Adaptive Systems
EXERCISE 1

PROBLEM 1.1

a

i) $\underline{A} = [\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n]$ (some signal) $\qquad \underline{a}_i = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

now: $size(\underline{A}) = (N \times 4)$ (row x col)

$\underline{A} = [\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4]$

$\underline{A} \cdot \underline{x} \overset{!}{=} x_1 \cdot \underline{a}_1 + x_2 \cdot \underline{a}_2 + x_3 \cdot \underline{a}_3 + x_4 \cdot \underline{a}_4 \Rightarrow \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $\qquad \underline{\underline{size(\underline{x}) = (4 \times 1)}}$

$(N \times 4) \cdot (4 \times 1) = (N \times 1)$

ii) $\underset{(3 \times 1)}{\underline{a}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

we want: $\underset{(3 \times 1)}{\begin{bmatrix} b_1 \cdot a_1 \\ b_2 \cdot a_2 \\ b_3 \cdot a_3 \end{bmatrix}} = \underline{B} \cdot \underline{a} = \underset{(3 \times 3)}{\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}} \cdot \underset{(3 \times 1)}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} = (3 \times 1) \Rightarrow \cancel{\underline{B} = eye(3) = \underline{I}_{(3 \times 3)}}$

$\underline{B} = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$

$\underset{(N \times 3)}{\underline{D}} = [\underline{d}_1 \ \ \underline{d}_2 \ \ \underline{d}_3]$

we want $\underset{(N \times 3)}{[b_1 \cdot \underline{d}_1 \ \ b_2 \cdot \underline{d}_2 \ \ b_3 \cdot \underline{d}_3]} \overset{!}{=} \underline{D} \cdot \underline{B} = [\underline{d}_1 \ \ \underline{d}_2 \ \ \underline{d}_3] \cdot \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$

$= \underline{\underline{[b_1 \cdot \underline{d}_1 \ \ b_2 \cdot \underline{d}_2 \ \ b_3 \cdot \underline{d}_3]}}$

iii) proof: $det(\alpha \cdot \underline{A}) = \alpha^N \cdot det(\underline{A})$; $size(\underline{A}) = (N \times N)$

e.g. $N=2$: $det\left(\alpha \cdot \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\right) = det\left(\begin{bmatrix} \alpha \cdot b_1 & \alpha \cdot b_2 \\ \alpha \cdot b_3 & \alpha \cdot b_4 \end{bmatrix}\right) = \alpha^2 \cdot b_1 \cdot b_4 - \alpha^2 \cdot b_2 \cdot b_3 =$

$= \alpha^2 \cdot (b_1 \cdot b_4 - b_2 \cdot b_3) = \underline{\underline{\overset{N}{\alpha^2} \cdot det(\underline{A})}}$ ✓

(iv) $J_{MSE} = E\{|e[n]|^2\} = E\{e[n] \cdot e^*[n]\} =$

$= E\{(d[n] - y[n]) \cdot (d^*[n] - y^*[n])\}$ // $(a-b)^* = a^* - b^*$

$= E\{|d[n]|^2 - d[n] \cdot y^*[n] - y[n] \cdot d^*[n] + |y[n]|^2\}$

$= E\{|d[n]|^2 - d[n] \cdot \left(\sum_{k=0}^{N-1} x[n-k] \cdot c_k[n]\right)^* - d^*[n] \cdot \left(\sum_{k=0}^{N-1} x[n-k] \cdot c_k[n]\right) + \left|\sum_{k=0}^{N-1} x[n-k] \cdot c_k[n]\right|^2\}$ // $|a|^2 = |a^2|$

↳ assuming: $\frac{d}{dc}(f(c))^* = \left(\frac{d}{dc} f(c)\right)^*$ // probably wrong, but I don't know what else I could do

makes this assumption obsolete

$|(1-1j)|^2 = \sqrt{2}^2$

$|(1-1j)^2| =$
$|1-2j+j^2| =$
$|-2j| = 2$

→ For Minimum: $\nabla_k J_{MSE} \stackrel{!}{=} 0$; $\nabla_k = \frac{d}{dc_k}$

$\nabla_k J_{MSE} \stackrel{!}{=} 0 = \nabla_k E\{ \text{———} \;\text{''}\; \text{———} \}$

$= E\{\nabla_k(|d[n]|^2) - \nabla_k\left(d[n] \cdot \left(\sum x[n-k] \cdot c_k[n]\right)^*\right) - \nabla_k d^*[n] \cdot \left(\sum x[n-k] \cdot c_k[n]\right) + \nabla_k \left|\sum x[n-k] \cdot c_k[n]\right|^2\}$

$= E\{0 - x^*[n-k] \cdot d[n] - d^*[n] \cdot x[n-k] + 2 \cdot (\underbrace{\sum x[n-k] \cdot c_k[n]}_{y[n]}) \cdot x[n-k]\}$ $\boxed{?}$

Assuming: $e[n] \in \mathbb{R} \rightarrow e[n] = e^*[n]$; $x[n] = x^*[n]$; $d[n] = d^*[n]$ // no clue how to solve with the complex conj. $()^*$-Operator

$= E\{-x[n-k] \cdot \underbrace{[d[n] + d[n]]}_{2d[n]} + 2 \cdot y[n] \cdot x[n-k]\}$

$= E\{-2 \cdot x[n-k] \cdot [\underbrace{d[n] - y[n]}_{e[n]}]\}$

$0 = -2 \cdot E\{e[n] \cdot x[n-k]\}$

$\underline{\underline{0 = E\{e[n] \cdot \underbrace{x[n-k]}_{\underline{x}[n]}\}}}$ // Principle of orthogonality

(v) $0 = E\{e[n] \cdot x[n-k]\} = E\{e[n] \cdot \underline{x}[n]\}$

$= E\{(d[n] - \underline{c}^T \cdot \underline{x}[n]) \cdot \underline{x}[n]\}$

$= E\{\underline{x}[n] \cdot (d[n] - \underline{c}^T \cdot \underline{x}[n])\}$ scalar: $(a)^T = a$

$= E\{\underline{x}[n] \cdot d[n] - \underline{x}[n] \cdot \underline{c}^T \cdot \underline{x}[n]\}$

$= \underbrace{E\{\underline{x}[n] \cdot d[n]\}}_{\underline{p}} - \underbrace{E\{\underline{x}[n] \cdot \underline{x}^T[n]\}}_{\underline{R}_{xx}} \cdot \underline{c}\}$

$0 = \underline{p} - \underline{R}_{xx} \cdot \underline{c}$

$\underline{R}_{xx} \cdot \underline{c}^* = \underline{p}$

$\underline{\underline{\underline{c} = \underline{R}_{xx}^{-1} \cdot \underline{p}}}$ // Wiener Hopf

(vi) $\underline{R}_{xx}$ needs to be invertible $\Rightarrow \underline{R}_{xx}$ semipositive definite or $\det(\underline{R}_{xx}) > 0$

c)

$$x[n] = a \cdot e^{j(\theta_0 \cdot n + \varphi)} + w[n] \qquad a, \theta_0 \in \mathbb{R}$$

(vii)

$\varphi \sim \mathcal{U}[-\pi, \pi]$

$w[n]$... zero-mean Gaussian noise with variance σ_w^2

$\hookrightarrow E\{w[n]\} = 0$

$w[n]$ and φ are independent $\rightarrow E\{w[n] \cdot \varphi\} = E\{w[n]\} \cdot E\{\varphi\}$

$a^* = a$

$$r_{xx}[n,k] = E\{x[n+k] \cdot x^*[n]\}$$

$$= E\{(a \cdot e^{j[\theta_0 \cdot (n+k) + \varphi]} + w[n+k]) \cdot (a \cdot e^{-j[\theta_0 \cdot (n) + \varphi]} + w^*[n])\} \quad // (a+b)^* = a^* + b^*$$

$$= E\{a^2 \cdot e^{j\theta_0 \cdot k} + a \cdot e^{j[\theta_0 \cdot (n+k) + \varphi]} \cdot w^*[n] + a \cdot e^{-j[\theta_0 \cdot n + \varphi]} \cdot w[n+k] + w[n+k] \cdot w^*[n]\} \quad // \sigma_w^2 \delta[k]$$

det

$\hookrightarrow E\{w[n+k] \cdot w^*[n]\} = r_{ww}[k]$

$$= a^2 \cdot e^{j\theta_0 k} + a \cdot e^{j\theta_0 \cdot (n+k)} \cdot E\{e^{j\varphi} \cdot w^*[n]\} + a \cdot e^{-j\theta_0 \cdot n} \cdot E\{e^{-j\varphi} w[n+k]\} + \sigma_w^2 \cdot \delta[k]$$

independent

$$= a^2 \cdot e^{j\theta_0 \cdot k} + a \cdot e^{j\theta_0 (n+k)} \cdot E\{e^{j\varphi}\} \cdot \underbrace{E\{w^*[n]\}}_{0} + a \cdot e^{-j\theta_0 n} \cdot E\{e^{j\varphi}\} \cdot \underbrace{E\{w[n+k]\}}_{0} + \sigma_w^2 \cdot \delta[k]$$

$$\underline{r_{xx}[k] = a^2 \cdot e^{j\theta_0 \cdot k} + \sigma_w^2 \cdot \delta[k]} \quad // \text{depends only on time shift } k$$

$\hookrightarrow$ Kronecker delta

vii) for WSS: first and second order moment are not dependet on time n

1) $m_x[n] = E\{x[n]\} = m_x$

2) $r_{xx}[n,k] = E\{x[n+k] \cdot x^*[n]\} = r_{xx}[k]$ ✓

1) $$m_x[n] = E\{a \cdot e^{j[\theta_0 n + \varphi]} + w[n]\}$$

$$= a \cdot e^{j\theta_0 \cdot n} \cdot E\{e^{j\varphi}\} + \underbrace{E\{w[n]\}}_{0}$$

$$= a \cdot e^{j\theta_0 \cdot n} \cdot E\{e^{j\varphi}\}$$ ~~// depends on n → NOT WSS~~

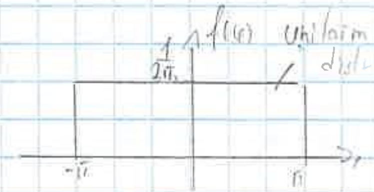
Q?

NR: $E\{g(x)\} = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$

$\hookrightarrow$ probability distribution func.

$\varphi \sim \mathcal{U}(-\pi, \pi)$

$f(\varphi) =$



$$E\{e^{j\varphi}\} = \int_{-\infty}^{\infty} e^{j\varphi} \cdot f(\varphi) \cdot d\varphi$$

$$= \int_{-\pi}^{\pi} e^{j\varphi} \cdot \frac{1}{2\pi} \cdot d\varphi = \left[\frac{1}{j} \cdot e^{j\varphi} \cdot \frac{1}{2\pi}\right]_{-\pi}^{\pi}$$

$$f(\varphi) = \begin{cases} \frac{1}{2\pi} & -\pi \le \varphi \le \pi \\ 0 & \text{else} \end{cases}$$

$$= \frac{1}{2\pi j} \cdot [e^{j\pi} - e^{-j\pi}] = \frac{1}{2\pi} \cdot [-1 - (-1)] = \underline{\underline{0}}$$

$$\Rightarrow \underline{m_x[n]} = a \cdot e^{j\theta_0 \cdot n} \cdot 0 = \underline{\underline{0}} \quad \Rightarrow \text{WSS}$$

viii) $\underline{R}_{xx} = ?$ hermitian $\hat{=}$ complex conj and transposed $x^H = (x^T)^*$

$$\underline{R}_{xx} = E\{\underline{x}[n] \cdot \underline{x}^H[n]\} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & \cdots & r_{xx}[N-1] \\ r_{xx}[-1] & r_{xx}[0] & & \vdots \\ \vdots & & \ddots & \vdots \\ r_{xx}[-N+1] & \cdots & \cdots & r_{xx}[0] \end{bmatrix}$$

$$\underline{\underline{r_{xx}[k] = a^2 \cdot e^{j\vartheta_0 \cdot k} + \sigma_w^2 \cdot \delta[k]}}$$

$\underline{N=1:}$ $\underline{R}_{xx} = r_{xx}[0] = a^2 \cdot e^0 + \sigma_w^2 = \underline{a^2 + \sigma_w^2}$ // invertible for $|a|$ OR $|\sigma_w| > 0$

$\underline{N=2:}$ $\underline{R}_{xx} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] \\ r_{xx}[-1] & r_{xx}[0] \end{bmatrix} = \begin{bmatrix} a^2 + \sigma_w^2 & a^2 \cdot e^{j\vartheta_0} \\ a^2 \cdot e^{-j\vartheta_0} & a^2 + \sigma_w^2 \end{bmatrix}$

$$\det(\underline{R}_{xx}) = (a^2 + \sigma_w^2)^2 - a^4 \cdot e^{j\vartheta_0} \cdot e^{-j\vartheta_0} = \cancel{a^4} + 2a^2\sigma_w^2 + \sigma_w^4 - \cancel{a^4} =$$
$$= 2a^2 \cdot \sigma_w^2 + \sigma_w^4 = \underline{\underline{\sigma_w^2 \cdot [2a^2 + \sigma_w^2]}}$$

$\hookrightarrow$ invertible for $|\sigma_w| > 0$

$\underline{N=3:}$ $\underline{R}_{xx} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] \\ r_{xx}[-1] & r_{xx}[0] & r_{xx}[1] \\ r_{xx}[-2] & r_{xx}[-1] & r_{xx}[0] \end{bmatrix} = \begin{bmatrix} a^2+\sigma_w^2 & a^2 e^{j\vartheta_0} & a^2 e^{2j\vartheta_0} \\ a^2 e^{-j\vartheta_0} & a^2+\sigma_w^2 & a^2 e^{j\vartheta_0} \\ a^2 e^{-2j\vartheta_0} & a^2 e^{-j\vartheta_0} & a^2+\sigma_w^2 \end{bmatrix} \begin{matrix} a^2+\sigma_w^2 & a^2 e^{j\vartheta_0} \\ a^2 e^{-j\vartheta_0} & a^2+\sigma_w^2 \\ a^2 e^{-2j\vartheta_0} & a^2 e^{-j\vartheta_0} \end{matrix}$

Sarrus:
$$\det(\underline{R}_{xx}) = (a^2 + \sigma_w^2)^3 + a^2 e^{j\vartheta_0} a^2 e^{j\vartheta_0} a^2 e^{-2j\vartheta_0} + a^2 e^{2j\vartheta_0} a^2 e^{-j\vartheta_0} a^2 e^{-j\vartheta_0}$$
$$- a^2 e^{-2j\vartheta_0} (a^2 + \sigma_w^2) \cdot a^2 \cdot e^{2j\vartheta_0} - a^2 e^{-j\vartheta} \cdot a^2 e^{j\vartheta} (a^2 + \sigma_w^2) - (a^2 + \sigma_w^2) \cdot a^2 e^{-j\vartheta} a^2 e^{j\vartheta}$$

$$= (a^2 + \sigma_w^2)^3 + a^6 \cdot 2 - [3a^4] \cdot (a^2 + \sigma_w^2)$$
$$= \cancel{a^6} + \cancel{3a^4\sigma_w^2} + 3a^2\sigma_w^4 + \sigma_w^6 + \cancel{2a^6} - [\cancel{3a^6} + \cancel{3a^4\sigma_w^2}]$$

Aufteilen Matrizen

$$\underline{\underline{= 3a^2 \cdot \sigma_w^4 + \sigma_w^6}}$$

andere Möglichkeit

The $\det(\underline{R}_{xx})$ for $N = 4:10$ was calculated in MATLAB (example_1_c.m

$\underline{N=4:}$ $\det(\underline{R}_{xx}) = 4a^2 \cdot \sigma_w^6 + \sigma_w^8$

$\underline{N=5:}$ $\det(\underline{R}_{xx}) = 5a^2 \cdot \sigma_w^8 + \sigma_w^{10}$

$\underline{\text{generalize:}}$ $\underline{\underline{\det(\underline{R}_{xx})|_N = N \cdot a^2 \cdot \sigma_w^{2\cdot(N-1)} + \sigma_w^{2N}}}$ // controlled for $N = 6:10$

$\rightarrow \det(\underline{R}_{xx}) > 0$ for $\sigma_w^2 \neq 0$ $\rightarrow$ $\underline{\text{invertible for } \sigma_w \neq 0}$

ix)

$$S_{xx}(\theta) = \sum_{k=-\infty}^{\infty} r_{xx}[k] \cdot e^{-j\theta \cdot k}$$

$$= \sum_{k=-\infty}^{\infty} \Big(a^2 e^{j\theta_0 k} + \sigma_w^2 \cdot \underbrace{\delta[k]}_{=0 \text{ für } k \neq 0} \Big) \cdot e^{-j\theta k}$$

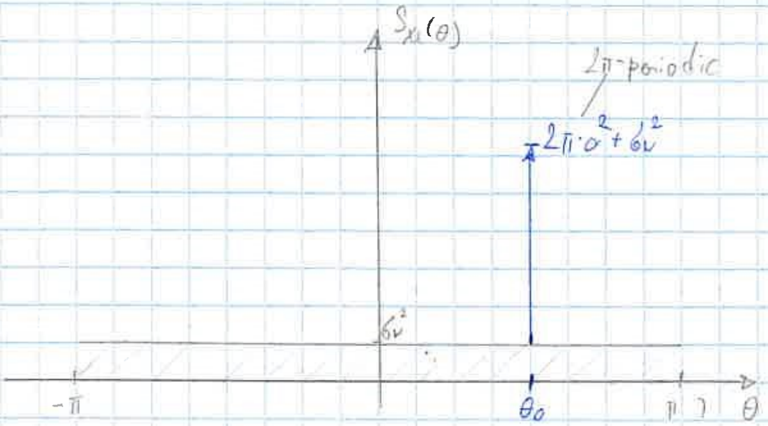
$$= \sigma_w^2 \cdot 1 \cdot e^{-j0} + \sum_{k=-\infty}^{\infty} a^2 \cdot e^{j\theta_0 \cdot k} \cdot e^{-j\theta \cdot k}$$

$$\left(= \underline{\underline{\sigma_w^2}} + \sum_{k=-\infty}^{\infty} a^2 \underline{\underline{e^{-j(\theta-\theta_0)\cdot k}}} \right) \text{ // } ~~\text{„Geometrische Reihe" does not converge}~~$$

$$= \sigma_w^2 + DTFT_k\left(a^2 \cdot e^{j\theta_0 \cdot k} \right) \text{ // „Formelsammlung"}$$

$$= \underline{\underline{\sigma_w^2 + 2\pi \cdot \delta_{\pi}(\theta - \theta_0)}} \cdot a^2$$

$\hookrightarrow$ 2π-periodic dirac delta



|| if $x[n]$ has a line spectrum with L lines, then R_{xx} is only invertible for $N \leq L$

$\rightarrow$ due to the noise floor it is not a simple line spectrum (or has infinite many lines)
therefore R_{xx} is invertible for $N \in \mathbb{N}$ as long as $\sigma_w^2 \neq 0$
if $\sigma_w = 0$, then there would be only 1 line $\rightarrow$ $N \leq 1$ ✓ // is true for calculated formulas

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