Phys 5405
HW 3

2.3 (a) Using method of image, substitute the intersecting planes with a straight-line charge with constant linear charge density λ', located perpendicular to the $x-y$ plane at point (a, b). Since the system is translation invariant along z axis. We write the potential as a function of x and y. By symmetry, we can simply write down the answer,

$$\begin{aligned}\Phi(x,y) = & \frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{(x-x_0)^2+(y-y_0)^2} - \frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{(x+x_0)^2+(y-y_0)^2} \\ & - \frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{(x-x_0)^2+(y+y_0)^2} + \frac{\lambda}{4\pi\epsilon_0}\ln\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\ .\end{aligned}$$

Cancelling the constant terms, we obtain

$$\begin{aligned}\Phi(x,y) = & -\frac{\lambda}{4\pi\epsilon_0}\ln\left((x-x_0)^2+(y-y_0)^2\right) + \frac{\lambda}{4\pi\epsilon_0}\ln\left((x+x_0)^2+(y-y_0)^2\right) \\ & + \frac{\lambda}{4\pi\epsilon_0}\ln\left((x-x_0)^2+(y+y_0)^2\right) - \frac{\lambda}{4\pi\epsilon_0}\ln\left((x+x_0)^2+(y+y_0)^2\right)\ .\end{aligned}$$

It is easily verified that the potential vanishes on the boundary. Now consider the tangential electric field. At $x = 0$, the tangential electric field should be along y axis.

$$\begin{aligned}E_y(x,y) = & -\partial_y\Phi(x,y) \\ = & \frac{\lambda}{4\pi\epsilon_0}\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{\lambda}{4\pi\epsilon_0}\frac{2(y-y_0)}{(x+x_0)^2+(y-y_0)^2} \\ & - \frac{\lambda}{4\pi\epsilon_0}\frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2} + \frac{\lambda}{4\pi\epsilon_0}\frac{2(y+y_0)}{(x+x_0)^2+(y+y_0)^2}\end{aligned} \tag{1}$$

It is easily seen that at $x = 0$,

$$E_y(x=0,y) = 0\ . \tag{2}$$

Similarly, at $y = 0$, the tangential electric field should be along x axis.

$$\begin{aligned}E_x(x,y) = & -\partial_x\Phi(x,y) \\ = & \frac{\lambda}{4\pi\epsilon_0}\frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2} - \frac{\lambda}{4\pi\epsilon_0}\frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2} \\ & - \frac{\lambda}{4\pi\epsilon_0}\frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2} + \frac{\lambda}{4\pi\epsilon_0}\frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2}\end{aligned} \tag{3}$$

It is easily seen that at $y = 0$,

$$E_x(x, y = 0) = 0 \ . \tag{4}$$

2.3 (c) At $y = 0$, the surface charge density is

$$\sigma(x) = -\epsilon_0 \frac{\partial \Phi}{\partial y}\bigg|_{y=0} \tag{5}$$

Now using result in (1), we have

$$\sigma(x) = -\frac{\lambda}{\pi}\left(\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x+x_0)^2+y_0^2}\right) \ . \tag{6}$$

Then the total charge, per unit length in z on the plane $y = 0, x \geq 0$ is

$$\begin{aligned} Q_x &= \int_0^\infty dx\, \sigma(x) = -\frac{\lambda}{\pi}\int_0^\infty dx \left(\frac{y_0}{(x-x_0)^2+y_0^2} - \frac{y_0}{(x+x_0)^2+y_0^2}\right) \\ &= -\frac{\lambda}{\pi}\left(\tan^{-1}\left(\frac{x-x_0}{y_0}\right)\bigg|_0^\infty - \tan^{-1}\left(\frac{x+x_0}{y_0}\right)\bigg|_0^\infty\right) \\ &= -\frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right) \ . \end{aligned} \tag{7}$$

2.3 (d) At position far from the origin $\rho \gg \rho_0$, where

$$\rho = \sqrt{x^2 + y^2}, \quad \rho_0 = \sqrt{x_0^2 + y_0^2} \tag{8}$$

We can expand the potential

$$\begin{aligned}
\Phi(x, y) = & -\frac{\lambda}{4\pi\epsilon_0} \ln\left((x - x_0)^2 + (y - y_0)^2\right) + \frac{\lambda}{4\pi\epsilon_0} \ln\left((x + x_0)^2 + (y - y_0)^2\right) \\
& + \frac{\lambda}{4\pi\epsilon_0} \ln\left((x - x_0)^2 + (y + y_0)^2\right) - \frac{\lambda}{4\pi\epsilon_0} \ln\left((x + x_0)^2 + (y + y_0)^2\right) \ . \\
= & -\frac{\lambda}{4\pi\epsilon_0} \ln\left(1 + \frac{\rho_0^2 - 2x_0 x - 2y_0 y}{\rho^2}\right) + \frac{\lambda}{4\pi\epsilon_0} \ln\left(1 + \frac{\rho_0^2 + 2x_0 x - 2y_0 y}{\rho^2}\right) \\
& + \frac{\lambda}{4\pi\epsilon_0} \ln\left(1 + \frac{\rho_0^2 - 2x_0 x + 2y_0 y}{\rho^2}\right) - \frac{\lambda}{4\pi\epsilon_0} \ln\left(1 + \frac{\rho_0^2 + 2x_0 x + 2y_0 y}{\rho^2}\right) \ .
\end{aligned}$$

Now using $\ln(1 + x) \approx x - \frac{x^2}{2}$ when x is very small, we have

$$\begin{aligned}
\Phi(x, y) \approx & -\frac{\lambda}{4\pi\epsilon_0} \left(\frac{\rho_0^2 - 2x_0 x - 2y_0 y}{\rho^2} - \frac{\rho_0^2 + 2x_0 x - 2y_0 y}{\rho^2} \right. \\
& \qquad \left. - \frac{\rho_0^2 - 2x_0 x + 2y_0 y}{\rho^2} + \frac{\rho_0^2 + 2x_0 x + 2y_0 y}{\rho^2} \right) \\
& + \frac{\lambda}{8\pi\epsilon_0} \left[\left(\frac{\rho_0^2 - 2x_0 x - 2y_0 y}{\rho^2} \right)^2 - \left(\frac{\rho_0^2 + 2x_0 x - 2y_0 y}{\rho^2} \right)^2 \right. \\
& \qquad \left. - \left(\frac{\rho_0^2 - 2x_0 x + 2y_0 y}{\rho^2} \right)^2 + \left(\frac{\rho_0^2 + 2x_0 x + 2y_0 y}{\rho^2} \right)^2 \right] \\
= & \frac{\lambda}{8\pi\epsilon_0 \rho^4} \times 32 x_0 y_0 x y = \frac{4\lambda}{\pi\epsilon_0} \frac{(x_0 y_0)(xy)}{\rho^4} \ .
\end{aligned} \tag{9}$$

2.7 (a) The general Green function is of the form

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}') \ , \tag{10}$$

where $\nabla^2 F(\vec{x},\vec{x}') = 0$. And the Dirichlet boundary condition implies that the Green function has to vanish on $z=0$ surface. We can view the problem as a unit charge at x', with a conducting plane at $z=0$. Suppose $\vec{x}' = (x',y',z')$, we can then put the image charge at $(x',y',-z')$ with negative unit charge. The potential is then given by

$$\begin{aligned}\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0}\Bigg(&\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ &-\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\Bigg) \ .\end{aligned} \tag{11}$$

And the Green function should be

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}''|} \ , \tag{12}$$

where $\vec{x}'' = (x',y',-z')$.

2.7 (b) We have

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}da' \tag{13}$$

Since there is no charge distribution, we have

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}da' \ . \tag{14}$$

The normal vector direction is along the $-z'$ direction. So

$$\frac{\partial G_D}{\partial n'} = -\frac{\partial G_D}{\partial z'} = -\frac{z-z'}{|\vec{x}-\vec{x}'|^3} - \frac{z+z'}{|\vec{x}-\vec{x}''|^3} \ . \tag{15}$$

When restricted to the surface, we have $\vec{x}' = \vec{x}'' = (x', y', 0)$ and

$$\frac{\partial G_D}{\partial n'} = -\frac{2z}{|\vec{x}-\vec{x}'|^3} \ . \tag{16}$$

Now we change to cylindrical coordinates with

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad x' = \rho'\cos\phi', \quad y' = \rho'\sin\phi' \ .$$

Then in cylindrical coordinates,

$$\frac{\partial G_D}{\partial n'} = -\frac{2z}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{3/2}} \tag{17}$$

and

$$\Phi(\rho,\phi,\theta) = -\frac{V}{4\pi}\int_0^a d\rho' \int_0^{2\pi} \rho' d\phi' \, \frac{\partial G_D}{\partial n'} \tag{18}$$

After simplification,

$$\Phi(\rho,\phi,\theta) = \frac{Vz}{2\pi}\int_0^a d\rho' \int_0^{2\pi} \rho' d\phi' \, (\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{-3/2} \ . \tag{19}$$

2.7 (c) Using (19), at $\rho = 0$, we have

$$
\begin{aligned}
\Phi &= \frac{Vz}{2\pi} \int_0^a d\rho' \int_0^{2\pi} \rho' d\phi' \, (\rho'^2 + z^2)^{-3/2} \\
&= Vz \int_0^a d\rho' \frac{\rho'}{(\rho'^2 + z^2)^{3/2}} \\
&= -Vz \int_0^a d\rho' \frac{\partial}{\partial \rho'} \left(\frac{1}{\sqrt{\rho'^2 + z^2}} \right) \\
&= V \left(1 - \frac{z}{\sqrt{a^2 + z^2}} \right) \, .
\end{aligned}
\tag{20}
$$

2.7 (d) For $\rho^2 + z^2 \gg a^2$, we have

$$\begin{aligned}
\Phi &= \frac{Vz}{2\pi} \int_0^a d\rho' \int_0^{2\pi} \rho' d\phi' \, (\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') + z^2)^{-3/2} \\
&= \frac{Vz}{2\pi} \frac{1}{(\rho^2 + z^2)^{3/2}} \int_0^a d\rho' \int_0^{2\pi} \rho' d\phi' \left(1 + \frac{\rho'^2 - 2\rho\rho' \cos(\phi - \phi')}{\rho^2 + z^2}\right)^{-3/2} \\
&= \frac{Vz}{2\pi} \frac{1}{(\rho^2 + z^2)^{3/2}} \int_0^a d\rho' \int_0^{2\pi} \rho' d\phi' \left(1 - \frac{3}{2}u + \frac{15}{8}u^2 + \cdots\right) \,, \qquad (21)
\end{aligned}$$

where

$$u = \frac{\rho'^2 - 2\rho\rho' \cos(\phi - \phi')}{\rho^2 + z^2} \,. \qquad (22)$$

Evaluating the integrals terms by terms

$$\begin{aligned}
&\int_0^a d\rho' \int_0^{2\pi} \rho' d\phi' \, 1 = \pi a^2 \\
&\int_0^a d\rho' \int_0^{2\pi} \rho' d\phi' \left(-\frac{3}{2}u\right) = -\frac{3}{2} \frac{1}{\rho^2 + z^2} \left(\frac{\pi}{2} a^4\right) \\
&\int_0^a d\rho' \int_0^{2\pi} \rho' d\phi' \left(\frac{15}{8}u^2\right) = \frac{15}{8} \frac{1}{(\rho^2 + z^2)^2} 2\pi \left(\frac{1}{6}a^6 + \frac{1}{2}\rho^2 a^4\right) \,. \qquad (23)
\end{aligned}$$

Collecting these terms, we have

$$\Phi = \frac{Va^2}{2} \frac{z}{(\rho^2 + z^2)^{3/2}} \left[1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + \cdots\right] \qquad (24)$$

Check: from (24), when $\rho = 0$, we have

$$\Phi = \frac{Va^2}{2z} \left(1 - \frac{3}{4}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^4}{z^4} + \cdots\right) \,. \qquad (25)$$

From (19), when $z \gg a$, the potential

$$\Phi = V \left(1 - \frac{1}{\sqrt{1 + \frac{a^2}{z^2}}}\right) \,. \qquad (26)$$

Since for small u,

$$\frac{1}{\sqrt{1 + u}} = 1 - \frac{1}{2}u + \frac{3}{8}u^2 - \frac{5}{16}x^3 + \cdots \qquad (27)$$

Therefore, we can expand (26) as

$$
\begin{aligned}
\Phi &= V\left(1 - 1 + \frac{1}{2}\frac{a^2}{z^2} - \frac{3}{8}\frac{a^4}{z^4} + \frac{5}{16}\frac{a^6}{z^6} + \cdots\right) \\
&= \frac{Va^2}{2z^2}\left(1 - \frac{3}{4}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^4}{z^4} + \cdots\right),
\end{aligned}
\tag{28}
$$

in agreement with (25).