Theory/Computation Problems

Problem 1 (20 points)

Show that the stationary point (zero gradient) of the function

$$f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

is a saddle (with indefinite Hessian). Find the directions of downslopes away from the saddle. Hint: Use Taylor's expansion at the saddle point. Find directions that reduce f.

Problem 2 (50 points)

- (10 points) Find the point in the plane $x_1 + 2x_2 + 3x_3 = 1$ in $\mathbb{R}^3$ that is nearest to the point $(-1, 0, 1)^T$. Is this a convex problem? Hint: Convert the problem into an unconstrained problem using $x_1 + 2x_2 + 3x_3 = 1$.
- (40 points) Implement the gradient descent and Newton's algorithm for solving the problem. Attach your codes along with a short summary including (1) the initial points tested, (2) corresponding solutions, (3) a log-linear convergence plot.

Problem 3 (10 points)

Let $f(x)$ and $g(x)$ be two convex functions defined on the convex set $\mathcal{X}$.

- (5 points) Prove that $af(x) + bg(x)$ is convex for $a > 0$ and $b > 0$.
- (5 points) In what conditions will $f(g(x))$ be convex?

Problem 4 (bonus 10 points)

Show that $f(\mathbf{x}_1) \geq f(\mathbf{x}_0) + \mathbf{g}_{\mathbf{x}_0}^T(\mathbf{x}_1 - \mathbf{x}_0)$ for a convex function $f(\mathbf{x}) : \mathcal{X} \to \mathbb{R}$ and for $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{X}$.

Design Problems

Problem 5 (20 points)

Consider an illumination problem: There are n lamps and m mirrors fixed to the ground. The target reflection intensity level is I_t. The actual reflection intensity level on the kth mirror can be computed as $\mathbf{a}_k^T\mathbf{p}$, where $\mathbf{a}_k$ is given by the distances between all lamps to the mirror, and $\mathbf{p} := [p_1, \ldots, p_n]^T$ are the power output of the lamps. The objective is to keep the actual intensity levels as close to the target as possible by tuning the power output $\mathbf{p}$.

- (5 points) Formulate this problem as an optimization problem.
- (5 points) Is your problem convex?
- (5 points) If we require the overall power output of any of the n lamps to be less than p^*, will the problem have a unique solution?
- (5 points) If we require no more than half of the lamps to be switched on, will the problem have a unique solution?

Note

For this homework, you may want to attach sketches as means to explain your ideas. Here is how you can attach images.

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