Problem 1

In the polar coordinates, we have

$$
\begin{aligned}
x =& r\sin\theta\cos\phi, \\
y =& r\sin\theta\sin\phi, \\
z =& r\cos\theta.
\end{aligned}
\tag{1}
$$

And

$$
\begin{aligned}
dx =& \sin\theta\cos\phi\ dr + r\cos\theta\cos\phi\ d\theta - r\sin\theta\sin\phi\ d\phi, \\
dy =& \sin\theta\sin\phi\ dr + r\cos\theta\sin\phi\ d\theta + r\sin\theta\cos\phi\ d\phi, \\
dz =& \cos\theta\ dr - r\sin\theta\ d\theta.
\end{aligned}
\tag{2}
$$

Therefore,

$$
\begin{aligned}
ds^2 =& dt^2 - dx^2 - dy^2 - dz^2 \\
=& dt^2 - (\sin\theta\cos\phi\ dr + r\cos\theta\cos\phi\ d\theta - r\sin\theta\sin\phi\ d\phi)^2 \\
& - (\sin\theta\sin\phi\ dr + r\cos\theta\sin\phi\ d\theta + r\sin\theta\cos\phi\ d\phi)^2 \\
& - (\cos\theta\ dr - r\sin\theta\ d\theta)^2 \\
=& dt^2 - dr^2 - r^2\ d\theta^2 - r^2\sin^2\theta\ d\phi^2.
\end{aligned}
\tag{3}
$$

So, the induced metric,

$$
g' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}. \tag{4}
$$

Problem 2

1. Christoffel symbols are not components of a tensor

Suppose we have two different local coordinates $\{x^\mu\}, \{y^\mu\}$ whose bases are $\{e_\mu\} = \{\partial/\partial x^\mu\}$ and $\{f_\mu\} = \{\partial/\partial y^\mu\}$ respectively. Denote the Christoffel symbols with respect to y-coordinates by $\widetilde{\Gamma}^\mu{}_{\alpha\beta}$. The basis vector f_μ satisfies

$$
\nabla_{f_\alpha} f_\beta = \widetilde{\Gamma}^\mu{}_{\alpha\beta} f_\mu. \tag{5}
$$

If we write $f_\alpha = (\partial x^\sigma/\partial y^\alpha)e_\sigma$, $f_\beta = (\partial x^\rho/\partial y^\beta)e_\rho$, the LHS becomes

$$
\begin{aligned}
\nabla_{f_\alpha} f_\beta =& \nabla_{f_\alpha}(\frac{\partial x^\rho}{\partial y^\beta} e_\rho) = \frac{\partial^2 x^\rho}{\partial y^\alpha \partial y^\beta} e_\rho + \frac{\partial x^\sigma}{\partial y^\alpha}\frac{\partial x^\rho}{\partial y^\beta}\nabla_{e_\sigma} e_\rho \\
=& \left(\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\sigma}{\partial y^\alpha}\frac{\partial x^\rho}{\partial y^\beta}\Gamma^\nu{}_{\sigma\rho} \right) e_\nu.
\end{aligned}
\tag{6}
$$

Since the RHS of (5) is equal to $\widetilde{\Gamma}^{\mu}{}_{\alpha\beta}(\partial x^\nu/\partial y^\mu)e_\nu$, the Christoffel symbols must transform as

$$\widetilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{\partial x^\sigma}{\partial y^\alpha}\frac{\partial x^\rho}{\partial y^\beta}\frac{\partial y^\mu}{\partial x^\nu}\Gamma^{\nu}{}_{\sigma\rho} + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta}\frac{\partial y^\mu}{\partial x^\nu}. \tag{7}$$

Therefore, Christoffel symbols are not components of a tensor.

2. Christoffel symbols of Minkowski metric vanish

Since the Christoffel symbols only involve the derivatives of the metric. The Minkowski metric is a constant metric, so its Christoffel symbols vanish.

3. Christoffel symbols in polar coordinates

Recall the definition of the Christoffel symbols,

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right). \tag{8}$$

Using metric in (4), after some calculation, we can easily get,

$$\begin{gathered}\Gamma^{r}{}_{\theta\theta} = -r, \quad \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta, \quad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \\ \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r}, \quad \Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r}, \quad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta.\end{gathered} \tag{9}$$

Problem 3

The metric in polar coordinates is given by

$$g_{\mu\nu} = \begin{pmatrix} e^{N(r)} & 0 & 0 & 0 \\ 0 & e^{-L(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}. \tag{10}$$

Its inverse

$$g^{\mu\nu} = \begin{pmatrix} e^{-N(r)} & 0 & 0 & 0 \\ 0 & e^{L(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{11}$$

After some calculation, we can get the Christoffel symbols as

$$\begin{gathered}\Gamma^{r}{}_{tt} = -\frac{1}{2}e^{N(r)+L(r)}N'(r), \quad \Gamma^{r}{}_{rr} = -\frac{1}{2}L'(r), \quad \Gamma^{r}{}_{\theta\theta} = re^{L(r)}, \\ \Gamma^{r}{}_{\phi\phi} = r\sin^2\theta e^{L(r)}, \quad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{t}{}_{tr} = \Gamma^{t}{}_{rt} = \frac{1}{2}N'(r), \\ \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r}, \quad \Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r}, \quad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta.\end{gathered} \tag{12}$$

Problem 4

1. covariant derivative of vectors

The covariant derivative of a vector is defined as

$$A^{\mu}{}_{;\beta} = \frac{\partial A^{\mu}}{\partial x^{\beta}} + \Gamma^{\mu}{}_{\alpha\beta} A^{\alpha}. \tag{13}$$

Suppose we have two different local coordinates $\{x^{\mu}\}$, $\{y^{\mu}\}$. Denote the vector components and the Christoffel symbols with respect to y-coordinates by $\widetilde{A}^{\mu}$ and $\widetilde{\Gamma}^{\mu}{}_{\alpha\beta}$. They transform as

$$\widetilde{A}^{\mu} = \frac{\partial y^{\mu}}{\partial x^{\nu}} A^{\nu}, \quad \widetilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{\partial x^{\sigma}}{\partial y^{\alpha}} \frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial y^{\mu}}{\partial x^{\nu}} \Gamma^{\nu}{}_{\sigma\rho} + \frac{\partial^2 x^{\nu}}{\partial y^{\alpha} \partial y^{\beta}} \frac{\partial y^{\mu}}{\partial x^{\nu}}. \tag{14}$$

First, we derive an alternative equation for the transformation law of Christoffel symbols. Using chain rule,

$$\frac{\partial y^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial y^{\alpha}} = \frac{\partial y^{\mu}}{\partial y^{\alpha}} = \delta^{\mu}_{\alpha}. \tag{15}$$

Since the Kronecker delta doesn't depend on the local coordinate. We have

$$\frac{\partial}{\partial y^{\beta}} \left(\frac{\partial y^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial y^{\alpha}} \right) = 0. \tag{16}$$

Now using Leibniz rule,

$$\frac{\partial^2 y^{\mu}}{\partial y^{\beta} \partial x^{\nu}} \frac{\partial x^{\nu}}{\partial y^{\alpha}} + \frac{\partial y^{\mu}}{\partial x^{\nu}} \frac{\partial^2 x^{\nu}}{\partial y^{\alpha} \partial y^{\beta}} = 0, \tag{17}$$

or

$$\frac{\partial y^{\mu}}{\partial x^{\nu}} \frac{\partial^2 x^{\nu}}{\partial y^{\alpha} \partial y^{\beta}} = -\frac{\partial^2 y^{\mu}}{\partial y^{\beta} \partial x^{\nu}} \frac{\partial x^{\nu}}{\partial y^{\alpha}} = -\frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial^2 y^{\mu}}{\partial x^{\rho} \partial x^{\nu}} \frac{\partial x^{\nu}}{\partial y^{\alpha}}. \tag{18}$$

Therefore, the Christoffel symbols transform as

$$\widetilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{\partial x^{\sigma}}{\partial y^{\alpha}} \frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial y^{\mu}}{\partial x^{\nu}} \Gamma^{\nu}{}_{\sigma\rho} - \frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial^2 y^{\mu}}{\partial x^{\rho} \partial x^{\nu}} \frac{\partial x^{\nu}}{\partial y^{\alpha}}. \tag{19}$$

Now, the ordinary derivative transforms as

$$\begin{aligned} \frac{\partial \widetilde{A}^{\mu}}{\partial y^{\beta}} &= \frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial}{\partial x^{\rho}} \left(\frac{\partial y^{\mu}}{\partial x^{\nu}} A^{\nu} \right) \\ &= \left(\frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial y^{\mu}}{\partial x^{\nu}} \right) \frac{\partial A^{\nu}}{\partial x^{\rho}} + \frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial^2 y^{\mu}}{\partial x^{\rho} \partial x^{\nu}} A^{\nu}. \end{aligned} \tag{20}$$

The rest part transform as

$$\begin{aligned} \widetilde{\Gamma}^{\mu}{}_{\alpha\beta} \widetilde{A}^{\alpha} &= \left(\frac{\partial x^{\sigma}}{\partial y^{\alpha}} \frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial y^{\mu}}{\partial x^{\nu}} \Gamma^{\nu}{}_{\sigma\rho} - \frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial^2 y^{\mu}}{\partial x^{\rho} \partial x^{\nu}} \frac{\partial x^{\nu}}{\partial y^{\alpha}} \right) \left(\frac{\partial y^{\alpha}}{\partial x^{\tau}} A^{\tau} \right) \\ &= \left(\frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial y^{\mu}}{\partial x^{\nu}} \right) \Gamma^{\nu}{}_{\sigma\rho} A^{\sigma} - \frac{\partial x^{\rho}}{\partial y^{\beta}} \frac{\partial^2 y^{\mu}}{\partial x^{\rho} \partial x^{\nu}} A^{\nu}. \end{aligned} \tag{21}$$

Adding them up yields

$$\frac{\partial \widetilde{A}^\mu}{\partial y^\beta} + \widetilde{\Gamma}^\mu{}_{\alpha\beta}\widetilde{A}^\alpha = \left(\frac{\partial x^\rho}{\partial y^\beta}\frac{\partial y^\mu}{\partial x^\nu}\right)\left(\frac{\partial A^\nu}{\partial x^\rho} + \Gamma^\nu{}_{\sigma\rho}A^\sigma\right). \tag{22}$$

Therefore, the covariant derivative of vectors transform like tensors.

2. contravariant derivatives of vectors

The contravariant derivative of a vector is defined as

$$B_{\mu;\beta} = \frac{\partial B_\mu}{\partial x^\beta} - \Gamma^\alpha{}_{\mu\beta}B_\alpha. \tag{23}$$

We have

$$\widetilde{B}_\mu = \frac{\partial x^\nu}{\partial y^\mu}B_\nu, \quad \widetilde{\Gamma}^\alpha{}_{\mu\beta} = \frac{\partial x^\sigma}{\partial y^\mu}\frac{\partial x^\rho}{\partial y^\beta}\frac{\partial y^\alpha}{\partial x^\nu}\Gamma^\nu{}_{\sigma\rho} + \frac{\partial^2 x^\nu}{\partial y^\mu \partial y^\beta}\frac{\partial y^\alpha}{\partial x^\nu}. \tag{24}$$

The ordinary derivative transforms as

$$\begin{aligned}\frac{\partial \widetilde{B}_\mu}{\partial y^\beta} =& \frac{\partial x^\rho}{\partial y^\beta}\frac{\partial}{\partial x^\rho}\left(\frac{\partial x^\sigma}{\partial y^\mu}B_\sigma\right)\\ =& \left(\frac{\partial x^\rho}{\partial y^\beta}\frac{\partial x^\sigma}{\partial y^\mu}\right)\frac{\partial B_\sigma}{\partial x^\rho} + \frac{\partial^2 x^\sigma}{\partial y^\beta \partial y^\mu}B_\sigma.\end{aligned} \tag{25}$$

The rest part transforms as

$$\begin{aligned}-\widetilde{\Gamma}^\alpha{}_{\mu\beta}\widetilde{B}_\alpha =& -\left(\frac{\partial x^\sigma}{\partial y^\mu}\frac{\partial x^\rho}{\partial y^\beta}\frac{\partial y^\alpha}{\partial x^\nu}\Gamma^\nu{}_{\sigma\rho} + \frac{\partial^2 x^\nu}{\partial y^\mu \partial y^\beta}\frac{\partial y^\alpha}{\partial x^\nu}\right)\left(\frac{\partial x^\tau}{\partial y^\alpha}B_\tau\right)\\ =& -\left(\frac{\partial x^\sigma}{\partial y^\mu}\frac{\partial x^\rho}{\partial y^\beta}\right)\Gamma^\nu{}_{\sigma\rho}B_\nu - \frac{\partial^2 x^\nu}{\partial y^\mu \partial y^\beta}B_\nu.\end{aligned} \tag{26}$$

Adding them up yields

$$\frac{\partial \widetilde{B}_\mu}{\partial y^\beta} - \widetilde{\Gamma}^\alpha{}_{\mu\beta}\widetilde{B}_\alpha = \left(\frac{\partial x^\rho}{\partial y^\beta}\frac{\partial x^\sigma}{\partial y^\mu}\right)\left(\frac{\partial B_\sigma}{\partial x^\rho} - \Gamma^\nu{}_{\sigma\rho}B_\nu\right). \tag{27}$$

Therefore, the contravariant derivatives of vectors also transform like tensors.

Problem 5

$$\begin{aligned}g^{\mu\nu}{}_{;\beta} =& \frac{\partial g^{\mu\nu}}{\partial x^\beta} + \Gamma^\mu{}_{\alpha\beta}g^{\alpha\nu} + \Gamma^\nu{}_{\alpha\beta}g^{\mu\alpha}\\ =& \frac{\partial g^{\mu\nu}}{\partial x^\beta} + \frac{1}{2}g^{\mu\rho}\left(\frac{\partial g_{\alpha\rho}}{\partial x^\beta} + \frac{\partial g_{\rho\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\rho}\right)g^{\alpha\nu}\\ &+ \frac{1}{2}g^{\nu\rho}\left(\frac{\partial g_{\alpha\rho}}{\partial x^\beta} + \frac{\partial g_{\rho\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\rho}\right)g^{\alpha\mu}.\end{aligned} \tag{28}$$

Since α and ρ are dumb indices, interchange them in the third term and obtain

$$\begin{aligned}
g^{\mu\nu}{}_{;\beta} =& \frac{\partial g^{\mu\nu}}{\partial x^\beta} + \frac{1}{2} g^{\mu\rho} \left(\frac{\partial g_{\alpha\rho}}{\partial x^\beta} + \frac{\partial g_{\rho\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\rho} \right) g^{\alpha\nu} \\
&+ \frac{1}{2} g^{\nu\alpha} \left(\frac{\partial g_{\rho\alpha}}{\partial x^\beta} + \frac{\partial g_{\alpha\beta}}{\partial x^\rho} - \frac{\partial g_{\rho\beta}}{\partial x^\alpha} \right) g^{\rho\mu} \\
=& \frac{\partial g^{\mu\nu}}{\partial x^\beta} + g^{\mu\rho} g^{\alpha\nu} \frac{\partial g_{\alpha\rho}}{\partial x^\beta}.
\end{aligned} \tag{29}$$

Since $\delta^\nu_\rho = g_{\rho\alpha} g^{\alpha\nu}$, we have

$$\frac{\partial}{\partial x^\beta} \left(g_{\rho\alpha} g^{\alpha\nu} \right) = g^{\alpha\nu} \frac{\partial g_{\rho\alpha}}{\partial x^\beta} + g_{\rho\alpha} \frac{\partial g^{\alpha\nu}}{\partial x^\beta} = 0. \tag{30}$$

Therefore

$$g^{\mu\nu}{}_{;\beta} = \frac{\partial g^{\mu\nu}}{\partial x^\beta} - g^{\mu\rho} g_{\rho\alpha} \frac{\partial g^{\alpha\nu}}{\partial x^\beta} = \frac{\partial g^{\mu\nu}}{\partial x^\beta} - \delta^\mu_\alpha \frac{\partial g^{\alpha\nu}}{\partial x^\beta} = 0. \tag{31}$$

Similarly,

$$\begin{aligned}
g_{\mu\nu;\beta} =& \frac{\partial g_{\mu\nu}}{\partial x^\beta} - \Gamma^\alpha{}_{\mu\beta} g_{\alpha\nu} - \Gamma^\alpha{}_{\nu\beta} g_{\mu\alpha} \\
=& \frac{\partial g_{\mu\nu}}{\partial x^\beta} - \frac{1}{2} g^{\alpha\rho} \left(\frac{\partial g_{\rho\beta}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\beta} - \frac{\partial g_{\mu\beta}}{\partial x^\rho} \right) g_{\alpha\nu} \\
&- \frac{1}{2} g^{\alpha\rho} \left(\frac{\partial g_{\rho\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\beta} - \frac{\partial g_{\nu\beta}}{\partial x^\rho} \right) g_{\alpha\mu} \\
=& \frac{\partial g_{\mu\nu}}{\partial x^\beta} - \frac{1}{2} \delta^\rho_\nu \left(\frac{\partial g_{\rho\beta}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\beta} - \frac{\partial g_{\mu\beta}}{\partial x^\rho} \right) \\
&- \frac{1}{2} \delta^\rho_\mu \left(\frac{\partial g_{\rho\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\beta} - \frac{\partial g_{\nu\beta}}{\partial x^\rho} \right) = 0.
\end{aligned} \tag{32}$$

Problem 6

1. properties of Riemann tensor

Riemann tensor is given by

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\sigma{}_{\beta\nu} \Gamma^\alpha{}_{\sigma\mu} - \Gamma^\sigma{}_{\beta\mu} \Gamma^\alpha{}_{\sigma\nu}. \tag{33}$$

We lower the first index

$$\begin{aligned}
R_{\alpha\beta\mu\nu} =& g_{\alpha\tau} R^\tau{}_{\beta\mu\nu} \\
=& g_{\alpha\tau} \Gamma^\tau{}_{\beta\nu,\mu} - g_{\alpha\tau} \Gamma^\tau{}_{\beta\mu,\nu} + g_{\alpha\tau} \Gamma^\sigma{}_{\beta\nu} \Gamma^\tau{}_{\sigma\mu} - g_{\alpha\tau} \Gamma^\sigma{}_{\beta\mu} \Gamma^\tau{}_{\sigma\nu}.
\end{aligned} \tag{34}$$

Also, in a local inertial frame, the curvature can vanish, though its derivative doesn't vanish. So we can ignore the last two terms in (34) and further expand

(34) as

$$\begin{aligned} R_{\alpha\beta\mu\nu} =& \frac{1}{2}\partial_\mu(\partial_\beta g_{\alpha\nu} + \partial_\nu g_{\beta\alpha} - \partial_\alpha g_{\beta\nu}) - \frac{1}{2}\partial_\nu(\partial_\beta g_{\alpha\mu} + \partial_\mu g_{\beta\alpha} - \partial_\alpha g_{\beta\mu}) \\ =& \frac{1}{2}(\partial_\mu\partial_\beta g_{\alpha\nu} - \partial_\mu\partial_\alpha g_{\beta\nu} - \partial_\nu\partial_\beta g_{\alpha\mu} + \partial_\nu\partial_\alpha g_{\beta\mu}) \end{aligned} \tag{35}$$

Then, it's easy to see that

$$\begin{gathered} R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}, \quad R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \\ R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}, \quad R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0. \end{gathered} \tag{36}$$

2. Independent components in 4-dimensional spacetime

Consider the tensor $R_{\alpha\beta\mu\nu}$ in general n-dimensional spacetime, since the tensor is antisymmetric in the first two indices, then the choice of α and β is limited to $n(n-1)/2$. Similarly, the choice of μ and ν is also $n(n-1)/2$. Since the Riemann tensor is symmetric under the exchange of the pair of indices. So we can calculate the independent components as

$$\frac{1}{2}\left(\frac{1}{2}n(n-1)\right)\left(\frac{1}{2}n(n-1)+1\right) = \frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n). \tag{37}$$

However, we haven't considered the cyclic identity $R_{\alpha[\beta\mu\nu]} = 0$. A consequence of this identity is that the totally antisymmetric part of the Riemann tensor vanishes

$$R_{[\alpha\beta\mu\nu]} = 0. \tag{38}$$

In fact, this equation, plus other identities, are enough to imply the cyclic identity. So we only need to consider this equation instead. The number of independent components of a totally antisymmetric 4-index tensor is $n(n-1)(n-2)(n-3)/4!$. So we are left with

$$\frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) - \frac{1}{24}n(n-1)(n-2)(n-3) = \frac{1}{12}n^2(n^2-1) \tag{39}$$

independent components of the Riemann tensor. Now, plug in $n = 4$ and we get 20 independent components in 4-dimensional spacetime.

3. Independent components in 2d and 3d

Since we have obtained the formula for general n-dimensional spacetime, we can see that there are 6 independent components in 3d and only 1 independent component in 2d.

Problem 7

For a covariant vector B_α, the parallel transport from P to P_1 is

$$B_\alpha - \Gamma^\beta_{\alpha\mu}(P) B_\beta d\xi^\mu. \tag{40}$$

The parallel transport form P_1 tp P' is

$$B_\alpha - \Gamma^\beta_{\alpha\mu}(P) B_\beta d\xi^\mu - \Gamma^\sigma_{\alpha\nu}(P_1)(B_\sigma - \Gamma^\beta_{\alpha\mu}(P) B_\beta d\xi^\mu) d\zeta^\nu. \tag{41}$$

Since $\Gamma^\sigma_{\alpha\nu}(P_1) = \Gamma^\sigma_{\alpha\nu}(P) + \Gamma^\sigma_{\alpha\nu,\mu}(P) d\xi^\mu$, neglecting third order terms, we obtain

$$B_\alpha - \Gamma^\beta_{\alpha\mu} B_\beta d\xi^\mu - \Gamma^\sigma_{\alpha\nu} B_\sigma d\zeta^\nu + \Gamma^\sigma_{\alpha\nu}\Gamma^\beta_{\alpha\mu} B_\beta d\xi^\mu d\zeta^\nu - \Gamma^\sigma_{\alpha\nu,\mu} B_\sigma d\xi^\mu d\zeta^\nu. \tag{42}$$

Repeating the same procedure, passing first through P_2 and then through P_1, we obtain

$$B_\alpha - \Gamma^\beta_{\alpha\mu} B_\beta d\zeta^\mu - \Gamma^\sigma_{\alpha\nu} B_\sigma d\xi^\nu + \Gamma^\sigma_{\alpha\nu}\Gamma^\beta_{\alpha\mu} B_\beta d\zeta^\mu d\xi^\nu - \Gamma^\sigma_{\alpha\nu,\mu} B_\sigma d\zeta^\mu d\xi^\nu. \tag{43}$$

Subtracting the two quantities, we find

$$(\Gamma^\beta_{\alpha\mu,\nu} - \Gamma^\beta_{\alpha\nu,\mu} + \Gamma^\sigma_{\alpha\nu}\Gamma^\beta_{\alpha\mu} - \Gamma^\sigma_{\alpha\mu}\Gamma^\beta_{\alpha\nu}) B_\beta d\xi^\mu d\zeta^\nu. \tag{44}$$

Therefore,

$$\Delta B_\alpha = R^\beta{}_{\alpha\nu\mu} B_\beta d\xi^\mu d\zeta^\nu = R_\alpha{}^\beta{}_{\mu\nu} B_\beta d\xi^\mu d\zeta^\nu. \tag{45}$$