The power and limitations of self-testing

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The notion of self-testing was first introduced by Mayers and Yao [7]. It is one way to certify that quantum devices perform according to their specification. By self-testing methods, we can derive a quantum mechanical description of the device from classical observations. More precisely, if Alice and Bob want to know the state and measurements of their quantum devices, they can play a nonlocal game and check the output probabilities. If the game self-tests those output probabilities, then Alice and Bob can be sure that certain states and measurements are present in their device. Therefore, it is an important question to ask which games are self-tests for certain output probabilities. See also [11] for a review on self-testing.

It was shown by Bell [2] that quantum mechanics cannot be fully described using local hidden-variable theories. He obtained the result by violating so called *Bell inequalities* using local measurements on entangled states. Our definition of self-testing will be given in the setting of *nonlocal games*, which were first introduced in [4] and can be used to obtain violations of Bell inequalities. A nonlocal game is played by two players, Alice and Bob, and a referee. The referee gives questions to Alice and Bob and they have to respond with an answer. They win according to a function on questions and answers, which is known to the players beforehand. Crucially, Alice and Bob are not allowed to communicate after they received their questions. They can agree on a strategy ahead of time though. In a *quantum strategy*, Alice and Bob additionally share an entangled state and are able to make local measurements. Finding a quantum strategy for a nonlocal game which exceeds the winning probability of any classical strategy especially yields a violation of a Bell inequality. We call quantum strategies with the highest winning probability in a nonlocal game *optimal* quantum strategies.

Self-testing now asks whether the state and the local measurements used in an optimal quantum strategy are unique, up to local isometries. This can be used in the following way. If we want to know whether certain states and measurements are present in a quantum device, we play the nonlocal game that self-tests those states and measurements. Achieving optimal winning probability in the nonlocal game then ensures that up to local isometries, the states and measurements are present. We will now give a more precise definition of self-testing.

Definition 1. Let $\mathcal{G}$ be a nonlocal game and $\tilde{S} = (|\tilde{\psi}\rangle \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B, \{\tilde{E}_{xa}\}_x, \{\tilde{E}_{yb}\}_y)$ an optimal quantum strategy. Then $\mathcal{G}$ is a *self-test* for $\tilde{S}$ if for any optimal quantum strategy $S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{E_{xa}\}_x, \{F_{yb}\}_y)$ there exist Hilbert spaces $\mathcal{H}_{A,aux}$ and $\mathcal{H}_{B,aux}$, a state $|aux\rangle \in \mathcal{H}_{A,aux} \otimes \mathcal{H}_{B,aux}$ and isometries $U_A : \mathcal{H}_A \to \tilde{\mathcal{H}}_A \otimes \mathcal{H}_{A,aux}$, $U_B : \mathcal{H}_B \to \tilde{\mathcal{H}}_B \otimes \mathcal{H}_{B,aux}$ such that with $U := U_A \otimes U_B$ it holds

$$U|\psi\rangle = |\tilde{\psi}\rangle \otimes |aux\rangle, \tag{1}$$

$$U(E_{xa} \otimes F_{yb})|\psi\rangle = [(\tilde{E}_{xa} \otimes \tilde{F}_{yb})|\tilde{\psi}\rangle] \otimes |aux\rangle. \tag{2}$$

If only Equation (1) holds for every optimal quantum strategy S, we say that the game $\mathcal{G}$ *self-tests* the state $|\tilde{\psi}\rangle$.

Note that in real experiments, one will obtain optimal winning probabilities in nonlocal games just up to some error. Thus, it makes sense to ask for a stronger form of self-testing, called *robust self-testing*. A nonlocal game is a robust self-test for an optimal quantum strategy, if it is a self-test and additionally, almost optimal quantum strategies consist of states and measurements that are close to the states and measurements in the self-tested optimal quantum strategy. Since the introduction of self-testing by Mayers and Yao [7], quite a few different settings and definitions have been explored.

There have been a large number of positive results of games that gives rises to self-tests for their optimal strategies. One of the most well known games is the CHSH game, for which a statement of self-testing was proven even before the term was coined [3, 8, 10, 12]. Another example of a game that self-tests is the magic square game [13]. Both the CHSH game [1] and the magic square game [13] are furthermore robust self-tests. The magic square game is of particular interest here, since it has a perfect quantum strategy but not a classical one.

Contributions. We will start with looking at how mixed states can be incorporated in the definition of self-testing. It is well known that you cannot self-test mixed canonical states [11, Section 3.5]. If we were to consider a mixed state as the reference state, then we have a purification of this state which can have the exact same game value. But there cannot exist local isometries that extract a mixed state from a pure state, implying the definition of self-testing would not hold.

The next step is to consider what happens if we modify the definition of self-testing such that we allow the arbitrary state to be mixed. It is straight forward to split these definitions with a qualifier, depending on whether we allow the arbitrary strategy to be mixed. This gives rise to the two different definitions of self-testing, pure self-testing and mixed self-testing. Among these, the definition seen in Definition 1 is equivalent to pure self-testing, while you can make small changes to the definition to make it equivalent to mixed self-testing.

From the way we defined pure and mixed self-testing, it is clear that every strategy that can be mixed self-tested can be pure self-tested, but the converse is not clear. This relationship between pure self-testing and mixed self-testing is exactly one of the questions we aim to answer. It is worth noting that many, or even most, known self-tests are proven using the pure self-testing definition. For certain applications, assuming that the state in strategy S in Def. 1 is pure is a limitation, as such statements do not characterize the form of strategies employing a mixed state. If instead one claims, that the goal is to characterize the purified strategies, then this might be problematic, for example, in cryptographic applications where the purifying system might be held by an adversary. So we are faced with the following question:

Question 2. ***Does pure self-testing imply mixed self-testing?***

We answer this question with the following theorem, which proves for most practical cases that this is indeed true.

Theorem 3. *Let $\mathcal{G}$ be a game that pure self-tests the strategy $\tilde{S} = (|\tilde{\psi}\rangle, \{\tilde{E}_{xa}\}_x, \{\tilde{E}_{yb}\}_y)$. If $|\tilde{\psi}\rangle$ has full Schmidt rank then $\mathcal{G}$ mixed self-tests $\tilde{S}$.*

This theorem gives a way in which one can simplify the process of proving a self-testing statement, since we for most practical purposes can assume the state of the arbitrary strategy to be pure without loss of generality. The result of this theorem is that pure self-testing and mixed self-testing is almost equivalent, with the only missing part being what happens if the canonical state is not of full Schmidt rank. We would like to note that it easy to come up with an example, where we can mixed self-test a strategy that does not have full Schmidt rank. The general idea is that you take a game that we know mixed self-tests a strategy, for example the CHSH game, and then you give Alice access to an additional 2-dimensional space. With this new space, we simply tensor the state of the reference strategy with a basis vector from the new space and let all operators act with identity on it. This new construction does not have full Schmidt rank, but we can extract it from the old reference strategy, so this new strategy is a mixed self-test that does not have full Schmidt rank.

Another assumption that is commonly made when proving self-tests is that the arbitrary strategy S employs only projective measurements (PVMs) rather than general POVMs. Again, depending on the application, this might be problematic or not. Just like in the case of pure (mixed) self-testing, we can use Def. 1 to come up with two variants of this definition that correspond to whether or not we characterize strategies S that employ POVMs. Then we are faced with the following question:

Question 4. *Is every PVM self-test also a POVM self-test?*

Towards answering this question, we exhibit the first example of a correlation, p, which can only be implemented using POVMs if we assume that the strategy employs a state of full Schmidt rank. In fact, this correlation p is a POVM self-test if we only consider strategies with full Schmidt rank. This result suggests that *in general* the answer to the above question might be negative. If this is the case, then the two commonly made assumptions (purity of the state & projective measurements) have a different footing. Specifically, purity can be assumed without loss of generality while PVMs cannot.

Recall that we have three forms of pure self-tests: robust self-tests, exact self-tests and state self-tests. By definition it holds

$$\mathcal{G} \text{ robustly self-tests } S \Rightarrow \mathcal{G} \text{ self-tests } S \Rightarrow \mathcal{G} \text{ self-tests } |\psi\rangle$$

It is a natural question to ask whether or not any of those forms of self-testing are equivalent. Combining the results of [5] and [6], one sees that the glued magic square game is not an exact self-test, but a state self-test. We thus ask the remaining question.

Question 5. ***Does exact self-testing imply robust self-testing?***

Curiously, all known exact self-tests are also robust. Therefore, one might *hope* that the answer to the above question is positive. It would indeed be very convenient just having to prove exact self-testing as most of the times the robustness is the difficult part. Unfortunately, we show that this is not the case by exhibiting the first example of a non-robust self-test.

Theorem 6. *There exists a nonlocal game $\mathcal{G}$ that exactly self-tests a quantum strategy S, but this self-test is not robust.*

For obtaining this example, we introduce the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game for nonlocal games $\mathcal{G}_1$ and $\mathcal{G}_2$. In this game, the players obtain pairs of questions, one from each game. They choose which question they answer and win if they answered a question from the same game and additionally win this game with their answers.

The idea of the construction of a non-robust self-test is the following. Let $\mathcal{G}_1$ be a game that has no perfect quantum strategy, but a sequence of strategies whose winning probabilities converge to one. Note that Slofstra constructed such a game in [9]. If we now let $\mathcal{G}_2$ be the magic square game, then the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game still self-tests the strategy in which Alice and Bob always choose game $\mathcal{G}_2$ and play the game with the strategy that is self-tested by $\mathcal{G}_2$. But this self-test is not robust, as there are almost optimal strategies of the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game coming from the almost optimal strategies of $\mathcal{G}_1$.

Question 7. ***Does every nonlocal game self-test some state?***

One can derive from the self-testing definition that a game with two optimal quantum strategies that use states with coprime Schmidt ranks will not self-test any state. We construct such a game one again using the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game. Here, $\mathcal{G}_1$ is the magic square game which has a perfect quantum strategy that uses a state with Schmidt rank 4. For $\mathcal{G}_2$ we choose the (G, t)-independent set game, where G is the orthogonality graph of a 3-dimensional weak Kochen-Specker set and some $t \in \mathbb{N}$. This game has a perfect quantum strategy using a state with Schmidt rank 3. One can now conclude that the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game has perfect quantum strategies that use states with coprime Schmidt ranks and thus does not self-test any state.

Theorem 8. *There exists a game that does not self-test any state.*

Impact. The results of this submission are of foundational importance to the field of self-testing. We examine what common assumptions can or cannot be made without loss of generality. Our results highlight the intricacies of the adopted self-testing definition and represent a step towards a rigorous theory of self-testing.

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Counterexamples in self-testing

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Abstract

In the recent years self-testing has grown into a rich and active area of study with applications ranging from practical verification of quantum devices to deep complexity theoretic results. Self-testing allows a classical verifier to deduce which quantum measurements and on what state are used, for example, by provers Alice and Bob in a nonlocal game. Hence, self-testing as well as its noise-tolerant cousin—robust self-testing—are desirable features for a nonlocal game to have.

Contrary to what one might expect, we have a rather incomplete understanding of if and how self-testing could fail to hold. In particular, could it be that every 2-party nonlocal game or Bell inequality with a quantum advantage certifies the presence of a specific quantum state? Also, is it the case that every self-testing result can be turned robust with enough ingeniuty and effort? We answer these questions in the negative by providing simple and fully explicit counterexamples. To this end, given two nonlocal games $\mathcal{G}_1$ and $\mathcal{G}_2$, we introduce the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game, in which the players get pairs of questions and choose which game they want to play. The players win if they choose the same game and win it with the answers they have given. Our counterexamples are based on this game.

1 Introduction

The notion of self-testing was first introduced by Mayers and Yao [20] and it has since developed into an active and rich area of study (see [30] for a review). One major motivation behind self-testing is that it can be used by a classical verifier to certify that untrusted quantum devices perform according to their specification. This is accomplished by deriving a quantum mechanical description of a quantum device merely from classical observations. More precisely, if Alice and Bob want to know the state and measurements of their quantum devices, they can play a nonlocal game and check the output probabilities. If this game is a self-test then after observing the desired output probabilities, Alice and Bob can conclude that their devices must be implementing certain measurements on a certain quantum state.

In addition to the early applications of self-testing to certification and device-independent protocols, in the recent years it has been a key ingredient for important results in quantum complexity theory [8, 21, 22, 12]. The most notable among these results is the recent breakthrough [12] establishing that MIP*=RE and resolving Connes' Embedding Problem which had resisted all attempts for over 50 years.

It was shown by Bell [2] that the predictions of quantum mechanics are not compatible with any local hidden-variable theory. He obtained this result by showing that quantum mechanics

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predict violations of what are now known as *Bell inequalities*. Our definition of self-testing will be given in the setting of *nonlocal games* [6] which are another way to explore Bell inequalities and their violations. A (2-player) nonlocal game is played by two collaborating players, Alice and Bob, and a referee. The referee gives questions to Alice and Bob and they each have to respond with an answer. The players win according to a function on questions and answers, which is known to the players beforehand. Crucially, Alice and Bob are not allowed to communicate after they have received their questions. They can, however, agree on a strategy ahead of time. In a *quantum strategy* Alice and Bob can share an entangled state and use local measurements to come up with their answers. Finding a quantum strategy for a nonlocal game which exceeds the winning probability of any classical strategy yields a violation of a Bell inequality. We call quantum strategies with the highest winning probability in a nonlocal game *optimal* quantum strategies.

Motivation and results. In a nutshell, *self-testing* says that any optimal quantum strategy for some game $\mathcal{G}$ is equal to a reference strategy, up to local isometries. There are different forms of self-testing. A game can self-test an optimal quantum strategy or just the shared state of this quantum strategy. Also, a game can *robustly* self-test a strategy in which case we in addition require that any near-optimal quantum strategy must be close to an optimal reference strategy. For a nonlocal game $\mathcal{G}$, an optimal quantum strategy S, and state $|\psi\rangle$ used in S, we have the following relation between the above forms of self-testing:

$$\mathcal{G} \text{ robustly self-tests } S \Rightarrow \mathcal{G} \text{ self-tests } S \Rightarrow \mathcal{G} \text{ self-tests } |\psi\rangle. \tag{1}$$

So we see that robust self-tests are the strongest while self-tests of states are the weakest form of self-testing. It is natural to ask for counterexamples showing that there are no equivalences above.

Unsurprisingly, it is often easier to prove that a certain game self-tests a reference strategy S but more care and effort is needed to show that this self-test is in fact robust. In most known cases, however, the same argument that is used to establish a self-testing result can also be turned robust by simply arguing that all the desired relations hold approximately rather than exactly. In fact, there are no known examples of self-tests that are not robust! Moreover, if a self-test is proven using a specific technique that leverages representation theory of finite groups then robustness of such self-tests follows from the Gowers-Hatami theorem [11, 33]. The idea here is that all representations of finite groups are stable in a certain precise mathematical sense. Hence, one might hope that the same holds true for self-tests and any (exact) self-test is necessarily robust. Regretfully, we show that this is not the case, *i.e.*, the converse of the first implication from (1) fails to hold:

Theorem A (Theorem 5.2 & Example 5.3)**.** *There exists a nonlocal game $\mathcal{G}$ that exactly self-tests a quantum strategy S, but this self-test is not robust.*

The second question we address in this article is whether every 2-party nonlocal game with quantum advantage self-tests some quantum state. The only known non-trivial set-ups that do not self-test any quantum state are known in the many party case [10]. We answer our second question in the negative:

Theorem B (Theorem 6.2 & Example 6.13)**.** *There exists a game $\mathcal{G}$ that does not self-test any state.*

Discussion and outlook. Another way to get different forms of self-testing is to vary the classical observations one has access to. In this article we have focused on self-testing where we assume that a quantum strategy achieves optimal or near-optimal winning probability in a nonlocal game. Another common form is where we derive self-testing from optimal or near-optimal Bell inequality violations. Since every nonlocal game can be cast as a Bell inequality, our counterexamples also hold in that setting. Finally, rather than self-testing from a single numerical value (*e.g.* Bell violation or winning probability), we can prove a self-testing result for quantum strategies that produce a certain quantum correlation (collection of probabilities). As we explain in more detail below, our counterexamples do not carry over to the setting of correlations. We summarize the known counterexamples to self-testing, including results of this article, in the table below:

	Non-robust self-test	Not a measurement self-test	Not a state self-test
Nonlocal games	Theorem A	[7], [27]	Theorem B
Bell inequalities		[13], [14]	[1], [9], [10]
Extreme correlations	?	[29], [31]	?

Table 1: Counterexamples for equivalence of different forms of self-testing in the three settings: nonlocal games, Bell inequalities, and extreme correlations. Note that every counterexample for nonlocal games also yields a counterexample for Bell inequalities.

We now comment on Table 1. We present the first example of a game that self-tests a quantum strategy, but this self-test is not robust (Theorem A) in this article. Since the correlations of the approximate optimal strategies of our game do not necessarily converge to the self-tested strategy, we do not get a non-robust self-test for correlations. We leave this as an open question. For obtaining an example of a game that does not self-test measurements, one needs to construct games with inequivalent optimal strategies. In [7] it was shown that the glued magic square game does not self-test measurements. It turns out that this game still self-tests a quantum state [16]. This gives an example of a game that self-tests a state but not the whole strategy. A similar result was obtained by Kaniewski [13] in the framework of Bell inequalities. An extreme correlation obtained by inequivalent strategies was first presented in [31] and studied further in [29]. In this article, we construct the first examples of two-party games that do not self-test any state (Theorem B). As shown in the table above, an example of a three-party Bell inequaltiy that does not self-test any state has been identified in [10]. Furthermore, Bell inequalities that are maximally violated by states in an entangled subspace are studied in [1] and [9], but all their examples use more than two parties. We do not know an extreme correlation that does not self-test any state.

Proof ideas. We now explain the key ideas behind our two main theorems. For obtaining the examples of non-robust self-tests and games that do not self-test any state, we introduce the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game for nonlocal games $\mathcal{G}_1$ and $\mathcal{G}_2$. In this game the players receive pairs of questions, one from each game. They choose which question they answer and win if they answered a question from the same game and additionally win this game with their answers.

For the non-robust self-test (Theorem A), we let $\mathcal{G}_1$ be a game that has no perfect quantum strategy, but a sequence of strategies whose winning probabilities converge to one. Note that Slofstra constructed such a game in [28]. We choose $\mathcal{G}_2$ to be a pseudo-telepathy game (*i.e.* a game with perfect quantum strategy, but no perfect classical strategy) that self-tests some strategy S_2. Then the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game self-tests a strategy in which Alice and Bob always choose game $\mathcal{G}_2$ and play according to S_2. This self-test is not robust, however, as there are near-optimal strategies for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game coming from the near-optimal strategies for $\mathcal{G}_1$.

To obtain nonlocal games that do not self-test any state (Theorem B), it suffices to construct a game with two optimal quantum strategies that use states with coprime Schmidt ranks (see Lemma 6.1). In our case, those will be perfect quantum strategies. Note that we can get perfect strategies for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game from perfect quantum strategies for $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, by Alice and Bob always choosing the same game and playing the perfect quantum strategy of this game. Thus, to get a game that does not self-test any state, it suffices to find two pseudo-telepathy games that have perfect strategies using states of coprime Schmidt ranks. Then the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game will not self-test any state.

The magic square game is a pseudo-telepathy game with a strategy that uses a state with Schmidt rank 4. Thus, we are left with finding a pseudo-telepathy game that has a perfect quantum strategy with a state of odd Schmidt rank. Our example of such a game is a (G, t)-independent set game, where G is the orthogonality graph of a 3-dimensional weak Kochen-Specker set and some $t \in \mathbb{N}$. The perfect quantum strategy uses a state of Schmidt rank 3.

Structure of the article. In Section 2 we collect background material on nonlocal games. We then define all forms of self-tests in Section 3 and provide some known results we will use later on. Furthermore, we show that the synchronous magic square game self-tests a quantum strategy. In Section 4 we introduce the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game and establish some of its properties in case when both $\mathcal{G}_1$ and $\mathcal{G}_2$ are pseudo-telepathy games. Using the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game, we construct a non-robust self-test in Section 5. Finally, we obtain games that do not self-test states in Section 6.

2 Preliminaries

We state some basic facts and notions we will use throughout this article. We write $[m] := \{1, \dots, m\}$. Each Hilbert space $\mathcal{H}$ considered in this article is **finite-dimensional** which means that we have $\mathcal{H} \cong \mathbb{C}^d$ for some $d \in \mathbb{N}$. We let $\| |\xi\rangle \| = (\langle \xi|\xi\rangle)^{\frac{1}{2}}$ for $|\xi\rangle \in \mathcal{H}$. An operator $X \in B(\mathcal{H})$ is *positive*, if $\langle \xi|X|\xi\rangle \geq 0$ for all $|\xi\rangle \in \mathcal{H}$. We write $X \leq Y$ if $Y - X$ is positive.

A *positive operator valued measure* (POVM) consists of a family of positive operators $\{M_i \in B(\mathcal{H}) \,|\, i \in [m]\}$ such that $\sum_{i=1}^m M_i = 1_{B(\mathcal{H})}$. If all positive operators are projections ($M_i = M_i^* = M_i^2$), then we call $\{M_i \in B(\mathcal{H}) \,|\, i \in [m]\}$ with $\sum_{i=1}^m M_i = 1_{B(\mathcal{H})}$ a *projective measurement* (PVM). A *state* $|\psi\rangle$ is a unit vector in a Hilbert space $\mathcal{H}$. Each state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ admits the so-called *Schmidt decomposition* $|\psi\rangle = \sum_{i=1}^m \lambda_i a_i \otimes b_i$, where $\{a_i \,|\, i \in [m]\}$ and $\{b_i \,|\, i \in [m]\}$ are orthonormal sets in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and $\lambda_i \geq 0$ for all i. The strictly positive values $\lambda_i > 0$ are called *Schmidt coefficients*. The *Schmidt rank* of a state $|\psi\rangle$ is the number n of Schmidt coefficients λ_i (counted with multiplicity) in a Schmidt decomposition. We say that $|\psi\rangle$ has *full Schmidt rank* if $n = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$. For $|\psi\rangle = \sum_{i=1}^m \lambda_i a_i \otimes b_i$, we let $\mathrm{supp}_A(|\psi\rangle) = \mathrm{span}\{a_i\}_{i=1}^m$ and $\mathrm{supp}_B(|\psi\rangle) = \mathrm{span}\{b_i\}_{i=1}^m$. Furthermore, we say that a subspace $\mathcal{K} \subseteq \mathcal{H}$ is an *invariant subspace* of $X \in B(\mathcal{H})$ if $X(\mathcal{K}) \subseteq \mathcal{K}$.

In the following, we describe the framework of nonlocal games [6]. A two-player *nonlocal game* $\mathcal{G}$ is played between two collaborating players, Alice and Bob, and a referee. It is specified by finite input sets I_A, I_B and finite output sets O_A, O_B for Alice and Bob, respectively as well as a *verification function* $V : I_A \times I_B \times O_A \times O_B \to \{0, 1\}$ and a probability distribution π on $I_A \times I_B$. In the game, the referee samples a pair $(x, y) \in I_A \times I_B$ using the distribution π and sends x to Alice and y to Bob. Alice and Bob respond with $a \in O_A$ and $b \in O_B$, respectively. They win if $V(x, y, a, b) = 1$. In our examples, we always use the uniform distribution π. The players are not allowed to communicate during the game, but they can agree on a strategy beforehand. A *classical strategy* for a nonlocal game is a strategy in which Alice and Bob only have access to shared randomness. In a *quantum strategy*, the players are allowed to perform local measurements on a shared entangled state. Thus, a quantum strategy may be written as $S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{E_{xa}\}_x, \{F_{yb}\}_y)$, where $|\psi\rangle$ is the shared entangled state, $\{E_{xa}\}_x$ are the POVMs for each $x \in I_A$ for Alice and $\{F_{yb}\}_y$ are the POVMs for each $y \in I_B$ for Bob. A nonlocal game $\mathcal{G}$ is called *synchronous* [23] if we have $I_A = I_B$, $O_A = O_B$ and if Alice and Bob receive identical inputs, they must answer with identical outputs to win, i.e. $V(x, x, a, b) = 0$ for $a \neq b$.

The *classical value* $\omega(\mathcal{G})$ of a nonlocal game $\mathcal{G}$ is the greatest probability of winning the game with a classical strategy. The *quantum value* $\omega^*(\mathcal{G})$ is the supremum over the winning probabilities of the quantum strategies for the game. An *optimal quantum strategy* is a quantum strategy achieving this quantum value. In contrary to the classical value, the quantum value is not always attained [28]. In general, the quantum value is bigger than the classical value. We are especially interested in *pseudo-telepathy* games, which are games having a *perfect* quantum strategy (a strategy that allows the players to win with probability one), but no perfect classical strategy.

3 Self-testing

In this section, we formally introduce the notion of self-testing. The idea of self-testing is the following. A game self-tests an optimal quantum strategy $\tilde{S}$ if the strategy is unique, in the sense that every other optimal strategy S is related the strategy $\tilde{S}$ by a local isometry. As in [19], we will say that $\tilde{S}$ is a local dilation of S.

Definition 3.1. Let $\tilde{S} = (|\tilde{\psi}\rangle \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B, \{\tilde{E}_{xa}\}_x, \{\tilde{F}_{yb}\}_y)$, $S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{E_{xa}\}_x, \{F_{yb}\}_y)$ be quantum strategies. We say that $\tilde{S}$ is a *local dilation* of S if there exist Hilbert spaces $\mathcal{H}_{A,aux}$ and $\mathcal{H}_{B,aux}$, a state $|aux\rangle \in \mathcal{H}_{A,aux} \otimes \mathcal{H}_{B,aux}$ and isometries $U_A : \mathcal{H}_A \to \tilde{\mathcal{H}}_A \otimes \mathcal{H}_{A,aux}$, $U_B : \mathcal{H}_B \to \tilde{\mathcal{H}}_B \otimes \mathcal{H}_{B,aux}$ such that with $U := U_A \otimes U_B$ it holds

$$U|\psi\rangle = |\tilde{\psi}\rangle \otimes |aux\rangle, \tag{2}$$

$$U(E_{xa} \otimes 1)|\psi\rangle = [(\tilde{E}_{xa} \otimes 1)|\tilde{\psi}\rangle] \otimes |aux\rangle, \tag{3}$$

$$U(1 \otimes F_{yb})|\psi\rangle = [(1 \otimes \tilde{F}_{yb})|\tilde{\psi}\rangle] \otimes |aux\rangle. \tag{4}$$

Note that if $\tilde{S}$ is a local dilation of S, then they induce the same correlation and thus the same winning probability for a game $\mathcal{G}$.

Remark 3.2. There is a slight abuse of notation in the previous Definition. The lefthand side of equations (2), (3) and (4) is an element of $\tilde{\mathcal{H}}_A \otimes \mathcal{H}_{A,aux} \otimes \tilde{\mathcal{H}}_B \otimes \mathcal{H}_{B,aux}$, whereas the righthand side is an element of $\tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B \otimes \mathcal{H}_{A,aux} \otimes \mathcal{H}_{B,aux}$. We identify those Hilbert spaces via the unitary that flips the second and the third tensor factors.

Remark 3.3. The equations (3) and (4) are equivalent to

$$U(E_{xa} \otimes F_{yb})|\psi\rangle = [(\tilde{E}_{xa} \otimes \tilde{F}_{yb})|\tilde{\psi}\rangle] \otimes |aux\rangle.$$

Definition 3.4. Let $\mathcal{G}$ be a nonlocal game and $\tilde{S} = (|\tilde{\psi}\rangle \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B, \{\tilde{E}_{xa}\}_x, \{\tilde{F}_{yb}\}_y)$ be an optimal quantum strategy. We say that $\mathcal{G}$ *self-tests* the strategy $\tilde{S}$ if $\tilde{S}$ is a local dilation of any optimal quantum strategy $S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{E_{xa}\}_x, \{F_{yb}\}_y)$. If only equation (2) holds for all optimal quantum strategies S, we say that $\mathcal{G}$ *self-tests the state* $|\tilde{\psi}\rangle$.

Remark 3.5. Some authors only impose that $\tilde{S}$ is a local dilation of any projective strategy S. This is a priori a weaker notion of self-testing. We will see, however, that in some cases (e.g. the magic square game) the PVM self-test can be used to prove a POVM self-test for the synchronous version of the game.

We will now state some results of [19] on local dilations. Those will mostly help to reduce proving self-testing for general quantum strategies to quantum strategies having states with full Schmidt rank.

Lemma 3.6. *[19, Lemma 4.8] Let $X \in B(\mathcal{H}_A)$, $Y \in B(\mathcal{H}_B)$ and $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that $(X \otimes 1)|\psi\rangle = (1 \otimes Y)|\psi\rangle$. Then $\operatorname{supp}_A(|\psi\rangle)$ is invariant under X and $\operatorname{supp}_B(|\psi\rangle)$ is invariant under Y.*

The next lemma and its corollary split [19, Lemma 4.9] into two parts, making the lemma slightly more general. We do this since we need to use the lemma in its more general form later on.

Lemma 3.7. *Let $\mathcal{G}$ be a nonlocal game. Let $S = (|\psi\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}, \{E_{xa}\}_x, \{F_{yb}\}_y)$ be a quantum strategy such that $\operatorname{supp}_A(|\psi\rangle)$ and $\operatorname{supp}_B(|\psi\rangle)$ are invariant under each E_{xa} and F_{yb}, respectively. Then there exists a quantum strategy $S' = (|\psi'\rangle, \{E'_{xa}\}_x, \{F'_{yb}\}_y)$ such that $|\psi'\rangle$ has full Schmidt rank and S' is a local dilation of S.*

Proof. We follow the proof of [19, Lemma 4.9]. Consider the Schmidt decomposition $|\psi\rangle = \sum_{i=1}^r \alpha_i \xi_i \otimes \eta_i$ and let $U_A : \mathbb{C}^r \to \mathbb{C}^{d_A}$, $U_B : \mathbb{C}^r \to \mathbb{C}^{d_B}$ be isometries given by

$$U_A = \sum_{i=1}^r \xi_i e_i^*, \quad U_B = \sum_{i=1}^r \eta_i e_i^*.$$

Then

$$E'_{xa} = U_A^* E_{xa} U_A, \quad F'_{yb} = U_B^* F_{yb} U_B$$

are positive operators and $\{E'_{xa} \,|\, x \in [r]\}$, $\{F'_{yb} \,|\, y \in [r]\}$ are POVMs. With $|\psi'\rangle = (U_A^* \otimes U_B^*)|\psi\rangle$, we get a quantum strategy S', where $|\psi'\rangle$ has full Schmidt rank.

We will now show that S' is a local dilation of S. Set $H_{A,aux} = \mathbb{C}^{d_A}$, $H_{B,aux} = \mathbb{C}^{d_B}$ and define isometries $V_A : \mathbb{C}^{d_A} \to \mathbb{C}^r \otimes \mathbb{C}^{d_A}$, $V_B : \mathbb{C}^{d_B} \to \mathbb{C}^r \otimes \mathbb{C}^{d_B}$ by

$$\begin{aligned} V_A(v) &= U_A^*(v) \otimes \xi_1 + e_1 \otimes (1_{d_A} - U_A U_A^*)(v), \\ V_B(w) &= U_B^*(w) \otimes \eta_1 + e_1 \otimes (1_{d_B} - U_B U_B^*)(w), \end{aligned}$$

where $v \in \mathbb{C}^{d_A}$, $w \in \mathbb{C}^{d_B}$. Since $\operatorname{supp}_A(|\psi\rangle)$ and $\operatorname{supp}_B(|\psi\rangle)$ are invariant under each E_{xa} and F_{yb}, respectively, and $U_A U_A^*$ and $U_B U_B^*$ are the projections onto $\operatorname{supp}_A(|\psi\rangle)$ and $\operatorname{supp}_B(|\psi\rangle)$, respectively, we have

$$(V_A \otimes V_B)(E_{xa} \otimes F_{yb})|\psi\rangle = (U_A^* \otimes U_B^*)((E_{xa} \otimes F_{yb})|\psi\rangle) \otimes (\xi_1 \otimes \eta_1).$$

Furthermore, it holds

$$\begin{aligned} (U_A^* \otimes U_B^*)(E_{xa} \otimes F_{yb})|\psi\rangle &= (U_A^* \otimes U_B^*)(E_{xa} \otimes F_{yb})(U_A U_A^* \otimes U_B U_B^*)|\psi\rangle \\ &= (E'_{xa} \otimes F'_{yb})|\psi'\rangle, \end{aligned}$$

since $U_A U_A^*$ and $U_B U_B^*$ are the projections onto $\operatorname{supp}_A(|\psi\rangle)$ and $\operatorname{supp}_B(|\psi\rangle)$, respectively. This concludes the proof. □

Corollary 3.8. *Let $\mathcal{G}$ be a synchronous game and let $S = (|\psi\rangle, \{E_{xa}\}_x, \{F_{yb}\}_y)$ a perfect quantum strategy of $\mathcal{G}$. Then there exists a perfect quantum strategy $S' = (|\psi'\rangle, \{E'_{xa}\}_x, \{F'_{yb}\}_y)$ of $\mathcal{G}$ such that $|\psi'\rangle$ has full Schmidt rank and S' is a local dilation of S.*

Proof. By [19, Corollary 3.6 (a)], we know $(E_{xa} \otimes 1)|\psi\rangle = (1 \otimes F_{xa})|\psi\rangle$. Lemma 3.6 yields that $\operatorname{supp}_A(|\psi\rangle)$ and $\operatorname{supp}_B(|\psi\rangle)$ are invariant under each E_{xa} and F_{yb}, respectively. We obtain the result by Lemma 3.7. □

We also have a transitivity result for local dilations. One can use it for proving self-testing results in the follwing way. First show that a general optimal quantum strategy dilates to a quantum strategy having a state of full Schmidt rank. Then check if the reference strategy is a local dilation of any optimal strategy with a state of full Schmidt rank.

Lemma 3.9. *[19, Lemma 4.7] Let S_1, S_2 and S_3 be quantum strategies. If S_1 is a local dilation of S_2 and S_2 is a local dilation of S_3, then S_1 is a local dilation of S_3.*

Finally, we give the definition of robust self-testing. Roughly speaking, a game robustly self-tests a quantum strategy $\tilde{S}$ if it self-tests the strategy and additionally, for every almost optimal strategy S, a local dilation of S is close to $\tilde{S}$. For $\delta > 0$, we say that a quantum strategy is *δ-optimal* for a game $\mathcal{G}$ if the winning probability is greater or equal to $\omega^*(\mathcal{G}) - \delta$, where $\omega^*(\mathcal{G})$ is the quantum value of $\mathcal{G}$.

Definition 3.10. Let $\mathcal{G}$ be a nonlocal game, $\tilde{S} = (|\tilde{\psi}\rangle \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B, \{\tilde{E}_{xa}\}_x, \{\tilde{F}_{yb}\}_y)$ an optimal quantum strategy. Then $\mathcal{G}$ is a *robust self-test* for $\tilde{S}$ if $\mathcal{G}$ is a self-test for $\tilde{S}$ and for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any δ-optimal strategy $S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{E_{xa}\}_x, \{B_{yb}\}_y)$, there exists Hilbert spaces $\mathcal{H}_{A,aux}$ and $\mathcal{H}_{B,aux}$, a state $|aux\rangle \in \mathcal{H}_{A,aux} \otimes \mathcal{H}_{B,aux}$ and isometries $U_A : \mathcal{H}_A \to \tilde{\mathcal{H}}_A \otimes \mathcal{H}_{A,aux}$, $U_B : \mathcal{H}_B \to \tilde{\mathcal{H}}_B \otimes \mathcal{H}_{B,aux}$ such that with $U := U_A \otimes U_B$ it holds

$$\begin{aligned} \|U|\psi\rangle - |\tilde{\psi}\rangle \otimes |aux\rangle\| &\leq \varepsilon, \\ \|U(E_{xa} \otimes 1)|\psi\rangle - [(\tilde{E}_{xa} \otimes 1)|\tilde{\psi}\rangle] \otimes |aux\rangle\| &\leq \varepsilon, \\ \|U(1 \otimes \tilde{F}_{yb})|\psi\rangle - [(1 \otimes \tilde{F}_{yb})|\tilde{\psi}\rangle] \otimes |aux\rangle\| &\leq \varepsilon. \end{aligned}$$

3.1 The synchronous magic square game self-tests a quantum strategy

In this subsection, we show that the synchronous magic square game self-tests a quantum strategy $\tilde{S}$ with maximally entangled state $|\psi_4\rangle$. In [34] it was shown that the magic square game self-tests a strategy S. Note that their general strategies consist of projective measurements, the case of strategies with POVMs is not considered. We will deduce a POVM self-testing result for the synchronous magic square game from the PVM self-test of the magic square.

We will first describe the magic square game of [5, Section 5] and then its synchronous version. Note that there are also other versions of the magic square game. We choose the one with minimal input and output sets. Consider the following set of equations

$$\begin{aligned} &x_1x_2x_3 = 1 \quad (r_1), && x_1x_4x_7 = -1 \quad (c_1), \\ &x_4x_5x_6 = 1 \quad (r_2), && x_2x_5x_8 = -1 \quad (c_2), \\ &x_7x_8x_9 = 1 \quad (r_3), && x_3x_6x_9 = -1 \quad (c_3). \end{aligned}$$

In the magic square game, the referee sends one of the three equations on the left hand side to Alice and one of the three equations on the right hand side to Bob. Alice answers with a $\{-1, 1\}$-assignment of the variables such that their product is 1 and Bob answers with a $\{-1, 1\}$-assignment of the variables such that their product is -1. The players win the game, if their assignments coincide in the common variable of the equations they received. More formally, we have

$$\begin{aligned} &I_A = R := \{r_1, r_2, r_3\}, \\ &I_B = C := \{c_1, c_2, c_3\}, \\ &O_A = O_1 := \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}, \\ &O_B = O_{-1} := \{(-1, -1, -1), (-1, 1, 1), (1, -1, 1), (1, 1, -1)\}, \\ &V_{MS}(r_i, c_j, a, b) = \begin{cases} 1 \text{ if } a_j = b_i, \\ 0 \text{ otherwise.} \end{cases} \end{aligned}$$

Here $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$. Note that it is shown in [5, Section 5] that this game is a pseudo-telepathy game.

The synchronous version of the magic square game is played as follows. The referee sends one of the six equations above to Alice and one of them to Bob. They answer with $\{-1, 1\}$-assignments such that the equations are fulfilled. Alice and Bob win the game if their assignments coincide on the common variables of the equations they received. Here, we have

$$I_A = I_B = R \cup C, \quad O_A = O_B = O_1 \cup O_{-1},$$
$$V(x, y, a, b) = \begin{cases} V_{MS}(x, y, a, b) \text{ if } x \in R, y \in C, a \in O_1, b \in O_{-1}, \\ V_{MS}(y, x, b, a) \text{ if } x \in C, y \in R, a \in O_{-1}, b \in O_1, \\ 1 \text{ if } (x, y \in R, x \neq y, a, b \in O_1) \text{ or } (x, y \in C, x \neq y, a, b \in O_{-1}), \\ \delta_{ab} \text{ if } x = y \in R, a, b \in O_1 \text{ or } x = y \in C, a, b \in O_{-1}, \\ 0 \text{ otherwise.} \end{cases}$$

The following self-testing statement for the magic square game was shown in [34]. We will use this theorem to deduce a self-testing statement for the synchronous magic square game. Recall the Pauli matrices

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Theorem 3.11 ([34]). *Restricting to projective measurements, the magic square game self-tests a perfect quantum strategy $S = (|\psi_4\rangle, \{E_{xa}\}_x, \{F_{yb}\}_y)$. Here $|\psi_4\rangle = \frac{1}{2}\sum_{i=1}^4 e_i \otimes e_i$ and $E_{xa} = \frac{1}{8}(1 + a_1X_1)(1 + a_2X_2)(1 + a_3X_3)$, $F_{yb} = \frac{1}{8}(1 + b_1Y_1^\intercal)(1 + b_2Y_2^\intercal)(1 + b_3Y_3^\intercal)$, where X_i and Y_i are the i-th entries in row x and column y of*

$I \otimes \sigma_Z$	$\sigma_Z \otimes I$	$\sigma_Z \otimes \sigma_Z$
$\sigma_X \otimes I$	$I \otimes \sigma_X$	$\sigma_X \otimes \sigma_X$
$-\sigma_X \otimes \sigma_Z$	$-\sigma_Z \otimes \sigma_X$	$\sigma_Y \otimes \sigma_Y$

Corollary 3.12. *Let S be as in Theorem 3.11. The synchronous magic square game self-tests the perfect quantum strategy $\tilde{S} = (|\psi_4\rangle, \{\tilde{E}_{xa}\}_x, \{\tilde{F}_{yb}\}_y)$, where*

$$\tilde{E}_{xa} = \begin{cases} E_{xa} \text{ if } x \in R, a \in O_1 \\ F_{xa}^\intercal \text{ if } x \in C, a \in O_{-1}, \\ 0 \text{ otherwise,} \end{cases} \qquad \tilde{F}_{yb} = \begin{cases} F_{yb} \text{ if } y \in C, b \in O_{-1} \\ E_{yb}^\intercal \text{ if } y \in R, b \in O_1, \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Let $S' = (|\psi'\rangle, \{E'_{xa}\}, \{F'_{yb}\})$ be a perfect quantum strategy for the synchronous magic square game. By Corollary 3.8, there exists a perfect quantum strategy $\hat{S} = (|\hat{\psi}\rangle, \{\hat{E}_{xa}\}, \{\hat{F}_{yb}\})$ that is a local dilation of S', where $|\hat{\psi}\rangle$ has full Schmidt rank, $|\hat{\psi}\rangle = \sum_{i=1}^d \hat{\lambda}_i e_i \otimes e_i$ with $\hat{\lambda}_i > 0$ for all $i \in [d]$. Furthermore, by [19, Corollary 3.6], the operators $\hat{E}_{xa}$ and $\hat{F}_{yb}$ are projections.

Let $\hat{\varphi} := \operatorname{diag}(\hat{\lambda}_i) \in \mathbb{C}^{d\times d}$, i.e. $\hat{\varphi}$ is a diagonal matrix with entries $\hat{\lambda}_i$. Let $x \in R$ and $a \in O_{-1}$. Then

$$\operatorname{Tr}(\hat{E}_{xa}\hat{\varphi}^2) = \langle\hat{\psi}|\hat{E}_{xa} \otimes 1|\hat{\psi}\rangle = \langle\hat{\psi}|\hat{E}_{xa} \otimes \sum_b \hat{F}_{yb}|\hat{\psi}\rangle = 0,$$

since $\langle\hat{\psi}|\hat{E}_{xa} \otimes \hat{F}_{yb}|\hat{\psi}\rangle = 0$ for all y, b as $a \in O_{-1}$. Since $\hat{\varphi}$ is invertible, we conclude $\hat{E}_{xa} = 0$. We similarly get $\hat{E}_{xa} = 0$ for $x \in C, a \in O_1$ and $\hat{F}_{yb} = 0$ for $y \in C, b \in O_1$ or $y \in R, b \in O_{-1}$.

By the previous argument, we get that $\{\hat{E}_{xa} \,|\, a \in O_1\}$ and $\{\hat{F}_{yb} \,|\, b \in O_{-1}\}$ are PVMs for $x \in R, y \in C$. Together with $|\hat{\psi}\rangle$, they form a perfect quantum strategy for the magic square game. By Theorem 3.11, there exists Hilbert spaces $\mathcal{H}_{A,aux}$ and $\mathcal{H}_{B,aux}$, a state $|aux\rangle \in \mathcal{H}_{A,aux} \otimes \mathcal{H}_{B,aux}$

and isometries $U_A : \hat{\mathcal{H}}_A \to \mathcal{H}_A \otimes \mathcal{H}_{A,aux}$, $U_B : \hat{\mathcal{H}}_B \to \mathcal{H}_B \otimes \mathcal{H}_{B,aux}$ such that with $U := U_A \otimes U_B$ it holds

$$\begin{aligned} U|\hat{\psi}\rangle &= |\psi_4\rangle \otimes |aux\rangle, \\ U(\hat{E}_{xa} \otimes 1)|\hat{\psi}\rangle &= [(E_{xa} \otimes 1)|\psi_4\rangle] \otimes |aux\rangle, \\ U(1 \otimes \hat{F}_{yb})|\hat{\psi}\rangle &= [(1 \otimes F_{xa})|\psi_4\rangle] \otimes |aux\rangle \end{aligned} \tag{5}$$

for $x \in R$, $y \in C$ and $a \in O_1$, $b \in O_{-1}$.

Now, let $x \in C$ and $a \in O_{-1}$. By [19, Corollary 3.6 (a)], we have

$$U(\hat{E}_{xa} \otimes 1)|\hat{\psi}\rangle = U(1 \otimes \hat{F}_{xa})|\hat{\psi}\rangle.$$

Furthermore, it holds

$$U(1 \otimes \hat{F}_{xa})|\hat{\psi}\rangle = [(1 \otimes F_{yb})|\psi_4\rangle] \otimes |aux\rangle$$

by (5). We conclude

$$U(\hat{E}_{xa} \otimes 1)|\hat{\psi}\rangle = [(1 \otimes F_{yb})|\psi_4\rangle] \otimes |aux\rangle = [(F_{xa}^{\mathsf{T}} \otimes 1)|\psi_4\rangle] \otimes |aux\rangle \tag{6}$$

for all $x \in C$ and $a \in O_{-1}$. One similarly obtains

$$U(1 \otimes \hat{F}_{yb})|\hat{\psi}\rangle = [(1 \otimes E_{yb}^{\mathsf{T}})|\psi_4\rangle] \otimes |aux\rangle \tag{7}$$

for all $y \in R$ and $a \in O_1$. Since we know $\hat{E}_{xa} = 0 = E_{xa}$ and $\hat{F}_{yb} = 0 = F_{yb}$ for the remaining x, y, a, b, we deduce from (5), (6) and (7) that the synchronous magic square game self-tests the perfect quantum strategy $\tilde{S}$. □

4 The $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be nonlocal games. The $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game is played as follows: The referee sends Alice and Bob a pair of questions (x_1, x_2) and (y_1, y_2), respectively, where x_1, y_1 are questions in $\mathcal{G}_1$ and x_2, y_2 in $\mathcal{G}_2$. Each of them chooses one of the questions they received and responds with an answer from the corresponding game. To win the game, two conditions have to be fulfilled:

(1) Alice and Bob have to choose questions from the same game,

(2) their answers have to win the corresponding game.

More formally, suppose the nonlocal games $\mathcal{G}_1$ and $\mathcal{G}_2$ have input sets $I_{A,i}$, $I_{B,i}$, output sets $O_{A,i}$, $O_{B,i}$, verification functions $V_{\mathcal{G}_i}$ and probability distributions π_i on $I_{A,i} \times I_{B,i}$ for $i = 1, 2$. Then the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game has input sets $I_{A,1} \times I_{A,2}$, $I_{B,1} \times I_{B,2}$, output sets $O_{A,1} \dot{\cup} O_{A,2}$, $O_{B,1} \dot{\cup} O_{B,2}$ and verification function

$$V((x_1, x_2), (y_1, y_2), a, b) = \begin{cases} V_{\mathcal{G}_i}(x_i, y_i, a, b) & \text{if } a \in O_{A,i} \text{ and } b \in O_{B,i} \text{ for some } i = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

For the probability distribution, we take $\pi = \pi_1 \times \pi_2$ on $(I_{A,1} \times I_{A,2}) \times (I_{B,1} \times I_{B,2})$, *i.e.* $\pi((x_1, x_2), (y_1, y_2)) = \pi_1(x_1, y_1)\pi_2(x_2, y_2)$. In this article, we assume that the probability distributions of $\mathcal{G}_1$ and $\mathcal{G}_2$ are uniform, so this will also be the case for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game.

We first check that the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game keeps its quantum advantage if one of the games is a pseudo-telepathy game and the other one does not have a perfect quantum strategy.

Lemma 4.1. *Let $\mathcal{G}_1$ be a pseudo-telepathy game and let $\mathcal{G}_2$ be a game with $\omega(G) < 1$. Then the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game is a pseudo-telepathy game.*

Proof. It is easy to see that the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game has a perfect strategy. Indeed, Alice and Bob can always choose to answer the question from $\mathcal{G}_1$ and then use a perfect quantum strategy for $\mathcal{G}_1$ to come up with their answers.

It remains to show that there is no perfect classical strategy for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game. First note that it suffices to show that there is no perfect deterministic strategy. For contradiction assume that there is a perfect deterministic strategy S. If in this strategy Alice chooses a question x_1 for some pair (x_1, x_2), then we know that Bob has to choose y_1 for all pairs (y_1, y_2), as otherwise there are questions for which the players lose the game. A similar argument shows that Alice also always has to choose the questions from $\mathcal{G}_1$. This shows that in strategy S the players always choose to answer questions from $\mathcal{G}_1$. Hence, S can be used to construct a perfect classical strategy for game $\mathcal{G}_1$ which is a contradiction. Since $\mathcal{G}_2$ also does not admit a perfect classical strategy, a similar argument works if in strategy S Alice chooses a question x_2 for some tuple (x_1, x_2). $\square$

The following lemma will be used to show that resticting a POVM measurement with a projection yields a POVM measurement on a smaller Hilbert space.

Lemma 4.2. *[4, Proposition II, 3.3.2] Let A be a C^*-algebra, let $a \in A$ be a positive element and $p \in A$ a projection. If $a \leq p$, then $ap = pa = a$.*

The next lemma shows that if we have a perfect quantum strategy for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game, where Alice and Bob have non-zero probability of answering with some outputs of $\mathcal{G}_1$, then there is a perfect quantum strategy for $\mathcal{G}_1$. In particular, this shows that if $\mathcal{G}_2$ has a perfect quantum strategy and $\mathcal{G}_1$ does not, the players will always choose to play $\mathcal{G}_2$ and never $\mathcal{G}_1$.

Lemma 4.3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be nonlocal games and consider the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game as above. If there is a perfect quantum strategy $S = (|\psi\rangle, \{E_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{F_{(y_1,y_2)b}\}_{(y_1,y_2)})$ of the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game, where additionally*

$$\langle\psi|E_{(x_1,x_2)a} \otimes F_{(y_1,y_2)b}|\psi\rangle > 0 \tag{8}$$

for some $a \in O_{A,1}$ and $b \in O_{B,1}$, then $\mathcal{G}_1$ has a perfect quantum strategy.

Proof. Let $S' = (|\psi'\rangle \in \mathcal{H}'_A \otimes \mathcal{H}'_B, \{E'_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{F'_{(y_1,y_2)b}\}_{(y_1,y_2)})$ be a perfect quantum strategy of the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game. By restricting the state and the operators from S' to $\mathrm{supp}_A(|\psi'\rangle) \otimes \mathrm{supp}_B(|\psi'\rangle)$, we get a perfect quantum strategy $S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{E_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{F_{(y_1,y_2)b}\}_{(y_1,y_2)})$ such that

$$\langle\psi|E_{(x_1,x_2)a} \otimes F_{(y_1,y_2)b}|\psi\rangle = \langle\psi'|E'_{(x_1,x_2)a} \otimes F'_{(y_1,y_2)b}|\psi'\rangle \tag{9}$$

and $|\psi\rangle$ has full Schmidt rank, *i.e.* $|\psi\rangle = \sum_{i=1}^{d} \lambda_i e_i \otimes e_i$ with $\lambda_i > 0$ for all $i \in [d]$. We will first show

$$\sum_{a \in O_{A,i}} E_{(x_1,x_2)a} = \sum_{a \in O_{A,i}} E_{(s_1,s_2)a}, \qquad \sum_{b \in O_{B,j}} F_{(y_1,y_2)b} = \sum_{b \in O_{B,j}} F_{(t_1,t_2)b}$$

for all $(x_1, x_2), (s_1, s_2) \in I_{A,1} \times I_{A,2}$, $i = 1, 2$ and $(y_1, y_2), (t_1, t_2) \in I_{B,1} \times I_{B,2}$, $j = 1, 2$. Note that we have

$$\sum_{a \in O_{A,1} \dot\cup O_{A,2}} E_{(x_1,x_2)a} = 1 = \sum_{b \in O_{B,1} \dot\cup O_{B,2}} F_{(y_1,y_2)b} \tag{10}$$

and $\langle\psi|E_{(x_1,x_2)a}\otimes F_{(y_1,y_2)b}|\psi\rangle = 0$ for all $a\in O_{A,i}$, $b\in O_{B,j}$ with $i\neq j$, since S is a perfect quantum strategy. Let $\varphi := \mathrm{diag}(\lambda_i)\in\mathbb{C}^{d\times d}$, *i.e.* φ is the diagonal matrix with entries λ_i. Let $p_{(x_1,x_2)i} := \sum_{a\in O_{A,i}} E_{(x_1,x_2)a}$, $i=1,2$ and $q_{(y_1,y_2)j} := \sum_{b\in O_{B,j}} F_{(y_1,y_2)b}$, $j=1,2$. It holds

$$\begin{aligned}\mathrm{Tr}(p_{(x_1,x_2)i}\varphi(q_{(y_1,y_2)j})^{\mathsf{T}}\varphi) &= \mathrm{Tr}(\varphi^* p_{(x_1,x_2)i}\varphi(q_{(y_1,y_2)j})^{\mathsf{T}})\\ &= \langle\psi|\sum_{a\in O_{A,i}} E_{(x_1,x_2)a}\otimes\sum_{b\in O_{B,j}} F_{(y_1,y_2)b}|\psi\rangle = 0\end{aligned}$$

for $i\neq j$, since S is a perfect quantum strategy for the $(\mathcal{G}_1\vee\mathcal{G}_2)$-game. Thus, we have

$$p_{(x_1,x_2)i}\varphi(q_{(y_1,y_2)j})^{\mathsf{T}}\varphi = 0 \tag{11}$$

for all $i\neq j$. Using equations (10) and (11) several times, we get

$$\begin{aligned}p_{(x_1,x_2)i}\varphi^2 &= p_{(x_1,x_2)i}\varphi((q_{(y_1,y_2)i})^{\mathsf{T}} + (q_{(y_1,y_2)j})^{\mathsf{T}})\varphi\\ &= p_{(x_1,x_2)i}\varphi(q_{(y_1,y_2)i})^{\mathsf{T}}\varphi\\ &= (p_{(x_1,x_2)i} + p_{(x_1,x_2)j})\varphi(q_{(y_1,y_2)i})^{\mathsf{T}}\varphi\\ &= \varphi(q_{(y_1,y_2)i})^{\mathsf{T}}\varphi\\ &= p_{(s_1,s_2)i}\varphi(q_{(y_1,y_2)i})^{\mathsf{T}}\varphi\\ &= p_{(s_1,s_2)i}\varphi^2\end{aligned}$$

for all $(x_1,x_2),(s_1,s_2)\in I_{A,1}\times I_{A,2}$ and $i,j\in\{1,2\}, i\neq j$. Since $|\psi\rangle$ has full Schmidt rank, φ is invertible and we obtain $p_{(x_1,x_2)i} = p_{(s_1,s_2)i}$ for all $(x_1,x_2),(s_1,s_2)\in I_{A,1}\times I_{A,2}$. One similarly shows $q_{(y_1,y_2)j} = q_{(t_1,t_2)j}$. From now on, we let $p_i := p_{(x_1,x_2)i}$, $q_j := q_{(y_1,y_2)j}$ for $i=1,2$, $j=1,2$.

We will now show that p_i and q_j are projections. Using equations (10) and (11), we obtain

$$\begin{aligned}p_1p_2\varphi^2 &= p_1p_2\varphi(q_1^{\mathsf{T}} + q_2^{\mathsf{T}})\varphi\\ &= p_1p_2\varphi(q_2^{\mathsf{T}})\varphi\\ &= p_1(p_1+p_2)\varphi(q_2^{\mathsf{T}})\varphi\\ &= p_1\varphi(q_2^{\mathsf{T}})\varphi\\ &= 0.\end{aligned}$$

Since φ is invertible, we obtain $p_1p_2 = 0$. In particular, we have $p_1 = p_1(p_1+p_2) = p_1^2$ and since we already know $p_1 = p_1^*$, we get that p_1 is a projection. It immediately follows that $p_2 = 1-p_1$ is a projection. One can similarly show that q_1 and q_2 are projections.

In the next step, we show $p_1\neq 0\neq q_1$. By (8) and (9), we have

$$\langle\psi|E_{(x_1,x_2)a}\otimes F_{(y_1,y_2)b}|\psi\rangle = \langle\psi'|E'_{(x_1,x_2)a}\otimes F'_{(y_1,y_2)b}|\psi'\rangle > 0$$

for some $a\in O_{A,1}$ and $b\in O_{B,1}$. Since all $E_{(x_1,x_2)a}$, $F_{(y_1,y_2)b}$ are positive, it holds

$$\langle\psi|p_1\otimes q_1|\psi\rangle \geq \langle\psi|E_{(x_1,x_2)a}\otimes F_{(y_1,y_2)b}|\psi\rangle > 0$$

which implies $p_1\neq 0\neq q_1$.

In the last step, we construct a perfect quantum strategy for $\mathcal{G}_1$. Fix some $x_2\in O_{A,2}$ and $y_2\in O_{B,2}$ and define the quantum strategy $\tilde{S}$ for $\mathcal{G}_1$ as follows.

$$\begin{aligned}|\tilde{\psi}\rangle &:= (p_1\otimes q_1)|\psi\rangle \in p_1\mathcal{H}_A\otimes q_1\mathcal{H}_B,\\ \tilde{E}_{x_1a} &:= E_{(x_1,x_2)a},\\ \tilde{F}_{y_1b} &:= F_{(y_1,y_2)b}\end{aligned}$$

for $x_1 \in I_{A,1}$, $y_1 \in I_{B,1}$ and $a \in O_{A,1}$, $b \in O_{B,1}$. Note that p_1 is the identity in $B(p_1\mathcal{H}_A)$ and similarly q_1 is the identity in $B(q_1\mathcal{H}_B)$. Using Lemma 4.2, we see that $\tilde{E}_{x_1a} \in B(p_1\mathcal{H}_A)$ and $\tilde{F}_{y_1b} \in B(q_1\mathcal{H}_B)$. Thus $\{\tilde{E}_{x_1a} \,|\, a \in O_{A,1}\}$ and $\{\tilde{F}_{y_1b} \,|\, b \in O_{B,1}\}$ are POVMs on the space $p_1\mathcal{H}_A \otimes q_1\mathcal{H}_B$ for all x_1, y_1. Furthermore, we have

$$\begin{aligned}\langle\tilde{\psi}|\tilde{E}_{x_1a} \otimes \tilde{F}_{y_1b}|\tilde{\psi}\rangle &= \langle\psi|p_1E_{(x_1,x_2)a}p_1 \otimes q_1F_{(y_1,y_2)b}q_1|\psi\rangle \\ &= \langle\psi|E_{(x_1,x_2)a} \otimes F_{(y_1,y_2)b}|\psi\rangle\end{aligned}$$

by Lemma 4.2. Since $S = (|\psi\rangle, \{E_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{F_{(y_1,y_2)b}\}_{(y_1,y_2)})$ is a perfect quantum strategy for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game, we know that

$$\langle\psi|E_{(x_1,x_2)a} \otimes F_{(y_1,y_2)b}|\psi\rangle = 0$$

whenever $V((x_1,x_2),(y_1,y_2),a,b) = 0$. But since we have $a \in O_{A,1}$ and $b \in O_{B,1}$, this is by definition equivalent to $V_{\mathcal{G}_1}(x_1,y_1,a,b) = 0$. We conclude that $\tilde{S}$ is a perfect quantum strategy for $\mathcal{G}_1$. ☐

We will use the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game in the next sections to construct non-robust self-tests and games that do not self-test any states.

5 A non-robust self-test

In this section, we construct a game that non-robustly self-tests a perfect quantum strategy. The idea is to consider the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game with the following games. We let $\mathcal{G}_1$ be a game that has no perfect quantum strategy, but a sequence of quantum strategies whose winning probabilities converge to 1. Note that such a game was constructed by Slofstra in [28]. For $\mathcal{G}_2$ we take a pseudo-telepathy game that self-tests a quantum strategy. Then the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game still self-tests this strategy, because $\mathcal{G}_1$ has no perfect quantum strategy. This self-test is not robust, however, since we can construct a near-optimal strategies of $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game from the ones of $\mathcal{G}_1$ and these strategies are not close to the self-tested strategy.

For our proof technique to go through, we need to choose a game $\mathcal{G}_2$ that is synchronous. In this case we can ensure that if one of the players gets a pair of questions (x_1,x_2) and chooses to play game $\mathcal{G}_1$, the question x_2 does not matter for the output (see part (ii) below).

Lemma 5.1. *Let $\mathcal{G}_1$ be a nonlocal game and let $\mathcal{G}_2$ be a synchronous nonlocal game. Furthermore, let $S = (|\psi\rangle, \{E_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{F_{(y_1,y_2)b}\}_{(y_1,y_2)})$ be a perfect quantum strategy for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game.*

(i) It holds $(E_{(x_1,x_2)a} \otimes 1)|\psi\rangle = (1 \otimes F_{(y_1,x_2)a})|\psi\rangle$ for all $x_1 \in I_{A,1}, y_1 \in I_{B,1}$, $x_2 \in I_2$ and $a \in O_2$.

(ii) For all $x_1, x_3 \in I_{A,1}$, $x_2 \in I_2$, $a \in O_2$ and $y_1, y_3 \in I_{B,1}$, $y_2 \in I_2$, $b \in O_2$, we have

$$(E_{(x_1,x_2)a} \otimes 1)|\psi\rangle = (E_{(x_3,x_2)a} \otimes 1)|\psi\rangle \text{ and } (1 \otimes F_{(y_1,y_2)b})|\psi\rangle = (1 \otimes F_{(y_3,y_2)b})|\psi\rangle.$$

(iii) If $|\psi\rangle$ has full Schmidt rank, then $E_{(x_1,x_2)a} = E_{(x_3,x_2)a}$ for $x_1, x_3 \in I_{A,1}$, $x_2 \in I_2$, $a \in O_2$ and $F_{(y_1,y_2)b} = F_{(y_3,y_2)b}$ for $y_1, y_3 \in I_{B,1}$, $y_2 \in I_2$, $b \in O_2$.

Proof. Let S be as above and $x_1 \in I_{A,1}$, $x_2 \in I_2$, $a \in O_2$. Since $\mathcal{G}_2$ is synchronous, it holds

$$\langle\psi|E_{(x_1,x_2)a} \otimes 1|\psi\rangle = \langle\psi|E_{(x_1,x_2)a} \otimes F_{(y_1,x_2)a}|\psi\rangle = \langle\psi|1 \otimes F_{(y_1,x_2)a}|\psi\rangle$$

for all $y_1 \in I_{B,1}$. Using the equation above and the fact that $E^2_{(x_1,x_2)a} \leq E_{(x_1,x_2)a}$ and $F^2_{(y_1,x_2)a} \leq F_{(y_1,x_2)a}$, we obtain

$$\begin{aligned}
\|(E_{(x_1,x_2)a} \otimes 1)|\psi\rangle - (1 \otimes F_{(y_1,x_2)a})|\psi\rangle\|^2 &= \langle\psi|E^2_{(x_1,x_2)a} \otimes 1|\psi\rangle + \langle\psi|1 \otimes F^2_{(y_1,x_2)a}|\psi\rangle \\
&\quad - 2\langle\psi|E_{(x_1,x_2)a} \otimes F_{(y_1,x_2)a}|\psi\rangle \\
&\leq \langle\psi|E_{(x_1,x_2)a} \otimes 1|\psi\rangle + \langle\psi|1 \otimes F_{(y_1,x_2)a}|\psi\rangle \\
&\quad - 2\langle\psi|E_{(x_1,x_2)a} \otimes F_{(y_1,x_2)a}|\psi\rangle \\
&= 0.
\end{aligned}$$

This yields $(E_{(x_1,x_2)a} \otimes 1)|\psi\rangle = (1 \otimes F_{(y_1,x_2)a})|\psi\rangle$ and thus proves part (i). Since this equation holds for every $x_1 \in I_{A,1}$, we get $(E_{(x_1,x_2)a} \otimes 1)|\psi\rangle = (E_{(x_3,x_2)a} \otimes 1)|\psi\rangle$, $x_1, x_3 \in I_{A,1}$ and similarly $(1 \otimes F_{(y_1,x_2)a})|\psi\rangle = (1 \otimes F_{(y_3,x_2)a})|\psi\rangle$ for $y_1, y_3 \in I_{B,1}$ which yields (ii). Part (iii) follows from (ii) since $|\psi\rangle\langle\psi|$ is invertible if $|\psi\rangle$ has full Schmidt rank. □

We can now present the non-robust self-test.

Theorem 5.2. *Let $\mathcal{G}_1$ be a nonlocal game that has no perfect quantum strategy, but there exists a sequence of quantum strategies whose winning probabilities converge to 1. Let $\mathcal{G}_2$ be a synchronous pseudo-telepathy game that self-tests a strategy $S_2 = (|\tilde{\psi}\rangle \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B, \{\hat{E}_{x_2a}\}_{x_2}, \{\hat{F}_{y_2b}\}_{y_2})$. In this case the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game is a non-robust self-test for the strategy $\tilde{S}_2 = (|\tilde{\psi}\rangle, \{\tilde{E}_{(x_1,x_2)a}\}_{(x_1,x_2)}$, $\{\tilde{F}_{(y_1,y_2)b}\}_{(y_1,y_2)})$, where*

$$\tilde{E}_{(x_1,x_2)a} = \begin{cases} \hat{E}_{x_2a} \text{ for } a \in O_{A,2}, \\ 0 \text{ otherwise,} \end{cases} \qquad \tilde{F}_{(y_1,y_2)b} = \begin{cases} \hat{F}_{y_2b} \text{ for } b \in O_{B,2}, \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Let $S' = (|\psi'\rangle, \{E'_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{F'_{(y_1,y_2)b}\}_{(y_1,y_2)})$ be a perfect quantum strategy for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game. To show that the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game is a self-test for the strategy $\tilde{S}_2$, we have to prove that $\tilde{S}_2$ is a local dilation of S'.

By assumption, the game $\mathcal{G}_1$ has no perfect quantum strategy, thus we get $\langle\psi'|E'_{(x_1,x_2)a} \otimes F'_{(y_1,y_2)b}|\psi'\rangle = 0$ for all $a \in O_{A,1}$, $b \in O_{B,1}$ by Lemma 4.3. Summing over all $b \in O_{B,1}\dot{\cup}O_{B,2}$ and $a \in O_{A,1}\dot{\cup}O_{A,2}$, respectively, yields $\langle\psi'|E'_{(x_1,x_2)a} \otimes 1|\psi'\rangle = 0$, $\langle\psi'|1 \otimes F'_{(y_1,y_2)b}|\psi'\rangle = 0$ for $a \in O_{A,1}$, $b \in O_{B,1}$. Therefore, using $(E'_{(x_1,x_2)a})^2 \leq E'_{(x_1,x_2)a}$, we have

$$\begin{aligned}
\|(E'_{(x_1,x_2)a} \otimes 1)|\psi'\rangle\|^2 &= \langle\psi'|(E'_{(x_1,x_2)a})^2 \otimes 1)|\psi'\rangle \\
&\leq \langle\psi'|E'_{(x_1,x_2)a} \otimes 1)|\psi'\rangle \\
&= 0.
\end{aligned}$$

We conclude $(E'_{(x_1,x_2)a} \otimes 1)|\psi'\rangle = 0$ for all $a \in O_{A,1}$. We similarly get $(1 \otimes F'_{(y_1,y_2)b})|\psi'\rangle = 0$ for all $b \in O_{B,1}$. This especially yields that $\mathrm{supp}_A(|\psi'\rangle)$ and $\mathrm{supp}_B(|\psi'\rangle)$ are invariant under those $E'_{(x_1,x_2)a}$ and $F'_{(y_1,y_2)b}$, respectively. By Lemma 5.1 (i), we know $(E'_{(x_1,x_2)a} \otimes 1)|\psi'\rangle = (1 \otimes F'_{(y_1,x_2)a})|\psi'\rangle$ for all $x_1 \in I_{A,1}, y_1 \in I_{B,1}$, $x_2 \in I_2$ and $a \in O_2$. Therefore, Lemma 3.6 yields that $\mathrm{supp}_A(\psi')$ is invariant under $E'_{(x_1,x_2)a}$ and $\mathrm{supp}_B(\psi')$ is invariant under $F'_{(y_1,y_2)b}$ in those cases. We conclude that $\mathrm{supp}_A(|\psi'\rangle)$ and $\mathrm{supp}_B(|\psi'\rangle)$ are invariant under all $E'_{(x_1,x_2)a}$ and $F'_{(y_1,y_2)b}$, respectively. Lemma 3.7 now yields that there exists a perfect quantum strategy $S = (|\psi\rangle, \{E_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{F_{(y_1,y_2)b}\}_{(y_1,y_2)})$ such that S is a local dilation of S' and $|\psi\rangle$ has full Schmidt rank, *i.e.* $|\psi\rangle = \sum_{i=1}^d \lambda_i e_i \otimes e_i$ with $\lambda_i > 0$ for all $i \in [d]$.

Let $\varphi = \mathrm{diag}(\lambda_i)$, where λ_i are the Schmidt coefficients of $|\psi\rangle$. Once again, by Lemma 4.3, we have $\langle\psi|E_{(x_1,x_2)a} \otimes F_{(y_1,y_2)b}|\psi\rangle = 0$ for all $a \in O_{A,1}$, $b \in O_{B,1}$. Summing over all

$b \in O_{B,1} \dot\cup O_{B,2}$ and $a \in O_{A,1} \dot\cup O_{A,2}$, respectively, yields

$$\mathrm{Tr}(E_{(x_1,x_2)a}\varphi^2) = \langle\psi|E_{(x_1,x_2)a} \otimes 1|\psi\rangle = 0,$$
$$\mathrm{Tr}((\varphi^*)^2 (F_{(y_1,y_2)b})^\intercal) = \langle\psi|1 \otimes F_{(y_1,y_2)b}|\psi\rangle = 0$$

for $a \in O_{A,1}$, $b \in O_{B,1}$. Since $|\psi\rangle$ has full Schmidt rank, we obtain $E_{(x_1,x_2)a} = 0$ and $F_{(y_1,y_2)b} = 0$ for all $a \in O_{A,1}$, $b \in O_{B,1}$. Therefore, $\{E_{(x_1,x_2)a} \,|\, a \in O_{A,2}\}$ and $\{F_{(y_1,y_2)b} \,|\, b \in O_{B,2}\}$ are POVMs. For fixed $x_1 \in I_{A,1}$, $y_1 \in I_{B,1}$, we get that $(|\psi\rangle, \{E_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{F_{(y_1,y_2)b}\}_{(y_1,y_2)})$ is a perfect quantum strategy for $\mathcal{G}_2$. Because $\mathcal{G}_2$ is a self-test for S_2, we know that for fixed x_1, y_1 there exists Hilbert spaces $\mathcal{H}_{A,aux}$ and $\mathcal{H}_{B,aux}$, a state $|aux\rangle \in \mathcal{H}_{A,aux} \otimes \mathcal{H}_{B,aux}$ and isometries $U_A : \mathcal{H}_A \to \tilde{\mathcal{H}}_A \otimes \mathcal{H}_{A,aux}$, $U_B : \mathcal{H}_B \to \tilde{\mathcal{H}}_B \otimes \mathcal{H}_{B,aux}$ such that with $U := U_A \otimes U_B$ it holds

$$U|\psi\rangle = |\tilde{\psi}\rangle \otimes |aux\rangle,$$
$$U(E_{(x_1,x_2)a} \otimes 1)|\psi\rangle = [(\hat{E}_{x_2a} \otimes 1)|\tilde{\psi}\rangle] \otimes |aux\rangle,$$
$$U(1 \otimes F_{(y_1,y_2)b})|\psi\rangle = [(1 \otimes \hat{F}_{(y_1,y_2)b})|\tilde{\psi}\rangle] \otimes |aux\rangle.$$

Note that from Lemma 5.1 (iii), we have $E_{(x_1,x_2)a} = E_{(x_3,x_2)a}$ for $x_1, x_3 \in I_{A,1}$, $x_2 \in I_2$, $a \in O_2$ and $F_{(y_1,y_2)b} = F_{(y_3,y_2)b}$ for $y_1, y_3 \in I_{B,1}$, $y_2 \in I_2$, $b \in O_2$. Thus, we get

$$\begin{aligned} U(E_{(x_3,x_2)a} \otimes 1)|\psi\rangle &= U(E_{(x_1,x_2)a} \otimes 1)|\psi\rangle \\ &= [(\hat{E}_{x_2a} \otimes 1)|\tilde{\psi}\rangle] \otimes |aux\rangle \\ &= [(\tilde{E}_{(x_3,x_2)a} \otimes 1)|\tilde{\psi}\rangle] \otimes |aux\rangle \end{aligned}$$

for all $(x_3, x_2) \in I_{A,1} \times I_{A,2}$, $a \in O_2$. We similarly get $U(1 \otimes F_{(y_3,y_2)b})|\psi\rangle = [(1 \otimes \tilde{F}_{(y_3,y_2)b})|\tilde{\psi}\rangle] \otimes |aux\rangle$ for all $(y_3, y_2) \in I_{B,1} \times I_{B,2}$, $a \in O_2$. Since we know $E_{(x_3,x_2)a} = 0$ and $F_{(y_3,y_2)b} = 0$ for all $a \in O_1$, $b \in O_1$, we deduce that $\tilde{S}_2$ is a local dilation of S. Summarizing, we have that $\tilde{S}_2$ is a local dilation of S and S is a local dilation of S'. By Lemma 3.9, we get that $\tilde{S}_2$ is a local dilation of S', thus the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game is a self-test for $\tilde{S}_2$.

It remains to show that this self-test is not robust. Since there exists a sequence of quantum strategies for $\mathcal{G}_1$ whose winning probability converges to 1, for every $\delta > 0$ there is a quantum strategy $\hat{S}_\delta = (|\psi^{(\delta)}\rangle, \{\hat{E}^{(\delta)}_{x_1a}\}_{x_1}, \{\hat{F}^{(\delta)}_{y_1b}\}_{y_1}))$ with winning probability at least $1 - \delta$. By defining

$$E^{(\delta)}_{(x_1,x_2)a} := \begin{cases} \hat{E}^{(\delta)}_{x_1a} \text{ for } a \in O_{A,1}, \\ 0 \text{ otherwise}, \end{cases} \qquad F^{(\delta)}_{(y_1,y_2)b} := \begin{cases} \hat{F}^{(\delta)}_{y_1b} \text{ for } b \in O_{B,1}, \\ 0 \text{ otherwise}, \end{cases}$$

we obtain a strategy $S_\delta = (|\psi^{(\delta)}\rangle, \{E^{(\delta)}_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{F^{(\delta)}_{(y_1,y_2)b}\}_{(y_1,y_2)}))$ for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game with the same winning probability as $\hat{S}^{(\delta)}$ (*i.e.* at least $1 - \delta$). Since we have $E^{(\delta)}_{(x_1,x_2)a} = 0$ for all $a \in O_{A,2}$ and all $\delta > 0$, we see that

$$\|U(E^{(\delta)}_{(x_1,x_2)a} \otimes 1)|\psi^{(\delta)}\rangle - [(\tilde{E}_{(x_1,x_2)a} \otimes 1)|\tilde{\psi}\rangle] \otimes |aux'\rangle\| = \|[(\tilde{E}_{(x_1,x_2)a} \otimes 1)|\tilde{\psi}\rangle] \otimes |aux'\rangle\|$$

for all $a \in O_{A,2}$, all $\delta > 0$, all suitable isometries U and all auxiliary states $|aux'\rangle$. Since we have $\tilde{E}_{(x_1,x_2)a} = 0$ for all $a \in O_{A,1}$, we know that there is $a_0 \in O_{A,2}$ such that $\tilde{E}_{(x_1,x_2)a_0} \neq 0$ and thus

$$\|[(\tilde{E}_{(x_1,x_2)a_0} \otimes 1)|\tilde{\psi}\rangle] \otimes |aux'\rangle\| > \varepsilon'$$

for some $\varepsilon' > 0$. Summarizing, we found an $\varepsilon' > 0$ such that for all $\delta > 0$, we have a δ-optimal strategy $S^{(\delta)}$ such that

$$\|U(E^{(\delta)}_{(x_1,x_2)a_0} \otimes 1)|\psi^{(\delta)}\rangle - [(\tilde{E}_{(x_1,x_2)a_0} \otimes 1)|\tilde{\psi}\rangle] \otimes |aux'\rangle\| > \varepsilon'$$

for all isometries U. This shows that the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game is a non-robust self-test for $\tilde{S}_2$. $\square$

Example 5.3. By [28], there exists a linear constraint system game that has no perfect quantum strategy, but a sequence of strategies whose winning probabilities converge to 1. The proof is constructive, the linear system has 184 equations and 235 variables. Let $\mathcal{G}_1$ be this linear constraint system game. We have $|I_{A,1}| = 184$, $|I_{B,1}| = 235$ and $|O_{A,1}| = 8$, $|O_{B,1}| = 2$. We let $\mathcal{G}_2$ be the synchronous version of the magic square game (see Subsection 3.1), for which we know that it is synchronous pseudo-telepathy game. We have $|I_{A,2}| = |I_{B,2}| = 6$ and $|O_{A,2}| = |O_{B,2}| = 8$. By Corollary 3.12, we know that it self-tests the perfect quantum strategy S_2. Theorem 5.2 shows that the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game is a non-robust self-test for the strategy $\tilde{S}_2$. For this game, we have $|I_A| = 1104$, $|I_B| = 1410$ and $|O_A| = 16$, $|O_B| = 10$.

6 Games that do not self-test states

We will now construct games that do not self-test any state. We will once more use the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game. We first show that a game does not self-test any state if it has two optimal strategies using states of coprime Schmidt rank.

Lemma 6.1. *Let $\mathcal{G}$ be a nonlocal game such that $\omega^*(\mathcal{G}) > \omega(\mathcal{G})$. Let*

$$S_1 = \left(|\psi_1\rangle \in \mathcal{H}_A^{(1)} \otimes \mathcal{H}_B^{(1)}, \{E_{xa}^{(1)}\}_x, \{F_{yb}^{(1)}\}_y\right), \quad S_2 = \left(|\psi_2\rangle \in \mathcal{H}_A^{(2)} \otimes \mathcal{H}_B^{(2)}, \{E_{xa}^{(2)}\}_x, \{F_{yb}^{(2)}\}_y\right)$$

be two optimal quantum strategies. If the Schmidt ranks of $|\psi_1\rangle$ and $|\psi_2\rangle$ are coprime (i.e. $\gcd(n_1, n_2) = 1$ for Schmidt ranks n_1, n_2), then $\mathcal{G}$ does not self-test any state $|\tilde{\psi}\rangle$ of an optimal quantum strategy.

Proof. We prove this lemma by contradiction. Assume that $\mathcal{G}$ self-tests the state $|\tilde{\psi}\rangle \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B$. The state $|\tilde{\psi}\rangle$ has Schmidt rank $d > 1$ as otherwise the classical value and the quantum value of $\mathcal{G}$ coincide. Let n_1, n_2 be the Schmidt ranks of the states $|\psi_1\rangle \in \mathcal{H}_A^{(1)} \otimes \mathcal{H}_B^{(1)}$ and $|\psi_2\rangle \in \mathcal{H}_A^{(2)} \otimes \mathcal{H}_B^{(2)}$. Since $\mathcal{G}$ self-tests $|\tilde{\psi}\rangle$, we get isometries $U_A^{(i)} : \mathcal{H}_A^{(i)} \to \tilde{\mathcal{H}}_A \otimes \mathcal{H}_{A,aux}^{(i)}$, $U_B : \mathcal{H}_B^{(i)} \to \tilde{\mathcal{H}}_B \otimes \mathcal{H}_{B,aux}^{(i)}$ and states $|aux_i\rangle \in \mathcal{H}_{A,aux}^{(i)} \otimes \mathcal{H}_{B,aux}^{(i)}$, $i = 1, 2$, such that with $U_i = U_A^{(i)} \otimes U_B^{(i)}$ it holds that

$$U_1|\psi_1\rangle = |\tilde{\psi}\rangle \otimes |aux_1\rangle$$

and

$$U_2|\psi_2\rangle = |\tilde{\psi}\rangle \otimes |aux_2\rangle.$$

Note that since we have $U_i = U_A^{(i)} \otimes U_B^{(i)}$, the states $U_i|\psi_i\rangle$ have the same the Schmidt rank with respect to the bipartition $\tilde{\mathcal{H}}_A \otimes \mathcal{H}_{A,aux}^{(i)}$, $\tilde{\mathcal{H}}_B \otimes \mathcal{H}_{B,aux}^{(i)}$ as $|\psi_i\rangle$ with respect to $\mathcal{H}_A^{(i)}$, $\mathcal{H}_B^{(i)}$. Furthermore, the Schmidt rank of $|\tilde{\psi}\rangle \otimes |aux_i\rangle$ with respect to the bipartition $\tilde{\mathcal{H}}_A \otimes \mathcal{H}_{A,aux}^{(i)}$, $\tilde{\mathcal{H}}_B \otimes \mathcal{H}_{B,aux}^{(i)}$ is equal to dc_i for c_i being the Schmidt rank of $|aux_i\rangle$. Thus, comparing the Schmidt ranks with respect to $\tilde{\mathcal{H}}_A \otimes \mathcal{H}_{A,aux}^{(i)}$, $\tilde{\mathcal{H}}_B \otimes \mathcal{H}_{B,aux}^{(i)}$ in the equations above yields $n_1 = dc_1$ and $n_2 = dc_2$. This shows that $d > 1$ is a common divisor of n_1 and n_2 which contradicts our assumption that n_1 and n_2 are coprime. □

The next theorem shows that the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game has two perfect quantum strategies using states with coprime Schmidt rank if the games $\mathcal{G}_1$ and $\mathcal{G}_2$ have perfect quantum strategies with states of coprime Schmidt rank. Thus we are just left with finding two nonlocal games whose perfect quantum strategies have states with coprime Schmidt rank.

Theorem 6.2. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be pseudo-telepathy games with perfect quantum strategies*

$$S_1 = \left(|\psi_1\rangle \in \mathcal{H}_A^{(1)} \otimes \mathcal{H}_B^{(1)}, \{E_{xa}^{(1)}\}_x, \{F_{yb}^{(1)}\}_y\right), S_2 = \left(|\psi_2\rangle \in \mathcal{H}_A^{(2)} \otimes \mathcal{H}_B^{(2)}, \{E_{xa}^{(2)}\}_x, \{F_{yb}^{(2)}\}_y\right),$$

respectively. Assume that $|\psi_1\rangle$ and $|\psi_2\rangle$ have coprime Schmidt ranks. Then the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game does not self-test any state.

Proof. We get two perfect quantum strategies $\tilde{S}_1 = (|\psi_1\rangle, \{\tilde{E}^{(1)}_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{\tilde{F}^{(1)}_{(y_1,y_2)b}\}_{(y_1,y_2)})$ and $\tilde{S}_2 = (|\psi_2\rangle, \{\tilde{E}^{(2)}_{(x_1,x_2)a}\}_{(x_1,x_2)}, \{\tilde{F}^{(2)}_{(y_1,y_2)b}\}_{(y_1,y_2)})$ for the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game by defining

$$\tilde{E}^{(i)}_{(x_1,x_2)a} := \begin{cases} E_{x_i a} \text{ for } a \in O_{A,i}, \\ 0 \text{ otherwise,} \end{cases} \qquad \tilde{F}^{(i)}_{(y_1,y_2)b} := \begin{cases} F_{y_i a} \text{ for } a \in O_{B,i}, \\ 0 \text{ otherwise,} \end{cases}$$

for $i = 1, 2$. Since $|\psi_1\rangle$ and $|\psi_2\rangle$ have coprime Schmidt rank, Lemma 6.1 shows that the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game does not self-test any state. □

In the following, we will construct an explicit game fulfilling the conditions of Theorem 6.2. Note that if we look at strategies involving maximally entangled states on $\mathbb{C}^{d_i \times d_i}$, then it suffices to find two such strategies with coprime d_1 and d_2. In our case, we will have $d_1 = 3$, $d_2 = 4$. We can get a perfect quantum strategy with $d_2 = 4$ from the magic square game (see Subsection 3.1). For $d_1 = 3$, we will use an independent set game with a graph coming from a 3-dimensional weak Kochen-Specker set. This will be explained in the next subsections.

6.1 Independent set game

In this subsection we discuss the independent set game, which was introduced in [17]. We will see the connection between quantum independent sets and perfect quantum strategies for the game. For us, a graph G is always finite, simple and undirected. Thus, it consists of a finite vertex set $V(G)$ and an edge set $E(G)$ which is a set of unordered pairs of vertices.

Definition 6.3. Let G be a graph. An independent set of size t in the graph G is a set of vertices $\{v_1, \dots, v_t\} \in V(G)$ such that $(v_i, v_j) \notin E(G)$ for all $i \neq j$. The *independence number* $\alpha(G)$ denotes the size of the largest independent set in G.

For a natural number $t \in \mathbb{N}$ and a graph G, the (G, t)*-independent set game* is played with two players Alice and Bob, and a referee. Alice and Bob try to convince the referee that they know an independent set of size t of the graph G. The game is played as follows. The referee sends the players natural numbers $x_A, x_B \in [t]$ and the players answer with vertices $v_A, v_B \in V(G)$. In order to win the (G, t)-independent set game, the following conditions must be met:

(1) If $x_A = x_B$, then $v_A = v_B$,

(2) If $x_A \neq x_B$, then $v_A \neq v_B$ and $(v_A, v_B) \notin E(G)$.

The players can agree on a strategy beforehand, but cannot communicate during the game.

Note that the (G, t)-independent set game can be thought of as the $(K_t, \bar{G})$-homomorphism game, see [17].

Definition 6.4. Let G be a graph. A *quantum independent set* of size t in G is a collection $P = \{P_{xu}\}_{x \in [t], u \in V(G)}$ of projections $P_{xu} \in \mathbb{C}^{d \times d}$ such that

(i) $\sum_{u \in V(G)} P_{xu} = 1_{\mathbb{C}^{d \times d}}$ for all $x \in [t]$,

(ii) $P_{xu} P_{yv} = 0$ for $(u, v) \in E(G)$ and all $x, y \in [t]$,

(iii) $P_{xu} P_{yu} = 0$ for all $x \neq y$, $u \in V(G)$.

The quantum independence number $\alpha_q(G)$ denotes the maximum number t such that there exists a quantum independent set of size t in G.

The following lemma follows from [17, Section 2.1& 2.2].

Lemma 6.5. *Let G be a graph and let $P = \{P_{xu}\}_{x\in[t],u\in V(G)}$ be a quantum independent set of size t in G of projections $P_{xu} \in \mathbb{C}^{d\times d}$.*

(i) The strategy $(\frac{1}{\sqrt{d}}\sum_{i=1}^{d} e_i \otimes e_i, \{P_{xu}\}_x, \{(P_{yv})^{\mathsf{T}}\}_y)$ is a perfect quantum strategy of the (G,t)-independent set game.

(ii) If $t > \alpha(G)$, then the (G,t)-independent set game is a pseudo-telepathy game.

By the previous lemma, we see that quantum independent sets yield perfect quantum strategies with maximally entangled state for the independent set game. Thus, our goal is to construct a graph that has quantum independent sets with projections in odd dimension. This will be done in the next subsection using odd-dimensional Kochen-Specker sets.

6.2 Kochen-Specker sets

To get a counterexample for state self-testing, we use Kochen-Specker sets to construct an explicit independent set game having a perfect quantum strategy with a state of Schmidt rank 3. Kochen-Specker sets are sets of vectors that provide proofs of the (Bell-)Kochen-Specker theorem [3, 15]. Let $S \subseteq \mathbb{C}^n$ be a set of vectors. A function $f : S \to \{0,1\}$ is a *marking function* for S if for all orthonormal bases $B \subseteq S$, we have $\sum_{v\in B} f(v) = 1$.

Definition 6.6. Let $S \subseteq \mathbb{C}^n$ be a set of unit vectors.

(i) The set S is a *Kochen-Specker set* if there is no marking function for S.

(ii) The set S is a *weak Kochen-Specker set* [25] if for all marking functions f for S there exist orthogonal vectors $u, v \in S$ such that $f(u) = f(v) = 1$.

The above notions were generalized in [18] to sets of projections. Let $\mathcal{Q}_n \subseteq \mathbb{C}^{n\times n}$ be the set of all $n \times n$ projections. A marking function f for $S \subsetneq \mathcal{Q}_n$ is a function $f : S \to \{0,1\}$ such that for all $M \subseteq S$ with $\sum_{p\in M} p = 1_{\mathbb{C}^{n\times n}}$, we have $\sum_{p\in M} f(p) = 1$.

Definition 6.7. [18] A set $S \subsetneq \mathcal{Q}_n$ is a *projective Kochen-Specker set* if for all marking functions f for S, there exists $p, p' \in S$ for which $pp' = 0$ and $f(p) = f(p') = 1$.

Let $\{v_1, \dots, v_k\} \subseteq \mathbb{C}^n$ be a weak Kochen-Specker set. Then we get a projective Kochen-Specker set by considering the rank one projections $\{v_1v_1^\dagger, \dots, v_kv_k^\dagger\} \subseteq \mathbb{C}^{n\times n}$.

Definition 6.8. Let S be a projective Kochen-Specker set. Let $S_1 = \{p_{11}, \dots, p_{1i_1}\}, \dots, S_k = \{p_{k1}, \dots, p_{ki_k}\}$ be all subsets of S such that $\sum_{b\in[i_a]} p_{ab} = 1$. We define the graph G_S as follows: Let $V(G_S) = \{(a,b) \,|\, a \in [k], b \in [i_a]\}$, where $((a,b),(c,d)) \in E(G_S)$ if and only $p_{ab}p_{cd} = 0$. Note that G_S is the orthogonality graph of the multiset $S_1\dot{\cup}\dots\dot{\cup}S_k$.

Remark 6.9. Note that we may have $p_{ab} = p_{cd}$ for some a, b, c, d in the above definition.

The next lemma shows that given a projective Kochen-Specker set S we can construct a quantum independent set in the orthogonality graph G_S.

Lemma 6.10. *Let $S \subsetneq \mathcal{Q}_n$ be a projective Kochen-Specker set. Let $S_1 = \{p_{11}, \dots, p_{1i_1}\}, \dots, S_k = \{p_{k1}, \dots, p_{ki_k}\}$ be all subsets of S such that $\sum_{b\in[i_a]} p_{ab} = 1$ and G_S as in Definition 6.8. Then the collection $Q = \{Q_{j(a,b)}\}_{j\in[k],(a,b)\in V(G_S)}$ with $Q_{j(a,b)} := \delta_{aj}p_{ab}$ is a quantum independent set of size k of G_S.*

Proof. We check conditions (i)-(iii) of Definition 6.4. For (i), we compute

$$\sum_{(a,b)\in V(G_S)} Q_{j(a,b)} = \sum_{b\in[i_j]} p_{jb} = 1$$

by definition of Q and choice of S_j. Condition (ii) is fulfilled since for $((a,b),(c,d)) \in E(G_S)$, we know $Q_{j(a,b)}Q_{l(c,d)} = \delta_{aj}\delta_{cl}p_{ab}p_{cd} = 0$, since $p_{ab}p_{cd} = 0$ by definition of G_S. For (iii), we have $Q_{j(a,b)}Q_{l(a,b)} = \delta_{aj}\delta_{al}p_{ab} = 0$ for $j \neq l$, since $\delta_{aj}\delta_{al} = 0$ for $j \neq l$. ☐

From previous work it is already known that the size of the quantum independent set Q from Lemma 6.10 is larger than the independence number of the orthogonality graph.

Theorem 6.11. *[26, Theorem 3.4.4] Let S be a projective Kochen-Specker set and let $S_1, \dots, S_k$ and G_S be as in Definition 6.8. Then $k > \alpha(G_S)$.*

Combining Lemma 6.5, Lemma 6.10 and Theorem 6.11, we have the following corollary.

Corollary 6.12. *Let $S \subsetneq \mathcal{Q}_n$ be a projective Kochen-Specker set and G_S as in Definition 6.8. Then the (G_S, k)-independent set game is a pseudo-telepathy game with a perfect quantum strategy using the maximally entangled state $|\psi_n\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i$.*

We will now give an example of a $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game that does not self-test any state. Note that a 3-dimensional weak Kochen-Specker set yields a projective Kochen-Specker set $S \subsetneq \mathcal{Q}_3$ and thus the (G_S, k)-independent set game has a quantum strategy using the state $|\psi_3\rangle$.

Example 6.13. Consider Peres' 3-dimensional weak Kochen-Specker set S_1 with 33 vectors forming 16 bases [24]. We get a projective Kochen-Specker set by considering the associated rank-1 projections. The orthogonality graph G_{S_1} has 48 vertices. Furthermore, we get $\alpha(G_{S_1}) = 15$, $\alpha_q(G_{S_1}) \geq 16$ by using Sage [32] to compute $\alpha(G_{S_i})$ and Lemma 6.10 for a lower bound on $\alpha_q(G_{S_i})$. By Corollary 6.12, we know that the $(G_{S_1}, 16)$-independent set game has a perfect quantum strategy with a state of Schmidt rank 3. We let $\mathcal{G}_1$ be the $(G_{S_1}, 16)$-independent set game and let $\mathcal{G}_2$ be the magic square game considered in Subsection 3.1. We know that $\mathcal{G}_2$ has a perfect quantum strategy with a state of Schmidt rank 4 from Theorem 3.11. Therefore, the $(\mathcal{G}_1 \vee \mathcal{G}_2)$-game does not self-test any state by Theorem 6.2. In this case, we have $|I_A| = |I_B| = 48$, $|O_A| = |O_B| = 52$.

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The role of classical randomness in self-testing

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1 Introduction

The concept of *self-testing* was first introduced by Mayers and Yao [4]. A statement of self-testing is in its essence a statement that for a given nonlocal game, the strategy that optimally plays this game is unique up to local isometries.

There are already a significant number of self-testing results that has been shown previously. Examples of games for which a self-testing statement has been shown includes the CHSH game [1, 5, 7, 9] and the magic square game [10].

In this paper, to build upon the concept of self-testing we add multiple qualifiers to describe under which assumptions the self-testing property has been proved. We specifically augment the definition by specifying the following two assumptions that could have been made:

- Whether we allow the state of the arbitrary strategy to be mixed, or if we restrict to pure states.
- Whether we allow the measurements to be POVMs, or if we restrict to projective measurements.

We will mostly focus on how self-testing with pure states and self-testing with mixed states are related. Our main result shows that these two are equivalent as long as the self-tested state has full Schmidt rank. One reason for why such a result is of interest arises from what mixed states can be used to represent. They both be used to model the players having access to classical randomness, but also that there are properties the players do not know about the state. We are however able to show that access to mixed states does not create new nonequivalent strategies for most practical purposes.

To gain a bit of intuition into how we prove the equivalence between pure and mixed self-testing, recall that we can use the process of purification to convert mixed states to pure states that act identically, except for an additional purification space. The general idea of the proof is that we for an arbitrary strategy using a mixed state considers the purification of this strategy. By combining two local isometries, whose existence is guaranteed by the self-testing statement, we can create a strategy that is equivalent to the reference strategy in multiple characteristics. This includes the outcome of the game, the measurements of the strategy and the space it acts on. Finally, we prove that the reference strategy is unique in having these characteristics.

1.1 Structure of the paper

In this this section we have briefly introduced the main topic of this paper, namely pure vs. mixed self-testing. In section 2, we formally define the concept of self-testing. In section 3, we build the different parts that are needed to prove the central theorem, which we finally prove in section 3.1.

2 Preliminaries

2.1 Notation

Unless stated otherwise, we assume all Hilbert spaces to be finite-dimensional. For a quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, we can consider its Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^{k} \alpha_i |e_i\rangle |f_i\rangle \tag{1}$$

where $\alpha_i > 0$ and both $\{|e_i\rangle\}_{i=1}^k \subseteq \mathcal{H}_A$ and $\{|f_i\rangle\}_{i=1}^k \subseteq \mathcal{H}_B$ are orthonormal sets. Note that (1) only includes terms with positive Schmidt coefficients. We refer to number k as the Schmidt rank of state $|\psi\rangle$ and if $k = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$ then we say that $|\psi\rangle$ has full Schmidt rank. We also define $\operatorname{supp}_A |\psi\rangle := \operatorname{span}\{|e_1\rangle, \dots |e_k\rangle\} \subseteq \mathcal{H}_A$. Similarly, we define $\operatorname{supp}_B |\psi\rangle := \operatorname{span}\{|f_1\rangle, \dots |f_k\rangle\} \subseteq \mathcal{H}_B$ and $\operatorname{supp} |\psi\rangle := \operatorname{span}\{|e_1, f_1\rangle, \dots, |e_k, f_k\rangle\} \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$.

For a set $S \subseteq \mathcal{H}$, we write $\dim S = \dim \operatorname{span} S$. Furthermore, we will often need to discuss particular subsets of Hilbert spaces; namely all those corresponding to normalised states. For this, we use the notation $\operatorname{St}(\mathcal{H}) := \{|\psi\rangle \in \mathcal{H} \mid \||\psi\rangle\| = 1\} \subseteq \mathcal{H}$. Observe that $\operatorname{span}(\operatorname{St}(\mathcal{H})) = \mathcal{H}$, and furthermore that for every $0 \neq |\psi\rangle \in \mathcal{H}$, it holds that $\frac{|\psi\rangle}{\||\psi\rangle\|} \in \operatorname{St}(\mathcal{H})$.

We will in this work be using both POVM and PVM measurements. We will be working from the following definition.

Definition 2.1 (POVM)**.** *A POVM measurement is a set of operators $\{E_i\}_i$ called POVM elements such that*

$$\sum_i E_i = \mathbb{1}$$

Furthermore, if for all $i \neq j$, $E_i E_j = 0$, then we call the measurement for a PVM measurement.

2.2 Self-testing

Definition 2.2 (Nonlocal game)**.** *A nonlocal game G is a tuple $(\mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{B}, \pi, \mathcal{V})$ of a probability distribution of the questions p and a scoring function $\mathcal{V}$ defined over*

$$\pi : \mathcal{S} \times \mathcal{T} \to [0, 1]$$

$$\mathcal{V} : \mathcal{A} \times \mathcal{B} \times \mathcal{S} \times \mathcal{T} \to \{0, 1\}$$

where $\mathcal{S}$ and $\mathcal{T}$ are sets of questions for Alice and Bob respectively, and $\mathcal{A}$ and $\mathcal{B}$ are the sets of answers for Alice and Bob respectively.

A nonlocal game G is a cooperative game played by two players, which we typically call Alice and Bob, and a referee. The game is played the following way: Before the game starts, Alice and Bob agree on some strategy, including possibly sharing a quantum state. Then during the game the players are not allowed to communicate, but they are allowed to perform measurements on their own parts of the shared state. Alice and Bob each receives a question s and t respectively from predetermined sets of questions $\mathcal{S}$ and $\mathcal{T}$, determined by the probability distribution π. Each of them then answers their question, such that Alice answers some a from a set of possible answers $\mathcal{A}$ and Bob an answer b from a set of possible answers $\mathcal{B}$. They then win if they answered correctly according to the scoring function $\mathcal{V}$.

Definition 2.3 (Strategy). *A (tensor-product) quantum strategy for a game $G = (\mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{B}, \pi, \mathcal{V})$ is a tuple*

$$S = (\rho_{AB}, \{A_{sa}\}_{s\in\mathcal{S},a\in\mathcal{A}}, \{B_{tb}\}_{t\in\mathcal{T},b\in\mathcal{B}}), \tag{2}$$

consisting of a shared density operator $\rho_{AB} \in \mathrm{End}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\mathcal{H}_A$ is the state space of Alice and $\mathcal{H}_B$ is the state space of Bob. Furthermore, for each $s \in \mathcal{S}$, the set $\{A_{sa}\}_{a\in\mathcal{A}} \subset \mathrm{End}(\mathcal{H}_A)$ is a POVM on $\mathcal{H}_A$, and for each $t \in \mathcal{T}$, the set $\{B_{tb}\}_{b\in\mathcal{B}} \subset \mathrm{End}(\mathcal{H}_B)$ is a POVM on $\mathcal{H}_B$. We furthermore identify the following special cases (which are not mutually exclusive):

- *If $\rho_{AB} = |\psi\rangle\langle\psi|$ for some pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, we refer to the strategy as* pure*. In this case, we may replace ρ_{AB} with $|\psi\rangle$ in (2).*
- *If all POVM elements, A_{sa} and B_{tb}, are projectors, then we refer to the strategy as* projective*.*

We will often suppress the subscript of the measurements.We will write $\{A_{sa}\}_{s\in\mathcal{S},a\in\mathcal{A}}$ as $\{A_{sa}\}$ when from the context it is clear that the set is indexed over the sets $\mathcal{S}$ and $\mathcal{A}$. We will use analogous notation for Bob's measurements $\{B_{tb}\}$.

It is easy to compute the winning probability when using a particular strategy S for game G. Sometimes, it will be useful to collect all the information regarding game G and the employed measurements in a single operator W.

Lemma 2.4. *Let $G = (\mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{B}, \pi, \mathcal{V})$ be a nonlocal game and $S = (\rho_{AB}, \{A_{sa}\}, \{B_{tb}\})$ a strategy for G. Define W as*

$$W := \sum_{a,b,s,t} \pi(s,t)\mathcal{V}(a,b|s,t)(A_{sa} \otimes B_{tb}).$$

Then the probability of winning the game $\omega(S, G)$ using strategy S can be found as

$$\omega(S, G) = \mathrm{tr}(W\rho_{AB})$$

Proof. We show this by simply rewriting $\omega(S, G)$ using the definition of W and linearity of the trace,

$$\begin{aligned}
\omega(S,G) &= \sum_{s,t} \pi(s,t) \sum_{a,b} \mathcal{V}(a,b|s,t)\,\mathrm{tr}((A_{st} \otimes B_{tb})\rho_{AB}) \\
&= \mathrm{tr}\left(\left(\sum_{s,t}\sum_{a,b} \pi(s,t)\mathcal{V}(a,b|s,t)(A_{st} \otimes B_{tb})\right)\rho_{AB}\right) \\
&= \mathrm{tr}(W\rho)
\end{aligned}$$

□

Furthermore, we define $\omega_q(G) := \sup_S \omega(S, G)$, where the supremum is taken over all (tensor-product) quantum strategies S which are compatible with G. We refer to $\omega_q(G)$ as the *optimal* or *maximal* quantum value of G. Note that in general it may not be possible to obtain $\omega_q(G)$ with any tensor-product strategy [6].

The main topic of this work, *self-testing*, asks whether the state and the measurements used in an optimal quantum strategy for some game are unique, up to local isometries. The reason this is useful is that it enables us to make conclusions about the strategy used, using only an observed probability of winning. Consider a situation where we make Alice and Bob play a nonlocal game that self-test an optimal quantum strategy. If in this setting we observe that Alice and Bob win with the same probability as some optimal quantum strategy, then we can guarantee that Alice and Bob must have used a strategy that is equivalent to the optimal one up to local isometries.

Usually in the literature quantum strategies are assumed to use pure states (and often also projective measurements), and so common formulations of self-testing reflect this fact. We will be working with a definition of self-testing that is very similar to the one presented by [8, Definition 2], though augmented with additional qualifiers that are relevant to the presentation of our results.

To start discussing the concept of self-testing, we first introduce the concept of local dilations. Informally, if $\tilde{S}$ is a local dilation of S, then it should be understood that we through local isometries are able to extract $\tilde{S}$ from S.

Definition 2.5 (Local dilation)**.** *Given two strategies*

$$S = (\rho_{AB} \in \mathrm{End}(\mathcal{H}_A \otimes \mathcal{H}_B), \{A_{sa}\}_{s\in\mathcal{S},a\in\mathcal{A}}, \{B_{tb}\}_{t\in\mathcal{T},b\in\mathcal{B}}) \text{ and}$$
$$\tilde{S} = (\,|\tilde{\psi}\rangle \in \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\tilde{B}}, \{\tilde{A}_{sa}\}_{s\in\mathcal{S},a\in\mathcal{A}}, \{\tilde{B}_{tb}\}_{t\in\mathcal{T},b\in\mathcal{B}})$$

we say that $\tilde{S}$ is a local dilation of S *and write* $S \hookrightarrow \tilde{S}$ *if for any purification* $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_P$ *of* ρ_{AB} *there exist spaces* $\mathcal{H}_{\hat{A}}, \mathcal{H}_{\hat{B}}$*, a local isometry* $U = U_A \otimes U_B$*, with* $U_A : \mathcal{H}_A \to \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\hat{A}}$*,* $U_B : \mathcal{H}_B \to \mathcal{H}_{\tilde{B}} \otimes \mathcal{H}_{\hat{B}}$ *and a state* $|\mathrm{aux}\rangle \in \mathrm{End}(\mathcal{H}_{\hat{A}} \otimes \mathcal{H}_{\hat{B}} \otimes \mathcal{H}_P)$ *such that for all* s, t, a, b *we have*

$$(U \otimes \mathbb{1}_P)(A_{sa} \otimes B_{tb} \otimes \mathbb{1}_P)\,|\psi\rangle = (\tilde{A}_{sa} \otimes \tilde{B}_{tb})\,|\tilde{\psi}\rangle \otimes |\mathrm{aux}\rangle\,. \tag{3}$$

In case we want to name the local isometry and the auxiliary state from (3)*, we write* $S \xhookrightarrow{U,|\mathrm{aux}\rangle} \tilde{S}$*. We will use this notation only when* ρ_{AB} *is pure to avoid ambiguity.*

As an additional note, observe that local dilations are transitive. That is if $S_X \hookrightarrow S_Y$ and $S_Y \hookrightarrow S_Z$, then $S_X \hookrightarrow S_Z$.

If the state ρ_{AB} in strategy S is pure, we do not need to concern ourselves with purifications of ρ_{AB} in the above definition. More precisely, we have that $S \hookrightarrow \tilde{S}$ is equivalent to us being able to find local isometry $U = U_A \otimes U_B$ such that

$$U(A_{sa} \otimes B_{tb})\,|\psi\rangle = (\tilde{A}_{sa} \otimes \tilde{B}_{tb})\,|\tilde{\psi}\rangle \otimes |\mathrm{aux}\rangle\,.$$

Intuitively, self-testing allows us to say that *any* optimal strategy S for a game G can be mapped to a chosen canonical strategy $\tilde{S}$. Different sources impose different restrictions on the strategy S regarding the type of state (pure, mixed) or the type of measurements

(PVM, POVM) it is allowed to employ. This results in a priori different definitions of self-testing.

Definition 2.6 (Self-testing)**.** *Let $t \in \{$pure PVM, pure POVM, mixed PVM, mixed POVM$\}$ and let $\tilde{S}$ be a pure strategy. We say that a nonlocal game G t-self-tests a (reference) strategy $\tilde{S}$ if $S \hookrightarrow \tilde{S}$ for every optimal t-strategy S for game G.*

Rather than considering local dilation condition (3) in a "vector form" one could instead consider a "matrix-form" condition (see Appendix C in [3]). We show that these definitions indeed are equivalent in appendix A.

At this point, it may seem entirely appropriate to distinguish between mixed and pure self-testing, as any mixed self-test is clearly also a pure self-test. At first glance it appears obvious that mixed self-testing is more general than pure self-testing. However, it is not at all obvious how to show this, and it turns out that in practical cases, the two concepts are identical. We show this in Theorem 3.8 and Corollary 3.9.

3 From pure self-testing to mixed

Given an optimal strategy $S = (|\psi\rangle, \{A_{sa}\}, \{B_{tb}\})$ for G, we want to understand the set, Q, of all quantum states that yield optimal quantum strategies when measured using the measurements from S. The following lemma characterizes this set Q in terms of the W operator.

Lemma 3.1. *Let $S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{A_{sa}\}, \{B_{tb}\})$ be any optimal strategy for some game $G = (\mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{B}, \pi, \mathcal{V})$. Define the set*

$$Q := \{|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B : \omega(|\phi\rangle, (\{A_{sa}\}, \{B_{tb}\}), G) = \omega_q(G)\}.$$

Define the operator

$$W := \sum_{a,b,s,t} \pi(s,t)\mathcal{V}(a,b|s,t)(A_{sa} \otimes B_{tb}).$$

Then $Q = \mathrm{St}(V_{\max(\sigma(W))})$, where V_λ denotes the λ-eigenspace of W and $\sigma(W)$ is the spectrum of W.

Proof. Let $\lambda_{\max} := \max(\sigma(W))$. A key thing to observe is that $\langle\psi'| W |\psi'\rangle$ is the success probability of strategy $S' = (\{A_{sa}\}, \{B_{tb}\}, |\psi'\rangle)$ (see Lemma 2.4). It now follows that

$$\langle\psi'| W |\psi'\rangle = \omega_q(G) \iff S' \text{ is an optimal strategy for } G \iff |\psi'\rangle \in Q. \quad (4)$$

Since the unit vectors $|\psi\rangle$ that maximize expression $\langle\psi'| W |\psi'\rangle$ are precisely the quantum states in $\mathrm{St}(V_{\lambda_{\max}})$, the desired statement follows. □

Our main use for this lemma is that it for pure self-tested strategies allows us to characterise Q in relation to a vector space. In particular, $Q = \mathrm{St}(\mathrm{span}(Q))$.

While the above result already characterises the state space of optimal states for a given set of measurement operators, it turns out that in some cases, it is actually possible to say more; namely that if all those states have full Schmidt rank, then that space necessarily has dimension 1. The statement and proof of this result due to Cubitt, Montanaro and Winter [2], though it is restated her for the sake of completeness.

Lemma 3.2 ([2]). *Let S be a subspace of the bipartite space $\mathcal{H}_A \otimes \mathcal{H}_B$, where* $\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$. *If every nonzero state in S has Schmidt rank d, then* $\dim S = 1$.

Proof. Consider any two-dimensional subspace spanned by unit vectors $|\varphi\rangle, |\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. We want to show that at least one superposition $|\phi_x\rangle = |\varphi\rangle + x|\psi\rangle$ has Schmidt rank less than d. The crucial observation is that we can arrange the coefficients of a state vector $|\phi\rangle$ in the computational basis $\{|i\rangle |j\rangle\}_{i,j=1,\dots,d}$, into a $d \times d$ matrix $M(\phi)$, and that the Schmidt rank of the state vector equals the linear rank of the associated matrix. In other words, the statement that $|\psi_x\rangle$ has Schmidt rank less than d is captured by the vanishing of the determinant $\det M(\phi_x)$. But the latter is a non-constant polynomial in x of degree d. Hence, it must have a root in the complex field, and the corresponding $|\phi_x\rangle$ has Schmidt rank $d-1$ or less. $\square$

Apart from this, we will need another result characterising the Schmidt rank of optimal states; namely that they all have minimal Schmidt rank. This will be useful for proving that the conditions of the above result holds.

Lemma 3.3. *Let G be nonlocal game that pure self-tests the strategy $\tilde{S} = (|\tilde{\psi}\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\})$. Then $|\tilde{\psi}\rangle$ has minimum Schmidt rank across the states of all optimal pure strategies.*

Proof. Let t be the Schmidt rank of $|\tilde{\psi}\rangle$. For contradiction, assume that there exists a strategy $S = (|\phi\rangle, \{A_{sa}\}, \{B_{tb}\})$ with Schmidt rank $s < t$. By the definition of pure self-testing, there exist local isometries V_A, V_B and a state $|\text{aux}\rangle$ such that

$$(V_A \otimes V_B)|\phi\rangle = |\tilde{\psi}\rangle \otimes |\text{aux}\rangle$$

Since local isometries preserve Schimdt rank, the Schmidt rank (with respect to Alice/Bob partition) of $|\tilde{\psi}\rangle \otimes |\text{aux}\rangle$ is s. This is a contradiction, since tensoring with a state $|\text{aux}\rangle$ cannot decrease the Schmidt rank of $|\tilde{\psi}\rangle$. $\square$

Next, we make a simple observation that, in a certain restricted sense, to "undo" partial traces.

Lemma 3.4. *Let $\rho \in \mathrm{End}(\mathcal{H}_S \otimes \mathcal{H}_T)$ be a density matrix. If $\mathrm{tr}_T\, \rho_{ST} = |\psi\rangle\langle\psi|$ for some pure state $|\psi\rangle \in \mathcal{H}_S$ then $\rho_{ST} = |\psi\rangle\langle\psi| \otimes \sigma_T$, for some state σ_T.*

Proof. Consider the spectral decomposition, $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, and observe that each $|\phi_i\rangle$ has some Schmidt decomposition,

$$|\phi_i\rangle = \sum_j \alpha_{ij} |e_{ij}\rangle_S |f_{ij}\rangle_T .$$

This implies $\mathrm{tr}_T |\phi_i\rangle\langle\phi_i| = \sum_j \alpha_{ij}^2 |e_{ij}\rangle\langle e_{ij}|$, and so we have by our assumption that

$$|\psi\rangle\langle\psi| = \mathrm{tr}_T(\rho_{ST}) = \mathrm{tr}_T \sum_i p_i |\phi_i\rangle\langle\phi_i|_{ST} = \sum_{i,j} p_i \alpha_{ij}^2 |e_{ij}\rangle\langle e_{ij}| .$$

Now, we can use that $\||\psi\rangle\| = 1$ along with the above result to obtain that

$$1 = \langle\psi|\psi\rangle \langle\psi|\psi\rangle = \langle\psi| \sum_{i,j} p_i \alpha_{ij}^2 |e_{ij}\rangle\langle e_{ij}| |\psi\rangle = \sum_{i,j} p_i \alpha_{ij}^2 |\langle\psi|e_{ij}\rangle|^2 .$$

Now Cauchy-Schwarz gives us $|\langle\psi|e_{ij}\rangle|^2 \leq 1$, and since $\sum_j \alpha_{ij}^2 = 1$ for all i, and $\sum_i p_i = 1$, we must therefore have $|\langle\psi|e_{ij}\rangle|^2 = 1$, implying $|\psi\rangle = |e_{ij}\rangle$ for all i,j. This implies $|\phi_i\rangle\langle\phi_i| = |\psi\rangle\langle\psi| \otimes |\eta_i\rangle\langle\eta_i|_T$ for some state $|\eta_i\rangle \in \mathcal{H}_T$, and therefore $\rho = |\psi\rangle\langle\psi| \otimes \sigma_T$. $\square$

Our argument will rely upon certain states having full Schmidt rank. This can be ensured, for example, by restricting the appropriate spaces to the support of our state. A quantum strategy also involves two sets of operators. The restrictions of these operators is only well-defined if these split into a direct sum with the support of the state being one of the summands. This is difficult to ensure in general, but if G self-tests a strategy $\tilde{S} = \Big(|\tilde{\psi}\rangle \in \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\tilde{B}}, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\}\Big)$, where $|\tilde{\psi}\rangle$ has full Schmidt rank, then all optimal strategies of G can be assumed to have full Schmidt rank. To show that this, we prove a slightly more general statement:

Lemma 3.5. *Suppose* $S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{A_{sa}\}, \{B_{tb}\})$ *and* $\tilde{S} = \Big(|\tilde{\psi}\rangle \in \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\tilde{B}}, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\}\Big)$ *are two quantum strategies such that* $S \hookrightarrow \tilde{S}$ *Then it holds for all* s *and* a *that*

$$\operatorname{supp}_A \left((A_{sa} \otimes \mathbb{1}_B) |\psi\rangle \right) \subseteq \operatorname{supp}_A(|\psi\rangle) \iff \operatorname{supp}_{\tilde{A}} \Big((\tilde{A}_{sa} \otimes \mathbb{1}_{\tilde{B}}) |\tilde{\psi}\rangle \Big) \subseteq \operatorname{supp}_{\tilde{A}}(|\tilde{\psi}\rangle) \quad (5)$$

and similarly for all t *and* b

$$\operatorname{supp}_B \left((\mathbb{1}_A \otimes B_{tb}) |\psi\rangle \right) \subseteq \operatorname{supp}_B(|\psi\rangle) \iff \operatorname{supp}_{\tilde{B}} \Big((\mathbb{1}_{\tilde{A}} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle \Big) \subseteq \operatorname{supp}_{\tilde{B}}(|\tilde{\psi}\rangle). \quad (6)$$

Proof. Assume that $S \xhookrightarrow{V_A \otimes V_B, |\mathrm{aux}\rangle} \tilde{S}$ holds. Then for all s, t, a, b it holds that

$$(V_A \otimes V_B)(A_{sa} \otimes B_{tb}) |\psi\rangle = \Big((\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle \Big) \otimes |\mathrm{aux}\rangle \,. \quad (7)$$

Sum (7) over b to get

$$(V_A \otimes V_B)(A_{sa} \otimes \mathbb{1}_B) |\psi\rangle = \Big((\tilde{A}_{sa} \otimes \mathbb{1}_{\tilde{B}}) |\tilde{\psi}\rangle \Big) \otimes |\mathrm{aux}\rangle \,. \quad (8)$$

Consider the the left-hand-side statement of (5), namely

$$\operatorname{supp}_A \left((A_{sa} \otimes \mathbb{1}_B) |\psi\rangle \right) \subseteq \operatorname{supp}_A(|\psi\rangle). \quad (9)$$

Observe that applying isometry V_A to both sides of (9) gives the statement

$$\operatorname{supp}_{\tilde{A},\hat{A}}((\tilde{A}_{sa} \otimes \mathbb{1}_{\tilde{B}}) |\tilde{\psi}\rangle) \otimes |\mathrm{aux}\rangle) \subseteq \operatorname{supp}_{\tilde{A},\hat{A}}(|\tilde{\psi}\rangle \otimes |\mathrm{aux}\rangle), \quad (10)$$

where we have used (8) and the fact that $U_X \operatorname{supp}_X(|\phi\rangle_{XY}) = \operatorname{supp}_X((U_X \otimes \mathbb{1}_Y) |\phi\rangle_{XY}) = \operatorname{supp}_X((U_X \otimes U_Y) |\phi\rangle_{XY})$ for any isometries U_X, U_Y. Hence, (9) holds if and only if (10) holds. From the definition of supp_X, note that $\operatorname{supp}_{X_1,X_2} \left(|\phi_1\rangle_{X_1Y_1} \otimes |\phi_2\rangle_{X_2Y_2} \right) = \left(\operatorname{supp}_{X_1}(|\phi_1\rangle\right) \otimes \left(\operatorname{supp}_{X_2}(|\phi_2\rangle\right)$. Using this property of supp, we conclude that (10) is equivalent to the statement that

$$\operatorname{supp}_{\tilde{A}} \Big((\tilde{A}_{sa} \otimes \mathbb{1}_{\tilde{B}}) |\tilde{\psi}\rangle \Big) \subseteq \operatorname{supp}_{\tilde{A}}(|\tilde{\psi}\rangle). \quad (11)$$

We have now shown that (9) $\iff$ (10) $\iff$ (11) hence proving (5). The equivalence in (6) follows from a similar argument. ☐

We will now state a simple lemma that will allow us to say when two measurements have *essentially* the same POVM elements. Suppose that $(A \otimes \mathbb{1}_B) |\psi\rangle = (A' \otimes \mathbb{1}_B) |\psi\rangle$ for some operators A, A'. It would be convenient to conclude that this necessarily implies that $A = A'$. Yet, of course, this is not always the case. For example, if $|\psi\rangle$ does not have full Schmidt rank, then A and A' can act differently on the space orthogonal to $\mathrm{supp}(|\psi\rangle)$. The following lemma implies that when it comes to $\mathrm{supp}_A(|\psi\rangle)$, the two operators must act the same. In particular, if $|\psi\rangle$ has full Schmidt rank, then knowing that $A \otimes \mathbb{1}_B$ and $A' \otimes \mathbb{1}_B$ act the same on a single vector, allows us to conclude that $A_1 = A_2$ and hence act the same on *all* vectors.

Lemma 3.6. *Suppose X, Y are two finite sets, that we have collections $\{A_x\}_{x\in X}, \{A'_x\}_{x\in X} \subseteq \mathcal{B}(\mathcal{H}_A)$, and $\{B_y\}_{y\in Y}, \{B'_y\}_{y\in Y} \subseteq \mathcal{B}(\mathcal{H}_B)$, for some separable Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Furthermore, suppose $\sum_{x\in X} A_x = I_A = \sum_{x\in X} A'_x$, and symmetrically $\sum_{y\in Y} B_y = I_B = \sum_{y\in Y} B'_y$. Then, for any state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ which satisfies*

$$A_x \otimes B_y |\psi\rangle = A'_x \otimes B'_y |\psi\rangle \qquad \textit{for all } x \in X, \textit{ and } y \in Y,$$

it holds that

$$\begin{aligned} A_x |\phi\rangle_A &= A'_x |\phi\rangle_A, \quad \textit{for all } |\phi\rangle_A \in \mathrm{supp}_A(|\psi\rangle), \quad \textit{and} \\ B_y |\phi\rangle_B &= B'_y |\phi\rangle_B, \quad \textit{for all } |\phi\rangle_B \in \mathrm{supp}_B(|\psi\rangle). \end{aligned}$$

Proof. Initially observe that

$$A_x \otimes I_B |\psi\rangle = \sum_{y\in Y} A_x \otimes B_y |\psi\rangle = \sum_{y\in Y} A'_x \otimes B'_y |\psi\rangle = A'_x \otimes I_B |\psi\rangle .$$

Now, Schmidt decompose $|\psi\rangle = \sum_i \lambda_i |v_i\rangle |w_i\rangle$, which is possible as the spaces are separable, and observe that

$$\sum_i \lambda_i (A_x |v_i\rangle) |w_i\rangle = A_x \otimes I_B = A'_x \otimes I_B = \sum_i \lambda_i (A'_x |v_i\rangle) |w_i\rangle ,$$

which by orthogonality of $\{|w_i\rangle\}_i$ implies that $A_x |v_i\rangle = A'_x |v_i\rangle$ for all i. However, $\{|v_i\rangle\}_i$ constitutes a basis for $\mathrm{supp}_A(|\psi\rangle)$, and so we easily obtain that $A_x |\phi\rangle_A = A'_x |\phi\rangle_A$, for all $|\phi\rangle_A \in \mathrm{supp}_A(|\psi\rangle)$. The result for the operators on $\mathcal{H}_B$ follows similarly. $\square$

3.1 Main Result

While we are technically already able to show that pure self-testing implies mixed self-testing, at least if the pure self-testing result in question uses a state of full Schmidt rank in the reference strategy, we still choose to postpone that slightly. Instead, we show how to extract a pure self-testing result with a state of full Schmidt rank from one which does not necessarily satisfy this criterion, and this can be used to extend our main result to apply in more generality.

Lemma 3.7. *Let G be a nonlocal game that pure POVM self-tests a strategy $\tilde{S} = (|\tilde{\psi}\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\})$, $|\tilde{\psi}\rangle \in \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\tilde{B}}$. Then there exists a POVM strategy $S' = (|\psi'\rangle, \{A'_{sa}\}, \{B'_{tb}\})$, such that $\tilde{S} \hookrightarrow S'$ and $|\psi'\rangle$ has full Schmidt rank.*

Proof. We show this by constructing such a strategy. Consider the Schmidt decomposition of $|\tilde{\psi}\rangle$,

$$|\tilde{\psi}\rangle = \sum_{i=0}^{d-1} \lambda_i |e_i\rangle |f_i\rangle$$

We will then consider the coisometries

$$U_A = \sum_{i=0}^{d-1} |i\rangle\langle e_i|_A \qquad \text{and} \qquad U_B = \sum_{i=0}^{d-1} |i\rangle\langle f_i|_B$$

onto the space $\mathbb{C}^d$. We will from this define POVMs $\{A'_{sa}\}$ and $\{B'_{tb}\}$, and a new state by

$$\begin{aligned} A'_{sa} &= U_A \tilde{A}_{sa} U_A^* \\ B'_{tb} &= U_B \tilde{B}_{tb} U_B^* \\ |\psi'\rangle &= U_A \otimes U_B |\tilde{\psi}\rangle \end{aligned}$$

We first show that this is indeed a well-defined POVM strategy. For any s, the set $\{A'_{sa}\}_{a\in\mathcal{A}}$ is indeed a POVM since

$$\begin{aligned} \sum_{a\in\mathcal{A}} A'_{sa} &= \sum_{a\in\mathcal{A}} U_A \tilde{A}_{sa} U_A^* \\ &= U_A \left(\sum_{a\in\mathcal{A}} \tilde{A}_{sa} \right) U_A^* \\ &= U_A \mathbb{1}_{\mathcal{H}_{\tilde{A}}} U_A^* = \mathbb{1}_d \end{aligned}$$

using that U_A is a coisometry. This is symmetric for U_B and $\{B'_{tb}\}_{b\in\mathcal{B}}$.

We now show that the strategy S' exhibits the same behaviour as $\tilde{S}$.We first note that

$$U_A^* U_A = \sum_{i=0}^{d-1} |e_i\rangle\langle e_i| \qquad \text{and} \qquad U_B^* U_B = \sum_{i=0}^{d-1} |f_i\rangle\langle f_i|,$$

and therefore

$$(U_A^* U_A) \otimes \mathbb{1}_B |\psi\rangle = |\psi\rangle, \tag{12}$$

$$\mathbb{1}_A \otimes (U_B^* U_B) |\psi\rangle = |\psi\rangle, \tag{13}$$

since we act with identity on the support of the state in each case. Fixing a pair of questions (s, t) we see that the probability of attaining answers (a, b) by using this strategy is

$$\begin{aligned} \langle \psi' | A'_{sa} \otimes B'_{tb} | \psi' \rangle &= \langle \psi | (U_A \otimes U_B)^* \left((U_A \tilde{A}_{sa} U_A^*) \otimes (U_B \tilde{B}_{tb} U_B^*) \right) (U_A \otimes U_B) |\psi\rangle \\ &= \langle \psi | (U_A^* U_A \tilde{A}_{sa} U_A^* U_A) \otimes (U_B^* U_B \tilde{B}_{tb} U_B^* U_B) |\psi\rangle \\ &= \langle \psi | \tilde{A}_{sa} \otimes \tilde{B}_{tb} | \psi \rangle \end{aligned}$$

using (12) and (13) for the last equality. This shows $\tilde{S}$ and S both have the same behaviour, and thus in particular they have the same winning probability. This means that by pure POVM self-testing of $\tilde{S}$, $S' \hookrightarrow \tilde{S}$. As $|\psi'\rangle$ has full Schmidt rank, $\mathrm{supp}_A(A'_{sa} \otimes \mathbb{1}_B |\psi'\rangle) \subseteq \mathrm{supp}_A(|\psi'\rangle)$ and symmetrically for Bob's system. By Lemma 3.5 we therefore have

$$\mathrm{supp}_{\tilde{A}}(\tilde{A}_{sa} \otimes \mathbb{1}_{\tilde{B}} |\tilde{\psi}\rangle) \subseteq \mathrm{supp}_{\tilde{A}}(|\tilde{\psi}\rangle). \tag{14}$$

We can now define a new map. We extend $\{|e_i\rangle\}_i$ to be a basis for $\mathcal{H}_{\tilde{A}}$ such that the first d vectors corresponds to a basis for the support of $|\tilde{\psi}\rangle$ on $\mathcal{H}_{\tilde{A}}$, and do symmetrically for

$\{|f_i\rangle\}_i$ and $\mathcal{H}_{\tilde{B}}$. We then define the local isometry $W_A \otimes W_B$ by:

$$W_A = \sum_{i=0}^{\dim \mathcal{H}_{\tilde{A}}-1} (|i \bmod d\rangle \, |\lfloor i/d \rfloor\rangle) \langle e_i|$$
$$W_B = \sum_{i=0}^{\dim \mathcal{H}_{\tilde{B}}-1} (|i \bmod d\rangle \, |\lfloor i/d \rfloor\rangle) \langle f_i|$$

Observe that for a basis vector $|e_i\rangle \in \mathrm{supp}_A(|\tilde{\psi}\rangle)$, it holds that $W_A |e_i\rangle = |i\rangle |0\rangle = (U_A |e_i\rangle) |0\rangle$. Recalling (14), we can apply this and a similar relation between W_B and U_B to obtain:

$$\begin{aligned} W_A \otimes W_B(\tilde{A}_{st} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle &= \Big[U_A \otimes U_B(\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle\Big] \otimes |00\rangle \\ &= \Big[(U_A \otimes U_B)(\tilde{A}_{sa} \otimes \tilde{B}_{tb})(U_A \otimes U_B)^*(U_A \otimes U_B) |\tilde{\psi}\rangle\Big] \otimes |00\rangle \\ &= A'_{sa} \otimes B'_{tb} |\psi'\rangle \otimes |00\rangle \end{aligned}$$

by using (12) and (13) for the last equality. This implies $\tilde{S} \hookrightarrow S'$. □

With this in mind, we will now move on to state our main theorem, namely that under the weak assumption that the state in the reference strategy has full Schmidt rank, pure self-testing implies mixed self-testing. Note that we do not specify whether the strategies are additionally assumed projective or POVM – the result holds in both cases, and our proof preserves projectiveness of the strategies.

Theorem 3.8. *Let $t \in \{PVM, POVM\}$. Let G be a nonlocal game that pure t-self-tests a quantum strategy $\tilde{S} = \Big(|\tilde{\psi}\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\}\Big)$. If $|\tilde{\psi}\rangle$ has full Schmidt rank, then G mixed t-self-tests $\tilde{S}$.*

Proof. Let $S = (\rho_{AB}, \{A_{sa}\}, \{B_{tb}\})$, with $\rho_{AB} \in \mathrm{End}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a strategy that achieves the optimal value of G. Consider a purification of ρ, $|\psi\rangle_{ABP} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_P$.

Observe that we performed a purification on the state from the original strategy, and therefore we have a slight leeway in which of the provers we give access to the purification space. Thus, we define two new strategies S_A and S_B, with the subscript indicating which prover has access to the purification space.

$$\begin{aligned} S_A &:= (|\psi\rangle_{ABP}, \{A_{sa} \otimes \mathbb{1}_P\}, \{B_{tb}\}) \\ S_B &:= (|\psi\rangle_{ABP}, \{A_{sa}\}, \{B_{tb} \otimes \mathbb{1}_P\}) \end{aligned}$$

From being purifications, these strategies exhibit the same behaviour as our original one, and in particular they are both optimal.

Since each of S_A and S_B achieves the optimal quantum value of G, by pure t-self-testing $S_A \xhookrightarrow{V_{AP} \otimes V_B, |\mathrm{aux}_1\rangle} \tilde{S}$ and $S_B \xhookrightarrow{W_A \otimes W_{BP}, |\mathrm{aux}_2\rangle} \tilde{S}$.

Now consider any $|\phi\rangle_{\tilde{A},\breve{A}} \in \mathrm{supp}_{\tilde{A},\breve{A}}((W_A \otimes \mathbb{1}_{BP}) |\psi\rangle_{ABP})$, and any

$|\phi\rangle_{\tilde{B},\hat{B}} \in \mathrm{supp}_{\tilde{B},\hat{B}}(\mathbb{1}_{AP} \otimes V_B) |\psi\rangle_{ABP})$. By Lemma 3.6 it then holds that

$$W_A A_{sa} W_A^* |\phi\rangle_{\tilde{A},\breve{A}} = (\tilde{A}_{sa} \otimes \mathbb{1}_{\breve{A}}) |\phi\rangle_{\tilde{A},\breve{A}}, \tag{15}$$
$$V_B B_{tb} V_B^* |\phi\rangle_{\tilde{B},\hat{B}} = (\tilde{B}_{tb} \otimes \mathbb{1}_{\hat{B}}) |\phi\rangle_{\tilde{B},\hat{B}}. \tag{16}$$

For ease of notation, we combine the isometries associated with the different strategies; namely we set $X := W_A \otimes V_B$. Observe that $X : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\tilde{B}} \otimes \mathcal{H}_{\check{A}} \otimes \mathcal{H}_{\hat{B}}$.

In other words, X mixes the isometries from the t-self-testing of the two pure strategies such that the isometries appearing in X for each of the provers is the isometries which does not include the purification space. From this, X does not act on the purification space.

Now we have

$$\begin{aligned}(X \otimes \mathbb{1}_P)(A_{sa} \otimes B_{tb} \otimes \mathbb{1}_P)\left|\psi\right\rangle_{ABP} &= (X(A_{sa} \otimes B_{tb})X^* \otimes \mathbb{1}_P)(X \otimes \mathbb{1}_P)\left|\psi\right\rangle_{ABP} \\ &= (\tilde{A}_{sa} \otimes \tilde{B}_{tb} \otimes \mathbb{1}_{\check{A}\hat{B}P})(X \otimes \mathbb{1}_P)\left|\psi\right\rangle_{ABP} \qquad (17)\end{aligned}$$

where we used (15) and (16) in the last line. From this we are motivated to define

$$\rho'_{\tilde{A}\tilde{B}} := \operatorname{tr}_{\check{A}\hat{B}P}\left((X \otimes \mathbb{1}_P)\left|\psi\right\rangle\!\left\langle\psi\right|_{ABP}(X \otimes \mathbb{1}_P)^*\right). \qquad (18)$$

Observe that while $\rho'_{\tilde{A}\tilde{B}}$ depends upon the original mixed state and the isometries constructed above from the strategy, it does not depend upon any single combination of elements s, t, a, b. Thus, we will now consider the strategy $S' = (\rho'_{\tilde{A}\tilde{B}}, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\})$. For any s, t, a, b, it holds that

$$\begin{aligned}&\operatorname{tr}\Big((\tilde{A}_{sa} \otimes \tilde{B}_{tb})\rho'_{\tilde{A}\tilde{B}}\Big) &(19)\\ &= \operatorname{tr}((X \otimes \mathbb{1}_P)(A_{sa} \otimes B_{tb} \otimes \mathbb{1}_P)\left|\psi\right\rangle\!\left\langle\psi\right|_{ABP}(X \otimes \mathbb{1}_P)^*) &(20)\\ &= \operatorname{tr}((A_{sa} \otimes B_{tb} \otimes \mathbb{1}_P)\left|\psi\right\rangle\!\left\langle\psi\right|_{ABP}) &(21)\end{aligned}$$

by using that X is an isometry and cyclicity of trace in the last equality. This implies S' exhibits the same behaviour as S. We can therefore by optimality of S, conclude that S' is also optimal. Writing up the spectral decomposition of $\rho'_{\tilde{A}\tilde{B}}$,

$$\rho'_{\tilde{A}\tilde{B}} = \sum_{i=0}^{\operatorname{rank}\rho'_{\tilde{A}\tilde{B}}-1} p_i \left|\psi'_i\right\rangle\!\left\langle\psi'_i\right|_{\tilde{A}\tilde{B}},$$

we find that since S' has optimal probability of winning G using the measurements $\{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\}$, then so must each of the $\left|\psi'_i\right\rangle$.

We now consider the set Q of states that achieves the optimal value of G using the measurements $\{\tilde{A}_{sa}\}$ and $\{\tilde{B}_{tb}\}$. This set is defined as

$$Q := \{\left|\phi\right\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B : \omega(\left|\phi\right\rangle, (\{A_{sa}\}, \{B_{tb}\}), G) = \omega_q(G)\}.$$

and claim that $\dim(\operatorname{span}(Q)) = 1$. Observe that by Lemma 3.1, $\operatorname{St}(\operatorname{span}(Q)) = Q$, and so Q contains all the normalised non-zero vectors of $\operatorname{span} Q$. Note that $|\tilde{\psi}\rangle$ has full Schmidt rank. By Lemma 3.3, every state in Q must then have full Schmidt rank. From this, we observe that every non-zero vector in $\operatorname{span} Q$ then also must have full Schmidt rank. By Lemma 3.2, since $\operatorname{span} Q$ is a vector space where all elements have full Schmidt rank, it has dimension 1, and so it is spanned by $|\tilde{\psi}\rangle$.

This implies that $\left|\psi'_i\right\rangle\!\left\langle\psi'_i\right| = |\tilde{\psi}\rangle\!\langle\tilde{\psi}|$ for all $i \in \{0, \dots, \operatorname{rank}\rho'_{\tilde{A}\tilde{B}}-1\}$. Thus, $\rho'_{\tilde{A}\tilde{B}} = |\tilde{\psi}\rangle\!\langle\tilde{\psi}|_{\tilde{A}\tilde{B}}$. By Lemma 3.4, this gives $(X \otimes \mathbb{1}_P)\left|\psi\right\rangle\!\left\langle\psi\right|_{ABP}(X \otimes \mathbb{1}_P)^* = |\tilde{\psi}\rangle\!\langle\tilde{\psi}| \otimes \left|\text{aux}'\right\rangle\!\left\langle\text{aux}'\right|_{\check{A}\hat{B}P}$, for some state $\left|\text{aux}'\right\rangle\!\left\langle\text{aux}'\right|_{\check{A}\hat{B}P}$. It then follows that

$$(X \otimes \mathbb{1}_P)\left|\psi\right\rangle_{ABP} = |\tilde{\psi}\rangle \otimes \left|\text{aux}'\right\rangle_{\check{A}\hat{B}P} \qquad (22)$$

where we have absorbed a potential global phase into $|\mathrm{aux}'\rangle$. Substitute (22) into (17) and we get

$$(X \otimes \mathbb{1})(A_{sa} \otimes B_{tb}) |\psi\rangle_{ABP} = \left[(\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle_{\tilde{A}\tilde{B}}\right] \otimes \left|\mathrm{aux}'\right\rangle_{\hat{A}\hat{B}P},$$

which shows that G mixed self-tests $\tilde{S}$. □

While Theorem 3.8 does have a condition of full Schmidt rank, we can in fact get rid of this condition if we look at games that pure POVM self-test strategies. The reason we can get rid of the restriction is that the statement of pure POVM self-testing a strategy enables us to essentially restrict a strategy to the support of its state through local isometries, which is not necessarily possible when pure PVM self-testing a strategy.

Corollary 3.9. *If G is a nonlocal game which pure POVM self-tests a strategy $\tilde{S} = \left(|\tilde{\psi}\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\}\right)$, then G mixed POVM self-tests $\tilde{S}$.*

Proof. Apply Lemma 3.7 to obtain the strategy $S' = (|\psi'\rangle, \{A'_{sa}\}, \{B'_{tb}\})$ such that $\tilde{S} \hookrightarrow S'$. By transitivity of local dilations, G pure POVM self-tests S'. As $|\psi'\rangle$ has full Schmidt rank Theorem 3.8 implies G mixed POVM self-tests S'. Finally, since G pure self-tests $\tilde{S}$, by definition of pure self-testing $S' \hookrightarrow \tilde{S}$. Together with transitivity of local dilations this implies G mixed POVM self-tests $\tilde{S}$. □

As a final remark, while we have mostly focused on self-testing nonlocal games, the same result (see Theorem B.4) holds for self-testing of extremal (in the quantum set) probability distributions. We elaborate on this in appendix B.

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A Equivalence of alternative definitions of self-testing

Rather than considering local dilation condition (3) in a "vector form" one could instead consider a "matrix-form" condition (see Appendix C in [3]):

Definition A.1 (Local dilation (alternative))**.** *Given two strategies*

$$S = (\rho_{AB} \in \mathrm{End}(\mathcal{H}_A \otimes \mathcal{H}_B), \{A_{sa}\}_{s\in\mathcal{S},a\in\mathcal{A}}, \{B_{tb}\}_{t\in\mathcal{T},b\in\mathcal{B}}) \text{ and}$$
$$\tilde{S} = (|\tilde{\psi}\rangle \in \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\tilde{B}}, \{\tilde{A}_{sa}\}_{s\in\mathcal{S},a\in\mathcal{A}}, \{\tilde{B}_{tb}\}_{t\in\mathcal{T},b\in\mathcal{B}})$$

we write $S \hookrightarrow_1 \tilde{S}$ if there exist spaces $\mathcal{H}_{\hat{A}}, \mathcal{H}_{\hat{B}}$, a local isometry $U = U_A \otimes U_B$, with $U_A : \mathcal{H}_A \to \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\hat{A}}$, $U_B : \mathcal{H}_B \to \mathcal{H}_{\tilde{B}} \otimes \mathcal{H}_{\hat{B}}$ and a state $|\mathrm{aux}\rangle \in \mathrm{End}(\mathcal{H}_{\hat{A}} \otimes \mathcal{H}_{\hat{B}})$ such that for all s, t, a, b we have

$$U(A_{sa} \otimes B_{tb})\rho_{AB}U^* = (\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle\langle\tilde{\psi}| \otimes \sigma_{\mathrm{aux}}. \tag{23}$$

This kind of definition of local dilation can also be used to construct an alternative definition for self-testing. We will in this appendix prove that the two definitions for local dilations in fact are equivalent.

Lemma A.2. *Let $|\phi\rangle \in \mathcal{H}_S$ and $|\psi\rangle \in \mathcal{H}_T$ be states and let $\{S_i\}_i \subset \mathrm{End}(\mathcal{H}_S)$ and $\{T_i\}_i \subseteq \mathrm{End}(\mathcal{H}_T)$ be POVMs. Let $U : \mathcal{H}_S \to \mathcal{H}_T$ be an isometry. If*

$$US_i |\phi\rangle\langle\phi| U^* = (T_i |\psi\rangle\langle\psi|) \otimes |\mathrm{aux}\rangle\langle\mathrm{aux}| \tag{24}$$

for all i, then there exists an state $|\mathrm{aux}'\rangle$, such that

$$US_i |\phi\rangle = (T_i |\psi\rangle) |\mathrm{aux}'\rangle \tag{25}$$

and

$$|\mathrm{aux}\rangle\langle\mathrm{aux}| = |\mathrm{aux}'\rangle\langle\mathrm{aux}'| \tag{26}$$

Proof. Sum (24) over i to get

$$U |\phi\rangle\langle\phi| U^* = |\psi\rangle\langle\psi| \otimes |\mathrm{aux}\rangle\langle\mathrm{aux}| \tag{27}$$

using the fact that $\{S_i\}_i$ and $\{T_i\}_i$ are POVMs. Both of these operators are have rank one, and therefore have a single non-zero eigenvalue. Observe that the eigenspace with the non-zero eigenvalue of the left-hand side is spanned by $U\,|\phi\rangle$ and the eigenspace with the non-zero eigenvalue of the right-hand side is spanned by $|\psi\rangle\,|\text{aux}\rangle$. Furthermore, these two spaces are equivalent. We can therefore conclude that

$$U\,|\phi\rangle = e^{-i\gamma}\,|\psi\rangle\,|\text{aux}\rangle\,, \tag{28}$$

for some $\gamma \in \mathbb{R}$. They are in other words they are equal up to global phase. This phase change can simply be absorbed into $|\text{aux}\rangle$, creating a new state $|\text{aux}'\rangle = e^{i\gamma}\,|\text{aux}\rangle$. Right multiply (24) with (28) and we get the desired equation of

$$VS_i\,|\phi\rangle = (T_i\,|\psi\rangle)\,\big|\text{aux}'\big\rangle\,, \tag{29}$$

proving (25). Finally, (26) follows directly from the definition of $|\text{aux}'\rangle$. □

We are also going to use the following observation, proven in [3] as Observation C.1:

Lemma A.3 ([3])**.** *Let $\rho^0_{ST}, \rho^1_{ST} \in \mathrm{End}(\mathcal{H}_S \otimes \mathcal{H}_T)$ be positive semidefinite operators. If*

$$\rho^0_{ST} + \rho^1_{ST} = |\psi\rangle\!\langle\psi|_S \otimes \sigma_T$$

for some $|\psi\rangle\!\langle\psi|_S \in \mathrm{End}(\mathcal{H}_S)$ and $\sigma_T \in \mathrm{End}(\mathcal{H}_T)$, then

$$\rho^i_{ST} = |\psi\rangle\!\langle\psi|_S \otimes \sigma^i_T$$

for $i \in \{0, 1\}$ and some $\sigma^i_T \in \mathrm{End}(\mathcal{H}_T)$

The next lemma shows that the "matrix-form" condition (23) is in fact equivalent to the vector-form condition (3). The take-away here is that it does not matter which of the two variants of local dilation we base our self-testing definition on.

Lemma A.4. *Let S and $\tilde{S}$ be two strategies. Then*

$$S \hookrightarrow \tilde{S} \quad \Longleftrightarrow \quad S \hookrightarrow_1 \tilde{S}$$

Proof. Let $\tilde{S} = (\,|\tilde{\psi}\rangle\,, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\})$ and $S = (\rho_{AB}, \{A_{sa}\}, \{B_{tb}\})$.

We start by showing $S \hookrightarrow \tilde{S} \Rightarrow S \hookrightarrow_1 \tilde{S}$. $S \hookrightarrow \tilde{S}$ implies that for any purification $|\psi\rangle_{ABP}$ of ρ_{AB} there exists local isometries $U = U_A \otimes U_B$ and a state $|\text{aux}\rangle$ such that

$$(U \otimes \mathbb{1}_P)(A_{sa} \otimes B_{tb} \otimes \mathbb{1}_P)\,|\psi\rangle_{ABP} = (\tilde{A}_{sa} \otimes \tilde{B}_{tb})\,|\tilde{\psi}\rangle_{\tilde{A}\tilde{B}} \otimes |\text{aux}\rangle_{\hat{A}\hat{B}}\,. \tag{30}$$

If we sum over a, b, using $\{A_{sa} \otimes B_{tb}\}_{a,b}$ is a POVM, we get

$$(U \otimes \mathbb{1}_P)\,|\psi\rangle_{ABP} = \,|\tilde{\psi}\rangle_{\tilde{A}\tilde{B}} \otimes |\text{aux}\rangle_{\hat{A}\hat{B}}\,. \tag{31}$$

We can now combine (30) and (31) through an outer product to get

$$(U\otimes\mathbb{1})(A_{sa}\otimes B_{tb}\otimes\mathbb{1}_P)\,|\psi\rangle\!\langle\psi|_{ABP}\,(U\otimes\mathbb{1}_P)^* = (\tilde{A}_{sa}\otimes\tilde{B}_{tb})\,|\tilde{\psi}\rangle\!\langle\tilde{\psi}|_{\tilde{A}\tilde{B}}\otimes|\text{aux}\rangle\!\langle\text{aux}|_{\hat{A}\hat{B}P}\,. \tag{32}$$

Finally, by using that $|\psi\rangle\!\langle\psi|_{ABP}$ is a purification of ρ_{AB} and tracing out the purification space, we get

$$U(A_{sa} \otimes B_{tb})\rho_{AB}U^* = (\tilde{A}_{sa} \otimes \tilde{B}_{tb})\,|\tilde{\psi}\rangle\!\langle\tilde{\psi}|_{\tilde{A}\tilde{B}} \otimes \mathrm{tr}_P(|\text{aux}\rangle\!\langle\text{aux}|_{\hat{A}\hat{B}P}). \tag{33}$$

which shows $S \hookrightarrow_1 \tilde{S}$.

Secondly, we will show $S \hookrightarrow_1 \tilde{S} \Rightarrow S \hookrightarrow \tilde{S}$.

$S \hookrightarrow_1 \tilde{S}$ implies there exist an local isometry $U = U_A \otimes U_B$ such that

$$U(A_{sa} \otimes B_{tb})\rho_{AB}U^* = (\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle\langle\tilde{\psi}| \otimes \sigma_{\text{aux}}. \tag{34}$$

for some state $\sigma_{\text{aux}} \in \text{End}(\mathcal{H}_{\hat{A}} \otimes \mathcal{H}_{\hat{B}})$.

If we sum both sides over a, b, using $\{A_{sa} \otimes B_{tb}\}_{a,b}$ is a POVM, then we get

$$U\rho_{AB}U^* = |\tilde{\psi}\rangle\langle\tilde{\psi}| \otimes \sigma_{\text{aux}}. \tag{35}$$

Now, consider any purification $|\psi\rangle_{ABP}$ of ρ_{AB}. Consider the Schmidt decomposition of $|\psi\rangle_{ABP}$ over the spaces $(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\mathcal{H}_P$. This gives

$$|\psi\rangle_{ABP} = \sum_i \lambda_i |\alpha_i\rangle_{AB} |\beta_i\rangle_P . \tag{36}$$

If we trace out the purification space of this state, using that it indeed is a purification, we get that

$$\rho_{AB} = \sum_i \lambda_i^2 |\alpha_i\rangle\langle\alpha_i| . \tag{37}$$

If we now substitute (37) into (35) we arrive at

$$\sum_i \lambda_i^2 U |\alpha_i\rangle\langle\alpha_i| U^* = |\tilde{\psi}\rangle\langle\tilde{\psi}| \otimes \sigma_{\text{aux}}. \tag{38}$$

Observe that this is a sum of positive semidefinite operators, and therefore by Lemma A.3 we have

$$U |\alpha_i\rangle\langle\alpha_i| U^* = |\tilde{\psi}\rangle\langle\tilde{\psi}| \otimes |\text{aux}_i\rangle\langle\text{aux}_i| , \tag{39}$$

for some pure state $|\text{aux}_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Observe that if we substitute (39) into (40), we get

$$|\tilde{\psi}\rangle\langle\tilde{\psi}| \otimes \sigma_{\text{aux}} = |\tilde{\psi}\rangle\langle\tilde{\psi}| \otimes \left(\sum_i \lambda_i^2 |\text{aux}_i\rangle\langle\text{aux}_i| \right),$$

implying that $\sigma_{\text{aux}} = \sum_i \lambda_i^2 |\text{aux}_i\rangle\langle\text{aux}_i|$.

By Lemma A.2, we can conclude that there exists a state $|\text{aux}'_i\rangle$ such that

$$U |\alpha_i\rangle = |\tilde{\psi}\rangle |\text{aux}'_i\rangle . \tag{40}$$

and $|\text{aux}'_i\rangle\langle\text{aux}'_i| = |\text{aux}_i\rangle\langle\text{aux}_i|$.

We now fix i and right-multiply (34) with (40), after simplifying both sides, this gives

$$U(A_{sa} \otimes B_{tb}) |\alpha_i\rangle = \left((\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle \right) |\text{aux}'_i\rangle \tag{41}$$

where we have used the fact that $|\alpha_i\rangle$ and $|\alpha_j\rangle$ are orthogonal and $|\text{aux}'_i\rangle$ and $|\text{aux}'_j\rangle$ are othogonal when $i \neq j$. Finally, apply $U \otimes \mathbb{1}$ to $|\psi\rangle_{ABP}$ as

$$\begin{aligned}(U \otimes \mathbb{1})(A_{sa} \otimes B_{tb} \otimes \mathbb{1}_P) |\psi\rangle_{ABP} &= \sum_i \lambda_i U(A_{sa} \otimes B_{tb}) |\alpha_i\rangle_{AB} |\beta_i\rangle_P \\ &= (\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle \otimes \left(\sum_i \lambda_i |\text{aux}'_i\rangle |\beta_i\rangle_P \right)\end{aligned}$$

where the first equality used the Schmidt decomposition of $|\psi\rangle_{ABP}$ and the second equality substituted in (41). Setting

$$|\text{aux}'\rangle = \left(\sum_i \lambda_i |\text{aux}'_i\rangle |\beta_i\rangle_P\right)$$

implies $S \hookrightarrow \tilde{S}$. □

B Self-testing probability distributions

In the main paper we have mostly focused on self-testing from the perspective of non-local games. A different way to define self-testing is from the perspective of probability distributions. In this setting, instead of having a game self-test a strategy $\tilde{S}$, we have a probability distribution p that self-tests a strategy $\tilde{S}$. We will denote the probability distribution induced by the strategy S as p_S.

We say that the strategy $S = (|\psi\rangle, \{A_{sa}\}, \{B_{tb}\})$ induces the probability distribution p if for all a, b, s, t

$$p(a, b|s, t) = \text{tr}((A_{sa} \otimes B_{tb}) |\psi\rangle\langle\psi|).$$

This is the definition of self-testing that is used in [8], though again augmented with the same qualifiers as in Definition 2.6.

Definition B.1 (Self-Testing (Probability Distribution))**.** *Let $t \in \{$pure PVM, pure POVM, mixed PVM, mixed POVM$\}$ and let $\tilde{S}$ be a pure strategy. We say that a probability distribution p t-self-tests a (reference) strategy $\tilde{S}$ if $S \hookrightarrow \tilde{S}$ for every t-strategy S where $p_S(a, b|s, t) = p(a, b|s, t)$.*

Our results for self-testings games does not translate directly into this setting, as there are two of significant points where they differ. The first one is that for it is not clear whether the set

$$Q := \{|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B : p_{(|\phi\rangle,(\{A_{sa}\},\{B_{tb}\})} = p_S\}.$$

is equal to $\text{St}(V)$ for some vector space V. In the game setting, the analogous set is guaranteed to be the normalised states of a vector space by Lemma 3.1. This means we sadly do not directly get the nice properties of having Q be closely related to a vector space. This is however a central part of the proof of Theorem 3.8.

The second major difference is related how we define strategies that can be self-tested. In Definition 2.6 we require the strategies to be optimal, which gives us a extremality conditions on the game value. In the probability distribution case, we would instead have the probability distribution be extremal in the set of realisable probability distributions. While we could define this into Definition B.1, it would not line up very well with the definition of self-testing in other works. Furthermore, with games we only can only self-test optimal strategies, except in some degenerate cases. This property is however not clear with self-testing using probability distributions.

There have been done some work on the topic of whether it is only possible to self-test extremal probability distributions. It has been shown that if some probability distribution

p mixed self-tests a strategy $\tilde{S}$ then p is extremal in the set of realisable probability distributions [3]. This result is however not easily extendible to pure self-testing, which would be needed in the following results.

For the following Lemma we will use Lemma 3.3. While this Lemma is not stated in terms of probability distributions, the proof itself does not use any properties inherent to non-local games. The same proof would therefore also holds for probability distribution based self-testing.

Lemma B.2. *Let $t \in \{PVM, POVM\}$. Let $\tilde{S} = (|\tilde{\psi}\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\})$ and $S = (|\psi\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\})$. Furthermore let $p_{\tilde{S}}(a,b|s,t) = p_S(a,b|s,t)$ for all a,b,s,t. If $|\tilde{\psi}\rangle$ and $|\psi\rangle$ has full Schmidt rank and $p_{\tilde{S}}(a,b|s,t)$ pure t-self-tests $\tilde{S}$ then there exists a local unitary $U = U_A \otimes U_B$ such that for all s,t,a,b*

$$U(A_{sa} \otimes B_{tb})|\psi\rangle = (A_{sa} \otimes B_{tb})|\tilde{\psi}\rangle$$

Proof. By t-self-testing, $S \xhookrightarrow{V,|\mathrm{aux}\rangle} \tilde{S}$. This implies

$$V(A_{sa} \otimes B_{tb})|\psi\rangle = (A_{sa} \otimes B_{tb})|\tilde{\psi}\rangle_{AB} \otimes |\mathrm{aux}\rangle_{\hat{A}\hat{B}}. \tag{42}$$

Since $|\psi\rangle$ and $|\tilde{\psi}\rangle$ both have full Schmidt rank and live in the same space, they must have the same Schmidt rank. This means $|\mathrm{aux}\rangle$ has Schmidt rank 1, and so is a product state, $|\mathrm{aux}\rangle_{\hat{A}\hat{B}} = |\mathrm{aux}\rangle_{\hat{A}} \otimes |\mathrm{aux}\rangle_{\hat{B}}$.

Consider the Schmidt decomposition of $|\psi\rangle = \sum_i \lambda_i |\alpha_i\rangle |\beta_i\rangle$. If we now trace out $\mathcal{H}_{\tilde{B}} \otimes \mathcal{H}_{\hat{B}}$ from (42), and sum over a,b we get

$$\sum_i \lambda_i V_A |\alpha_i\rangle\langle\alpha_i| V_A^* = \mathrm{tr}_{\tilde{B}}(|\tilde{\psi}\rangle\langle\tilde{\psi}|_{\tilde{A}\tilde{B}}) \otimes |\mathrm{aux}\rangle\langle\mathrm{aux}|_{\hat{A}}. \tag{43}$$

By Lemma A.3 we can conclude that

$$V_A |\alpha_i\rangle\langle\alpha_i| V_A^* = |\phi_i\rangle\langle\phi_i|_{\tilde{A}} \otimes |\mathrm{aux}\rangle\langle\mathrm{aux}|_{\hat{A}}$$

for some state $|\phi_i\rangle \in \mathcal{H}_{\tilde{A}}$ for all i. This implies that $V_A|\alpha_i\rangle = |\phi_i\rangle_{\tilde{A}} \otimes |\mathrm{aux}\rangle_{\hat{A}}$ where we have absorbed a potential global phase into $|\phi_i\rangle$. Since $|\psi\rangle$ has full Schmidt rank, $\mathrm{span}(\{\alpha_i\}_i) = \mathcal{H}_A$. This implies for all $|v\rangle \in \mathcal{H}_A$, $V_A|v\rangle = |v'\rangle |\mathrm{aux}\rangle_{\hat{A}}$. This directly implies that $(\mathbb{1}_{\tilde{A}} \otimes \langle\mathrm{aux}|_{\hat{A}})U_A$ is unitary. A similar argument for $\mathcal{H}_B$ shows that $(\mathbb{1}_{\tilde{B}} \otimes \langle\mathrm{aux}|_{\hat{B}})U_B$ is unitary, completing the proof. □

What we are going show in this appendix is the following lemma, which can be used to prove Theorem 3.8 under additional constraints in the setting of probability distributions.

Lemma B.3. *Let $t \in \{PVM, POVM\}$. Let $\tilde{S} = (|\tilde{\psi}\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\})$ and $S = (|\psi\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\})$ be two strategies such that $p_{\tilde{S}} = p_S =: p$. If $|\tilde{\psi}\rangle$ has full Schmidt rank and $p_{\tilde{S}}$ pure t-self-tests $\tilde{S}$ then $|\tilde{\psi}\rangle\langle\tilde{\psi}| = |\psi\rangle\langle\psi|$.*

Proof. Since p pure t-tests $\tilde{S}$, $S \hookrightarrow \tilde{S}$. By Lemma 3.3, $|\psi\rangle$ has at least as large Schmidt rank as $|\tilde{\psi}\rangle$. The fact that they live on the same space and $|\tilde{\psi}\rangle$ has full Schmidt rank implies $|\psi\rangle$ also has full Schmidt rank. By Lemma B.2, there exists a local isometry $U = U_A \otimes U_B$ such that

$$U(\tilde{A}_{sa} \otimes \tilde{B}_{tb})|\psi\rangle = (A_{sa} \otimes B_{tb})|\tilde{\psi}\rangle \tag{44}$$

Since $|\tilde{\psi}\rangle$ has full Schmidt rank, $\mathrm{supp}_{\tilde{A}}(|\psi\rangle) = \mathcal{H}_{\tilde{A}}$. By Lemma 3.6, this implies that for all $|v\rangle \in \mathcal{H}_{\tilde{A}}$, $U_A\tilde{A}_{sa}U_A^* |v\rangle = \tilde{A}_{sa} |v\rangle$ for all s, a. This directly gives that $U_A\tilde{A}_{sa}U_A^* = \tilde{A}_{sa}$, and so $[\tilde{A}_{sa}, U_A] = 0$ for all s, a. By Lemma 3.3 the state $|\tilde{\psi}\rangle$ has minimum Schmidt rank across all states that can give rise to p using some local measurements. Hence, strategy $\tilde{S}$ has the minimum (local) dimension among all strategies that give rise to p. It now follows that the matrix algebra generated by all the A_{sa} is irreducible and thus $\langle \tilde{A}_{sa}\rangle_{sa} = \mathbb{M}_{d\times d}$, where $d = \dim(\mathcal{H}_{\tilde{A}})$. The only matrix that commutes with all $d \times d$ matrices is $U_A = c\mathbb{1}$. After applying a similar argument to Bob, we obtain that $(U_A \otimes U_B) |\psi\rangle = |\tilde{\psi}\rangle$, where both U_A and U_B are proportional to identity. This establishes the desired statement. □

From this Lemma, we can state the primary result of this appendix. We are not going to fully prove this, since the proof is essentially the same as Theorem 3.8. We are instead going to state the places where they differ.

Theorem B.4. *Let $t \in \{PVM, POVM\}$. Let p be a probability distribution that pure t-self-tests a quantum strategy $\tilde{S} = \left(|\tilde{\psi}\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\}\right)$. If $|\tilde{\psi}\rangle$ has full Schmidt rank and p is extremal in the set of realisable probability distributions, then p mixed t-self-tests $\tilde{S}$.*

Sketch. The proof is very similar to the one in Theorem 3.8, and so we will only mention were they differ. They only differ in two places.

The first one is when the optimality of self-testing games is used to claim in the expression

$$\rho'_{\tilde{A}\tilde{B}} = \sum_{i=0}^{\mathrm{rank}\,\rho'_{\tilde{A}\tilde{B}}-1} p_i |\psi_i'\rangle\langle\psi_i'|_{\tilde{A}\tilde{B}},$$

that each of $|\psi_i'\rangle_{\tilde{A}\tilde{B}}$ must have optimal probability of winning G. Instead observe that by the extremality of p, each of $|\psi_i'\rangle_{\tilde{A}\tilde{B}}$ must have the same probability distribution using the same measurements as $\rho'_{\tilde{A}\tilde{B}}$.

The second one is when the proof claims that $|\psi_i'\rangle\langle\psi_i'| = |\tilde{\psi}\rangle\langle\tilde{\psi}|$ using Lemma 3.1. Here instead use Lemma B.3 to make the same conclusion.

This proves proves that p mixed t-self-tests $\tilde{S}$. □

As a final remark, it is possible to use the method presented in Lemma 3.7 to remove the need for full Schmidt rank for pure POVM-self-testing, like it is done in Corollary 3.9. The arguments are essentially the same.

C Measurement-based Self-Testing

In this appendix we will look at a slightly different definition for self-testing. The idea is that we focus more on the measurements instead of the state.

Definition C.1 (Local dilation (Measurements))**.** *Given two pure full-rank strategies*

$$S = (|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \{A_{sa}\}_{s\in\mathcal{S},a\in\mathcal{A}}, \{B_{tb}\}_{t\in\mathcal{T},b\in\mathcal{B}}) \text{ and}$$
$$\tilde{S} = (|\tilde{\psi}\rangle \in \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\tilde{B}}, \{\tilde{A}_{sa}\}_{s\in\mathcal{S},a\in\mathcal{A}}, \{\tilde{B}_{tb}\}_{t\in\mathcal{T},b\in\mathcal{B}})$$

we write $S \hookrightarrow_2 \tilde{S}$ *if there exist a local unitary* $U = U_A \otimes U_B$, *with* $U_A : \mathcal{H}_A \to \mathcal{H}_{\tilde{A}} \otimes \mathcal{H}_{\hat{A}}$, $U_B : \mathcal{H}_B \to \mathcal{H}_{\tilde{B}} \otimes \mathcal{H}_{\hat{B}}$ *such that for all* s, t, a, b *we have*

$$U |\psi\rangle = |\tilde{\psi}\rangle \otimes |\mathrm{aux}\rangle, \tag{45}$$

$$U_A A_{sa} U_A^* = \tilde{A}_{sa} \otimes \mathbb{1}_{\hat{A}}, \tag{46}$$

$$U_B B_{tb} U_B^* = \tilde{B}_{tb} \otimes \mathbb{1}_{\hat{B}}. \tag{47}$$

In case we want to name the local isometry and the auxiliary state from (45), *we write* $S \xhookrightarrow{U,|\mathrm{aux}\rangle}_2 \tilde{S}$.

We show that for full rank strategies, this type of local dilations is in fact equivalent to the one presented in Definition 2.5. One thing that is important to note in the following lemma is that $S \hookrightarrow_2 \tilde{S}$ is only defined when both S and $\tilde{S}$ are pure full-rank strategies.

Lemma C.2. *Let* S *and* $\tilde{S}$ *be two pure full-rank strategies. Then*

$$S \hookrightarrow \tilde{S} \quad \iff \quad S \hookrightarrow_2 \tilde{S}$$

Proof. Let $\tilde{S} = (|\tilde{\psi}\rangle, \{\tilde{A}_{sa}\}, \{\tilde{B}_{tb}\})$ and $S = (\rho_{AB}, \{A_{sa}\}, \{B_{tb}\})$.

We start by showing $S \xhookrightarrow{U,|\mathrm{aux}\rangle}_2 \tilde{S} \Rightarrow S \hookrightarrow \tilde{S}$. Tensor (46) and (47), and right-multiply with (45) to get

$$(U_A A_{sa} U_A^* \otimes U_B B_{tb} U_B^*) U |\psi\rangle = (\tilde{A}_{sa} \otimes \mathbb{1}_{\hat{A}} \otimes \tilde{B}_{tb} \otimes \mathbb{1}_{\hat{B}}) |\tilde{\psi}\rangle \otimes |\mathrm{aux}\rangle.$$

By U being a unitary, implying $U^*U = \mathbb{1}$, we have

$$U(A_{sa} \otimes B_{tb}) |\psi\rangle = \left[(\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle\right] \otimes |\mathrm{aux}\rangle.$$

implying $S \hookrightarrow \tilde{S}$.

We then show $S \xhookrightarrow{V,|\mathrm{aux}\rangle} \tilde{S} \Rightarrow S \hookrightarrow_2 \tilde{S}$. Consider the Schmidt decomposition

$$|\mathrm{aux}\rangle_{\hat{A}\hat{B}} = \sum_{i=0}^{r-1} \lambda_i |\alpha_i\rangle |\beta_i\rangle \tag{48}$$

where r is the Schmidt rank of $|\mathrm{aux}\rangle_{\hat{A}\hat{B}}$. Furthermore observe that $\dim(\mathcal{H}_A) = r \cdot \dim(\mathcal{H}_{\tilde{A}})$. Define the isometries

$$T_{\hat{A}} := \sum_{i=0}^{r-1} |\alpha_i\rangle\langle i|, \qquad T_{\hat{B}} := \sum_{i=0}^{r-1} |\beta_i\rangle\langle i|$$

where $|i\rangle \in \mathbb{C}^r$, and observe that $T_{\hat{A}} T_{\hat{A}}^*$ is a projection onto $\mathrm{supp}_{\hat{A}}(|\mathrm{aux}\rangle_{\hat{A}\hat{B}})$ and $T_{\hat{B}} T_{\hat{B}}^*$ is a projection onto $\mathrm{supp}_{\hat{B}}(|\mathrm{aux}\rangle_{\hat{A}\hat{B}})$.

Next note that

$$|\tilde{\psi}\rangle |\mathrm{aux}\rangle = VV^*V |\psi\rangle = VV^* |\tilde{\psi}\rangle |\mathrm{aux}\rangle$$

and so $V_A V_A^*$ act with identity on $\mathrm{supp}_{\tilde{A}\hat{A}}(|\tilde{\psi}\rangle |\mathrm{aux}\rangle) = \mathcal{H}_{\tilde{A}} \otimes \mathrm{supp}_{\hat{A}}(|\mathrm{aux}\rangle)$, and similarly for $\mathcal{H}_{\tilde{B}}$. Define

$$W_A := (\mathbb{1}_{\tilde{A}} \otimes T_{\hat{A}}^*) V_A, \qquad W_B := (\mathbb{1}_{\tilde{B}} \otimes T_{\hat{B}}^*) V_B, \qquad |\mathrm{aux}'\rangle := (T_{\hat{A}}^* \otimes T_{\hat{B}}^*) |\mathrm{aux}\rangle$$

Observe that $|\mathrm{aux}'\rangle \in \mathbb{C}^r \otimes \mathbb{C}^r$ has full Schmidt rank and that W_A and W_B are square matrices. We claim that W_A is unitary. This can be seen by

$$W_A W_A^* = (\mathbb{1}_{\tilde{A}} \otimes T_{\hat{A}}^*) U_A U_A^* (\mathbb{1}_{\tilde{A}} \otimes T_{\hat{A}}) = (\mathbb{1}_{\tilde{A}} \otimes T_{\hat{A}}^*)(\mathbb{1}_{\tilde{A}} \otimes T_{\hat{A}}) = \mathbb{1}_{\tilde{A}} \otimes \mathbb{1}_r$$

using that $U_A U_A^*$ act with identity on $\mathcal{H}_{\tilde{A}} \otimes \mathrm{supp}_{\hat{A}}(|\mathrm{aux}\rangle$ and that $T_{\hat{A}}$ is an isometry. A similar argument shows that W_B is unitary. Now we have

$$(W_A \otimes W_B)(A_{sa} \otimes B_{tb}) |\psi\rangle = (\mathbb{1}_{\tilde{A}\tilde{B}} \otimes T_{\hat{A}\hat{B}}) \left[(\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle \right] \otimes |\mathrm{aux}\rangle = \left[(\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle \right] \otimes \left|\mathrm{aux}'\right\rangle .$$

Hence

$$(W_A A_{sa} W_A^* \otimes W_B B_{tb} W_B^*)(W_A \otimes W_B) |\psi\rangle = \left[(\tilde{A}_{sa} \otimes \tilde{B}_{tb}) |\tilde{\psi}\rangle \right] \otimes \left|\mathrm{aux}'\right\rangle .$$

Since $|\tilde{\psi}\rangle \otimes |\mathrm{aux}'\rangle$ has full Schmidt rank, it follows that

$$W_A A_{sa} W_A^* = \tilde{A}_{sa}, \qquad W_B B_{tb} W_B^* = \tilde{B}_{tb}$$

From this, we can conclude $S \hookrightarrow_2 \tilde{S}$. □

Self-testing a non-projective realisation

Jed Kaniewski Laura Mančinska

Let P be a probability point in the quantum set $\mathcal{Q}$. Clearly, we can always find a quantum realisation where the measurements of Alice and Bob are projective: given any realisation we simply apply the Naimark's dilation to all the measurements.

Similarly, we can always find a quantum realisation where the marginal states are full-rank: given any realisation we simply truncate the local Hilbert spaces to the support of the marginal states (this does not affect the statistics).

However, it is not clear that one can always find a realisation which is both projective and locally full-rank, because the transformations suggested above might destroy the other property. In fact, this is not true as shown by the example below.

Consider a bipartite Bell scenario where Alice has two binary observables: A_0, A_1, while Bob has two binary observables B_0, B_1 and one three-outcome measurement $\{F_j\}_{j=0}^2$. Define the following two functionals:

$$\beta_1 := \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle, \tag{1}$$

$$\beta_2 := \langle A_0 F_0 \rangle - \frac{1}{2}\langle A_0 F_1 \rangle + \frac{\sqrt{3}}{2}\langle A_1 F_1 \rangle - \frac{1}{2}\langle A_0 F_2 \rangle - \frac{\sqrt{3}}{2}\langle A_1 F_2 \rangle. \tag{2}$$

Lemma 0.1. *There exists a unique probability point P satisfying $\beta_1 = 2\sqrt{2}$ and $\beta_2 = 1$. Therefore, P is an extremal point of the quantum set.*

Proof. For the purpose of studying the quantum set, we can restrict ourselves to realisations where the marginal states are full-rank. Then, achieving $\beta_1 = 2\sqrt{2}$ implies that

$$A_0 = \mathsf{Z} \otimes \mathbb{1}, \tag{3}$$

$$A_1 = \mathsf{X} \otimes \mathbb{1}, \tag{4}$$

$$\rho = \Phi^+_{AB} \otimes \sigma_{A'B'}. \tag{5}$$

Let us now consider the three-outcome measurement and define its effective action on the qubit:

$$G_j := \operatorname{tr}_{B'} \big[F_j \, \mathbb{1}_B \otimes \sigma_{B'} \big]. \tag{6}$$

It is easy to see that the effective operators G_j fully determine the statistics, since the observables of Alice completely ignore the A' system.

Let us also define $\{T_j\}_{j=0}^2$ and note that they can be computed explicitly:

$$T_0 := \operatorname{tr}_{AA'B'} \big[A_0 \otimes \mathbb{1}_{BB'} \rho \big] = \frac{1}{2}\mathsf{Z}, \tag{7}$$

$$T_1 := \operatorname{tr}_{AA'B'} \Big[\big(-\frac{1}{2} A_0 + \frac{\sqrt{3}}{2} A_1 \big) \otimes \mathbb{1}_{BB'} \rho \Big] = \frac{1}{2}\Big(-\frac{1}{2}\mathsf{Z} + \frac{\sqrt{3}}{2}\mathsf{X} \Big), \tag{8}$$

$$T_2 := \operatorname{tr}_{AA'B'} \Big[\big(-\frac{1}{2} A_0 - \frac{\sqrt{3}}{2} A_1 \big) \otimes \mathbb{1}_{BB'} \rho \Big] = \frac{1}{2}\Big(-\frac{1}{2}\mathsf{Z} - \frac{\sqrt{3}}{2}\mathsf{X} \Big). \tag{9}$$

It is easy to verify that the functional β_2 can be rewritten as:

$$\beta_2 = \sum_j \operatorname{tr}(T_j G_j). \tag{10}$$

Each term can be upper-bounded using Holder's inequality to give:

$$\beta \leq \sum_j \|T_j\|_\infty \|G_j\|_1 = \frac{1}{2} \sum_j \operatorname{tr} G_j = 1, \tag{11}$$

where we used the fact that $\|T_j\|_\infty = \frac{1}{2}$. It easy to determine the conditions under which these inequalities hold as equalities: since for every T_j the positive part is one-dimensional, the G_j operator must be proportional to these rank-1 projectors. The completeness condition allows us to deduce the proportionality constants and finally we conclude that:

$$G_0 = \frac{1}{3}(\mathbb{1} + \mathsf{Z}), \tag{12}$$

$$G_1 = \frac{1}{3}(\mathbb{1} - \frac{1}{2}\mathsf{Z} + \frac{\sqrt{3}}{2}\mathsf{X}), \tag{13}$$

$$G_2 = \frac{1}{3}(\mathbb{1} - \frac{1}{2}\mathsf{Z} - \frac{\sqrt{3}}{2}\mathsf{X}). \tag{14}$$

This allows us to fully compute the statistics, which means that indeed there is a unique probability point satisfying the conditions of the lemma.

This point is an exposed point of the $\beta_1 = 2\sqrt{2}$ face of the quantum set. Therefore, it must be (at least) extremal within the entire quantum set. □

The statistics can be computed explicitly:

$$\langle \mathbb{1} \otimes F_j, \rho \rangle = \frac{1}{3}, \tag{15}$$

$$\langle A_0 \otimes F_0, \rho \rangle = \frac{1}{3}, \tag{16}$$

$$\langle A_0 \otimes F_1, \rho \rangle = \langle A_0 \otimes F_2, \rho \rangle = -\frac{1}{6}, \tag{17}$$

$$\langle A_1 \otimes F_1, \rho \rangle = -\langle A_1 \otimes F_2, \rho \rangle = \frac{1}{2\sqrt{3}}. \tag{18}$$

Lemma 0.2. *For quantum realisations whose marginal states are full-rank, the measurement operators F_j are given by $F_j = G_j \otimes \mathbb{1}_{B'}$. In other words, the device performs a non-projective measurement on the qubit, while acting trivially on the auxiliary system B'.*

Proof. Let us start with the characterisation of the G_j operators given above. Define

$$H_j := (\mathbb{1} \otimes \sigma_{B'}^{1/2}) F_j (\mathbb{1} \otimes \sigma_{B'}^{1/2}). \tag{19}$$

and note that $G_j = \operatorname{tr}_{B'} H_j$. Since G_j are rank-1 PSD operators, we must have

$$H_j = G_j \otimes K_j, \tag{20}$$

for some $K_j \geq 0$ satisfying $\operatorname{tr} K_j = 1$. Now, if $\sigma_{B'}$ is full-rank we can actually reconstruct the original measurement operators:

$$F_j = G_j \otimes (\sigma_{B'}^{-1/2} K_j \sigma_{B'}^{-1/2}). \tag{21}$$

Using the completeness relation $\sum_j F_j = \mathbb{1}$ and the fact that the G_j operators correspond to an extremal three-outcome measurement on a qubit, we find that the only solution is $K_j = \sigma_{B'}$, which gives use precisely the statement of the lemma. □

Observation 0.1. *Clearly, the conclusion above is not valid without the full-rank assumption. Indeed, it is easy to verify that if we take the realisation above and enlarge Bob's space to a qutrit (spanned by $\{|0\rangle, |1\rangle, |2\rangle\}$, while the reduced state is only supported on $\{|0\rangle, |1\rangle\}$), then the following rank-1 projective measurement will lead to the same statistics as before: $F_j := |e_j\rangle\langle e_j|$ for*

$$|e_0\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix}, \tag{22}$$

$$|e_1\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -\sqrt{3} \\ \sqrt{2} \end{pmatrix}, \tag{23}$$

$$|e_2\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ \sqrt{3} \\ \sqrt{2} \end{pmatrix}. \tag{24}$$

This means that P is yet another example of a probability point which does not have a unique NPA completion (the completion for projective and non-projective measurements will necessarily differ).

Implications for standard self-testing: Note that for "standard" points solving the non-full rank problem is sufficient. Suppose we can prove that all full-rank realisations employ a fixed set of projective measurements. Then, as shown in the handwritten note we can lift this argument to any realisation and we will see that the measurements must be projective on the support. Once we know they are projective on the support, they must be a direct sum with some measurement on the rest of the space.

The argument above shows that such a lifting does not work if the measurements are not projective. Then, the total measurement on the full space might not be simply a direct sum.