**Phys 5405**
HW 11
5.13, 5.17, 5.19(a), 5.20, 5.21, 5.26, 5.27

**5.13** We can write down the current density,

$$\mathbf{J}(\mathbf{x}) = \sigma\omega a \sin\theta\delta(r-a)\hat{\phi} \ . \tag{1}$$

Then the vector potential is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{\mu_0\sigma}{4\pi}\omega a \int r^2 dr \sin\theta' d\theta' d\phi' \frac{\sin\theta'\delta(r'-a)\hat{\phi}'}{|\mathbf{x}-\mathbf{x}'|} \ . \tag{2}$$

We can expand the Green function in spherical coordinates

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \ , \tag{3}$$

where $r_< = \min(r,r')$ and $r_> = \max(r,r')$. Therefore, we get

$$\mathbf{A}(\mathbf{x}) = \mu_0\sigma\omega a^3 \sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\int d\Omega' Y_{lm}^*(\theta',\phi')\sin\theta'\hat{\phi}' \ , \tag{4}$$

where $r_< = \min(r,a)$ and $r_> = \max(r,a)$. Now we decompose $\hat{\phi}'$ in Cartesian coordinates,

$$\hat{\phi}' = \cos\phi'\hat{i} + \sin\phi'\hat{j} \ . \tag{5}$$

We can evaluate the integrals,

$$\begin{aligned}\int d\Omega' Y_{lm}^*(\theta',\phi')\sin\theta'\cos\phi' &= \sqrt{\frac{2\pi}{3}}\int d\Omega' Y_{lm}^*(\theta',\phi')(-Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')) \\ &= \sqrt{\frac{2\pi}{3}}(-\delta_{l1}\delta_{m1} + \delta_{l1}\delta_{m,-1}) \ . \end{aligned} \tag{6}$$

$$\begin{aligned}\int d\Omega' Y_{lm}^*(\theta',\phi')\sin\theta'\sin\phi' &= \sqrt{\frac{2\pi}{3}}i\int d\Omega' Y_{lm}^*(\theta',\phi')(Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')) \\ &= \sqrt{\frac{2\pi}{3}}i(\delta_{l1}\delta_{m1} + \delta_{l1}\delta_{m,-1}) \ . \end{aligned} \tag{7}$$

Therefore, we have

$$\mathbf{A}(\mathbf{x}) = \frac{1}{3}\mu_0\sigma\omega a^3\frac{r_<}{r_>^2}\sin\theta(\cos\phi\hat{i} + \sin\phi\hat{j}) = \frac{1}{3}\mu_0\sigma\omega a^3\frac{r_<}{r_>^2}\sin\theta\hat{\phi} \ . \tag{8}$$

Then, inside the sphere $r_< = r$ and $r_> = a$,

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{1}{3}\mu_0\sigma\omega a r\sin\theta\hat{\phi}} \tag{9}$$

Outside the sphere, $r_< = a$ and $r_> = r$,

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{1}{3}\mu_0 \sigma \omega \frac{a^4}{r^2} \sin\theta \hat{\phi}} \tag{10}$$

The magnetic flux density is given by

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta} \ . \tag{11}$$

So inside the sphere, we have

$$\boxed{\mathbf{B} = \frac{2}{3}\mu_0 \sigma \omega a(\cos\theta \hat{r} - \sin\theta\hat{\theta})} \tag{12}$$

Outside the sphere, we have

$$\boxed{\mathbf{B} = \frac{1}{3}\mu_0 \sigma \omega \frac{a^4}{r^3}(2\cos\theta \hat{r} + \sin\theta\hat{\theta})} \tag{13}$$

**5.17** For $z > 0$, the magnetic induction is generated by the current $\mathbf{J}$ and the image current $\mathbf{J}^*$,

$$\mathbf{B}^+(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{(\mathbf{J}(\mathbf{x}') + \mathbf{J}^*(\mathbf{x}')) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \ . \tag{14}$$

It is an integration over the whole region. For $z < 0$, the magnetic induction is generated by the current $k\mathbf{J}$, where $k$ is a scaling constant because of different permeability,

$$\mathbf{B}^-(\mathbf{x}) = \frac{\mu_0 \mu_r k}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \ . \tag{15}$$

It is an integration over the region where $z' > 0$. Now we want to transform the first integral such that these two integrals have the same integration domain. Suppose the component of $\mathbf{x}'$ is $(x', y', z')$, then we define $\mathbf{x}'' = (x', y', -z')$ and we can write

$$\mathbf{B}^+(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} + \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}^*(\mathbf{x}'') \times (\mathbf{x} - \mathbf{x}'')}{|\mathbf{x} - \mathbf{x}''|^3} \ . \tag{16}$$

Now this integral is defined in the region where $z' > 0$. And especially, when $z = 0$, there is no difference between $|\mathbf{x} - \mathbf{x}'|$ and $|\mathbf{x} - \mathbf{x}''|$.

Now the boundary conditions are given by

$$\mathbf{B}_z^+(z=0) = \mathbf{B}_z^-(z=0) \ , \quad \mathbf{B}_{x,y}^+(z=0) = \frac{1}{\mu_r}\mathbf{B}_{x,y}^-(z=0) \ . \tag{17}$$

For the first equation, we can equate the numerator in the integrand. When $z = 0$,

$$\hat{z} \cdot [\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] + \hat{z} \cdot [\mathbf{J}^*(\mathbf{x}'') \times (\mathbf{x} - \mathbf{x}'')] = \mu_r k \hat{z} \cdot (\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')) \ , \tag{18}$$

from which we can get that at $z = 0$,

$$(\mathbf{x} - \mathbf{x}') \cdot [\hat{z} \times \mathbf{J}(\mathbf{x}')] + (\mathbf{x} - \mathbf{x}'') \cdot [\hat{z} \times \mathbf{J}^*(\mathbf{x}'')] = \mu_r k (\mathbf{x} - \mathbf{x}') \cdot (\hat{z} \times \mathbf{J}(\mathbf{x}')) \ , \tag{19}$$

and expanding it in components we can get

$$\boxed{J_y(\mathbf{x}') + J_y^*(\mathbf{x}'') = \mu_r k J_y(\mathbf{x}'), \quad J_x(\mathbf{x}') + J_x^*(\mathbf{x}'') = \mu_r k J_x(\mathbf{x}')} \tag{20}$$

Another equation gives that at $z = 0$,

$$\hat{z} \times [\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] + \hat{z} \times [\mathbf{J}^*(\mathbf{x}'') \times (\mathbf{x} - \mathbf{x}'')] = k \hat{z} \times [\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')] \ . \tag{21}$$

Now using,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \ , \tag{22}$$

we can further simplify the above equation to,

$$J_z^*(\mathbf{x}'')(\mathbf{x} - \mathbf{x}'') - z' \mathbf{J}^*(\mathbf{x}'') = (k-1) J_z(\mathbf{x}')(\mathbf{x} - \mathbf{x}') + (k-1) z' \mathbf{J}(\mathbf{x}') \ . \tag{23}$$

Expanding it into components and using the fact that the equation holds for arbitrary $\mathbf{x}$, we have,

$$\boxed{J_x^*(\mathbf{x}'') = (1-k)J_x(\mathbf{x}'), \quad J_y^*(\mathbf{x}'') = (1-k)J_y(\mathbf{x}'), \quad J_z^*(\mathbf{x}'') = (k-1)J_z(\mathbf{x}')} \tag{24}$$

From (20) and (24), we can solve for $k = 2/(1+\mu_r)$. Plug it back into (24), we have the image current distribution $\mathbf{J}^*$, with components,

$$\boxed{\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_x(x, y, -z), \quad \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_y(x, y, -z), \quad -\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_z(x, y, -z)}$$

Since $k = 2/(1+\mu_r)$, we have stated that for $z < 0$, the magnetic induction is due to a current distribution $k\mathbf{J}$ in a medium of relative permeability $\mu_r$. We can also consider it due to a current distribution

$$\boxed{k\mu_r\mathbf{J} = \frac{2\mu_r}{1+\mu_r}\mathbf{J}} \tag{25}$$

in a medium of unit relative permeability.

**5.19 (a)** Since $\mathbf{J}=0$, we can use the magnetic scalar potential $\Phi_M$. Since the magnetization is uniform, we have

$$\Phi_M(\mathbf{x}) = \frac{1}{4\pi}\oint_S \frac{\mathbf{n}'\cdot\hat{z}\,M_0\,da'}{|\mathbf{x}-\mathbf{x}'|} \tag{26}$$

Since the magnetization points along the $z$-direction, we only need to consider the top boundary (say, at $z=L$) and the bottom boundary (at $z=0$) and let the axis of the cylinder lying in the $z$-axis. Then when $\mathbf{x}$ is on the axis, at top, $|\mathbf{x}-\mathbf{x}'| = \sqrt{x'^2+y'^2+(z-L)^2}$ and at bottom, $|\mathbf{x}-\mathbf{x}'| = \sqrt{x'^2+y'^2+z^2}$. For the two dimensional surface integral, we can also use polar coordinates. Then, we have,

$$\begin{aligned}
\Phi_M(\mathbf{x}) &= \frac{M_0}{4\pi}\int_0^a \rho' d\rho' \int_0^{2\pi} d\phi \left(\frac{1}{\sqrt{\rho'^2+(z-L)^2}} - \frac{1}{\sqrt{\rho'^2+z^2}}\right) \\
&= \frac{M_0}{2}\int_0^a d\rho' \left(\frac{\rho'}{\sqrt{\rho'^2+(z-L)^2}} - \frac{\rho'}{\sqrt{\rho'^2+z^2}}\right) \\
&= \frac{M_0}{2}\left(\sqrt{\rho'^2+(z-L)^2} - \sqrt{\rho'^2+z^2}\right)\Big|_{\rho'=0}^{\rho'=a} \\
&= \frac{M_0}{2}\left(\sqrt{a^2+(z-L)^2} - |z-L| - \sqrt{a^2+z^2} + |z|\right) \ .
\end{aligned} \tag{27}$$

Therefore,

$$\Phi_M(z) = \begin{cases} \frac{M_0}{2}\left(\sqrt{a^2+(z-L)^2} - \sqrt{a^2+z^2} - L\right) & z<0 \\ \frac{M_0}{2}\left(\sqrt{a^2+(z-L)^2} - \sqrt{a^2+z^2} + 2z - L\right) & 0<z<L \\ \frac{M_0}{2}\left(\sqrt{a^2+(z-L)^2} - \sqrt{a^2+z^2} + L\right) & z>L \end{cases} \tag{28}$$

The magnetic field $\mathbf{H}$ is given by $\mathbf{H} = -\nabla\Phi_M = -\hat{z}\,\partial\Phi_M/\partial z$. Then

$$\mathbf{H}_{\text{in}} = -\frac{M_0}{2}\left(\frac{z-L}{\sqrt{a^2+(z-L)^2}} - \frac{z}{\sqrt{a^2+z^2}} + 2\right)\hat{z} \tag{29}$$

$$\mathbf{H}_{\text{out}} = -\frac{M_0}{2}\left(\frac{z-L}{\sqrt{a^2+(z-L)^2}} - \frac{z}{\sqrt{a^2+z^2}}\right)\hat{z} \tag{30}$$

The magnetic induction is given by $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$. So we have,

$$\mathbf{B}_{\text{in}} = -\frac{\mu_0 M_0}{2}\left(\frac{z-L}{\sqrt{a^2+(z-L)^2}} - \frac{z}{\sqrt{a^2+z^2}}\right)\hat{z} \tag{31}$$

$$\mathbf{B}_{\text{out}} = -\frac{\mu_0 M_0}{2}\left(\frac{z-L}{\sqrt{a^2+(z-L)^2}} - \frac{z}{\sqrt{a^2+z^2}}\right)\hat{z} \tag{32}$$

**5.20** We start from the force equation

$$\mathbf{F} = \int \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})\, d^3x \ . \tag{33}$$

Using the fact that a magnetization $\mathbf{M}$ inside a volume $V$ bounded by a surface $S$ is equivalent to a volume current density $\mathbf{J}_M = \nabla \times \mathbf{M}$ and a surface current density $\mathbf{M} \times \mathbf{n}$, we can write

$$\mathbf{F} = \int_V (\nabla \times \mathbf{M}) \times \mathbf{B}_e\, d^3x + \int_S (\mathbf{M} \times \mathbf{n}) \times \mathbf{B}_e\, da \tag{34}$$

Since

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \ . \tag{35}$$

In our case, we have

$$(\nabla \times \mathbf{M}) \times \mathbf{B}_e = -\nabla(\mathbf{M} \cdot \mathbf{B}) + (\mathbf{M} \cdot \nabla)\mathbf{B}_e + (\mathbf{B}_e \cdot \nabla)\mathbf{M} + \mathbf{M} \times (\nabla \times \mathbf{B}_e) \ . \tag{36}$$

Since there's no external current, $\nabla \times \mathbf{B}_e = 0$, we have

$$\boxed{(\nabla \times \mathbf{M}) \times \mathbf{B}_e = -\nabla(\mathbf{M} \cdot \mathbf{B}_e) + (\mathbf{M} \cdot \nabla)\mathbf{B}_e + (\mathbf{B}_e \cdot \nabla)\mathbf{M}} \tag{37}$$

Also, we have

$$\boxed{(\mathbf{M} \times \mathbf{n}) \times \mathbf{B}_e = (\mathbf{B}_e \cdot \mathbf{M})\mathbf{n} - (\mathbf{B}_e \cdot \mathbf{n})\mathbf{M}} \tag{38}$$

For volume integration of the first term in the right hand side of (37), we can use Stokes theorem, which will cancel the surface integral of the first term in the right hand side of (38). So we are left with

$$\mathbf{F} = \int_V (\mathbf{M} \cdot \nabla)\mathbf{B}_e\, d^3x + \int_V (\mathbf{B}_e \cdot \nabla)\mathbf{M}\, d^3x - \int_S (\mathbf{B}_e \cdot \mathbf{n})\mathbf{M}\, da \ . \tag{39}$$

Now we want to do integration by parts, we have

$$\int_V (\mathbf{A} \cdot \nabla)\mathbf{B}\, d^3x = -\int_V (\nabla \cdot \mathbf{A})\mathbf{B}\, d^3x + \int_S (\mathbf{A} \cdot \mathbf{n})\mathbf{B}\, da \ . \tag{40}$$

Then we have

$$\boxed{\mathbf{F} = -\int_V (\nabla \cdot \mathbf{M})\mathbf{B}_e\, d^3x + \int_S (\mathbf{M} \cdot \mathbf{n})\mathbf{B}_e\, da} \ , \tag{41}$$

where we have used the fact that $\nabla \cdot \mathbf{B}_e = 0$.

Now we have the external magnetic field, given in components,

$$B_x = B_0(1 + \beta y) \ , \quad B_y = B_0(1 + \beta x) \ , \tag{42}$$

and the magnetization points to $\theta_0$, $\phi_0$. And since it's uniform, $\nabla \cdot \mathbf{M} = 0$, the volume integral vanishes. For the surface integral, we can first evaluate,

$$\mathbf{M} \cdot \mathbf{n} = M_0(\cos\theta \cos\theta_0 + \sin\theta \sin\theta_0 \cos(\phi - \phi_0)) \ . \tag{43}$$

Then the $x$-component of the force is

$$\begin{aligned} F_x &= M_0 B_0 \int R^2 \sin\theta d\theta d\phi (\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0))(1+\beta R\sin\theta\sin\phi) \\ &= M_0 B_0 \beta R^3 \sin\theta_0 \int d(\cos\theta) d\phi (1-\cos^2\theta)\sin\phi\cos(\phi-\phi_0) \\ &= \frac{4}{3}\pi R^3 \beta M_0 B_0 \sin\theta_0 \sin\phi_0 \ . \end{aligned} \tag{44}$$

Similarly, the $y$-component of the force is

$$\begin{aligned} F_y &= M_0 B_0 \int R^2 \sin\theta d\theta d\phi (\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0))(1+\beta R\sin\theta\cos\phi) \\ &= M_0 B_0 \beta R^3 \sin\theta_0 \int d(\cos\theta) d\phi (1-\cos^2\theta)\cos\phi\cos(\phi-\phi_0) \\ &= \frac{4}{3}\pi R^3 \beta M_0 B_0 \sin\theta_0 \cos\phi_0 \ . \end{aligned} \tag{45}$$

Therefore, the force is given by

$$\boxed{\mathbf{F} = \frac{4}{3}\pi R^3 \beta M_0 B_0 \sin\theta_0 (\sin\phi_0 \hat{x} + \cos\phi_0 \hat{y})} \tag{46}$$

**5.21** Since there is no free current, we have

$$\nabla \times \mathbf{H} = 0 \tag{47}$$

Now we evaluate

$$\int \mathbf{B} \cdot \mathbf{H} \, d^3x = \int (\nabla \times \mathbf{A}) \cdot \mathbf{H} \, d^3x \tag{48}$$

Since

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A} \ , \tag{49}$$

so we have

$$\begin{aligned} \int \mathbf{B} \cdot \mathbf{H} \, d^3x &= \int \nabla \cdot (\mathbf{A} \times \mathbf{H}) \, d^3x + \int (\nabla \times \mathbf{H}) \cdot \mathbf{A} \, d^3x \\ &= \int \nabla \cdot (\mathbf{A} \times \mathbf{H}) \, d^3x = 0 \ . \end{aligned} \tag{50}$$

In the last step, we can use the Stokes theorem. Since the magnetostatic field is due to a localized distribution of permanent magnetization, the fields vanish at infinity.

The potential energy of a dipole in an external field is given by

$$U = -\mathbf{m} \cdot \mathbf{B} \ . \tag{51}$$

For a continuous distribution of permanent magnetization, the magnetostatic energy can be written as

$$\begin{aligned} W &= -\frac{1}{2} \int \mathbf{M} \cdot \mathbf{B} \, d^3x \\ &= -\frac{\mu_0}{2} \int \mathbf{M} \cdot (\mathbf{M} + \mathbf{H}) \, d^3x \\ &= -\frac{\mu_0}{2} \int \mathbf{M} \cdot \mathbf{M} \, d^3x - \frac{\mu_0}{2} \int \mathbf{M} \cdot \mathbf{H} \, d^3x \ . \end{aligned} \tag{52}$$

We can drop the first term, which is a constant independent of the orientation or position of the various constituent magnetized bodies. So

$$\begin{aligned} W &= \boxed{-\frac{\mu_0}{2} \int \mathbf{M} \cdot \mathbf{H} \, d^3x} \\ &= -\frac{\mu_0}{2} \int \left( \frac{\mathbf{B}}{\mu_0} - \mathbf{H} \right) \cdot \mathbf{H} \, d^3x \\ &= -\frac{1}{2} \int \mathbf{B} \cdot \mathbf{H} \, d^3x + \frac{\mu_0}{2} \int \mathbf{H} \cdot \mathbf{H} \, d^3x \\ &= \boxed{\frac{\mu_0}{2} \int \mathbf{H} \cdot \mathbf{H} \, d^3x} \ , \end{aligned} \tag{53}$$

where in the last step we used result from part a.

**5.26** The magnetic energy per unit length is given by

$$W = \frac{1}{2} L I^2 \ . \tag{54}$$

We can also calculate the magnetic energy, using

$$W = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A} \, d^3x \ . \tag{55}$$

Therefore, the self-inductance per unit length is

$$L = \frac{1}{I^2} \int \mathbf{J} \cdot \mathbf{A} \, d^3x = \frac{1}{I^2} \int \mathbf{J}_a \cdot \mathbf{A}_a \, d^3x + \frac{1}{I^2} \int \mathbf{J}_b \cdot \mathbf{A}_b \, d^3x \ . \tag{56}$$

For a single wire of radius $a$, it is easy to evaluate the magnetic field. Define $\rho$ as the distance from the symmetry axis of the wire, then

$$\mathbf{B} = \begin{cases} \frac{\mu_0 I \rho}{2\pi a^2} \hat{\phi} & \rho < a \\ \frac{\mu_0 I}{2\pi \rho} \hat{\phi} & \rho > a \end{cases} \ . \tag{57}$$

Since $\mathbf{B} = \nabla \times \mathbf{A}$, we can find one configuration of vector potential such that it is continuous at the boundary of the wire,

$$\mathbf{A} = \begin{cases} -\frac{\mu_0 I \rho^2}{4\pi a^2} \hat{z} & \rho < a \\ -\frac{\mu_0 I}{4\pi} (1 + 2 \ln \rho - 2 \ln a) \hat{z} & \rho > a \end{cases} \ . \tag{58}$$

Now in our case, define $\rho$ the distance to wire $a$ and $\rho'$ the distance to wire $b$. Suppose the current in $a$ is along $\hat{z}$ direction and the current in $b$ is along the $-\hat{z}$ direction, then we have

$$\mathbf{A}_a = -\frac{\mu_0 I \rho^2}{4\pi a^2} \hat{z} + \frac{\mu_0 I}{4\pi} (1 + 2 \ln(\rho'/b)) \hat{z} \tag{59}$$

$$\mathbf{A}_b = \frac{\mu_0 I \rho'^2}{4\pi b^2} \hat{z} - \frac{\mu_0 I}{4\pi} (1 + 2 \ln(\rho/a)) \hat{z} \tag{60}$$

Therefore,

$$\int \mathbf{J}_a \cdot \mathbf{A}_a \, d^3x = -\frac{\mu_0 I^2}{4\pi^2 a^2} \int \left( -1 + \frac{\rho^2}{a^2} - \ln \left( \frac{\rho'^2}{b^2} \right) \right) \rho d\rho d\phi \ , \tag{61}$$

where $\rho'^2$ is given by

$$\rho'^2 = d^2 + \rho^2 - 2 d \rho \cos \phi \ . \tag{62}$$

Then we have

$$\int \mathbf{J}_a \cdot \mathbf{A}_a \, d^3x = -\frac{\mu_0 I^2}{4\pi^2 a^2} \left( -\frac{1}{2} \pi a^2 + 2\pi a \ln b^2 - \int_0^a d\rho \int_0^{2\pi} d\phi \rho \ln(d^2 + \rho^2 - 2 d \rho \cos \phi) \right) \tag{63}$$

Since $d \ll \rho$, we have

$$\ln(d^2 + \rho^2 - 2d\rho\cos\phi) = \ln d^2 + \left(-2\frac{\rho}{d}\cos\phi + (1 - 2\cos^2\phi)\frac{\rho^2}{d^2}\right) \ . \tag{64}$$

The integral of $\cos\phi$ and $1 - 2\cos^2\phi$ over $\phi$ vanish. Therefore, we have

$$\int \mathbf{J}_a \cdot \mathbf{A}_a \, d^3x = \frac{\mu_0 I^2}{8\pi} + \frac{\mu_0 I^2}{4\pi} \ln\left(\frac{d^2}{b^2}\right) \ . \tag{65}$$

Similarly, we can get

$$\int \mathbf{J}_b \cdot \mathbf{A}_b \, d^3x = \frac{\mu_0 I^2}{8\pi} + \frac{\mu_0 I^2}{4\pi} \ln\left(\frac{d^2}{a^2}\right) \ . \tag{66}$$

Adding them up and dividing by $I^2$, we obtain

$$\boxed{L = \frac{\mu_0}{4\pi}\left[1 + 2\ln\left(\frac{d^2}{ab}\right)\right]} \tag{67}$$

**5.27** Suppose the current in the inner wire is pointing along the $\hat{z}$ direction. Using Ampere's law, it is easy to write down the magnetic induction,

$$\mathbf{B} = \begin{cases} \frac{\mu I \rho}{2\pi b^2}\hat{\phi} & \rho < b \\ \frac{\mu_0 I}{2\pi\rho}\hat{\phi} & b < \rho < a \\ 0 & \rho > a \end{cases} \tag{68}$$

Now we use the following equation to calculate the magnetic energy,

$$\begin{aligned} W &= \frac{1}{2}\int \frac{\mathbf{B}\cdot\mathbf{B}}{\mu(x)} d^3x = \frac{\mu I^2}{8\pi^2 b^4}\int \rho^2\, \rho d\rho d\phi + \frac{\mu_0 I^2}{8\pi^2}\int \frac{1}{\rho^2}\, \rho d\rho d\phi \\ &= \frac{\mu I^2}{16\pi} + \frac{\mu_0 I^2}{4\pi}\ln(a/b) \end{aligned} \tag{69}$$

Therefore, the self-inductance per unit length is given by

$$\boxed{L = \frac{2W}{I^2} = \frac{\mu}{8\pi} + \frac{\mu_0}{2\pi}\ln(a/b)} \tag{70}$$

Now if the inner conductor is a thin hollow tube, the only non-vanishing magnetic induction is in $b < \rho < a$, given by,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}, \quad b < \rho < a \tag{71}$$

Following the same procedure, we can get

$$\boxed{L = \frac{\mu_0}{2\pi}\ln(a/b)} \tag{72}$$