Densities of the Fat Cantor Set

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1 Introduction

In this project I will investigate the densities of Fat Cantor set.
I examine Fat Cantor sets which are built by symmetric intersection of intervals and which have positive measure.
Let the fat Cantor set be defined on $[0, 1]$, the complement to the fat Cantor set is a countable union of open intervals and hence is open. Therefore the density of every point in the complement of the fat Cantor is 0.
The fat Cantor sets contains no interior points. However we can split the points of the fat Cantor set into two disjoint sets.
1) Those which are on the closure of some removed interval (set A).
2) Those which are not. (set B)
Let k-voids be refer to the intervals removed on the k^{th} iteration of constructing the fat Cantor set.
Let C_k denote one of the 2^k intervals left over after the k^{th} iteration for $k > 0$ and let $C_0 = [0, 1]$. Let $C_k^{'}$ denote a C_k unioned with its neighbouring k-void.
Let V_k for $k \geq 1$ denote one of the 2^{k-1} voids which are removed on the k^{th} iteration.
Let D_k denote the disjoint union of all possible C_k.
Let U_k denote the disjoint union of all possible V_k.

2 Opening Lemmas

Lemma 2.1.
Given $p \in A$, and if there exists $q \in [0,1]$ and $n \in \mathbb{N}$ such that $p, q \in C_n$, with p an endpoint of C_n. Let E_i denote some C_i or a null set. Then there exists a sequence $(E_i)_{i=n}^{\infty}$ s.t. $\forall \delta > 0, \exists M \in \mathbb{N}$ s.t.

$$F \cap (p, q-\delta) \subset \bigcup_{i=n}^{M} E_i \cap F \subset F \cap (p,q)$$

Proof. Without loss of generality say that p is a left endpoint (symmetry).
Also without loss of generality say that n is the largest integer such that $p, q \in C_n$.
Build the sequence. For each man choose E_i and C_M such that $\forall M < n, M \in \mathbb{N}$

$$\bigcup_{i=n+1}^{M} E_i \cap F \subset F \cap (p,q) \subset (\bigcup_{i=n+1}^{M} E_i \cap F\ ) \cup (C_M \cap F)$$

This can be show to be true by induction.
Base case: Choose $n \in \mathbb{N}$ (let p_n=p) and C_n so $F \cap (p,q) \subset F \cap C_n$ (one of the endpoints of C_n is p) and there is no such C_{n+1} for which this is true (this must be true for some n). Therefore there exists a $C_{n+1} \subset C_n$ (with $C^1_{n+1} \subset C_n$ the other interval) s.t. $F \cap C_{n+1} \subset F \cap (p,q)$ and $F \cap (p,q) \subset (F \cap C_{n+1}) \cup (F \cap C^1_{n+1})$ and let p_{n+1} be defined such that $(F \cap (p,q)) \setminus F \cap C_{n+1} = F \cap (p_{n+1}, q)$, this implies that p_{n+1} is an endpoint of C^1_{n+1}. Set $E_i = C_i$
Inductive Step:
Assume for $M \in \mathbb{N}$ (1)-

$$\bigcup_{i=n+1}^{M} E_i \cap F \subset F \cap (p,q) \subset (\bigcup_{i=n+1}^{M} E_i \cap F\ ) \cup (C_M \cap F)$$

and (2)-

$$R_M = (F \cap (p_n, q)) \setminus \bigcup_{i=n+1}^{M} E_i \cap F \subset F \cap (p_M, q)$$

where p_M is an endpoint of some $C^1_M \neq E_M$ and
(3)-$F \cap (p_M, q) \subset C^1_M$.
With those assumptions, want to show the same 3 points for M+1.
Assuming some C_{M+1} completely contains R_M, let $E_{M+1} = \emptyset$, $C^1_{M+1} = C_{M+1}$, then $p_M = p_{M+1}$ and the original points still hold for $M+1$.
Assuming no C_{M+1} completely contains R_M, let C_{M+1} denote the cantor interval in C^1_M which is completely contained and also has p_M an endpoint. Letting $E_{M+1} = C_{M+1}$ and C^1_{M+1} be the other Cantor interval in C_M and defining p_{M+1} to be the endpoint of C^1_{M+1} nearest p and the original points still hold for $M+1$
Thus inductively the proposition is true for all $M \in \mathbb{N}$

Let I denote the set s.t. $i \in I \iff E_i = C_i$

$\square$

Lemma 2.2. *The following statements are true:*

$$\text{(a) } \frac{m(F)}{2^i m(C_i)} \to 1 \text{ as } i \to \infty$$

$$\text{(b) } \frac{m(C_i')}{m(C_i)} \to 1 \text{ as } i \to \infty$$

Proof.
(a)

$$D_k \setminus D_{k+1} = U_{k+1}$$

$$\text{But } F = D_k \setminus \bigsqcup_k^\infty U_{k+1}$$

$$\text{Where } D_k \subset \bigsqcup_k^\infty U_{k+1}$$

$$\text{Therefore by measurability: } m(F) = m(D_k) - m(\bigsqcup_k^\infty U_{k+1})$$

$$m(D_k) = m(F) + m(\bigsqcup_k^\infty U_{k+1})$$

$$2^k m(C_k) = m(F) + \sum_k^\infty m(U_{k+1}) \to m(F) \text{ as sum from 0 to infinity is finite.}$$

$$\text{So } \frac{m(F)}{2^k m(C_k)} \to 1$$

(b)

$$\frac{m(C_i')}{m(C_i)} = \frac{m(C_i) + m(V_i)}{m(C_i)}$$

$$\frac{m(C_i')}{m(C_i)} = \frac{m(D_i) + 2m(U_i)}{m(D_i)} \text{ multiplying above and below by } 2^i$$

But $m(D_i) \geq m(F) > 0$ whereas $m(U_i) \to 0$

$$\text{Therefore } \frac{m(D_i) + 2m(U_i)}{m(D_i)} \to 1$$

$$\frac{m(C_i')}{m(C_i)} \to 0$$

$\square$

3 Body

3.1 Look at set A

By symmetry all $C_k^{(i)} \cap F$ for a fixed k are merely translates of one another.

Lemma 3.1.
Let $p \in A$,

$$\lim_{q \downarrow p} \frac{m(F \cap (p,q))}{m((p,q))} = 1 \textit{ if } p \textit{ is a left endpoint.}$$

$$\lim_{q \uparrow p} \frac{m(F \cap (p,q))}{m((p,q))} = 1 \textit{ if } p \textit{ is a right endpoint.}$$

Proof.
Suppose WLOG that p is a left endpoint.
Therefore need only show

$$\lim_{q \downarrow p} \frac{m(F \cap (p,q))}{m((p,q))} = 1$$

Assume for contradiction:

$$\lim_{q \downarrow p} \frac{m(F \cap (p,q))}{m((p,q))} \neq 1$$

$\exists$ sequence $(q_n) \downarrow p$ s.t. $\frac{m(F \cap (p,q_n))}{m((p,q_n))} \not\to 1$
If is obvious that $\frac{m(F \cap (p,q))}{m((p,q))} <= 1$
Importantly, we can assume without loss of generality that q_n is not in a void as if any q_n is in a void we can make the sequence at least as extreme in distance from 1 (each $\frac{m(F \cap (p,q_n))}{m((p,q_n))}$ no bigger) by moving q_n to the right hand side of the void it is in.
Let n be large enough so that $q_m \in C_p, \forall m > n$ where C_p is the largest Cantor interval for which p is an endpoint.
Let k_n be the largest number s.t. $\exists C_{k_n} \supset \{p, q_n\}$ ($k_n \to 0$ as $n \to \infty$). This means p is an endpoint of C_{k_n}
Let E_i denote some C_i or a null set.
$\forall \delta > 0, \exists M \in \mathbb{N}$ s.t. for some sequence of E_i

$$F \cap (p, q_n - \delta) \subset \bigcup_{k_n}^{M} E_i \cap F \subset F \cap (p, q_n) \text{by Lemma 2.1.}$$

Let I denote the set s.t. $i \in I \iff E_i = C_i, i \leq M$

$$\bigsqcup_{k_n}^{M} E_i \cap F = \bigsqcup_{i \in I} C_i \cap F$$

$$m(\bigsqcup_{k_n}^{M} E_i \cap F) = m(\bigsqcup_{i \in I} C_i \cap F) = \sum_{i \in I} m(C_i \cap F) = \sum_{i \in I} \frac{m(F)}{2^i} = \sum_{i \in I} \frac{m(C_i)m(F)}{m(C_i)2^i}$$

$$\text{but } \frac{m(F)}{2^i m(C_i)} \to 1 \text{ as } i \to \infty, \text{ by Lemma 2.2.}$$

$$\text{and } \frac{m(C_i^{'})}{m(C_i)} \to 1 \text{ as } i \to \infty, \text{ by Lemma 2.2.}$$

Therefore $\forall \epsilon > 0$ can pick N s.t.

$$n > N => |\frac{m(F)}{2^i m(C_{k_n})} - 1| < \epsilon \text{ and } |\frac{m(C_{k_n}^{'})}{m(C_{k_n})} - 1| < \epsilon \text{ and } |\frac{m(C_{k_n})}{m(C_{k_n}^{'})} - 1| < \epsilon$$

$$m(\bigsqcup_{k_n}^{M} E_i \cap F) = \sum_{i \in I} \frac{m(C_i)m(F)}{m(C_i)2^i} > \sum_{i \in I} m(C_i)(1 - \epsilon)$$

$$= \sum_{i \in I} \frac{m(C_i)m(C_i^{'})}{m(C_i^{'})}(1 - \epsilon) > \sum_{i \in I} m(C_i^{'})(1 - \epsilon)^2$$

$$m(F \cap (p, q_n)) \geq m(\bigsqcup_{k_n}^{M} E_i \cap F) > \sum_{i \in I} m(C_i^{'})(1 - \epsilon)^2 > m([p, q_n - \delta))(1 - \epsilon)^2$$

recalling that no q_n is in a void
take $\delta \to 0$, then:

$$m(F \cap (p, q_n)) \geq m(p, q_n)(1 - \epsilon)^2$$

$$\frac{m(F \cap (p, q_n))}{m(p, q_n)} \geq (1 - \epsilon)^2$$

$$\text{Taking } n \to \infty \text{ implies } (1 - \epsilon)^2 \to 1$$

$$\text{therefore } \lim_{n \to \infty} \frac{m(F \cap (p, q_n))}{m((p, q_n))} \geq 1$$

$$\text{but } \lim_{n \to \infty} \frac{m(F \cap (p, q_n))}{m((p, q_n))} \leq 1$$

$$\lim_{n \to \infty} \frac{m(F \cap (p, q_n))}{m((p, q_n))} = 1$$

Contradiction, therefore

$$\lim_{q \downarrow p} \frac{m(F \cap (p, q))}{m((p, q))} = 1$$

$\square$

This means that for every $p \in A$, the upper density of A at p is 1.
For every $p \in A$, the lower density of A at p is 0, as p is right next to a void of postive length.
Therefore for every $p \in A$, the balanced density of A at p is 1/2.

3.2 Look at set B

Moving to $b \in B$:

I will show that at every point the upper density is 1. Then at almost every point the lower density exists and is 1. Then I will outline how every there is a point with every lower density in $(0, 1)$

Upper density: Assume for contradiction that there exist a sequence (r_n) where $r_n \to 0$ such that,

$$\forall B_n : \text{ where } b \in B_n \text{ and } m(B_n) = r_n$$

$$\exists \epsilon > 0 : \frac{m(B_n \cap F)}{m(B_n)} < 1 - \epsilon$$

Then let k_n denote the smallest integer such that there exists a C_{k_n} such that $m(C_{k_n}) > r_n$. Let B_n have one endpoint which is at the nearest endpoint of C_{k_n} to b and it extends out from there in the direction of b. From how we defined n_k, $b \in B_n$ and $B_n \subset C_{n_k}$, however if $\frac{m(B_n \cap F)}{m(B_n)}$ does not go to 1 then $p = 0$ and $q_n = r_n$ would give a sequence such that $\lim_{n\to\infty} \frac{m(F\cap(p,q_n))}{m((p,q_n))} \neq 1$ as $m(F \cap (p, q_n)) = m(F \cap B_n)$ by symmetry (of C_{k_n}). Therefore $\frac{m(B_n \cap F)}{m(B_n)} \to 1$ - a contradiction. Therefore the upper density at all $p \in B$ is 1.

Now consider lower density. We must first prove a Lemma.

Lemma 3.2.
Assuming $B \setminus (\cup_{k=N}^{\infty} \frac{2}{1-a} U_k)$ *non-empty for some* $N \in \mathbb{N}$ *and* $a \in \mathbb{R}$.
For $b \in B \setminus (\cup_{k=N}^{\infty} \frac{2}{1-a} U_k)$*:*

$$\boldsymbol{\delta(b; F) \geq a}$$

Proof.
Let C_{N-1} contain b. Let $r \in \mathbb{R}$ be small enough so that b is a distance larger than r from both endpoints of C_{N-1}, this is possible as b is an interior point. Given any such r, let M be an largest integer such that $m(C_M) > r$, let C_M refer to such an interval containing b.
Recall:

$$\lim_{q \downarrow p} \frac{m(F \cap (p, q))}{m((p, q))} = 1 \text{ if } p \text{ is a left endpoint.}$$

This means that there exists a lower bound, l_r, which is a function of r:
$l_r \uparrow 1$ as $r \to 0$ such that $\frac{m(F\cap(0,r))}{m((0,r))} > l_r$. By symmetry this gives us a lower bound on the concentration of F in subintervals of C_M which start from an endpoint of C_M.
Let K_r denote an interval containing b of length r. Any such $K_r \subset C_M$ will have concentration lower bounded by some value greater than $2l_r - 1$.

The interesting K_r are those which are not subsets of C_M but go off the sides. Let V_O be one of the voids neighbouring to the which can be reached by such a K_r, therefore $O > N-1$. WLOG suppose V_O is to the left of b. Let k_1 and k_2 denote the left and right endpoints of K_r respectively. Let v_1 and v_2 denote the left and right endpoints of V_O respectively. Want to find the K_r with the lowest concentration of F. Therefore $v_1 \geq k_1$. $(\frac{1}{1-a})m(V_O) \leq r = m(K_r)$. But as the segment from k_1 to v_1 is contained in another C_M^1 then (interpret $\frac{0}{0}$ as 0)

$$\frac{m(F\cap K_r)}{m(K_r)} = 0 + \frac{m((k1,v1)\cap F)}{m((k1,v1))}\frac{m((k1,v1))}{m(K_r)} + \frac{m((k2,v2)\cap F)}{m((k2,v2))}\frac{m((k2,v2))}{m(K_r)}$$

$$\geq l_r\frac{m((k1,v1))+m((k2,v2))}{m(K_r)} = l_r\frac{m(K_r)-m(V_O)}{m(K_r)} \geq l_r a$$

Therefore given r, any K_r which contains b has $\frac{m(F\cap K_r)}{m(K_r)} \geq min(l_r a, 2l_r - 1)$. Taking $r\to\infty$, $\frac{m(F\cap K_r)}{m(K_r)} \geq l_r a \uparrow a$. So this proves that

$$\delta(b;F) \geq a$$

For such a $b \in B \setminus (\cup_{k=N}^{\infty}\frac{2}{1-a}U_k)$. □

Returning to general $b\in B$ Recall $\sum\limits_{k=N}^{\infty} m(U_k) \to 0$ as $N\to\infty$ (as all $V_i \cap V_j = \emptyset, i \neq j$).

Let $S_a = \{b \in B : \delta(b;F) \geq a\}$ for $a \in [0,1]$

Want to show $m(S_a) = M(B), \forall a \in (0,1)$, this would imply $m(S_a) = m(B)$.
Assume $\exists\epsilon > 0 : m(S_a) < M(B) - \epsilon$
But can pick $N \in \mathbb{N} : \sum\limits_{k=N}^{\infty} m(U_k) < \epsilon(1-a)/2$.
Let $\frac{1}{1-a}V_k$ denote the V_k scaled from its center by a factor of $\frac{1}{1-a}$.
Let $\frac{1}{1-a}U_k$ be the union of all of these scaled V_k
For $b \in B\setminus(\cup_{k=N}^{\infty}\frac{2}{1-a}U_k)$, $\delta(b;F) > a$. By Lemma 3.2.
Therefore $m(S_a) \geq m(B\setminus\cup_{k=N}^{\infty}\frac{2}{1-a}U_k) > m(B) - \epsilon$.
Therefore $m(S_a) = M(B)$, $\forall a \in (0,1)$.
But $S_1 = \cup_{n=1}^{\infty} S_{1-1/n}$, therefore $m(S_1) = M(B) = M(F)$ by upward measure continuity.

Therefore:
Upper density of F is 1 for $\forall b \in B$.
Lower density of F is 1 for almost all $b \in B$, and hence lower density of F is 1 for almost all points in F.
Therefore balanced density of F is 1 for almost all $b \in B$, and hence balanced density of F is 1 for almost all points in F.

4 Bonus!

Lemma 4.1.
$\forall a \in (0,1)$, there exists infinitely many points $b \in B$ which have lower density equal to a.

Proof.
Sketch:
Fix $a \in \mathbb{R}, a \in (0,1)$

$$S_k = \frac{2}{1-a} U_k$$

$$T_{k,r} = \frac{2}{1-(a+r)} U_k \setminus \frac{2}{1-a} U_k$$

Refer to interval components of these as S_k^i and T_k^i respectively.
Recall

$$\frac{m(C_i^{'})}{m(C_i)} \to 1 \text{ as } i \to \infty$$

$$\frac{m(V_i)}{m(C_i)} \to 1 \text{ as } i \to \infty$$

Let U_{k_n} be a sequence of void unions which is are each larger in measure than all remaining void unions (i.e. $m(U_{k_n}) \geq U_i, \forall i_n$) (this must be infinite).
Choose $N_r \in \mathbb{N}$ large enough such that $\forall \alpha > 0$, and β a bit more than 2 (function of upper limit of $\frac{m(V_{k_i})}{m(C_{k_i})}$, call that upper limit γ. $\beta = (\frac{1+\gamma}{2})^{-1}$ should work).

$$2^{-(floor(-log_\beta(\frac{1+(a+r)}{1-(a+r)}\frac{m(V_{k_i})}{m(C_{k_i})})))} < \alpha(\frac{2}{1-a} - \frac{2}{1-(a+r)}), \forall i > N_r$$

And also (of secondary importance, needed to limit overlap, could change $\frac{1}{4}$ to any smaller number),

$$\frac{1}{1-a}\frac{m(V_i)}{m(C_i)} < \frac{1}{4}, \forall i > N_r$$

.
Significance of 1^{st} inequality:
Let $x = \frac{2}{1-a}$ and $y = \frac{2}{1-(a+r)}$
Want $z(N_r) = \frac{m(V_i)}{m(C_i)}$ small enough so that it will take a lot of iterations for any void to get close to $\frac{2}{1-(a+r)}V_i$
Let $r(N_r)$ denote the proportion through the neighbouring C_i that the right endpoint of $\frac{2}{1-(a+r)}V_i$ would be
$r(N_r) = (\frac{2}{1-(a+r)} - 1)z(N_r) = (\frac{1+(a+r)}{1-(a+r)})z(N_r)$
$w = floor(-log_\beta(r(N_r)))$ this is the lower bound on how many more iterations until new voids can pass left of the right endpoint of $\frac{2}{1-(a+r)}V_i$

Want $p(N_r) = \alpha(y - x) > 2^{(-w)}$ so that any subpart of $T_{i,r}$ would cover completely any subpart of S_k which could come in contact with it.
Taking $N_r \to \infty$ brings $2^{-w} \to 0$. So this is allowed.
This means that if given $C_i \cap S_i^C$, can take $C_i \cap T_{i,r}^j$ (nonempty) such that any $S_k, k > i$ can only cover an arbitrarily small portion of this intersection. Therefore can fit C_l in one of these intersections such that the intersection $C_i \cap T_{i,r}^j \cap_{m=i}^{m=l-1} S_m^C \cap S_l^C$ contains a complete half of C_l less the bit taken away by S_l^C. Therefore we can repeat with $C_l \cap T_{l,r}^{j'}$ as if we had the original $C_i \cap S_i^C$ and again we will not run out of points. Or we could add $S_{(l+1)}^C$ to the intersection and do the same thing again.
From this it is apparent, if we start at a sufficiently high $M \in \mathbb{N}$ a set:

$$G = \cap_{l=M}^{\infty} H_l$$

where

$$H_l = S_l^C \text{ or } T_{l,r_l}$$

with $r_l \to 0$ and $H_l = T_{l,r_l}$ for infinitely many l.
Exists and is non-empty. However any point in this set will have a lower density of a (Lemma 3.2 gives lower bound and this lower bound is attained by taking a sequence intervals which have b as one endpoint and and the other endpoint is the far end of a void V_l which could be deduced from the part of T_{l,r_l} which b is in. As $l \to \infty, r_l \to 0$ and the concentration of F in the interval which contains b goes to the value "a" similarly to in proof of Lemma 3.2.).
This means that $\forall a \in (0, 1)$, there exists infinitely many points $b \in B$ which have lower density equal to a.

□

5 Conclusion

For every $p \in A$, the upper density of F at p is 1.
For every $p \in A$, the lower density of F at p is 0.
For every $p \in A$, the balanced density of F at p is 1/2.
Upper density of F is 1 for $\forall b \in B$.
Lower density of F is 1 for almost all $b \in B$, and hence lower density of F is 1 for almost all points in F.
Therefore balanced density of F is 1 for almost all $b \in B$, and hence balanced density of F is 1 for almost all points in F.
$\forall a \in (0, 1)$, there exists infinitely many points $b \in B$ which have lower density equal to a.