# Lecture 1 Assignment

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1. Recall the definition of a rational number, denoted as $\mathbb{Q}$. Prove that the Euler's number $e = \sum_{k=0}^{\infty} \frac{1}{k!} \notin \mathbb{Q}$. A factorial is defined as $k! = (k)(k-1)(k-2)(k-3)..., \forall k \in \mathbb{Z}^{+}$, note that $0! = 1$. Furthermore, a sum notation $\sum_{k=0}^{\infty} k = 0+1+2+3+...+...$

   Suppose that $e$ is rational, such that $e=\frac{n}{d}$, where $n$ and $d$ are positive integers.
$$\frac{n}{d}=e=\sum_{k=0}^{\infty}\frac{1}{k!}$$
   Multiply both sides by $d!$, we get the following equation
$$\begin{aligned}\frac{n}{d}d! &= d!\sum_{k=0}^{\infty}\frac{1}{k!}\\ n(d-1)! &= d!\left[\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{d!}+\frac{1}{(d+1)!}+\frac{1}{(d+2)!}+\frac{1}{(d+3)!}+...\right]\\ &= \left(\frac{d!}{0!}+\frac{d!}{1!}+\frac{d!}{2!}+\dots+\frac{d!}{d!}\right)+\left[\frac{d!}{(d+1)!}+\frac{d!}{(d+2)!}+\frac{d!}{(d+3)!}+...\right]\\ &= \sum_{k=0}^{d}\frac{d!}{k!}+\sum_{k=d+1}^{\infty}\frac{d!}{k!}\end{aligned}$$
   The term $n(d-1)!$ is clearly an integer. The first sum is also an integer, since $k \le d$. The second sum we can simplify as follows
$$\frac{1}{d+1}+\frac{1}{(d+1)(d+2)}+\frac{1}{(d+1)(d+2)(d+3)}+...$$
   The second sum is clearly greater than 0, and we can see (by considering respective terms) that
$$\left[\frac{1}{d+1}+\frac{1}{(d+1)(d+2)}+\frac{1}{(d+1)(d+2)(d+3)}+...\right]<\sum_{k=0}^{\infty}\left(\frac{1}{d+1}\right)^{(k+1)}$$
   The right-hand side of the inequality is a geometric series, and we used the formula to show that the second sum is going to be less than 1
$$\begin{aligned}\sum_{k=0}^{\infty}\left(\frac{1}{d+1}\right)^{(k+1)} &= \left[\frac{1}{d+1}+\frac{1}{(d+1)^2}+\frac{1}{(d+1)^3}+...\right]\\ &= \frac{\frac{1}{d+1}}{1-\frac{1}{d+1}}=\frac{1}{d}\le 1\end{aligned}$$
   With this logical statement, the second sum is greater than 0 and less than 1, which is not an integer. This is a contradiction, since $n(d-1)!$ is an integer, and the second sum is not an integer
$$n(d-1)! \neq \sum_{k=0}^{d}\frac{d!}{k!}+\sum_{k=d+1}^{\infty}\frac{d!}{k!}$$
   Therefore, $e$ is irrational since $\mathbb{Z}=\mathbb{Z}+\overline{\mathbb{Z}}$ is simply impossible.

2. Prove Minkowski's Inequality for sums, $\forall (p>1, (a_k, b_k)>0)$:
$$\left(\sum_{k=1}^{n}|a_k+b_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}|b_k|^p\right)^{\frac{1}{p}}$$

   Define:
$$q=\frac{p}{p-1}$$
   Then:
$$\frac{1}{p}+\frac{1}{q}=\frac{1}{p}+\frac{p-1}{p}=1$$

It follows that:

$$\begin{aligned}
\sum_{k=1}^{n}(a_k+b_k) &= \sum_{k=1}^{n} a_k (a_k+b_k)^{p-1} + \sum_{k=1}^{n} b_k (a_k+b_k)^{p-1} \\
&\leq \left(\sum_{k=1}^{n} {a_k}^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \left((a_k+b_k)^{p-1}\right)^q\right)^{\frac{1}{q}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} \left((a_k+b_k)^{p-1}\right)^q\right)^{\frac{1}{q}} \\
&\quad \text{(Holder's Inequality for Sums, applied twice)} \\
&= \left(\sum_{k=1}^{n} {a_k}^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (a_k+b_k)^p\right)^{\frac{1}{q}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} (a_k+b_k)^p\right)^{\frac{1}{q}} \\
&\quad \text{(by the power of power property and hypothesis: } (p-1)q=p) \\
&= \left(\left(\sum_{k=1}^{n} {a_k}^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} {b_k}^p\right)^{\frac{1}{p}}\right) \left(\sum_{k=1}^{n} (a_k+b_k)^p\right)^{\frac{1}{q}}
\end{aligned}$$

$$\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{k=1}^{n} {a_k}^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} {b_k}^p\right)^{\frac{1}{p}} \quad \text{(Dividing by:)} \left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{\frac{1}{q}}$$

$$\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} {a_k}^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} {b_k}^p\right)^{\frac{1}{p}} \quad \left(\text{as } 1-\frac{1}{q}=p\right)$$

3. Prove the triangle inequality $|x+y| \leq |x|+|y|, \forall (x,y) \in \mathbb{R}$.
   Proof of Triangle Inequality:
   For any real numbers $x$ and $y$, we want to prove that $|x+y| \leq |x|+|y|$.

   Consider the cases:

   Case 1: $x+y \geq 0$
   In this case, $|x+y| = x+y$, $|x| = x$, and $|y| = y$. Therefore, we have:
   $|x+y| = x+y = |x|+|y|$

   Case 2: $x+y < 0$
   In this case, $|x+y| = -(x+y)$, $|x| = -x$, and $|y| = -y$. Therefore, we have:
   $|x+y| = -(x+y) = -x-y = |x|+|y|$

   Conclusion:
   In both cases, we have $|x+y| \leq |x|+|y|$. Thus, the triangle inequality holds for all real numbers $x$ and $y$.
4. Prove Sedrakyan's Lemma $\forall u_i, v_i \in \mathbb{R}^+$:

$$\frac{(\sum_{i=1}^{n} u_i)^2}{\sum_{i=1}^{n} v_i} \leq \sum_{i=1}^{n} \frac{(u_i)^2}{v_i} \tag{1}$$

   To prove Sedrakyan's Lemma, we can use Cauchy-Schwarz Inequality.
   Let's denote $x_i = \sqrt{vi}$ and $y_i = u_i$. Then the given inequality becomes

$$\left(\frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} v_i}}\right)^2 \leq \sum_{i=1}^{n} \frac{(x_i)^2 (y_i)^2}{v_i} \tag{2}$$

   This is essentially the squared form of the Cauchy-Schwarz Inequality, which states

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \tag{3}$$

   where

$$a_i = x_i \sqrt{\frac{v_i}{\sum_{i=1}^n v_i}} \tag{4}$$

$$b_i = \frac{y_i}{\sqrt{\frac{v_i}{\sum_{i=1}^n v_i}}} \tag{5}$$

By applying the Cauchy-Schwarz Inequality to (4) and (5), we get

$$\left(\sum_{i=1}^n x_i y_i \sqrt{\frac{v_i}{\sum_{i=1}^n v_i}}\right)^2 \leq \left(\sum_{i=1}^n (x_i)^2 \frac{v_i}{\sum_{i=1}^n v_i}\right)\left(\sum_{i=1}^n (y_i)^2 \frac{1}{\frac{v_i}{\sum_{i=1}^n v_i}}\right) \tag{6}$$

Simplify

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \leq \left(\sum_{i=1}^n (x_i)^2\right)\left(\sum_{i=1}^n (y_i)^2\right) \tag{7}$$

Which is equivalent to

$$\left(\frac{\sum_{i=1}^n u_i}{\sqrt{\sum_{i=1}^n v_i}}\right)^2 \leq \sum_{i=1}^n \frac{(u_i)^2}{v_i} \tag{8}$$

Therefore, Sedrakyan's Lemma is proved.