Homework 6 (Due at class on April 23)

1 Partition functions

1.1 S^1 partition function of bosons

Consider two harmonic oscillators on S^1 (time direction is compactified and its period is β) with the Hamiltonian

$$H = a^\dagger a + b^\dagger b \ . \tag{1.1}$$

Their commutation relations are

$$[a, a^\dagger] = 1 \ , \quad [b, b^\dagger] = 2 \ , \tag{1.2}$$

otherwise zero. Derive the partition function of this theory:

$$Z = \mathrm{tr}\left[e^{-\beta H}\right] \tag{1.3}$$

1.2 S^1 partition function of fermions

Consider two fermions on S^1 with Hamiltonian

$$H = d^\dagger d + 2 f^\dagger f \ . \tag{1.4}$$

Non-trivial anti-commutation relations are

$$\{d, d^\dagger\} = 1 \ , \quad \{f, f^\dagger\} = 1 \ . \tag{1.5}$$

Derive the partition function of this theory with a periodic boundary condition

$$Z = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] \ , \tag{1.6}$$

as well as an anti-periodic boundary condition

$$Z = \mathrm{tr}\left[e^{-\beta H}\right] \ . \tag{1.7}$$

Note that

$$\{(-1)^F, d\} = 0, \quad \{(-1)^F, d^\dagger\} = 0, \quad \{(-1)^F, f\} = 0, \quad \{(-1)^F, f^\dagger\} = 0 \ , \tag{1.8}$$

and assume

$$(-1)^F |0\rangle = -|0\rangle \ . \tag{1.9}$$

1.3 Torus partition function of X

Derive the torus partition function of matter sector of the bosonic string $\langle 1 \rangle_X$.

1.4 Torus partition function of ghosts

Derive the torus partition function of holomorphic ghost sector of the bosonic string

$$\mathrm{tr}\left[(-1)^F b_0 c_0 q^{L_0-\frac{c_{gh}}{24}}\right] \ , \tag{1.10}$$

where $c_{gh} = -26$, and

$$L_0 = \left(\sum_{n\in\mathbb{Z}} n :b_{-n}c_n:\right) - 1 \ . \tag{1.11}$$

The non-trivial anti-commutation relations are

$$\{c_m, b_n\} = \delta_{m+n,0} \ . \tag{1.12}$$

Note that c_m ($m \le 0$) and b_n ($n < 0$) are creation op, and c_m ($m > 0$) and b_n ($n \ge 0$) are anihilation op.

1.5 Torus partition function of the bosonic string

- Derive the number of the second excited states of the bosonic string from an expansion of $|\eta(\tau)|^{-48}$.

- The modular transformations of the τ-function are given by

$$T: \quad \eta(\tau+1) = e^{i\pi/12}\eta(\tau) \ , \tag{1.13}$$
$$S: \quad \eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau) \ . \tag{1.14}$$

  Show the torus partition function of the bosonic string is modular invariant.

2 Wold-sheet supersymmetry

2.1 Supersymmetric transformation

Show that the action

$$S^{\mathrm{m}} = \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X\bar{\partial}X + \psi^\mu\bar{\partial}\psi_\mu + \bar{\psi}^\mu\partial\bar{\psi}_\mu\right) \tag{2.1}$$

is invariant under the supersymmetric transformation

$$\delta_{\epsilon,\bar{\epsilon}}X^\mu = -\sqrt{\frac{\alpha'}{2}}\left(\epsilon\psi^\mu + \bar{\epsilon}\bar{\psi}^\mu\right) \ , \qquad \delta_\epsilon\psi^\mu = \sqrt{\frac{2}{\alpha'}}\,\epsilon\partial X^\mu \ , \qquad \delta_{\bar{\epsilon}}\bar{\psi}^\mu = \sqrt{\frac{2}{\alpha'}}\,\bar{\epsilon}\bar{\partial}X^\mu \ .$$

Also, show that the combination of supersymmetric transformations provides derivatives

$$[\delta_{\epsilon_1,\bar{\epsilon}_1}, \delta_{\epsilon_2,\bar{\epsilon}_2}]\mathcal{O} = 2\epsilon_1\epsilon_2\partial\mathcal{O} + 2\bar{\epsilon}_1\bar{\epsilon}_2\bar{\partial}\mathcal{O} \ .$$

2.2 Superspace formalism

This theory can be formulated in terms of superspace $(z, \overline{z}, \theta, \overline{\theta})$ where $(\theta, \overline{\theta})$ are anti-commuting Grassmann coordinates. We can introduce the superfield

$$Y^\mu(z, \overline{z}, \theta, \overline{\theta}) = X^\mu(z, \overline{z}) + i\theta\psi^\mu(z, \overline{z}) + i\overline{\theta\psi}^\mu(z, \overline{z}) + \frac{1}{2}\overline{\theta}\theta F^\mu(z, \overline{z}) \ ,$$

as well as the superderivative

$$D = \frac{\partial}{\partial\theta} + \theta\partial_z \ , \quad \overline{D} = \frac{\partial}{\partial\overline{\theta}} + \overline{\theta}\partial_{\overline{z}} \ .$$

The field F^μ is called an **auxiliary field**. Then, show that the action (2.1) at $\alpha' = 2$ is equivalent to

$$S = \frac{1}{4\pi}\int d^2z d^2\theta \ \overline{D}Y^\mu DY_\mu \ ,$$

where the superspace integral is defined as $d^2\theta = d\theta d\overline{\theta}$, and

$$\int d\theta d\overline{\theta} \ \overline{\theta}\theta = 1 \ .$$

2.3 Supersymmetric ghost

Let us define the ghost superfields as

$$B = \beta + \theta b \ , \qquad C = c + \theta\gamma \ .$$

Show that the supersymmetric ghost action can be written as

$$S^{\text{gh}} = \frac{1}{2\pi}\int d^2z d^2\theta \ B\overline{D}C \ .$$

3 $\mathcal{N} = 1$ superconformal algebra

In general, a 2d superconformal field theory has the following OPEs for the stress-energy tensor $T(z)$ and the supercurrent $G(z)$

$$\begin{aligned}
T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \\
T(z)G(w) &\sim \frac{3/2}{(z-w)^2}G(w) + \frac{1}{z-w}\partial_w G(w) \\
G(z)G(w) &\sim \frac{2c/3}{(z-w)^3} + \frac{2}{z-w}T(w) \ .
\end{aligned} \tag{3.1}$$

As in the lecture, we carry out the mode expansion

$$T_B(z) = \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}} \ , \qquad G(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}}$$

where $\nu = 0$ and $\nu = \frac{1}{2}$ correspond to Ramond sector and Neveu-Schwarz sector, respectively. From the OPEs (3.1), derive $\mathcal{N} = 1$ superconformal algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}\,m(m^2-1)\delta_{m+n,0} \\
[L_m, G_r] &= \left(\frac{1}{2}\,m - r\right) G_{r+m} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}(r^2-\nu^2)\delta_{r+s,0}\ .
\end{aligned}$$