Problem 1

The metric is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{1}$$

From the definition of the Christoffel symbols,

$$\Gamma^{\mu}{}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\nu\sigma}}{\partial x^{\rho}} + \frac{\partial g_{\rho\nu}}{\partial x^{\sigma}} - \frac{\partial g_{\rho\sigma}}{\partial x^{\nu}} \right), \tag{2}$$

we can easily calculate that,

$$\begin{aligned}
\Gamma^{\theta}{}_{\phi\phi} =& -\frac{1}{2} g^{\theta\theta} \frac{\partial g_{\phi\phi}}{\partial\theta} = -\frac{1}{2}\partial_{\theta}(\sin^2\theta) = -\sin\theta\cos\theta, \\
\Gamma^{\phi}{}_{\theta\phi} =& \frac{1}{2} g^{\phi\phi} \frac{\partial g_{\phi\phi}}{\partial\theta} = \frac{1}{2}\frac{1}{\sin^2\theta}\partial_{\theta}(\sin^2\theta) = \cot\theta.
\end{aligned} \tag{3}$$

Problem 2

Given $ds^2 = dx^2 + dy^2 + dz^2$ and for a two-sphere with radius R, we have

$$\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}. \tag{4}$$

Then,

$$\begin{aligned}
dx &= R\cos\theta\cos\phi\ d\theta - R\sin\theta\sin\phi\ d\phi, \\
dy &= R\cos\theta\sin\phi\ d\theta + R\sin\theta\cos\phi\ d\phi, \\
dz &= -R\sin\theta\ d\theta.
\end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned}
ds^2 =& dx^2 + dy^2 + dz^2 \\
=& R^2 d\theta^2 + R^2\sin^2\theta d\phi^2.
\end{aligned} \tag{6}$$

Problem 3

In the polar coordinates, we have $\vec{x} = r\hat{r}$ and

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{7}$$

Therefore,

$$d\vec{x} = (dx, dy) = (\cos\theta dr - r\sin\theta d\theta, \sin\theta dr + r\cos\theta d\theta). \tag{8}$$

And

$$d\vec{x}\cdot d\vec{x} = dx^2 + dy^2 = dr^2 + r^2 d\theta^2. \tag{9}$$

Also

$$\vec{x}\cdot d\vec{x} = r\cos^2\theta dr - r^2\sin\theta\cos\theta d\theta + r\sin^2\theta dr + r^2\sin\theta\cos\theta d\theta \\ = rdr. \tag{10}$$

Problem 4

Pove

$$(\xi^n)_\alpha^{\ \beta} = K^n (xCx)^{n-1} C_{\alpha\rho} x^\rho x^\beta, \tag{11}$$

by induction, where the matrix $\xi_\alpha^{\ \nu} = KC_{\alpha\rho}x^\rho x^\nu$ and $xCx \equiv C_{\mu\nu}x^\mu x^\nu$.

First,

$$(\xi^2)_\alpha^{\ \beta} = \xi_\alpha^{\ \nu}\xi_\nu^{\ \beta} = K^2 C_{\alpha\rho}x^\rho x^\nu C_{\nu\sigma}x^\sigma x^\beta \\ = K^2(xCx)C_{\alpha\rho}x^\rho x^\beta. \tag{12}$$

Suppose equation (11) holds for $n-1$,

$$(\xi^{n-1})_\alpha^{\ \beta} = K^{n-1}(xCx)^{n-2}C_{\alpha\rho}x^\rho x^\beta, \tag{13}$$

then,

$$(\xi^n)_\alpha^{\ \beta} = (\xi^{n-1})_\alpha^{\ \nu}\xi_\nu^{\ \beta} = (K^{n-1}(xCx)^{n-2}C_{\alpha\rho}x^\rho x^\nu)(KC_{\nu\sigma}x^\sigma x^\beta) \\ = K^n(xCx)^{n-1}C_{\alpha\rho}x^\rho x^\beta. \tag{14}$$

Problem 5

Prove that

$$R_{\mu\rho} = \partial_\mu\partial_\rho \log\sqrt{-g} - \frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g}\ \Gamma^\sigma{}_{\mu\rho}) + \Gamma^\sigma{}_{\mu\alpha}\Gamma^\alpha{}_{\rho\sigma}. \tag{15}$$

From the definition of Ricci tensor, L.H.S. equals

$$R_{\mu\rho} = R^\alpha{}_{\mu\rho\alpha} = \Gamma^\alpha{}_{\mu\alpha,\rho} - \Gamma^\alpha{}_{\mu\rho,\alpha} + \Gamma^\sigma{}_{\mu\alpha}\Gamma^\alpha{}_{\sigma\rho} - \Gamma^\sigma{}_{\mu\rho}\Gamma^\alpha{}_{\sigma\alpha}. \tag{16}$$

Fisrt expand the Christoffel symbol of the form

$$\Gamma^\alpha{}_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial x^\beta}\right) = \frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}. \tag{17}$$

In the last step, the last two terms inside the bracket, when dotted with $g^{\alpha\beta}$, gives zero, because they are antisymmetric in α, β, while the metric $g^{\alpha\beta}$ is symmetric in α, β. Now consider

$$\partial_\mu\partial_\rho\log\sqrt{-g} = \partial_\rho\left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{\partial g}{\partial g_{\alpha\beta}}\frac{\partial}{\partial g}(\log\sqrt{-g})\right) \\ = \partial_\rho\left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu}(gg^{\alpha\beta})\left(\frac{1}{2g}\right)\right) \\ = \partial_\rho\left(\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right) = \Gamma^\alpha{}_{\mu\alpha,\rho}. \tag{18}$$

Next consider

$$
\begin{aligned}
-\frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g}\;\Gamma^{\sigma}{}_{\mu\rho}) =& -\frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g})\Gamma^{\sigma}{}_{\mu\rho} - \Gamma^{\sigma}{}_{\mu\rho,\sigma} \\
=& -\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\Gamma^{\sigma}{}_{\mu\rho} - \Gamma^{\alpha}{}_{\mu\rho,\alpha} \\
=& -\Gamma^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\mu\rho} - \Gamma^{\alpha}{}_{\mu\rho,\alpha}.
\end{aligned} \tag{19}
$$

Therefore, R.H.S. of (15) equals

$$
\Gamma^{\alpha}{}_{\mu\alpha,\rho} - \Gamma^{\alpha}{}_{\mu\rho,\alpha} + \Gamma^{\sigma}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\rho\sigma} - \Gamma^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\mu\rho}. \tag{20}
$$

Compare (16) and (20), we see that equation (15) is proved.

Problem 6

First consider

$$
\begin{aligned}
\partial_\mu\partial_\rho \log\sqrt{-g} =& -\frac{1}{2}\partial_\mu\partial_\rho \log(1-K(xCx)) = \partial_\mu\left(\frac{KC_{\rho\sigma}x^\sigma}{1-K(xCx)}\right) \\
=& \frac{KC_{\rho\mu}(1-K(xCx)) + 2K^2C_{\mu\nu}C_{\rho\sigma}x^\nu x^\sigma}{(1-K(xCx))^2} \\
=& \frac{KC_{\rho\mu}}{1-K(xCx)} + \frac{2K^2C_{\mu\nu}C_{\rho\sigma}x^\nu x^\sigma}{(1-K(xCx))^2} \\
=& \frac{K}{1-K(xCx)}g_{\mu\rho} + \frac{K^2}{(1-K(xCx))^2}C_{\mu\alpha}C_{\rho\beta}x^\alpha x^\beta.
\end{aligned} \tag{21}
$$

Next consider in N dimensional spacetime,

$$
\begin{aligned}
&-\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\;\Gamma^{\alpha}{}_{\mu\rho}) = \partial_\alpha\left(\frac{1}{\sqrt{-g}}\right)\sqrt{-g}\;\Gamma^{\alpha}{}_{\mu\rho} - \partial_\alpha\Gamma^{\alpha}{}_{\mu\rho} \\
=&\partial_\alpha(\sqrt{1-K(xCx)})\frac{Kx^\alpha g_{\mu\rho}}{\sqrt{1-K(xCx)}} - \partial_\alpha(Kx^\alpha g_{\mu\rho}) \\
=&-\frac{K^2(xCx)}{1-K(xCx)}g_{\mu\rho} - K\partial_\alpha(x^\alpha)g_{\mu\rho} - Kx^\alpha\partial_\alpha g_{\mu\rho} \\
=&-\frac{K^2(xCx)}{1-K(xCx)}g_{\mu\rho} - K\delta^\alpha_\alpha g_{\mu\rho} \\
&-Kx^\alpha\left(\frac{KC_{\mu\alpha}C_{\rho\beta}x^\beta + KC_{\rho\alpha}C_{\mu\beta}x^\beta}{1-K(xCx)} + \frac{2K^2C_{\mu\nu}C_{\rho\sigma}C_{\alpha\beta}x^\nu x^\sigma x^\beta}{(1-K(xCx))^2}\right) \\
=&-\frac{K^2(xCx)}{1-K(xCx)}g_{\mu\rho} - KNg_{\mu\rho} - \frac{2K^2}{(1-K(xCx))^2}C_{\mu\alpha}C_{\rho\beta}x^\alpha x^\beta
\end{aligned} \tag{22}
$$

Then consider

$$\begin{aligned}
{\Gamma^\sigma}_{\mu\alpha}{\Gamma^\alpha}_{\rho\sigma} &= K^2 x^\sigma x^\alpha g_{\mu\alpha} g_{\rho\sigma} \\
&= K^2 x^\sigma x^\alpha \left(C_{\mu\alpha} + \frac{K C_{\mu\nu} C_{\alpha\beta} x^\nu x^\beta}{1 - K(xCx)} \right) \left(C_{\rho\sigma} + \frac{K C_{\rho\nu} C_{\sigma\beta} x^\nu x^\beta}{1 - K(xCx)} \right) \\
&= K^2 C_{\mu\alpha} C_{\rho\sigma} x^\alpha x^\sigma + \frac{2K^3 (xCx) C_{\rho\alpha} C_{\mu\beta} x^\alpha x^\beta}{1 - K(xCx)} \\
&\quad + \frac{K^4 (xCx)^2 C_{\rho\alpha} C_{\mu\beta} x^\alpha x^\beta}{(1 - K(xCx))^2} \\
&= \left(K^2 + \frac{2K^3 (xCx)}{1 - K(xCx)} + \frac{K^4 (xCx)^2}{(1 - K(xCx))^2} \right) C_{\mu\alpha} C_{\rho\beta} x^\alpha x^\beta \\
&= \frac{K^2}{(1 - K(xCx))^2} C_{\mu\alpha} C_{\rho\beta} x^\alpha x^\beta. \qquad (23)
\end{aligned}$$

Now add them up and we obtain

$$R_{\mu\rho} = \frac{K}{1 - K(xCx)} g_{\mu\rho} - \frac{K^2 (xCx)}{1 - K(xCx)} g_{\mu\rho} - K N g_{\mu\rho} = K(1 - N) g_{\mu\rho}. \qquad (24)$$