Homework 10 (Due at class on May 23)

1 Fermionization

1.1

At the special radius $R = \sqrt{\frac{\alpha'}{2}}$ of the circle compactification, one can redefine a bosonic field $H(z) := \sqrt{\frac{2}{\alpha'}}X(z)$ so that the OPE are given by

$$H(z)H(0) \sim -\ln z \,.$$

Let us also consider two Majorana-Weyl fermions ψ^1, ψ^2 with OPE

$$\psi^i(z)\psi^j(0) \sim \frac{\delta^{ij}}{z} \,.$$

We can define the complex fermion

$$\psi(z) = 2^{-1/2}(\psi^1(z) + i\psi^2(z)) \,, \qquad \overline{\psi}(z) = 2^{-1/2}(\psi^1(z) - i\psi^2(z)) \,.$$

Show the equivalence of operators in boson and fermion

$$: e^{iH} :\cong \psi \,, \quad : e^{-iH} :\cong \overline{\psi} \,, \quad i\partial H \cong \psi\overline{\psi} \,, \quad T_H \cong T_\psi \,,$$

by calculating the OPEs of operators in both theories and comparing.

1.2

At the special radius $R = \sqrt{\frac{\alpha'}{2}}$, show that the torus partition function can be written as

$$Z^{25} = \frac{1}{2|\eta(q)|^2}\left[\Big|\sum_n q^{n^2/2}\Big|^2 + \Big|\sum_n (-1)^n q^{n^2/2}\Big|^2 + \Big|\sum_n q^{\frac{1}{2}(n+1/2)^2}\Big|^2\right] \,.$$

Verify that this is the torus partition function of a free complex fermion with all the possible boundary conditions (AA, AP, PA, PP). This is called the **diagonal modular invariant** partition function.

2 Toroidal compactifications

Let us consider the toroidal compactification in the presence of B-field

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma(\delta_{IJ}\eta^{\alpha\beta} + B_{IJ}\epsilon^{\alpha\beta})\partial_\alpha X^I \partial_\beta X^J \,,$$

where $\epsilon^{01} = -1$ and $\eta^{\alpha\beta} = \mathrm{diag}(-1, 1)$. The bosonic field is quantized as in the lecture note

$$X^I_R(z) = x^I - i\sqrt{\frac{\alpha'}{2}} p^I_R \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha^I_m}{m z^m},$$
$$\overline{X}^I_L(\bar{z}) = \bar{x}^I - i\sqrt{\frac{\alpha'}{2}} p^I_L \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\bar{\alpha}^I_m}{m \bar{z}^m}.$$

where the bosonic field $X^I(z,\bar{z}) = X^I_R(z) + \overline{X}^I_L(\bar{z})$ is periodically identified on the lattice $W^I \in \Lambda$

$$X^I(\sigma + 2\pi, \tau) = X^I(\sigma, \tau) + 2\pi W^I \, . \tag{2.1}$$

2.1

Show that the center of mass momentum

$$\pi_I := \int_0^{2\pi} d\sigma \, \Pi_I \, , \qquad \Pi_I = \frac{\delta S}{\delta \dot{X}^I} \, ,$$

together with (2.1) imply that

$$(p_I)_{L,R} := \delta_{IJ} p^J_{L,R} = \sqrt{\frac{\alpha'}{2}} \Big(\pi_I \pm \frac{1}{\alpha'} (\delta_{IJ} \mp B_{IJ}) W^J \Big) \, .$$

Here left-modes get the upper sign and right-modes get the lower sign whenever you find the notation $\pm$ and $\mp$.

2.2

It is π_I that generates translations and it must therefore lie on the lattice Λ^*_D, $\pi_I = e^{*i}_I m_i$ for $m_i \in \mathbb{Z}$. Writing $\mathbf{g} = g_{ij} = e^I_i \delta_{IJ} e^J_j$, $\mathbf{b} = b_{ij} = e^I_i B_{IJ} e^J_j$, show that the mass formula and the level-matching condition for closed strings are modified to (a matrix notation is employed in the expression below)

$$\begin{aligned} \alpha' M^2 =& \alpha' \mathbf{m}^T \mathbf{g}^{-1} \mathbf{m} + \frac{1}{\alpha'} \mathbf{n}^T (\mathbf{g} - \mathbf{b}\mathbf{g}^{-1}\mathbf{b}) \mathbf{n} + 2\mathbf{n}^T \mathbf{b} \mathbf{g}^{-1} \mathbf{m} + 2(N + \overline{N} - 2) \\ N - \overline{N} =& \boldsymbol{\pi} \cdot \mathbf{W} = \mathbf{m}^T \mathbf{n} \, . \end{aligned} \tag{2.2}$$

2.3

Use (2.2) to show that the spectrum is invariant under the map

$$\mathbf{m} \leftrightarrow \mathbf{n} \, , \quad \alpha' \mathbf{g}^{-1} \leftrightarrow \frac{1}{\alpha'} (\mathbf{g} - \mathbf{b}\mathbf{g}^{-1}\mathbf{b}) \, , \quad \mathbf{b}\mathbf{g}^{-1} \leftrightarrow -\mathbf{g}^{-1}\mathbf{b} \, .$$

The second and the third are indeed equivalent to

$$\frac{1}{\alpha'} (\mathbf{g} + \mathbf{b}) \leftrightarrow \alpha' (\mathbf{g} + \mathbf{b})^{-1} \, ,$$

and this is the T-duality in the presence of B-field.

2.4

A non-trivial instructive example is the 2-torus $T^2 = \mathbb{R}^2/\Lambda$ where $\mathbf{e}_1, \mathbf{e}_2$ are the two generators of Λ and its metric is $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. The torus has one Kähler modulus $\sqrt{\det g_{ij}}$ and one complex structure modulus

$$\tau = \frac{|\mathbf{e}_2|}{|\mathbf{e}_1|} e^{i\,\mathrm{angle}(\mathbf{e}_1,\mathbf{e}_2)} = \frac{g_{12} + i\sqrt{\det g_{ij}}}{g_{11}} = \tau_1 + i\tau_2\,.$$

In the presence of B-field, the Kähler modulus is complexified

$$\omega = \frac{1}{\alpha'}(B + i\sqrt{\det g_{ij}}) = \omega_1 + i\omega_2\,.$$

This leads to the expression

$$g_{ij} = \alpha' \frac{\omega_2}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}\,.$$

Using these expressions, show that the momentum contribution to the mass formula is given by

$$\mathbf{p}_L^2 = \frac{1}{2\omega_2\tau_2} |m_2 - \tau m_1 + \overline{\omega}(n^1 + \tau n^2)|^2$$
$$\mathbf{p}_R^2 = \frac{1}{2\omega_2\tau_2} |m_2 - \tau m_1 + \omega(n^1 + \tau n^2)|^2\,.$$

Show that, if we transform the momentum and winding numbers appropriately, the string spectrum is then invariant under the following two $\mathrm{SL}(2,\mathbb{Z})$ transformations

• T-dualities

$$\omega \to \frac{a\omega + b}{c\omega + d}$$

• Torus modular transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

• **mirror symmetry** (symmetry between complexified Kähler modulus and complex structure modulus)

$$\omega \leftrightarrow \tau$$

How do the momentum and winding numbers transform?