$$\boxed{V = x\sigma_x + y\sigma_y + z\sigma_z}$$

Exercise - show that:

$$V^2 = (x^2 + y^2 + z^2)\mathbb{1}$$

$$= \|\vec{v}\|^2\mathbb{1}$$

$$V^2 = \begin{bmatrix} z & x - yi \\ x + yi & -z \end{bmatrix} \begin{bmatrix} z & x - yi \\ x + yi & -z \end{bmatrix}$$

$$= \begin{bmatrix} (z)(z) + (x - yi)(x + yi) & (z)(x - yi) + (x - yi)(-z) \\ (x + yi)(z) + (-z)(x + yi) & (x + yi)(x - yi) + (-z)(-z) \end{bmatrix}$$

$$= \begin{bmatrix} z^2 + x^2 + y^2 & 0 \\ 0 & z^2 + x^2 + y^2 \end{bmatrix}$$

$$= (x^2 + y^2 + z^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (x^2 + y^2 + z^2)\mathbb{1}$$

Exercise: prove that $UU^\dagger = \mathbb{1}$ for below:

$$\left(\cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2}\sigma_x\sigma_y\right) \rightarrow \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

$$\left(\cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2}\sigma_z\sigma_x\right) \rightarrow \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$\left(\cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2}\sigma_y\sigma_z\right) \rightarrow \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

Step 1: Note that $(\sigma_i \sigma_j)^2 = -\mathbb{1}$ for $i \neq j$:

$$(\sigma_i \sigma_j)^2 = \sigma_i \sigma_j \sigma_i \sigma_j = \sigma_i(-\sigma_i \sigma_j)\sigma_j$$
$$= -\sigma_i(\sigma_i \sigma_j)\sigma_j = -\sigma_i \sigma_i \sigma_j \sigma_j = -(\mathbb{1})(\mathbb{1}) = -\mathbb{1}$$

Step 2: Note that $(\sigma_i \sigma_j)^\dagger = -\sigma_i \sigma_j$ for $i \neq j$:

$$(\sigma_i \sigma_j)^\dagger = \sigma_j^\dagger \sigma_i^\dagger = \sigma_j^\dagger \sigma_i^\dagger = -\sigma_i \sigma_j$$

Step 3: Proof

$$\left(\cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2}\sigma_x\sigma_y\right)\left(\cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2}\sigma_x\sigma_y\right)^\dagger$$

$$\left(\cos\frac{\theta}{2}\mathbb{1} - \sin\frac{\theta}{2}\sigma_x\sigma_y\right)\left(\cos\frac{\theta}{2}\mathbb{1} + \sin\frac{\theta}{2}\sigma_x\sigma_y\right)$$

$$\left(\cos\frac{\theta}{2}\right)^2\mathbb{1} + \left(\cos\frac{\theta}{2}\mathbb{1}\right)\left(\sin\frac{\theta}{2}\sigma_x\sigma_y\right)$$

$$+\left(-\sin\frac{\theta}{2}\sigma_x\sigma_y\right)\left(\cos\frac{\theta}{2}\mathbb{1}\right) - \left(\sin\frac{\theta}{2}\right)^2\left(\sigma_x\sigma_y\right)^2$$

$$\left(\cos\frac{\theta}{2}\right)^2\mathbb{1} + 0 - \left(\sin\frac{\theta}{2}\right)^2\mathbb{1} \quad = \quad \mathbb{1}$$