Homework 1: Due at class on Mar 12

1 Canonical quantization of free scalar

Using the mode expansions in the lecture note, derive from the canonical commutation relation for X^μ and Π_μ

$$
\begin{aligned}
&[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i\delta(\sigma-\sigma')\,\delta^\mu{}_\nu \quad , \\
&[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [\Pi_\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = 0 \quad .
\end{aligned}
$$

commutation relations for the Fourier modes x^μ, p^μ, α^μ_n and $\widetilde{\alpha}^\mu_n$

$$
[x^\mu, p_\nu] = i\delta^\mu{}_\nu \quad \text{and} \quad [\alpha^\mu_n, \alpha^\nu_m] = [\widetilde{\alpha}^\mu_n, \widetilde{\alpha}^\nu_m] = n\,\eta^{\mu\nu}\delta_{n+m,0} \ ,
$$

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Problem 2.3 in Becker-Becker-Schwarz.

3 Open string spectra

An open string has boundaries so that one needs to impose a boundary condition. There are two boundary conditions one can impose:

- **Neumann boundary condition**: $\partial_\sigma X^\mu = 0 \quad$ at $\sigma = 0, \pi$
- **Dirichlet boundary condition**: $X^\mu = c^\mu$ (constant) at $\sigma = 0, \pi$

Like the close string, we take the mode expansion for the open string $X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$ by

$$
\begin{aligned}
X^\mu_L(\sigma^+) &= \frac{1}{2}x^\mu + \alpha' p^\mu\,\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\widetilde{\alpha}^\mu_n\,e^{-in\sigma^+} \ , \\
X^\mu_R(\sigma^-) &= \frac{1}{2}x^\mu + \alpha' p^\mu\,\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\,e^{-in\sigma^-} \ .
\end{aligned} \tag{1}
$$

Note that the second term differs from the closed string by factor of 2. Show that the boundary conditions impose the following requirements

- Neumann boundary condition requires $\alpha^a_n = \widetilde{\alpha}^a_n$.
- Dirichlet boundary condition requires $x^I = c^I, \quad p^I = 0, \quad \alpha^I_n = -\widetilde{\alpha}^I_n$.

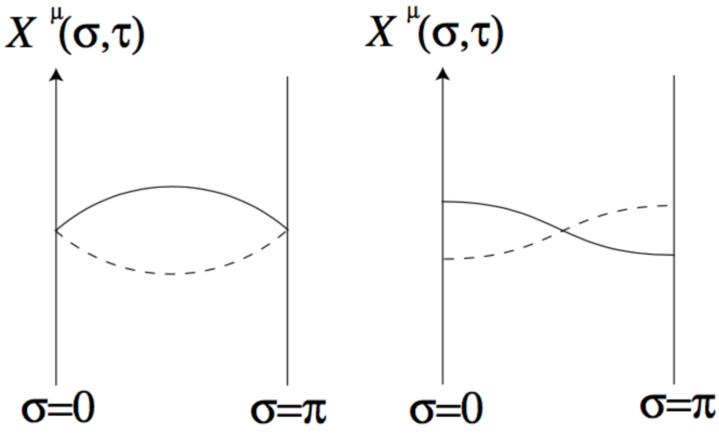

Figure 1: Dirichlet (left) and Neumann (right) boundary conditions

Actually, this is an essence of the previous probelm.

Now let us study open string mass spectrum in the quantum theory. In the case of open strings, we can define the momentum $\alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu$. Show that the light-cone gauge quantization for (1) gives

$$2\alpha_n^- = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{D-2}\alpha_{n-m}^i\alpha_m^i \ .$$

Check that $n=0$ can be read off

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^ip^i = \frac{1}{\alpha'}\left(\sum_{n>0}\sum_{i=1}^{D-2}\alpha_{-n}^i\alpha_n^i + \frac{D-2}{2}\sum_{n>0} n\right) \ ,$$

and explain the reason why there is the difference of factor 4 from the closed string. Again, in $D=26$, the open string mass spectrum becomes

$$M^2 = \frac{1}{\alpha'}(N-1) \ .$$

so that there is the tachyon for $N=0$. The massless states

$$\alpha_{-1}^i|0;k\rangle \qquad i=1,\ldots,D-2$$

for $N=1$ correspond to a vector boson.