Phys 5405
HW 4 2.12 2.13 2.17 2.24

2.12 Starting with

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n)\ . \quad (1)$$

Also,

$$\sin(n\phi + \alpha_n) = \cos\alpha_n \sin(n\phi) + \sin\alpha_n \cos(n\phi)\ , \quad (2)$$

$$\sin(n\phi + \beta_n) = \cos\beta_n \sin(n\phi) + \sin\beta_n \cos(n\phi)\ . \quad (3)$$

For potential inside the cylinder, we have to set $b_n = 0$ for $n \geq 0$. Then,

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \left[\cos\alpha_n \sin(n\phi) + \sin\alpha_n \cos(n\phi)\right] \quad (4)$$

Then evaluating,

$$\int_0^{2\pi} \Phi(\rho,\phi)\sin(n\phi) d\phi = \pi a_n \rho^n \cos\alpha_n\ , \quad (5)$$

$$\int_0^{2\pi} \Phi(\rho,\phi)\cos(n\phi) d\phi = \pi a_n \rho^n \sin\alpha_n\ . \quad (6)$$

Therefore, we obtain

$$a_n = \frac{\rho^{-n}}{\pi\cos\alpha_n}\int_0^{2\pi}\Phi(\rho,\phi)\sin(n\phi)d\phi = \frac{\rho^{-n}}{\pi\sin\alpha_n}\int_0^{2\pi}\Phi(\rho,\phi)\cos(n\phi)d\phi \quad (7)$$

Plug this back into the potential, we obtain Then evaluating,

$$\int_0^{2\pi} \Phi(\rho,\phi)\sin(n\phi) d\phi = \pi a_n \rho^n \cos\alpha_n + \pi b_n \rho^{-n}\cos\beta_n\ , \quad (8)$$

$$\int_0^{2\pi} \Phi(\rho,\phi)\cos(n\phi) d\phi = \pi a_n \rho^n \sin\alpha_n + \pi b_n \rho^{-n}\sin\beta_n\ . \quad (9)$$

Since we have specified the potential on the surface of the cylinder of radius b, we can set $\rho = b$, and obtain,

$$b^n a_n(\sin\alpha_n\cos\beta_n - \cos\alpha_n\sin\beta_n) = \frac{1}{\pi}\int_0^{2\pi}\Phi(b,\phi)\left[\cos(n\phi)\cos\beta_n - \sin(n\phi)\sin\beta_n\right] d\phi$$

$$b^{-n} b_n(\sin\beta_n\cos\alpha_n - \cos\beta_n\sin\alpha_n) = \frac{1}{\pi}\int_0^{2\pi}\Phi(b,\phi)\left[\cos(n\phi)\cos\alpha_n - \sin(n\phi)\sin\alpha_n\right] d\phi$$

Therefore, we can derive, for $\alpha_n \neq \beta_n$,

$$a_n = \frac{b^{-n}}{\pi \sin(\alpha_n - \beta_n)} \int_0^{2\pi} \Phi(b, \phi) \cos(n\phi + \beta_n) d\phi \tag{10}$$

$$b_n = \frac{b^{n}}{\pi \sin(\beta_n - \alpha_n)} \int_0^{2\pi} \Phi(b, \phi) \cos(n\phi + \alpha_n) d\phi \tag{11}$$

For potential inside the cylinder, we have to set $b_n = 0$ for $n \geq 0$, and neglecting the constant term,

$$\Phi(\rho, \phi) = \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) \tag{12}$$

$$= \frac{1}{\pi} \int_0^{2\pi} d\phi' \Phi(b, \phi') \sum_{n=1}^{\infty} \frac{b^{-n} \rho^n \sin(n\phi + \alpha_n)}{\sin(\alpha_n - \beta_n)} \cos(n\phi' + \beta_n)$$

$$= \frac{1}{\pi} \int_0^{2\pi} d\phi' \Phi(b, \phi') \mathrm{Im} \sum_{n=1}^{\infty} b^{-n} \rho^n \tag{13}$$