Phys 5405
HW 4
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2.12 Starting with

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n) \ . \quad (1)$$

For potential inside the cylinder, we have to set $b_n = 0$ for $n \geq 0$. Then,

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \left[\cos\alpha_n \sin(n\phi) + \sin\alpha_n \cos(n\phi)\right] \ . \quad (2)$$

Since we have specified the potential on the surface of the cylinder of radius b, we can set $\rho = b$, and obtain,

$$\Phi(b,\phi) = a_0 + \sum_{n=1}^{\infty} a_n b^n \left[\cos\alpha_n \sin(n\phi) + \sin\alpha_n \cos(n\phi)\right] \ . \quad (3)$$

Then for $n \geq 1$, evaluating,

$$\int_0^{2\pi} \Phi(b,\phi) \sin(n\phi) d\phi = \pi a_n b^n \cos\alpha_n \ , \quad (4)$$

$$\int_0^{2\pi} \Phi(b,\phi) \cos(n\phi) d\phi = \pi a_n b^n \sin\alpha_n \ . \quad (5)$$

Therefore, we obtain for $n \geq 1$,

$$a_n = \frac{b^{-n}}{\pi \cos\alpha_n} \int_0^{2\pi} \Phi(\rho,\phi) \sin(n\phi) d\phi = \frac{b^{-n}}{\pi \sin\alpha_n} \int_0^{2\pi} \Phi(\rho,\phi) \cos(n\phi) d\phi \ . \quad (6)$$

For a_0, we have

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \Phi(b,\phi) \cos(0\phi) d\phi = \frac{1}{2\pi} \int_0^{2\pi} \Phi(b,\phi) d\phi \ . \quad (7)$$

we obtain

$$\begin{aligned}\Phi(\rho,\phi) = & \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')d\phi' \\ & + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}\int_0^{2\pi}\Phi(b,\phi')\left[\sin(n\phi')\sin(n\phi)+\cos(n\phi')\cos(n\phi)\right]d\phi' \\ = & \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')d\phi' + \frac{1}{\pi}\int_0^{2\pi}d\phi'\Phi(b,\phi')\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}\cos[n(\phi-\phi')] \\ = & \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')d\phi' + \frac{1}{\pi}\int_0^{2\pi}d\phi'\Phi(b,\phi')\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{\rho^n}{b^n}e^{in(\phi-\phi')} \end{aligned} \tag{8}$$

We can evaluate the summation (using $\theta = \phi - \phi'$ for short),

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{\rho}{b}e^{i(\phi-\phi')}\right)^n = \frac{\rho e^{i(\phi-\phi')}}{b-\rho e^{i(\phi-\phi')}} &= \frac{\rho\cos\theta + i\rho\sin\theta}{b-\rho\cos\theta - i\rho\sin\theta} \\ &= \frac{(\rho\cos\theta+i\rho\sin\theta)(b-\rho\cos\theta+i\rho\sin\theta)}{(b-\rho\cos\theta)^2+\rho^2\sin^2\theta}\end{aligned} \tag{9}$$

Therefore its real part is given by

$$\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{\rho}{b}e^{i(\phi-\phi')}\right)^n = \frac{b\rho\cos\theta-\rho^2}{b^2+\rho^2-2b\rho\cos\theta}\ . \tag{10}$$

Therefore, for potential inside the cylinder,

$$\begin{aligned}\Phi(\rho,\phi) &= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\left(1+\frac{2b\rho\cos\theta-2\rho^2}{b^2+\rho^2-2b\rho\cos\theta}\right)d\phi' \\ &= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')}d\phi'\ .\end{aligned} \tag{11}$$

For potential outside the cylinder, we have to set $a_n = 0, n \geq 1$ and $b_0 = 0$, we just need to swap b and ρ in the fraction in the above expression. Therefore, for potential outside the cylinder,

$$\Phi(\rho,\phi) = -\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')}d\phi'\ . \tag{12}$$

2.13 (a) In last problem, we have computed the potential inside a cylinder is

$$\Phi(\rho,\phi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi' \tag{13}$$

Consider the integral of the form,

$$\begin{aligned}
&\int\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'\\
&=\int\frac{(b^2-\rho^2)(\sin^2\frac{\phi'-\phi}{2}+\cos^2\frac{\phi'-\phi}{2})d\phi'}{(b^2+\rho^2)(\sin^2\frac{\phi'-\phi}{2}+\cos^2\frac{\phi'-\phi}{2})-2b\rho(\cos^2\frac{\phi'-\phi}{2}-\sin^2\frac{\phi'-\phi}{2})}\\
&=\int\frac{2(b^2-\rho^2)d\tan\theta}{(b-\rho)^2+(b+\rho)^2\tan^2\theta}\ ,
\end{aligned} \tag{14}$$

where $\theta=\frac{\phi'-\phi}{2}$. Now let $u=\tan\theta$, we have

$$\begin{aligned}
\int\frac{2(b^2-\rho^2)du}{(b-\rho)^2+(b+\rho)^2u^2}&=\frac{2(b^2-\rho^2)}{(b-\rho)^2}\int\frac{du}{1+u^2(b+\rho)^2/(b-\rho)^2}\\
&=2\tan^{-1}\left(\frac{b+\rho}{b-\rho}u\right)\\
&=2\tan^{-1}\left(\frac{b+\rho}{b-\rho}\tan\left(\frac{\phi'-\phi}{2}\right)\right)
\end{aligned} \tag{15}$$

Therefore, in this case, the potential is

$$\begin{aligned}
\Phi(\rho,\phi)&=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}V_1\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'\\
&\quad+\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}V_2\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)}d\phi'\\
&=\frac{V_1}{\pi}\tan^{-1}\left(\frac{b+\rho}{b-\rho}\tan\left(\frac{\phi'-\phi}{2}\right)\right)\Bigg|_{-\pi/2}^{\pi/2}+\frac{V_2}{\pi}\tan^{-1}\left(\frac{b+\rho}{b-\rho}\tan\left(\frac{\phi'-\phi}{2}\right)\right)\Bigg|_{\pi/2}^{3\pi/2}
\end{aligned}$$

Since

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}, \tag{16}$$

therefore,

$$\alpha - \beta + n\pi = \tan^{-1}\left(\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\right) \ . \tag{17}$$

Here we add a constant factor $n\pi$ where $n \in \mathbb{Z}$ so that $\alpha - \beta + n\pi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Further,

$$\tan^{-1} x - \tan^{-1} y + n\pi = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \ . \tag{18}$$

So

$$\frac{V_1}{\pi}\tan^{-1}\left(\frac{b+\rho}{b-\rho}\tan\left(\frac{\phi' - \phi}{2}\right)\right)\Bigg|_{-\pi/2}^{\pi/2} = \frac{V_1}{\pi}\tan^{-1}\left(\frac{x-y}{1+xy}\right) + nV_1 \ , \tag{19}$$

with

$$x = \frac{b+\rho}{b-\rho}\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) = \frac{b+\rho}{b-\rho}\frac{\cos\phi}{1+\sin\phi} \tag{20}$$

$$y = \frac{b+\rho}{b-\rho}\tan\left(-\frac{\pi}{4} - \frac{\phi}{2}\right) = \frac{b+\rho}{b-\rho}\frac{-\cos\phi}{1-\sin\phi} \tag{21}$$

Therefore,

$$\frac{x-y}{1+xy} = -\frac{b^2-\rho^2}{2b\rho\cos\phi} \tag{22}$$

Therefore,

$$\frac{V_1}{\pi}\tan^{-1}\left(\frac{b+\rho}{b-\rho}\tan\left(\frac{\phi' - \phi}{2}\right)\right)\Bigg|_{-\pi/2}^{\pi/2} = -\frac{V_1}{\pi}\tan^{-1}\left(\frac{b^2-\rho^2}{2b\rho\cos\phi}\right) + nV_1 \ , \tag{23}$$

Similarly, we can compute

$$\frac{V_2}{\pi}\tan^{-1}\left(\frac{b+\rho}{b-\rho}\tan\left(\frac{\phi' - \phi}{2}\right)\right)\Bigg|_{\pi/2}^{3\pi/2} = \frac{V_2}{\pi}\tan^{-1}\left(\frac{b^2-\rho^2}{2b\rho\cos\phi}\right) + mV_2 \tag{24}$$

Moreover, $\tan^{-1} x + \tan^{-1}(1/x) + k\pi = \pi/2$. Therefore,

$$\Phi(\rho,\phi) = \left(n + k_1 - \frac{1}{2}\right)V_1 + \left(m - k_2 + \frac{1}{2}\right)V_2 + \frac{V_1 - V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) \ .$$

Now using boundary condition, for $\cos\phi > 0$, $\Phi(\rho = b, \phi) = V_1$. For $\cos\phi < 0$, $\Phi(\rho = b, \phi) = V_2$. We can fix the integer constants and get

$$\Phi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1}\left(\frac{2b\rho}{b^2 - \rho^2}\cos\phi\right) . \tag{25}$$

2.13 (b) The surface charge density is given by

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial n}\bigg|_{\rho=b} = \epsilon_0 \frac{\partial \Phi}{\partial \rho}\bigg|_{\rho=b} \tag{26}$$

We can calculate

$$\begin{aligned}
\frac{\partial \Phi(\rho,\phi)}{\partial \rho} &= \frac{V_1 - V_2}{\pi} \partial_\rho \tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) \\
&= \frac{V_1 - V_2}{\pi} \frac{1}{1 + \frac{4b^2\rho^2}{(b^2-\rho^2)^2}\cos^2\phi} \cos\phi \, \partial_\rho \left(\frac{2b\rho}{b^2-\rho^2}\right) \\
&= \frac{V_1 - V_2}{\pi} \frac{\cos\phi}{1 + \frac{4b^2\rho^2}{(b^2-\rho^2)^2}\cos^2\phi} \frac{2b(b^2+\rho^2)}{(b^2-\rho^2)^2} \\
&= \frac{V_1 - V_2}{\pi} \frac{2b(b^2+\rho^2)\cos\phi}{(b^2-\rho^2)^2 + 4b^2\rho^2\cos^2\phi}
\end{aligned} \tag{27}$$

Therefore,

$$\sigma = \epsilon_0 \frac{V_1 - V_2}{\pi b \cos\phi} \ . \tag{28}$$

2.17 (a) The three-dimensional Green function is given by

$$G(\vec{x},\vec{x}') = \frac{1}{R} \equiv \frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{29}$$

Consider the integration

$$\begin{aligned}\int \frac{1}{R} d(z'-z) &= \int \frac{du}{\sqrt{(x-x')^2+(y-y')^2+u^2}} \\ &= \ln\left(\sqrt{u^2+(x-x')^2+(y-y')^2}+u\right)+C \end{aligned} \tag{30}$$

Therefore, write $a=(x-x')^2+(y-y')^2$

$$\begin{aligned} G(x,y;x',y') &= \lim_{Z\to\infty}\int_{-Z}^{Z}\frac{1}{R}d(z'-z) \\ &= \lim_{Z\to\infty}\ln\left(\frac{\sqrt{Z^2+a}+Z}{\sqrt{Z^2+a}-Z}\right) \\ &= \lim_{Z\to\infty}\ln\left(\frac{(\sqrt{Z^2+a}+Z)^2}{a}\right) \\ &\sim -\ln a = -\ln[(x-x')^2+(y-y')^2] \\ &= -\ln[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')]\ , \end{aligned} \tag{31}$$

where we have dropped the constant term.

2.17 (b) The Dirac delta in polar coordinates is

$$\frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \ . \tag{32}$$

The Green function should satisfy

$$\nabla^2 G(\rho-\rho',\phi-\phi') = -4\pi\frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \ . \tag{33}$$

The Laplacian in polar coordinates is

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} \ . \tag{34}$$

Since Green function is symmetric in $\vec{r}$ and $\vec{r}'$, I interchanged ρ, ρ' and ϕ, ϕ' in the following discussion. Consider a solution to the Green function equation of the form $g(\rho,\rho')h(\phi-\phi')$. For $\vec{r}\neq\vec{r}'$, consider $\nabla^2(gh)=0$,

$$\frac{h}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right) + \frac{g}{\rho^2}\frac{\partial^2 h}{\partial\phi^2} = 0 \tag{35}$$

Using separation of variables, we have

$$\frac{\rho}{g}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right) = -\frac{1}{h}\frac{\partial^2 h}{\partial\phi^2} = m^2 \ . \tag{36}$$

We can solve for $h(\phi-\phi') = Ae^{im(\phi-\phi')}$, since it should be a period function in ϕ, m has to be an integer. We can write down a general solution as the superposition

$$G = \sum_m a_m g_m(\rho,\rho')e^{im(\phi-\phi')}$$

and for $\rho\neq\rho'$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right) - \frac{m^2}{\rho^2}g_m = 0 \ . \tag{37}$$

Now calculating ∇^2 again,

$$\nabla^2 G = \sum_m a_m e^{im(\phi-\phi')}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right) - \frac{m^2}{\rho^2}g_m\right] = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi').$$

1. If we start from all the a_m's are $1/(2\pi)$, then

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right) - \frac{m^2}{\rho^2}g_m = 2\pi\int_0^{2\pi} d\phi \left(\frac{1}{2\pi}e^{-im(\phi-\phi')}\nabla^2 G\right)$$
$$= -4\pi\frac{\delta(\rho-\rho')}{\rho} \ , \tag{38}$$

where I have used the fact that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\phi}d\phi = \delta_{mn} \ . \tag{39}$$

2. If we start from

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right) - \frac{m^2}{\rho^2}g_m = -4\pi\frac{\delta(\rho-\rho')}{\rho} \ , \tag{40}$$

then

$$-4\pi\frac{\delta(\rho-\rho')}{\rho}a_m = \int_0^{2\pi} d\phi \left(\frac{1}{2\pi}e^{-im(\phi-\phi')}\nabla^2 G\right) \ . \tag{41}$$

and

$$a_m = -\frac{1}{4\pi}\int \rho d\rho \int d\phi \left(\frac{1}{2\pi}e^{-im(\phi-\phi')}\nabla^2 G\right) = \frac{1}{2\pi} \ . \tag{42}$$

2.17 (c) Still, I interchanged ρ and ρ'. First consider $m \neq 0$. For $\rho \neq \rho'$, the solution to

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right) - \frac{m^2}{\rho^2}g_m = 0 \tag{43}$$

is simply $A_m\rho^{|m|} + B_m\rho^{-|m|}$. Now consider finiteness. For $\rho < \rho'$, we would have

$$g_m(\rho, \rho') = A_m\rho^{|m|} \ . \tag{44}$$

For $\rho > \rho'$, we would have

$$g_m(\rho, \rho') = B_m\rho^{-|m|} \ . \tag{45}$$

In order for g_m to be continuos, we should have

$$A_m\rho'^{|m|} = B_m\rho'^{-|m|} = C_m \ . \tag{46}$$

Therefore,

$$\begin{aligned} g_m(\rho, \rho') &= \begin{cases} C_m\left(\frac{\rho}{\rho'}\right)^{|m|} & \rho < \rho' \\ C_m\left(\frac{\rho'}{\rho}\right)^{|m|} & \rho > \rho' \end{cases} \\ &= C_m\left(\frac{\rho_<}{\rho_>}\right)^{|m|} \ . \end{aligned} \tag{47}$$

When $\rho = \rho'$, there is a singularity. Integrate the differential equation

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right) - \frac{m^2}{\rho}g_m = -4\pi\delta(\rho - \rho') \ , \tag{48}$$

for ρ in a small interval from $\rho' - \epsilon$ to $\rho' + \epsilon$, and take the limit $\epsilon \to 0$, since the function g_m/ρ is finite, we obtain

$$\rho\frac{\partial g_m}{\partial\rho}\bigg|_{\rho'+\epsilon} - \rho\frac{\partial g_m}{\partial\rho}\bigg|_{\rho'-\epsilon} = -4\pi \tag{49}$$

and

$$(\rho' + \epsilon)g_m'\big|_{\rho'+\epsilon} - (\rho' - \epsilon)g_m'\big|_{\rho'-\epsilon} = -4\pi. \tag{50}$$

Taking $\epsilon \to 0$, we have

$$g'_m\big|_{\rho'+\epsilon} - g'_m\big|_{\rho'-\epsilon} = -\frac{4\pi}{\rho'} \tag{51}$$

$$-|m|\frac{C_m}{\rho'} - |m|\frac{C_m}{\rho'} = -\frac{4\pi}{\rho'} \tag{52}$$

We can obtain, $C_m = 2\pi/|m|$ for $m \neq 0$. Therefore,

$$g_m(\rho, \rho') = \frac{2\pi}{|m|}\left(\frac{\rho_<}{\rho_>}\right)^{|m|}, \quad m \neq 0 \tag{53}$$

For $m = 0$, consider g_0, if $\rho < \rho'$, the general solution is a constant A. If $\rho > \rho'$, the general solution is $B \ln \rho$. The continuity gives $A = B \ln \rho'$. The integration procedure would give

$$\frac{B}{\rho'} - 0 = -\frac{4\pi}{\rho'} \ . \tag{54}$$

Therefore,

$$\begin{aligned} g_0(\rho, \rho') &= \begin{cases} -4\pi \ln \rho' & \rho < \rho' \\ -4\pi \ln \rho & \rho > \rho' \end{cases} \\ &= -2\pi \ln(\rho_>^2) \ . \end{aligned} \tag{55}$$

Therefore the Green function is

$$\begin{aligned} G &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} g_m(\rho, \rho') e^{im(\phi-\phi')} \\ &= -\ln(\rho_>^2) + \sum_{m\neq 0} \frac{1}{|m|}\left(\frac{\rho_<}{\rho_>}\right)^{|m|} e^{im(\phi-\phi')} \\ &= -\ln(\rho_>^2) + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^{m} e^{im(\phi-\phi')} + \sum_{m=-1}^{-\infty} \frac{1}{-m}\left(\frac{\rho_<}{\rho_>}\right)^{-m} e^{im(\phi-\phi')} \\ &= -\ln(\rho_>^2) + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^{m} e^{im(\phi-\phi')} + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^{m} e^{-im(\phi-\phi')} \\ &= -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^{m} \left(e^{im(\phi-\phi')} + e^{-im(\phi-\phi')}\right)/2 \\ &= -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^{m} \cdot \cos[m(\phi-\phi')] \ . \end{aligned} \tag{56}$$

2.24 We want to show that, for $0 < \phi, \phi' < \beta$,

$$\delta(\phi - \phi') = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin(m\pi\phi/\beta) \sin(m\pi\phi'/\beta) \ .$$

Since $\sin(m\pi\phi/\beta)$ with integer m form a complete set for functions on the interval $[0, \beta]$ with Dirichlet boundary conditions. We can write for an arbitrary function $f(\phi)$ in ϕ,

$$f(\phi) = \sum_{m=1}^{\infty} a_m \sin(m\pi\phi/\beta) \tag{57}$$

For a positive integer n, consider the integral

$$\int_0^\beta f(\phi) \sin(n\pi\phi/\beta) d\phi = \sum_{m=1}^{\infty} a_m \int_0^\beta \sin(n\pi\phi/\beta) \sin(m\pi\phi/\beta) d\phi \tag{58}$$

Since

$$\sin(n\pi\phi/\beta) \sin(m\pi\phi/\beta) = \frac{1}{2}[\cos((n-m)\pi\phi/\beta) - \cos((n+m)\pi\phi/\beta)] \tag{59}$$

Integrate ϕ over $[0, \beta]$, we obtain

$$\begin{aligned} \int_0^{2\pi} \sin(n\pi\phi/\beta) \sin(m\pi\phi/\beta) d\phi &= \frac{\beta}{2}(\delta_{n,m} - \delta_{n,-m}) \\ &= \frac{\beta}{2}\delta_{n,m} \ , \end{aligned} \tag{60}$$

since we are assuming both n and m are positive. Plug this back into (58), we obtain

$$a_n = \frac{2}{\beta} \int_0^\beta f(\phi) \sin(n\pi\phi/\beta) \ . \tag{61}$$

Plug the formula of a_m back to (57), we obtain

$$\begin{aligned} f(\phi) &= \sum_{m=1}^{\infty} a_m \sin(m\pi\phi/\beta) \\ &= \int_0^\beta \left(\frac{2}{\beta} \sum_{m=1}^{\infty} \sin(m\pi\phi'/\beta) \sin(m\pi\phi/\beta) \right) f(\phi') \end{aligned} \tag{62}$$

Therefore what's inside the bracket is a delta function,

$$\delta(\phi - \phi') = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin(m\pi\phi/\beta) \sin(m\pi\phi'/\beta) \ . \tag{63}$$