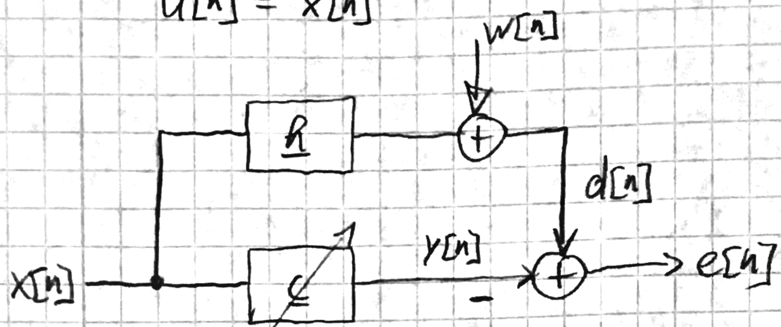
Problem 1.2

ad a) $S_1(z) = S_2(z) = 1 \qquad \sigma_u^2 = 1 \qquad \sigma_w^2 = \frac{1}{4}$

$u[n] = x[n]$

$$\underline{h} = \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}$$

$$\underline{c} = [c_0 \; c_1 \; c_2]^T$$



a1) $r_{xx}[k] = E(x[n+k]\,x^*[n]) = \sigma_w^2 \cdot \delta[k]$

$k=0$:

$$r_{xx}[0] = E(x[n]\,x^*[n]) = \sigma_w^2$$

$k=1$:

$$r_{xx}[1] = E(x[n+1]\,x^*[n]) = 0$$

$\vdots$

$$R_{xx} = E(\underline{x}[n]\,\underline{x}^H[n])$$

$$= E\left(\begin{bmatrix} x[n] \\ x[n-1] \\ x[n-2] \\ \vdots \\ x[n-N+1] \end{bmatrix} \cdot [x^*[n] \; x^*[n-1] \; \cdots \; x^*[n-N+1]] \right)$$

$$= E\left(\begin{bmatrix} x[n]x^*[n] & x[n]x^*[n-1] & \cdots & x[n]x^*[n-N+1] \\ x[n-1]x^*[n] & x[n-1]x^*[n-1] & \cdots & x[n-1]x^*[n-N+1] \\ \vdots & & & \\ x[n-N+1]x^*[n] & x[n-N+1]x^*[n-1] & \cdots & x[n-N+1]x^*[n-N+1] \end{bmatrix} \right)$$

$$\underline{R}_{xx} = \begin{bmatrix} \sigma_u^2 & 0 & 0 & \\ 0 & \sigma_u^2 & 0 & \\ 0 & 0 & \sigma_u^2 & \\ \vdots & & & \ddots \end{bmatrix}$$

a2) $d[n] = \underline{h}^T \underline{x}[n] + w[n]$

$$\underline{p} = E\left(d[n]\,\underline{x}^*[n]\right) = E\left(\underline{x}[n]\,d^*[n]\right)$$

$$= E\left(\underline{x}[n]\,\underbrace{\underline{h}^T \underline{x}[n]}_{(\cdot)^T} + x[n]\,w[n]\right)$$

$$= E\left(\underline{x}[n]\,\underline{x}^T[n]\,\underline{h}\right) + \overbrace{E(x[n])\,E(w[n])}^{=0}$$

each uncorrelated and of zero-mean

$$= \underline{R}_{xx} \cdot \underline{h}$$

$$= \sigma_u^2\,\underline{I} \cdot \underline{h} \qquad \underline{p} = \begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \end{bmatrix}$$

$$= \sigma_u^2\,\underline{a} = \underline{h}$$

a3) $\underline{C}_{MSE} = \underline{R}_{xx}^{-1}\,\underline{p} = \underline{R}_{xx}^{-1}\,\underline{R}_{xx} \cdot \underline{h} = \underline{h}$

$$\underline{C}_{MSE} = \underline{h} = \begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \end{bmatrix}$$

ad a 4) $e[n] = d[n] - y[n]$

$$= \underline{h}^T \underline{u}[n] + w[n] - \underline{c}^T \underline{u}[n]$$

$$e[n]^2 = (\underline{h}^T \underline{u}[n] + w[n] - \underline{c}^T \underline{u}[n])(\underline{h}^T \underline{u}[n] + w[n] - \underline{c}^T \underline{u}[n])$$

$$= \underline{h}^T \underline{u}[n] \underline{u}^T \underline{h} + \underline{h}^T \underline{u}[n] w[n] - \underline{h}^T \underline{u}[n] \underline{u}^T[n] \underline{c} + \underline{h}^T \underline{u}[n] w[n] + w[n] w[n]$$

$$- \underline{c}^T \underline{u}[n] w[n] - \underline{c}^T \underline{u}[n] \underline{u}^T[n] \underline{h} - \underline{c}^T \underline{u}[n] w[n] + \underline{c}^T \underline{u}[n] \underline{u}^T[n] \underline{c}$$

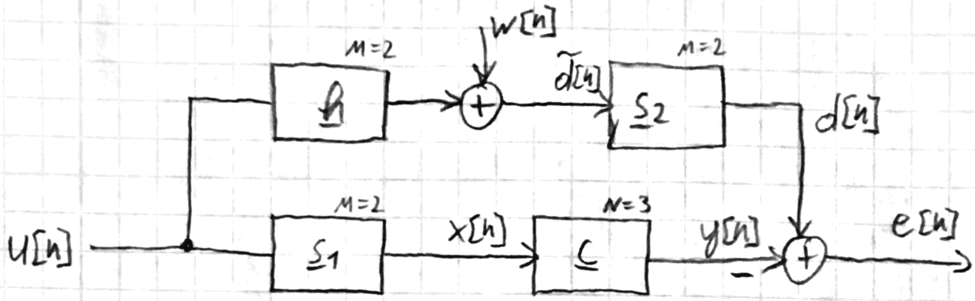
$$E(|e[n]|^2) = \underline{h}^T \sigma_u^2 \underline{I} \underline{h} - \underline{h}^T \sigma_u^2 \underline{I} \underline{c} + \sigma_w^2 - \underline{c}^T \sigma_u^2 \underline{I} \underline{h} + \underline{c}^T \sigma_u^2 \underline{I} \underline{c}$$

$\underline{h} = \underline{c}$ -> all terms cancle out

$$E(|e[n]|^2) = \sigma_w^2 = \underline{\underline{\frac{1}{4}}} = J_{min}$$

ad h) $\quad S_1(z) = S_2(z) = 1 + 0.5\,z^{-1}$

$\underline{s}_1 = [1 \;\; 0.5]^T \qquad \underline{s}_2 = [1 \;\; 0.5]^T$



$\tilde{d}[n] = \underline{h}^T \underline{u}[n] + w[n] \qquad\qquad x[n] = \underline{s}_1^T \underline{u}[n]$

$d[n] = \underline{s}_2^T \underline{\tilde{d}}[n] \qquad\qquad y[n] = \underline{c}^T \underline{x}[n]$

$$r_{xx}[k] = E\left(x[n+k]\, x^*[n]\right)$$
$$= E\left(\underline{s}_1^T \underline{u}[n+k]\, \underline{s}_1^T \underline{u}[n]\right) = E\left(\underline{s}_1^T \underline{u}[n+k]\, \underline{u}^T[n]\, \underline{s}_1\right)$$
$$= \underline{s}_1^T\, E\left(\underline{u}[n+k]\, \underline{u}^T[n]\right) \underline{s}_1^T$$

$k=0$:

$$r_{xx}[0] = \underline{s}_1^T \underbrace{E\left(\underline{u}[n]\, \underline{u}^T[n]\right)}_{\sigma_u^2 \underline{I}} \underline{s}_1 = \underline{s}_1^T \underline{s}_1 \sigma_u^2 = \|\underline{s}_1\|_2^2\, \sigma_u^2$$

$$\|\underline{s}_1\|_2^2 = [1 \;\; 0.5] \cdot \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = 1 + 0.25 = \frac{5}{4} \qquad \sigma_u^2 = 1$$

$$r_{xx}[0] = \frac{5}{4}$$

$k=1$:

$$r_{xx}[k] = \underline{s}_1^T\, E\big(\underline{u}[\overbrace{n+k}^{m}]\, \underline{u}^T[n]\big)\, \underline{s}_1 \qquad n = m-k$$

$$r_{xx}[1] = \underline{s}_1^T\, E\left(\underline{u}[m]\, \underline{u}^T[m-k]\right) \underline{s}_1$$

$$= \underline{s}_1^T \cdot E\left(\begin{bmatrix} u[m] \\ u[m-1] \\ u[m-2] \\ \vdots \\ u[m-N+1] \end{bmatrix} \cdot \begin{bmatrix} u[m-1] & u[m-2] & \cdots & u[m-N] \end{bmatrix} \right) \underline{s}_1$$

$$r_{xx}[1] = \underline{s}_1^T \cdot E\left(\begin{bmatrix} u[m]\,u[m-1] & u[m]\,u[m-2] & \cdots & u[m]\,u[m-N] \\ u[m-1]\,u[m-1] & u[m-1]\,u[m-2] & & u[m-1]\,u[m-N] \\ u[m-2]\,u[m-1] & u[m-2]\,u[m-2] & & \vdots \\ \vdots & \vdots \quad \vdots & & u[m-N]\,u[m-N] \\ u[m-N+1]\,u[m-1] & u[m-N+1]\,u[m-2] & & u[m-N+1]\,u[m-N] \end{bmatrix}\right) \underline{s}_1$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 & 0 \\ 0 & \sigma_u^2 & 0 & 0 \\ 0 & 0 & \sigma_u^2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sigma_u^2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

$$r_{xx}[1] = \frac{1}{2}\sigma_u^2 = \frac{1}{2}$$

$k = 2$:

$$r_{xx}[2] = \underline{s}_1^T\, E\left(\begin{bmatrix} u[m]\,u[m-2] & \cdots & \\ u[m-1]\,u[m-2] & & \\ u[m-2]\,u[m-2] & & \\ \vdots & u[m-3]\,u[m-3] & \\ \vdots & & \ddots \end{bmatrix}\right) \underline{s}_1$$

$$= \underline{s}_1^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 & 0 \\ 0 & \sigma_u^2 & 0 & 0 \end{bmatrix} \underline{s}_1 = 0$$

$$\underline{R}_{xx} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] \\ r_{xx}[-1] & r_{xx}[0] & r_{xx}[1] \\ r_{xx}[-2] & r_{xx}[-1] & r_{xx}[0] \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{5}{2} & 1 & 0 \\ 1 & \frac{5}{2} & 1 \\ 0 & 1 & \frac{5}{2} \end{bmatrix}$$

$$\underline{R}_{xx}^{-1} = \begin{bmatrix} 0.9882 & -0.4706 & 0.1882 \\ -0.4706 & 1.1765 & -0.4706 \\ 0.1882 & -0.4706 & 0.9882 \end{bmatrix}$$

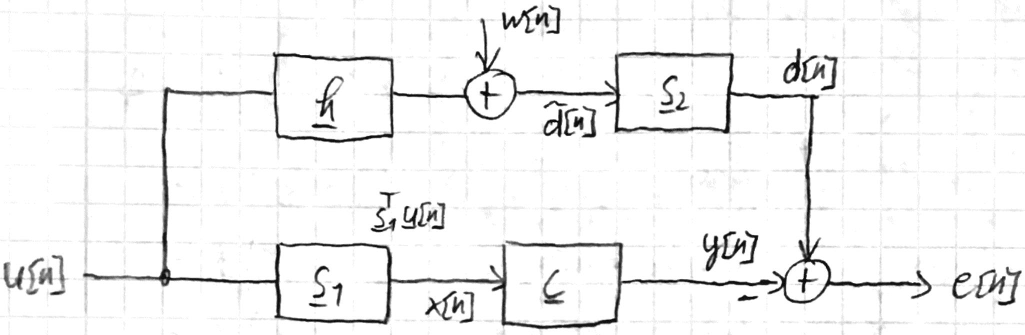
↳ calculated with Matlab

b2) $\underline{p} = E(\underline{x}[n]\, d^*[n])$

$d[n] = \underline{s}_2^T \underline{\tilde{d}}[n]$ $\qquad \underline{s}_1 = \underline{s}_2 = \left[1 \;\; \tfrac{1}{2}\right]^T$

$\tilde{d}[n] = \underline{h}^T \underline{u}[n] + w[n]$ $\qquad \underline{h} = \left[1 \;\; \tfrac{1}{4}\right]^T$

$x[n] = \underline{s}_1^T \underline{u}[n]$



$$x[n] = \underline{s}_1^T \begin{bmatrix} u[n] \\ u[n-1] \end{bmatrix} \qquad x[n+1] = \underline{s}_1^T \begin{bmatrix} u[n+1] \\ u[n] \end{bmatrix}$$

$$\underline{x}^T[n] = \underline{s}_1^T \begin{bmatrix} u[n] & u[n+1] & u[n+2] & \cdots & u[n+N] \\ u[n-1] & u[n] & u[n+1] & \cdots & u[n+N-1] \end{bmatrix}$$

$$\underline{x}[n] = \underbrace{\begin{bmatrix} u[n] & u[n-1] \\ u[n+1] & u[n] \\ u[n+2] & u[n+1] \\ \vdots & \vdots \\ u[n+N] & u[n+N-1] \end{bmatrix}}_{\underline{U}[n]} \underline{s}_1 \qquad \Bigg| \qquad \underline{\tilde{d}}[n] = \underbrace{\begin{bmatrix} u[n] & u[n-1] \\ u[n+1] & u[n] \\ \vdots & \vdots \\ u[n+N] & u[n+N-1] \end{bmatrix}}_{\underline{U}[n]} \underline{h} + \underline{w}[n]$$

$\dim(\underline{U}) = N \times$ coeff. size

$$d[n] = \underline{s}_2^T \underline{\tilde{d}}[n] = \underline{s}_2^T \begin{bmatrix} \tilde{d}[n] \\ \tilde{d}[n-1] \end{bmatrix} = \underline{s}_2^T \begin{bmatrix} [u[n] \;\; u[n-1] \;\; u[n-2]]\, \underline{h} + \underline{w}[n] \\ [[u[n-1] \;\; u[n] \;\; u[n-1]]\, \underline{h} + w[n-1] \end{bmatrix}$$

$$= \underline{s}_2^T (\underline{U}[n]\, \underline{h}) + \underline{s}_2^T \underline{w}[n]$$

$$\underline{d}[n] = \underbrace{\begin{bmatrix} \tilde{d}[n] & \tilde{d}[n-1] \\ \tilde{d}[n+1] & \tilde{d}[n] \\ \vdots & \vdots \\ \tilde{d}[n+N] & \tilde{d}[n+N-1] \end{bmatrix}}_{\tilde{D}[n]} \underline{s}_2 = \left[\underline{U}[n]\, \underline{h} + \underline{w}[n] \quad \underline{U}[n-1]\, \underline{h} + w[n-1]\right] \underline{s}_2$$

$$p = E(\underline{x}[n]\, d[n]) = E\left(\begin{bmatrix} x[n] \cdot d[n] \\ x[n-1] \cdot d[n] \\ x[n-2] \cdot d[n] \end{bmatrix}\right)$$

$$x[n] = \underline{s}_1^T \underline{u}[n] = \begin{bmatrix} s_{10} & s_{11} & s_{12} \end{bmatrix} \cdot \begin{bmatrix} u[n] \\ u[n-1] \\ u[n-2] \end{bmatrix} = s_{10} u[n] + s_{11} u[n-1] + s_{12} u[n-2]$$

$$\tilde{d}[n] = \underline{h}^T \underline{u}[n] + w[n] = \begin{bmatrix} h_0 & h_1 & h_2 \end{bmatrix} \begin{bmatrix} u[n] \\ u[n-1] \\ u[n-2] \end{bmatrix} + w[n]$$

$$d[n] = \underline{s}_2^T \underline{\tilde{d}}[n] = \underline{s}_2^T \begin{bmatrix} \tilde{d}[n] \\ \tilde{d}[n-1] \\ \tilde{d}[n-2] \end{bmatrix} = \underline{s}_2^T \begin{bmatrix} \underline{h}^T \underline{u}[n] + w[n] \\ \underline{h}^T \underline{u}[n-1] + w[n-1] \\ \underline{h}^T \underline{u}[n-2] + w[n-2] \end{bmatrix} = s_{20} \underline{h}^T \underline{u}[n] + w[n] s_{20} + s_{21} \underline{h}^T \underline{u}[n-1] + s_{21} w[n-1] + s_{22} \underline{h}^T \underline{u}[n-2] + s_{22} w[n-2]$$

$$p = E\left(\left(s_{20} \underline{h}^T \underline{u}[n] + w[n] s_{20} + s_{21} \underline{h}^T \underline{u}[n-1] + s_{21} w[n-1] + s_{22} \underline{h}^T \underline{u}[n-2] + s_{22} w[n-2]\right) \cdot \begin{bmatrix} \underline{s}_1^T \underline{u}[n] \\ \underline{s}_1^T \underline{u}[n-1] \\ \underline{s}_1^T \underline{u}[n-2] \end{bmatrix} \right)$$

$w[n]$ ... zero mean, uncorrel. to $u[n-k]$

all terms with $w[n]u[n-k]$ can be neglected

$$= E\left(\begin{bmatrix} s_{20} \underline{h}^T \underline{u}[n] \underline{u}^T[n] \underline{s}_1 + w[n] s_{20} \underline{u}^T[n] \underline{s}_1 + s_{21} \underline{h}^T \underline{u}[n-1] \underline{u}^T[n] \underline{s}_1 + s_{21} w[n] \underline{u}^T[n] \underline{s}_1 + s_{22} \underline{h}^T \underline{u}[n-2] \underline{u}^T[n] \underline{s}_1 + s_{22} w[n-2] \underline{u}^T[n] \underline{s}_1 \\ s_{20} \underline{h}^T \underline{u}[n] \underline{u}^T[n-1] \underline{s}_1 + w[n] s_{20} \underline{u}^T[n-1] \underline{s}_1 + s_{21} \underline{h}^T \underline{u}[n-1] \underline{u}^T[n-1] \underline{s}_1 + s_{21} w[n] \underline{u}^T[n-1] \underline{s}_1 + s_{22} \underline{h}^T \underline{u}[n-2] \underline{u}^T[n-1] \underline{s}_1 + s_{22} w[n-2] \underline{u}^T[n-1] \underline{s}_1 \\ s_{20} \underline{h}^T \underline{u}[n] \underline{u}^T[n-2] \underline{s}_1 + w[n] s_{20} \underline{u}^T[n-2] \underline{s}_1 + s_{21} \underline{h}^T \underline{u}[n-1] \underline{u}^T[n-2] \underline{s}_1 + s_{21} w[n] \underline{u}^T[n-2] \underline{s}_1 + s_{22} \underline{h}^T \underline{u}[n-2] \underline{u}^T[n-2] \underline{s}_1 + s_{22} w[n-2] \underline{u}^T[n-2] \underline{s}_1 \end{bmatrix}\right)$$

$$p = \begin{bmatrix} \begin{bmatrix} 1 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} \sigma_u^2 & & 0 \\ & \sigma_u^2 & 0 \\ 0 & & \sigma_u^2 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} \sigma_u^2 & & 0 \\ & \sigma_u^2 & \\ 0 & & \sigma_u^2 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 1 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{4} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \\ 0 + \frac{1}{2} \begin{bmatrix} \frac{1}{4} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{11}{8} \\ \frac{13}{16} \\ \frac{1}{8} \end{bmatrix}$$

Autocorrelation matrices for different vectors $\underline{u}[n]$:

$$E(\underline{u}[n]\underline{u}^T[n]) = E\left(\begin{bmatrix} u[n]u[n] & u[n]u[n-1] & u[n]u[n-2] \\ u[n-1]u[n] & u[n-1]u[n-1] & u[n-1]u[n-2] \\ u[n-2]u[n] & u[n-2]u[n-1] & u[n-2]u[n-2] \end{bmatrix}\right) = \begin{bmatrix} \sigma_u^2 & & 0 \\ & \sigma_u^2 & \\ 0 & & \sigma_u^2 \end{bmatrix}$$

$$E(\underline{u}[n-1]\underline{u}^T[n]) = E\left(\begin{bmatrix} u[n-1]u[n] & u[n-1]u[n-1] & u[n-1]u[n-2] \\ u[n-2]u[n] & u[n-2]u[n-1] & u[n-2] \text{ -"- } \\ u[n-3]u[n] & u[n-3] \text{ -"- } & u[n-3] \text{ -"- } \end{bmatrix}\right) = \begin{bmatrix} 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E(\underline{u}[n-2]\underline{u}^T[n]) = E\left(\begin{bmatrix} u[n-2]u[n] & u[n-2]u[n-1] & u[n-2]u[n-2] \\ u[n-3] \text{ -"- } & u[n-3] \text{ -"- } & \vdots \quad \text{ -"- } \\ u[n-4] \text{ -"- } & u[n-4] \text{ -"- } & \vdots \quad \text{ -"- } \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E(\underline{u}[n]\underline{u}^T[n-1]) = \begin{bmatrix} 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \end{bmatrix} = E((\underline{u}[n-1]\underline{u}^T[n])^T)$$

$$E(\underline{u}[n]\underline{u}^T[n-2]) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \end{bmatrix} \qquad E(\underline{u}[n-1]\underline{u}[n-1]) = \begin{bmatrix} \sigma_u^2 & & 0 \\ & \sigma_u^2 & \\ 0 & & \sigma_u^2 \end{bmatrix}$$

$$E(\underline{u}[n-2]\underline{u}^T[n-1]) = E(u[n-1]u^T[n]) = \begin{bmatrix} 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \end{bmatrix}$$

↳ „one step below"

ad b3)

$$\underline{c}_{MSE} = \underline{R}_{xx}^{-1}\,\underline{p}$$

$$= \left(\frac{1}{2} \begin{bmatrix} \frac{5}{2} & 1 & 0 \\ 1 & \frac{5}{2} & 1 \\ 0 & 1 & \frac{5}{2} \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} \frac{11}{8} \\ \frac{13}{16} \\ \frac{1}{8} \end{bmatrix} = \begin{bmatrix} 0.9882 & -0.4706 & 0.1882 \\ -0.4706 & 1.1765 & -0.4706 \\ 0.1882 & -0.4706 & 0.9882 \end{bmatrix} \cdot \underline{p}$$

calculated with Matlab

$$\underline{c}_{MSE} = \begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \end{bmatrix} \xrightarrow{\text{compare to } \underline{h}} \underline{h} = \begin{bmatrix} 1 \\ 0.25 \\ 0 \end{bmatrix}$$

ad b4) $e[n] = d[n] - y[n]$

$$y[n] = \underline{c}^T \underline{x}[n] = \underline{c}^T \begin{bmatrix} \underline{s}_1^T \underline{u}[n] \\ \underline{s}_1^T \underline{u}[n-1] \\ \underline{s}_1^T \underline{u}[n-2] \end{bmatrix} = c_0 \underline{s}_1^T \underline{u}[n] + c_1 \underline{s}_1^T \underline{u}[n-1] + c_2 \underline{s}_1^T \underline{u}[n-2]$$

$$e[n] = s_{20}\, \underline{h}^T \underline{u}[n] + w[n] s_{20} + s_{21} \underline{h}^T \underline{u}[n-1] + s_{21} w[n-1] + s_{22} \underline{h}^T \underline{u}[n-2] + s_{22} w[n-2]$$
$$- c_0 \underline{s}_1^T \underline{u}[n] - c_1 \underline{s}_1^T \underline{u}[n-1] - c_2 \underline{s}_1^T \underline{u}[n-2]$$

$\rightarrow$ set $c_2 = 0;\ s_{20} = 1 \quad s_{22} = 0$

$$e[n] = \underline{h}^T \underline{u}[n] + w[n] + s_{21} \underline{h}^T \underline{u}[n-1] + s_{21} w[n-1] - c_0 \underline{s}_1^T \underline{u}[n] - c_1 \underline{s}_1^T \underline{u}[n-1]$$

$$|e[n]|^2 = \left(\underline{h}^T \underline{u}[n] + w[n] + s_{21} \underline{h}^T \underline{u}[n-1] + s_{21} w[n-1] - c_0 \underline{s}_1^T \underline{u}[n] - c_1 \underline{s}_1^T \underline{u}[n-1] \right) \cdot$$
$$\left(\underline{h}^T \underline{u}[n] + w[n] + s_{21} \underline{h}^T \underline{u}[n-1] + s_{21} w[n-1] - c_0 \underline{s}_1^T \underline{u}[n] - c_1 \underline{s}_1^T \underline{u}[n-1] \right)$$

$$= \underline{h}^T \underline{u}[n]\, \underline{h}^T \underline{u}[n] + \underline{h}^T \underline{u}[n]\, w[n] + \underline{h}^T \underline{u}[n]\, s_{21} \underline{h}^T \underline{u}[n-1] + \underline{h}^T \underline{u}[n]\, s_{21} w[n-1] - \underline{h}^T \underline{u}[n]\, c_0 \underline{s}_1^T \underline{u}[n]$$
$$- \underline{h}^T \underline{u}[n]\, c_1 \underline{s}_1^T \underline{u}[n-1] + w[n]\, \underline{h}^T \underline{u}[n] + w[n]\, w[n] + w[n]\, s_{21} \underline{h}^T \underline{u}[n-1] + s_{21} w[n]\, w[n-1]$$
$$- c_0 w[n]\, \underline{s}_1^T \underline{u}[n] - c_1 w[n]\, \underline{s}_1^T \underline{u}[n-1] + s_{21} \underline{h}^T \underline{u}[n-1]\, \underline{h}^T \underline{u}[n] + s_{21} \underline{h}^T \underline{u}[n-1]\, w[n]$$
$$+ s_{21} \underline{h}^T \underline{u}[n-1]\, s_{21} \underline{h}^T \underline{u}[n-1] + s_{21} \underline{h}^T \underline{u}[n-1]\, s_{21} w[n-1] - s_{21} \underline{h}^T \underline{u}[n-1]\, c_0 \underline{s}_1^T \underline{u}[n]$$
$$- s_{21} \underline{h}^T \underline{u}[n-1]\, c_1 \underline{s}_1^T \underline{u}[n-1] + s_{21} w[n-1]\, \underline{h}^T \underline{u}[n] + s_{21} w[n-1]\, w[n] + s_{21} w[n-1]\, s_{21} \underline{h}^T \underline{u}[n-1]$$

$$+ s_{21} w[n-1] s_{21} w[n-1] - s_{21} w[n-1] c_0 \mathbf{s}_1^T \mathbf{u}[n] - s_{21} w[n-1] c_1 \mathbf{s}_1^T \mathbf{u}[n-1] - c_0 \mathbf{s}_1^T \mathbf{u}[n] \mathbf{h}^T \mathbf{u}[n]$$
$$- c_0 \mathbf{s}_1^T \mathbf{u}[n] w[n] - c_0 \mathbf{s}_1^T \mathbf{u}[n] s_{21} \mathbf{h}^T \mathbf{u}[n-1] - c_0 \mathbf{s}_1^T \mathbf{u}[n] s_{21} w[n-1] + c_0^2 \mathbf{s}_1^T \mathbf{u}[n] \mathbf{s}_1^T \mathbf{u}[n]$$
$$+ c_0 c_1 \mathbf{s}_1^T \mathbf{u}[n] \mathbf{s}_1^T \mathbf{u}[n-1] - c_1 \mathbf{s}_1^T \mathbf{u}[n-1] \mathbf{h}^T \mathbf{u}[n] - c_1 \mathbf{s}_1^T \mathbf{u}[n-1] w[n] - c_1 \mathbf{s}_1^T \mathbf{u}[n-1] s_{21} \mathbf{h}^T \mathbf{u}[n-$$
$$- c_1 \mathbf{s}_1^T \mathbf{u}[n-1] s_{21} w[n-1] + c_0 c_1 \mathbf{s}_1^T \mathbf{u}[n-1] \mathbf{u}^T[n] \mathbf{s}_1 + c_1^2 \mathbf{s}_1^T \mathbf{u}[n-1] \mathbf{u}^T[n-1] \mathbf{s}_1$$

$$E(|e[n]|^2) = \mathbf{h}^T E(\mathbf{u}[n]\mathbf{u}^T[n]) \mathbf{h} + \mathbf{h}^T E(\mathbf{u}[n] w[n]) + s_{21} \mathbf{h}^T E(\mathbf{u}[n]\mathbf{u}^T[n-1]) \mathbf{h}$$
$$+ s_{21} \mathbf{h}^T E(\mathbf{u}[n] w[n-1]) - c_0 \mathbf{h}^T E(\mathbf{u}[n]\mathbf{u}^T[n]) \mathbf{s}_1 - c_1 \mathbf{h}^T E(\mathbf{u}[n]\mathbf{u}^T[n-1]) \mathbf{s}_1$$
$$+ E(w[n]\mathbf{u}^T[n]) \mathbf{h} + E(w[n]w[n]) + s_{21} E(w[n]\mathbf{u}^T[n-1]) \mathbf{h} + s_{21} E(w[n]w[n-1])$$
$$- c_0 E(w[n]\mathbf{u}^T[n]) \mathbf{s}_1 - c_1 E(w[n]\mathbf{u}^T[n-1]) \mathbf{s}_1 + s_{21} \mathbf{h}^T E(\mathbf{u}[n-1]\mathbf{u}^T[n]) \mathbf{h}$$
$$+ s_{21} \mathbf{h}^T E(\mathbf{u}[n-1] w[n]) + s_{21}^2 \mathbf{h}^T E(\mathbf{u}[n-1]\mathbf{u}^T[n-1]) \mathbf{h} + s_{21}^2 \mathbf{h}^T E(\mathbf{u}[n-1] w[n-1])$$
$$- s_{21} \mathbf{h}^T E(\mathbf{u}[n-1]\mathbf{u}^T[n]) \mathbf{s}_1 c_0 - s_{21} c_1 \mathbf{h}^T E(\mathbf{u}[n-1]\mathbf{u}^T[n-1]) \mathbf{s}_1 + s_{21} E(w[n-1]\mathbf{u}[n])$$
$$+ s_{21} E(w[n-1]w[n]) + s_{21}^2 E(w[n-1]\mathbf{u}^T[n-1]) \mathbf{h} + s_{21}^2 E(w[n-1]w[n-1])$$
$$- s_{21} c_0 E(w[n-1]\mathbf{u}^T[n]) \mathbf{s}_1 - s_{21} c_1 E(w[n-1]\mathbf{u}^T[n-1]) \mathbf{s}_1 - c_0 \mathbf{s}_1^T E(\mathbf{u}[n]\mathbf{u}^T[n]) \mathbf{h}$$
$$- c_0 \mathbf{s}_1^T E(\mathbf{u}[n] w[n]) - c_0 s_{21} \mathbf{s}_1^T E(\mathbf{u}[n]\mathbf{u}^T[n-1]) \mathbf{h} - c_0 s_{21} \mathbf{s}_1^T E(\mathbf{u}[n] w[n-1])$$
$$+ c_0^2 \mathbf{s}_1^T E(\mathbf{u}[n]\mathbf{u}^T[n]) \mathbf{s}_1 + c_0 c_1 \mathbf{s}_1^T E(\mathbf{u}[n]\mathbf{u}^T[n-1]) \mathbf{s}_1 - c_1 \mathbf{s}_1^T E(\mathbf{u}[n-1]\mathbf{u}^T[n]) \mathbf{h}$$
$$- c_1 \mathbf{s}_1^T E(\mathbf{u}[n-1] w[n]) - c_1 s_{21} \mathbf{s}_1^T E(\mathbf{u}[n-1]\mathbf{u}^T[n-1]) \mathbf{h} - c_1 s_{21} \mathbf{s}_1^T E(\mathbf{u}[n-1] w[n-1])$$
$$+ c_0 c_1 \mathbf{s}_1^T E(\mathbf{u}[n-1]\mathbf{u}^T[n]) \mathbf{s}_1 + c_1^2 \mathbf{s}_1^T E(\mathbf{u}[n-1]\mathbf{u}^T[n-1]) \mathbf{s}_1$$

$\Rightarrow$ $u[n]$ & $w[n]$ are independent and white noise processes

$\rightarrow$ zero-mean

$$= \mathbf{h}^T \sigma_u^2 \mathbf{I} \mathbf{h} + s_{21} \mathbf{h}^T E(\mathbf{u}[n]\mathbf{u}^T[n-1]) \mathbf{h} - c_0 \mathbf{h}^T \sigma_u^2 \mathbf{I} \mathbf{s}_1 - c_1 \mathbf{h}^T E(\mathbf{u}[n]\mathbf{u}^T[n-1]) \mathbf{s}_1$$
$$+ \sigma_w^2 + s_{21} \mathbf{h}^T E(\mathbf{u}[n-1]\mathbf{u}^T[n]) \mathbf{h} + s_{21}^2 \mathbf{h}^T \sigma_u^2 \mathbf{I} \mathbf{h} - s_{21} \mathbf{h}^T E(\mathbf{u}[n-1]\mathbf{u}^T[n]) \mathbf{s}_1 c_0$$
$$- s_{21} c_1 \mathbf{h}^T \sigma_u^2 \mathbf{I} \mathbf{s}_1 + s_{21}^2 \sigma_w^2 - c_0 \mathbf{s}_1^T \sigma_u^2 \mathbf{I} \mathbf{h} - c_0 s_{21} \mathbf{s}_1^T E(\mathbf{u}[n]\mathbf{u}^T[n-1]) \mathbf{h}$$
$$+ c_0^2 \mathbf{s}_1^T \sigma_u^2 \mathbf{I} \mathbf{s}_1 + c_0 c_1 \mathbf{s}_1^T E(\mathbf{u}[n]\mathbf{u}^T[n-1]) \mathbf{s}_1 - c_1 \mathbf{s}_1^T E(\mathbf{u}[n-1]\mathbf{u}^T[n]) \mathbf{h}$$
$$- c_1 s_{21} \mathbf{s}_1^T \sigma_u^2 \mathbf{I} \mathbf{h} + c_0 c_1 \mathbf{s}_1^T E(\mathbf{u}[n-1]\mathbf{u}^T[n]) \mathbf{s}_1 + c_1^2 \mathbf{s}_1^T \sigma_u^2 \mathbf{I} \mathbf{s}_1$$

$$E(|e[n]|^2) = \begin{bmatrix}1 & \frac{1}{4} & 0\end{bmatrix}\begin{bmatrix}\sigma_u^2 & & 0\\ & \sigma_u^2 & \\ 0 & & \sigma_u^2\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix} + \frac{1}{2}\cdot\begin{bmatrix}1 & \frac{1}{4} & 0\end{bmatrix}\cdot\begin{bmatrix}0 & 0 & 0\\ \sigma_u^2 & 0 & 0\\ 0 & \sigma_u^2 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix}$$

$$- \begin{bmatrix}1 & \frac{1}{4} & 0\end{bmatrix}\begin{bmatrix}\sigma_u^2 & & 0\\ & \sigma_u^2 & \\ 0 & & \sigma_u^2\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} - \frac{1}{4}\begin{bmatrix}1 & \frac{1}{4} & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ \sigma_u^2 & 0 & 0\\ 0 & \sigma_u^2 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} + \frac{1}{4}$$

$$+ \frac{1}{2}\begin{bmatrix}1 & \frac{1}{4} & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_u^2 & 0\\ 0 & 0 & \sigma_u^2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1 & \frac{1}{4} & 0\end{bmatrix}\begin{bmatrix}\sigma_u^2 & & 0\\ & \sigma_u^2 & \\ 0 & & \sigma_u^2\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix}$$

$$- \frac{1}{2}\begin{bmatrix}1 & \frac{1}{4} & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_u^2 & 0\\ 0 & 0 & \sigma_u^2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} - \frac{1}{8}\begin{bmatrix}1 & \frac{1}{4} & 0\end{bmatrix}\begin{bmatrix}\sigma_u^2 & & 0\\ & \ddots & \\ 0 & & \end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} + \frac{1}{16}$$

$$- \begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}\sigma_u^2 & & 0\\ & \ddots & \\ 0 & & \end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ \sigma_u^2 & 0 & 0\\ 0 & \sigma_u^2 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix}$$

$$+ \begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}\sigma_u^2 & & 0\\ & \ddots & \\ 0 & & \end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} + \frac{1}{4}\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ \sigma_u^2 & 0 & 0\\ 0 & \sigma_u^2 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix}$$

$$- \frac{1}{4}\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_u^2 & 0\\ 0 & 0 & \sigma_u^2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix} - \frac{1}{8}\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix}$$

$$+ \frac{1}{4}\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}0 & \sigma_u^2 & 0\\ 0 & 0 & \sigma_u^2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} + \begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix}\frac{1}{16}$$

$$= 1 + \frac{1}{16} + \frac{1}{2}\begin{bmatrix}\frac{1}{4} & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix} - \left(1 + \frac{1}{8}\right) - \frac{1}{4}\begin{bmatrix}\frac{1}{4} & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} + \frac{1}{4}$$

$$+ \frac{1}{2}\begin{bmatrix}0 & 1 & \frac{1}{4}\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix} + \frac{1}{4}\left(1 + \frac{1}{16}\right) - \frac{1}{2}\begin{bmatrix}0 & 1 & \frac{1}{4}\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} - \frac{1}{8}\left(1 + \frac{1}{8}\right) + \frac{1}{16}$$

$$- 1 - \frac{1}{8} - \frac{1}{2}\begin{bmatrix}\frac{1}{2} & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix} + 1 + \frac{1}{4} + \frac{1}{4}\begin{bmatrix}\frac{1}{2} & 0 & 0\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix}$$

$$- \frac{1}{4}\begin{bmatrix}0 & 1 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{4}\\ 0\end{bmatrix} - \frac{1}{8}\left(1 + \frac{1}{8}\right) + \frac{1}{4}\begin{bmatrix}0 & 1 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\ \frac{1}{2}\\ 0\end{bmatrix} + \left(1 + \frac{1}{4}\right)\frac{1}{16}$$

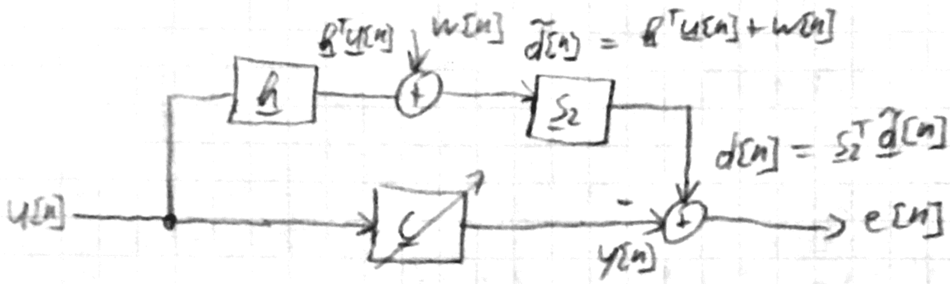
$$= \cancel{1} + \cancel{\tfrac{1}{16}} + \cancel{\tfrac{1}{8}} - \cancel{1} - \cancel{\tfrac{1}{8}} - \cancel{\tfrac{1}{16}} + \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{8}} + \cancel{\tfrac{1}{4}} + \cancel{\tfrac{1}{64}} - \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{8}} - \cancel{\tfrac{1}{64}} + \cancel{\tfrac{1}{16}}$$

$$- \cancel{1} - \cancel{\tfrac{1}{8}} - \cancel{\tfrac{1}{4}} + \cancel{1} + \underline{\tfrac{1}{4}} + \cancel{\tfrac{1}{8}} - \cancel{\tfrac{1}{16}} - \cancel{\tfrac{1}{8}} - \underline{\cancel{\tfrac{1}{64}}} + \cancel{\tfrac{1}{8}} + \frac{1}{16} + \cancel{\tfrac{1}{64}} = \frac{4}{16} + \frac{1}{16}$$

$$= \frac{5}{16} = J_{min}$$

$J_{min} = J_{MSE}(\underline{c}_{opt}) = \frac{5}{16}$

ad c1) $\underline{s}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \qquad \underline{s}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$



$r_{xx}[k] = E(\underline{u}[n+k]\, \underline{u}^*[n])$

$$\left.\begin{aligned} r_{xx}[0] &= E(u[n]\, u[n]) = \sigma_u^2 \\ r_{xx}[1] &= E(u[n+1]\, u[n]) = 0 \\ r_{xx}[2] &= E(u[n+2]\, u[n]) = 0 \end{aligned}\right\} \rightarrow \underline{R}_{xx} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots \\ r_{xx}[-1] & r_{xx}[0] & \\ \vdots & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_u^2 & 0 & 0 & \cdots \\ 0 & \sigma_u^2 & & \\ 0 & & \ddots & \\ \vdots & & & \end{bmatrix}$$

ad c2) $\underline{p} = E(d[n]\, \underline{u}[n])$

$$d[n] = \underline{s}_2^T \begin{bmatrix} \tilde{d}[n] \\ \tilde{d}[n-1] \\ \tilde{d}[n-2] \end{bmatrix} = \underline{s}_2^T \begin{bmatrix} \underline{h}^T \underline{u}[n] + w[n] \\ \underline{h}^T \underline{u}[n-1] + w[n-1] \\ \underline{h}^T \underline{u}[n-2] + w[n-2] \end{bmatrix} = \underline{h}^T \underline{u}[n-1] + w[n-1]$$

$$\underline{p} = E\left(\begin{bmatrix} \underline{h}^T \underline{u}[n-1]\, u[n] + w[n-1]\, u[n] \\ \underline{h}^T \underline{u}[n-1]\, u[n-1] + w[n-1]\, u[n-1] \\ \underline{h}^T \underline{u}[n-1]\, u[n-2] + w[n-1]\, u[n-2] \end{bmatrix} \right)$$

$$= E\left(\begin{bmatrix} \underline{h}^T \begin{bmatrix} u[n-1]\, u[n] \\ u[n-2]\, u[n] \\ u[n-3]\, u[n] \end{bmatrix} \\ \underline{h}^T \begin{bmatrix} u[n-1]\, u[n-1] \\ u[n-2]\ \text{--}\,\text{''}\,\text{--} \\ u[n-3]\ \text{--}\,\text{''}\,\text{--} \end{bmatrix} \\ \underline{h}^T \begin{bmatrix} u[n-1]\, u[n-2] \\ u[n-2]\, u[n-2] \\ u[n-3]\ \text{--}\,\text{''}\,\text{--} \end{bmatrix} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \sigma_u^2 \\ \frac{1}{4}\sigma_u^2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 \\ 1 \\ \frac{1}{4} \end{bmatrix}}}$$

ad c3) $\underline{c}_{opt} = \underline{R}_{xx}^{-1}\underline{p} = (\sigma_u^2 \underline{\underline{I}})^{-1}\underline{p} = \underline{p} = \underline{\begin{bmatrix} 0 \\ 1 \\ \frac{1}{4} \end{bmatrix}}$

ad c4) $J_{min} = J_{MSE}(\underline{c}_{opt}) = E(|e[n]|^2)$

$e[n] = d[n] - y[n]$ $\qquad \underline{c} = \underline{c}_{opt} = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{4} \end{bmatrix}$

$d[n] = \underline{h}^T \underline{u}[n-1] + w[n-1]$

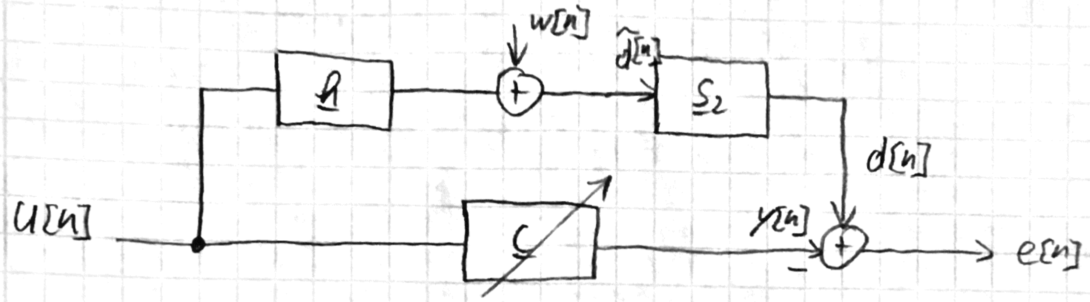
$y[n] = \underline{c}^T \underline{u}[n] \longrightarrow u[n-1] + c_2 u[n-2]$

$e[n] = \underline{h}^T \underline{u}[n-1] + w[n-1] - u[n-1] - c_2 u[n-2]$

$$\begin{aligned}
e^2[n] &= (\underline{h}^T \underline{u}[n-1] + w[n-1] - u[n-1] - c_2 u[n-2]) \cdot (\underline{h}^T \underline{u}[n-1] + w[n-1] \\
&\quad - u[n-1] - c_2 u[n-2]) \\
&= \underline{h}^T \underline{u}[n-1]\underline{u}^T[n-1]\underline{h} + \underline{h}^T\underline{u}[n-1]w[n-1] - \underline{h}^T\underline{u}[n-1]u[n-1] \\
&\quad - c_2\underline{h}^T\underline{u}[n-1]u[n-2] + \underline{h}^T\underline{u}[n-1]w[n-1] + w[n-1]w[n-1] \\
&\quad - u[n-1]w[n-1] - c_2 u[n-2]w[n-1] - \underline{h}^T\underline{u}[n-1]u[n-1] - u[n-1]w[n-1] \\
&\quad + u[n-1]u[n-1] + c_2 u[n-1]u[n-2] - c_2\underline{h}^T\underline{u}[n-1]u[n-2] - c_2 u[n-2]w[n-1] \\
&\quad + c_2 u[n-1]u[n-2] + c_2^2 u[n-2]u[n-2]
\end{aligned}$$

$$\begin{aligned}
E(|e[n]|^2) &= \underline{h}^T E(\underline{u}[n-1]\underline{u}^T[n-1])\underline{h} - \underline{h}^T E(\underline{u}[n-1]u[n-1]) - c_2\underline{h}^T E(\underline{u}[n-1]u[n-2]) \\
&\quad + E(w[n-1]w[n-1]) - \underline{h}^T E(\underline{u}[n-1]u[n-1]) + E(u[n-1]u[n-1]) \\
&\quad + c_2 E(u[n-1]u[n-2]) - c_2\underline{h}^T E(\underline{u}[n-1]u[n-2]) + c_2 E(u[n-1]u[n-2]) \\
&\quad + c_2^2 E(u[n-2]u[n-2]) \\
&= \underline{h}^T \sigma_u^2 \underline{\underline{I}}\,\underline{h} - \underline{h}^T \cdot \begin{bmatrix} \sigma_u^2 \\ 0 \\ 0 \end{bmatrix} - c_2\underline{h}^T \begin{bmatrix} 0 \\ \sigma_u^2 \\ 0 \end{bmatrix} + \sigma_w^2 - \underline{h}^T \begin{bmatrix} \sigma_u^2 \\ 0 \\ 0 \end{bmatrix} + \sigma_u^2 \\
&\quad - c_2\underline{h}^T \begin{bmatrix} 0 \\ \sigma_u^2 \\ 0 \end{bmatrix} + c_2^2\sigma_u^2 \\
&= \begin{bmatrix} 1 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \end{bmatrix} - 1 - \frac{1}{16} + \underline{\frac{1}{4}} - 1 + 1 - \frac{1}{16} + \frac{1}{16} \\
&= \frac{1}{4} = \sigma_w^2 = J_{min}
\end{aligned}$$

ad d1) $\underline{s}_1 = [1\ 0\ 0]^T \qquad \underline{s}_2 = [1\ 0\ 1]^T$



$$r_{xx}[k] = E(u[n+k]\,u[n])$$

↳ unchanged system behaviour at the input of the adaptive filter $\underline{c}$

$$\Rightarrow \underline{R}_{xx} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots \\ r_{xx}[-1] & r_{xx}[0] & \\ \vdots & & \ddots \end{bmatrix} = \sigma_u^2 \underline{I}$$

ad d2) $\underline{p} = E(d[n]\,\underline{u}[n])$

$$\tilde{d}[n] = \underline{h}^T \underline{u}[n] + w[n]$$

$$d[n] = \underline{s}_2^T \underline{\tilde{d}}[n] = \underline{s}_2^T \cdot \begin{bmatrix} \underline{h}^T \underline{u}[n] + w[n] \\ \underline{h}^T \underline{u}[n-1] + w[n-1] \\ \underline{h}^T \underline{u}[n-2] + w[n-2] \end{bmatrix}$$

$$= s_{20}\, \underline{h}^T \underline{u}[n] + s_{20}\, w[n] + s_{22}\, \underline{h}^T \underline{u}[n-2] + s_{22}\, w[n-2]$$

$$\underline{p} = E\left(\begin{bmatrix} s_{20}\, \underline{h}^T \underline{u}[n]\, u[n] + s_{20}\, w[n]\, u[n] + s_{22}\, \underline{h}^T \underline{u}[n-2]\, u[n] + s_{22}\, w[n-2]\, u[n] \\ s_{20}\, \underline{h}^T \underline{u}[n]\, u[n-1] + s_{20}\, w[n]\, u[n-1] + s_{22}\, \underline{h}^T \underline{u}[n-2]\, u[n-1] + s_{22}\, w[n-2]\, u[n-1] \\ s_{20}\, \underline{h}^T \underline{u}[n]\, u[n-2] + s_{20}\, w[n]\, u[n-1] + s_{22}\, \underline{h}^T \underline{u}[n-2]\, u[n-2] + s_{22}\, w[n-2]\, u[n-2] \end{bmatrix}\right)$$

$$\underline{p} = \begin{bmatrix} s_{20}\, \sigma_u^2 \\ \frac{1}{4} s_{20} \\ s_{22}\, \sigma_u^2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 \\ \frac{1}{4} \\ 1 \end{bmatrix}}}$$

ad d3)
$$\underline{c}_{MSE} = \underline{R}_{xx}^{-1}\, \underline{p}$$
$$= \left(\sigma_u^2 \underline{I}\right)^2 \underline{p} = \begin{bmatrix} 1 \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

ad d4)
$$J_{min} = J(\underline{c}_{MSE}) = E\left(|e[n]|^2\right)$$

$$y[n] = \underline{c}^T \underline{u}[n]$$

$$e[n] = d[n] - y[n]$$
$$= s_{20}\, \underline{h}^T \underline{u}[n] + s_{20}\, w[n] + s_{22}\, \underline{h}^T \underline{u}[n-2] + s_{22}\, w[n-2] - \underline{c}^T \underline{u}[n]$$

$$s_{20} = s_{22} = 1$$

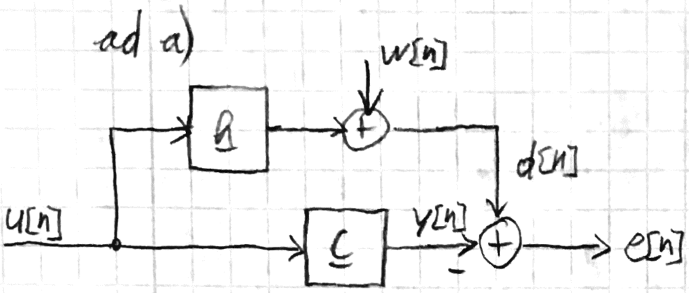
$$e[n]^2 = \left(\underline{h}^T\underline{u}[n] + w[n] + \underline{h}^T\underline{u}[n-2] + w[n-2] - \underline{c}^T\underline{u}[n]\right)\Big(\underline{h}^T\underline{u}[n] + w[n]$$
$$+ \underline{h}^T\underline{u}[n-2] + w[n-2] - \underline{c}^T\underline{u}[n]\Big)$$
$$= \underline{h}^T\underline{u}[n]\underline{u}^T[n]\underline{h} + \underline{h}^T\underline{u}[n]w[n] + \underline{h}^T\underline{u}[n]\underline{u}^T[n-2]\underline{h} + \underline{h}^T\underline{u}[n]w[n-2]$$
$$+ \underline{h}^T\underline{u}[n]w[n] + w[n]w[n] + \underline{h}^T\underline{u}[n-2]w[n] + w[n-2]w[n] - \underline{c}^T\underline{u}[n]w[n]$$
$$- \underline{h}^T\underline{u}[n]\underline{u}^T[n]\underline{c} + \underline{h}^T\underline{u}[n-2]\underline{u}^T[n]\underline{h} + \underline{h}^T\underline{u}[n-2]w[n] + \underline{h}^T\underline{u}[n-2]\underline{u}^T[n-2]\underline{h}$$
$$+ \underline{h}^T\underline{u}[n-2]w[n-2] - \underline{h}^T\underline{u}[n-2]\underline{u}^T[n]\underline{c} + \underline{h}^T\underline{u}[n]w[n-2] + w[n-2]w[n]$$
$$+ \underline{h}^T\underline{u}[n-2]w[n-2] + w[n-2]w[n-2] - \underline{c}^T\underline{u}[n]w[n-2] - \underline{c}^T\underline{u}[n]\underline{u}^T[n]\underline{h}$$
$$- \underline{c}^T\underline{u}[n]w[n] - \underline{c}^T\underline{u}[n]\underline{u}^T[n-2]\underline{h} - \underline{c}^T\underline{u}[n]w[n-2] + \underline{c}^T\underline{u}[n]\underline{u}[n]\underline{c}$$

$$E\left(|e[n]|^2\right) = \underline{h}^T\sigma_u^2\underline{I}\,\underline{h} + \underline{h}^T\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \end{bmatrix}\underline{h} + \sigma_w^2 - \underline{h}^T\sigma_u^2\underline{I}\,\underline{c} + \underline{h}^T\begin{bmatrix} 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{h}$$
$$+ \underline{h}^T\sigma_u^2\underline{I}\,\underline{h} - \underline{h}^T\begin{bmatrix} 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{c} + \sigma_w^2 - \underline{c}^T\sigma_u^2\underline{I}\,\underline{h} - \underline{c}^T\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \end{bmatrix}\underline{h}$$
$$+ \underline{c}^T\sigma_u^2\underline{I}\,\underline{c}$$

$$= \cancel{1} + \cancel{\tfrac{1}{16}} + \tfrac{1}{4} - \cancel{1} - \cancel{\tfrac{1}{16}} + \cancel{1} + \cancel{\tfrac{1}{16}} - \cancel{1} + \tfrac{1}{4} - \cancel{1} - \cancel{\tfrac{1}{16}} - \cancel{1}$$
$$+ \cancel{1} + \tfrac{1}{16} + \cancel{1} = \underline{\underline{\tfrac{9}{16}}}$$

System identification:

ad a)



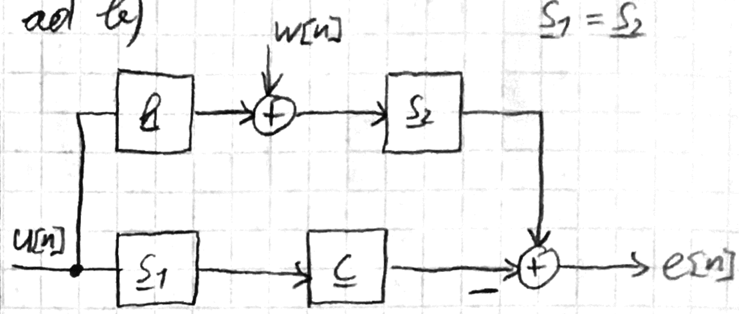
$\underline{c}_{MSE} = \underline{h}$

-> system can be identified

-> signal path of both systems are identical, except the added noise

ad b)

$\underline{S}_1 = \underline{S}_2$



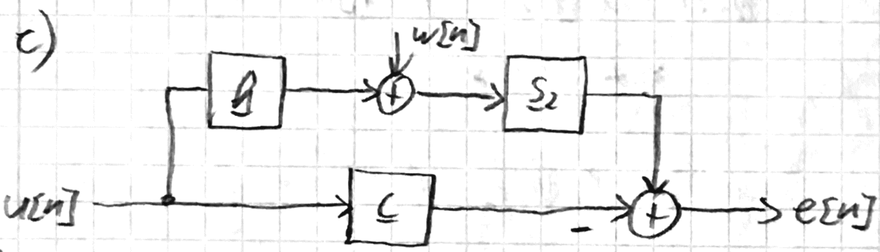
$\underline{c}_{MSE} = \underline{h}$

-> system can be identified

-> Due to the fact that the systems $\underline{S}_1$ and $\underline{S}_2$ are identical the system $\underline{h}$ can be successfully identified

-> The linearity property of LTI system allows to exchange the systems, so again both signal path are identically and can therefore be identified

ad c)



$$\underline{c}_{MSE} = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{4} \end{bmatrix} \neq \underline{h}$$

-> the system can be identified, but with some restrictions.

-> the adapted filter $\underline{c}$ is a delayed version of $\underline{h}$

-> this is caused by system $\underline{S}_2$, which represents a simple delay, so both signal path are not equal anymore

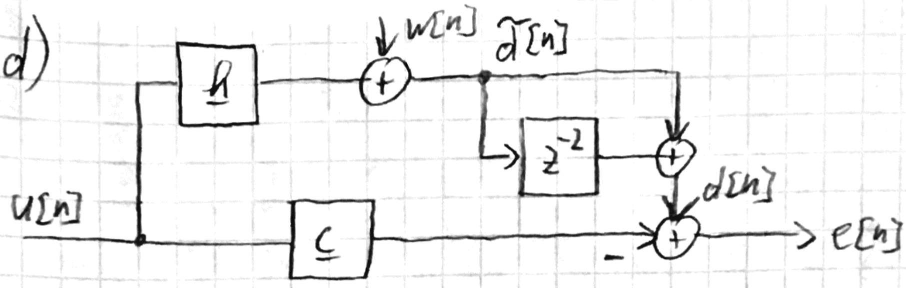
-> $\underline{c}$ aims to match the whole signal path, because this is the information we get: a delayed version of $\underline{h}$

If $\underline{S}_2$ is known than two options are possible as counter measure. 1) compensate $\underline{S}_2$ with z^1 (differentiator)

2) delay also underneath sig. path by z^{-1} *

* approche 2) would be the best, because a differentiator can cause difficulties in real applications

ad d)



$$\underline{c}_{MSE} = \begin{bmatrix} 1 \\ \frac{1}{4} \\ 1 \end{bmatrix} \neq \underline{h}$$

→ system can <u>not</u> be identified!

→ for a better analysation the system $\underline{S}_2$ has been drawn in more detail

→ the problem is caused by $\underline{h}$ and its filter order $N=2$

→ after $u[0]$ has been through $\underline{h}$ it will be delayed again by the filter zize $\underline{h}$

→ it seems that $\underline{h}$ is repeated, there fore the 3^{rd} coefficient of $\underline{c}$ is again 1.

$$\tilde{d}[0] = h_0 u[0] + h_1 u[-1]$$

$$\tilde{d}[1] = h_0 u[1] + h_1 u[0]$$

$$\underline{h}_4 = \begin{bmatrix} \underline{h} \\ \underline{h} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{4} \\ 1 \\ \frac{1}{4} \end{bmatrix}$$ look identical

with $\underline{S}_2$:

$$d[2] = h_0 u[2] + h_1 u[1] + \tilde{d}[0] = \tilde{d}[2] + \tilde{d}[0]$$

$$d[3] = h_0 u[3] + h_1 u[2] + \tilde{d}[1] = \tilde{d}[3] + \tilde{d}[1]$$

$$d[2] = \underline{h}^T \underline{u}[2] + \underline{h}^T \underline{u}[0] = \underline{h}_4^T \underline{u}[2]$$

without $\underline{S}_2$:

$$d[2] = h_0 u[2] + h_1 u[1] + h_2 u[0] + h_3 u[-1]$$

where $h_0 = h_2$, $h_1 = h_3$

note: for simplicity noise has been neglected

ad e) if the system of the sensors are known than the desired system $\underline{h}$ can be identified. As already discussed at the previous page under ad c), a delay (or differentiator) can be added to compensate the sensor systems.

→ In more general, because adding a delay is the solution for this special case, the inverse $S_2^{-1}(z)$ has to be applied, a so called 'equalizer'.

for ad b) it would be sufficient if $\underline{h}$ increases its filter order and there fore $\underline{c}$ also increases its filter order.