Signal Processing and Speech Communication Lab.
Graz University of Technology

AS

Adaptive Systems—Homework Assignment 1

v1.0

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Your solutions to the problems (your calculations, the answers to each task as well as the OCTAVE/MATLAB plots) have to be uploaded to the TeachCenter as a single `*.pdf` file, no later than **2019/12/6**. Use **this page** as the title page and fill in your **name(s) and matriculation number(s)**. Submitting your homework as a LaTeX document can earn you **up to 3 bonus points!**
All scripts needed for your OCTAVE/MATLAB solutions (all `*.m` files) have to be uploaded to the TeachCenter as a single `*.zip` archive, no later than **2019/12/6**.
All filenames consist of the assignment number and your matriculation number(s) such as `Assignment1_MatrNo1_MatrNo2.*`, for example,

Problem solutions:	`Assignment1_01312345_01312346.pdf`
OCTAVE/MATLAB files:	`Assignment1_01312345_01312346.zip`

Please make sure that your approaches, procedures, and results are clearly presented. Justify your answers! A single upload of the files per group is sufficient.

Ethics

All tasks (analytical and practical) have been completely elaborated from scratch on a stand-alone basis. Nevertheless I have talked/interchanged/discussed with and shown my results to Patrick Schuster and vice-versa. We still believe that we have not founded a group, because of the loosely coupled way of work. Thank you very much Ziad and Thomas for your great help, when I was stuck in a calculation!

1.1.a.i)

$$\underline{A} = \begin{pmatrix} \underline{a}_0 & \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{pmatrix}$$

$$\underbrace{\underset{(N\times4)}{\underline{A}} \cdot \underset{(4\times1)}{\underline{x}}}_{N\times1} = x_0 \cdot \underline{a}_0 + x_1 \cdot \underline{a}_1 + x_2 \cdot \underline{a}_2 + x_3 \cdot \underline{a}_3 \qquad \underline{x} = \begin{pmatrix} x_0 \\ \vdots \\ x_3 \end{pmatrix}$$

$\dim(\underline{x}) = 4$

1.1.a.ii)

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \xrightarrow{\underset{?}{B}} \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{pmatrix}$$

$$\begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \cdot b_1 \\ a_2 \cdot b_2 \\ a_3 \cdot b_3 \end{pmatrix}$$

$$\underbrace{\underset{(3\times3)}{\underline{B}} \cdot \underset{(3\times1)}{\underline{a}}}_{3\times1}$$

$$\underline{D} = (\underline{d}_1 \ \underline{d}_2 \ \underline{d}_3)^T \overset{?}{\rightarrow} (b_1\underline{d}_1 \ b_2\underline{d}_2 \ b_3\underline{d}_3)^T$$

$$\underline{D} \cdot \underline{B} = (\underline{d}_1 \ \underline{d}_2 \ \underline{d}_3)^T \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} = (b_1 \cdot \underline{d}_1 \ \ b_2\underline{d}_2 \ \ b_3\underline{d}_3)^T$$

$$\underline{D} \cdot \underline{B}$$

1.1.a.iii)

$$\det(a \cdot \underline{A}) = a^N \cdot \det(\underline{A}) \qquad \underline{A} \ldots N\times N$$

$N=2$

$$\det(a\underline{A}) = \det\begin{pmatrix} a a_{00} & a a_{01} \\ a a_{10} & a a_{11} \end{pmatrix} = a^2 a_{00} \cdot a_{11} - a^2 a_{01} a_{10} = \underline{a^2(a_{00} \cdot a_{11} - a^2 a_{01} a_{10})}$$

$$a^2 \cdot (\det(\underline{A})) = a^2 \cdot \det\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \underline{a^2(a_{00} a_{11} - a_{01} a_{10})}$$

1.1.b.iv) $E(e(n)\cdot \underline{x}(n))=0$

$$J_{MSE}=E(e^2(n))$$

$$e(n)=d(n)-y(n)=d(n)-\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)=d(n)-\sum_{k=0}^{N-1}c(k)\cdot x(n-k)$$ real valued signals

$$e^2(n)=\left(d(n)-\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)\right)^2=\left(d(n)-\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)\right)\left(d(n)-\sum_{k=0}^{N-1}c^*(k)x(n-k)\right)=$$

$$d^2(n)-d(n)\cdot\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)-d(n)\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)+\left(\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)\right)^2=$$

$$d^2(n)-2d(n)\cdot\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)+\left(\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)\right)^2\Rightarrow$$

$$E(e^2(n))=E\left(d^2(n)-2d(n)\cdot\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)+\left(\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)\right)^2\right)=$$

$$E(d^2(n))-2E\left(d(n)\cdot\left(\sum_{\substack{k=0\\k\neq i}}^{N-1}c^*(k)\cdot x(n-k)+c^*(i)\cdot x(n-i)\right)\right)+E\left(\left(\sum_{k=0}^{N-1}c^*(k)\cdot x(n-k)\right)^2\right)$$

$$\frac{\partial E}{\partial c_i}=-2E_i\left(d(n)\cdot x(n-i)\right)+E\left(2\sum_{k=0}^{N-1}c^*x(n-k)\cdot\left(x(n-i)\right)\right)$$

$$-2E_i(d(n)\cdot x(n-i))+2E\left(\underbrace{\sum_{k=0}^{N-1}c^*x(n-k)}_{y(n)}\cdot x(n-i)\right)=0$$

$$E_i(d(n)\cdot x(n-i))=E\left(y(n)\cdot x(n-i)\right)$$

$$E_i(d(n)\cdot x(n-i)-y(n)x(n-i))=0$$

$$E_i(x(n-i)\,(d(n)-y(n)))=0$$

$$E_i(x(n-i)\,e(n))=0$$

$$\begin{pmatrix}E_0\,(x(n)\cdot(d(n)-y(n))\\E_1\,(x(n-1)\cdot(d(n)-y(n))\\\vdots\quad\vdots\\E_{N-1}\,(x(n-N+1)\,(d(n)-y(n))\end{pmatrix}=\underline{E(\underline{x}(n)\cdot e(n))=0}$$ $\rightarrow$ principle of orthogonality

1.1.b.v) $E(\underline{x}(n) \cdot e(n)) = 0$

$E(\underline{x}(n)(d(n) - \underline{c}^T \underline{x}(n))) = E(\underline{x}(n) \cdot d(n) - \underline{x}(n)\,\underline{c}^T \underline{x}(n)) =$

$E(\underline{x}(n)\,d(n)) - E(\underline{x}(n) \cdot \underline{x}^T(n)) \cdot \underline{c} = 0$

$\underbrace{E(\underline{x}(n) d(n))}_{\underline{p}} = \underbrace{E(\underline{x}(n) \cdot \underline{x}^T(n))}_{\underline{R}_{xx}} \cdot \underline{c}$ ~~tra~~

$\underline{p} = \underline{R}_{xx} \cdot \underline{c}$

$\underline{c}_{opt} = \underline{R}_{xx}^{-1} \cdot \underline{p}$ Wiener Hopf Solution

1.1.b.vi) $\underline{R}_{xx}$ invertible ? $\rightarrow$ $\det(\underline{R}_{xx}) > 0$ (eigenvalues > 0)

or

$\underline{a}^H \cdot \underline{R}_{xx} \cdot \underline{a} > 0$ (positive semidefinite)

1.1.c.vii)

$$x(n) = \alpha \cdot e^{j(\theta_0 n + \varphi)} + w(n)$$

$$r_{xx}(n,k) = E\left(x(n+k) \cdot x^*(n)\right)$$

$$E\left(\left(\alpha \cdot e^{j(\theta_0 (n+k) + \varphi)} + w(n+k)\right) \cdot \left(\alpha e^{-j(\theta_0 n + \varphi)} + w(n)\right)\right) =$$

$$E\left(\alpha^2 e^{j(\theta_0 (n+k) + \varphi)} e^{-j(\theta_0 n + \varphi)} + \alpha w(n) \cdot e^{+j(\theta_0 (n+k) + \varphi)} + w(n+k) \alpha \cdot e^{-j(\theta_0 n + \varphi)} + w(n+k) w(n)\right) =$$

$$E\left(\alpha^2 \cdot e^{\cancel{j\theta_0 n} + j\theta_0 k \cancel{+ j\varphi} \cancel{- j\theta_0 n} \cancel{- j\varphi}} + \alpha w(n) \cdot e^{+j(\theta_0 (n+k) + \varphi)} + w(n+k) \cdot \alpha \cdot e^{-j(\theta_0 n + \varphi)} + w(n+k) w(n)\right) =$$

$$E\left(\alpha^2 e^{j\theta_0 k}\right) + E\left(\alpha \cdot w(n) \cdot e^{j(\theta_0 (n+k))} e^{j\varphi}\right) + \alpha \cdot E\left(w(n+k) \cdot e^{-j(\theta_0 n + \varphi)}\right) + E\left(w(n+k) \cdot w(n)\right) =$$

$$\alpha^2 \cdot e^{j\theta_0 k} + \alpha e^{j(\theta_0 (n+k))} \cdot E\left(w(n) \cdot e^{j\varphi}\right) + \alpha \cdot e^{-j\theta_0 n} \cdot E\left(w(n+k) \cdot e^{-j\varphi}\right) + E\left(w(n+k) \cdot w(n)\right) =$$

$$\alpha^2 \cdot e^{j\theta_0 k} + \alpha \cdot e^{j(\theta_0 (n+k))} \cdot \underbrace{E(w(n))}_{\mu_w = 0} \cdot E\left(e^{j\varphi}\right) + \alpha \cdot e^{-j\theta_0 n} \cdot E(w(n+k)) \cdot \underbrace{E\left(e^{-j\varphi}\right)}_{(1)\ = 0} + \underbrace{E\left(w(n+k) \cdot w(n)\right)}_{(2)}$$

$$\underline{\alpha^2 \cdot e^{j\theta_0 k} + \sigma_w^2 \cdot \delta(k) = r_{xx}(n,k) = r_{xx}(k)} \qquad \underline{r_{xx} = f(k) \rightarrow \text{Wide Sense Stationary}}$$

(1) $$E\left(e^{-j\varphi}\right) = \int_{-\pi}^{+\pi} e^{-j\varphi} \cdot \frac{1}{2\pi} d\varphi = \frac{1}{2\pi} \cdot \frac{e^{-j\varphi}}{-j} \Bigg|_{-\pi}^{+\pi} = \frac{1}{2\pi}\left(\frac{e^{-j\pi}}{-j} - \frac{e^{+j\pi}}{-j}\right) = -\frac{1}{2\pi j}(1 - 1) = 0$$

(2) $$E\left(w(n+k) \cdot w(n)\right) = \sigma_w^2 \cdot \delta(k)$$

$k = 0$: $E(w^2(n)) = \sigma_w^2$

$k \neq 0$: $E\left(w(n+k) w(n)\right) = E(w(n+k)) \cdot \underbrace{E(w(n))}_{\mu = 0} = 0$

1.1.c.viii) $r_{xx}(k) = \alpha^2 \cdot e^{j\theta_0 k} + \delta(k) \cdot \sigma_w^2$

$k=0$: $r_{xx}(0) = \alpha^2 + \sigma_w^2$

$k \neq 0$: $r_{xx}(k) = \alpha^2 \cdot e^{j\theta_0 k}$

$$\underline{R}_{xx} = \begin{pmatrix} \alpha^2+\sigma_w^2 & \alpha^2 \cdot e^{j\theta_0} & \alpha^2 \cdot e^{2j\theta_0} & \cdots \\ \alpha^2 \cdot e^{-j\theta_0} & \alpha^2+\sigma_w^2 & \alpha^2 \cdot e^{j\theta_0} & \\ \alpha^2 \cdot e^{-2j\theta_0} & \alpha^2 \cdot e^{j\theta_0} & \alpha^2+\sigma_w^2 & \\ \vdots & & & \end{pmatrix}$$

$\underline{R}_{xx}^{-1}$? $\Rightarrow \det(\underline{R}_{xx}) \overset{!}{\neq} 0$

$x(n) = \alpha \cdot e^{j(\theta_0 n + \varphi)} + w(n) = \alpha \cdot e^{j\theta_0 n} e^{j\varphi} + w(n) = s(n) + w(n)$

$$\underline{R}_{xx} = E(\underline{x}(n) \cdot \underline{x}^T(n)) = E\left((\underline{s}(n) + \underline{w}(n))(\underline{s}(n) + \underline{w}(n))^T\right) = E\Big(\underbrace{\underline{s}(n)\cdot\underline{s}^T(n)}_{\underline{R}_{ss}} + \underbrace{\underline{w}(n)\cdot\underline{s}^T(n)}_{① = 0} + \underbrace{\underline{s}(n)\cdot\underline{w}^T(n)}_{② = 0} + \underbrace{\underline{w}(n)\cdot\underline{w}^T(n)}_{\underline{R}_{ww}}\Big)$$

① $E(\underline{w}(n)\cdot\underline{s}^T(n))$: $r_{ws} = E(w(n)\cdot s(n-k)) = E(w(n)\cdot\alpha\cdot e^{j\theta_0(n-k)} \cdot e^{j\varphi}) = \underbrace{E(w(n))}_{\mu_w = 0} \cdot \underbrace{E(e^{j\varphi})}_{=0} \cdot \alpha \cdot e^{j\theta_0(n-k)} = \underline{0}$

② $E(\underline{s}(n)\cdot\underline{w}^T(n))$: $r_{sw} = E(s(n)\cdot w(n-k)) = E(\alpha\cdot e^{j\theta_0 n} e^{j\varphi}\cdot w(n-k)) = \alpha \cdot e^{j\theta_0 n} E(e^{j\varphi}\cdot w(n-k)) =$

$= E(e^{j\varphi}) \cdot E(w(n-k)) \cdot \alpha e^{j\theta_0 n} = 0 = r_{sw}$

$$E(e^{j\varphi}) = \int_{-\pi}^{+\pi} e^{j\varphi} \cdot \frac{1}{2\pi} \, d\varphi = \frac{1}{2\pi} \cdot \frac{1}{j}\left(e^{j\pi} - e^{-j\pi}\right) = \frac{1}{2\pi j}(1-1) = \underline{0}$$

$\Rightarrow \underline{R}_{xx} = \underline{R}_{ss} + \underline{R}_{ww}$

$\det(\underline{R}_{xx}) = \det(\underline{R}_{ss} + \underline{R}_{ww}) \geq \det(\underline{R}_{ss}) + \det(\underline{R}_{ww})$.

$$\det\begin{pmatrix} \sigma_w^2 & & \\ & \ddots & \\ & & \sigma_w^2 \end{pmatrix} = \prod_{i=0}^{N-1} \sigma_w^2 > 0$$

$\rightarrow \det(\underline{R}_{xx}) \geq \det(\underline{R}_{ss}) + \prod_{i=0}^{N-1} \sigma_w^2 > 0$

$\underline{R}_{xx}$ must exist $\forall \sigma_w^2 > 0$

if $\sigma_w^2 = 0$: not invertible (depends on $\underline{R}_{ss}$)

1.1.c, ix)

$$S_{xx}(\theta) = \sum_{k=-\infty}^{+\infty} r_{xx}(k) \cdot e^{-j\theta k}$$

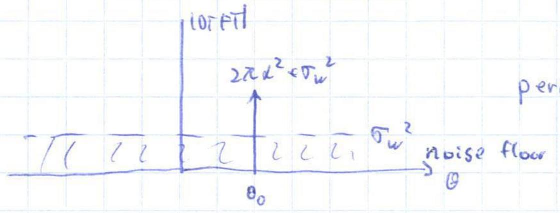
$$r_{xx}(0) = \alpha^2 \cdot e^{j\theta_0 \cdot 0} + \sigma_w^2$$

$$r_{xx}(k) = \alpha^2 \cdot e^{j\theta_0 k} \qquad k \neq 0$$

$$S_{xx}(\theta) = \sum_{k=-\infty}^{+\infty} \alpha^2 \cdot e^{j\theta_0 k} \cancel{\sigma_w^2} \cdot e^{-j\theta k} + \sigma_w^2$$

$$S_{xx}(\theta) = \underbrace{\sum_{k=-\infty}^{+\infty} \alpha^2 \cdot e^{j\theta_0 k} \cdot e^{-j\theta k}}_{F(\cdot)} + \sigma_w^2$$

$$= \alpha^2 \cdot \delta(\theta - \theta_0) + \sigma_w^2$$



periodicity 2π

∞ Lines in PSD $\quad N_1 < \infty \rightarrow \underline{R}_{xx}^{-1}$ must exist $\forall N < \infty$

$\sigma_w^2 \rightarrow 0$: $\sigma_w^2 = 0 \rightarrow 1$ Line in PSD $\rightarrow$ Filter of order 1 must exist $\rightarrow \underline{R}_{xx}^{-1}$ exists for $N=1$

1.2. preamble)

$$\underline{x}(n) = \underline{s}_1^T \underline{u}(n) \qquad \underline{s}_1 = (s_{10} \quad s_{11} \quad s_{12})^T \qquad \sigma_w^2 = 1/4$$

$$\tilde{d}(n) = \underline{h}^T \cdot \underline{u}(n) + w(n) \qquad \underline{s}_2 = (s_{20} \quad s_{21} \quad s_{22})^T \qquad \sigma_u^2 = 1$$

$$\underline{h} = (h_0 \quad h_1 \quad 0)^T$$

$$d(n) = \underline{s}_2^T \cdot \underline{\tilde{d}}(n) = s_{20}(\underline{h}^T \underline{u}(n) + w(n)) + s_{21}(\underline{h}^T \underline{u}(n-1) + w(n-1)) + s_{22}(\underline{h}^T \underline{u}(n-2) + w(n-2))$$

$$y(n) = c_0 \cdot \underline{s}_1^T(\underline{u}(n)) + c_1 \cdot \underline{s}_1^T \underline{u}(n-1) + c_2 \cdot \underline{s}_1^T \underline{u}(n-2) = \underline{c}^T \underline{x}(n)$$

$$\underline{p} = E(d(n) \cdot \underline{x}(n)) = E\begin{pmatrix} d(n) \cdot \underline{s}_1^T \underline{u}(n) \\ d(n) \cdot \underline{s}_1^T \underline{u}(n-1) \\ d(n) \cdot \underline{s}_1^T \underline{u}(n-2) \end{pmatrix} \quad \text{see below}$$

$$r_{xx}(k) = E(x(n+k) \cdot x(n)) = E(\underline{s}_1^T \underline{u}(n+k) \cdot \underline{s}_1^T \underline{u}(n)) = E(\underline{s}_1^T \cdot \underline{u}(n+k) \cdot \underline{u}^T(n) \cdot \underline{s}_1) =$$

$$\underline{r_{xx}(k) = \underline{s}_1^T \cdot E(\underline{u}(n+k) \cdot \underline{u}^T(n)) \cdot \underline{s}_1}$$

$$E(\underline{u}(n) \cdot \underline{u}^T(n-1)) = E\begin{pmatrix} u(0) \cdot u(-1) & u(0)\,u(-2) & u(0)\,u(-3) \\ u(-1)\,u(-1) & u(-1)\,u(-2) & u(-1)\,u(-3) \\ u(-2) \cdot u(-1) & u(-2)\,u(-2) & u(-2)\,u(-3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \end{pmatrix}$$

$$E(\underline{u}(n-1) \cdot \underline{u}^T(n-1)) = E\begin{pmatrix} u(-1)\,u(-1) & u(-1)\,u(-2) & u(-1)\,u(-3) \\ u(-2)\,u(-1) & u(-2)\,u(-2) & u(-2)\,u(-3) \\ u(-3)\,u(-1) & u(-3)\,u(-2) & u(-3)\,u(-3) \end{pmatrix} = \begin{pmatrix} \sigma_u^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_u^2 \end{pmatrix}$$

$$E(\underline{u}(n-2) \cdot \underline{u}^T(n)) = E\begin{pmatrix} u(-2)\,u(0) & u(-2)\,u(-1) & u(-2)\,u(-2) \\ u(-3)\,u(0) & u(-3)\,u(-1) & u(-3)\,u(-2) \\ u(-4)\,u(0) & u(-4)\,u(-1) & u(-4)\,u(-2) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E(\underline{u}(n-1) \cdot \underline{u}^T(n)) = E\begin{pmatrix} u(-1)\,u(0) & u(-1)\,u(-1) & u(-1)\,u(-2) \\ u(-2)\,u(0) & u(-2)\,u(-1) & u(-2)\,u(-2) \\ u(-3)\,u(0) & u(-3)\,u(-1) & u(-3)\,u(-2) \end{pmatrix} = \begin{pmatrix} 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E(\underline{u}(n+1)\,\underline{u}^T(n)) = E\begin{pmatrix} u(1)\,u(0) & u(1)\,u(-1) & u(1)\,u(-2) \\ u(0)\,u(0) & u(0)\,u(-1) & u(0)\,u(-2) \\ u(-1)\,u(0) & u(-1)\,u(-1) & u(-1)\,u(-2) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \end{pmatrix}$$

$$E(\underline{u}(n+2) \cdot \underline{u}^T(n)) = E\begin{pmatrix} u(2)\,u(0) & u(2)\,u(-1) & u(2)\,u(-2) \\ u(1)\,u(0) & u(1)\,u(-1) & u(1)\,u(-2) \\ u(0)\,u(0) & u(0)\,u(-1) & u(0)\,u(-2) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \end{pmatrix}$$

$$E(\underline{u}(n)\,\underline{u}^T(n-2)) = E\begin{pmatrix} u(0)\,u(-2) & u(0)\,u(-3) & u(0)\,u(-4) \\ u(-1)\,u(-2) & u(-1)\,u(-3) & u(-1)\,u(-4) \\ u(-2)\,u(-2) & u(-2)\,u(-3) & u(-2)\,u(-4) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \end{pmatrix}$$

$$E(\underline{u}(n-1) \cdot \underline{u}^T(n-2)) = E\begin{pmatrix} u(-1)\,u(-2) & u(-1)\,u(-3) & u(-1)\,u(-4) \\ u(-2)\,u(-2) & u(-2)\,u(-3) & u(-2)\,u(-4) \\ u(-3)\,u(-2) & u(-3)\,u(-3) & u(-3)\,u(-4) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \sigma_u^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \end{pmatrix}$$

$$E(\underline{u}(n-2) \cdot \underline{u}^T(n-1)) = E\begin{pmatrix} u(-2)\,u(-1) & u(-2)\,u(-2) & u(-2)\,u(-3) \\ u(-3)\,u(-1) & u(-3)\,u(-2) & u(-3)\,u(-3) \\ u(-4)\,u(-1) & u(-4)\,u(-2) & u(-4)\,u(-3) \end{pmatrix} = \begin{pmatrix} 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_u^2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{p} = E\begin{pmatrix} d(n) \cdot \underline{s}_1^T \underline{u}(n) \\ d(n) \cdot \underline{s}_1^T \underline{u}(n-1) \\ d(n) \cdot \underline{s}_1^T \underline{u}(n-2) \end{pmatrix} = \begin{pmatrix} s_{20} \cdot \underline{h}^T R_{uu}\, \underline{s}_1 + s_{21} \cdot \underline{h}^T E(\underline{u}(n-1) \cdot \underline{u}^T(n)) \cdot \underline{s}_1 + s_{22}\, \underline{h}^T E(\underline{u}(n-2)\, \underline{u}^T(n)) \cdot \underline{s}_1 \\ s_{20} \cdot \underline{h}^T E(\underline{u}(n) \cdot \underline{u}^T(n-1)) \cdot \underline{s}_1 + s_{21}\, \underline{h}^T R_{uu} \cdot \underline{s}_1 + s_{22}\, \underline{h}^T E(\underline{u}(n-2)\, \underline{u}^T(n-1)) \cdot \underline{s}_1 \\ s_{20} \cdot \underline{h}^T E(\underline{u}(n) \cdot \underline{u}^T(n-2)) \cdot \underline{s}_1 + s_{21} \cdot \underline{h}^T E(\underline{u}(n-1)\, \underline{u}^T(n-2)) \cdot \underline{s}_1 + s_{22} \cdot \underline{h}^T R_{uu} \cdot \underline{s}_1 \end{pmatrix}$$

1.2. preamble) $\underline{p} = E(d(n)\cdot\underline{x}(n)) = E\left(d(n)\cdot\begin{pmatrix}\underline{s}_1^T\underline{u}(n)\\ \underline{s}_1^T\underline{u}(n-1)\\ \underline{s}_1^T\underline{u}(n-2)\end{pmatrix}\right) = E\begin{pmatrix}d(n)\cdot\underline{s}_1^T\underline{u}(n)\\ d(n)\cdot\underline{s}_1^T\underline{u}(n-1)\\ d(n)\cdot\underline{s}_1^T\underline{u}(n-2)\end{pmatrix}$ $\begin{matrix}a=0\\ a=1\\ a=2\end{matrix}$

$$E\left(\left(s_{20}(\underline{h}^T\underline{u}(n)+w(n)) + s_{21}(\underline{h}^T\underline{u}(n-1)+w(n-1)) + s_{22}(\underline{h}^T\underline{u}(n-2)+w(n-2))\right)\cdot\underline{s}_1^T\underline{u}(n-a)\right)$$

$$s_{20}\underline{h}^T\underline{u}(n)\underline{s}_1^T\underline{u}(n-a) + s_{20}\cdot w(n)\cdot\underline{s}_1^T\underline{u}(n-a) + s_{21}\cdot\underline{h}^T\underline{u}(n-1)\cdot\underline{s}_1^T\underline{u}(n-a) + s_{21}\cdot w(n-1)\underline{s}_1^T\underline{u}(n-a) + s_{22}\underline{h}^T\underline{u}(n-2)\cdot\underline{s}_1^T(\underline{u}(n-a)) + s_{22}w(n-2)\,\underline{s}_1^T\underline{u}(n-a)$$

$a=0$: $s_{20}\cdot\underline{h}^T\underline{u}(n)\underline{u}^T(n)\underline{s}_1 + s_{20}w(n)\cdot\underline{s}_1^T\underline{u}(n) + s_{21}\cdot\underline{h}^T\underline{u}(n-1)\cdot\underline{u}^T(n)\cdot\underline{s}_1 + s_{21}\cdot w(n-1)\cdot\underline{s}_1^T\underline{u}(n) + s_{22}\cdot\underline{h}^T\underline{u}(n-2)\cdot\underline{u}^T(n)\cdot\underline{s}_1 + s_{22}w(n-2)\cdot\underline{s}_1^T(\underline{u}(n))$

$$E(\%) = s_{20}\cdot\underline{h}^T\cdot\underline{R}_{uu}\cdot\underline{s}_1 + s_{21}\cdot\underline{h}^T E(\underline{u}(n-1)\cdot\underline{u}^T(n))\cdot\underline{s}_1 + s_{22}\cdot\underline{h}^T E(\underline{u}(n-2)\cdot\underline{u}^T(n))\,\underline{s}_1$$

$a=1$: $s_{20}\cdot\underline{h}^T\underline{u}(n)\underline{s}_1^T\underline{u}(n-1) + s_{20}\cdot w(n)\cdot\underline{s}_1^T\underline{u}(n-1) + s_{21}\cdot\underline{h}^T\underline{u}(n-1)\underline{u}^T(n-1)\,\underline{s}_1 + s_{21}\cdot w(n-1)\cdot\underline{s}_1^T\underline{u}(n-1) + s_{22}\cdot\underline{h}^T\underline{u}(n-2)\underline{u}^T(n-1)\cdot\underline{s}_1 + s_{22}\cdot w(n-2)\cdot\underline{s}_1^T\underline{u}(n-1)$

$$E(\%) = s_{20}\cdot\underline{h}^T E(\underline{u}(n)\underline{u}^T(n-1))\cdot\underline{s}_1 + s_{21}\cdot\underline{h}^T\underline{R}_{u_{n-1}u_{n-1}}\cdot\underline{s}_1 + s_{22}\underline{h}^T E(\underline{u}(n-2)\underline{u}^T(n-1))\cdot\underline{s}_1$$

$a=2$: $s_{20}\cdot\underline{h}^T\underline{u}(n)\underline{s}_1^T\underline{u}(n-2) + s_{20}\cdot w(n)\cdot\underline{s}_1^T\underline{u}(n-2) + s_{21}\cdot\underline{h}^T\cdot\underline{u}(n-1)\cdot\underline{u}^T(n-2)\,\underline{s}_1 + s_{21}w(n-1)\cdot\underline{s}_1^T\underline{u}(n-2) + s_{22}\cdot\underline{h}^T\underline{u}(n-2)\cdot\underline{u}^T(n-2)\cdot\underline{s}_1 + s_{22}w(n-2)\cdot\underline{s}_1^T\underline{u}(n-2)$

$$E(\%) = s_{20}\,\underline{h}^T E(\underline{u}(n)\cdot\underline{u}^T(n-2))\cdot\underline{s}_1 + s_{21}\cdot\underline{h}^T E(\underline{u}(n-1)\,\underline{u}^T(n-2))\cdot\underline{s}_1 + s_{22}\underline{h}^T\underline{R}_{u_{n-2}u_{n-2}}\cdot\underline{s}_1$$

$$\underline{p} = \begin{pmatrix} s_{20}\cdot\underline{h}^T\underline{R}_{uu}\underline{s}_1 + s_{21}\cdot\underline{h}^T E(\underline{u}(n-1)\,\underline{u}^T(n))\cdot\underline{s}_1 + s_{22}\cdot\underline{h}^T E(\underline{u}(n-2)\cdot\underline{u}^T(n))\cdot\underline{s}_1 \\ s_{20}\cdot\underline{h}^T E(\underline{u}(n)\,\underline{u}^T(n-1))\cdot\underline{s}_1 + s_{21}\underline{h}^T\underline{R}_{uu}\,\underline{s}_1 + s_{22}\cdot\underline{h}^T E(\underline{u}(n-2)\underline{u}^T(n-1))\cdot\underline{s}_1 \\ s_{20}\cdot\underline{h}^T E(\underline{u}(n)\cdot\underline{u}^T(n-2))\cdot\underline{s}_1 + s_{21}\cdot\underline{h}^T E(\underline{u}(n-1)\underline{u}^T(n-2))\cdot\underline{s}_1 + s_{22}\cdot\underline{h}^T\cdot\underline{R}_{uu}\cdot\underline{s}_1 \end{pmatrix}$$

$E(e^2(n))$ $\qquad e(n) = d(n) - y(n)$

$$e^2(n) = (d(n)-y(n))^2 = (d(n)-\underline{c}^T\underline{x}(n))^2 = d^2(n) - 2d(n)\cdot\underline{c}^T\underline{x}(n) + \underline{c}^T\underline{x}(n)\,\underline{c}^T\underline{x}(n)$$

$$E(e^2(n)) = E(d^2(n) - 2d(n)\cdot\underline{c}^T\underline{x}(n) + \underline{c}^T\underline{x}(n)\cdot\underline{x}^T(n)\cdot\underline{c}) =$$

$$\sigma_d^2 - 2E(d(n)\cdot\underline{c}^T\underline{x}(n)) + \underline{c}^T E(\underline{x}(n)\cdot\underline{x}^T(n))\cdot\underline{c}$$

$$\sigma_d^2 - 2E(d(n)\,\underline{x}^T(n))\cdot\underline{c} + \underline{c}^T\underline{R}_{xx}\cdot\underline{c}$$

$$\nabla_{\underline{c}} E(e^2) = -2E(d(n)\cdot\underline{x}^T(n)) + 2\underline{R}_{xx}\cdot\underline{c} = 0$$

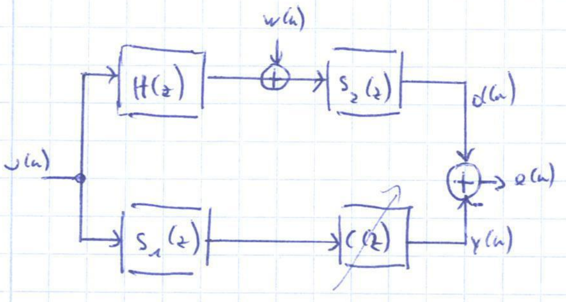
$$E(d(n)\cdot\underline{x}(n)) = \underline{R}_{xx}\underline{c}$$

$$\underline{R}_{xx}^{-1}\underbrace{E(d(n)\cdot\underline{x}(n))}_{\underline{p}} = \underline{c}_{opt}$$

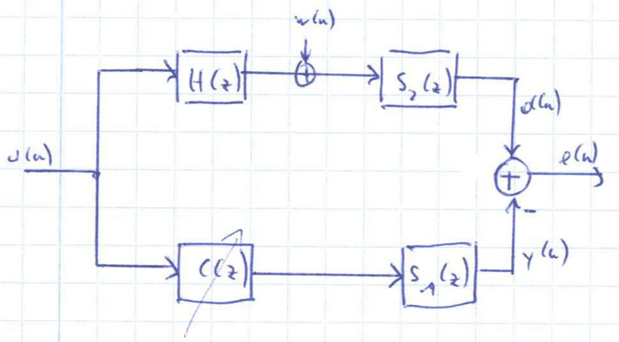
$$\underline{\underline{R}_{xx}^{-1}\cdot\underline{p} = \underline{c}_{opt}}$$

1.2) System identification: As long as $S_1(z)$ and $S_2(z)$ are unknown, no statement can be given preamble regarding system identification success / failure.

Depends on filter order, impulse response, ...



↓ rearrange due to linearity



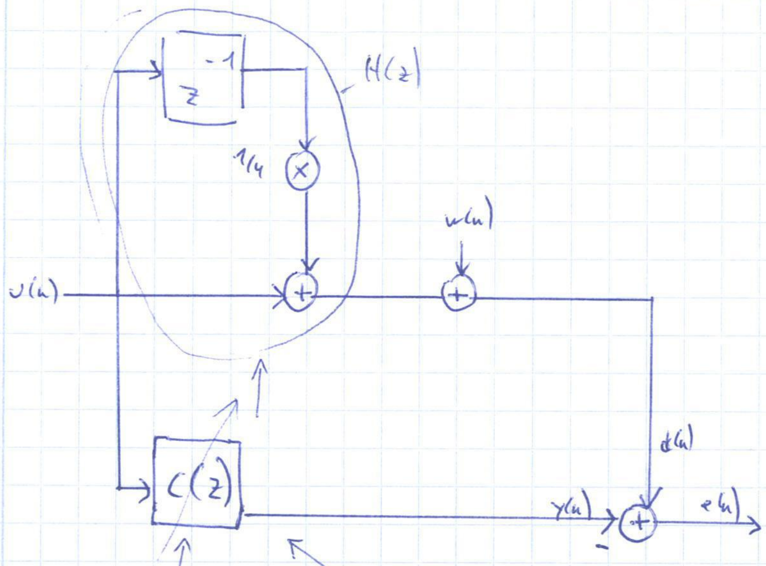
- input u(n) synchronized for upper and lower branch

→ ? - filter order upper / lower branch same?

→ - transfer function / impulse response same?

1.2.a) System can be identified
→e)

$\underline{c}_{MSE} = \underline{h}$, same impulseresponse , $C(z) = H(z)$



$\underline{c}_{MSE} = \begin{pmatrix} 1 \\ 1/4 \\ 0 \end{pmatrix}$ same as above

1.2.a.1) $\underline{r_{xx}(k) = \underline{s}_1^T \, E(\underline{v}(n+k) \cdot \underline{v}^T(n)) \cdot \underline{s}_1}$ $\quad s_1 = 1,\ s_2 = 1,\ \sigma_v^2 = 1$

$r_{xx}(0) = \underline{s}_1^T \cdot E(\underline{v}(n) \cdot \underline{v}^T(n)) \cdot \underline{s}_1 = \underline{s}_1^T \cdot \underline{R}_{vv} \cdot \underline{s}_1 = 1 \cdot \sigma_v^2 \cdot 1 = \underline{1}$

$r_{xx}(1) = \underline{s}_1^T \cdot E(\underline{v}(n+1) \cdot \underline{v}^T(n)) \cdot \underline{s}_1 = 1 \cdot 0 \cdot 1 = 0$

$r_{xx}(2) = \underline{s}_1^T \cdot E(\underline{v}(n+2) \cdot \underline{v}^T(n)) \cdot \underline{s}_1 = 1 \cdot 0 \cdot 1 = 0$

$$\underline{\underline{R}_{xx} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \qquad \underline{R}_{xx} = \begin{pmatrix} r_{xx}(0) & r_{xx}(1) & r_{xx}(2) \\ r_{xx}(-1) & r_{xx}(0) & r_{xx}(1) \\ r_{xx}(-2) & r_{xx}(-1) & r_{xx}(0) \end{pmatrix}$$

1.2.a.2)

$$\underline{p} = E\begin{pmatrix} d(n)\cdot \underline{s}_1^T \underline{v}(n) \\ d(n)\cdot \underline{s}_1^T \underline{v}(n-1) \\ d(n)\cdot \underline{s}_1^T \underline{v}(n-2) \end{pmatrix} = E\begin{pmatrix} \underline{h}^T \underline{v}(n)\cdot \underline{v}^T(n)\underline{s}_1 + w(n)\cdot \underline{s}_1^T \underline{v}(n) \\ \underline{h}^T \underline{v}(n)\cdot \underline{s}_1^T \underline{v}(n-1) + w(n)\cdot \underline{s}_1^T \underline{v}(n-1) \\ \underline{h}^T \underline{v}(n)\cdot \underline{s}_1^T \underline{v}(n-2) + w(n)\cdot \underline{s}_1^T \underline{v}(n-2) \end{pmatrix} = \begin{pmatrix} \underline{h}^T \underline{R}_{vv} \underline{s}_1 \\ \underline{h}^T \cdot E(\underline{v}(n)\underline{v}^T(n-1))\cdot \underline{s}_1 \\ \underline{h}^T \cdot E(\underline{v}(n)\cdot \underline{v}^T(n-2))\cdot \underline{s}_1 \end{pmatrix}$$

(1)

$$\underline{p} = \begin{pmatrix} (1 \quad 1/4)\begin{pmatrix}1\\0\end{pmatrix} \\ (1 \quad 1/4)\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \\ (1 \quad 1/4)\begin{pmatrix}0&0\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ (1 \quad 1/4)\begin{pmatrix}0\\1\end{pmatrix} \\ (1 \quad 1/4)\cdot 0 \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ 1/4 \\ 0 \end{pmatrix} = \underline{p}}$$

(1) $E(w(n)\cdot \underline{v}^T(n-k)) = \underbrace{E(w(n))}_{\mu_w=0}\cdot \underbrace{E(\underline{v}^T(n-k))}_{\mu_v=0} = 0$

1.2.a.3) $\underline{c}_{MSE} = \underline{R}_{rr}^{-1} \cdot \underline{r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/4 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 1/4 \\ 0 \end{pmatrix} = \underline{c}_{MSE}}}$

1.2.a.4)

$$J_{MSE}(\underline{c}_{OPT}) = E(e^2(n))$$

$$\begin{array}{cccc} & s_{10} & s_{11} & s_{12} \\ \underline{s}_1 = & (1 & 0 & 0)^T \\ \underline{s}_2 = & (1 & 0 & 0)^T \\ & s_{20} & s_{21} & s_{22} \end{array} \qquad \underline{c} = (c_0 \quad c_1 \quad c_2)^T = (1 \quad 1/4 \quad 0)^T$$

$$e(n) = d(n) - y(n)$$

$$e(n) = s_{20}\left(\underline{h}^T\underline{v}(n) + w(n)\right) + s_{21}\left(\underline{h}^T\underline{v}(n-1) + w(n-1)\right) + s_{22}\left(\underline{h}^T\underline{v}(n-2) + w(n-2)\right)$$

$$- c_0\,\underline{s}_1^T\underline{v}(n) - c_1\,\underline{s}_1^T\underline{v}(n-1) - c_2\cdot\underline{s}_1^T\underline{v}(n-2)$$

$$e^2(n) = \left(\underline{h}^T\underline{v}(n) + w(n) - c_0\cdot\underline{s}_1^T\underline{v}(n) - c_1\cdot\underline{s}_1^T\underline{v}(n-1)\right)^2 =$$

$$= \underline{h}^T\underline{v}(n)\cdot\underline{h}^T\underline{v}(n) + w(n)\cdot\underline{h}^T\underline{v}(n) - \underline{h}^T\underline{v}(n)\cdot c_0\cdot\underline{s}_1^T\underline{v}(n) - \underline{h}^T\underline{v}(n)\cdot c_1\cdot\underline{s}_1^T\underline{v}(n-1) +$$

$$\underline{h}^T\underline{v}(n)\cdot w(n) + w^2(n) - w(n)\,c_0\cdot\underline{s}_1^T\underline{v}(n) - w(n)\cdot c_1\,\underline{s}_1^T\underline{v}(n-1) +$$

$$- c_0\cdot\underline{s}_1^T\underline{v}(n)\cdot\underline{h}^T\underline{v}(n) - c_0\cdot\underline{s}_1^T\underline{v}(n)w(n) + c_0^2\cdot\underline{s}_1^T\underline{v}(n)\cdot\underline{v}^T(n)\cdot\underline{s}_1 + c_0\cdot\underline{s}_1^T\underline{v}(n)\cdot c_1\cdot\underline{s}_1^T\underline{v}(n-1) +$$

$$- c_1\cdot\underline{s}_1^T\underline{v}(n-1)\cdot\underline{h}^T\underline{v}(n) - c_1\cdot\underline{s}_1^T\underline{v}(n-1)\cdot w(n) + c_1\underline{s}_1^T\underline{v}(n-1)\cdot c_0\cdot\underline{s}_1^T\underline{v}(n) + c_1\underline{s}_1^T\underline{v}(n-1)\cdot c_1\underline{s}_1^T\underline{v}(n-1)$$

$$E(e^2(n)) = \underline{h}^T\cdot R_{vv}\cdot\underline{h} - c_0\cdot\underline{h}^T E(\underline{v}(n)\underline{v}^T(n))\cdot\underline{s}_1 - \underline{h}^T c_1 E(\underline{v}(n)\cdot\underline{v}^T(n-1))\cdot\underline{s}_1 + E(w^2(n)) +$$

$$- c_0\cdot\underline{s}_1^T\cdot R_{vv}\cdot\underline{h} + c_0^2\cdot\underline{s}_1^T\cdot\underline{s}_1 + c_0\cdot c_1\cdot\underline{s}_1^T E(\underline{v}(n)\underline{v}^T(n-1))\cdot\underline{s}_1 +$$

$$- c_1\underline{s}_1^T E(\underline{v}(n-1)\underline{v}^T(n))\cdot\underline{h} + c_0\cdot c_1\cdot\underline{s}_1^T E(\underline{v}(n-1)\cdot\underline{v}^T(n))\cdot\underline{s}_1 + c_1^2\,\underline{s}_1^T E(\underline{v}(n-1)\underline{v}^T(n-1))\cdot\underline{s}_1 =$$

$$= \|\underline{h}\|^2 - c_0\cdot\underline{h}^T\cdot\underline{s}_1 - \underline{h}^T c_1\cdot E(\underline{v}(n)\cdot\underline{v}^T(n-1))\cdot\underline{s}_1 + \sigma_w^2$$

$$- c_0\cdot\underline{s}_1^T\cdot\underline{h} + c_0^2\cdot\|\underline{s}_1\|^2 + c_0\cdot c_1\cdot\underline{s}_1^T E(\underline{v}(n)\underline{v}^T(n-1))\cdot\underline{s}_1$$

$$- c_1\cdot\underline{s}_1^T E(\underline{v}(n-1)\cdot\underline{v}^T(n))\cdot\underline{h} + c_0\cdot c_1\cdot\underline{s}_1^T\cdot\underset{E(\underline{v}(n-1)\cdot\underline{v}^T(n))}{\cancel{E(\underline{v}(n)\ldots}}\cdot\underline{s}_1 + c_1^2\|\underline{s}_1\|^2 =$$

$$= 1 + 1/16 - 1\,(1 \quad 1/4)^T\cdot\begin{pmatrix}1\\0\end{pmatrix} - (1 \quad 1/4)^T\cdot 1/4\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} + \sigma_w^2$$

$$- 1\cdot(1 \quad 0)^T\begin{pmatrix}1\\1/4\end{pmatrix} + 1^2\cdot 1 + 1\cdot 1/4\cdot(1 \quad 0)^T\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} +$$

$$- 1/4\,(1 \quad 0)^T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1\\1/4\end{pmatrix} + 1\cdot 1/4\cdot(1 \quad 0)^T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + 1/16 =$$

$$= 1 + 1/16 - 1 - (1 \quad 1/4)^T\cdot 1/4\begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{4}$$

$$- 1 + 1 + 1/4\,(1 \; 0)^T\begin{pmatrix}0\\1\end{pmatrix} +$$

$$- 1/4\,(1 \; 0)^T\begin{pmatrix}1/4\\0\end{pmatrix} + 1/4\,(1 \; 0)^T\begin{pmatrix}0\\0\end{pmatrix} + 1/16 =$$

$$= 1/16 - 1/4\cdot 1\cdot 1/4 + 1/4 - 1/4\cdot 1/4 + 1/16 =$$

$$1/4 - 1/16 + 1/16 = \underline{1/4 = J_{MSE}(\underline{c}_{OPT})}$$

1.2.b.1)

$$\underline{s}_1 = (1 \quad 1/2)^T$$
$$\underline{s}_2 = (1 \quad 1/2)^T$$

$$r_{xx}(k) = \underline{s}_1^T \, E(\underline{v}(n+k) \cdot \underline{v}^T(n)) \cdot \underline{s}_1$$

$$r_{xx}(0) = (1 \quad 1/2)^T \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = 1 + \frac{1}{4} = \frac{5}{4}$$

$$r_{xx}(1) = \begin{pmatrix} 1 & 1/2 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = (1 \quad 1/2)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1/2$$

$$r_{xx}(2) = \begin{pmatrix} 1 & 1/2 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = 0$$

$$\underline{R}_{xx} = \begin{pmatrix} 5/4 & 1/2 & 0 \\ 1/2 & 5/4 & 1/2 \\ 0 & 1/2 & 5/4 \end{pmatrix}$$

1.2.b.2)

$$\underline{s}_1 = \begin{array}{ccc} s_{10} & s_{11} & \text{zero padding} \\ (1 & 1/2)^T & 0 \end{array}$$

$$\underline{s}_2 = \begin{array}{ccc} (1 & 1/2)^T & 0 \\ s_{20} & s_{21} & \end{array}$$

$$R = \begin{pmatrix} 1\,(1 \;\; 1/4)^T \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + \frac{1}{2}(1 \;\; 1/4)^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \\ 1\,(1 \;\; 1/4)^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + \frac{1}{2}(1 \;\; 1/4)^T \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \\ 1 \cdot (1 \;\; 1/4)^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + \frac{1}{2}(1 \;\; 1/4)^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{9}{8} + \frac{1}{2}(1 \;\; 1/4)^T \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \\ (1 \;\; 1/4)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2} \cdot \frac{9}{8} \\ 1/2\,(1 \;\; 1/4)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{8} + \frac{1}{2} \cdot \frac{1}{2} \\ \frac{1}{4} + \frac{9}{16} \\ \frac{1}{2} \cdot \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{9}{8} + \frac{1}{4} \\ \frac{4+9}{16} \\ 1/8 \end{pmatrix} = \underline{\begin{pmatrix} 11/8 \\ 13/16 \\ 1/8 \end{pmatrix}}$$

1.2.b.3) $\underline{c}_{MSE} = \underline{R}_{xx}^{-1} \cdot \underline{p}$

$$\underline{R}_{xx} = \begin{pmatrix} 5/4 & 1/2 & 0 \\ 1/2 & 5/4 & 1/2 \\ 0 & 1/2 & 5/4 \end{pmatrix}$$

$$\underline{R}_{xx}^{-1} = \begin{pmatrix} 0{,}9882 & -0{,}4706 & 0{,}1882 \\ -0{,}4706 & 1{,}1765 & -0{,}4706 \\ 0{,}1882 & -0{,}4706 & 0{,}9882 \end{pmatrix} \qquad \underline{p} = \begin{pmatrix} 11/8 \\ 13/16 \\ 1/8 \end{pmatrix}$$

$$\underline{\underline{c_{MSE} = \underline{R}_{xx}^{-1} \cdot \underline{p} = \begin{pmatrix} 1 \\ 1/4 \\ 0 \end{pmatrix}}}$$

1.2.b.4) $J_{min} = J_{MSE}(c_{opt}) = E(e^2(n))$

$$e(n) = \underline{h}^T \underline{u}(n) + w(n) + \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) + \tfrac{1}{2} w(n-1) - c_0 \underline{s}_1^T \underline{u}(n) - c_1 \underline{s}_1^T \underline{u}(n-1) - c_2 \underline{s}_1^T \underline{u}(n-2)$$

$$e^2(n) = \left(\underline{h}^T \underline{u}(n) + w(n) + \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) + \tfrac{1}{2} w(n-1) - c_0 \underline{s}_1^T \underline{u}(n) - c_1 \underline{s}_1^T \underline{u}(n-1) - c_2 \underline{s}_1^T \underline{u}(n-2)\right)\left(\underline{h}^T \underline{u}(n) + w(n) + \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) + \tfrac{1}{2} w(n-1) - c_0 \underline{s}_1^T \underline{u}(n) - c_1 \underline{s}_1^T \underline{u}(n-1) - c_2 \underline{s}_1^T \underline{u}(n-2)\right)$$

$$= \underline{h}^T \underline{u}(n) \underline{h}^T \underline{u}(n) + \cancel{w(n) \underline{h}^T \underline{u}(n)} + \underline{h}^T \underline{u}(n) \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) + \cancel{\underline{h}^T \underline{u}(n) \tfrac{1}{2} w(n-1)} - \underline{h}^T \underline{u}(n) c_0 \underline{s}_1^T \underline{u}(n) - \underline{h}^T \underline{u}(n) c_1 \underline{s}_1^T \underline{u}(n-1) - \underline{h}^T \underline{u}(n) c_2 \underline{s}_1^T \underline{u}(n-2) +$$

$$+ \cancel{w(n) \underline{h}^T \underline{u}(n)} + w^2(n) + \cancel{w(n) \tfrac{1}{2} \underline{h}^T \underline{u}(n-1)} + w(n) \tfrac{1}{2} w(n-1) - \cancel{w(n) c_0 \underline{s}_1^T \underline{u}(n)} - \cancel{w(n) c_1 \underline{s}_1^T \underline{u}(n-1)} - \cancel{w(n) c_2 \underline{s}_1^T \underline{u}(n-2)} +$$

$$+ \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) \underline{h}^T \underline{u}(n) + \cancel{\tfrac{1}{2} \underline{h}^T \underline{u}(n-1) w(n)} + \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) + \cancel{\tfrac{1}{2} \underline{h}^T \underline{u}(n-1) \tfrac{1}{2} w(n-1)} - \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) c_0 \underline{s}_1^T \underline{u}(n) - \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) c_1 \underline{s}_1^T \underline{u}(n-1) - \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) c_2 \underline{s}_1^T \underline{u}(n-2) +$$

$$+ \cancel{\tfrac{1}{2} w(n-1) \underline{h}^T \underline{u}(n)} + \tfrac{1}{2} w(n-1) w(n) + \cancel{\tfrac{1}{2} w(n-1) \tfrac{1}{2} \underline{h}^T \underline{u}(n-1)} + \tfrac{1}{2} w^2(n-1) - \cancel{\tfrac{1}{2} w(n-1) c_0 \underline{s}_1^T \underline{u}(n)} - \cancel{\tfrac{1}{2} w(n-1) c_1 \underline{s}_1^T \underline{u}(n-1)} - \cancel{\tfrac{1}{2} w(n-1) c_2 \underline{s}_1^T \underline{u}(n-2)} +$$

$$- c_0 \underline{s}_1^T \underline{u}(n) \underline{h}^T \underline{u}(n) - \cancel{c_0 \underline{s}_1^T \underline{u}(n) w(n)} - c_0 \underline{s}_1^T \underline{u}(n) \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) - \cancel{c_0 \underline{s}_1^T \underline{u}(n) \tfrac{1}{2} w(n-1)} + c_0^2 \underline{s}_1^T \underline{u}(n) \underline{s}_1^T \underline{u}(n) + c_0 \underline{s}_1^T \underline{u}(n) c_1 \underline{s}_1^T \underline{u}(n-1) + c_0 \underline{s}_1^T \underline{u}(n) c_2 \underline{s}_1^T \underline{u}(n-2) +$$

$$- c_1 \underline{s}_1^T \underline{u}(n-1) \underline{h}^T \underline{u}(n) - \cancel{c_1 \underline{s}_1^T \underline{u}(n-1) w(n)} - c_1 \underline{s}_1^T \underline{u}(n-1) \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) - c_1 \underline{s}_1^T \underline{u}(n-1) \tfrac{1}{2} w(n-1) + c_1 \underline{s}_1^T \underline{u}(n-1) c_0 \underline{s}_1^T \underline{u}(n) + c_1^2 \underline{s}_1^T \underline{u}(n-1) \underline{s}_1^T \underline{u}(n-1) + c_1 \underline{s}_1^T \underline{u}(n-1) c_2 \underline{s}_1^T \underline{u}(n-2) +$$

$$- c_2 \underline{s}_1^T \underline{u}(n-2) \underline{h}^T \underline{u}(n) - \cancel{c_2 \underline{s}_1^T \underline{u}(n-2) w(n)} - c_2 \underline{s}_1^T \underline{u}(n-2) \tfrac{1}{2} \underline{h}^T \underline{u}(n-1) - \cancel{c_2 \underline{s}_1^T \underline{u}(n-2) \tfrac{1}{2} w(n-1)} + c_2 \underline{s}_1^T \underline{u}(n-2) c_0 \underline{s}_1^T \underline{u}(n) + c_2 \underline{s}_1^T \underline{u}(n-2) c_1 \underline{s}_1^T \underline{u}(n-1) + c_2 \underline{s}_1^T \underline{u}(n-2) c_2 \underline{s}_1^T \underline{u}(n-2)$$

$$E(e^2(n)) = \underline{h}^T \underline{h} + \underline{h}^T \tfrac{1}{2} E(\underline{u}(n) \underline{u}^T(n-1)) \underline{h} - c_0 \underline{h}^T \underline{s}_1 - c_1 \underline{h}^T E(\underline{u}(n) \underline{u}^T(n-1)) \underline{s}_1 - c_2 \underline{h}^T E(\underline{u}(n) \underline{u}^T(n-2)) \underline{s}_1 +$$

$$+ E(w^2(n)) + \tfrac{1}{2} E(w(n) w(n-1)) +$$

$$+ \tfrac{1}{2} \underline{h}^T E(\underline{u}(n-1) \underline{u}^T(n)) \underline{h} + \tfrac{1}{4} \underline{h}^T \underline{h} - \tfrac{1}{2} c_0 \underline{h}^T E(\underline{u}(n-1) \underline{u}^T(n)) \underline{s}_1 - \tfrac{1}{2} c_1 \underline{h}^T E(\underline{u}(n-1) \underline{u}^T(n-1)) \underline{s}_1 - \tfrac{1}{2} c_2 \underline{h}^T E(\underline{u}(n-1) \underline{u}^T(n-2)) \underline{s}_1 +$$

$$+ \tfrac{1}{2} E(w(n-1) w(n)) + \tfrac{1}{2} E(w^2(n-1))$$

$$- c_0 \underline{s}_1^T \underline{h} + \cancel{\ldots} - c_0 \underline{s}_1^T E(\underline{u}(n) \underline{u}^T(n-1)) \underline{h} \tfrac{1}{2} + c_0^2 \underline{s}_1^T \underline{s}_1 + c_0 \underline{s}_1^T E(\underline{u}(n) \underline{u}^T(n-1)) \underline{s}_1 c_1 + c_0 c_2 \underline{s}_1^T E(\underline{u}(n) \underline{u}^T(n-2)) \underline{s}_1 +$$

$$- c_1 \underline{s}_1^T E(\underline{u}(n-1) \underline{u}^T(n)) \underline{h} - \tfrac{1}{2} c_1 \underline{s}_1^T E(\underline{u}(n-1) \underline{u}^T(n-1)) \underline{h} + c_0 c_1 \underline{s}_1^T E(\underline{u}(n-1) \underline{u}^T(n)) \underline{s}_1 + c_1^2 \underline{s}_1^T \underline{s}_1 + c_1 c_2 \underline{s}_1^T E(\underline{u}(n-1) \underline{u}^T(n-2)) \underline{s}_1 +$$

$$- c_2 \underline{s}_1^T E(\underline{u}(n-2) \underline{u}^T(n)) \underline{h} - c_2 \tfrac{1}{2} \underline{s}_1^T E(\underline{u}(n-2) \underline{u}^T(n-1)) \underline{h} + \cancel{\ldots}\ c_0 c_2 \underline{s}_1^T E(\underline{u}(n-2) \underline{u}^T(n)) \underline{s}_1 + c_1 c_2 \underline{s}_1^T E(\underline{u}(n-2) \underline{u}^T(n-1)) \underline{s}_1 + c_2^2 \|\underline{s}_1\|^2$$

$\rightarrow$ next page

1.2-6.4) Fortsetzung 1

$$c = \begin{pmatrix} 1 & 1/4 & 1/10 \end{pmatrix}^T \quad h = \begin{pmatrix} 1 & 1/4 \end{pmatrix}^T \quad s_1 = \begin{pmatrix} 1 & 1/2 \end{pmatrix}^T \quad s_2 = \begin{pmatrix} 1 & 1/2 \end{pmatrix}^T$$

Ref: $E(\underline{v}(n)\underline{v}^T(n-1)) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad E(\underline{v}(n)\underline{v}^T(n-2)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad E(\underline{v}(n-1)\underline{v}^T(n)) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad E(\underline{v}(n-1)\underline{v}^T(n-2)) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$E(\underline{v}(n-2)\underline{v}^T(n)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$= E(e^2(n)) = \frac{17}{16} + (1\ 1/4)^T \cdot \frac{1}{2} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} - c_0 \cdot (1\ 1/4)^T \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} - c_1 \cdot (1\ 1/4)^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} - c_2 \cdot (1\ 1/4)^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} +$$

$$+ \sigma_w^2 +$$

$$+ \frac{1}{2} \cdot (1\ 1/4)^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} + \frac{1}{4} \cdot \frac{17}{16} - \frac{1}{2} c_0 \cdot (1\ 1/4)^T \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} - \frac{1}{2} \cdot c_1 \cdot (1\ 1/4)^T \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} - \frac{1}{2} \cdot c_2 \cdot \cancel{(1\ 1/4)^T} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} +$$

$$+ \frac{1}{2} \sigma_w^2 +$$

$$- c_0 \cdot (1\ 1/2)^T \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} - c_0 \cdot (1\ 1/2)^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} \cdot \frac{1}{2} + c_0^2 \cdot \frac{5}{4} + c_0 \cdot (1\ 1/2)^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \cdot c_1 + c_0 \cdot c_2 \cdot (1\ \cancel{1/2)^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} +$$

$$- c_1 \cdot (1\ 1/2)^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} - \frac{1}{2} c_1 \cdot (1\ 1/2)^T \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} + c_0 c_1 (1\ 1/2)^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + c_1^2 \frac{5}{4} + c_1 c_2 \cdot (1\ 1/2)^T \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} +$$

$$- c_2 \cdot \cancel{(1\ 1/2)^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} - \frac{1}{2} c_2 \cdot (1\ 1/2)^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} + c_0 c_2 \cdot \cancel{(1\ 1/2)^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + c_1 c_2 \cdot (1\ 1/2)^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} + c_2^2 \cdot \frac{5}{4} =$$

$$= \frac{17}{16} + 1/2\ (1\ 1/4)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} - c_0 \cdot \frac{9}{8} - c_1 \cdot (1\ 1/4)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} - c_2 \cancel{(1\ 1/4)^T \cdot \emptyset} +$$

$$\frac{1}{4} +$$

$$\frac{1}{2} (1\ 1/4)^T \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + \frac{1}{4} \cdot \frac{17}{16} - \frac{1}{2} c_0 (1\ 1/4)^T \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} - \frac{1}{2} c_1 (1\ 1/4)^T \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} - \frac{1}{2} c_2 (1\ 1/4)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} +$$

$$\frac{1}{2} \cdot \frac{1}{4} +$$

$$- c_0 \cdot \frac{9}{8} - c_0 \cdot (1\ 1/2)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \frac{1}{2} + c_0^2 \cdot \frac{5}{4} + c_0 \cdot (1\ 1/2)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot c_1 + \cancel{c_0 c_2 (1\ 1/2)^T \cdot \emptyset} +$$

$$- c_1\ (1\ 1/2)^T \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} - \frac{1}{2} c_1 (1\ 1/2)^T \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} + c_0 c_1 (1\ 1/2)^T \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + c_1^2 \cdot \frac{5}{4} + c_1 c_2 (1\ 1/2)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} +$$

$$\cancel{c_2} - \frac{1}{2} c_2 (1\ 1/2)^T \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + \cancel{c_0 c_2} + c_1 c_2 (1\ 1/2)^T \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} + c_2^2 \cdot \frac{5}{4} =$$

$$= \frac{17}{16} + \frac{1}{8} - c_0 \cdot \frac{9}{8} - c_1 \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{17}{16} - c_0 \frac{1}{4} - c_1 \cdot \frac{9}{16} - c_2 \cdot \frac{1}{8} + \frac{1}{8} - c_0 \cdot \frac{9}{8} - c_0 \frac{1}{4} + c_0^2 \frac{5}{4} + c_0 c_1 \frac{1}{2} - \cancel{c_0 c_2} - c_1 \frac{1}{4} - \frac{1}{2} c_1 \cdot \frac{9}{8} + c_0 c_1 \frac{1}{2} + c_1^2 \frac{5}{4} + c_1 c_2 \frac{1}{2} - \frac{1}{8} c_2 + \frac{1}{2} c_1 c_2 + c_2^2 \frac{5}{4} =$$

$$= \frac{17}{16} + \frac{2}{16} + \frac{4}{16} + \frac{2}{16} + \frac{17}{64} + \frac{2}{16} - c_0 \cdot \frac{9}{8} - \frac{1}{4} c_0 - \frac{9}{8} c_0 - \frac{1}{4} c_0 - \frac{1}{4} c_1 - \frac{9}{16} c_1 - \frac{1}{4} c_1 - \frac{9}{16} c_1 - \frac{1}{8} c_2 - \frac{1}{8} c_2 + \frac{5}{4} c_0^2 + \frac{1}{2} c_0 c_1 + \frac{1}{2} c_0 c_1 + \frac{5}{4} c_1^2 + c_1 c_2 \cdot \frac{1}{2} + c_1 c_2 \cdot \frac{1}{2} + c_2^2 \cdot \frac{5}{4} =$$

$$= \frac{27}{16} + \frac{17}{64} - c_0 \cdot \left(\frac{9}{8} + \frac{2}{8} + \frac{9}{8} + \frac{2}{8}\right) - c_1 \left(\frac{4}{16} + \frac{9}{16} + \frac{4}{16} + \frac{9}{16}\right) - c_2 \left(\frac{1}{8} + \frac{1}{8}\right) + \frac{5}{4} c_0^2 + c_0 c_1 + \frac{5}{4} c_1^2 + c_1 c_2 + c_2^2 \cdot \frac{5}{4} =$$

→ next page

$$\cancel{= \frac{108+17}{64} - c_0 \cdot \frac{22}{8} - c_1 \cdot \frac{26}{16} - c_2 \cdot \frac{1}{4} + \frac{5}{4} c_0^2 + c_0 c_1 + \frac{5}{4} c_1^2 + c_1 c_2 + \frac{5}{4} c_2^2 = \frac{125}{64} - \frac{22}{8} - \frac{26}{64} - \frac{1}{40} + \frac{5}{4} + \frac{1}{4} + \frac{5}{64} + \frac{1}{40} + \frac{5}{400} = \frac{125}{64} - \frac{176}{64} - \frac{26}{64} + \frac{5}{64} + \frac{16}{64} + \frac{80}{64} + \frac{5}{400} = \frac{24}{64} + \frac{5}{400} = \frac{600+20}{1600} = \frac{620}{1600}}$$

$$\cancel{\boxed{= 0{,}3875}} = \boxed{\frac{31}{80}}$$

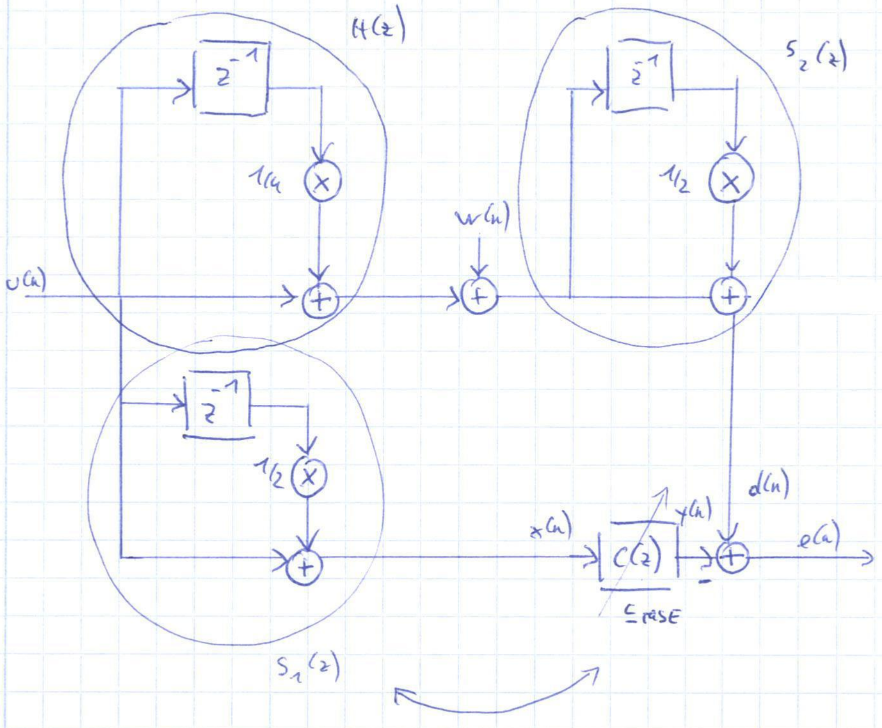
1.2.b.4) Fortsetzung 2 $\underline{c} = (1 \quad 1/4 \quad 0)^T$

$$\frac{27}{16} + \frac{17}{64} - c_0\left(\frac{9}{8} + \frac{2}{8} + \frac{9}{8} + \frac{2}{8}\right) - c_1\left(\frac{9}{16} + \frac{9}{16} + \frac{4}{16} + \frac{4}{16}\right) \cancel{- c_2\left(\frac{1}{8} + \frac{1}{8}\right)} + \frac{5}{4}c_0^2 + c_0c_1 + \frac{5}{4}c_1^2 \cancel{+ c_1c_2} \cancel{+ c_2^2 \cdot \frac{5}{4}} =$$

$$\frac{108+17}{64} - c_0 \cdot \frac{22}{8} - c_1 \cdot \frac{26}{16} + \frac{5}{4} + \frac{1}{4} + \frac{5}{4} \cdot \frac{1}{16} =$$

$$\frac{125}{64} - \frac{22}{8} - \frac{26}{64} + \frac{80}{64} + \frac{16}{64} + \frac{5}{64} = \frac{125 - \overset{176}{\cancel{22}} - 26 + 80 + 16 + 5}{64} = \frac{24}{64} = \underline{\underline{\frac{3}{8} = J_{min}}}$$

1.2.b)
→ e)



$C(z) = H(z)$: interchange $S_1(z)$ with $C(z)$ → identification ok
→ same system

$$\underline{c}_{MSE} = \begin{pmatrix} 1 \\ 1/4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/4 \\ 0 \end{pmatrix}$$

1.2.c.1) $r_{xx}(k) = \underline{s}_1^T \, E(\underline{v}(n+k) \cdot \underline{v}^T(n)) \cdot \underline{s}_1$

$\underline{s}_1 = (1 \;\; 0)^T$
$\underline{s}_2 = (0 \;\; 1)^T$
$\underline{h} = (1 \;\; 1/4)^T$

$r_{xx}(0) = (1 \;\; 0)^T \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{1}$

$r_{xx}(1) = (1 \;\; 0)^T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \;\; 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{0}$

$r_{xx}(2) = (1 \;\; 0)^T \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{0}$

$$\underline{R}_{xx} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{R}_x^{-1}$$

1.2.c.2) $\underline{p} = E \begin{pmatrix} d(n) \cdot \underline{s}_1^T \underline{v}(n) \\ d(n) \cdot \underline{s}_1^T \underline{v}(n-1) \\ d(n) \cdot \underline{s}_1^T \underline{v}(n-2) \end{pmatrix}$

$$= \begin{pmatrix} (1 \;\; 1/4) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (1 \;\; 1/4) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (1 \;\; 1/4) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (1 \;\; 1/4) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 1 \\ (1 \;\; 1/4) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1/4 \end{pmatrix} = \underline{p}$$

1.2.c.3) $\underline{c}_{opt} = \underline{R}_{xx}^{-1} \cdot \underline{p} = \underline{\underline{\begin{pmatrix} 0 \\ 1 \\ 1/4 \end{pmatrix}}}$

1.2.c.4) $J_{min} = J_{MSE}(c_{opt}) = E(e^2(n))$

$s_1 = (1 \quad 0)^T 0$

$s_2 = (0 \quad 1)^T 0$

$c = (0 \quad 1 \quad 1/4)^T$

$h = (1 \quad 1/4)^T$

$$e(n) = d(n) - y(n)$$

$$d(n) = s_{20} \cdot (\underline{h}^T \underline{u}(n) + w(n)) + s_{21} (\underline{h}^T \underline{u}(n-1) + w(n-1)) + s_{22} (\underline{h}^T \underline{u}(n-2) + w(n-2))$$

$$d(n) = \underline{h}^T \underline{u}(n-1) + w(n-1)$$

$$y(n) = \underline{s}_1^T \underline{u}(n-1) + 1/4 \cdot \underline{s}_1^T \underline{u}(n-2)$$

$$e(n) = \underline{h}^T \underline{u}(n-1) + w(n-1) - \underline{s}_1^T \underline{u}(n-1) - \frac{1}{4} \cdot \underline{s}_1^T \underline{u}(n-2)$$

$$e^2(n) = \underline{h}^T \underline{u}(n-1) \cdot \underline{h}^T \underline{u}(n-1) + \cancel{w(n-1)\, \underline{h}^T \underline{u}(n-1)} - \underline{h}^T \underline{u}(n-1)\, \underline{s}_1^T \underline{u}(n-1) - \underline{h}^T \underline{u}(n-1) \cdot \frac{1}{4} \cdot \underline{s}_1^T \underline{u}(n-2)$$

$$+ w(n-1) (\sim + w(n-1) + \; ---) +$$

$$- \underline{s}_1^T \underline{u}(n-1) \cdot \underline{h}^T \underline{u}(n-1) - \cancel{w(n-1) \cdot \underline{s}_1^T \underline{u}(n-1)} + \underline{s}_1^T \underline{u}(n-1) \cdot \underline{s}_1^T \underline{u}(n-1) + \underline{s}_1^T \underline{u}(n-1) \cdot \frac{1}{4} \cdot \underline{s}_1^T \underline{u}(n-2)$$

$$- \frac{1}{4} \cdot \underline{s}_1^T \underline{u}(n-2) \cdot \underline{h}^T \underline{u}(n-1) - \cancel{w(n-1) \cdot \frac{1}{4} \cdot \underline{s}_1^T \underline{u}(n-2)} + \underline{s}_1^T \underline{u}(n-1) \cdot \frac{1}{4} \cdot \underline{s}_1^T \underline{u}(n-2) + \frac{1}{16} \cdot \underline{s}_1^T \underline{u}(n-2)\, \underline{s}_1^T \underline{u}(n-2)$$

$$E(e^2(n)) = \|h\|^2 - \underline{h}^T \cdot \underline{s}_1 - \underline{h}^T \cdot \frac{1}{4} E(\underline{u}(n-1) \cdot \underline{u}^T(n-2)) \cdot \underline{s}_1 +$$

$$E(w^2(n-1)) +$$

$$- \underline{s}_1^T \cdot \underline{h} + \underline{s}_1^T \cdot \underline{s}_1 + \frac{1}{4} \cdot \underline{s}_1^T E(\underline{u}(n-1) \underline{u}^T(n-2)) \cdot \underline{s}_1$$

$$- \frac{1}{4} \cdot \underline{s}_1^T E(\underline{u}(n-2) \underline{u}^T(n-1)) \cdot \underline{h} + \frac{1}{4} \cdot \underline{s}_1^T E(\underline{u}(n-1) \underline{u}^T(n-2)) \underline{s}_1 + \frac{1}{16} \|s_1\|^2$$

zeropadding

$$= \frac{17}{16} - (1 \quad 1/4)^T \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - (1 \quad 1/4 \quad 0)^T \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1/4$$

$$- (1 \quad 0)^T \cdot \begin{pmatrix} 1 \\ 1/4 \end{pmatrix} + 1 + \frac{1}{4} \cdot (1 \quad 0 \quad 0)^T \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$- \frac{1}{4} (1 \quad 0 \quad 0)^T \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1/4 \\ 0 \end{pmatrix} + 1/4 \cdot (1 \quad 0 \quad 0)^T \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{16} \cdot 1 =$$

$$= \frac{17}{16} - 1 - (1 \quad 1/4 \quad 0)^T \cdot 1/4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1/4$$

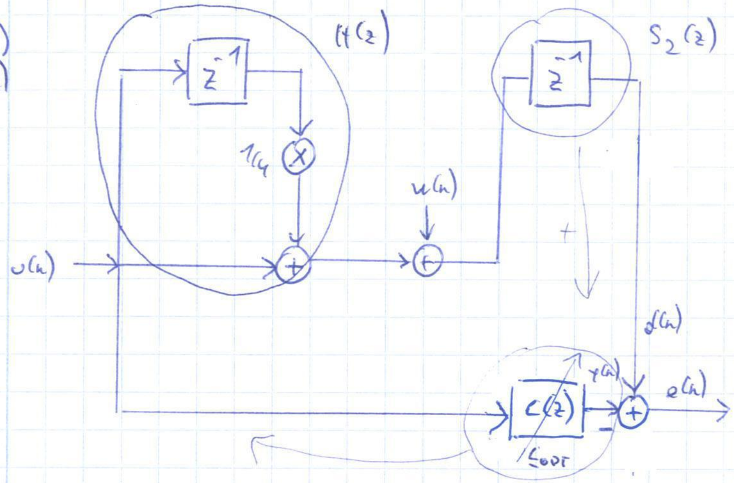
$$- 1 + 1 + 1/4 (1 \quad 0 \quad 0)^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} +$$

$$- \frac{1}{4} (1 \quad 0 \quad 0)^T \begin{pmatrix} 1/4 \\ 0 \\ 0 \end{pmatrix} + 1/4 (1 \quad 0 \quad 0)^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{16} =$$

$$= \cancel{\frac{17}{16}} - \cancel{1} - \cancel{\frac{1}{4} \cdot \frac{1}{4}} + \cancel{1/4} - \cancel{1} + \cancel{1} + \cancel{1/4 \cdot 0} - \cancel{\frac{1}{4} \cdot \frac{1}{4}} + \cancel{\frac{1}{4} \cdot 0} + \cancel{\frac{1}{16}} =$$

$$= \frac{17}{16} - \cancel{\frac{16}{16}} - \cancel{\frac{1}{16}} + \frac{4}{16} - \cancel{\frac{1}{16}} + \cancel{\frac{1}{16}} = \frac{4}{16} = \underline{\frac{1}{4} = J_{min}}$$

1.2.c)
>e)



for perfect system identification z^{-1} would be required after $C(z)$ to compensate z^{-1} in $S_2(z)$

$$\underline{c_{opt}} = \begin{pmatrix} 0 \\ 1 \\ 1/4 \end{pmatrix}$$

1.2.d.1) $r_{xx}(k) = \underline{s}_1^T \, E(\underline{u}(n+k)\cdot\underline{u}^T(n))\cdot \underline{s}_1$ $\qquad s_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

$s_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$

$$r_{xx}(0) = (1\ 0\ 0)^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$r_{xx}(1) = (1\ 0\ 0)^T \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1\ 0\ 0)^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$r_{xx}(2) = (1\ 0\ 0)^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1\ 0\ 0)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\underline{R}_{xx} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.2.d.2)

$$\underline{p} = \begin{pmatrix} (1\ \ 1/4\ \ 0)^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (1\ \ 1/4\ \ 0)^T \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ (1\ \ 1/4\ \ 0)^T \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (1\ \ 1/4\ \ 0) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ (1\ \ 1/4\ \ 0)^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (1\ \ 1/4\ \ 0)^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$\underline{p} = \begin{pmatrix} 1 + (1\ \ 1/4\ \ 0)^T \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ (1\ \ 1/4\ \ 0)^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \\ (1\ \ 1/4\ \ 0)^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1\ \ 1/4\ \ 0)^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 1/4 \\ 1 \end{pmatrix} = \underline{p}$$

1.2.d.3)

$$\underline{c}_{opt} = \underline{R}_{xx}^{-1} \cdot \underline{p} = \begin{pmatrix} 1 \\ 1/4 \\ 1 \end{pmatrix}$$

1.2.d.4) $J_{min} = J_{MSE}(\underline{c}_{opt}) = E(e^2(n))$

$$\underline{c} = \begin{array}{ccc} c_0 & c_1 & c_2 \\ (1 & 1/4 & 1) \end{array}$$

$s_1 = (1 \quad 0 \quad 0)^T$
$s_2 = (1 \quad 0 \quad 1)^T$

$e(n) = d(n) - y(n)$

$d(n) = \underline{h}^T \underline{v}(n) + w(n) + \underline{h}^T \underline{v}(n-2) + w(n-2)$

$y(n) = \underline{s}_1^T \underline{v}(n) \cdot c_0 + c_1 \cdot \underline{s}_1^T \underline{v}(n-1) + c_2 \cdot \underline{s}_1^T \underline{v}(n-2)$

$$e(n) = \underbrace{\underline{h}^T \underline{v}(n)}_{a} + \underbrace{w(n)}_{b} + \underbrace{\underline{h}^T \underline{v}(n-2)}_{c} + \underbrace{w(n-2)}_{d} - \underbrace{\underline{s}_1^T \underline{v}(n) c_0}_{e} + \underbrace{c_1 \cdot \underline{s}_1^T \underline{v}(n-1)}_{f} + \underbrace{c_2 \cdot \underline{s}_1^T \underline{v}(n-2)}_{g}$$

$$(a+b+c+d+e+f+g)^2 =$$

$$\begin{aligned} e^2(n) = {} & a^2 + \cancel{ab} + \cancel{ac} + \cancel{ad} + \cancel{ae} + \cancel{af} + \cancel{ag} + \\ & ba + \cancel{b^2} + \cancel{bc} + \cancel{bd} + \cancel{be} + \cancel{bf} + \cancel{bg} + \\ & \cancel{ca} + \cancel{cb} + \cancel{c^2} + \cancel{cd} + \cancel{ce} + \cancel{cf} + \cancel{cg} + \\ & da + \cancel{db} + \cancel{dc} + \cancel{d^2} + \cancel{de} + \cancel{df} + \cancel{dg} + \\ & \cancel{ea} + \cancel{eb} + \cancel{ec} + \cancel{ed} + \cancel{e^2} + \cancel{ef} + \cancel{eg} + \\ & \cancel{fa} + \cancel{fb} + \cancel{fc} + \cancel{fd} + \cancel{fe} + \cancel{f^2} + \cancel{fg} + \\ & \cancel{ga} + \cancel{gb} + \cancel{gc} + \cancel{gd} + \cancel{ge} + \cancel{gf} + g^2 = \end{aligned}$$

$$\begin{aligned} e^2(n) = {} & a^2 + \cancel{2ab} + 2ac + \cancel{2ad} + 2ae + 2af + 2ag + \\ & b^2 + \cancel{2bc} + 2bd + \cancel{2be} + \cancel{2bf} + \cancel{2bg} + \\ & c^2 + \cancel{2cd} + 2ce + 2cf + 2cg + \\ & d^2 + \cancel{2de} + \cancel{2df} + \cancel{2dg} + \\ & e^2 + 2ef + 2ge + \\ & f^2 + 2gf + \\ & g^2 \end{aligned}$$

$E(ab), E(bc), E(be), E(bf), E(bg) = 0$
$E(da), E(dc), E(de), E(df), E(dg) = 0$

$$\begin{aligned} e^2(n) \Rightarrow {} & a^2 + 2ac + 2ae + 2af + 2ag + \\ & b^2 + 2bd + \\ & c^2 + 2ce + 2cf + 2cg + \\ & d^2 + \\ & e^2 + 2ef + 2ge + \\ & f^2 + 2gf + \\ & g^2 \end{aligned}$$

→ next page

1.2.d.4) Fortsetzung 1

$a = \underline{h}^T \underline{u}(n) \quad b = w(n) \quad c = \underline{h}^T \underline{u}(n-2) \quad d = w(n-2) \quad e = -\underline{s}_1^T \underline{u}(n) c_0 \quad f = -c_1 \underline{s}_1^T \underline{u}(n-1) \quad g = -c_2 \underline{s}_1^T \underline{u}(n-2)$

$a^2 + 2ac + 2ae + 2af + 2ag +$
$b^2 + 2bd +$
$c^2 + 2ce + 2cf + 2cg +$
$d^2 +$
$e^2 + 2ef + 2ge +$
$f^2 + 2gf +$
g^2

$$e^2(n) = \underline{h}^T\underline{u}(n) \cdot \underline{h}^T\underline{u}(n) + 2\underline{h}^T\underline{u}(n)\underline{h}^T\underline{u}(n-2) - 2\underline{h}^T\underline{u}(n)\underline{s}_1^T\underline{u}(n)c_0 - 2\underline{h}^T\underline{u}(n)c_1\underline{s}_1^T\underline{u}(n-1) - 2\underline{h}^T\underline{u}(n)c_2\underline{s}_1^T\underline{u}(n-2) +$$

$$w^2(n) + 2w(n)\cdot w(n-2) +$$

$$\underline{h}^T\underline{u}(n-2)\underline{h}^T\underline{u}(n-2) - 2\underline{h}^T\underline{u}(n-2)\underline{s}_1^T\underline{u}(n)c_0 - 2\underline{h}^T\underline{u}(n-2)c_1\underline{s}_1^T\underline{u}(n-1) - 2\underline{h}^T\underline{u}(n-2)c_2\underline{s}_1^T\underline{u}(n-2) +$$

$$w^2(n-2) +$$

$$c_0^2\underline{s}_1^T\underline{u}(n)\underline{s}_1^T\underline{u}(n) + 2\underline{s}_1^T\underline{u}(n)c_1\underline{s}_1^T\underline{u}(n-1)c_0 + 2c_2\underline{s}_1^T\underline{u}(n-2)\underline{s}_1^T\underline{u}(n)c_0 +$$

$$c_1\underline{s}_1^T\underline{u}(n-1)\cdot c_1\underline{s}_1^T\underline{u}(n-1) + 2c_1\underline{s}_1^T\underline{u}(n-1)\cdot c_2\underline{s}_1^T\underline{u}(n-2) +$$

$$c_2^2\underline{s}_1^T\underline{u}(n-2)\cdot\underline{s}_1^T\underline{u}(n-2) \Rightarrow$$

$$E(\%) = \|\underline{h}\|^2 + 2\underline{h}^T E(\underline{u}(n)\underline{u}^T(n-2))\cdot\underline{h} - 2c_0\cdot\underline{h}^T\cdot\underline{s}_1 - 2c_1\underline{h}^T E(\underline{u}(n)\cdot\underline{u}^T(n-1))\underline{s}_1 - 2c_2\underline{h}^T E(\underline{u}(n)\cdot\underline{u}^T(n-2))\cdot\underline{s}_1$$

$$+ E(w^2(n)) + 2E(w(n)\cdot w(n-2)) +$$

$$\|\underline{h}\|^2 - 2c_0\cdot\underline{h}^T E(\underline{u}(n-2)\cdot\underline{u}^T(n))\cdot\underline{s}_1 - 2c_1\cdot\underline{h}^T E(\underline{u}(n-2)\underline{u}^T(n-1))\cdot\underline{s}_1 - 2\underline{h}^T\cdot\underline{s}_1\cdot c_2 +$$

$$+ E(w^2(n-2)) +$$

$$c_0^2\cdot\|\underline{s}_1\|^2 + 2c_1c_0\underline{s}_1^T E(\underline{u}(n)\underline{u}^T(n-1))\cdot\underline{s}_1 + 2c_0c_2\underline{s}_1^T E(\underline{u}(n-2)\cdot\underline{u}^T(n))\underline{s}_1 +$$

$$c_1^2\|\underline{s}_1\|^2 + 2c_1c_2\underline{s}_1^T E(\underline{u}(n-1)\cdot\underline{u}^T(n-2))\cdot\underline{s}_1 +$$

$$c_2^2\|\underline{s}_1\|^2$$

$s_1 = (1 \;\; 0 \;\; 0)^T$
$s_2 = (1 \;\; 0 \;\; 1)^T \quad \rightarrow E(\%) = \frac{17}{16} + 2\cdot(1 \;\; 1/4 \;\; 0)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\cdot$
$h = (1 \;\; 1/4 \;\; 0)^T$
$c = (1 \;\; 1/4 \;\; 1)^T$
$c_0 \;\; c_1 \;\; c_2$

$$E(\%) = \frac{17}{16} + 2(1 \;\; 1/4)^T\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1/4 \end{pmatrix} - 2\cdot(1 \;\; 1/4)^T\begin{pmatrix} 1 \\ 0 \end{pmatrix}c_0 - 2\cdot\frac{1}{4}(1 \;\; 1/4)^T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\cdot(1 \;\; 1/4)^T\cdot\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} +$$

$$\frac{1}{4} +$$

$$\frac{17}{16} - 2(1 \;\; 1/4)^T\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\cdot 1/4\cdot(1 \;\; 1/4)^T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\cdot(1 \;\; 1/4)^T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cdot 1 +$$

$$\frac{1}{4} +$$

$$1 + 2\cdot\frac{1}{4}\cdot(1 \;\; 0)^T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot(1 \;\; 1/4)^T\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} +$$

$$\frac{1}{16}\cdot 1 + 2\cdot\frac{1}{4}\cdot(1 \;\; 0)^T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 =$$

1.2 d.4) Fortsetzung 2

$$E(e^2(u)) = \frac{17}{16} + 2\left(1 \;\; \tfrac{1}{4}\right)^T \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 2 - \frac{1}{2}\left(1 \;\; \tfrac{1}{4}\right)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 2\,(0) +$$

$$\frac{1}{4} +$$

$$\frac{17}{16} - 2\left(1 \;\; \tfrac{1}{4}\right)^T \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 2 \cdot \frac{1}{4}\left(1 \;\; \tfrac{1}{4}\right)^T \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 2 \cdot 1 +$$

$$\frac{1}{4} +$$

$$1 + 2 \cdot \frac{1}{4}\left(1 \;\; 0\right)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2\left(1 \;\; 0\right)^T \begin{pmatrix} 0 \\ 0 \end{pmatrix} +$$

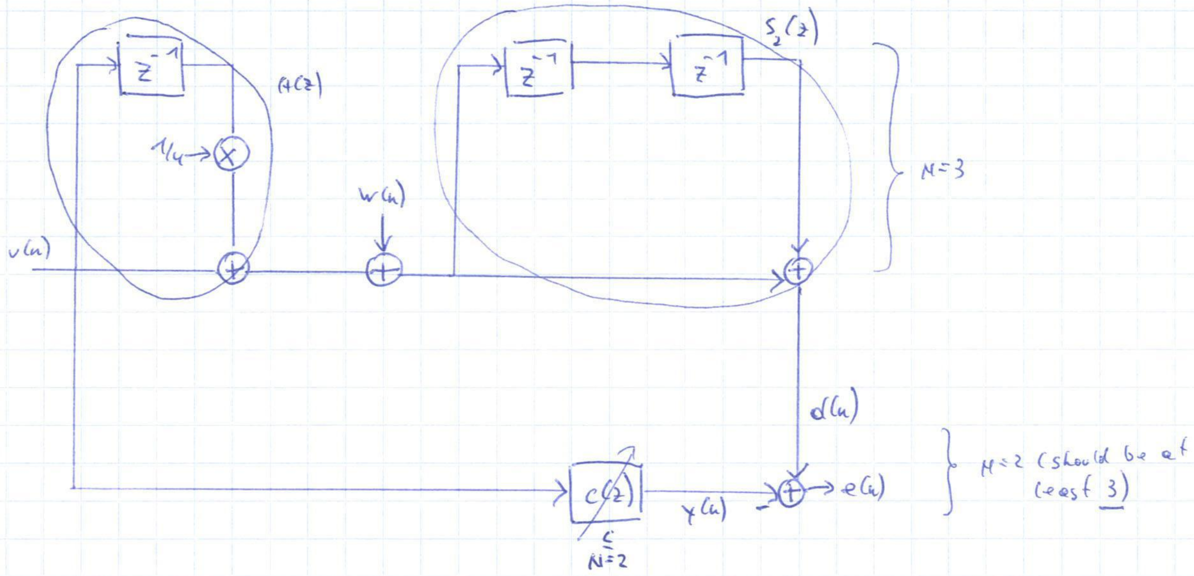
$$\frac{1}{16} + 2 \cdot \frac{1}{4}\left(1 \;\; 0\right)^T \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1 =$$

$$\cancel{\frac{17}{16}} - \cancel{2} - \cancel{\frac{1}{2}} \cdot \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} + \cancel{\frac{17}{16}} - \cancel{2} + \cancel{\frac{1}{4}} + \cancel{1} + \frac{1}{2} \cdot 0 + \cancel{\frac{1}{16}} + \cancel{1} =$$

$$\cancel{\frac{35}{16}} - \cancel{2} - \frac{1}{8} + \cancel{\frac{1}{4}} - \cancel{2} + \cancel{\frac{1}{4}} + \cancel{1} + \cancel{1} =$$

$$\frac{3}{16} - \frac{1}{8} + \frac{2}{4} = \frac{3}{16} - \frac{2}{16} + \frac{8}{16} = \underline{\frac{9}{16} = J_{min}}$$

1.2.d)



$C(z)$ = Filter order $N=2$ too less for this configuration (z^{-3} in total compared to z^{-2} of $C(z)$.

IDentification fails.

$\underline{c} = \begin{pmatrix} 1 \\ 1/4 \end{pmatrix}$

Task 1.4 a)

$$J_{WLS}(\underline{c}, n) = \sum_{k=n-M+1}^{n} g(n-k)\, |e(k)|^2 = \sum_k g(n-k) \cdot e^2(k) = \sum_k e(k) \cdot g(n-k) \cdot e(k)$$

$$\sum_k e(k) \cdot g(n-k) \cdot e(k)$$

$$\downarrow$$

$$\underbrace{(e(n-M+1) \; \dots \; e(n))}_{\underline{e}^T(n)} \underbrace{\begin{pmatrix} g(M-1) & & 0 \\ & \ddots & \\ 0 & & g(0) \end{pmatrix}}_{G} \underbrace{\begin{pmatrix} e(n-M+1) \\ \vdots \\ e(n) \end{pmatrix}}_{\underline{e}(n)}$$

$$e(n) = d(n) - y(n) = d(n) - \underline{c}^T \cdot \underline{x}(n)$$

$$e(0) = d(0) - \underline{c}^T \cdot \underline{x}(0) = d(0) - (c_0 \dots c_{M-1}) \begin{pmatrix} x(0) \\ \vdots \\ x(-M+1) \end{pmatrix}$$

$$\vdots$$

$$e(M-1) = d(M-1) - \underline{c}^T \underline{x}(M-1) = d(M-1) - (c_0 \dots c_{M-1}) \begin{pmatrix} x(M-1) \\ \vdots \\ x(M-M) \end{pmatrix}$$

$$\underline{e} = \underline{d} - \underline{X} \cdot \underline{c} \qquad \underline{X} = \begin{pmatrix} x(0) & & x(-M+1) \\ \vdots & & \vdots \\ x(M-1) & & x(M-M) \end{pmatrix}$$

$$J_{WLS} = \underline{e}^T \cdot G \cdot \underline{e} = (\underline{d} - \underline{X}\underline{c})^T \cdot G \cdot (\underline{d} - \underline{X}\underline{c})$$

$$= (\underline{d}^T - (\underline{X}\underline{c})^T)\, G\, (\underline{d} - \underline{X}\underline{c}) =$$

$$= (\underline{d}^T - \underline{c}^T \underline{X}^T)\, G\, (\underline{d} - \underline{X}\underline{c}) = (\underline{d}^T G - \underline{c}^T \underline{X}^T G)(\underline{d} - \underline{X}\underline{c}) =$$

$$= \underline{d}^T G \underline{d} - \underline{d}^T G \underline{X}\underline{c} - \underline{c}^T \underline{X}^T G \underline{d} + \underline{c}^T \underline{X}^T G \underline{X}\underline{c} =$$

$$= \underline{d}^T G \underline{d} - \underline{d}^T G \underline{X}\underline{c} - (\underline{X}^T G \underline{d})^T \cdot \underline{c} + \underline{c}^T \underline{X}^T G \underline{X}\underline{c} =$$

$$= \underline{d}^T G \underline{d} - \underline{d}^T G \underline{X}\underline{c} - \underline{d}^T (\underline{X}^T G)^T \cdot \underline{c} + \underline{c}^T \underline{X}^T G \underline{X}\underline{c} =$$

$$= \underline{d}^T G \underline{d} - \underline{d}^T G \underline{X}\underline{c} - \underline{d}^T G^T (\underline{X}^T)^T \cdot \underline{c} + \underline{c}^T \underline{X}^T G \underline{X}\underline{c} =$$

$$= \underline{d}^T G \underline{d} - 2\underline{d}^T G \underline{X}\underline{c} + \underline{c}^T \underline{X}^T G \underline{X}\underline{c}$$

$$\nabla_{\underline{c}}(J_{WLS}) = -2\,\underline{d}^T G \underline{X} + 2\,\underline{X}^T G \underline{X}\underline{c} = 0$$

$$\underline{d}^T G \underline{X} = \underline{X}^T G \underline{X}\underline{c}$$

$$\underline{(\underline{X}^T G \underline{X})^{-1}\, \underline{d}^T G \underline{X} = \underline{c}}$$

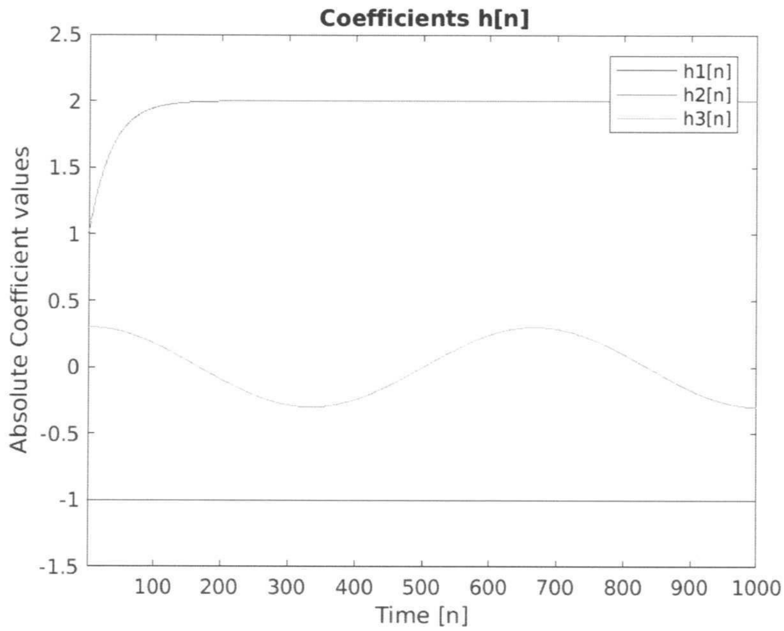
Adaptive Systems, Exercise

Winter term 2019/20

Harald Stiegler, 9330054

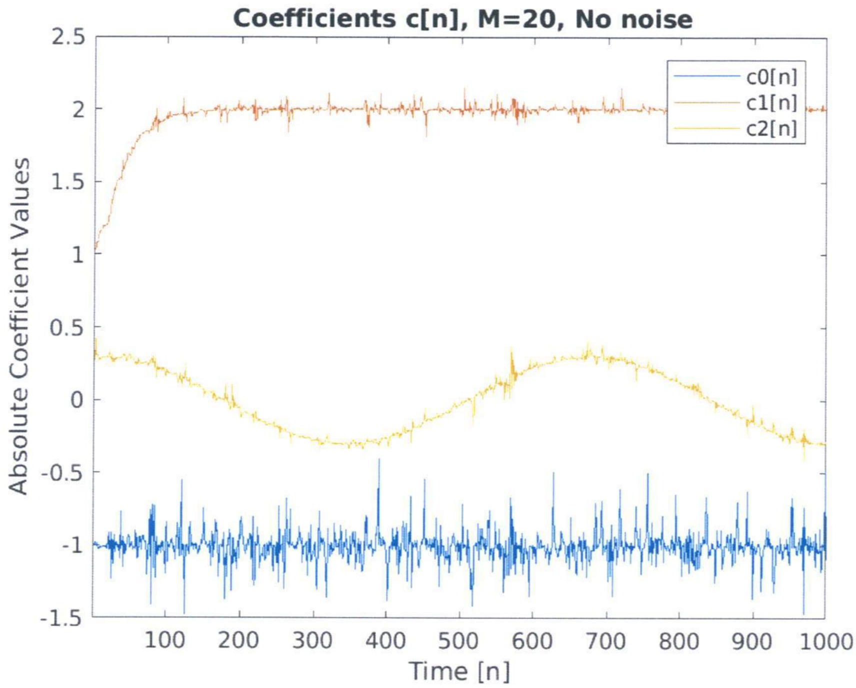
Task 1.3c: Impulse response of an "unknown" system is given, adaptive filter will try to approximate it (N=3 coefficients both "unknown" filter and adaptive filter).

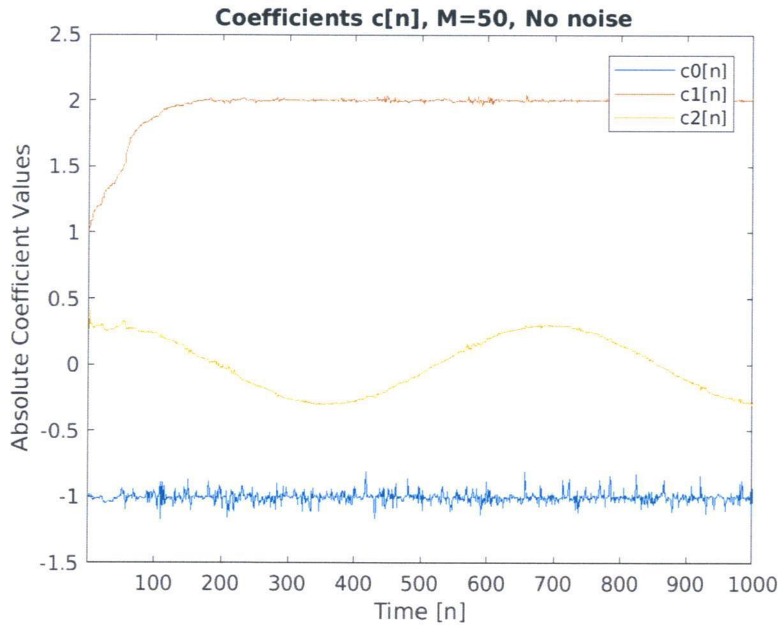
Impulse response



Approximated coefficients

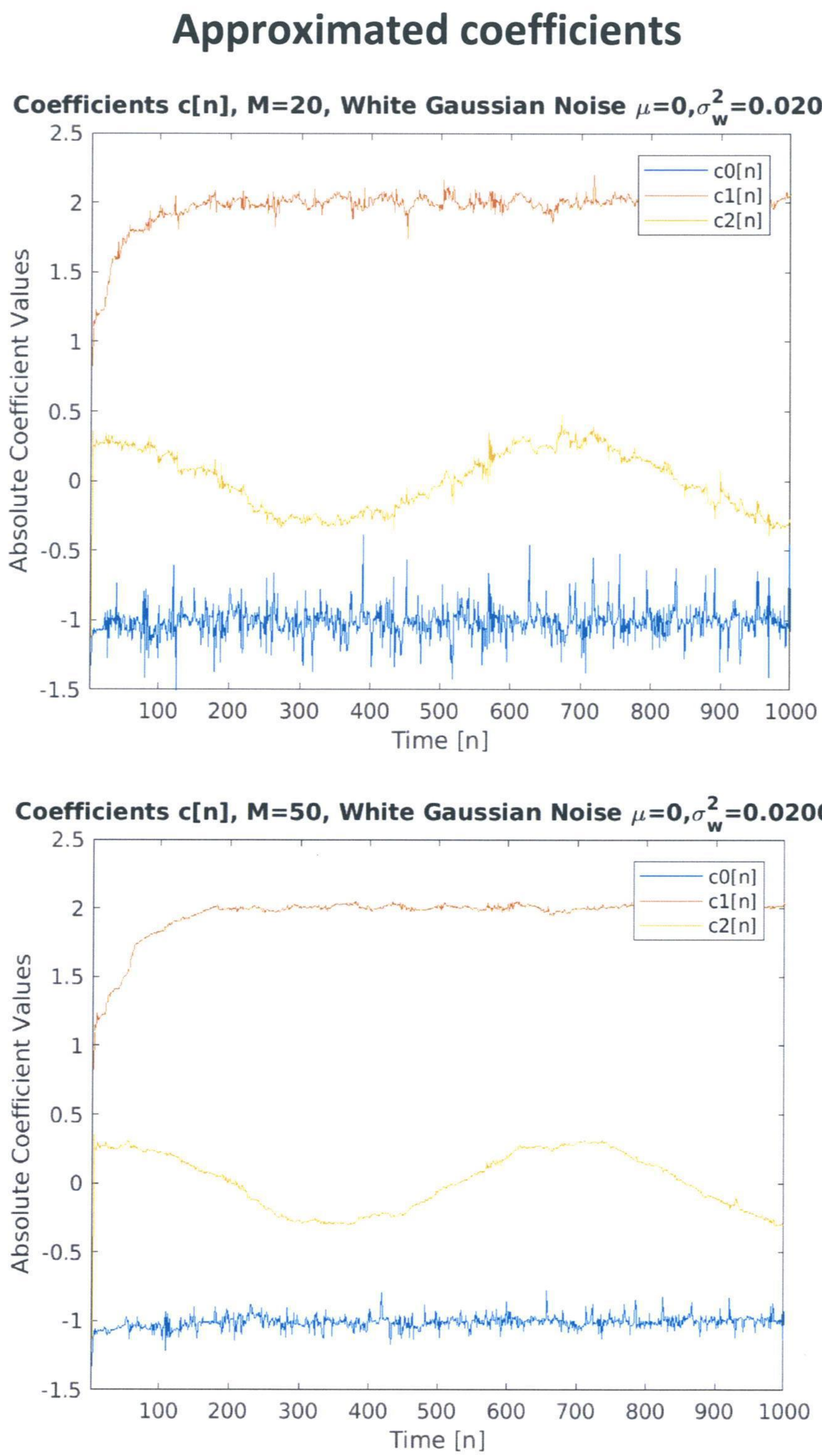
Scenario: No additive noise, comparison of different sliding window lengths used for coefficient computation of adaptive filter: M=20 vs M=50





Conclusion: A larger M results in smoother c-coefficients over time, as the coefficients have been computed to be "optimal" for a longer period of time. Within this longer time period, white noise fluctuations seem to cancel themselves a bit more by the various linear combinations of x(n) when computing the optimal coefficients.

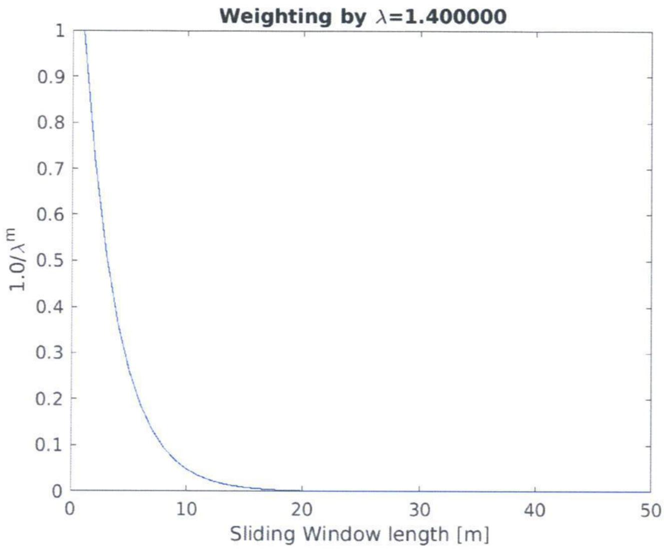
Task 1.3.d: Scenario: Same impulse response, but white gaussian noise has been added (σ^2=0.02). Compare approximated coefficients for different sliding window lengths, M=20, M=50



Conclusion: Same effect, a larger M leads to smoother c-coefficients over time, as the coefficients have been computed to be "optimal" for a longer time period. But coefficients fluctuate more in general due to the additional noise.

Task 1.4

Error function is weighted by giving more "attention" to latest errors and "ignoring" old errors. Weighting function as specified in task 1.4.c for an arbitrarily chosen λ=1.4:

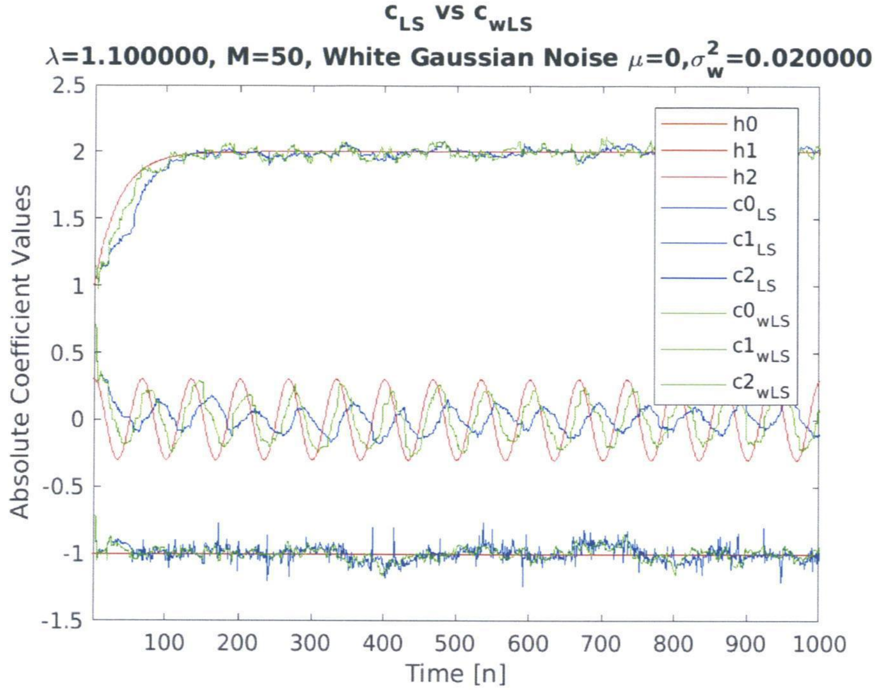


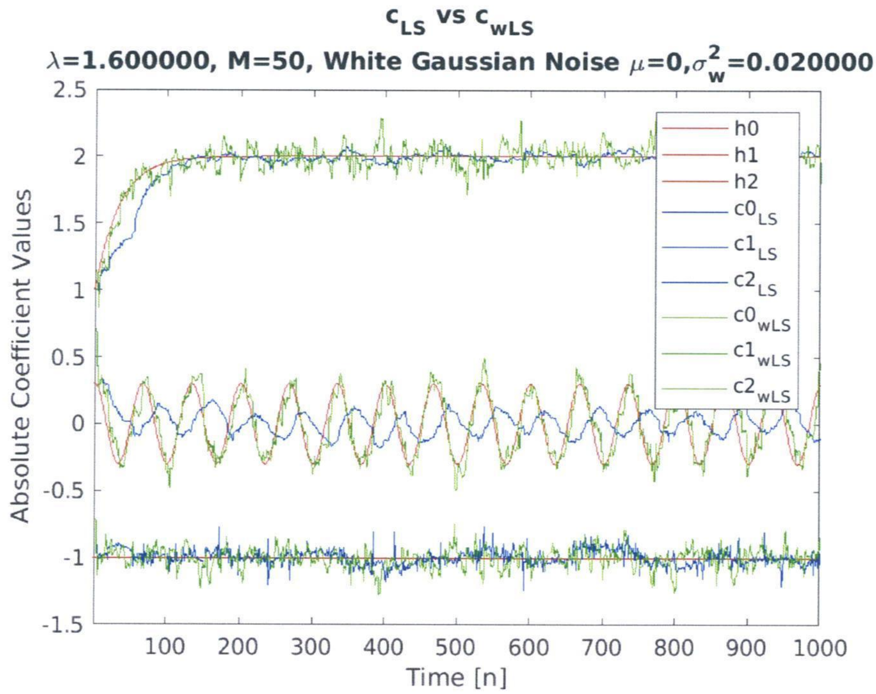
If λ>1.0 weighting suppresses "old" errors and gives more weights to the latest errors. The filter coefficients adapt much faster, the coefficient time lag almost vanishes (best seen at cosine-wave below when comparing the weighted with the unweighted coefficients, green vs. blue).

The larger λ, the more and more weight is given to lesser and lesser of the latest errors until only e(0) is weighted with maximum and all other e(i) are set to zero. In the figure above, the weighting curve gets steeper and steeper and closer and closer to the vertical axis.

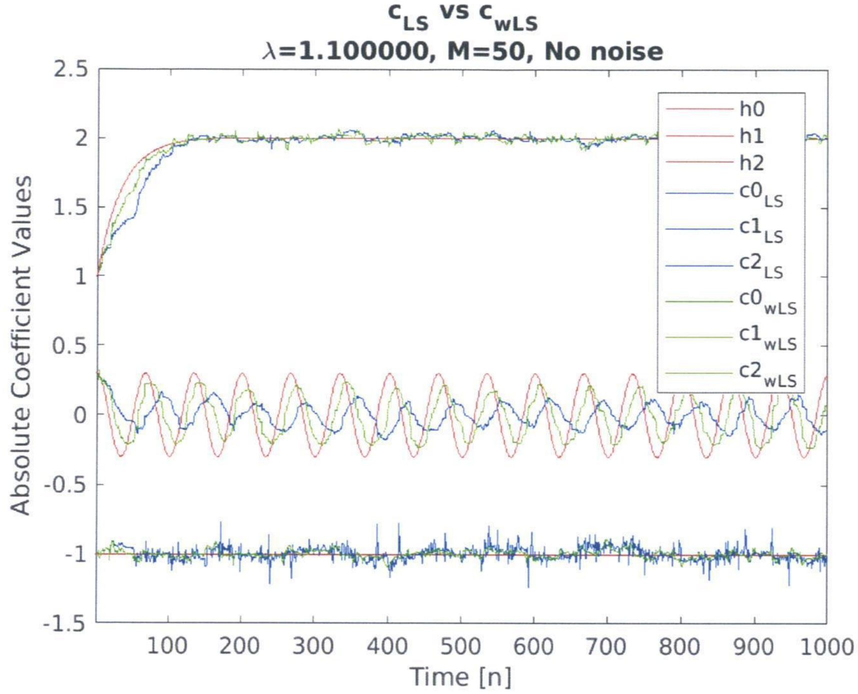
If λ<1.0 weighting has the reverse effect. "Old" errors are given more attention, whereas "latest" errors are ignored. This scenario most likely does not make sense.

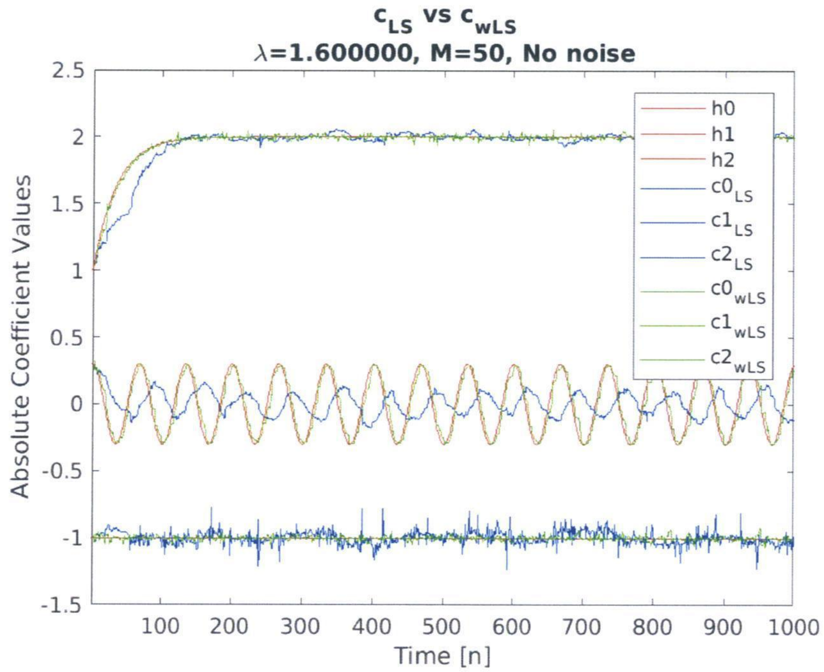
A rough comparison for different λ (>1.0) values shows a trade fast coefficient adaption off making them more and more noisy, next two plots show the approximated coefficients over time for λ =1.1 compared to 1.6. These plots contain also the unweighted coefficients c_{LS} for comparison.



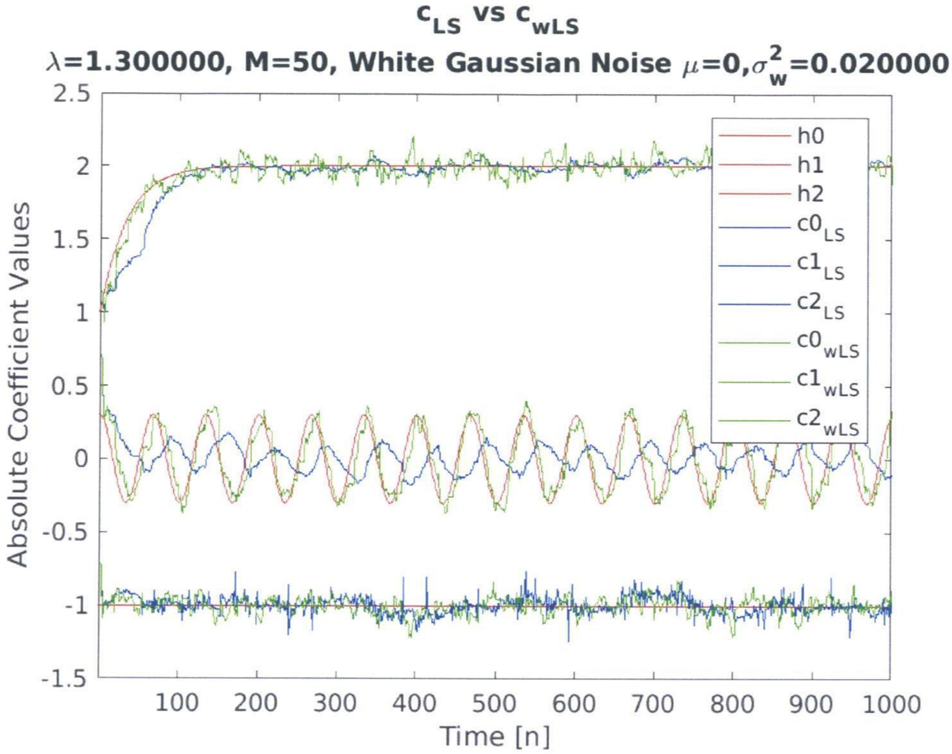


For the sake of completeness the following plots show again the same scenario but without additional white Gaussian noise, λ=1.1 vs 1.6, M=50.





Choosing an optimal value for lambda will depend on its task. On a coarse grid of λ evaluations with step size 0.1 from one λ value to next, within the range of 1.1 up to 1.6, a value of 1.3 for λ has been arbitrarily chosen.



c_{LS} vs c_{wLS}
λ=1.300000, M=50, White Gaussian Noise μ=0,σ_w^2=0.020000
Absolute Coefficient Values
Time [n]
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
100
200
300
400
500
600
700
800
900
1000
h0
h1
h2
$c0_{LS}$
$c1_{LS}$
$c2_{LS}$
$c0_{wLS}$
$c1_{wLS}$
$c2_{wLS}$