

# Homework 5: Due at class on April 16

## 1 Ghost number anomaly

### 1.1 Ghost number current

Derive the conserved current for the transformation

$$\delta_g c = \epsilon_g c, \quad \delta_g b = -\epsilon_g b. \quad (1)$$

Holomorphic part of the ghost action is given by

$$S_{gh} = \frac{1}{2\pi} \int d^2 z b \bar{\partial} c. \quad (2)$$

(You can forget about the anti-holomorphic part for the problem.)

### 1.2 OPE of EM tensor and the current

Calculate the OPE between ghost EM tensor and the current derived above. The EM tensor is given as follows.

$$T(z) = - : (2b\partial c + \partial b c)(z) :. \quad (3)$$

Furthermore, write down an infinitesimal conformal transformation (with parameter  $\epsilon(z)$ ) of the ghost number current from the OPE result. (Only holomorphic part is enough.)

### 1.3 Ghost number anomaly from curved WS

Use the assumption  $\nabla^a j_a = \kappa R^{(2)}$  derive the current  $j_z = -4\kappa\partial\omega - j(z)$ . (You can assume that  $j_{\bar{z}} = 0$ .) Metric is given by

$$ds^2 = e^{2\omega} dz d\bar{z}. \quad (4)$$

Conformal transformation laws for  $j_z$  and  $\omega$  are given as follows.

$$\tilde{j}_z = \left( \frac{\partial \tilde{z}}{\partial z} \right)^{-1} j_z, \quad (5)$$

$$\tilde{\omega} = \omega - \frac{1}{2} \log \left| \frac{\partial \tilde{z}}{\partial z} \right|^2, \quad (6)$$

where  $\tilde{z} = z - \epsilon(z)$ . Using the transformations, derive the infinitesimal transformation for  $j(z) = -j_z - 4\kappa\partial\omega$ , and confirm  $\kappa = \frac{3}{4}$  by comparing the infinitesimal transformation derived here and the one derived from the OPE calculation.

## 2 Tree amplitude

There are a few approach for derivation of the string amplitude. One is path integral method we discussed in the lecture, Here we use holomorphicity and operator method to derive the string amplitude.

### 2.1 Conformal Killing Vectors

Show that

$$(P \cdot \epsilon)_{ab} = 0, \quad \nabla^a \theta_{ab} = 0, \quad (7)$$

reduces to

$$\partial \bar{\epsilon} = \bar{\partial} \epsilon = 0, \quad \partial \bar{\theta} = \bar{\partial} \theta = 0, \quad (8)$$

for conformal gauge  $ds^2 = e^{2\omega} dz d\bar{z}$ , where  $\epsilon = \epsilon^z$ ,  $\theta = \theta_{zz}$ , and similar for their anti-holomorphic part. Note that the position of the index is determined so because  $\epsilon^a$  is a vector and  $\theta_{ab}$  is a 2-form.

Let us consider sphere case. On sphere we can consider 2 patches, on which the coordinates are  $z$  and  $u$ , respectively, and the metrics are flat. Their transition is defined as

$$u = \frac{1}{z}. \quad (9)$$

Derive  $\epsilon$  and  $\theta$  by requiring that they are globally defined holomorphic vector and holomorphic 2-form, respectively.

### 2.2 Ghost sector

Let us consider the ghost number current  $j = :cb:$  on sphere. Finite conformal transformation of  $j$  is given by

$$j(z) = \left( \frac{\partial \tilde{z}}{\partial z} \right) \tilde{j}(\tilde{z}) - \frac{3}{2} \partial_z \log \left( \frac{\partial \tilde{z}}{\partial z} \right). \quad (10)$$

Show that ghost number operator

$$\int_C \frac{dz}{2\pi i} j(z) \quad (11)$$

should be equal to 3 by considering the other patch  $\tilde{z} = u$ .

Derive the expression for  $\langle c(z_1)c(z_2)c(z_3) \rangle$  on sphere up to over all constant, from the features:  $c(z)$  is fermionic so  $\lim_{z \rightarrow w} c(z)c(w) = 0$ , and  $c(z)$  has weight  $-1$ . (forget about the anomaly for the anti-holomorphic part otherwise this is zero).

## 2.3 Matter sector

Here we use operator method to derive tachyon 4pt expectation value:

$$\langle :e^{ik_1 \cdot X(z_1)} :: e^{ik_2 \cdot X(z_2)} :: e^{ik_3 \cdot X(z_3)} :: e^{ik_4 \cdot X(z_4)} : \rangle . \quad (12)$$

Set  $(z_1, z_2, z_3)$  to  $(0, 1, \infty)$  and then the expectation value simplifies to

$$\begin{aligned} & \langle :e^{ik_3 \cdot X(\infty)} : T [ :e^{ik_4 \cdot X(z_4)} :: e^{ik_2 \cdot X(1)} : ] :e^{ik_1 \cdot X(0)} : \rangle \\ &= \langle 0, k_3 | T [ :e^{ik_4 \cdot X(z_4)} :: e^{ik_2 \cdot X(1)} : ] | 0, k_1 \rangle , \end{aligned} \quad (13)$$

where  $T[\ ]$  is a radial ordering, namely,

$$T [ :e^{ik_4 \cdot X(z_4)} :: e^{ik_2 \cdot X(1)} : ] = :e^{ik_4 \cdot X(z_4)} :: e^{ik_2 \cdot X(1)} : \quad \text{if } |z_4| > 1 . \quad (14)$$

Note that  $::$  can be understood as the normal ordering of the operators, namely,

$$:e^{ik \cdot X(z)} : = e^{ik \cdot X_+(z)} e^{ik \cdot X_-(z)} , \quad (15)$$

where

$$X_+^\mu(z) = x^\mu - i\sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^\mu z^n + \tilde{\alpha}_{-n}^\mu \bar{z}^n) , \quad (16)$$

$$X_-^\mu(z) = -i\frac{\alpha'}{2} p^\mu \log |z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_n^\mu z^{-n} + \tilde{\alpha}_n^\mu \bar{z}^{-n}) . \quad (17)$$

Assume  $|z_4| > 1$  and compute

$$\langle 0, k_3 | :e^{ik_4 \cdot X(z_4)} :: e^{ik_2 \cdot X(1)} : | 0, k_1 \rangle . \quad (18)$$

Guess what is the result for  $|z_4| < 1$  case. Campbell-Baker-Hausdorff formula is useful.

$$e^A e^B = e^B e^A e^{[A, B]} . \quad (19)$$

## 2.4 Another info from the amplitude

Show that residue of the Shapiro-Virasoro amplitude at  $s$ -channel pole can be written as a polynomial of  $t - u$ . What is the relation between the order of the polynomial and the maximum spin of the  $s$ -channel pole state? What is the slope of the spin versus mass-squared plot?