Fractals and the Mandelbrot set

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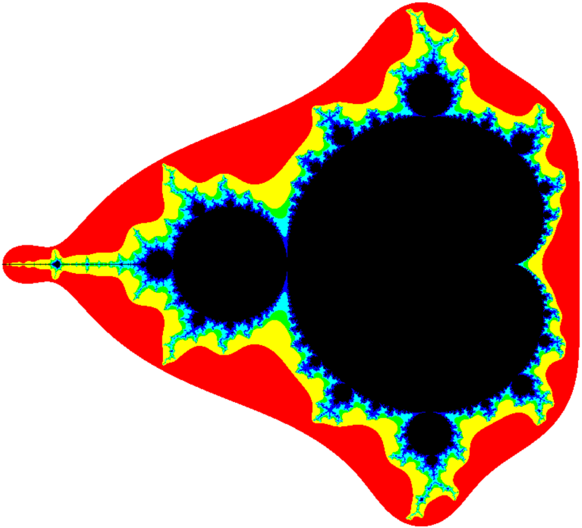
MATH 1040

The Mandelbrot set



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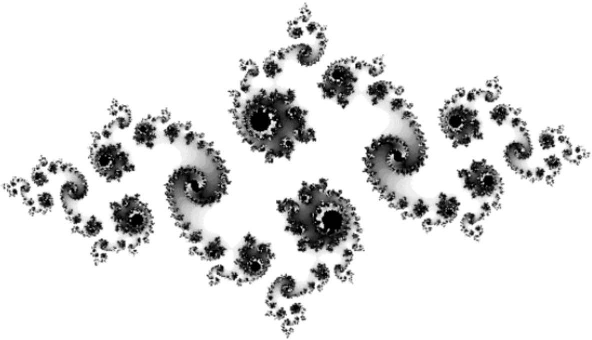

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Definition (Minkowski-Bouligand dimension)

If a self-similar set of size 1 *can be divided into* N *congruent sets of size* ϵ*, then the Minkowski-Bouligand dimension*

$$D = \frac{\log(N)}{\log(1/\epsilon)}.$$

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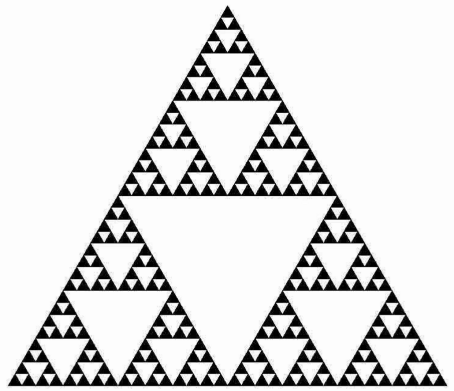
This makes sense, because we can divide a line into 2 congruent segments each of length $1/2$. Then the Hausdorff dimension $D = \frac{\log(2)}{\log(2)} = 1$ is the same as the topological dimension. Equality holds because the straight line is <u>not</u> a fractal.

Proving the Sierpinski triangle is a fractal



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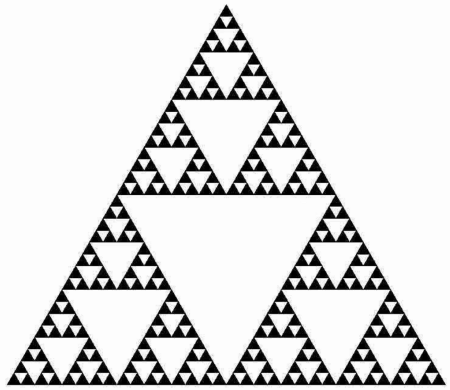

We can divide the Sierpinski triangle into 3 congruent sets each of size 1/2, so the Hausdorff dimension is

$$D = \frac{\log(3)}{\log(2)} \approx 1.585$$

which is greater than the topological dimension 1. Hence the Sierpinski triangle is a fractal.

The Mandelbrot set

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Definition (Mandelbrot set)

The set of complex numbers S such that $\forall c \in S$, the sequence defined by

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$$z_0 = 0$$

is bounded as $n \to \infty$.

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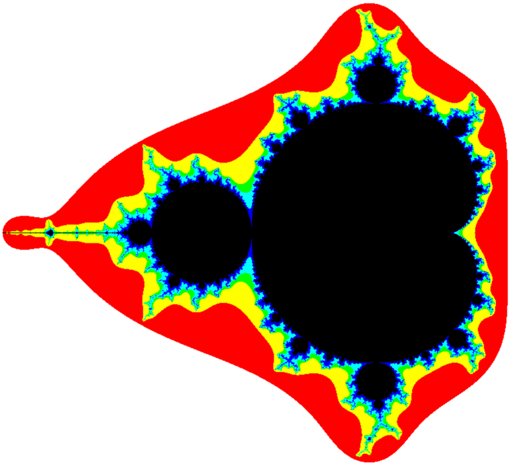
For example, $c = -1$ produces the sequence $-1, 0, -1, 0, -1, 0, \ldots$ which is bounded. Therefore -1 is a member of the Mandelbrot set.

The Mandelbrot set

Red $= z_5$, Yellow $= z_8$, Green $= z_{12}$
Light Blue $= z_{15}$, Dark Blue $= z_{25}$, Black $= z_{100}$.



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Since the Mandelbrot set has topological dimension 2, the set itself is not a fractal. However, the border of the Mandelbrot set has topological dimension 1 and Hausdorff dimension 2, so the border of the Mandelbrot set is a fractal.

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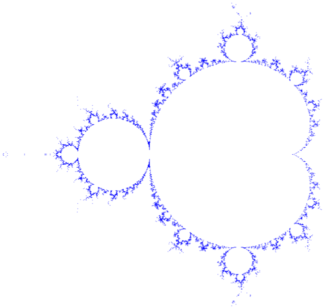

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