**Phys 5405**

HW 9

4.2 4.7(a,b) 4.8(a)

**4.2** A point dipole with dipole moment $\mathbf{p}$ is located at the point $\mathbf{x}_0$. From the properties of the derivative of a Dirac delta function, show that for calculation of the potential $\Phi$ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{\text{eff}}(\mathbf{x}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{x}-\mathbf{x}_0)$$

**4.2** The potential from the dipole is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \vec{p}(\vec{x}') \cdot \nabla' \left( \frac{1}{|\vec{x} - \vec{x}'|} \right) \ . \tag{1}$$

The distribution of the dipole is a delta function,

$$\vec{p}(\vec{x}') = \vec{p}\,\delta(\vec{x}' - \vec{x}_0) \ . \tag{2}$$

Plug it into the formula of potential and integrate by parts,

$$\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0} \int d^3x' \vec{p} \cdot \nabla' \delta(\vec{x}' - \vec{x}_0) \frac{1}{|\vec{x} - \vec{x}'|} \equiv \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho_{\text{eff}}(\vec{x}')}{|\vec{x} - \vec{x}'|} \ . \tag{3}$$

Therefore, the dipole can be described by an effective charge density,

$$\boxed{\rho_{\text{eff}}(\vec{x}) = -\vec{p} \cdot \nabla \delta(\vec{x} - \vec{x}_0) \ .} \tag{4}$$

**4.7** A localized distribution of charge has a charge density

$$\rho(\mathbf{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta$$

**(a)** Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.
**(b)** Determine the potential explicitly at any point in space, and show that near the origin, correct to $r^2$ inclusive,

$$\Phi(\mathbf{r}) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120} P_2(\cos\theta)\right]$$

**4.7 (a)** The multipole expansion is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\ , \tag{5}$$

with multipole moments,

$$\begin{aligned} q_{lm} &= \int Y^*_{lm}(\theta,\phi)r^l\rho(\vec{x})d^3x \\ &= \int Y^*_{lm}(\theta,\phi)r^l\left(\frac{1}{64\pi}r^2e^{-r}\sin^2\theta\right)r^2\sin\theta dr d\theta d\phi \end{aligned} \tag{6}$$

The spherical harmonics are related to the associated Legendre polynomials as

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}\ . \tag{7}$$

Consider the integral $\int_0^{2\pi}e^{im\phi}d\phi$, it vanishes for $m \neq 0$. Therefore,

$$\begin{aligned} q_{lm} &= \delta_{m,0}\int\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)r^l\left(\frac{1}{64\pi}r^2e^{-r}\sin^2\theta\right)r^2\sin\theta dr d\theta d\phi \\ &= \frac{2\pi}{64\pi}\sqrt{\frac{2l+1}{4\pi}}\delta_{m,0}\left(\int dr\, r^{l+4}e^{-r}\right)\left(\int d\theta\, P_l(\cos\theta)\sin^3\theta\right) \end{aligned} \tag{8}$$

By the definition of Gamma functions,

$$\boxed{\int_0^{\infty}dr\, r^{l+4}e^{-r} = \Gamma(l+5) = (l+4)!} \tag{9}$$

For another integral, we change the variable as $x = \cos\theta$, and we also make use of the identity, $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$, then the integral is now given by

$$\boxed{\int_{-1}^{1}dx\, P_l(x)(1-x^2) = \frac{2}{3}\int_{-1}^{1}dx\, P_l(x)(1-P_2(x)) = \frac{2}{3}\left(2\delta_{l,0}-\frac{2}{5}\delta_{l,2}\right)\ .} \tag{10}$$

Therefore, the only non-zero multipole moments are

$$\boxed{q_{00} = \frac{1}{2}\frac{1}{\sqrt{\pi}}\ , \quad q_{20} = -3\sqrt{\frac{5}{\pi}}\ .} \tag{11}$$

Therefore, the potential is given by

$$\boxed{\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}-\frac{6P_2(\cos\theta)}{r^3}\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{P_0(\cos\theta)}{r}-\frac{6P_2(\cos\theta)}{r^3}\right]\ .} \tag{12}$$

**4.8** A very long, right circular, cylindrical shell of dielectric constant $\epsilon/\epsilon_0$ and inner and outer radii $a$ and $b$, respectively, is placed in a previously uniform electric field $E_0$ with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

**(a)** Determine the potential and electric field in the three regions, neglecting end effects.