1 Analytical Problem 2.3 - LMS Algorithm

(a)

$$
\begin{aligned}
\underline{c}[n] &= \underline{c}[n-1] \ + \ \mu e[n]\underline{x}[n] \\
E[\underline{c}[n]] &= E[\underline{c}[n-1]] = E[\underline{c}_\infty] \\
e[n] &= \underline{h}^T \underline{x}[n] - \underline{c}^T \underline{x}[n] + \omega[n]
\end{aligned}
$$

$$
\begin{aligned}
\underline{c}[n] &= \underline{c}[n-1] + \mu \ (\underline{h}^T \ \underline{x}[n]) \ \underline{x}[n] + \mu \ \omega[n] \ \underline{x}[n] - \mu \ (\underline{c}[n]^T \ \underline{x}[n]) \ \underline{x}[n] \\
\underline{c}[n] &= \underline{c}[n-1] + \mu \ \underline{x}[n] \ \underline{h}^T \ \underline{x}[n] + \mu \ \omega[n] \ \underline{x}[n] - \mu \ \underline{x}[n] \ \underline{c}[n]^T \ \underline{x}[n] \\
\underline{c}[n] &= \underline{c}[n-1] + \mu \ \underline{x}[n] \ \underline{x}[n]^T \ \underline{h} + \mu \ \omega[n] \ \underline{x}[n] - \mu \ \underline{x}[n] \ \underline{x}[n]^T \ \underline{c}[n]
\end{aligned}
$$

$$
\begin{aligned}
E[\underline{c}[n]] &= E[\ \underline{c}[n-1] + \mu \ \underline{x}[n] \ \underline{x}[n]^T \ \underline{h} + \mu \ \omega[n] \ \underline{x}[n] - \mu \ \underline{x}[n] \ \underline{x}[n]^T \ \underline{c}[n] \ ] \\
\cancel{E[\underline{c}[n]]} &= \cancel{E[\ \underline{c}[n-1]]} + \mu \ E[\underline{x}[n] \ \underline{x}[n]^T \ ] \ \underline{h} \\
&\quad + \mu \ E[\ \omega[n] \ \underline{x}[n] \ ] - \mu \ E[\ \underline{x}[n] \ \underline{x}[n]^T \ ] \ \underline{c}[n] \\
\mu \ E[\ \underline{x}[n] \ \underline{x}[n]^T \ ] \ \underline{c}[n] &= \mu \ E[\underline{x}[n] \ \underline{x}[n]^T \ ] \ \underline{h} + \mu \ E[\ \omega[n] \ \underline{x}[n] \ ] \\
E[\ \underline{x}[n] \ \underline{x}[n]^T \ ] \ \underline{c}[n] &= E[\underline{x}[n] \ \underline{x}[n]^T \ ] \ \underline{h} + E[\ \omega[n] \ \underline{x}[n] \ ] \\
\underline{R}_{xx} \ E[\underline{c}[n]] &= \underline{R}_{xx} \ \underline{h} + 0 \\
E[\underline{c}[n]] &= \underline{h}
\end{aligned}
$$

$$
\begin{aligned}
\underline{c}[n] &= (1-\mu\alpha)\underline{c}[n-1] \ + \ \mu e[n]\underline{x}[n] \\
E[\underline{c}[n]] &= E[\underline{c}[n-1]] = E[\underline{c}_\infty] \\
e[n] &= \underline{h}^T \underline{x}[n] - \underline{c}^T \underline{x}[n] + \omega[n]
\end{aligned}
$$

$$
\begin{aligned}
\underline{c}[n] &= (1-\mu\alpha)\underline{c}[n-1] + \mu \ (\underline{h}^T \ \underline{x}[n]) \ \underline{x}[n] + \mu \ \omega[n] \ \underline{x}[n] - \mu \ (\underline{c}[n]^T \ \underline{x}[n]) \ \underline{x}[n] \\
\underline{c}[n] &= (1-\mu\alpha)\underline{c}[n-1] + \mu \ \underline{x}[n] \ \underline{h}^T \ \underline{x}[n] + \mu \ \omega[n] \ \underline{x}[n] - \mu \ \underline{x}[n] \ \underline{c}[n]^T \ \underline{x}[n] \\
\underline{c}[n] &= \underline{c}[n-1] \ - \ \mu\alpha \ \underline{c}[n-1] + \mu \ \underline{x}[n] \ \underline{x}[n]^T \ \underline{h} + \mu \ \omega[n] \ \underline{x}[n] \\
&\quad - \mu \ \underline{x}[n] \ \underline{x}[n]^T \ \underline{c}[n]
\end{aligned}
$$

$$
\begin{aligned}
E[\underline{c}[n]] &= E[\ \underline{c}[n-1] - \ \mu\alpha \ \underline{c}[n-1] + \mu \ \underline{x}[n] \ \underline{x}[n]^T \ \underline{h} \\
&\quad + \mu \ \omega[n] \ \underline{x}[n] - \mu \ \underline{x}[n] \ \underline{x}[n]^T \ \underline{c}[n] \ ] \\
\cancel{E[\underline{c}[n]]} &= \cancel{E[\ \underline{c}[n-1]]} \ - \mu\alpha \ E[\ \underline{c}[n-1]] + \mu \ E[\underline{x}[n] \ \underline{x}[n]^T \ ] \ \underline{h} \\
&\quad + \mu \ E[\ \omega[n] \ \underline{x}[n] \ ] - \mu \ E[\ \underline{x}[n] \ \underline{x}[n]^T \ ] \ E[\underline{c}[n]] \\
\mu \ E[\ \underline{x}[n] \ \underline{x}[n]^T \ ] \ E[\underline{c}[n]] + \mu\alpha \ E[\ \underline{c}[n-1]] &= \mu \ E[\underline{x}[n] \ \underline{x}[n]^T \ ] \ \underline{h} + \mu \ E[\ \omega[n] \ \underline{x}[n] \ ] \\
E[\ \underline{x}[n] \ \underline{x}[n]^T \ ] \ E[\underline{c}[n]] + \alpha \ E[\ \underline{c}[n-1]] &= E[\underline{x}[n] \ \underline{x}[n]^T \ ] \ \underline{h} + \ E[\ \omega[n] \ \underline{x}[n] \ ] \\
\underline{R}_{xx} \ E[\underline{c}[n]] + \alpha \underbrace{E[\underline{c}[n-1]}_{=E[\underline{c}[n]]} &= \underline{R}_{xx} \ \underline{h} + 0 \\
E[\underline{c}[n]] &= (\underline{R}_{xx} + \alpha \ \underline{I})^{-1} (\underline{R}_{xx} \ \underline{h})
\end{aligned}
$$

(b) Through the normalization of the LMS the step size parameter gets independent of the energy of the input signal.
α is a small positive constant to avoid division by zero.