

# Rotor routing on 2-manifolds

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# Outline

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- Rotor routing on  $\mathbb{Z}^2$
- Recurrence and transience
- Rotor routing on other 2-manifolds
- Visualizations and experimental results

# What is rotor routing?



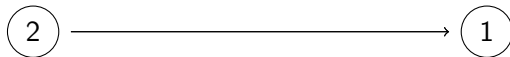
# What is rotor routing?

Rotor routing is a deterministic version of the random walk.

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## Definition (Rotor)

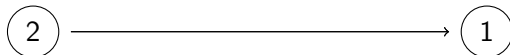
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*An assignment of a single rotor to every vertex on a graph.*

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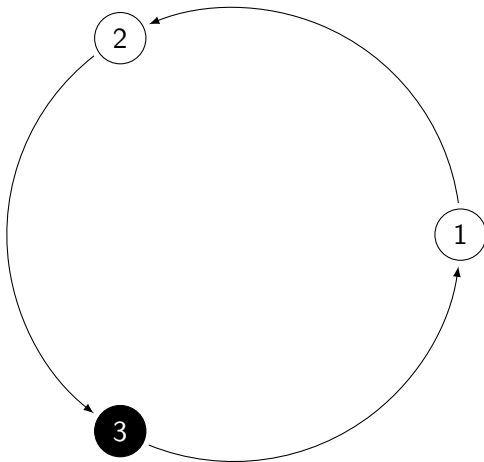
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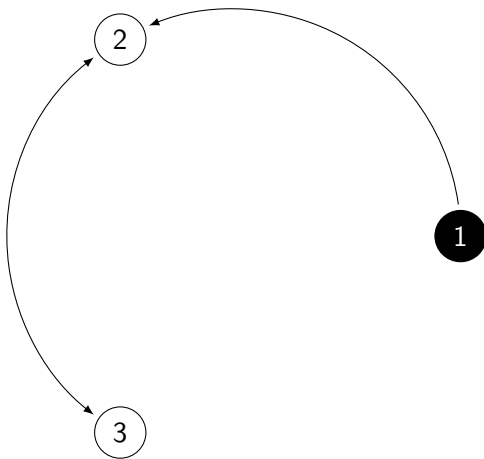
## Definition (Rotor Walk)

*A traversal of a particle along a initial rotor configuration. The particle follows the direction of rotors, which then change direction according to the rotor mechanism.*

# Rotor walk example

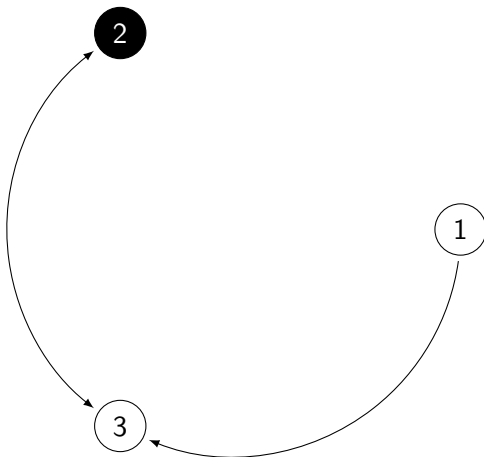


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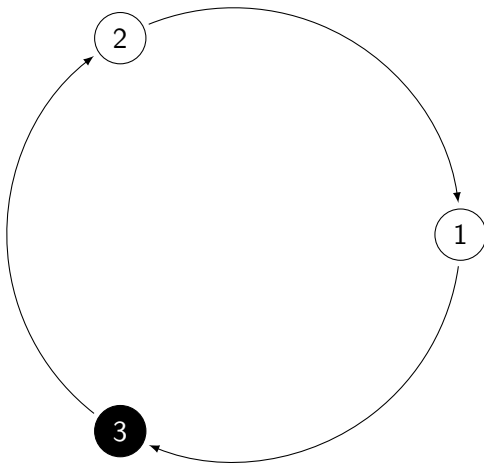




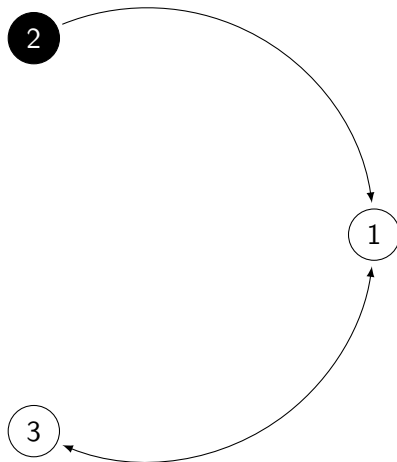
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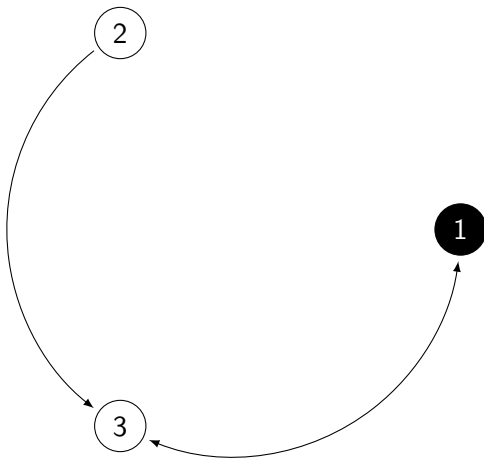
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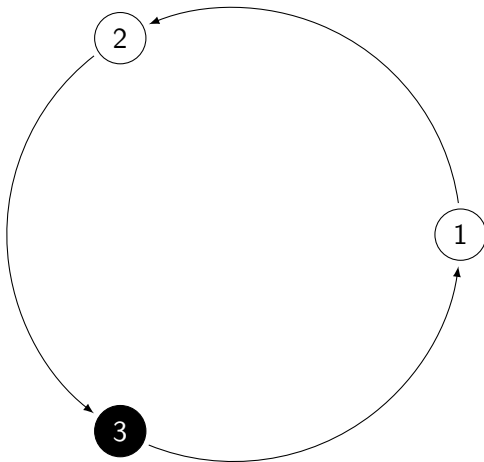
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*There are four possible rotors starting at a given point:*

$$\mathcal{E} = \{\text{North, East, South, West}\}.$$

*Rotors turn counter-clockwise, repeatedly cycling through  $\mathcal{E}$ .*

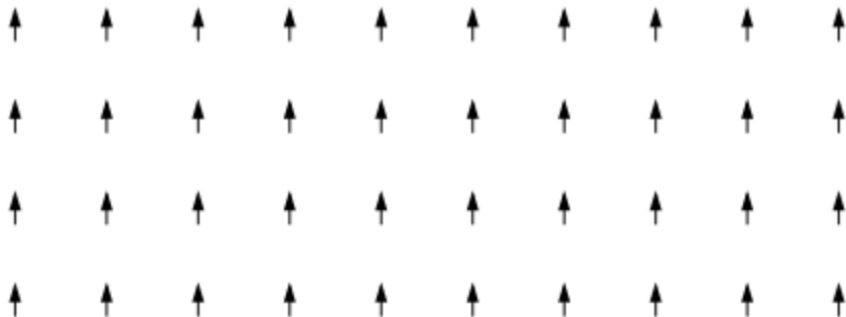
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## Definition (Transience)

*A walk is transient if it visits each point finitely many times.*

- Random walk on  $\mathbb{Z}^d$  where  $d \geq 3$
- Repeated rotor walks on  $\mathbb{Z}^3$  with single-direction configuration

# Theorems about recurrence and transience

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## Theorem (Angel-Holroyd 2011)

*Given a rotor mechanism and a configuration, a single rotor walk is either recurrent for every starting vertex, or transient for every starting vertex.*

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## Corollary

*Given a rotor mechanism, two rotor configurations differing at only finitely many vertices are either both recurrent or both transient.*



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## Proof

We need only consider two rotor configurations  $r, r'$  differing at only one vertex  $v$ , such that a single rotation of the rotor at  $v$  produces  $r'$  from  $r$ . If  $r$  is recurrent, a rotor walk started at  $v$  produces  $r'$  with the particle moved to some other vertex. By the aforementioned theorem,  $r'$  must also be recurrent. □

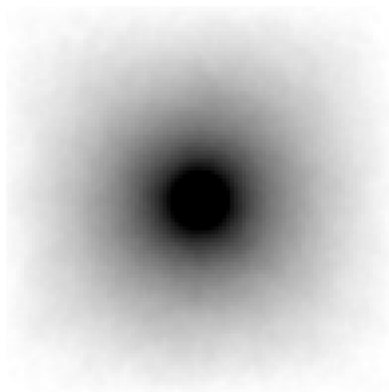
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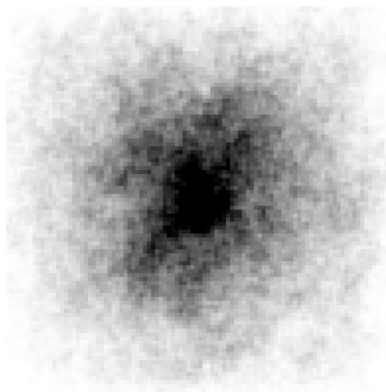
## Conjecture (Levine-Peres 2015)

*Given a random configuration of rotors on  $\mathbb{Z}^2$ , a rotor walk on that configuration is recurrent with probability 1.*

# Random configuration on $\mathbb{Z}^2$



Random Rotor Configuration



Random Walk

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Some 2-manifolds:

- Cylinder
- Möbius strip
- Torus
- Sphere
- Klein bottle
- Real projective plane

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*If a space is compact, every open cover has a finite sub-cover.*



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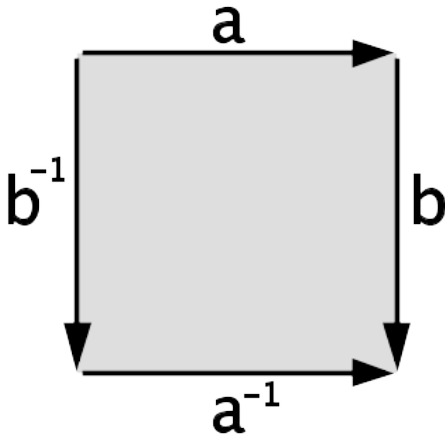
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Compact 2-manifolds:

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- Sphere
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## Rotor routing on other 2-manifolds

A gluing diagram shows how edges of a polygon are connected to create a surface. When a particle hits an edge that is associated to another, the particle "reappears" on the associated edge.



# Experimental results

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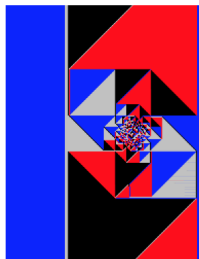
## Conjecture

*Repeated rotor walks on the Möbius strip, cylinder, and plane will eventually create congruent octagonal patterns for the single-direction configuration.*

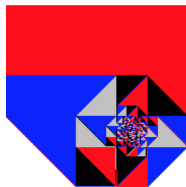
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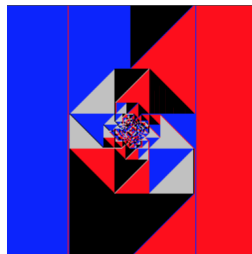
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Cylinder



Plane



Möbius Strip

# Experimental results

## Observation

*For any rotor configuration and rotor mechanism, a rotor walk on a compact surface eventually results in a finite loop of configurations.*

# Future work

- Testing alternate configurations, including different lattices and transient configurations.
- Experimenting with rotor routing in higher dimensions.
- Developing a predictive model for surface type based on loop length on compact surfaces.