Phys 5405

HW 10

5.1, 5.3, 5.4(a), 5.7(a,b,c,d), 5.8, 5.10(a,b)

5.1 We have the differential expression

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\, d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\ . \tag{1}$$

Then

$$\mathbf{B} = \int d\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\ . \tag{2}$$

It suffices to show that

$$\oint d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla\Omega\ . \tag{3}$$

First, we can make use of the identity,

$$\boxed{\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \equiv \mathbf{V}\ .} \tag{4}$$

Then, consider each components

$$\boxed{\hat{\mathbf{x}}_i \cdot \oint d\mathbf{l}' \times \mathbf{V} = \oint d\mathbf{l}' \cdot (\mathbf{V} \times \hat{\mathbf{x}}_i) = \int_S d\mathbf{a}' \cdot (\nabla' \times (\mathbf{V} \times \hat{\mathbf{x}}_i))\ .} \tag{5}$$

Now we evaluate $\nabla' \times (\mathbf{V} \times \hat{\mathbf{x}}_i)$, using the identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}\ , \tag{6}$$

so we have

$$\boxed{\nabla' \times (\mathbf{V} \times \hat{\mathbf{x}}_i) = -\hat{\mathbf{x}}_i(\nabla' \cdot \mathbf{V}) + (\hat{\mathbf{x}}_i \cdot \nabla')\mathbf{V} = (\hat{\mathbf{x}}_i \cdot \nabla')\mathbf{V} = \frac{\partial}{\partial x_i'}\mathbf{V}\ ,} \tag{7}$$

where we have used the fact that for $\mathbf{x} \neq \mathbf{x}'$,

$$\nabla' \cdot \mathbf{V} = \nabla'^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = 0\ . \tag{8}$$

Finally, we have achieved that

$$\hat{\mathbf{x}}_i \cdot \oint d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = \int_S d\mathbf{a}' \cdot \frac{\partial}{\partial x_i'}\left(\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\right) = -\frac{\partial}{\partial x_i}\int_S d\mathbf{a}' \cdot \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \tag{9}$$

From Jackson, the equation below (1.25), we have

$$d\mathbf{a}' \cdot \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -d\Omega\ . \tag{10}$$

Therefore,

$$\boxed{\hat{\mathbf{x}}_i \cdot \oint d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \frac{\partial}{\partial x_i} \int_S d\Omega \ ,} \tag{11}$$

which implies that

$$\oint d\mathbf{l}' \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \nabla \Omega \ . \tag{12}$$

5.3 In Jackson section 5.5, it already gives the magnetic induction for a circular current loop, For the position located on the symmetric axis of the loop, the magnetic induction is given by

$$B_z = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \ , \tag{13}$$

where a is the radius of the loop and z is the distance between the center of the loop and the position where we measure the magnetic induction.

In this case, we have

$$B_z = \frac{\mu_0 N I}{2} a^2 \int dz \frac{1}{(a^2 + z^2)^{3/2}} \ . \tag{14}$$

Instead of using variable z, we use the variable θ such that

$$\tan\theta = a/z \ . \tag{15}$$

Then

$$\boxed{B_z = \frac{\mu_0 N I}{2} \int_{\pi-\theta_1}^{\theta_2} d(\cos\theta) = \frac{\mu_0 N I}{2} (\cos\theta_2 + \cos\theta_1) \ .} \tag{16}$$

5.4 (a) In a current-free region, the magnetic induction satisfies

$$\nabla \cdot \mathbf{B} = 0 \ , \quad \nabla \times \mathbf{B} = 0 \ . \tag{17}$$

Written in components, we have

$$\frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{\partial B_z}{\partial z} = 0 \ , \quad \frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} = 0 \ . \tag{18}$$

Near the axis, the axial component of the magnetic induction can be expanded as

$$B_z(\rho, z) \approx B_z(0, z) + \frac{\partial B_z(0,z)}{\partial \rho}\rho + \frac{1}{2}\frac{\partial^2 B_z(0,z)}{\partial \rho^2}\rho^2 + \frac{1}{6}\frac{\partial^3 B_z(0,z)}{\partial \rho^3}\rho^3 + \cdots \tag{19}$$

The radial component of the magnetic induction can also be expanded as

$$B_\rho(\rho, z) \approx B_\rho(0, z) + \frac{\partial B_\rho(0,z)}{\partial \rho}\rho + \frac{1}{2}\frac{\partial^2 B_\rho(0,z)}{\partial \rho^2}\rho^2 + \frac{1}{6}\frac{\partial^3 B_\rho(0,z)}{\partial \rho^3}\rho^3 + \cdots \tag{20}$$

Next we just plug these two expansions into (18) and equating the coefficients.

$$\frac{\partial B_\rho}{\partial z} = \frac{\partial B_\rho(0,z)}{\partial z} + \frac{\partial^2 B_\rho(0,z)}{\partial z \partial \rho}\rho + \frac{1}{2}\frac{\partial^3 B_\rho(0,z)}{\partial z \partial \rho^2}\rho^2 + \frac{1}{6}\frac{\partial^4 B_\rho(0,z)}{\partial z \partial \rho^3}\rho^3 + \cdots \tag{21}$$

$$\frac{\partial B_z}{\partial \rho} = \frac{\partial B_z(0,z)}{\partial \rho} + \frac{\partial^2 B_z(0,z)}{\partial \rho^2}\rho + \frac{1}{2}\frac{\partial^3 B_z(0,z)}{\partial \rho^3}\rho^2 + \cdots \tag{22}$$

and

$$-\frac{\partial B_z}{\partial z} = -\frac{\partial B_z(0,z)}{\partial z} - \frac{\partial^2 B_z(0,z)}{\partial z \partial \rho}\rho - \frac{1}{2}\frac{\partial^3 B_z(0,z)}{\partial z \partial \rho^2}\rho^2 - \frac{1}{6}\frac{\partial^4 B_z(0,z)}{\partial z \partial \rho^3}\rho^3 + \cdots \tag{23}$$

$$\frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} = \frac{B_\rho(0,z)}{\rho} + 2\frac{\partial B_\rho(0,z)}{\partial \rho} + \frac{3}{2}\frac{\partial^2 B_\rho(0,z)}{\partial \rho^2}\rho + \frac{2}{3}\frac{\partial^3 B_\rho(0,z)}{\partial \rho^3}\rho^2 + \cdots \tag{24}$$

We have

$$B_\rho(0,z) = 0 \ , \tag{25}$$

$$\frac{\partial B_\rho(0,z)}{\partial \rho} = -\frac{1}{2}\frac{\partial B_z(0,z)}{\partial z} \ , \tag{26}$$

$$\frac{\partial^2 B_\rho(0,z)}{\partial \rho^2} = -\frac{2}{3}\frac{\partial^2 B_z(0,z)}{\partial z \partial \rho} = -\frac{2}{3}\frac{\partial^2 B_\rho(0,z)}{\partial z^2} = 0 \ , \tag{27}$$

$$\frac{\partial^3 B_\rho(0,z)}{\partial \rho^3} = -\frac{3}{4}\frac{\partial^3 B_z(0,z)}{\partial z \partial \rho^2} = -\frac{3}{4}\frac{\partial^3 B_\rho(0,z)}{\partial z^2 \partial \rho} = \frac{3}{8}\frac{\partial^3 B_z(0,z)}{\partial z^3} \ . \tag{28}$$

We also have

$$\frac{\partial B_z(0,z)}{\partial \rho} = \frac{\partial B_\rho(0,z)}{\partial z} = 0 \ , \tag{29}$$

$$\frac{\partial^2 B_z(0,z)}{\partial \rho^2} = \frac{\partial^2 B_\rho(0,z)}{\partial z \partial \rho} = -\frac{1}{2}\frac{\partial^2 B_z(0,z)}{\partial z^2} \ , \tag{30}$$

$$\frac{\partial^3 B_z(0,z)}{\partial \rho^3} = \frac{\partial^3 B_\rho(0,z)}{\partial z \partial \rho^2} = -\frac{2}{3}\frac{\partial^3 B_\rho(0,z)}{\partial z^3} = 0 \ . \tag{31}$$

Since $B_\rho(0,z) = 0$, all its derivatives with respect to z gives zero. Therefore,

$$\boxed{\begin{aligned} B_z(\rho, z) &\approx B_z(0,z) - \frac{1}{4}\frac{\partial^2 B_z(0,z)}{\partial z^2}\rho^2 + \cdots \\ B_\rho(\rho, z) &\approx -\frac{1}{2}\frac{\partial B_z(0,z)}{\partial z}\rho - \frac{1}{16}\frac{\partial^3 B_z(0,z)}{\partial z^3}\rho^3 + \cdots \end{aligned}} \tag{32, 33}$$

5.7 (a) We have

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \tag{34}$$

For $\mathbf{x}$ on the z axis, the only non-vanishing component is the z component

$$B_z(z) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\theta \frac{a^2}{(a^2 + z^2)^{3/2}} = \boxed{\frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}}} . \tag{35}$$

5.7 (b) From (a), we can write down the magnetic induction at the axis as

$$B_z(z) = \frac{\mu_0 I}{2}\left(\frac{a^2}{(a^2+(z+b/2)^2)^{3/2}} + \frac{a^2}{(a^2+(z-b/2)^2)^{3/2}}\right) \ . \tag{36}$$

When we look near the origin where z is very small,

$$\begin{aligned} B_z(z) &= \frac{\mu_0 I a^2}{2}\left(\frac{1}{(a^2+b^2/4+z^2+bz)^{3/2}} + \frac{1}{(a^2+b^2/4+z^2-bz)^{3/2}}\right) \\ &= \frac{\mu_0 I a^2}{2d^3}\left[\left(1+\frac{z^2+bz}{d^2}\right)^{-3/2} + \left(1+\frac{z^2-bz}{d^2}\right)^{-3/2}\right] \end{aligned} \tag{37}$$

Define

$$f(z) = \frac{1}{2}\left(1+\frac{z^2+bz}{d^2}\right)^{-3/2} + \frac{1}{2}\left(1+\frac{z^2-bz}{d^2}\right)^{-3/2} \ . \tag{38}$$

Now we calculate its Taylor expansion around $z=0$,

$$f(0) = 1 \tag{39}$$

$$f'(0) = 0 \tag{40}$$

$$f''(0) = \frac{3(5b^2-4d^2)}{4d^4} = \frac{3(b^2-a^2)}{d^4} \tag{41}$$

$$f^{(3)} = 0 \tag{42}$$

$$f^{(4)} = \frac{45(21b^4-56b^2d^2+16d^4)}{16d^8} = \frac{45(b^4-6b^2a^2+2a^4)}{2d^8} \tag{43}$$

Therefore,

$$B_z(z) = \left(\frac{\mu_0 I a^2}{d^3}\right)\left[1+\frac{3(b^2-a^2)z^2}{2d^4} + \frac{15(b^4-6b^2a^2+2a^4)z^4}{16d^8} + \cdots\right] \tag{44}$$

5.7(c) Using results from 5.4 (a), copied here for convenience, for position slightly off the axis, we have

$$B_z(\rho, z) \approx B_z(0, z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z(0, z)}{\partial z^2}\right] + \cdots \tag{45}$$

$$B_\rho(\rho, z) \approx -\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z(0, z)}{\partial z}\right] \tag{46}$$

Define

$$\sigma_0 \equiv \frac{\mu_0 I a^2}{d^3} \ , \quad \sigma_2 \equiv \frac{3(b^2 - a^2)}{2d^4}\sigma_0 \tag{47}$$

we have

$$\frac{\partial B_z(0, z)}{\partial z} = 2\sigma_2 z + \cdots \ , \quad \frac{\partial^2 B_z(0, z)}{\partial z^2} = 2\sigma_2 + \cdots \tag{48}$$

So correct to second order in coordinates

$$B_z(\rho, z) = \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right) \ , \quad B_\rho = -\sigma_2 z \rho \ . \tag{49}$$

5.7 (d) We start from

$$\begin{aligned} B_z(z) &= \frac{\mu_0 I a^2}{2}\left(\frac{1}{(a^2+(z+b/2)^2)^{3/2}}+\frac{1}{(a^2+(z-b/2)^2)^{3/2}}\right) \\ &= \frac{\mu_0 I a^2}{2|z|^3}\left[\left(1+bz^{-1}+d^2z^{-2}\right)^{3/2}+\left(1-bz^{-1}+d^2z^{-2}\right)^{3/2}\right] \end{aligned} \tag{50}$$

This is the same form as (37), and following the same procedure, we can expand it around large z as

$$B_z(z) = \frac{\mu_0 I a^2}{|z|^3}\left[1+\frac{3}{2}\frac{(b^2-a^2)}{z^2}+\frac{15}{16}\frac{(b^4-6b^2a^2+2a^4)}{z^4}+\cdots\right] \ . \tag{51}$$

It is just obtained from the small z expansion by the formal substitution $d \to |z|$.

5.8 The vector potential is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{\mu_0}{4\pi}\int d^3x' \frac{J(r,\theta)\hat{\phi}'}{|\mathbf{x}-\mathbf{x}'|} \ . \tag{52}$$

Decompose the spherical coordinates basis in Cartesian coordinates,

$$\hat{r} = \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{53}$$

$$\hat{\theta} = \frac{1}{r}\frac{\partial}{\partial\theta} = \frac{1}{r}\left(\frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z}\right) = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \tag{54}$$

$$\hat{\phi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} = \frac{1}{r\sin\theta}\left(\frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z}\right) = (-\sin\phi, \cos\phi, 0) \tag{55}$$

Therefore,

$$\hat{\phi}' = (\hat{\phi}'\cdot\hat{r})\hat{r} + (\hat{\phi}'\cdot\hat{\theta})\hat{\theta} + (\hat{\phi}'\cdot\hat{\phi})\hat{\phi} = \sin\theta\sin(\phi-\phi')\hat{r} + \cos\theta\sin(\phi-\phi')\hat{\theta} + \cos(\phi-\phi')\hat{\phi} \ . \tag{56}$$

Moreover, we can expand the Green's function in terms of spherical harmonics,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int r'^2 dr' d\Omega' \hat{\phi}' J(r',\theta') \sum_L \sum_m \frac{4\pi}{2L+1}\left(\frac{r_<^L}{r_>^{L+1}}\right) Y^*_{Lm}(\theta',\phi') Y_{L,m}(\theta,\phi) \tag{57}$$

For $m=0$, the integral over ϕ' vanishes. For $m \neq 0$, since

$$Y^*_{Lm}(\theta',\phi')Y_{Lm}(\theta,\phi) = Y_{Lm}(\theta',0)Y_{Lm}(\theta,0)e^{im(\phi-\phi')} \tag{58}$$

$$Y^*_{L,-m}(\theta',\phi')Y_{L,-m}(\theta,\phi) = Y_{Lm}(\theta',0)Y_{Lm}(\theta,0)e^{-im(\phi-\phi')} \ , \tag{59}$$

so

$$\mathbf{A} = \frac{\mu_0}{2\pi}\int r'^2 dr' d\Omega' \hat{\phi}' J(r',\theta') \sum_L \sum_{m=1}^{\infty} \frac{4\pi}{2L+1}\left(\frac{r_<^L}{r_>^{L+1}}\right) Y^*_{Lm}(\theta',0) Y_{L,m}(\theta,0)\cos(m(\phi-\phi')) \tag{60}$$

Performing the ϕ' integral, the only non-vanishing contribution is the $\hat{\phi}$ component with $m=1$, therefore,

$$\mathbf{A} = \hat{\phi}A_\phi = \hat{\phi}\frac{\mu_0}{2}\int r'^2 dr' \sin\theta' d\theta' J(r',\theta') \sum_L \frac{1}{L(L+1)}\left(\frac{r_<^L}{r_>^{L+1}}\right) P_L^1(\cos\theta)P_L^1(\cos\theta') \ . \tag{61}$$

In the interior, $r_< = r$ and $r_> = r'$, so we have

$$A_\phi = -\frac{\mu_0}{4\pi}\sum_L m_L r^L P_L^1(\cos\theta) \ , \tag{62}$$

where

$$m_L = -2\pi \frac{1}{L(L+1)} \int r'^2 dr' \sin\theta' d\theta' J(r',\theta') r'^{-L-1} P_L^1(\cos\theta')$$
$$= -\frac{1}{L(L+1)} \int d^3x r^{-L-1} P_L^1(\cos\theta) J(r,\theta) \ . \tag{63}$$

Similarly, outside the current distribution, $r_< = r'$, $r_> = r$, so we have

$$A_\phi(r,\theta) = -\frac{\mu_0}{4\pi} \sum_L \mu_L r^{-L-1} P_L^1(\cos\theta) \ , \tag{64}$$

where

$$\mu_L = -2\pi \frac{1}{L(L+1)} \int r'^2 dr' \sin\theta' d\theta' J(r',\theta') r'^L P_L^1(\cos\theta')$$
$$= -\frac{1}{L(L+1)} \int d^3x r^L P_L^1(\cos\theta) J(r,\theta) \ . \tag{65}$$

5.10 (a) We have to show that

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos kz I_1(k\rho_<) K_1(k\rho_>) \ . \tag{66}$$

The current density is given by

$$\mathbf{J}(\mathbf{x}') = \hat{\phi}' I \delta(\rho' - a)\delta(z') \ . \tag{67}$$

As we have derived in the last problem,

$$\hat{\phi}' = \sin\theta \sin(\phi - \phi')\hat{r} + \cos\theta \sin(\phi - \phi')\hat{\theta} + \cos(\phi - \phi')\hat{\phi} \ . \tag{68}$$

Using Jackson (3.149),

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{4}{\pi} \int_0^\infty dk \cos[k(z - z')] \left(\frac{1}{2} I_0(k\rho_<) K_0(k\rho_>) + \sum_{m=1}^\infty \cos[m(\phi - \phi')] I_m(k\rho_<) K_m(k\rho_>) \ , \right)$$

where $\rho_< = \min(\rho, \rho')$ and $\rho_> = \max(\rho, \rho')$. Therefore, from

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \ , \tag{69}$$

we can see that for $\hat{r}$ and $\hat{\theta}$ component, the θ integral vanishes and the only non-vanishing component is

$$\begin{aligned} A_\phi(\rho, z) &= \frac{\mu_0 I}{\pi^2} \int d^3x' \int dk \cos(\phi - \phi') \delta(\rho' - a) \delta(z') \cos[k(z - z')] \\ &\quad \times \left(\frac{1}{2} I_0(k\rho_<) K_0(k\rho_>) + \sum_{m=1}^\infty \cos[m(\phi - \phi')] I_m(k\rho_<) K_m(k\rho_>) \right) \\ &= \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos(kz) I_1(k\rho_<) K_1(k\rho_>) \ , \end{aligned} \tag{70}$$

where in the last step, only $m = 1$ term survives and $\rho_< = \min(a, \rho)$ and $\rho_> = \max(a, \rho)$.

5.10 (b) We want to show that an alternative expression for A_ϕ is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2} \int_0^\infty dk e^{-k|z|} J_1(ka) J_1(k\rho) \ . \tag{71}$$

From Jackson problem 3.16 (b), we have

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k(z_> - z_<)} \ . \tag{72}$$

Therefore,

$$\begin{aligned} A_\phi(\rho, z) &= \frac{\mu_0 I}{4\pi} \int d^3x' \int dk \delta(\rho' - a) \delta(z') \cos(\phi - \phi') \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') e^{-k(z_> - z_<)} \\ &= \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} d\phi' \int_0^\infty dk \cos(\phi - \phi') \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} J_m(k\rho) J_m(ka) e^{-k|z|} \end{aligned} \tag{73}$$

Since $J_{-m}(x) = (-1)^m J_m(x)$

$$\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} J_m(k\rho) J_m(ka) = J_0(k\rho) J_0(ka) + 2 \sum_{m=1}^{\infty} J_m(k\rho) J_m(ka) \cos(\phi - \phi') \ . \tag{74}$$

Only the $m = 1$ term contributes to the integral, so

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2} \int_0^\infty e^{-k|z|} J_1(k\rho) J_1(ka) \ . \tag{75}$$