Homework 11: Due at class on May 30

1 Electromagnetic duality

1.1 Differential form

An n-form field is defined by

$$A_n = \frac{1}{n!} A_{\mu_1,\mu_2,\cdots,\mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} \ . \tag{1.1}$$

Hodge dual in a D-dimensional curved space is defined by

$$*A_n = \frac{\sqrt{|G|}}{n!(D-n)!} \epsilon_{\mu_1,\cdots,\mu_{D-n}}{}^{\mu_{D-n+1},\cdots,\mu_D} A_{\mu_{D-n+1},\cdots,\mu_D} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{D-n}} \ , \tag{1.2}$$

where G is the determinant of a metric G_{MN}, and $\epsilon_{\mu_1,\cdots,\mu_D}$ is the totally anti-symmetric tensor and normalized as $\epsilon_{0,1,\cdots,D-1} = 1$.

- Confirm that

$$\frac{1}{2g^2} \int F_2{}^2 \equiv \frac{1}{2g^2} \int F_2 \wedge *F_2 = \frac{1}{4g^2} \int \sqrt{|G|} d^D x \, F_{\mu\nu} F^{\mu\nu} \ . \tag{1.3}$$

  (For example, compare the coefficients of $F_{01}F^{01}$ term in both side. If it is still difficult consider $D = 4$.)

Comment: this is the convention used in the lecture for kinetic terms of anti-symmetric tensors in SUGRA.

1.2 Electromagnetic duality

Let us consider a following action

$$S = -\frac{1}{2g^2} \int F_{n+1} \wedge *F_{n+1} - \int f_{D-n-1} \wedge (F_{n+1} - dA_n) \ , \tag{1.4}$$

where F_{n+1} and A_n are independent each other.

- Consider an equation of motion for f_{D-n-1} and derive the solution for the E.O.M. Then, what is the physical meaning of F_{n+1} and S with the solution ?
- Consider an E.O.M. for F_{n+1} and derive the solution. Rewrite the action S in terms of f_{D-n-1} using the solution.
- Consider an equation of motion for A_n with the action $S(f_{D-n-1})$ and derive the solution for the E.O.M. Then, what is the physical meaning of f_{D-n-1} and $S(f_{D-n-1})$ with the solution ?
- Define $\widetilde{S} = (\text{sign})S$, where you should properly choose the (sign) so that the kinetic term has the usual sign. If we write $\widetilde{S} \equiv -\frac{1}{2\widetilde{g}^2} \int \widetilde{F}_{D-n-1} \wedge *\widetilde{F}_{D-n-1}$, what is the value of $\widetilde{g}$ in terms of g ?

(Again, if you are confused with the differential form convention get back to normal convention and consider $D = 4$.)

2 Dirac monopole

2.1 Wu-Yang monopole

Let us consider so called Dirac monopole, which is a field configuration of $\mathbf{B}$ that satisfies

$$\nabla \cdot \mathbf{B} = q_m \delta^3(\mathbf{r}) \, , \tag{2.1}$$

where $\delta^3(\mathbf{r})$ is a three-dimensional delta function and $\mathbf{r}$ is a three-dimensional space point vector.

- Derive a solution of $\mathbf{B}$.
- Explain that $\mathbf{B}$ cannot be expressed by space components of a gauge field $\mathbf{A}$ globally.

Consider the polar coordinate (r, θ, ϕ) and a following gauge field

$$\mathbf{A}^N = \frac{q_m(1-\cos\theta)}{4\pi r \sin\theta}\mathbf{e}_\phi, \tag{2.2}$$

$$\mathbf{A}^S = -\frac{q_m(1+\cos\theta)}{4\pi r \sin\theta}\mathbf{e}_\phi, \tag{2.3}$$

where $\mathbf{A}^N$ is defined in a region $r \neq 0$ and $\theta \neq \pi$, and $\mathbf{A}^S$ is defined in a region $r \neq 0$ and $\theta \neq 0$. (Comments: The singular lines are called Dirac string. Those solutions are called Wu-Yang monopole. The point is that a gauge field A_μ is a section of fiber bundle and is not necessarily defined globally.)

- Show that $\mathbf{A}^N - \mathbf{A}^S$ is a gauge transformation in the region $r \neq 0$ and $\theta \neq 0, \pi$.
- Show that both $\mathbf{A}^N$ and $\mathbf{A}^S$ lead the $\mathbf{B}$ derived above in the region where they are defined.
- Show that magnetic flux Φ from the monopole is q_m, using $\mathbf{A}^N$ and $\mathbf{A}^S$.

The statements so far are based on vector analysis. More properly, a gauge field is a 1-form A_1 and magnetic flux density is a 2-form B_2. Let us rewrite the statements in terms of differential forms.

- Explain that Eq. (2.1) can be expressed as $dF_2 = q_m \delta_3(\mathbf{r})$, where $\delta_3(\mathbf{r})$ is a 3-form delta function: $\delta_3(\mathbf{r}) = \delta^3(\mathbf{r}) d^3\mathbf{r}$ $(= \delta(x)\delta(y)\delta(z) dx \wedge dy \wedge dz)$.
- Explain the Wu-Yang monopole can be written by

$$A_N = \frac{q_m}{4\pi}(1-\cos\theta)d\phi \, , \qquad A_S = -\frac{q_m}{4\pi}(1+\cos\theta)d\phi \, . \tag{2.4}$$

- Show that $A_N - A_S$ is a gauge transformation.
- Show that magnetic flux Φ from the monopole is q_m, using A_N and A_S in terms of differential forms.

2.2 Dirac quantization condition

The previous argument in differential forms can be easily generalized to higher dimension. Let us consider a $(p+1)$-form gauge field C_{p+1}, which satisfies following equation of motion.

$$\frac{1}{2\kappa^2} dG_{p+2} = q_m \delta_{p+3}(M_{D-p-3}) \,, \tag{2.5}$$

where G_{p+2} is a field strength of C_{p+1}, $\delta_{p+3}(M_{D-p-3})$ is a $(p+3)$-form delta function, and M_{D-p-3} is a world-sheet of an object that is magnetically coupled to C_{p+1} (Consider $D=4$ and $p=0$, which is the monopole case).

- (Optional) Explain the origin of $2\kappa^2$.
- Show that G_{p+2} integrated over a sphere enclosing the magnetic object is $2\kappa^2 q_m$.

Let us consider an object that is electrically coupled to C_{p+1}, whose coupling is

$$S_E = q_e \int_{E_{p+1}} C_{p+1} = q_e \int_{\widehat{E}_{p+2}} G_{p+2} \equiv S_E(\widehat{E}_{p+2}) \,, \tag{2.6}$$

where E_{p+1} is a world-sheet that starts from $t=-\infty$ and ends at $t=\infty$ or starts at $\mathbf{x}_0$ and ends at $\mathbf{x}_0$ (closed path), and $\widehat{E}_{p+2}$ is a manifold whose boundary is E_{p+1}.

- Since a choice of $\widehat{E}_{p+2}$ is arbitrary one can consider a deformation of $\delta\widehat{E}_{p+2} = \widehat{E}^N_{p+2} - \widehat{E}^S_{p+2}$, and then, $e^{iS_E(\delta\widehat{E}_{p+2})} = 1$. Show that this leads Dirac quantization condition.

3 $SL(2,\mathbb{R})$ invariance of type IIB SUGRA

- Show that the $SL(2,\mathbb{R})$ invariance of the type IIB SUGRA.

4 Kaluza-Klein theory

Let us consider a $D+1$-dimensional real scalar Kaluza-Klein theory:

$$\mathcal{L}^{(D+1)} = -\frac{1}{2} G^{MN} \partial_M \phi \partial_N \phi \,. \tag{4.1}$$

The metric is given by

$$ds^2 = G_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma} \left(dy + A_\mu dx^\mu \right)^2 \,, \tag{4.2}$$

where M, N run from 0 to D, μ, ν run from 0 to $D-1$, and $y \equiv x^D$, which is compactified to S^1 with radius r.

- Rewrite the Lagrangian in terms of μ, ν, and y, rather than M, N.

- Calculate the determinant of G_{MN} (i.e. $G \equiv \det G_{MN}$) and express it in terms of $g \equiv \det g_{\mu\nu}$.
- Assume that the fields $(g_{\mu\nu}, A_\mu, \sigma)$ are independent of y, except ϕ, which can be expanded as $\phi(\mathbf{x}, y) = \sum_{n\in\mathbb{Z}} \phi_n(\mathbf{x}) e^{iny/r}$, $(\phi_n^*(\mathbf{x}) = \phi_{-n}(\mathbf{x}))$. Write down the effective Lagrangian of D-dimensional theory:

$$\sqrt{-g}\mathcal{L}^{(D)} = \int dy \sqrt{-G}\mathcal{L}^{(D+1)} \,. \tag{4.3}$$

- Derive the masses of the fields ϕ_n, and their coupling constants to the KK-gauge field A_μ.

Let us consider a gravi-dilaton KK-theory:

$$S_{EH}^{(D+1)} \simeq \frac{1}{2\kappa^2} \int d^{D+1}x \sqrt{-G} e^{-2\Phi} \Big(R + \cdots \Big) \,, \tag{4.4}$$

which, with the same metric Eq. (4.2), reduces to

$$S_{EH}^{(D)} \simeq \frac{2\pi L}{2\kappa^2} \int d^D x \sqrt{-g} e^{-2\Phi} \Big(R + \cdots \Big) \,, \tag{4.5}$$

where we defined $L = re^{-\sigma}$, and $\cdots$ includes the dilaton kinetic term and other terms that are not important here.

- Define an effective coupling $\frac{1}{2\kappa_{\text{eff}}^2} = \frac{2\pi L}{2\kappa^2} e^{-2\langle\Phi\rangle}$, and compare it with that of T-dual theory. $\frac{1}{2\kappa_{\text{eff}}^2}$ should be the same after the T-dual. Derive the value of $\langle\widetilde{\Phi}\rangle$ in terms of $\langle\Phi\rangle$, where $\langle\widetilde{\Phi}\rangle$ is a dilaton vev of the T-dual theory.