Phys 5405

HW 10

5.1, 5.3, 5.4(a), 5.7(a,b,c,d), 5.8, 5.10(a,b)

5.1 We have the differential expression

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\, d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \ . \tag{1}$$

Then

$$\mathbf{B} = \int d\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \ . \tag{2}$$

It suffices to show that

$$\oint d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla\Omega \ . \tag{3}$$

First, we can make use of the identity,

$$\boxed{\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \equiv \mathbf{V} \ .} \tag{4}$$

Then, consider each components

$$\boxed{\hat{\mathbf{x}}_i \cdot \oint d\mathbf{l}' \times \mathbf{V} = \oint d\mathbf{l}' \cdot (\mathbf{V} \times \hat{\mathbf{x}}_i) = \int_S d\mathbf{a}' \cdot (\nabla' \times (\mathbf{V} \times \hat{\mathbf{x}}_i)) \ .} \tag{5}$$

Now we evaluate $\nabla' \times (\mathbf{V} \times \hat{\mathbf{x}}_i)$, using the identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \ , \tag{6}$$

so we have

$$\boxed{\nabla' \times (\mathbf{V} \times \hat{\mathbf{x}}_i) = -\hat{\mathbf{x}}_i(\nabla' \cdot \mathbf{V}) + (\hat{\mathbf{x}}_i \cdot \nabla')\mathbf{V} = (\hat{\mathbf{x}}_i \cdot \nabla')\mathbf{V} = \frac{\partial}{\partial x_i'}\mathbf{V} \ ,} \tag{7}$$

where we have used the fact that for $\mathbf{x} \neq \mathbf{x}'$,

$$\nabla' \cdot \mathbf{V} = \nabla'^2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = 0 \ . \tag{8}$$

Finally, we have achieved that

$$\hat{\mathbf{x}}_i \cdot \oint d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \int_S d\mathbf{a}' \cdot \frac{\partial}{\partial x_i'}\left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right) = -\frac{\partial}{\partial x_i}\int_S d\mathbf{a}' \cdot \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{9}$$

From Jackson, the equation below (1.25), we have

$$d\mathbf{a}' \cdot \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -d\Omega \ . \tag{10}$$

Therefore,

$$\boxed{\hat{\mathbf{x}}_i \cdot \oint d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \frac{\partial}{\partial x_i} \int_S d\Omega \ ,} \tag{11}$$

which implies that

$$\oint d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla\Omega \ . \tag{12}$$

5.3 In Jackson section 5.5, it already gives the magnetic induction for a circular current loop, For the position located on the symmetric axis of the loop, the magnetic induction is given by

$$B_z = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \ , \tag{13}$$

where a is the radius of the loop and z is the distance between the center of the loop and the position where we measure the magnetic induction.

In this case, we have

$$B_z = \frac{\mu_0 N I}{2} a^2 \int dz \frac{1}{(a^2 + z^2)^{3/2}} \ . \tag{14}$$

Instead of using variable z, we use the variable θ such that

$$\tan\theta = a/z \ . \tag{15}$$

Then

$$\boxed{B_z = \frac{\mu_0 N I}{2} \int_{\pi-\theta_1}^{\theta_2} d(\cos\theta) = \frac{\mu_0 N I}{2} (\cos\theta_2 + \cos\theta_1) \ .} \tag{16}$$

5.4 (a) In a current-free region, the magnetic induction satisfies

$$\nabla \cdot \mathbf{B} = 0 \ , \quad \nabla \times \mathbf{B} = 0 \ . \tag{17}$$

Written in components, we have

$$\frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial\rho} + \frac{\partial B_z}{\partial z} = 0 \ , \quad \frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial\rho} = 0 \ . \tag{18}$$

Near the axis, the axial component of the magnetic induction can be expanded as

$$B_z(\rho, z) \approx B_z(0, z) + \frac{\partial B_z(0, z)}{\partial\rho}\rho + \frac{1}{2}\frac{\partial^2 B_z(0, z)}{\partial\rho^2}\rho^2 + \frac{1}{6}\frac{\partial^3 B_z(0, z)}{\partial\rho^3}\rho^3 + \cdots \tag{19}$$

The radial component of the magnetic induction can also be expanded as

$$B_\rho(\rho, z) \approx B_\rho(0, z) + \frac{\partial B_\rho(0, z)}{\partial\rho}\rho + \frac{1}{2}\frac{\partial^2 B_\rho(0, z)}{\partial\rho^2}\rho^2 + \frac{1}{6}\frac{\partial^3 B_\rho(0, z)}{\partial\rho^3}\rho^3 + \cdots \tag{20}$$

Next we just plug these two expansions into (18) and equating the coefficients.

$$\frac{\partial B_\rho}{\partial z} = \frac{\partial B_\rho(0, z)}{\partial z} + \frac{\partial^2 B_\rho(0, z)}{\partial z\partial\rho}\rho + \frac{1}{2}\frac{\partial^3 B_\rho(0, z)}{\partial z\partial\rho^2}\rho^2 + \frac{1}{6}\frac{\partial^4 B_\rho(0, z)}{\partial z\partial\rho^3}\rho^3 + \cdots \tag{21}$$

$$\frac{\partial B_z}{\partial\rho} = \frac{\partial B_z(0, z)}{\partial\rho} + \frac{\partial^2 B_z(0, z)}{\partial\rho^2}\rho + \frac{1}{2}\frac{\partial^3 B_z(0, z)}{\partial\rho^3}\rho^2 + \cdots \tag{22}$$

and

$$-\frac{\partial B_z}{\partial z} = -\frac{\partial B_z(0, z)}{\partial z} - \frac{\partial^2 B_z(0, z)}{\partial z\partial\rho}\rho - \frac{1}{2}\frac{\partial^3 B_z(0, z)}{\partial z\partial\rho^2}\rho^2 - \frac{1}{6}\frac{\partial^4 B_z(0, z)}{\partial z\partial\rho^3}\rho^3 + \cdots \tag{23}$$

$$\frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial\rho} = \frac{B_\rho(0, z)}{\rho} + 2\frac{\partial B_\rho(0, z)}{\partial\rho} + \frac{3}{2}\frac{\partial^2 B_\rho(0, z)}{\partial\rho^2}\rho + \frac{2}{3}\frac{\partial^3 B_\rho(0, z)}{\partial\rho^3}\rho^2 + \cdots \tag{24}$$

We have

$$B_\rho(0, z) = 0 \ , \tag{25}$$

$$\frac{\partial B_\rho(0, z)}{\partial\rho} = -\frac{1}{2}\frac{\partial B_z(0, z)}{\partial z} \ , \tag{26}$$

$$\frac{\partial^2 B_\rho(0, z)}{\partial\rho^2} = -\frac{2}{3}\frac{\partial^2 B_z(0, z)}{\partial z\partial\rho} = -\frac{2}{3}\frac{\partial^2 B_\rho(0, z)}{\partial z^2} = 0 \ , \tag{27}$$

$$\frac{\partial^3 B_\rho(0, z)}{\partial\rho^3} = -\frac{3}{4}\frac{\partial^3 B_z(0, z)}{\partial z\partial\rho^2} = -\frac{3}{4}\frac{\partial^3 B_\rho(0, z)}{\partial z^2\partial\rho} = \frac{3}{8}\frac{\partial^3 B_z(0, z)}{\partial z^3} \ . \tag{28}$$

We also have

$$\frac{\partial B_z(0, z)}{\partial\rho} = \frac{\partial B_\rho(0, z)}{\partial z} = 0 \ , \tag{29}$$

$$\frac{\partial^2 B_z(0, z)}{\partial\rho^2} = \frac{\partial^2 B_\rho(0, z)}{\partial z\partial\rho} = -\frac{1}{2}\frac{\partial^2 B_z(0, z)}{\partial z^2} \ , \tag{30}$$

$$\frac{\partial^3 B_z(0, z)}{\partial\rho^3} = \frac{\partial^3 B_\rho(0, z)}{\partial z\partial\rho^2} = -\frac{2}{3}\frac{\partial^3 B_\rho(0, z)}{\partial z^3} = 0 \ . \tag{31}$$

Since $B_\rho(0,z)=0$, all its derivatives with respect to z gives zero. Therefore,

$$\boxed{\begin{aligned} B_z(\rho,z) &\approx B_z(0,z) - \frac{1}{4}\frac{\partial^2 B_z(0,z)}{\partial z^2}\rho^2 + \cdots \qquad (32)\\ B_\rho(\rho,z) &\approx -\frac{1}{2}\frac{\partial B_z(0,z)}{\partial z}\rho - \frac{1}{16}\frac{\partial^3 B_z(0,z)}{\partial z^3}\rho^3 + \cdots \qquad (33)\end{aligned}}$$

5.7 (a)