PROBLEM 1.2

a) 1)

$$r_{xx}[n,k] = E\{x[n+k] \cdot x^*[n]\} \qquad \underline{h}^T = [1 \ \tfrac{1}{4} \ 0]$$

$$x[n] = S_1 * u[n] = u[n]$$

$$r_{xx}[n,k] = E\{u[n+k] \cdot u^*[n]\}$$

$$= \sigma_u^2 \cdot \delta[k] = 1 \cdot \delta[k]$$

$$\underline{R}_{xx} = \begin{pmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] \\ r_{xx}[-1] & r_{xx}[0] & r_{xx}[1] \\ r_{xx}[-2] & r_{xx}[-1] & r_{xx}[0] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2)

$$\underline{p} = E\{d[n] \ \underline{x}[n]\}$$

$$d[n] = S_2 * (\underline{h}^T \underline{u}[n] + w[n]) = \underline{h}^T \underline{u}[n] + w[n]$$

$$\underline{p} = E\{(\underline{h}^T \cdot \underline{u}[n] + w[n]) \cdot \underline{u}[n]\}$$

$$\underline{p} = E\{\underline{h}^T \cdot \underline{u}[n] \cdot \underline{u}[n]\} + \underbrace{E\{w[n] \cdot \underline{u}[n]\}}_{0}$$

$$\underline{p} = E\{[1 \ \tfrac{1}{4} \ 0] \begin{bmatrix} u[n] \\ u[n-1] \\ u[n-1] \end{bmatrix} \cdot \underline{u}[n]\}$$

$$= E\{(u[n] + \tfrac{1}{4} u[n-1]) \cdot \underline{u}[n]\}$$

$$= E\left\{\begin{bmatrix} u[n] \cdot u[n] + \tfrac{1}{4} u[n-1] \cdot u[n] \\ u[n] \cdot u[n-1] + \tfrac{1}{4} u[n-1] \cdot u[n-1] \\ u[n] \cdot u[n-2] + \tfrac{1}{4} u[n-1] \cdot u[n-2] \end{bmatrix}\right\}$$

$$\underline{p} = \begin{bmatrix} \sigma_u^2 \\ \tfrac{1}{4}\sigma_u^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \tfrac{1}{4} \\ 0 \end{bmatrix} = \underline{h}$$

3)

$$\underline{c} = \underline{R}_{xx}^{-1} \cdot \underline{p} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \tfrac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \tfrac{1}{4} \\ 0 \end{bmatrix} = \underline{h} = \underline{c}_{opt}$$

4)

$$J_{min} = J_{MSE}(\underline{c}_{opt}) \qquad y[n] = c^T \cdot \underline{u}[n]$$

$$e[n] = d[n] - y[n] = (\underline{h}^T - \underline{c}^T)\, u[n] - w[n]$$

$$J_{MSE} = E\{|e[n]|^2\} = E\{e[n] \cdot e^*[n]\}$$

$$J_{MSE}(c_{opt}) = E\{((\underline{h}^T - \underline{h}^T) \cdot u[n] - w[n]) \cdot ((\underline{h}^T - \underline{h}^T) \cdot u[n] - w[n])^*\}$$

$$= E\{(-w[n]) \cdot (-w[n])^*\} = \tfrac{1}{4} = \sigma_w^2$$

$$J_{min} = \underline{\underline{\tfrac{1}{4}}}$$

b) $r_{xx}[n,k] = E\{x[n+k]\ x^*[n]\}$

1) $x[n] = s_1 * u[n] = (\delta[n] + \delta[n-1]\cdot 0{,}5) * u[n]$

$$= \sum_{k=0}^{N-1} (\delta[n] + \delta[n-1]\cdot 0{,}5)\cdot u[n+k]$$

$$= u[n] + 0{,}5\, u[n-1]$$

$$r_{xx}[n,k] = E\{(u[n+k] + 0{,}5\, u[n+k-1])(u[n] + u[n-1]\cdot 0{,}5)^*\}$$

$$= E\{u[n+k]\cdot u^*[n] + u[n+k]\cdot u^*[n-1]\cdot 0{,}5 + 0{,}5\cdot u[n+k-1]\cdot u^*[n]$$
$$+ 0{,}5\, u[n+k-1]\cdot 0{,}5\, u^*[n+k-1]\}$$

$$= \delta[k] + 0{,}5\,\delta[k+1] + 0{,}5\,\delta[k-1] + 0{,}25\,\delta[k]$$

$$r_{xx}[k] = 1{,}25\,\delta[k] + 0{,}5\,\delta[k+1] + 0{,}5\,\delta[k-1]$$

$$\underline{R}_{xx} = \begin{pmatrix} 1{,}25 & 0{,}5 & 0 \\ 0{,}5 & 1{,}25 & 0{,}5 \\ 0 & 0{,}5 & 1{,}25 \end{pmatrix}$$

2) $d[n] = s_2 * (\underline{h}^T\cdot \underline{u}[n] + w[n])$

$$= \sum_{k=0}^{N-1} (\delta[n] + 0{,}5\,\delta[n-1])\cdot(\underline{h}^T\cdot \underline{u}[n+k] + w[n+k])$$

$$= \underline{h}^T \underline{u}[n] + w[n] + \underline{h}^T\cdot \underline{u}[n-1]\cdot 0{,}5 + w[n-1]\cdot 0{,}5$$

$$\underline{p} = E\{d[n]\cdot \underline{x}[n]\} = E\{(\underline{h}^T\underline{u}[n] + w[n] + \underline{h}^T\underline{u}[n-1]0{,}5 + w[n-1]0{,}5)$$
$$\cdot(\underline{u}[n] + 0{,}5\cdot \underline{u}[n-1])\}$$

$$\underline{p} = E\{\underline{h}^T\underline{u}[n]\cdot\underline{u}[n] + \underline{h}^T\underline{u}[n]\,\underline{u}[n-1]\cdot 0{,}5 + w[n]\cdot\underline{u}[n] + w[n]\,\underline{u}[n-1]\cdot 0{,}5$$
$$+ \underline{h}^T\underline{u}[n-1]\cdot 0{,}5\cdot\underline{u}[n] + \underline{h}^T\underline{u}[n-1]\cdot 0{,}5\cdot 0{,}5\,\underline{u}[n-1] + w[n-1]\cdot\underline{u}[n] + w[n-1]\,\underline{u}[n-1]\}$$

$$\underline{p} = E\{\underline{h}^T\underline{u}[n]\,\underline{u}[n]\} + E\{\underline{h}^T\underline{u}[n]\cdot\underline{u}[n-1]\}\,0{,}5 + 0 + 0 + E\{\underline{h}^T\underline{u}[n-1]\cdot 0{,}5\cdot\underline{u}[n]\}$$
$$+ E\{\underline{h}^T\underline{u}[n-1]\cdot 0{,}25\cdot\underline{u}[n-1]\} + 0 + 0$$

$$= E\{(u[n] + \tfrac{1}{4}u[n-1])\cdot\underline{u}[n]\} + 0{,}5\cdot E\{(u[n] + \tfrac{1}{4}u[n-1])\cdot\underline{u}[n-1]\}$$
$$+ 0{,}5\,E\{(u[n-1] + \tfrac{1}{4}u[n-2])\cdot\underline{u}[n]\} + 0{,}25\,E\{(u[n-1] + \tfrac{1}{4}u[n-2])\cdot\underline{u}[n-1]\}$$

$$= \begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \end{bmatrix} + 0{,}5\begin{bmatrix} \frac{1}{4} \\ 0 \\ 0 \end{bmatrix} + 0{,}5\begin{bmatrix} 0 \\ 1 \\ \frac{1}{4} \end{bmatrix} + 0{,}25\begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1{,}375 \\ 0{,}8125 \\ 0{,}125 \end{bmatrix} = \underline{p} = \begin{bmatrix} 11/8 \\ 13/16 \\ 1/8 \end{bmatrix}$$

3) $\underline{c} = \underline{R}_{xx}^{-1} \cdot \underline{p} = \begin{bmatrix} \frac{84}{85} & -\frac{8}{17} & \frac{16}{85} \\ -\frac{8}{17} & \frac{20}{17} & -\frac{8}{17} \\ \frac{16}{85} & -\frac{8}{17} & \frac{84}{85} \end{bmatrix} \cdot \begin{bmatrix} 11/8 \\ 13/16 \\ 1/8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \\ 0 \end{bmatrix} = \underline{h}$

4) $J_{min} = E\{ e[n] \cdot e^*[n] \}$

$$e[n] = \underbrace{(\underline{h}^T - \underline{c}^T)}_{0} u[n] + \underbrace{(\underline{h}^T - \underline{c}^T)}_{0} u[n-1] \cdot 0{,}5 + w[n] + w[n-1] \cdot 0{,}5)$$

if $\underline{c} = \underline{h}$

$$J_{min} = E \{ (w[n] + 0{,}5\, w[n-1]) \cdot (w[n] + 0{,}5\, w[n-1])^* \}$$

$$= E \{ w[n] \cdot w[n] + 0{,}5\, w[n] \cdot w[n-1] + 0{,}5\, w[n-1] \cdot w[n] + 0{,}25\, w[n-1] \cdot w[n-1] \}$$

$$= 0{,}25\, \sigma_w^2 + \sigma_w^2 = \frac{1}{4} + \frac{1}{16} = \underline{\underline{\frac{5}{16}}}$$

c) 1) is the same as a) 1

$r_{xx} = \delta[k] \qquad x[n] = u[n]$

$\underline{R}_{xx} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

2)

$d[n] = \underline{h}^T \underline{u}[n-1] + w[n-1]$

$\underline{p} = E\{(\underline{h}^T\underline{u}[n-1]+w[n-1])\cdot(\underline{u}[n])\}$

$= E\{\underline{h}^T\cdot\underline{u}[n-1]\cdot\underline{u}[n] + w[n-1]\cdot\underline{u}[n]\}$

$= E\{(u[n-1]+\frac{1}{4}u[n-2])\,\underline{u}[n])\} + E\{\underbrace{w[n-1]\cdot u[n]}_{0}\}$

$= \begin{bmatrix}0\\1\\\frac{1}{4}\end{bmatrix}$

3) $\underline{c} = \underline{R}_{xx}^{-1}\cdot\underline{p} = \begin{bmatrix}0\\1\\1/4\end{bmatrix}$

4) $J_{min} = E\{[e[n]\cdot e^*[n]\} \qquad e[n] = \underline{h}^T\cdot\underline{u}[n-1]+w[n-1] - \underline{c}^T\underline{u}[n]$

$J_{min} = E\{(\underline{h}^T\underline{u}[n-1]+w[n-1]-\underline{c}^T\underline{u}[n])\cdot(\underline{h}^T\underline{u}[n-1]+w[n-1]-\underline{c}^T\underline{u}[n])^*\}$

$= E\{\underline{h}^T\underline{u}[n-1]\,\underline{h}^T\underline{u}[n-1] + \underbrace{\underline{h}^T\underline{u}[n-1]\cdot w[n-1]}_{0} - \underline{h}^T\underline{u}[n-1]\,\underline{c}^T\underline{u}[n]$

$+ \underbrace{w[n-1]\cdot\underline{h}^T\underline{u}[n-1]}_{0} + w[n-1]\cdot w[n-1] - \underbrace{w[n-1]\cdot\underline{c}^T\underline{u}[n]}_{0}$

$- \underline{c}^T\underline{u}[n]\,\underline{h}^T\underline{u}[n-1] - \underbrace{\underline{c}^T\underline{u}[n]\cdot w[n-1]}_{0} + \underline{c}^T\underline{u}[n]\cdot\underline{c}^T\underline{u}[n]\}$

$= \underline{h}^T\cdot E\{\underline{u}[n-1]\cdot\underline{u}^T[n-1]\}\underline{h} - \underline{h}^T E\{\underline{u}[n-1]\,\underline{u}^T[n]\}\underline{c}$

$+ \sigma_w^2 - \underline{c}^T E\{\underline{u}[n]\cdot\underline{u}^T[n-1]\}\underline{h} + \underline{c}^T E\{\underline{u}[n]\cdot\underline{u}^T[n]\}\underline{c}$

$= \underline{h}^T\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\underline{h} - \underline{h}^T\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\underline{c} + \frac{1}{4} - \underline{c}^T\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}\underline{h} + \underline{c}^T\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\underline{c}$

$= 1+\frac{1}{16} - 1 - \frac{1}{16} + \frac{1}{4} - 1 - \frac{1}{16} + 1 + \frac{1}{16} = \underline{\underline{\frac{1}{4}}} = J_{min}$

d) 1) is the same as d) 1

$r_{xx} = \delta[k]$ $\quad x[n] = u[n]$

$$\underline{R}_{xx} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2) $d[n] = \underline{h}^T \underline{u}[n] + w[n] + \underline{h}^T \cdot \underline{u}[n-2] + w[n-2]$

$$\underline{p} = E\{(\underline{h}^T \underline{u}[n] + w[n] + \underline{h}^T \cdot \underline{u}[n-2] + w[n-2]) \cdot \underline{u}[n]\}$$

$$= E\{(u[n] + \tfrac{1}{4} u[n-1]) \cdot \underline{u}[n]\} + E\{(u[n-2] + \tfrac{1}{4} u[n-3]) \cdot \underline{u}[n]\}$$

$$= \begin{bmatrix} 1 \\ \frac{1}{4} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/4 \\ 1 \end{bmatrix} = \underline{p}$$

3) $\underline{c} = \underline{R}_{xx}^{-1} \cdot \underline{p} = \begin{bmatrix} 1 \\ 1/4 \\ 1 \end{bmatrix}$

4) $J_{Min} = J_{MSE}(\underline{c}_{opt})$ $\quad e[n] = \underline{h}^T u[n] + w[n] + \underline{h}^T u[n-2] + w[n-2] - \underline{c}^T u[n]$

$$\underline{a} = \underline{h} - \underline{c} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$J_{min} = E\{e[n] \cdot e^*[n]\} = E((\underline{a}^T \underline{u}[n] + w[n] + \underline{h}^T \underline{u}[n-2] + w[n-2])(\underline{a}^T \underline{u}[n] + w[n] + \underline{h}^T \underline{u}[n-2] + w[n-2])$$

$$= E\{\underline{a}^T \underline{u}[n]\, \underline{a}^T \underline{u}[n] + \underline{a}^T \underline{u}[n]\, w[n] + \underline{a}^T \underline{u}[n]\, \underline{h}^T \underline{u}[n-2] + \underline{a}^T \underline{u}[n]\, w[n-2]$$

$$+ \ldots\ldots$$

$$= \underline{a}^T E\{\underline{u}[n]\, \underline{u}^T[n]\}\, \underline{a} + E\{\underline{a}^T \underline{u}[n]\, \underline{u}^T[n-2]\, \underline{h}\} + \delta_w^2$$
$$+ \underline{h}^T E\{\underline{u}[n-2]\, \underline{u}^T[n]\}\, \underline{a} + \underline{h}^T E\{\underline{u}[n-1]\, \underline{u}^T[n-1]\}\, \underline{h} + \delta_w^2$$

$$= \underline{a}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{a} + \underline{a}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \underline{h} + \underline{h}^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{a} + \underline{h}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{h} + \frac{1}{2}$$

$$= \cancel{1} - \cancel{1} - \cancel{1} + \cancel{1} + \frac{1}{16} + \frac{1}{2} = \underline{\underline{\frac{9}{16}}}$$

e)

- if the two sensor respones are the same, the System gets identified correctly
- if the two sensor respones are the same, but one has a delay (e.g. $S_2(z) = S_1(z) \cdot z^{-1}$) The system won't be identified correctly, unless we introduce same delay on our own (e.g. delay path through S_1 by z^{-1})
- The system in d) cannot be identified correctly.