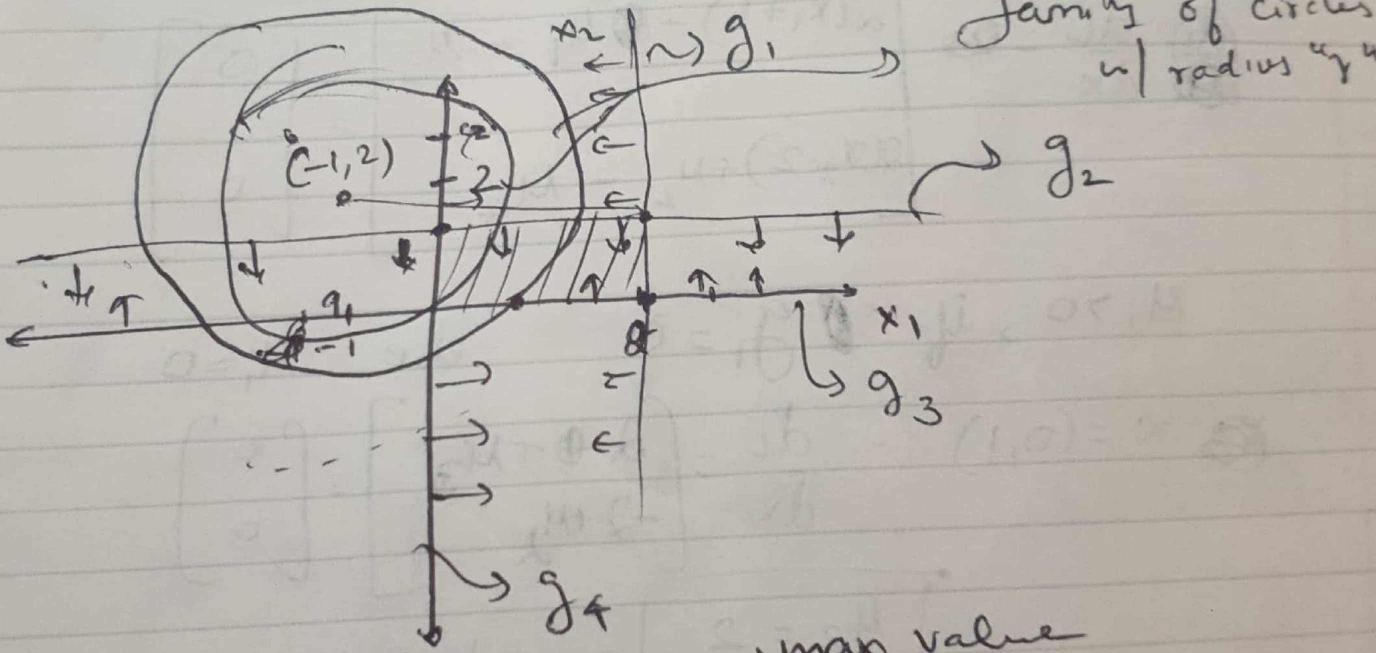
Homework 4

1) $\min_{x_1, x_2} f(x) = (x_1+1)^2 + (x_2-2)^2$

s.t.: $g_1 = x_1 - 2 \le 0, \quad g_3 = -x_1 \le 0$

$g_2 = x_2 - 1 \le 0, \quad g_4 = -x_2 \le 0$



@ $(2,1)$: $f(x) = 9 + 1 = \underline{\underline{10}}$ max value

min value:

~~@(1,0)~~ @ $(0,1)$: $f(x) = \underline{\underline{2}}$

KKT:

$$L(x,\lambda,\mu) = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_1-2) + \mu_2(x_2-1) + \mu_3(-x_1) + \mu_4(-x_2)$$

KKT conditions:

$$\frac{dL}{dx} = \begin{bmatrix} 2(x_1+1) + \mu_1 - \mu_3 \\ 2(x_2-2) + \mu_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mu_i > 0$, if $g_i = 0$ else $\mu_i = 0$

$x = (0,1)$:- $\frac{dL}{dx} = \begin{bmatrix} 2 - \mu_3 \\ -2 + \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\boxed{\begin{matrix} \mu_2 = 2 \\ \mu_3 = 2 \end{matrix}}$ → satisfies KKT condition $\mu_i > 0$

Sufficient condition:

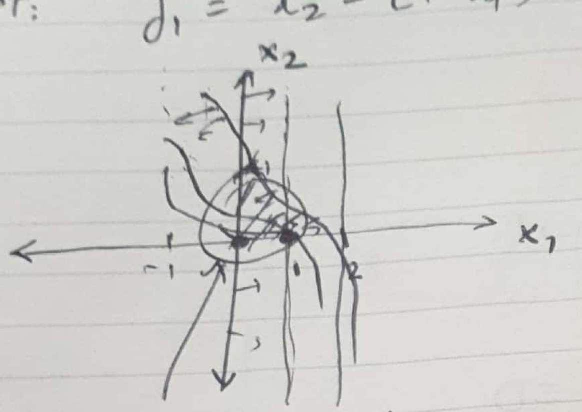
$$H = \nabla^2 L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix};$$

$dx^T \nabla^2 L dx > 0$ for all x.

Therefore $x = (0,1)$ satisfies the KKT condition and the second order condition

② $\min\limits_{x_1, x_2} \quad f = -x_1$

s.t.: $g_1 = x_2 - (1-x_1)^3 \leq 0, \quad x_2 \geq 0$



feasible region

$x_1 = 1; \; x_2 = 0; \; f^* = -1$

KKT condition: $\nabla L = 0 \Rightarrow L = f + \mu_1\left(x_2 - (1-x_1)^3\right) - \mu_2(x_2)$

$$\nabla L = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 3(1-x_1^2) \\ 1 \end{pmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

at $(1,0)$: both g_1, g_2 are active

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-1 = 0$ ✗ — ①

$\mu_1 = \mu_2$ — ②

This is not a KKT point due to ①.

③ $\min_{x_1, x_2, x_3} f = x_1 x_2 + x_2 x_3 + x_1 x_3$

St: $h = x_1 + x_2 + x_3 - 3 = 0$

1. Reduced gradient.

$s = x_1 \quad ; \underset{\sim\sim\sim}{d = x_2, x_3}$

$\frac{df}{dd} = \frac{\partial f}{\partial d} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial d}$ — ??

$h = 0 \Rightarrow \frac{dh}{dd} = 0$

$\frac{dh}{dd} = \frac{\partial h}{\partial d} + \frac{\partial h}{\partial s}\frac{\partial s}{\partial d} = 0$

$\frac{\partial s}{\partial d} = -\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d}$

$\frac{df}{dd} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\left(\frac{\partial h}{\partial d}\right)$

$= \frac{\partial f}{\partial d} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix}$

$\frac{\partial f}{\partial s} = x_2 + x_3$

$\left(\frac{\partial h}{\partial s}\right)^{-1} = 1$

$\frac{\partial h}{\partial d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\frac{df}{dd} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix} - (x_2 + x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{df}{dd} = \begin{bmatrix} x_1 - x_2 \\ x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = x_2 = x_3$$

$h = 0$:-

$$3x_1 = 3$$

$$\boxed{x_1 = 1 = x_2 = x_3}$$

$$\boxed{f^* = 3}$$

Lagrangean method:

$$L(x_1, x_2, x_3, \lambda) = -(x_1 x_2 + x_2 x_3 + x_1 x_3) + \lambda (x_1 + x_2 + x_3 - 3)$$

$$\frac{dL}{dx} = 0 = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_2 - x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = x_2 + x_3 = x_1 + x_2$$

$$\Rightarrow x_1 = x_2 = x_3 \;;\; h = 0 \Rightarrow \boxed{x_1 = x_2 = x_3 = 1}$$

$$-2x_1 = -\lambda$$

$$\boxed{\lambda = 2}$$

$$f^* = 3$$

Second order sufficiency :

$$\nabla^2 L = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

Eigenvalues:

$$\left| \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & -1 & -1 \\ -1 & -\lambda & -1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 - 1) + 1(\lambda - 1) - 1(1 - \lambda) = 0$$

$$-\lambda^3 + \lambda + \lambda - 1 - 1 + \lambda = 0$$

$$-\lambda^3 + 3\lambda - 2 = 0$$

Roots: $\lambda_1 = 1$; $\lambda_2 = 1$; $\lambda_3 = -2$

1 eigenvalue is negative. Depending on the feasibility of the direction, the point may be a global maximum

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$-x_2 - x_3 = -2x_1$$
$$-x_1 - x_3 = -2x_2$$
$$-x_1 - x_2 = -2x_3$$

$$2x_1 - x_2 - x_3 = 0$$
$$2x_2 - x_1 - x_3 = 0$$
$$2x_3 - x_1 - x_2 = 0$$

$$x_1 = \frac{x_2 + x_3}{2}$$

$$2x_2 - \left(\frac{x_2 + x_3}{2}\right) - x_3 = 0$$

$$4x_2 - x_2 - x_3 - 2x_3 = 0$$
$$3x_2 - 3x_3 = 0$$
$$x_2 = x_3$$

$$x_1 = x_2 = x_3 = 1$$

$$\vec{V}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \lambda_3 = -2$$

④ $\max_{x_1,x_2} f = 2x_1 + bx_2$; $\min_{x_1,x_2} f = -2x_1 - bx_2$

st: $g_1 = x_1^2 + x_2^2 - 5 \leq 0 \rightarrow$ active
$g_2 = x_1 - x_2 - 2 \leq 0 \rightarrow$ inactive } @ (1,2)

$s = x_1$; $d = x_2$

$$\frac{df}{dd} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d} = 0$$

$$\frac{\partial f}{\partial d} = b \; ; \; \left(\frac{\partial h}{\partial s}\right)^{-1} = \frac{1}{2x_1}$$

$$\frac{\partial f}{\partial s} = 2 \; ; \; \frac{\partial h}{\partial d} = 2x_2$$

$$\frac{df}{dd} = b - \frac{2x_2}{x_1} = 0$$

$$\underline{\underline{b = 4}}$$

Second order suffiviry:

$$\frac{d^2f}{dd^2} = \left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial^2 s}{\partial d^2}\right)$$

$$= \left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial h}{\partial s}\right)^{-1}\left(I, \frac{\partial s}{\partial d}\right)^{T} h_{xx} \left(I, \left(\frac{\partial s}{\partial d}\right)\right)^{T}$$

$$= -2\frac{1}{2x_2}$$