Problem 1

In the polar coordinates, we have

$$\begin{aligned} x =& r\sin\theta\cos\phi, \\ y =& r\sin\theta\sin\phi, \\ z =& r\cos\theta. \end{aligned} \tag{1}$$

And

$$\begin{aligned} dx =& \sin\theta\cos\phi\ dr + r\cos\theta\cos\phi\ d\theta - r\sin\theta\sin\phi\ d\phi, \\ dy =& \sin\theta\sin\phi\ dr + r\cos\theta\sin\phi\ d\theta + r\sin\theta\cos\phi\ d\phi, \\ dz =& \cos\theta\ dr - r\sin\theta\ d\theta. \end{aligned} \tag{2}$$

Therefore,

$$\begin{aligned} ds^2 =& dt^2 - dx^2 - dy^2 - dz^2 \\ =& dt^2 - (\sin\theta\cos\phi\ dr + r\cos\theta\cos\phi\ d\theta - r\sin\theta\sin\phi\ d\phi)^2 \\ & - (\sin\theta\sin\phi\ dr + r\cos\theta\sin\phi\ d\theta + r\sin\theta\cos\phi\ d\phi)^2 \\ & - (\cos\theta\ dr - r\sin\theta\ d\theta)^2 \\ =& dt^2 - dr^2 - r^2\ d\theta^2 - r^2\sin^2\theta\ d\phi^2. \end{aligned} \tag{3}$$

So, the induced metric,

$$g' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}. \tag{4}$$

Problem 2

1. Christoffel symbols are not components of a tensor

Suppose we have two different local coordinates $\{x^\mu\}, \{y^\mu\}$ whose bases are $\{e_\mu\} = \{\partial/\partial x^\mu\}$ and $\{f_\mu\} = \{\partial/\partial y^\mu\}$ respectively. Denote the Christoffel symbols with respect to y-coordinates by $\widetilde{\Gamma}^\mu{}_{\alpha\beta}$. The basis vector f_μ satisfies

$$\nabla_{f_\alpha} f_\beta = \widetilde{\Gamma}^\mu{}_{\alpha\beta} f_\mu. \tag{5}$$

If we write $f_\alpha = (\partial x^\sigma/\partial y^\alpha)e_\sigma$, $f_\beta = (\partial x^\rho/\partial y^\beta)e_\rho$, the LHS becomes

$$\begin{aligned} \nabla_{f_\alpha} f_\beta =& \nabla_{f_\alpha}(\frac{\partial x^\rho}{\partial y^\beta} e_\rho) = \frac{\partial^2 x^\rho}{\partial y^\alpha \partial y^\beta} e_\rho + \frac{\partial x^\sigma}{\partial y^\alpha}\frac{\partial x^\rho}{\partial y^\beta}\nabla_{e_\sigma} e_\rho \\ =& \left(\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\sigma}{\partial y^\alpha}\frac{\partial x^\rho}{\partial y^\beta}\Gamma^\nu{}_{\sigma\rho}\right) e_\nu. \end{aligned} \tag{6}$$

Since the RHS of (5) is equal to $\widetilde{\Gamma}^{\mu}{}_{\alpha\beta}(\partial x^{\nu}/\partial y^{\mu})e_{\nu}$, the Christoffel symbols must transform as

$$\widetilde{\Gamma}^{\mu}{}_{\alpha\beta} = \frac{\partial x^{\sigma}}{\partial y^{\alpha}}\frac{\partial x^{\rho}}{\partial y^{\beta}}\frac{\partial y^{\mu}}{\partial x^{\nu}}\Gamma^{\nu}{}_{\sigma\rho} + \frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}}\frac{\partial y^{\mu}}{\partial x^{\nu}}. \tag{7}$$

Therefore, Christoffel symbols are not components of a tensor.

2. Christoffel symbols of Minkowski metric vanish

Since the Christoffel symbols only involve the derivatives of the metric. The Minkowski metric is a constant metric, so its Christoffel symbols vanish.

3. Christoffel symbols in polar coordinates

Recall the definition of the Christoffel symbols,

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right). \tag{8}$$

Using metric in (4), after some calculation, we can easily get,

$$\begin{gathered}\Gamma^{r}{}_{\theta\theta} = -r, \quad \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta, \quad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \\ \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r}, \quad \Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r}, \quad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta.\end{gathered} \tag{9}$$

Problem 3

The metric in polar coordinates is given by

$$g_{\mu\nu} = \begin{pmatrix} e^{N(r)} & 0 & 0 & 0 \\ 0 & e^{-L(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}. \tag{10}$$

Its inverse

$$g^{\mu\nu} = \begin{pmatrix} e^{-N(r)} & 0 & 0 & 0 \\ 0 & e^{L(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{11}$$

After some calculation, we can get the Christoffel symbols as

$$\begin{gathered}\Gamma^{r}{}_{tt} = -\frac{1}{2}e^{N(r)+L(r)}N'(r), \quad \Gamma^{r}{}_{rr} = -\frac{1}{2}L'(r), \quad \Gamma^{r}{}_{\theta\theta} = re^{L(r)}, \\ \Gamma^{r}{}_{\phi\phi} = r\sin^2\theta e^{L(r)}, \quad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{t}{}_{tr} = \Gamma^{t}{}_{rt} = \frac{1}{2}N'(r), \\ \Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r}, \quad \Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r}, \quad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta.\end{gathered} \tag{12}$$