Problem 1

The metric is given by

$$ds^2 = (1+2V)dt^2 - dx^2 - dy^2 - dz^2. \tag{1}$$

From the definition of the Christoffel symbols,

$$\Gamma^{\mu}{}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\rho}} + \frac{\partial g_{\rho\nu}}{\partial x^{\sigma}} - \frac{\partial g_{\rho\sigma}}{\partial x^{\nu}} \right), \tag{2}$$

we can easily calculate that, for $i = 1, 2, 3$,

$$\Gamma^{i}{}_{00} = -\frac{1}{2} g^{ii} \frac{\partial g_{00}}{\partial x^i} = -\frac{1}{2}(-1)\partial_i(1+2V) = \partial_i V, \tag{3}$$

$$\Gamma^{0}{}_{i0} = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^i} = \frac{\partial_i V}{1+2V}. \tag{4}$$

In the weak field limit, V is very small and we have

$$\Gamma^{i}{}_{00} = \Gamma^{0}_{i0} = \partial_i V. \tag{5}$$

Problem 2

We have to show that

$$\Gamma^{\mu}{}_{\mu\alpha} = (\log\sqrt{-g})_{,\alpha}. \tag{6}$$

The determinant g of the matrix $g_{\mu\nu}$ is given by

$$g = \sum_{\mu} g_{\mu\nu}\Delta^{\mu\nu}, \tag{7}$$

where $\Delta^{\mu\nu}$ is the algebraic cofactor and the summation is only summed over μ. The inverse of $g_{\mu\nu}$ is related to the algebraic cofactor by

$$g^{\mu\nu} = \frac{\Delta^{\mu\nu}}{g}. \tag{8}$$

Therefore,

$$\frac{\partial g}{\partial g_{\mu\nu}} = \Delta^{\mu\nu} = g g^{\mu\nu}. \tag{9}$$

The R.H.S of (6) equals

$$\frac{\partial \log\sqrt{-g}}{\partial g} \frac{\partial g}{\partial x^{\alpha}} = \frac{1}{2g} \frac{\partial g}{\partial x^{\alpha}} = \frac{1}{2g} \frac{\partial g}{\partial g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}. \tag{10}$$

By definition, the L.H.S of (6) is

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} \right) = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}. \tag{11}$$

In the last step, the first term cancels the last term because both μ, ν are dummy indices and $g^{\mu\nu} = g^{\nu\mu}$. So we have proved,

$$\Gamma^{\mu}{}_{\mu\alpha} = \left(\log\sqrt{-g}\right)_{,\alpha}. \tag{12}$$

Problem 3

The action corresponding to the matter field is given by

$$S_{\text{scalar}} = \int \sqrt{-g} d^4x \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right). \tag{13}$$

The variation of the action with respect to the metric is given by

$$\frac{\delta S_{\text{scalar}}}{\delta g^{\alpha\beta}} = \frac{\delta \sqrt{-g}}{\delta g^{\alpha\beta}} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right) + \frac{1}{2} \sqrt{-g} \partial_\alpha \phi \partial_\beta \phi. \tag{14}$$

From equation (9) in the last problem, we have

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu}. \tag{15}$$

Therefore

$$\frac{\delta \sqrt{-g}}{\delta g^{\alpha\beta}} = -\frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\delta g}{\delta g^{\alpha\beta}} = -\frac{1}{2} \sqrt{-g} g_{\alpha\beta}, \tag{16}$$

and the stress energy tensor is given by

$$T_{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{scalar}}}{\delta g^{\alpha\beta}} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\alpha\beta} m^2 \phi^2. \tag{17}$$

Problem 4

We have to show that

$$\epsilon^{\mu\nu\rho\sigma} \equiv \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}, \tag{18}$$

is a tensor. First, consider a 4×4 matirx M, and from the definition of $\varepsilon^{\mu\nu\rho\sigma}$ and the definition of the determinant $\det M$, we have

$$\varepsilon^{\mu\nu\rho\sigma} M^0_\mu M^1_\nu M^2_\rho M^3_\sigma = \det M, \tag{19}$$

and further,

$$\varepsilon^{\mu\nu\rho\sigma} M^\alpha_\mu M^\beta_\nu M^\gamma_\rho M^\delta_\sigma = \varepsilon^{\alpha\beta\gamma\delta} \det M. \tag{20}$$

Now suppose we have two different local coordinates $\{x^\mu\}$ and $\{y^\mu\}$, and the Levi Civita symbol in those coordinates is denoted by ε and ε', metric denoted by g and g'. We choose M to be the Jacobian $(\partial y / \partial x)$. Then the above equation shows that

$$\varepsilon^{\mu\nu\rho\sigma} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \frac{\partial y^\gamma}{\partial x^\rho} \frac{\partial y^\delta}{\partial x^\sigma} = \varepsilon'^{\alpha\beta\gamma\delta} \det \left(\frac{\partial y}{\partial x} \right). \tag{21}$$

So $\varepsilon^{\mu\nu\rho\sigma}$ is a tensor density of weight $\omega = 1$. The metric transforms as

$$g_{\mu\nu} = g'_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}. \tag{22}$$

Therefore, the quare root of the determinant $\sqrt{-g}$ transforms as

$$\sqrt{-g} = \sqrt{-g'} \det\left(\frac{\partial y}{\partial x}\right). \tag{23}$$

So we have

$$\frac{\varepsilon'^{\alpha\beta\gamma\delta}}{\sqrt{-g'}} = \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \frac{\partial y^\gamma}{\partial x^\rho} \frac{\partial y^\delta}{\partial x^\sigma}, \tag{24}$$

i.e.,

$$\epsilon^{\alpha\beta\gamma\delta} = \epsilon^{\mu\nu\rho\sigma} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \frac{\partial y^\gamma}{\partial x^\rho} \frac{\partial y^\delta}{\partial x^\sigma}. \tag{25}$$

Therefore, $\epsilon^{\mu\nu\rho\sigma}$ is a tensor.

Problem 5

The generators of the group $SO(3,1)$ are given by

$$J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu. \tag{26}$$

The commutator

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] =& [x_\mu\partial_\nu - x_\nu\partial_\mu, x_\rho\partial_\sigma - x_\sigma\partial_\rho] \\ =& [x_\mu\partial_\nu, x_\rho\partial_\sigma] - [x_\mu\partial_\nu, x_\sigma\partial_\rho] - [x_\nu\partial_\mu, x_\rho\partial_\sigma] + [x_\nu\partial_\mu, x_\sigma\partial_\rho]. \end{aligned} \tag{27}$$

Now we take out one term and try to evaluate it,

$$\begin{aligned} [x_\mu\partial_\nu, x_\rho\partial_\sigma] =& x_\mu\partial_\nu(x_\rho\partial_\sigma) - x_\rho\partial_\sigma(x_\mu\partial_\nu) \\ =& \eta_{\nu\rho}x_\mu\partial_\sigma - \eta_{\sigma\mu}x_\rho\partial_\nu. \end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] =& \eta_{\nu\rho}x_\mu\partial_\sigma - \eta_{\sigma\mu}x_\rho\partial_\nu - \eta_{\nu\sigma}x_\mu\partial_\rho + \eta_{\rho\mu}x_\sigma\partial_\nu \\ & - \eta_{\mu\rho}x_\nu\partial_\sigma + \eta_{\sigma\nu}x_\rho\partial_\mu + \eta_{\mu\sigma}x_\nu\partial_\rho - \eta_{\rho\nu}x_\sigma\partial_\mu \\ =& \eta_{\nu\rho}(x_\mu\partial_\sigma - x_\sigma\partial_\mu) + \eta_{\mu\sigma}(x_\nu\partial_\rho - x_\rho\partial_\mu) \\ & - \eta_{\nu\sigma}(x_\mu\partial_\rho - x_\rho\partial_\mu) - \eta_{\mu\rho}(x_\nu\partial_\sigma - x_\sigma\partial_\nu) \\ =& \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\rho}J_{\nu\sigma}. \end{aligned} \tag{29}$$