Phys 5405
HW 11
5.13, 5.17, 5.19(a), 5.20, 5.21, 5.26, 5.27

5.13 We can write down the current density,

$$\mathbf{J}(\mathbf{x}) = \sigma\omega a \sin\theta\delta(r-a)\hat{\phi} \ . \tag{1}$$

Then the vector potential is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{\mu_0\sigma}{4\pi}\omega a \int r^2 dr \sin\theta' d\theta' d\phi' \frac{\sin\theta'\delta(r'-a)\hat{\phi}'}{|\mathbf{x}-\mathbf{x}'|} \ . \tag{2}$$

We can expand the Green function in spherical coordinates

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \ , \tag{3}$$

where $r_< = \min(r,r')$ and $r_> = \max(r,r')$. Therefore, we get

$$\mathbf{A}(\mathbf{x}) = \mu_0\sigma\omega a^3 \sum_{l,m} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi)\int d\Omega' Y_{lm}^*(\theta',\phi')\sin\theta'\hat{\phi}' \ , \tag{4}$$

where $r_< = \min(r,a)$ and $r_> = \max(r,a)$. Now we decompose $\hat{\phi}'$ in Cartesian coordinates,

$$\hat{\phi}' = \cos\phi'\hat{i} + \sin\phi'\hat{j} \ . \tag{5}$$

We can evaluate the integrals,

$$\begin{aligned}\int d\Omega' Y_{lm}^*(\theta',\phi')\sin\theta'\cos\phi' &= \sqrt{\frac{2\pi}{3}}\int d\Omega' Y_{lm}^*(\theta',\phi')(-Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')) \\ &= \sqrt{\frac{2\pi}{3}}(-\delta_{l1}\delta_{m1} + \delta_{l1}\delta_{m,-1}) \ . \end{aligned} \tag{6}$$

$$\begin{aligned}\int d\Omega' Y_{lm}^*(\theta',\phi')\sin\theta'\sin\phi' &= \sqrt{\frac{2\pi}{3}} i \int d\Omega' Y_{lm}^*(\theta',\phi')(Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')) \\ &= \sqrt{\frac{2\pi}{3}} i(\delta_{l1}\delta_{m1} + \delta_{l1}\delta_{m,-1}) \ . \end{aligned} \tag{7}$$

Therefore, we have

$$\mathbf{A}(\mathbf{x}) = \frac{1}{3}\mu_0\sigma\omega a^3 \frac{r_<}{r_>^2}\sin\theta(\cos\phi\hat{i} + \sin\phi\hat{j}) = \frac{1}{3}\mu_0\sigma\omega a^3\frac{r_<}{r_>^2}\sin\theta\hat{\phi} \ . \tag{8}$$

Then, inside the sphere $r_< = r$ and $r_> = a$,

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{1}{3}\mu_0\sigma\omega a r \sin\theta\hat{\phi}} \tag{9}$$

Outside the sphere, $r_< = a$ and $r_> = r$,

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{1}{3}\mu_0 \sigma \omega \frac{a^4}{r^2} \sin\theta \hat{\phi}} \tag{10}$$

The magnetic flux density is given by

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta} \ . \tag{11}$$

So inside the sphere, we have

$$\boxed{\mathbf{B} = \frac{2}{3}\mu_0 \sigma \omega a (\cos\theta \hat{r} - \sin\theta \hat{\theta})} \tag{12}$$

Outside the sphere, we have

$$\boxed{\mathbf{B} = \frac{1}{3}\mu_0 \sigma \omega \frac{a^4}{r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})} \tag{13}$$

5.17 For $z > 0$, the magnetic induction is generated by the current $\mathbf{J}$ and the image current $\mathbf{J}^*$,

$$\mathbf{B}^+(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{(\mathbf{J}(\mathbf{x}') + \mathbf{J}^*(\mathbf{x}')) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \ . \tag{14}$$

It is an integration over the whole region. For $z < 0$, the magnetic induction is generated by the current $k\mathbf{J}$, where k is a scaling constant because of different permeability,

$$\mathbf{B}^-(\mathbf{x}) = \frac{\mu_0 \mu_r k}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \ . \tag{15}$$

It is an integration over the region where $z' > 0$. Now we want to transform the first integral such that these two integrals have the same integration domain. Suppose the component of $\mathbf{x}'$ is (x', y', z'), then we define $\mathbf{x}'' = (x', y', -z')$ and we can write

$$\mathbf{B}^+(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} + \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}^*(\mathbf{x}'') \times (\mathbf{x} - \mathbf{x}'')}{|\mathbf{x} - \mathbf{x}''|^3} \ . \tag{16}$$

Now this integral is defined in the region where $z' > 0$. And especially, when $z = 0$, there is no difference between $|\mathbf{x} - \mathbf{x}'|$ and $|\mathbf{x} - \mathbf{x}''|$.

Now the boundary conditions are given by

$$\mathbf{B}_z^+(z = 0) = \mathbf{B}_z^-(z = 0) \ , \quad \mathbf{B}_{x,y}^+(z = 0) = \frac{1}{\mu_r}\mathbf{B}_{x,y}^-(z = 0) \ . \tag{17}$$

For the first equation, we can equate the numerator in the integrand. When $z = 0$,

$$\hat{z} \cdot [\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] + \hat{z} \cdot [\mathbf{J}^*(\mathbf{x}'') \times (\mathbf{x} - \mathbf{x}'')] = \mu_r k \hat{z} \cdot (\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')) \ , \tag{18}$$

from which we can get that at $z = 0$,

$$(\mathbf{x} - \mathbf{x}') \cdot [\hat{z} \times \mathbf{J}(\mathbf{x}')] + (\mathbf{x} - \mathbf{x}'') \cdot [\hat{z} \times \mathbf{J}^*(\mathbf{x}'')] = \mu_r k (\mathbf{x} - \mathbf{x}') \cdot (\hat{z} \times \mathbf{J}(\mathbf{x}')) \ , \tag{19}$$

and expanding it in components we can get

$$\boxed{J_y(\mathbf{x}') + J_y^*(\mathbf{x}'') = \mu_r k J_y(\mathbf{x}'), \quad J_x(\mathbf{x}') + J_x^*(\mathbf{x}'') = \mu_r k J_x(\mathbf{x}')} \tag{20}$$

Another equation gives that at $z = 0$,

$$\hat{z} \times [\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] + \hat{z} \times [\mathbf{J}^*(\mathbf{x}'') \times (\mathbf{x} - \mathbf{x}'')] = k\hat{z} \times [\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] \ . \tag{21}$$

Now using,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \ , \tag{22}$$

we can further simplify the above equation to,

$$J_z^*(\mathbf{x}'')(\mathbf{x} - \mathbf{x}'') - z'\mathbf{J}^*(\mathbf{x}'') = (k-1)J_z(\mathbf{x}')(\mathbf{x} - \mathbf{x}') + (k-1)z'\mathbf{J}(\mathbf{x}') \ . \tag{23}$$

Expanding it into components and using the fact that the equation holds for arbitrary $\mathbf{x}$, we have,

$$\boxed{J_x^*(\mathbf{x}'') = (1-k)J_x(\mathbf{x}'), \quad J_y^*(\mathbf{x}'') = (1-k)J_y(\mathbf{x}'), \quad J_z^*(\mathbf{x}'') = (k-1)J_z(\mathbf{x}')} \tag{24}$$

From (20) and (24), we can solve for $k = 2/(1+\mu_r)$. Plug it back into (24), we have the image current distribution $\mathbf{J}^*$, with components,

$$\boxed{\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_x(x, y, -z), \quad \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_y(x, y, -z), \quad -\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_z(x, y, -z)}$$

Since $k = 2/(1+\mu_r)$, we have stated that for $z < 0$, the magnetic induction is due to a current distribution $k\mathbf{J}$ in a medium of relative permeability μ_r. We can also consider it due to a current distribution

$$\boxed{k\mu_r\mathbf{J} = \frac{2\mu_r}{1+\mu_r}\mathbf{J}} \tag{25}$$

in a medium of unit relative permeability.

5.19 (a) Since $\mathbf{J} = 0$, we can use the magnetic scalar potential Φ_M. Since the magnetization is uniform, we have

$$\Phi_M(\mathbf{x}) = \frac{1}{4\pi} \oint_S \frac{\mathbf{n}' \cdot \hat{z}\, M_0\, da'}{|\mathbf{x} - \mathbf{x}'|} \tag{26}$$

Since the magnetization points along the z-direction, we only need to consider the top boundary (say, at $z = L$) and the bottom boundary (at $z = 0$) and let the axis of the cylinder lying in the z-axis. Then when $\mathbf{x}$ is on the axis, at top, $|\mathbf{x} - \mathbf{x}'| = \sqrt{x'^2 + y'^2 + (z - L)^2}$ and at bottom, $|\mathbf{x} - \mathbf{x}'| = \sqrt{x'^2 + y'^2 + z^2}$. For the two dimensional surface integral, we can also use polar coordinates. Then, we have,

$$\begin{aligned}
\Phi_M(\mathbf{x}) &= \frac{M_0}{4\pi} \int_0^a \rho' d\rho' \int_0^{2\pi} d\phi \left(\frac{1}{\sqrt{\rho'^2 + (z-L)^2}} - \frac{1}{\sqrt{\rho'^2 + z^2}} \right) \\
&= \frac{M_0}{2} \int_0^a d\rho' \left(\frac{\rho'}{\sqrt{\rho'^2 + (z-L)^2}} - \frac{\rho'}{\sqrt{\rho'^2 + z^2}} \right) \\
&= \frac{M_0}{2} \left(\sqrt{\rho'^2 + (z-L)^2} - \sqrt{\rho'^2 + z^2} \right) \Big|_{\rho'=0}^{\rho'=a} \\
&= \frac{M_0}{2} \left(\sqrt{a^2 + (z-L)^2} - |z - L| - \sqrt{a^2 + z^2} + |z| \right) \; .
\end{aligned} \tag{27}$$

Therefore,

$$\Phi_M(z) = \begin{cases} \frac{M_0}{2} \left(\sqrt{a^2 + (z-L)^2} - \sqrt{a^2 + z^2} - L \right) & z < 0 \\ \frac{M_0}{2} \left(\sqrt{a^2 + (z-L)^2} - \sqrt{a^2 + z^2} + 2z - L \right) & 0 < z < L \\ \frac{M_0}{2} \left(\sqrt{a^2 + (z-L)^2} - \sqrt{a^2 + z^2} + L \right) & z > L \end{cases} \tag{28}$$

The magnetic field $\mathbf{H}$ is given by $\mathbf{H} = -\nabla \Phi_M = -\hat{z}\, \partial \Phi_M / \partial z$. Then

$$\mathbf{H}_{\text{in}} = -\frac{M_0}{2} \left(\frac{z - L}{\sqrt{a^2 + (z-L)^2}} - \frac{z}{\sqrt{a^2 + z^2}} + 2 \right) \hat{z} \tag{29}$$

$$\mathbf{H}_{\text{out}} = -\frac{M_0}{2} \left(\frac{z - L}{\sqrt{a^2 + (z-L)^2}} - \frac{z}{\sqrt{a^2 + z^2}} \right) \hat{z} \tag{30}$$

The magnetic induction is given by $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. So we have,

$$\mathbf{B}_{\text{in}} = -\frac{\mu_0 M_0}{2} \left(\frac{z - L}{\sqrt{a^2 + (z-L)^2}} - \frac{z}{\sqrt{a^2 + z^2}} \right) \hat{z} \tag{31}$$

$$\mathbf{B}_{\text{out}} = -\frac{\mu_0 M_0}{2} \left(\frac{z - L}{\sqrt{a^2 + (z-L)^2}} - \frac{z}{\sqrt{a^2 + z^2}} \right) \hat{z} \tag{32}$$

5.20 We start from the force equation

$$\mathbf{F} = \int \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})\, d^3x \ . \tag{33}$$

Using the fact that a magnetization $\mathbf{M}$ inside a volume V bounded by a surface S is equivalent to a volume current density $\mathbf{J}_M = \nabla \times \mathbf{M}$ and a surface current density $\mathbf{M} \times \mathbf{n}$, we can write

$$\mathbf{F} = \int_V (\nabla \times \mathbf{M}) \times \mathbf{B}_e\, d^3x + \int_S (\mathbf{M} \times \mathbf{n}) \times \mathbf{B}_e\, da \tag{34}$$

Since

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \ . \tag{35}$$

In our case, we have

$$(\nabla \times \mathbf{M}) \times \mathbf{B}_e = -\nabla(\mathbf{M} \cdot \mathbf{B}) + (\mathbf{M} \cdot \nabla)\mathbf{B}_e + (\mathbf{B}_e \cdot \nabla)\mathbf{M} + \mathbf{M} \times (\nabla \times \mathbf{B}_e) \ . \tag{36}$$

Since there's no external current, $\nabla \times \mathbf{B}_e = 0$, we have

$$\boxed{(\nabla \times \mathbf{M}) \times \mathbf{B}_e = -\nabla(\mathbf{M} \cdot \mathbf{B}_e) + (\mathbf{M} \cdot \nabla)\mathbf{B}_e + (\mathbf{B}_e \cdot \nabla)\mathbf{M}} \tag{37}$$

Also, we have

$$\boxed{(\mathbf{M} \times \mathbf{n}) \times \mathbf{B}_e = (\mathbf{B}_e \cdot \mathbf{M})\mathbf{n} - (\mathbf{B}_e \cdot \mathbf{n})\mathbf{M}} \tag{38}$$

For volume integration of the first term in the right hand side of (37), we can use Stokes theorem, which will cancel the surface integral of the first term in the right hand side of (38). So we are left with

$$\mathbf{F} = \int_V (\mathbf{M} \cdot \nabla)\mathbf{B}_e\, d^3x + \int_V (\mathbf{B}_e \cdot \nabla)\mathbf{M}\, d^3x - \int_S (\mathbf{B}_e \cdot \mathbf{n})\mathbf{M}\, da \ . \tag{39}$$

Now we want to do integration by parts, we have

$$\int_V (\mathbf{A} \cdot \nabla)\mathbf{B}\, d^3x = -\int_V (\nabla \cdot \mathbf{A})\mathbf{B}\, d^3x + \int_S (\mathbf{A} \cdot \mathbf{n})\mathbf{B}\, da \ . \tag{40}$$

Then we have

$$\boxed{\mathbf{F} = -\int_V (\nabla \cdot \mathbf{M})\mathbf{B}_e\, d^3x + \int_S (\mathbf{M} \cdot \mathbf{n})\mathbf{B}_e\, da} \ , \tag{41}$$

where we have used the fact that $\nabla \cdot \mathbf{B}_e = 0$.

The force is then

5.21

5.26

5.27