

Problem 1

The metric is given by

$$ds^2 = (1 + 2V)dt^2 - dx^2 - dy^2 - dz^2. \quad (1)$$

From the definition of the Christoffel symbols,

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\rho} + \frac{\partial g_{\rho\nu}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^\nu} \right), \quad (2)$$

we can easily calculate that, for $i = 1, 2, 3$,

$$\Gamma^i_{00} = -\frac{1}{2}g^{ii}\frac{\partial g_{00}}{\partial x^i} = -\frac{1}{2}(-1)\partial_i(1 + 2V) = \partial_i V, \quad (3)$$

$$\Gamma^0_{i0} = \frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial x^i} = \frac{\partial_i V}{1 + 2V}. \quad (4)$$

In the weak field limit, V is very small and we have

$$\Gamma^i_{00} = \Gamma^0_{i0} = \partial_i V. \quad (5)$$

Problem 2

We have to show that

$$\Gamma^\mu_{\mu\alpha} = (\log \sqrt{-g})_{,\alpha}. \quad (6)$$

The determinant g of the matrix $g_{\mu\nu}$ is given by

$$g = \sum_{\mu} g_{\mu\nu} \Delta^{\mu\nu}, \quad (7)$$

where $\Delta^{\mu\nu}$ is the algebraic cofactor and the summation is only summed over μ . The inverse of $g_{\mu\nu}$ is related to the algebraic cofactor by

$$g^{\mu\nu} = \frac{\Delta^{\mu\nu}}{g}. \quad (8)$$

Therefore,

$$\frac{\partial g}{\partial g_{\mu\nu}} = \Delta^{\mu\nu} = g g^{\mu\nu}. \quad (9)$$

The R.H.S of (6) equals

$$\frac{\partial \log \sqrt{-g}}{\partial g} \frac{\partial g}{\partial x^\alpha} = \frac{1}{2g} \frac{\partial g}{\partial x^\alpha} = \frac{1}{2g} \frac{\partial g}{\partial g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\alpha}. \quad (10)$$

By definition, the L.H.S of (6) is

$$\Gamma^\mu_{\mu\alpha} = \frac{1}{2}g^{\mu\nu} \left(\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial x^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu} \right) = \frac{1}{2}g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\alpha}. \quad (11)$$

In the last step, the first term cancels the last term because both μ, ν are dummy indices and $g^{\mu\nu} = g^{\nu\mu}$. So we have proved,

$$\Gamma^\mu_{\mu\alpha} = (\log \sqrt{-g})_{,\alpha}. \quad (12)$$

Problem 3

The action corresponding to the matter field is given by

$$S_{\text{scalar}} = \int \sqrt{-g} d^4x \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right). \quad (13)$$

The variation of the action with respect to the metric is given by

$$\frac{\delta S_{\text{scalar}}}{\delta g^{\alpha\beta}} = \frac{\delta \sqrt{-g}}{\delta g^{\alpha\beta}} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right) + \frac{1}{2} \sqrt{-g} \partial_\alpha \phi \partial_\beta \phi. \quad (14)$$

From equation (9) in the last problem, we have

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu}. \quad (15)$$

Therefore

$$\frac{\delta \sqrt{-g}}{\delta g^{\alpha\beta}} = -\frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\delta g}{\delta g^{\alpha\beta}} = -\frac{1}{2} \sqrt{-g} g_{\alpha\beta}, \quad (16)$$

and the stress energy tensor is given by

$$T_{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{scalar}}}{\delta g^{\alpha\beta}} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\alpha\beta} m^2 \phi^2. \quad (17)$$

Problem 4

Problem 5