Qubits and Quantum Gates

Quantum Gates and Quantum Circuits (Binary Quantum Gates)

Up to now, you learned what is a Qubit. It is a 2 dimensional vector with two basis of and |0> and |1>

One can represent the states |0> and |1> with column vectors as shown below.

$|0>=\binom{1}{0}$, $|1>=\binom{0}{1}$

We will apply operators in the form of matrices to the vectors in the state space.

We represent a **superposition** of states **as the linear combination of computational bases.**

Using the **two computational basis vectors** in the case of **a single qubit**, two examples of **superpositions** of states are:

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

and

$$|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Now it is time to define what is a Quantum Gate.

A **Quantum Gate** is a **Matrix**.

A **matrix** is a rectangular **array of numbers** arranged in **rows** and **columns**. The array of numbers below is an example of a matrix.

$$\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} \quad \begin{bmatrix} 0&1\\1&0 \end{bmatrix} \quad \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}$$

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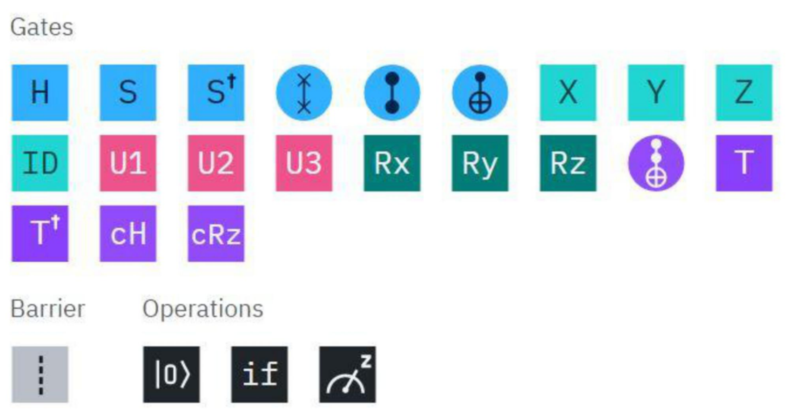
Quantum logic gates are represented by **unitary matrices**.



Pauli matrices

The **X-gate** is represented by the **Pauli-X matrix**:

To see the effect a gate on a qubit, we simply multiply the qubit's vector by the gate (matrix).

See here the effect of X-gate on a qubit.

$$X|0\rangle = \begin{bmatrix} 0&1\\1&0 \end{bmatrix}\begin{bmatrix} 1\\0 \end{bmatrix} = \begin{bmatrix} 0\\1 \end{bmatrix} = |1\rangle$$

The Y & Z-gates

$$Y = \begin{bmatrix} 0&-i\\i&0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1&0\\0&-1 \end{bmatrix}$$

The Hadamard Gate

The Hadamard gate (**H-gate**) is a fundamental quantum gate. It creates a superposition of |0> and |1>

$$H = \tfrac{1}{\sqrt{2}}\begin{bmatrix} 1&1\\1&-1 \end{bmatrix}$$

Let's see the effect of the **Hadamard** gate to state **|0>** :

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1&1\\1&-1 \end{pmatrix}\begin{pmatrix} 1\\0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+0\\1+0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1\\1 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Similarly, we apply the Hadamard gate to state |1> :

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1&1\\1&-1 \end{pmatrix}\begin{pmatrix} 0\\1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0+1\\0-1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1\\-1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

As one can see that the **H gate** takes a **computational basis** state and map it into a **superposition** of states.

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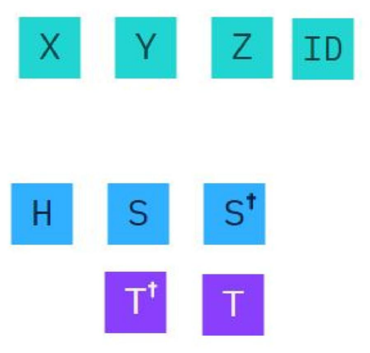
Basic **quantum gates and their matrix representations**

$$X = \begin{bmatrix} 0&1\\1&0 \end{bmatrix} \quad Y = \begin{bmatrix} 0&-i\\i&0 \end{bmatrix} \quad Z = \begin{bmatrix} 1&0\\0&-1 \end{bmatrix} \quad I = \begin{bmatrix} 1&0\\0&1 \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1&1\\1&-1 \end{bmatrix} \quad S = \begin{bmatrix} 1&0\\0&i \end{bmatrix} \quad T = \begin{bmatrix} 1&0\\0&\exp(i\pi/4) \end{bmatrix}$$



For the case of **two-qubits** system, we use the following **computational basis states**:

$$|00\rangle = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}$$

Controlled NOT gate

controlled NOT gate is a quantum logic gate and essential operator for quantum computing.

For this binary operator, we take the **first qubit** as the **control qubit** and the **second** as the **target qubit.**

$$CNOT := \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}$$

If the **control qubit** is in the state **|0>** then **do Nothing** to the **target qubit**.

If the **control qubit** is in the state **|1>** , then apply the **Not gate (X)** to the **target qubit**.

For example, apply **CNOT** to the state **|10>**, we get:

$$\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}\begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} = \begin{pmatrix} 0+0+0+0\\0+0+0+0\\0+0+0+0\\0+0+1+0 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} = |11\rangle$$

Table of Quantum Gates:

X	$\begin{pmatrix} 0&1\\1&0 \end{pmatrix}$	$\oplus$
Y	$\begin{pmatrix} 0&-i\\i&0 \end{pmatrix}$	$\boxed{Y}$
Z	$\begin{pmatrix} 1&0\\0&-1 \end{pmatrix}$	$\boxed{Z}$
H	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1&1\\1&-1 \end{pmatrix}$	$\boxed{H}$
$R_x(\theta)$	$\begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	$\boxed{R_x(\theta)}$
$R_y(\theta)$	$\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$	$\boxed{R_y(\theta)}$
R_φ	$\begin{pmatrix} 1&0\\0&e^{i\varphi} \end{pmatrix}$	$\boxed{R_\varphi}$

How the gates work is very important!

The **quantum** circuit is viewed as a sequence of **vertical** sections and **horizontal** levels

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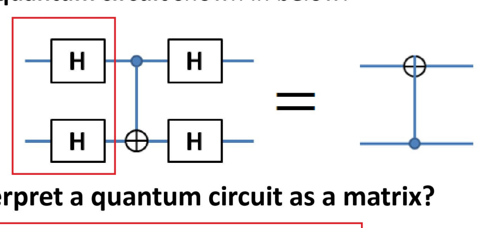
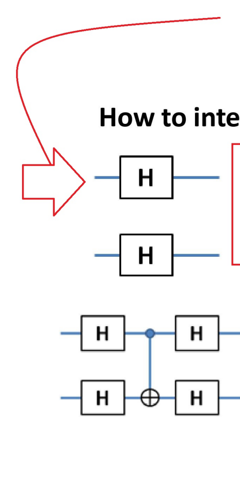
Consider the **quantum circuit** shown in below:



How to interpret a quantum circuit as a matrix?



$$\mathbf{H \otimes H} = \frac{1}{\sqrt{2}}\begin{bmatrix} H & H\\ H & -H \end{bmatrix}$$

Use Tensor Product of two Matrices to Calculate this

$$= \frac{1}{2}\begin{bmatrix} H & H\\ H & -H \end{bmatrix}\begin{bmatrix} I & 0\\ 0 & X \end{bmatrix}\begin{bmatrix} H & H\\ H & -H \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} H & HX\\ H & -HX \end{bmatrix}\begin{bmatrix} H & H\\ H & -H \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I+HXH & I-HXH\\ I-HXH & I+HXH \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} I+Z & I-Z\\ I-Z & I+Z \end{bmatrix} = \begin{bmatrix} 1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0 \end{bmatrix}$$

Exercise

Now based on the **explicit matrix forms** of **quantum gates** in the above table, prove the below identities.

$$HXH = Z$$
$$HZH = X$$
$$HYH = -Y$$
$$H^\dagger = H$$
$$X^2 = Y^2 = Z^2 = I$$
$$H = (X+Z)/\sqrt{2}$$
$$H^2 = I$$

Homework

Participants should send their assignments to the instructor and share and discuss their results.