Solving the low energy physics of (0,2) Landau-Ginzburg models

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In this paper, we discuss the Landau-Ginzburg models with $(0,2)$ supersymmetry. Techniques such as 't Hooft anomaly, c-extremization, elliptic genus and modular invariant partition functions are used to solve the low energy physics of the theory exactly. We will give detailed introduction to these methods and explain how to solve the low energy physics in a general setting. By solving a specific type of $\mathcal{N}=(0,2)$ Landau-Ginzburg theory, we will see its torus partition function goes beyond the ADE classification.

I. INTRODUCTION

This paper is based on our original paper [1] on $(0,2)$ Landau-Ginzburg models. $\mathcal{N}=(0,2)$ supersymmetry plays an important role in the compactifications of type II strings and heterotic strings. In [2], it is shown that $(0,2)$ Landau-Ginzburg (LG) theories describe certain phases of supersymmetric gauge theories. These theories are rich because of different choices of E-terms and J-terms, as we will explain later. We would like to identify the low energy physics of $(0,2)$ Landau-Ginzburg models exactly and see how the infrared (IR) conformal field theories (CFTs) incorporate the richness of $(0,2)$ LG models. We mainly follow [3], where they obtained the low energy fixed point for a number of classes of relatively simple Landau-Ginzburg models and generalize their results to other classes of $(0,2)$ LG models. To get the IR fixed point theory from the ultraviolet (UV) data, we used several tools and constraints, such as c-extremization, 't Hooft anomaly matching condition, elliptic genera, and modular invariant partition functions. In [4, 5], the correspondence between $\mathcal{N}=(2,2)$ LG models with quasi-homogeneous superpotential of ADE type and minimal models of the corresponding ADE type has been found. Therefore, in general, one would expect that $(0,2)$ LG models flow to heterotic CFTs with the right-moving part being the $\mathcal{N}=2$ minimal models and the left-moving part being a CFT without supersymmetry. However, in this paper, we will show that a certain $(0,2)$ LG model does not fit into the ADE classification.

II. $\mathcal{N}=(0,2)$ LANDAU-GINZBURG MODELS

First, we give a brief introduction to $\mathcal{N}=(0,2)$ Landau-Ginzburg models. It is a two-dimensional theory with supersymmetry, which is a spacetime symmetry between fermions and bosons. We usually work in the superspace $(x^0, x^1, \theta^+, \overline{\theta}^+)$, where $\theta^+, \overline{\theta}^+$ are Grassmann variables. We also collect fields into a superfield, which is a "function" of the superspace coordinates. The supersymmetry is generated by supercharges given by

$$Q_+ = \frac{\partial}{\partial\theta^+} + i\overline{\theta}^+(\partial_0+\partial_1) \ , \quad \overline{Q}_+ = -\frac{\partial}{\partial\overline{\theta}^+} - i\theta^+(\partial_0+\partial_1) \ . \tag{1}$$

The superderivatives are given by

$$D_+ = \frac{\partial}{\partial\theta^+} - i\overline{\theta}^+(\partial_0+\partial_1) \ , \quad \overline{D}_+ = -\frac{\partial}{\partial\overline{\theta}^+} + i\theta^+(\partial_0+\partial_1) \ , \tag{2}$$

which anti-commute with the supercharges [2]. There are two kinds of fields, chiral multiplets Φ_i, satisfying $\overline{D}_+\Phi_i = 0$, and fermi multiplets Ψ_a satisfying $\overline{D}_+\Psi_a = E_a(\Phi_i)$, where E_a is called an E-term and is a holomorphic function in the chiral multiplets Φ_i.

The Lagrangian of the theory is given by

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \int d\theta^+\, W + \int d\overline{\theta}^+\, \overline{W} \ , \tag{3}$$

where the term of interest is the second interaction term and W is called a superpotential and is given by

$$W = \sum_a \Psi_a J_a(\Phi_i) \ . \tag{4}$$

Here J_a is called a J-term and is a holomorphic function in the chiral multiplets Φ_i. Moreover supersymmetry requires that [2]

$$\sum_a E_a(\Phi_i) J_a(\Phi_i) = 0 \ . \tag{5}$$

For simplicity, in this paper, we only consider the cases in which the E-terms vanish.

There are two kinds of $U(1)$ symmetries. One is the $\mathcal{R}$-symmetry $U(1)_{\mathcal{R}}$, which is an internal symmetry of rotating the supercharges. Another one will be called a $U(1)_L$ symmetry. The $U(1)_L$ charge for θ^+ is zero and $U(1)_L$ charges of Φ_i and Ψ_a will be denoted as q_i and Q_a respectively. Thus, J_a has to be quasi-homogeneous,

$$J_a(t^{q_i}\Phi_i) = t^{-Q_a} J_a(\Phi_i) \ . \tag{6}$$

Note that in general, we can have several copies of $U(1)_L$ symmetry, leading to a $U(1)_L^m$ symmetry. These charges are defined up to a normalization fixed by c-extremization. Moreover, this flavor symmetry enables us to construct a UV candidate for the $\mathcal{R}$-symmetry of IR superconformal field theory [6].

In [7], $(0,2)$ LG models flowing to compact IR fixed points with $c = \bar{c} < 3$ are classified. In all these theories, there are two chiral multiplets Φ_1, Φ_2 and two fermi multiplets Ψ_1, Ψ_2. The E-terms are zero and the theories are classified according to the J-terms:

$$\begin{aligned} \text{type a}: J_1 &= \Phi_1^m + \Phi_2^n \ , J_2 = \Phi_1\Phi_2 \ , \\ \text{type b}: J_1 &= \Phi_1^k + \Phi_2^2 \ , J_2 = \Phi_1^2\Phi_2 \ , \\ \text{type c}: J_1 &= \Phi_1^3 + \Phi_2^2 \ , J_2 = \Phi_1^3\Phi_2 \ , \\ \text{type d}: J_1 &= \Phi_1^3 + \Phi_2^2 \ , J_2 = \Phi_1^2\Phi_2^2 \ , \\ \text{type e}: J_1 &= \Phi_1^3 + \Phi_2^2 \ , J_2 = \Phi_1\Phi_2^2 \ . \end{aligned} \tag{7}$$

Following the same line in [3], we study the low energy physics of these theories. In this paper, we will mainly focus on type b theory with $k = 4$.

III. METHODS

In this section, we introduce several methods we used to identify the IR CFT of a $(0,2)$ LG model.

A. c-extremization

In four-dimensional superconformal field theories (SCFTs), the $\mathcal{R}$-symmetry can be determined by a-maximization [8]. Similarly, in two-dimensional SCFTs, there's a procedure called c-extremization [9], which will be used to determine the $\mathcal{R}$-symmetries of two-dimensional $\mathcal{N} = (0,2)$ unitary SCFTs.

The main argument is that we can determine the $\mathcal{R}$-symmetry for a unitary SCFT by extremizing a function $c^{\text{tr}}_{\mathcal{R}}$. It is called a trial central charge and is a linear combination of the IR $U(1)_{\mathcal{R}}$ charges. Assuming there is no new abelian symmetry in IR, we can calculate this function from UV configurations with the help of 't Hooft anomaly.

Explicitly, if we assign trial $\mathcal{R}$-charges $\mathcal{R}[\Phi_i]$ to chiral multiplets Φ_i, and $\mathcal{R}[\Psi_a]$ to fermi multiplets Ψ_a, then the trial central charge is given by

$$c_{\mathcal{R}}^{\text{tr}} = 3\,\text{Tr}\,\gamma^3\mathcal{R}^2 = 3\sum_i(\mathcal{R}[\Phi_i]-1)^2 - 3\sum_a(\mathcal{R}[\Psi_a])^2\ . \tag{8}$$

We also have another constraint that the $\mathcal{R}$-charge of the superpotential should be one,

$$\mathcal{R}[J_a(\Phi_i)] + \mathcal{R}[\Psi_a] = 1\ . \tag{9}$$

We extremize the trial central charge subject to this constraint. Moreover, we can calculate the gravitational anomaly [10],

$$\text{Tr}\,\gamma^3 = c - \bar{c} = \#(\text{fermi}) - \#(\text{chiral})\ , \tag{10}$$

which equals the number of fermi multiplets minus the number of chiral multiplets. For the theories of interest (7), there are two chiral multiplets and two fermi multiplets. So the gravitational anomaly vanishes and $c = \bar{c}$. By this procedure we can obtain the $\mathcal{R}$-symmetry and the central charges of the fixed point, which would give us the IR structure.

B. 't Hooft anomaly matching

In [11], 't Hooft showed that chiral anomaly for the flavor symmetry does not depend on the energy scale so the 't Hooft anomaly calculated in UV is still valid in IR. In general, consider a theory with $U(1)^m$ flavor symmetry with conserved currents J_I^μ with $I = 1,\dots,m$. When the theory is coupled to a background gauge field A_μ^I with field strength $F_{\mu\nu}^I$, the current is not conserved at quantum level and the anomaly is given by

$$\partial_\mu J_I^\mu = \sum_J k_{IJ}F_{\mu\nu}^J\epsilon^{\mu\nu}\ , \tag{11}$$

where the 't Hooft anomaly matrix is given by

$$k_{IJ} = \sum_a Q_I^a Q_J^a - \sum_i q_I^i q_J^i\ , \tag{12}$$

where Q_I^a is the I-th $U(1)$ charge of the fermi multiplet Ψ_a and q_I^i is the I-th $U(1)$ charge of the chiral multiplet Φ_i. The anomaly matrix k is positive definite, meaning there should be a $U(1)_k^m$ affine algebra in IR in the left-moving sector.

C. Elliptic genus and modular invariant partition function

After identifying the IR structure, we can calculate the torus partition function of the IR CFT. We should check if the torus partition function is modular invariant. Also, the elliptic genus can be calculated from the UV information and is closely related to the torus partition function, which would give us a tight constraint on the torus partition function. So we can use the elliptic genus to verify if the IR structure is correct or not. The elliptic genus is defined by [12]

$$\mathcal{I}(\tau, z) = \mathrm{Tr}\,(-1)^F q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - c/24} y^{J_0} \ , \tag{13}$$

where L_0 and $\overline{L}_0$ are the Virasoro generators, J_0 is the $U(1)$ charge operator, $q = e^{2\pi i \tau}$ with τ parametrizing the torus, and $y = e^{2\pi i z}$ is the $U(1)$ fugacity. The trace is taken over the Ramond-Ramond sector where we apply periodic boundary conditions on fermions. The fermion number operator is given by

$$(-1)^F = \exp(i\pi(J_0 - \overline{J}_0)) \ . \tag{14}$$

For a $(0,2)$ supersymmetry, the $\overline{q}$-dependence drops out, though we keep it explicit here. We can also calculate the elliptic genus in the NS-NS sector from the UV data, and it takes a compact form [3],

$$\mathcal{I}(\tau, z) = q^{-\frac{c}{24}} \frac{\prod_a \vartheta(q^{\frac{1+R_a}{2}} y^{Q_a}; q)}{\prod_i \vartheta(q^{\frac{r_i}{2}} y^{q_i}; q)} \ . \tag{15}$$

The numerator receives contributions from fermi multiplets Ψ_a with $\mathcal{R}$-charges R_a and $U(1)_L$ charges Q_a, while the denominator receives contributions from chiral multiplets Φ_i with $\mathcal{R}$-charges r_i and $U(1)_L$ charges q_i. The function ϑ is given by

$$\vartheta(x; q) = \prod_{k=0}^{\infty} (1 - x q^k) \left(1 - \frac{q^{k+1}}{x}\right) \ . \tag{16}$$

Also, we have modular invariant partition function defined on torus,

$$Z = \mathrm{Tr}\, q^{L_0 - c/24} y^{J_0} \overline{q}^{\overline{L}_0 - c/24} \overline{x}^{\overline{J}_0} \ . \tag{17}$$

In general, the IR CFT can be constructed as coset models from affine $U(1)$ and $SU(2)$. And in practice we can write the torus partition function as a summation of product of the chiral algebra characters. Now we discuss how to obtain the elliptic genus from the partition function. First, only the chiral primary states satisfying $\overline{L}_0 = -\overline{J}_0/2$ in the right-moving sector would contribute to the elliptic genus. Then we have to insert $(-1)^F$ for each term.

IV. ONE EXAMPLE

The model we considered is called type b with $k = 4$ in (7). There are two chiral multiplets Φ_1, Φ_2 and two fermi multiplets Ψ_1, Ψ_2 and the superpotential is given by

$$W = \Psi_1(\Phi_1^4 + \Phi_2^2) + \Psi_2\Phi_1^2\Phi_2 \ . \tag{18}$$

From the gravitational anomaly and c-extremization, we obtain that,

$$c = \bar{c} = \frac{25}{9} \ . \tag{19}$$

The central charge for $\mathcal{N} = 2$ minimal model is given by [13],

$$c = \frac{3k}{k+2} \ , \tag{20}$$

with $k \geq 1$ being the level. We can easily solve for $k = 25$. So the right-moving sector is simply the $\mathcal{N} = 2$ minimal model with level $k = 25$.

The $U(1)$ charges are given by

	Φ_1	Φ_2	Ψ_1	Ψ_2
$U(1)_{\mathcal{R}}$	$\frac{5}{27}$	$\frac{10}{27}$	$\frac{7}{27}$	$\frac{7}{27}$
$U(1)_L$	1	2	-4	-4

TABLE I. $U(1)$ charges of multiplets

The elliptic genus is given by

$$\mathcal{I} = q^{-25/216}(1 + yq^{5/54} + 2y^2q^{5/27} + 2y^3q^{5/18} + \dots) \ . \tag{21}$$

The 't Hooft anomaly can be calculated as

$$\mathrm{Tr}\,\gamma^3\mathcal{R}^2 = (-4)^2 + (-4)^2 - 1^2 - 2^2 = 27 \ . \tag{22}$$

Hence part of the left-moving sector in IR is a $U(1)_{27}$ Wess-Zumino-Witten model with central charge one. The remaining part has central charge 16/9. A natural guess would be the parafermion model which is a coset model $SU(2)_k/U(1)_{2k}$ with level $k = 25$ since the parafermion has central charge $2(k-1)/(k+2)$ with k being the level [13].

Therefore, in conclusion, the IR structure of this model is given by

$$\left(\frac{SU(2)_{25}}{U(1)_{50}} \times U(1)_{27}\right) \otimes \overline{\left(\frac{SU(2)_{25} \times U(1)_4}{U(1)_{54}}\right)} \ . \tag{23}$$

For such a coset model, the modular invariant partition function can be written in terms of the affine $SU(2)$ and affine $U(1)$ characters,

$$Z = \sum_{\alpha,\beta,\mu,\lambda,\rho} N_{\overline{\alpha}\beta}\, C_{\mu\overline{\lambda}\overline{\rho}}\, \chi^{SU(2)_{25}/U(1)_{50}}_{\alpha,\overline{\mu}} \chi^{U(1)_{27}}_{\lambda} \chi^{SU(2)_{25}\times U(1)_4/U(1)_{54}}_{\overline{\beta}\rho} \ , \tag{24}$$

where χ's are the characters, $\alpha, \beta \in \mathbb{Z}_{26}$ are the $SU(2)$ weights and $\mu \in \mathbb{Z}_{50}, \lambda \in \frac{1}{2} + \mathbb{Z}_{27}, \rho \in \mathbb{Z}_{54}$ are the $U(1)$ weights. We use overlines over indices to suggest that they transform under the conjugate representation of the modular group $SL(2, \mathbb{Z})$. We sum over all the possible weights, which is encoded in the modular invariant tensor $N_{\overline{\alpha}\beta}$ and $C_{\mu\overline{\lambda}\overline{\rho}}$ that take values in either one or zero. The $U(1)$ tensor C can be obtained by a process dubbed "rational transformation" in [3]. Basically, there is a rational equivalence between the $U(1)$ levels, leading to identities among $U(1)$ characters [14]. Together with the modular invariance, this would determine the $U(1)$ tensor. We directly list the result that $C_{\mu\lambda\rho} = 1$, when $\mu = 5m, m \in \mathbb{Z}$ and $\lambda = (27m - 5\rho)/2$. Otherwise, C is zero. On the other hand, the $SU(2)$ tensor N should be the identity matrix according to the ADE classification [15]. Therefore, the modular invariant partition function takes the form,

$$Z = \sum_{\ell\in\mathbb{Z}_{26}} \sum_{m\in\mathbb{Z}_{10}} \sum_{n\in\mathbb{Z}_{54}} \chi^{SU(2)_{25}/U(1)_{50}}_{\ell,5m} \chi^{U(1)_{27}}_{(27m-5n)/2} \chi^{SU(2)_{25}\times U(1)_4/U(1)_{54}}_{\ell n} \ . \tag{25}$$

To obtain the corresponding elliptic genus from this expression, we follow the procedure discussed in §III C. For chiral primary states, we should have $n = -\ell$ and the corresponding conformal dimension J_0 is $-\ell/54$. And we also have to insert the factor $(-1)^F = (-1)^{J_0 - \overline{J}_0} = (-1)^{2L_0 + 2\overline{L}_0}$. So we get rid of the minimal model character in (25), set $n = -\ell$, and insert a factor $(-1)^{-\ell/27}$ which accounts for $(-1)^{2\overline{L}_0}$. For the remaining factor $(-1)^{2L_0}$, recall that in the left-moving sector the partition function looks like $\mathrm{Tr}\, q^{L_0 - c/24} y^{J_0}$. Up to a factor $q^{c/24}$, it is a series expansion in q with coefficients functions of y and the exponent being the conformal dimension L_0, so we define an action f on q-series such that

$$f\left(\sum_i \alpha_i q^{\beta_i}\right) = \sum_i (-1)^{2\beta_i} \alpha_i q^{\beta_i} \ . \tag{26}$$

Then, the corresponding elliptic genus is given by

$$\begin{aligned} \mathcal{I}' &= q^{-25/216} f\left(q^{25/216} \sum_{\ell=0}^{25} \sum_{m=0}^{9} (-1)^{-\ell/27} \chi^{SU(2)_{25}/U(1)_{50}}_{\ell,5m} \chi^{U(1)_{27}}_{(27m+5\ell)/2} \right) \\ &= q^{-25/216}(1 + 2yq^{5/54} + 2y^2 q^{5/27} + 2y^3 q^{5/18} + \dots) \ . \end{aligned} \tag{27}$$

However, this result does not agree with the elliptic genus (21), though they do look similar. By reverse engineering, we find that non-diagonal parts with fractional multiplicity should be added in the $SU(2)$ modular invariant tensor. The correct $SU(2)$ modular invariant tensor for this model takes the form

$$\begin{aligned} N_{\ell\bar{\ell}} = \delta_{\ell\bar{\ell}} &- \frac{1}{3}(\delta_{2,\ell} - \delta_{14,\ell} + \delta_{20,\ell})(\delta_{2,\bar{\ell}} - \delta_{14,\bar{\ell}} + \delta_{20,\bar{\ell}}) \\ &- \frac{1}{3}(\delta_{5,\ell} - \delta_{11,\ell} + \delta_{23,\ell})(\delta_{5,\bar{\ell}} - \delta_{11,\bar{\ell}} + \delta_{23,\bar{\ell}}) \ , \end{aligned} \tag{28}$$

which is beyond the usual ADE classification. This suggests that the left-moving sector contains a new CFT other than the parafermion. A detailed study of the Hilbert space of the new CFT can be found in the original paper [1].

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