Phys 5405
HW 6
3.13 3.14 3.16 b,c,d

3.13 The Green function for a spherical shell bounded by $r = a$ and $r = b$ is

$$G(\vec{x}, \vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) . \tag{1}$$

Since we have azimuthal symmetry, we only need to consider $m = 0$, therefore,

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) . \tag{2}$$

Now the potential is

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}') G(\vec{x}, \vec{x}') d^3x' + \frac{1}{4\pi} \int_S \left[G(\vec{x}, \vec{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \right] da' . \tag{3}$$

In between the two spheres, the charge density is zero, and the Green function vanishes on the boundary. We have

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \int_S \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} da' . \tag{4}$$

Then we have to calculate the derivative of the Green function. For the boundary at radius a, we have $r > r'$,

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left(r'^l - \frac{a^{2l+1}}{r'^{l+1}} \right) \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) . \tag{5}$$

Calculate its derivative and evaluate it at $r' = a$

$$\begin{aligned}
\left.\frac{\partial G(\vec{x}, \vec{x}')}{\partial n'}\right|_{r'=a} &= -\left.\frac{\partial G(\vec{x}, \vec{x}')}{\partial r'}\right|_{r'=a} \\
&= -\sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} \left(l r'^{l-1} + (l+1) \frac{a^{2l+1}}{r'^{l+2}} \right) \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \Bigg|_{r'=a} \\
&= -\sum_{l=0}^{\infty} \frac{P_l(\cos\theta') P_l(\cos\theta)}{1 - \left(\frac{a}{b}\right)^{2l+1}} a^{l-1} (2l+1) \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) .
\end{aligned} \tag{6}$$

Then

$$\begin{aligned}
\int_{S_a} \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} da' &= \int_0^{\pi/2} a^2 \sin\theta' d\theta' \int_0^{2\pi} d\phi' \, V \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \\
&= -2\pi V \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{1 - (a/b)^{2l+1}} a^{l+1} (2l+1) \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \int_0^{\pi/2} P_l(\cos\theta') \sin\theta' d\theta'
\end{aligned} \tag{7}$$

Now evaluate for $l \neq 0$,

$$\int_0^{\pi/2} P_l(\cos\theta')\sin\theta' d\theta' = \int_0^1 P_l(x)dx = \frac{1}{2l+1}\left[P_{l-1}(0) - P_{l+1}(0)\right] \tag{8}$$

For $l = 0$, we simply have

$$\int_0^{\pi/2} P_l(\cos\theta')\sin\theta' d\theta' = \int_0^1 P_l(x)dx = 1 \ . \tag{9}$$

Therefore,

$$\begin{aligned}
&\int_{S_a} \Phi(\vec{x}') \frac{\partial G(\vec{x},\vec{x}')}{\partial n'} da' \\
&= -2\pi V \frac{a}{1-(a/b)}\left(\frac{1}{r} - \frac{1}{b}\right) - 2\pi V \sum_{l=1}^{\infty} \frac{P_l(\cos\theta)a^{l+1}}{1-(a/b)^{2l+1}}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\left[P_{l-1}(0) - P_{l+1}(0)\right] \ .
\end{aligned}$$

Similarly, for boundary at radius b, we have $r < r'$,

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty} \frac{P_l(\cos\theta')P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) \ . \tag{10}$$

Calculate its derivative and evaluate it at $r' = b$

$$\begin{aligned}
\left.\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right|_{r'=b} &= \left.\frac{\partial G(\vec{x},\vec{x}')}{\partial r'}\right|_{r'=b} \\
&= \sum_{l=0}^{\infty} \frac{P_l(\cos\theta')P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left.\left(-(l+1)\frac{1}{r'^{l+2}} - l\frac{r'^{l-1}}{b^{2l+1}}\right)\right|_{r'=b} \\
&= -\sum_{l=0}^{\infty} \frac{P_l(\cos\theta')P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)(2l+1)b^{-l-2} \ .
\end{aligned} \tag{11}$$

Then

$$\begin{aligned}
&\int_{S_b} \Phi(\vec{x}') \frac{\partial G(\vec{x},\vec{x}')}{\partial n'} da' = \int_{\pi/2}^{\pi} b^2 \sin\theta' d\theta' \int_0^{2\pi} d\phi' \, V \frac{\partial G(\vec{x},\vec{x}')}{\partial n'} \\
&\quad = -2\pi V \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{1-(a/b)^{2l+1}} b^{-l}(2l+1)\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\int_{\pi/2}^{\pi} P_l(\cos\theta')\sin\theta' d\theta' \\
&\quad = -2\pi V \frac{1}{1-a/b}(1-a/r) - 2\pi V \sum_{l=1}^{\infty} (-1)^l \frac{P_l(\cos\theta)b^{-l}}{1-(a/b)^{2l+1}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\left[P_{l-1}(0) - P_{l+1}(0)\right]
\end{aligned}$$

Then the potential between the two spheres are simply given by

$$
\begin{aligned}
\Phi(\vec{x}) = & -\frac{1}{4\pi}\int_{S_a}\Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da' - \frac{1}{4\pi}\int_{S_b}\Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da' \\
= & \frac{V}{2}\frac{1}{1-a/b}\left(\frac{a}{r}-\frac{a}{b}\right) + \frac{V}{2}\frac{1}{1-a/b}\left(1-\frac{a}{r}\right) \\
& + \frac{V}{2}\sum_{l=1}^{\infty}[P_{l-1}(0)-P_{l+1}(0)]P_l(\cos\theta) \\
& \times\left[\frac{a^{l+1}}{1-(a/b)^{2l+1}}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right) + (-1)^l\frac{b^{-l}}{1-(a/b)^{2l+1}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\right] \\
= & \frac{V}{2} + \frac{V}{2}\sum_{l=1}^{\infty}(P_{l-1}(0)-P_{l+1}(0))P_l(\cos\theta)\left(\frac{(-1)^l b^{l+1}-a^{l+1}}{b^{2l+1}-a^{2l+1}}r^l + \frac{(-1)^l b^{-l}-a^{-l}}{b^{-(2l+1)}-a^{-(2l+1)}}r^{-(l+1)}\right) \; .
\end{aligned}
$$

This is exactly what I got in homework 5, Jackson problem 3.1.

3.14 (a) A line charge of length $2d$ with a total charge Q has linear charge density varying as $(d^2 - z^2)$, where z is the distance from the midpoint. A grounded, conducting, spherical shell of inner radius $b > d$ is centered at the midpoint of the line charge.

First, we write down the charge density as a function of z,

$$\rho(z) = k(d^2 - z^2), \tag{12}$$

where the constant k has to be determined. We have

$$Q = 2\int_0^d \rho(z)dz = 2\int_0^d k(d^2 - z^2)dz = \frac{4}{3}kd^3 \ . \tag{13}$$

Therefore, $k = 3Q/(4d^3)$ and

$$\rho(z) = \frac{3Q}{4d^3}(d^2 - z^2) \ . \tag{14}$$

Now we write the charge density as a function of the position vector, in terms of delta functions, for $|\vec{x}'| \leq d$,

$$\rho(\vec{x}') = \frac{3Q}{4d^3}(d^2 - r'^2)\frac{1}{2\pi r'^2}[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)] \ . \tag{15}$$

Otherwise, the charge density is zero. The Green function inside a sphere of radius b with azimuthal symmetry is given by,

$$G(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} P_l(\cos\theta')P_l(\cos\theta)r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \ . \tag{16}$$

Since on the boundary, the potential vanishes, we have

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x}, \vec{x}')d^3x' \ . \tag{17}$$

For $r > d$, we have $r_< = r'$ and $r_> = r$. Then the potential is

$$\begin{aligned}
\Phi(\vec{x}) = &\frac{1}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) \\
&\int_0^d dr' \int_{-1}^1 d(\cos\theta')(d^2 - r'^2)r'^l P_l(\cos\theta')[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)] \\
= &\frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^d dr'(d^2 - r'^2)r'^l \\
= &\frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\frac{2d^{l+3}}{(l+1)(l+3)} \\
= &\frac{3Q}{4\pi\epsilon_0}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\frac{d^l}{(l+1)(l+3)} \ .
\end{aligned} \tag{18}$$

For $r < d$, we have two regimes, $r' < r$ and $r < r' < d$. Then in the first regime, its contribution to the potential is

$$\Phi_1(\vec{x}) = \frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^r dr'(d^2 - r'^2)r'^l$$
$$= \frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\left(\frac{d^2 r^{l+1}}{l+1} - \frac{r^{l+3}}{l+3}\right) \ . \tag{19}$$

In the second regime, its contribution to the potential is

$$\Phi_2(\vec{x}) = \frac{2}{4\pi\epsilon_0}\frac{3Q}{4d^3}\sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta)r^l\int_r^d dr'(d^2 - r'^2)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) \ . \tag{20}$$

Denote the integral by

$$\mathcal{I}_l = \int_r^d dr'(d^2 - r'^2)\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right) \ . \tag{21}$$

When $l = 0$, it's

$$\mathcal{I}_0 = d^2 \ln\frac{d}{r} - \frac{d^2}{b}(d-r) - \frac{1}{2}(d^2 - r^2) + \frac{1}{3b}(d^3 - r^3) \ . \tag{22}$$

When $l = 2$, it's

$$\mathcal{I}_2 = -\ln\frac{d}{r} - \frac{d^2}{2}(d^{-2} - r^{-2}) - \frac{d^2}{3b^5}(d^3 - r^3) + \frac{1}{5b^5}(d^5 - r^5) \ . \tag{23}$$

When $l \geq 4$, it'

$$\mathcal{I}_l = -\frac{d^2}{l}(d^{-l} - r^{-l}) - \frac{d^2}{(l+1)b^{2l+1}}(d^{l+1} - r^{l+1}) + \frac{1}{l-2}(d^{2-l} - r^{2-l}) + \frac{1}{(l+3)b^{2l+1}}(d^{l+3} - r^{l+3}) \ . \tag{24}$$

Therefore, for $r < d$

$$\Phi(\vec{x}) = \Phi_1(\vec{x}) + \Phi_2(\vec{x}) \ . \tag{25}$$

3.14 (b) For potential near the boundary, we have $r > d$ and

$$\Phi(\vec{x}) = \frac{3Q}{4\pi\epsilon_0} \sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta) \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \frac{d^l}{(l+1)(l+3)} \ . \tag{26}$$

Then, the surface charge density is given by

$$\sigma = \epsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=b} = -\frac{3Q}{4\pi} \sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta) \frac{(2l+1)}{(l+1)(l+3)} \frac{d^l}{b^{l+2}} \ . \tag{27}$$

3.14 (c) In the limit that $d \ll b$, then also $d << r$. The potential in (a) is

$$\Phi(\vec{x}) = \frac{3Q}{4\pi\epsilon_0} \sum_{l=0,\ \text{even}}^{\infty} P_l(\cos\theta) \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \frac{d^l}{(l+1)(l+3)} \ . \tag{28}$$

Taking the limit, only the $l = 0$ term matters,

$$\Phi(\vec{x}) = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{b} \right) \ . \tag{29}$$

It consists the point charge potential and the induced surface charge potential. Similarly, the surface charge density is

$$\sigma = -\frac{Q}{4\pi b^2}, \tag{30}$$

from which we can see that the charge is uniformly distributed on the sphere with a total charge Q.