\frac{d^2}{dz^2}g - k^2 g = -4\pi\delta(z-z') \ . \tag{34}$$

For $z \neq z'$, we can solve that $g$ is proportional to $e^{\pm kz}$. Considering finiteness, we assume

$$g(k,z,z') = \begin{cases} Ae^{kz} \ , & z < z' \\ Be^{-kz} \ , & z > z' \end{cases} \ . \tag{35}$$

Continuity at $z = z'$ implies that

$$Ae^{kz'} = Be^{-kz'} \ . \tag{36}$$

Also, integrating (34) gives

$$\lim_{\epsilon\to 0}\left(\left.\frac{dg}{dz}\right|_{z=z'+\epsilon} - \left.\frac{dg}{dz}\right|_{z=z'-\epsilon}\right) = -4\pi \ , \tag{37}$$

which leads to

$$-kBe^{-kz'} - kAe^{kz'} = -4\pi \tag{38}$$

Now with (36) and (38), we can solve for

$$A = \frac{2\pi}{k}e^{-kz'}, \quad B = \frac{2\pi}{k}e^{kz'} \tag{39}$$

Therefore,

$$g(k, z, z') = \begin{cases} \frac{2\pi}{k}e^{-k(z'-z)} \ , & z < z' \\ \frac{2\pi}{k}e^{-k(z-z')} \ , & z > z' \end{cases} = \frac{2\pi}{k}e^{-k(z_> - z_<)} \ . \tag{40}$$

And now

$$\frac{1}{|\vec{x} - \vec{x}'|} = G(\vec{x}, \vec{x}') = \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\, e^{im(\phi - \phi')} J_m(k\rho) J_m(k\rho') e^{-k(z_> - z_<)} \ . \tag{41}$$

**3.16 (c)** The position vectors are given by

$$\vec{x} = (\rho\cos\phi, \rho\sin\phi, z), \quad \vec{x}' = (\rho'\cos\phi', \rho'\sin\phi', z') \tag{42}$$

Then,

$$\begin{aligned} |\vec{x}-\vec{x}'|^2 &= (\rho\cos\phi - \rho'\cos\phi')^2 + (\rho\sin\phi - \rho'\sin\phi')^2 + (z-z')^2 \\ &= \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi') + (z-z')^2 \end{aligned} \tag{43}$$

So we have

$$\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}} = \sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')e^{-k(z_>-z_<)}$$

Taking $\rho'=0, z'=0$ while $\phi'$ can be arbitrary, LHS is $\frac{1}{\sqrt{\rho^2+z^2}}$. In right hand side, because of azimuthal symmetry in $\phi'$, we only need to consider $m=0$. When $z>0$, $z_>=z, z_<=0$, when $z<0$, $z_>=0, z_<=z$. Therefore, $z_>-z_<=|z|$. So

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^\infty e^{-k|z|}J_0(k\rho)dk\ . \tag{44}$$

Now we replace $\rho^2$ by $\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')$, then we have

$$\sum_{m=-\infty}^{\infty}\int_0^\infty dk e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')e^{-k|z|} = \int_0^\infty e^{-k|z|}J_0(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}) \tag{45}$$

Therefore,

$$J_0(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}) = \sum_{m=-\infty}^{\infty} e^{im\phi}J_m(k\rho)J_m(k\rho')\ . \tag{46}$$

First, the asymptotic behavior of $J_\alpha(z)$ as $|z|$ goes to infinity is

$$J_\alpha(z) \to \frac{1}{\sqrt{2\pi z}}e^{-i\left(z-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)}\ , \quad 0<\arg z<\pi\ . \tag{47}$$

Therefore, we set $\rho'\to i\rho'$ and then let $\rho'\to\infty$, the right hand side of (46) now becomes

$$\text{RHS} \to \sum_{m=\infty}^{\infty} e^{im\phi}J_m(k\rho)\frac{1}{\sqrt{2\pi ik\rho'}}e^{-i\left(ik\rho'-\frac{m\pi}{2}-\frac{\pi}{4}\right)}\ . \tag{48}$$

And when $\rho'\to i\rho'$ and $\rho'\to\infty$,

$$k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi} \to k\rho'\left(i-\frac{\rho}{\rho'}\cos\phi\right) = ik\rho' - k\rho\cos\phi\ . \tag{49}$$

Then the left hand side becomes

$$\text{LHS} \rightarrow \frac{1}{\sqrt{2\pi(ik\rho' - k\rho\cos\phi)}} e^{-i\left(ik\rho' - k\rho\cos\phi - \frac{\pi}{4}\right)} \rightarrow \frac{1}{\sqrt{2\pi ik\rho'}} e^{-i\left(ik\rho' - k\rho\cos\phi - \frac{\pi}{4}\right)}. \tag{50}$$

Now equating the transformed LHS and RHS yields

$$e^{ik\rho\cos\phi} = \sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) e^{im\pi/2} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(k\rho)\ . \tag{51}$$

**3.16 (d)** Since now we have

$$e^{ix\cos\phi} = \sum_{n=-\infty}^{\infty} i^n e^{in\phi} J_n(x) \ . \tag{52}$$

Then

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\phi - im\phi} d\phi &= \sum_{n=-\infty}^{\infty} i^n \left(\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\phi} d\phi\right) J_n(x) \\ &= \sum_{n=-\infty}^{\infty} i^n \delta_{mn} J_n(x) = i^m J_m(x) \ . \end{aligned} \tag{53}$$

Therefore,

$$J_m(x) = \frac{1}{2\pi i^m}\int_0^{2\pi} e^{ix\cos\phi - im\phi} d\phi \ . \tag{54}$$