Phys 5405

HW 9

4.2 4.7(a,b) 4.8(a)

4.2 A point dipole with dipole moment $\mathbf{p}$ is located at the point $\mathbf{x}_0$. From the properties of the derivative of a Dirac delta function, show that for calculation of the potential Φ or the energy of a dipole in an external field, the dipole can be described by an effective charge density

$$\rho_{\text{eff}}(\mathbf{x}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{x} - \mathbf{x}_0)$$

4.2 The potential from the dipole is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \vec{p}(\vec{x}') \cdot \nabla' \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \ . \tag{1}$$

The distribution of the dipole is a delta function,

$$\vec{p}(\vec{x}') = \vec{p}\,\delta(\vec{x}' - \vec{x}_0) \ . \tag{2}$$

Plug it into the formula of potential and integrate by parts,

$$\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0} \int d^3x' \vec{p} \cdot \nabla' \delta(\vec{x}' - \vec{x}_0) \frac{1}{|\vec{x} - \vec{x}'|} \equiv \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho_{\text{eff}}(\vec{x}')}{|\vec{x} - \vec{x}'|} \ . \tag{3}$$

Therefore, the dipole can be described by an effective charge density,

$$\boxed{\rho_{\text{eff}}(\vec{x}) = -\vec{p} \cdot \nabla \delta(\vec{x} - \vec{x}_0) \ .} \tag{4}$$

4.7 A localized distribution of charge has a charge density

$$\rho(\mathbf{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta$$

(a) Make a multipole expansion of the potential due to this charge density and determine all the nonvanishing multipole moments. Write down the potential at large distances as a finite expansion in Legendre polynomials.
(b) Determine the potential explicitly at any point in space, and show that near the origin, correct to r^2 inclusive,

$$\Phi(\mathbf{r}) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120} P_2(\cos\theta)\right]$$

4.7 (a) The multipole expansion is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \ , \tag{5}$$

with multipole moments,

$$\begin{aligned} q_{lm} &= \int Y^*_{lm}(\theta,\phi) r^l \rho(\vec{x}) d^3x \\ &= \int Y^*_{lm}(\theta,\phi) r^l \left(\frac{1}{64\pi} r^2 e^{-r}\sin^2\theta\right) r^2 \sin\theta dr d\theta d\phi \end{aligned} \tag{6}$$

The spherical harmonics are related to the associated Legendre polynomials as

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi} \ . \tag{7}$$

Consider the integral $\int_0^{2\pi} e^{im\phi}d\phi$, it vanishes for $m \neq 0$. Therefore,

$$\begin{aligned} q_{lm} &= \delta_{m,0}\int \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta) r^l \left(\frac{1}{64\pi}r^2 e^{-r}\sin^2\theta\right) r^2\sin\theta dr d\theta d\phi \\ &= \frac{2\pi}{64\pi}\sqrt{\frac{2l+1}{4\pi}}\delta_{m,0}\left(\int dr\, r^{l+4}e^{-r}\right)\left(\int d\theta\, P_l(\cos\theta)\sin^3\theta\right) \end{aligned} \tag{8}$$

By the definition of Gamma functions,

$$\boxed{\int_0^\infty dr\, r^{l+4}e^{-r} = \Gamma(l+5) = (l+4)!} \tag{9}$$

For another integral, we change the variable as $x = \cos\theta$, and we also make use of the identity, $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$, then the integral is now given by

$$\boxed{\int_{-1}^{1} dx\, P_l(x)(1-x^2) = \frac{2}{3}\int_{-1}^{1} dx\, P_l(x)(1-P_2(x)) = \frac{2}{3}\left(2\delta_{l,0} - \frac{2}{5}\delta_{l,2}\right) \ .} \tag{10}$$

Therefore, the only non-zero multipole moments are

$$\boxed{q_{00} = \frac{1}{2}\frac{1}{\sqrt{\pi}} \ , \quad q_{20} = -3\sqrt{\frac{5}{\pi}} \ .} \tag{11}$$

Therefore, the potential is given by

$$\boxed{\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{6P_2(\cos\theta)}{r^3}\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{P_0(\cos\theta)}{r} - \frac{6P_2(\cos\theta)}{r^3}\right] \ .} \tag{12}$$

The potential is calculated as

$$\Phi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' = \frac{1}{4\pi\epsilon_0}\frac{1}{64\pi}\int \frac{r'^2 e^{-r'}\sin^2\theta'}{|\vec{x}-\vec{x}'|}r'^2\sin\theta' dr' d\theta' d\phi' \ . \tag{13}$$

Now we use Jackson (3.70),

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \ . \tag{14}$$

Due to azimuthal symmetry of the charge distribution, we only need to consider $m=0$,

$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)P_l(\cos\theta') \ .} \tag{15}$$

Therefore, the potential is

$$\begin{aligned}\Phi &= \sum_{l=0}^{\infty}\frac{1}{4\pi\epsilon_0}\frac{1}{64\pi}\int r'^2 e^{-r'}\sin^2\theta'\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)P_l(\cos\theta')r'^2\sin\theta' dr' d\theta' d\phi' \\ &= \sum_{l=0}^{\infty}\frac{1}{4\pi\epsilon_0}\frac{1}{32}P_l(\cos\theta)\left(\int r'^4 e^{-r'}\frac{r_<^l}{r_>^{l+1}}dr'\right)\left(\int_{-1}^{1}dx\, P_l(x)(1-x^2)\right) \ .\end{aligned} \tag{16}$$

The second integral is already calculated in (10), we copy it here for reference,

$$\int_{-1}^{1}dx\, P_l(x)(1-x^2) = \frac{2}{3}\left(2\delta_{l,0}-\frac{2}{5}\delta_{l,2}\right) \ . \tag{17}$$

Therefore,

$$\boxed{\Phi = \frac{1}{4\pi\epsilon_0}\frac{1}{24}\left(\int r'^4 e^{-r'}\frac{1}{r_>}dr'\right) - \frac{1}{4\pi\epsilon_0}\frac{1}{120}\left(\int r'^4 e^{-r'}\frac{r_<^2}{r_>^3}dr'\right)P_2(\cos\theta) \ .} \tag{18}$$

First,

$$\begin{aligned}\int_0^{\infty}dr'\, r'^4 e^{-r'}\frac{1}{r_>} &= \int_0^{r}dr'\, r'^4 e^{-r'}\frac{1}{r} + \int_r^{\infty}dr'\, r'^4 e^{-r'}\frac{1}{r'} \\ &= \boxed{\frac{1}{r}e^{-r}(-24+24e^r-18r-6r^2-r^3)} \ .\end{aligned} \tag{19}$$

Second,

$$\begin{aligned}\int_0^{\infty}dr'\, r'^4 e^{-r'}\frac{r_<^2}{r_>^3} &= \int_0^{r}dr'\, r'^4 e^{-r'}\frac{r'^2}{r^3} + \int_r^{\infty}dr'\, r'^4 e^{-r'}\frac{r^2}{r'^3} \\ &= \boxed{5\frac{1}{r^3}e^{-r}(-144+144e^r-144r-72r^2-24r^3-6r^4-r^5)} \ .\end{aligned} \tag{20}$$

Therefore, the exact potential is given by

$$\boxed{\begin{aligned}\Phi = &\frac{1}{4\pi\epsilon_0}\frac{e^{-r}}{r}\left(e^r - 1 - \frac{3}{4}r - \frac{1}{4}r^3 - \frac{1}{24}r^3\right)\\ &- \frac{1}{4\pi\epsilon_0}\frac{e^{-r}}{r^3}\left(6e^r - 6 - 6r - 3r^2 - r^2 - \frac{1}{4}r^4 - \frac{1}{24}r^5\right)P_2(\cos\theta)\ .\end{aligned}} \tag{21}$$

Near the origin, we expand,

$$e^{-r} = 1 - r + \frac{1}{2}r^2 - \frac{1}{6}r^3 + \frac{1}{24}r^4 - \frac{1}{120}r^5 + \cdots\ . \tag{22}$$

Then, up to r^2 order,

$$\Phi \approx \frac{1}{4\pi\epsilon_0}\frac{1}{4} - \frac{1}{4\pi\epsilon_0}\frac{1}{120}r^2 P_2(\cos\theta) = \boxed{\frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right]}\ . \tag{23}$$

4.8 A very long, right circular, cylindrical shell of dielectric constant ϵ/ϵ_0 and inner and outer radii a and b, respectively, is placed in a previously uniform electric field E_0 with its axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity.

(a) Determine the potential and electric field in the three regions, neglecting end effects.

4.8 (a) Neglecting end effects, we can think that the cylinder is infinitely long. Since the system is translation invariant along z-axis, we can treat it as a two dimensional problem. Denote the distance from the axis by ρ and the angle by ϕ. The general solution is

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n) \ . \tag{24}$$

Since this system satisfies $\Phi(\rho,\phi) = \Phi(\rho,-\phi)$, we can write the general solution as

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} (a_n \rho^n + b_n \rho^{-n}) \cos(n\phi) \ . \tag{25}$$

Now consider finiteness. For $\rho < a$,

$$\Phi_1(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \cos(n\phi) \ . \tag{26}$$

For $a < \rho < b$,

$$\Phi_2(\rho,\phi) = b_0 + c_0 \ln\rho + \sum_{n=1}^{\infty} (b_n \rho^n + c_n \rho^{-n}) \cos(n\phi) \ . \tag{27}$$

For $\rho > b$,

$$\Phi_3(\rho,\phi) = d_0 + \sum_{n=1}^{\infty} d_n \rho^{-n} \cos(n\phi) - E_0 \rho \cos\phi \ , \tag{28}$$

where the last term is added due to the constant electric field at infinity.

The boundary conditions are the tangential E is continuous,

$$-\frac{1}{a}\frac{\partial \Phi_1}{\partial \phi}\bigg|_{\rho=a} = -\frac{1}{a}\frac{\partial \Phi_2}{\partial \phi}\bigg|_{\rho=a} \ , \quad -\frac{1}{b}\frac{\partial \Phi_2}{\partial \phi}\bigg|_{\rho=b} = -\frac{1}{b}\frac{\partial \Phi_3}{\partial \phi}\bigg|_{\rho=b} \ . \tag{29}$$

Also, the normal D is continuous,

$$-\epsilon_0 \frac{\partial \Phi_1}{\partial \rho}\bigg|_{\rho=a} = -\epsilon \frac{\partial \Phi_2}{\partial \rho}\bigg|_{\rho=a} \ , \quad -\epsilon \frac{\partial \Phi_2}{\partial \rho}\bigg|_{\rho=b} = -\epsilon_0 \frac{\partial \Phi_3}{\partial \rho}\bigg|_{\rho=b} \ . \tag{30}$$

Now we calculate the derivatives of Φ_1,

$$\frac{\partial \Phi_1}{\partial \phi} = -\sum_{n=1}^{\infty} n a_n \rho^n \sin(n\phi) \ , \quad \frac{\partial \Phi_1}{\partial \rho} = \sum_{n=1}^{\infty} n a_n \rho^{n-1} \cos(n\phi) \ . \tag{31}$$

Also, the derivatives of Φ_2,

$$\frac{\partial \Phi_2}{\partial \phi} = -\sum_{n=1}^{\infty} n(b_n \rho^n + c_n \rho^{-n}) \sin(n\phi) \ , \quad \frac{\partial \Phi_2}{\partial \rho} = \frac{c_0}{\rho} + \sum_{n=1}^{\infty} (n b_n \rho^{n-1} - n c_n \rho^{-n-1}) \cos(n\phi) \ . \tag{32}$$

And the derivatives of Φ_3,

$$\frac{\partial \Phi_3}{\partial \phi} = -\sum_{n=1}^{\infty} n d_n \rho^{-n} \sin(n\phi) + E_0 \rho \sin\phi \ , \qquad \frac{\partial \Phi_3}{\partial \rho} = -\sum_{n=1}^{\infty} n d_n \rho^{-n-1} \cos(n\phi) - E_0 \cos\phi \ . \quad (33)$$

From (29), we can derive that,

$$\boxed{\begin{aligned} & c_n = (a_n - b_n) a^{2n} \ , \quad n \geq 1 \ . && (34) \\ & b_n = (d_n - c_n) b^{-2n} \ , \quad n \geq 2 \ . && (35) \\ & b_1 = (d_1 - c_1) b^{-2} - E_0 \ . && (36) \end{aligned}}$$

From (30), we can derive that,

$$\boxed{\begin{aligned} c_0 &= 0 \ . && (37) \\ a_n a^{2n} &= \frac{\epsilon}{\epsilon_0}(b_n a^{2n} - c_n) \ , \quad n \geq 1 \ . && (38) \\ -\epsilon(b_n b^{n-1} - c_n b^{-n-1}) &= \epsilon_0 d_n b^{-n-1} \ , \quad n \geq 2 \ . && (39) \\ -\epsilon(b_1 - c_1 b^{-2}) &= \epsilon_0 d_1 b^{-2} + \epsilon_0 E_0 \ . && (40) \end{aligned}}$$

For $n \geq 2$, we can derive that

$$\boxed{a_n = b_n = c_n = d_n = 0 \ .} \quad (41)$$

For $n = 1$, We can derive that

$$a_1 = \frac{2\epsilon}{\epsilon + \epsilon_0} b_1 \ , c_1 = \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} a^2 b_1 \ , d_1 = \frac{2\epsilon}{\epsilon + \epsilon_0} \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} a^2 b_1 + \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} E_0 b^2 \ , \quad (42)$$

with b_1 given by

$$b_1 = \left[\left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \right)^2 \frac{a^2}{b^2} - 1 \right]^{-1} \frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0 \ . \quad (43)$$

Therefore, we obtain

$$\boxed{\begin{aligned} a_1 &= \frac{4\epsilon\epsilon_0 b^2 E_0}{a^2(\epsilon - \epsilon_0)^2 - b^2(\epsilon + \epsilon_0)^2} \ . && (44) \\ b_1 &= \frac{2\epsilon_0(\epsilon + \epsilon_0) b^2 E_0}{a^2(\epsilon - \epsilon_0)^2 - b^2(\epsilon + \epsilon_0)^2} \ . && (45) \\ c_1 &= \frac{2\epsilon_0(\epsilon - \epsilon_0) a^2 b^2 E_0}{a^2(\epsilon - \epsilon_0)^2 - b^2(\epsilon + \epsilon_0)^2} \ . && (46) \\ d_1 &= \frac{(\epsilon^2 - \epsilon_0^2)(a^2 - b^2) b^2 E_0}{a^2(\epsilon - \epsilon_0)^2 - b^2(\epsilon + \epsilon_0)^2} \ . && (47) \end{aligned}}$$

Now the continuity of the potential leads

$$a_0 = b_0 = d_0 \ . \quad (48)$$

We can drop this constant in the potential. and now we have

$$\Phi(\rho,\phi) = \begin{cases} \frac{4\epsilon\epsilon_0 b^2 E_0}{a^2(\epsilon-\epsilon_0)^2-b^2(\epsilon+\epsilon_0)^2}\rho\cos\phi\ , & \rho < a \\ \\ \frac{2\epsilon_0(\epsilon+\epsilon_0)b^2 E_0}{a^2(\epsilon-\epsilon_0)^2-b^2(\epsilon+\epsilon_0)^2}\rho\cos\phi + \frac{2\epsilon_0(\epsilon-\epsilon_0)a^2 b^2 E_0}{a^2(\epsilon-\epsilon_0)^2-b^2(\epsilon+\epsilon_0)^2}\rho^{-1}\cos\phi\ , & a < \rho < b \\ \\ \frac{(\epsilon^2-\epsilon_0^2)(a^2-b^2)b^2 E_0}{a^2(\epsilon-\epsilon_0)^2-b^2(\epsilon+\epsilon_0)^2}\rho^{-1}\cos\phi - E_0\rho\cos\phi\ , & \rho > b \end{cases} \tag{49}$$