Problem 1

The metric is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{1}$$

From the definition of the Christoffel symbols,

$$\Gamma^{\mu}{}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\rho}} + \frac{\partial g_{\rho\nu}}{\partial x^{\sigma}} - \frac{\partial g_{\rho\sigma}}{\partial x^{\nu}} \right), \tag{2}$$

we can easily calculate that,

$$\begin{aligned}
\Gamma^{\theta}{}_{\phi\phi} &= -\frac{1}{2} g^{\theta\theta} \frac{\partial g_{\phi\phi}}{\partial \theta} = -\frac{1}{2}\partial_{\theta}(\sin^2\theta) = -\sin\theta\cos\theta, \\
\Gamma^{\phi}{}_{\theta\phi} &= \frac{1}{2} g^{\phi\phi} \frac{\partial g_{\phi\phi}}{\partial \theta} = \frac{1}{2}\frac{1}{\sin^2\theta}\partial_{\theta}(\sin^2\theta) = \cot\theta.
\end{aligned} \tag{3}$$

Problem 2

Given $ds^2 = dx^2 + dy^2 + dz^2$ and for a two-sphere with radius R, we have

$$\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}. \tag{4}$$

Then,

$$\begin{aligned}
dx &= R\cos\theta\cos\phi \; d\theta - R\sin\theta\sin\phi \; d\phi, \\
dy &= R\cos\theta\sin\phi \; d\theta + R\sin\theta\cos\phi \; d\phi, \\
dz &= -R\sin\theta \; d\theta.
\end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned}
ds^2 =& dx^2 + dy^2 + dz^2 \\
=& R^2 d\theta^2 + R^2\sin^2\theta d\phi^2.
\end{aligned} \tag{6}$$

Problem 3

In the polar coordinates, we have $\vec{x} = r\hat{r}$ and

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{7}$$

Therefore,

$$d\vec{x} = (dx, dy) = (\cos\theta dr - r\sin\theta d\theta, \sin\theta dr + r\cos\theta d\theta). \tag{8}$$

And

$$d\vec{x}\cdot d\vec{x} = dx^2 + dy^2 = dr^2 + r^2 d\theta^2. \tag{9}$$

Also

$$\begin{aligned}\vec{x}\cdot d\vec{x} &= r\cos^2\theta dr - r^2\sin\theta\cos\theta d\theta + r\sin^2\theta dr + r^2\sin\theta\cos\theta d\theta \\ &= rdr. \end{aligned} \tag{10}$$

Problem 4

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$$(\xi^n)_\alpha^{\ \beta} = K^n (xCx)^{n-1} C_{\alpha\rho} x^\rho x^\beta, \tag{11}$$

by induction, where the matrix $\xi_\alpha^{\ \nu} = K C_{\alpha\rho} x^\rho x^\nu$ and $xCx \equiv C_{\mu\nu} x^\mu x^\nu$.

First,

$$\begin{aligned}(\xi^2)_\alpha^{\ \beta} = \xi_\alpha^{\ \nu}\xi_\nu^{\ \beta} &= K^2 C_{\alpha\rho} x^\rho x^\nu C_{\nu\sigma} x^\sigma x^\beta \\ &= K^2 (xCx) C_{\alpha\rho} x^\rho x^\beta. \end{aligned} \tag{12}$$

Suppose equation (11) holds for $n-1$,

$$(\xi^{n-1})_\alpha^{\ \beta} = K^{n-1}(xCx)^{n-2} C_{\alpha\rho} x^\rho x^\beta, \tag{13}$$

then,

$$\begin{aligned}(\xi^n)_\alpha^{\ \beta} = (\xi^{n-1})_\alpha^{\ \nu}\xi_\nu^{\ \beta} &= (K^{n-1}(xCx)^{n-2} C_{\alpha\rho} x^\rho x^\nu)(K C_{\nu\sigma} x^\sigma x^\beta) \\ &= K^n (xCx)^{n-1} C_{\alpha\rho} x^\rho x^\beta. \end{aligned} \tag{14}$$