Phys 5405
HW 2

1.14 The Green's theorem is,

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) da \ . \tag{1}$$

Apply this with integration variable $\vec{y}$ and $\phi = G(\vec{x},\vec{y})$, $\psi = G(\vec{x}',\vec{y})$ with $\nabla_y^2 G(\vec{z},\vec{y}) = -4\pi\delta(\vec{y}-\vec{z})$. Suppose $\vec{x}$ and $\vec{x}'$ are inside the volume V,then we have

$$\begin{aligned}
\text{L.H.S.} &= \int_V \Big(G(\vec{x},\vec{y})\nabla_y^2 G(\vec{x}',\vec{y}) - G(\vec{x}',\vec{y})\nabla_y^2 G(\vec{x},\vec{y})\Big)d^3y \\
&= -4\pi\int_V \Big(G(\vec{x},\vec{y})\delta(\vec{y}-\vec{x}') - G(\vec{x}',\vec{y})\delta(\vec{y}-\vec{x})\Big)d^3y \\
&= -4\pi[G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x})] \ . \qquad (2)
\end{aligned}$$

Therefore,

$$G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x}) = \frac{1}{4\pi}\oint_S \left(G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n} - G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n}\right) da_y \ . \tag{3}$$

(a) For Dirichlet boundary conditions, we have

$$G_D(\vec{x}, \vec{y}) = 0 \ , \quad \text{for } \vec{y} \text{ on } S \ . \tag{4}$$

The surface integral vanishes and we have

$$G_D(\vec{x}, \vec{x}') - G_D(\vec{x}', \vec{x}) = 0 \ . \tag{5}$$

Therefore, $G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x})$ and $G_D(\vec{x}, \vec{x}')$ is symmetric in $\vec{x}$ and $\vec{x}'$.

(b) For Neumann boundary conditions, we have

$$\frac{\partial G_N}{\partial n'}(\vec{x},\vec{y}) = -\frac{4\pi}{S}, \quad \text{for } \vec{y} \text{ on } S \ . \tag{6}$$

Then,

$$G_N(\vec{x},\vec{x}') - G_N(\vec{x}',\vec{x}) = \frac{1}{S}\oint_S \Big(G_N(\vec{x},\vec{y}) - G_N(\vec{x}',\vec{y})\Big) da_y \ . \tag{7}$$

and

$$G_N(\vec{x},\vec{x}') - \frac{1}{S}\oint_S G_N(\vec{x},\vec{y}) da_y = G_N(\vec{x}',\vec{x}) - \frac{1}{S}\oint_S G_N(\vec{x}',\vec{y}) da_y \ . \tag{8}$$

Therefore, $G_N(\vec{x},\vec{x}')$ is not symmetric in general, but $G_N(\vec{x},\vec{x}') - F(\vec{x})$ is symmetric in $\vec{x}$ and $\vec{x}'$, where

$$F(\vec{x}) = \frac{1}{S}\oint_S G_N(\vec{x},\vec{y}) da_y \ . \tag{9}$$

(c) The Neumann boundary solution is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')d^3x' + \frac{1}{4\pi}\int_S G(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'}da' \\ + \frac{1}{S}\int_S \Phi(\vec{x}')d^3x' \; . \tag{10}$$

For a transformation, $G_N(\vec{x},\vec{x}') \to G_N(\vec{x},\vec{x}') + F(\vec{x})$,

$$\begin{aligned}\Phi'(\vec{x}) - \Phi(\vec{x}) &= \frac{1}{4\pi\epsilon_0}F(\vec{x})\int_V \rho(\vec{x}')d^3x' + \frac{1}{4\pi}F(\vec{x})\int_S \frac{\partial\Phi}{\partial n'}da' \\ &= \frac{F(\vec{x})}{4\pi}\left(\int_V \frac{\rho(\vec{x}')}{\epsilon_0}d^3x' + \int_S \nabla\Phi\cdot\hat{n}da;\right)\end{aligned} \tag{11}$$

Since $\vec{E} = -\nabla\Phi$ and $\int_S \vec{E}\cdot\hat{n}da = \int_V \frac{\rho(\vec{x}')}{\epsilon_0}d^3x'$, we have

$$\Phi'(\vec{x}) = \Phi(\vec{x}) \; . \tag{12}$$

Therefore, the addition of $F(\vec{x})$ to the Green function does not affect the potential $\Phi(\vec{x})$.

2.2 (a) Suppose the original charge is at $\vec{y}$. Substitute the conductor with an image charge at $\vec{y}'$ with charge q'. The potential can be written as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}'|}\right) . \tag{13}$$

The potential has to vanish on conductor, so $\Phi(|\vec{x}| = a) = 0$

$$\Phi(|\vec{x}| = a) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{a^2+y^2-2ay\cos\theta}} + \frac{q'}{\sqrt{a^2+y'^2-2ay'\cos\theta}}\right) . \tag{14}$$

Therefore, q and q' should have different signs and

$$\frac{q^2}{q'^2} = \frac{a^2+y^2-2ay\cos\theta}{a^2+y'^2-2ay'\cos\theta} , \tag{15}$$

which leads to

$$y' = \frac{a^2}{y}, \quad q' = -\frac{a}{y}q \tag{16}$$

By symmetry, $\vec{y}'$ has to be parallel with $\vec{y}$, therefore, for $|\vec{x}| < a$,

$$\Phi(\vec{x}') = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x}-\vec{y}|} - \frac{a/y}{|\vec{x}-(a^2/y^2)\vec{y}|}\right) . \tag{17}$$

(b) The surface charge density can be evaluated as

$$
\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial \Phi}{\partial n}\bigg|_{|\vec{x}|=a} = \epsilon_0 \frac{\partial \Phi}{\partial x}\bigg|_{|\vec{x}|=a} \\
&= \frac{q}{4\pi}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2-2xy\cos\theta}} - \frac{a/y}{\sqrt{x^2+a^4/y^2-2xa^2\cos\theta/y}}\right)\Bigg|_{|\vec{x}|=a} \\
&= \frac{q}{4\pi}\left(\frac{-a+y\cos\theta}{(a^2+y^2-2ay\cos\theta)^{3/2}} - \frac{y\cos\theta - y^2/a}{(a^2+y^2-2ay\cos\theta)^{3/2}}\right) \\
&= \frac{q}{4\pi}\frac{y^2/a-a}{(a^2+y^2-2ay\cos\theta)^{3/2}} = \frac{q}{4\pi a y}\frac{1-a^2/y^2}{(1+a^2/y^2-2(a/y)\cos\theta)^{3/2}}\ . \quad (18)
\end{aligned}
$$

(c) The force is then

$$\vec{F} = \frac{1}{4\pi\epsilon_0} qq' \frac{\vec{y} - \vec{y}'}{|\vec{y} - \vec{y}'|^3} = \frac{q^2 a}{4\pi\epsilon_0 y^3} \frac{\hat{y}}{\left(1 - \frac{a^2}{y^2}\right)^2} \ . \tag{19}$$