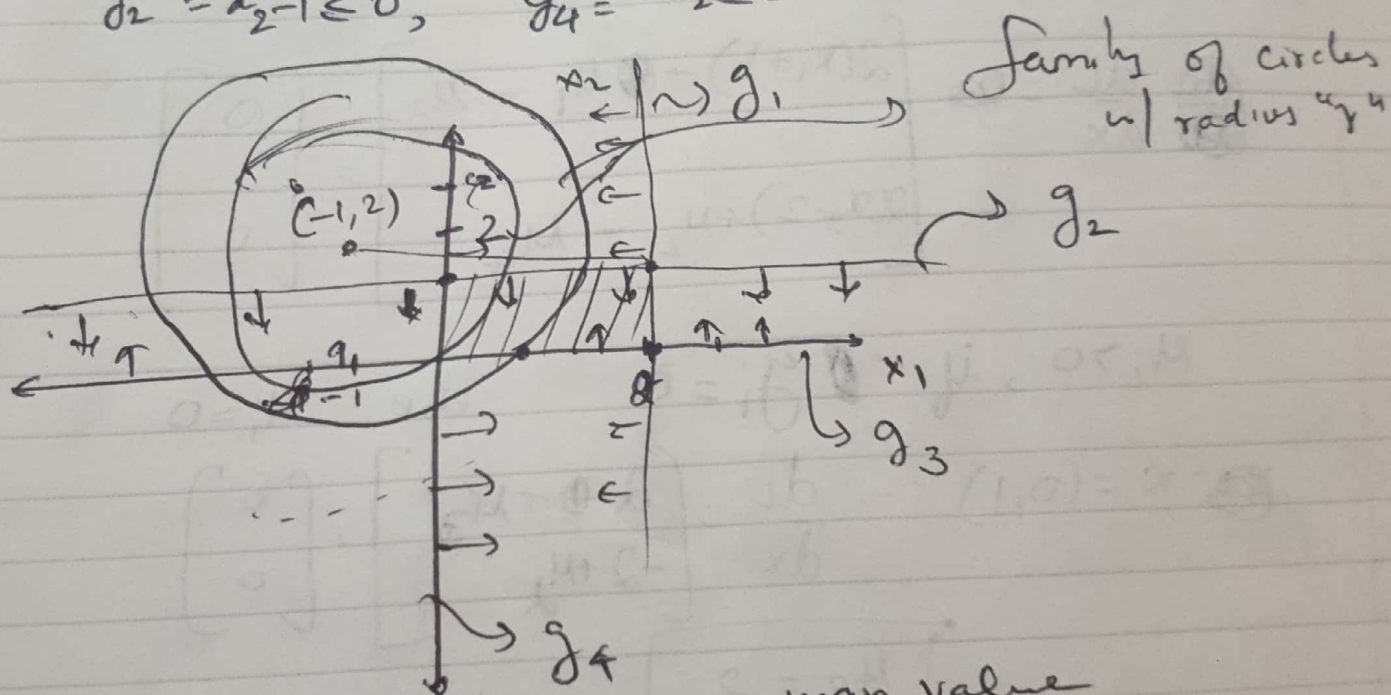


Homework 4

1) $\min_{x_1, x_2} f(x) = (x_1 + 1)^2 + (x_2 - 2)^2$

st: $g_1 = x_1 - 2 \leq 0, \quad g_3 = -x_1 \leq 0$

$g_2 = x_2 - 1 \leq 0, \quad g_4 = -x_2 \leq 0$



@ $(2, 1) : f(x) = 9 + 1 = \underline{10}$ max value

min value:

@ $(-1, 2) : f(x) = \underline{2}$

KKT:

~~$L(x, \lambda, \mu)$~~

$$L(x, \lambda, \mu) = (x_1 + 1)^2 + (x_2 - 2)^2 + \mu_1(x_1 - 2) + \mu_2(x_2 - 1) + \mu_3(-x_1) + \mu_4(-x_2)$$

KKT conditions:

$$\frac{dL}{dx} = \begin{bmatrix} 2(x_1 + 1) + \mu_1 - \mu_3 \\ 2(x_2 - 2) + \mu_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mu_i > 0, \text{ if } g_i = 0 \quad \text{else } \mu_i = 0$$

$$x = (0, 1) : - \frac{dL}{dx} = \begin{bmatrix} 2 - \mu_3 \\ -2 + \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \mu_2 = 2 \\ \mu_3 = 2 \end{bmatrix} \rightarrow \text{satisfies KKT condition } \mu_i > 0$$

Sufficient condition:

$$H = \nabla^2 L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

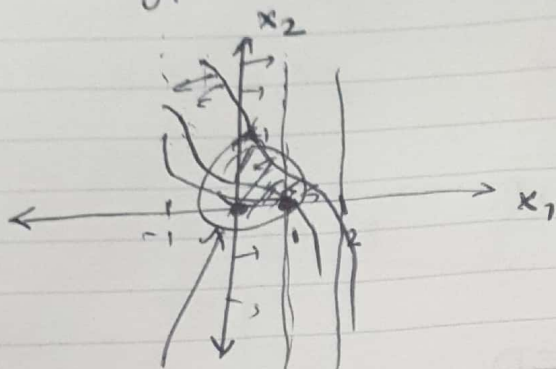
$$dx^T \nabla^2 L dx > 0 \quad \text{for all } x.$$

Therefore $x = (0, 1)$ satisfies the KKT condition and the second order condition.

②

$$\min_{x_1, x_2} f = -x_1$$

$$s.t.: g_1 = x_2 - (1 - x_1)^3 \leq 0, x_2 \geq 0$$



feasible region

• $x_1 = 1; x_2 = 0; f^* = -1$

KKT condition: $\nabla L = 0 \Rightarrow L = f + \mu_1(x_2 - (1 - x_1)^3)$

$$\nabla L = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 3(1 - x_1^2) \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

at $(1, 0)$: $\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
 both g_1, g_2 are active

$$-1 + 0 = 0 \quad \text{--- (1)}$$

$$\mu_1 = \mu_2 \quad \text{--- (2)}$$

This is not a KKT point due to (1).

$$\textcircled{3} \quad \min_{x_1, x_2, x_3} f = x_1 x_2 + x_2 x_3 + x_1 x_3$$

$$\text{St: } h = x_1 + x_2 + x_3 - 3 = 0$$

1. Reduced gradient.

$$s = x_1; \quad d = x_2, x_3$$

$$\frac{df}{dd} = \frac{\partial f}{\partial d} + \frac{\partial f}{\partial s} \left(\frac{\partial s}{\partial d} \right)^{-1} ?$$

$$h=0 \Rightarrow \frac{dh}{dd} = 0$$

$$\frac{dh}{dd} = \frac{\partial h}{\partial d} + \frac{\partial h}{\partial s} \frac{\partial s}{\partial d} = 0$$

$$\frac{\partial s}{\partial d} = \left(\frac{\partial h}{\partial s} \right)^{-1} \frac{\partial h}{\partial d}$$

$$\frac{df}{dd} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s} \left(\frac{\partial h}{\partial s} \right)^{-1} \left(\frac{\partial h}{\partial d} \right)$$

$$\frac{df}{dd} = \frac{\partial f}{\partial d} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix}$$

$$\frac{\partial f}{\partial s} = x_2 + x_3$$

$$\left(\frac{\partial h}{\partial s} \right)^{-1} = 1$$

$$\frac{\partial h}{\partial d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{df}{da} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix} - (x_2 + x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{df}{da} = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = x_2 = x_3$$

$$h=0:-$$

$$3x_1 = 3$$

$$x_1 = 1 = x_2 = x_3$$

$$f^* = 3$$

Lagrangian method:

$$L(x_1, x_2, x_3, \lambda) = -(x_1 x_2 + x_2 x_3 + x_1 x_3) + \lambda (x_1 + x_2 + x_3 - 3)$$

$$\frac{dL}{dx} = 0 = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_2 - x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = x_2 + x_3 = x_1 + x_2$$

$$\Rightarrow x_1 = x_2 = x_3 ; h=0 \Rightarrow x_1 = x_2 = x_3 = 1$$

$$-2x_1 = -\lambda$$

$$\lambda = 2$$

$$f^* = 3$$

~~Second order~~ Sufficiency :

$$D^2L = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

Eigenvalues: $\left| \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$

$$\begin{vmatrix} -\lambda & -1 & -1 \\ -1 & -\lambda & -1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

~~$\lambda^3 - 3\lambda + 2 = 0$~~

$$-\lambda(\lambda^2 - 1) + 1(\lambda - 1) - 1(1 - \lambda) = 0$$

$$-\lambda^3 + \lambda + \lambda - 1 - 1 + \lambda = 0$$

$$-\lambda^3 + 3\lambda - 2 = 0$$

~~$\lambda^3 - 3\lambda + 2 = 0$~~

~~$\lambda^3 - 3\lambda + 2 = 0$~~

Roots: $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -2$

1 eigenvalue is negative. Depending on the feasibility of the direction, the point may be a global maximum.

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$-x_2 - x_3 = -2x_1$$

$$-x_1 - x_3 = -2x_2$$

$$-x_1 - x_2 = -2x_3$$

$$2x_1 - x_2 - x_3 = 0$$

$$2x_2 - x_1 - x_3 = 0$$

$$2x_3 - x_1 - x_2 = 0$$

$$x_1 = \frac{x_2 + x_3}{2}$$

$$2x_2 - \left(\frac{x_2 + x_3}{2} \right) - x_3 = 0$$

$$4x_2 - x_2 - x_3 - 2x_3 = 0$$

$$3x_2 - 3x_3 = 0$$

$$x_2 = x_3$$

$$x_1 = x_2 = x_3 = 1$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \lambda_3 = -2$$

④

$$\max_{x_1, x_2} f = 2x_1 + bx_2 ; \min_{x_1, x_2} f = -2x_1 - bx_2$$

$$\text{st: } \begin{cases} g_1 = x_1^2 + x_2^2 - 5 \leq 0 \rightarrow \text{active} \\ g_2 = x_1 - x_2 - 2 \leq 0 \rightarrow \text{inactive} \end{cases} \quad @ (1, 2)$$

$$s = x_1 ; d = x_2$$

$$\frac{df}{dd} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial s} \left(\frac{\partial h}{\partial s} \right)' \frac{\partial h}{\partial d} = 0$$

$$\frac{\partial f}{\partial d} = b ; \left(\frac{\partial h}{\partial s} \right)' = \frac{1}{2x_1}$$

$$\frac{\partial f}{\partial s} = 2 ; \frac{\partial h}{\partial d} = 2x_2$$

$$\frac{df}{dd} = b + \frac{2x_2}{x_1} = 0$$

$$\underline{\underline{b = 4}}$$