

Problem 1

The metric is given by

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (1)$$

From the definition of the Christoffel symbols,

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\rho} + \frac{\partial g_{\rho\nu}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^\nu} \right), \quad (2)$$

we can easily calculate that,

$$\begin{aligned} \Gamma^\theta_{\phi\phi} &= -\frac{1}{2}g^{\theta\theta} \frac{\partial g_{\phi\phi}}{\partial \theta} = -\frac{1}{2}\partial_\theta(\sin^2 \theta) = -\sin \theta \cos \theta, \\ \Gamma^\phi_{\theta\phi} &= \frac{1}{2}g^{\phi\phi} \frac{\partial g_{\phi\phi}}{\partial \theta} = \frac{1}{2} \frac{1}{\sin^2 \theta} \partial_\theta(\sin^2 \theta) = \cot \theta. \end{aligned} \quad (3)$$

Problem 2

Given $ds^2 = dx^2 + dy^2 + dz^2$ and for a two-sphere with radius R , we have

$$\begin{cases} x = R \sin \theta \cos \phi \\ y = R \sin \theta \sin \phi \\ z = R \cos \theta \end{cases}. \quad (4)$$

Then,

$$\begin{aligned} dx &= R \cos \theta \cos \phi d\theta - R \sin \theta \sin \phi d\phi, \\ dy &= R \cos \theta \sin \phi d\theta + R \sin \theta \cos \phi d\phi, \\ dz &= -R \sin \theta d\theta. \end{aligned} \quad (5)$$

Therefore,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2. \end{aligned} \quad (6)$$

Problem 3

In the polar coordinates, we have $\vec{x} = r\hat{r}$ and

$$x = r \cos \theta, \quad y = r \sin \theta. \quad (7)$$

Therefore,

$$d\vec{x} = (dx, dy) = (\cos \theta dr - r \sin \theta d\theta, \sin \theta dr + r \cos \theta d\theta). \quad (8)$$

And

$$d\vec{x} \cdot d\vec{x} = dx^2 + dy^2 = dr^2 + r^2 d\theta^2. \quad (9)$$

Also

$$\begin{aligned}\vec{x} \cdot d\vec{x} &= r \cos^2 \theta dr - r^2 \sin \theta \cos \theta d\theta + r \sin^2 \theta dr + r^2 \sin \theta \cos \theta d\theta \\ &= r dr.\end{aligned}\tag{10}$$