Problem 1

The metric is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{1}$$

From the definition of the Christoffel symbols,

$$\Gamma^{\mu}{}_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\sigma}}{\partial x^{\rho}} + \frac{\partial g_{\rho\nu}}{\partial x^{\sigma}} - \frac{\partial g_{\rho\sigma}}{\partial x^{\nu}}\right), \tag{2}$$

we can easily calculate that,

$$\begin{aligned}\Gamma^{\theta}{}_{\phi\phi} &= -\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\phi\phi}}{\partial\theta} = -\frac{1}{2}\partial_{\theta}(\sin^2\theta) = -\sin\theta\cos\theta,\\ \Gamma^{\phi}{}_{\theta\phi} &= \frac{1}{2}g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial\theta} = \frac{1}{2}\frac{1}{\sin^2\theta}\partial_{\theta}(\sin^2\theta) = \cot\theta.\end{aligned} \tag{3}$$

Problem 2

Given $ds^2 = dx^2 + dy^2 + dz^2$ and for a two-sphere with radius R, we have

$$\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}. \tag{4}$$

Then,

$$\begin{aligned} dx &= R\cos\theta\cos\phi\ d\theta - R\sin\theta\sin\phi\ d\phi,\\ dy &= R\cos\theta\sin\phi\ d\theta + R\sin\theta\cos\phi\ d\phi,\\ dz &= -R\sin\theta\ d\theta.\end{aligned} \tag{5}$$

Therefore,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2\\ &= R^2 d\theta^2 + R^2\sin^2\theta d\phi^2.\end{aligned} \tag{6}$$

Problem 3

In the polar coordinates, we have $\vec{x} = r\hat{r}$ and

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{7}$$

Therefore,

$$d\vec{x} = (dx, dy) = (\cos\theta dr - r\sin\theta d\theta, \sin\theta dr + r\cos\theta d\theta). \tag{8}$$

And

$$d\vec{x}\cdot d\vec{x} = dx^2 + dy^2 = dr^2 + r^2 d\theta^2. \tag{9}$$

Also

$$\begin{aligned}\vec{x}\cdot d\vec{x} &= r\cos^2\theta dr - r^2\sin\theta\cos\theta d\theta + r\sin^2\theta dr + r^2\sin\theta\cos\theta d\theta \\ &= rdr. \end{aligned} \tag{10}$$

Problem 4

Pove

$$(\xi^n)_\alpha^{\ \beta} = K^n (xCx)^{n-1} C_{\alpha\rho} x^\rho x^\beta, \tag{11}$$

by induction, where the matrix $\xi_\alpha^{\ \nu} = KC_{\alpha\rho}x^\rho x^\nu$ and $xCx \equiv C_{\mu\nu}x^\mu x^\nu$.

First,

$$\begin{aligned}(\xi^2)_\alpha^{\ \beta} = \xi_\alpha^{\ \nu}\xi_\nu^{\ \beta} &= K^2 C_{\alpha\rho}x^\rho x^\nu C_{\nu\sigma}x^\sigma x^\beta \\ &= K^2 (xCx) C_{\alpha\rho}x^\rho x^\beta. \end{aligned} \tag{12}$$

Suppose equation (11) holds for $n-1$,

$$(\xi^{n-1})_\alpha^{\ \beta} = K^{n-1}(xCx)^{n-2}C_{\alpha\rho}x^\rho x^\beta, \tag{13}$$

then,

$$\begin{aligned}(\xi^n)_\alpha^{\ \beta} = (\xi^{n-1})_\alpha^{\ \nu}\xi_\nu^{\ \beta} &= (K^{n-1}(xCx)^{n-2}C_{\alpha\rho}x^\rho x^\nu)(KC_{\nu\sigma}x^\sigma x^\beta) \\ &= K^n(xCx)^{n-1}C_{\alpha\rho}x^\rho x^\beta. \end{aligned} \tag{14}$$

Problem 5

Prove that

$$R_{\mu\rho} = \partial_\mu\partial_\rho \log\sqrt{-g} - \frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g}\ \Gamma^\sigma_{\ \mu\rho}) + \Gamma^\sigma_{\ \mu\alpha}\Gamma^\alpha_{\ \rho\sigma}. \tag{15}$$

From the definition of Ricci tensor, L.H.S. equals

$$R_{\mu\rho} = R^\alpha_{\ \mu\rho\alpha} = \Gamma^\alpha_{\ \mu\alpha,\rho} - \Gamma^\alpha_{\ \mu\rho,\alpha} + \Gamma^\sigma_{\ \mu\alpha}\Gamma^\alpha_{\ \sigma\rho} - \Gamma^\sigma_{\ \mu\rho}\Gamma^\alpha_{\ \sigma\alpha}. \tag{16}$$

Fisrt expand the Christoffel symbol of the form

$$\Gamma^\alpha_{\ \mu\alpha} = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial x^\beta}\right) = \frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}. \tag{17}$$

In the last step, the last two terms inside the bracket, when dotted with $g^{\alpha\beta}$, gives zero, because they are antisymmetric in α, β, while the metric $g^{\alpha\beta}$ is symmetric in α, β. Now consider

$$\begin{aligned}\partial_\mu\partial_\rho\log\sqrt{-g} &= \partial_\rho\left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{\partial g}{\partial g_{\alpha\beta}}\frac{\partial}{\partial g}(\log\sqrt{-g})\right) \\ &= \partial_\rho\left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu}(gg^{\alpha\beta})\left(\frac{1}{2g}\right)\right) \\ &= \partial_\rho\left(\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right) = \Gamma^\alpha_{\ \mu\alpha,\rho}. \end{aligned} \tag{18}$$

Next consider

$$
\begin{aligned}
-\frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g}\;\Gamma^\sigma{}_{\mu\rho}) &= -\frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g})\Gamma^\sigma{}_{\mu\rho} - \Gamma^\sigma{}_{\mu\rho,\sigma} \\
&= -\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\Gamma^\sigma{}_{\mu\rho} - \Gamma^\alpha{}_{\mu\rho,\alpha} \\
&= -\Gamma^\alpha{}_{\sigma\alpha}\Gamma^\sigma{}_{\mu\rho} - \Gamma^\alpha{}_{\mu\rho,\alpha}. \qquad (19)
\end{aligned}
$$

Therefore, R.H.S. of (15) equals

$$
\Gamma^\alpha{}_{\mu\alpha,\rho} - \Gamma^\alpha{}_{\mu\rho,\alpha} + \Gamma^\sigma{}_{\mu\alpha}\Gamma^\alpha{}_{\rho\sigma} - \Gamma^\alpha{}_{\sigma\alpha}\Gamma^\sigma{}_{\mu\rho}. \qquad (20)
$$

Compare (16) and (20), we see that equation (15) is proved.