**Phys 5405**

HW 8

3.17 3.22

**3.17** The Dirichlet Green function for the unbounded space between the planes at $z = 0$ and $z = L$ allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential.

**(a)** Using cylindrical coordinates show that one form of the Green function is

$$G(\mathbf{x}, \mathbf{x}') = \frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right) \ . \tag{1}$$

**(b)** Show that an alternative form of the Green function is

$$G(\mathbf{x}, \mathbf{x}') = 2 \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<) \sinh[k(L-z_>)]}{\sinh(kL)} \ . \tag{2}$$

**3.17 (a)** Now consider the Green function

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta^3(\mathbf{x} - \mathbf{x}') = -\frac{4\pi}{\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z') \ . \tag{3}$$

The $\phi$ and $z$ delta functions can be written in terms of orthonormal functions

$$\delta(z - z') = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right), \quad \delta(\phi - \phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \ . \tag{4}$$

We expand the Green function in similar fashion,

$$\boxed{G(\mathbf{x}, \mathbf{x}') = \frac{1}{\pi L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) g_{mn}(\rho, \rho') \ .} \tag{5}$$

Now plug this back into (3), we obtain

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_{mn}}{d\rho}\right) - \left(\frac{m^2}{\rho^2} + \left(\frac{n\pi}{L}\right)^2\right) g_{mn} = -\frac{4\pi}{\rho}\delta(\rho - \rho') \ . \tag{6}$$

For $\rho \neq \rho'$, this is just equation for the modified Bessel function $I_m(n\pi\rho/L)$ and $K_m(n\pi\rho/L)$. Suppose that $\psi_1(n\pi\rho/L)$ is some linear combination of $I_m$ and $K_m$ for $\rho < \rho'$ and $\psi_2(n\pi\rho/L)$ is a a linearly independent combination for $\rho > \rho'$. Then the symmetry of the Green function in $\rho$ and $\rho'$ requires that

$$g_{mn}(\rho, \rho') = \psi_1(n\pi\rho_</L)\psi_2(n\pi\rho_>/L) \ . \tag{7}$$

The normalization of the product $\psi_1\psi_2$ is determined by the discontinuity in slope implied by the delta function in (6):

$$\left.\frac{dg_{mn}}{d\rho}\right|_+ - \left.\frac{dg_{mn}}{d\rho}\right|_- = \frac{n\pi}{L}(\psi_1\psi_2' - \psi_2\psi_1') = -\frac{4\pi}{\rho'} \ . \tag{8}$$

If there are no boundary surfaces, $g_{mn}$ must be finite at $\rho = 0$ and vanish at $\rho \to \infty$. Consequently $\psi_1(x) = AI_m(x)$ and $\psi_2(x) = K_m(x)$. And (8) implies that $A = 4\pi$. Therefore,

$$\boxed{G(\mathbf{x}, \mathbf{x}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right) \ .} \tag{9}$$

**3.17 (b)** Similarly,The $\phi$ and $\rho$ delta functions can be written in terms of orthonormal functions

$$\frac{1}{\rho}\delta(\rho-\rho') = \int_0^\infty kJ_m(k\rho)J_m(k\rho')dk, \quad \delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \ . \tag{10}$$

We expand the Green function in similar fashion,

$$\boxed{G(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dke^{im(\phi-\phi')}kJ_m(k\rho)J_m(k\rho')h(k,z,z') \ .} \tag{11}$$

Now plug this back into (3), we obtain

$$\frac{d^2h}{dz^2} - k^2h = -4\pi\delta(z-z') \tag{12}$$

For $z \neq z'$, we can solve that

$$h(k,z,z') \propto e^{\pm kx} \ . \tag{13}$$

Considering boundary conditions $h(k, z = 0, z') = h(k, z = L, z') = 0$, we can assume $h(k, z, z')$ is of the form

$$h(k,z,z') = \begin{cases} A\sinh(kz) \ , & z < z' \\ B\sinh(k(L-z)), & z > z' \end{cases} \ . \tag{14}$$

The continuity of $h$ at $z = z'$ gives

$$A\sinh(kz') = B\sinh(k(L-z')) \ . \tag{15}$$

The normalization is determined by the discontinuity in slope implied by the delta function in (12):

$$\left.\frac{dh}{dz}\right|_+ - \left.\frac{dh}{dz}\right|_- = -Bk\cosh(k(L-z')) - Ak\cosh(kz') = -4\pi \ . \tag{16}$$

We can derive that

$$A = 4\pi\frac{\sinh(k(L-z'))}{k\sinh(kL)} \ , \quad B = 4\pi\frac{\sinh(kz')}{k\sinh(kL)} \ . \tag{17}$$

Therefore,

$$\boxed{h(k,z,z') = 4\pi\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{k\sinh(kL)}} \tag{18}$$

Therefore,

$$\boxed{G(\mathbf{x},\mathbf{x}') = 2\sum_{m=-\infty}^{\infty}\int_0^\infty dke^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)} \ .} \tag{19}$$

**3.22** The geometry of a two-dimensional potential problem is defined in polar coordinates by the surfaces $\phi = 0, \phi = \beta$, and $\rho = a$. Using separation of variables in polar coordinates, show that the Green function can be written as

$$G(\rho, \phi; \rho', \phi') = \sum_{m=1}^{\infty} \frac{4}{m} \rho_<^{m\pi/\beta} \left( \frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}} \right) \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right) \tag{20}$$

**3.22** The delta function of $\phi$ can be expanded as

$$\delta(\phi-\phi')=\frac{2}{\beta}\sum_{m=1}^{\infty}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)\ . \tag{21}$$

Then the Green function can be expanded as

$$\boxed{G(\rho,\phi;\rho',\phi')=\frac{2}{\beta}\sum_{m=1}^{\infty}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)g_m(\rho,\rho')\ .} \tag{22}$$

Plug this into equation

$$\nabla^2 G=-\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\ , \tag{23}$$

we obtain

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right)-\frac{1}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2 g_m=-\frac{4\pi}{\rho}\delta(\rho-\rho') \tag{24}$$

When $\rho\neq\rho'$, we can obtain

$$g_m(\rho,\rho')\propto\rho^{\pm m\pi/\beta}\ . \tag{25}$$

Consider finiteness and also $g$ vanishing at $\rho=a$, we can assume $g$ is of the form

$$g_m(\rho,\rho')=\begin{cases}A\rho^{m\pi/\beta}\ , & \rho<\rho'\\ B\rho^{m\pi/\beta}-Ba^{2m\pi/\beta}\rho^{-m\pi/\beta}\ , & \rho>\rho'\end{cases}\ . \tag{26}$$

Then continuity at $\rho=\rho'$ implies that

$$A\rho'^{m\pi/\beta}=B\rho'^{m\pi/\beta}-Ba^{2m\pi/\beta}\rho'^{-m\pi/\beta}\ . \tag{27}$$

Integrating the differential equation over a small interval containing $\rho'$ gives

$$\left.\frac{dg_m}{d\rho}\right|_+-\left.\frac{dg_m}{d\rho}\right|_-=-\frac{4\pi}{\rho'}\ , \tag{28}$$

which leads to

$$(m\pi/\beta)B\rho'^{m\pi/\beta-1}+(m\pi/\beta)Ba^{2m\pi/\beta}\rho'^{-m\pi/\beta-1}-(m\pi/\beta)A\rho'^{m\pi/\beta-1}=-\frac{4\pi}{\rho'}\ . \tag{29}$$

We can solve for

$$A=\frac{2\pi}{m\pi/\beta}\left(\rho'^{-m\pi/\beta}-a^{-2m\pi/\beta}\rho'^{m\pi/\beta}\right)\ ,\quad B=-\frac{2\pi}{m\pi/\beta}a^{-2m\pi/\beta}\rho'^{m\pi/\beta} \tag{30}$$

Therefore, we obtain

$$\boxed{g_m(\rho,\rho')=\frac{2\beta}{m}\rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}}-\frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)\ .} \tag{31}$$

Therefore, the Green function can be written as

$$\boxed{G(\rho,\phi;\rho',\phi') = \sum_{m=1}^{\infty} \frac{4}{m}\rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}}\right)\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)\ .} \tag{32}$$