2.6 Analytical Problem

$J_{MSE}(\underline{c}) = E\{|e[n]|^2\} = E\{e[n]\cdot e^*[n]\} =$ // $e[n] = y[n] - d[n]$; $(a-b)^* = a^* - b^*$

$= E\{(y[n] - d[n])\cdot(y^*[n] - d^*[n])\} =$

$(a\cdot b)^* = a^*\cdot b^*$

$= E\{y[n]\cdot y^*[n] - d[n]\cdot y^*[n] - d^*[n]\cdot y[n] + d[n]\cdot d^*[n]\}$ // $y[n] = \underline{c}^H\cdot\underline{x}[n]$; don't sub $d[n]$

$= E\{\underline{c}^H\cdot\underline{x}[n]\cdot(\underbrace{\underline{c}^T\cdot\underline{x}^*[n]}_{\text{scalar}})^T - d[n]\cdot(\underline{c}^T\cdot\underline{x}^*[n])^T - d^*[n]\cdot(\underline{c}^H\cdot\underline{x}[n])^T + d[n]\cdot d^*[n]\} =$

$= E\{\underline{c}^H\cdot\underbrace{\underline{x}[n]\cdot\underline{x}^H[n]}_{R_{xx}}\cdot\underline{c}\} - E\{\underbrace{d[n]\cdot\underline{x}^H[n]}_{\underline{p}^H}\cdot\underline{c}\} - E\{\underbrace{d^*[n]\cdot\underline{x}^T[n]}_{\underline{p}^T}\cdot\underline{c}^*\} + E\{\underbrace{d[n]\cdot d^*[n]}_{\sigma_d^2}\} =$

$$= \underline{c}^H\cdot\underline{R}_{xx}\cdot\underline{c} - \underline{p}^H\cdot\underline{c} - \underline{p}^T\cdot\underline{c}^* + \sigma_d^2$$

Minimize -> derivate ($\underline{\nabla}_{\underline{c}}$)

$$\underline{\nabla}_{\underline{c}} = \begin{bmatrix} \frac{\partial}{\partial c_1} \\ \frac{\partial}{\partial c_2} \\ \vdots \\ \frac{\partial}{\partial c_N} \end{bmatrix} \text{ with } \frac{\partial}{\partial c_k} = \frac{\partial}{\partial a_k} + j\,\frac{\partial}{\partial b_k} \text{ for complex } c_k$$

$$\underline{\nabla}_{\underline{c}} J_{MSE}(\underline{c}) = \underline{\nabla}_{\underline{c}}\left[\underline{c}^H\cdot\underline{R}_{xx}\cdot\underline{c} - \underline{p}^H\cdot\underline{c} - \underline{p}^T\cdot\underline{c}^* + \overbrace{\sigma_d^2}^{0}\right]$$

$\hookrightarrow \underline{\nabla}_{\underline{c}}[\underbrace{\underline{p}^H\cdot\underline{c}}_{\text{scalar}}]^{T\,\text{(not necessary)}} = \underline{\nabla}_{\underline{c}}[\underline{c}^T\cdot\underline{p}^*] = \underline{\nabla}_{\underline{c}}[\sum_{k=0}^{N-1} c_k\cdot p_k^*] =$

$= \underline{\nabla}_{\underline{c}}\sum_k (a_k + j\,b_k)\cdot p_k^* =$

$\hookrightarrow \underline{\nabla}_{\underline{c}} = \begin{bmatrix} \frac{\partial}{\partial a_0} + j\cdot\frac{\partial}{\partial b_0} \\ \vdots \\ \frac{\partial}{\partial a_{N-1}} + j\cdot\frac{\partial}{\partial b_{N-1}} \end{bmatrix}$

$$= \begin{bmatrix} \left(\frac{\partial}{\partial a_0} + j\cdot\frac{\partial}{\partial b_0}\right)\cdot\sum_k (a_k + j\cdot b_k)\cdot p_k^* \\ \vdots \\ \left(\frac{\partial}{\partial a_{N-1}} + j\cdot\frac{\partial}{\partial b_{N-1}}\right)\cdot\sum_k (a_k + j\cdot b_k)\cdot p_k^* \end{bmatrix} = \begin{bmatrix} \frac{\partial\sum_k (a_k + j\cdot b_k)\cdot p_k^*}{\partial a_0} + j\cdot\frac{\partial\sum_k (a_k + j\cdot b_k)\cdot p_k^*}{\partial b_0} \\ \vdots \\ \frac{\partial\sum_k (a_k + j\cdot b_k)\cdot p_k^*}{\partial a_{N-1}} + j\cdot\frac{\partial\sum_k (a_k + j\cdot b_k)\cdot p_k^*}{\partial b_{N-1}} \end{bmatrix} =$$

$$= \begin{bmatrix} p_0^* + j\cdot(j\cdot p_0^*) \\ \vdots \\ p_{N-1}^* + j\cdot(j\cdot p_{N-1}^*) \end{bmatrix} = \begin{bmatrix} p_0^* - 1\cdot p_0^* \\ \vdots \\ p_{N-1} - 1\cdot p_{N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\boxed{\underline{\nabla}_{\underline{c}}[\underline{p}^H\cdot\underline{c}] = \underline{0}}$$

$$\hookrightarrow \nabla_c [\overbrace{p^T \cdot c^*}^{\text{scalar}}]^T = \nabla_c [c^H \cdot p] = \nabla_c \Big[\sum_{k=0}^{N-1} c_k^* \cdot p_k\Big] \quad // \; c_k^* = a_k - j \cdot b_k$$

$$= \nabla_c \Big[\sum_{k=0}^{N-1} (a_k - j \cdot b_k) \cdot p_k\Big]$$

$$= \begin{bmatrix} \left(\frac{\partial}{\partial a_0} + j \cdot \frac{\partial}{\partial b_0}\right) \cdot \sum_k (a_k - j \cdot b_k) \cdot p_k \\ \vdots \\ \left(\frac{\partial}{\partial a_{N-1}} + j \cdot \frac{\partial}{\partial b_{N-1}}\right) \cdot \sum_k (a_k - j \cdot b_k) \cdot p_k \end{bmatrix} = \begin{bmatrix} p_0 + j \cdot (-j \cdot p_0) \\ \vdots \\ p_{N-1} + j \cdot (-j \cdot p_{N-1}) \end{bmatrix} =$$

$$= \begin{bmatrix} p_0 + p_0 \\ \vdots \\ p_{N-1} + p_{N-1} \end{bmatrix} = \underline{\underline{2 \cdot p}}$$

$$\boxed{\nabla_c [p^T \cdot c^*] = 2 \cdot p}$$

$$\hookrightarrow \nabla_c [c^H \cdot R_{xx} \cdot c] \qquad R_{xx}^T = R_{xx} = \begin{bmatrix} r_{00} & r_{01} & \cdots & r_{0i} \\ r_{10} & r_{11} & & \\ \vdots & & \ddots & \\ r_{j0} & & & r_{ji} \end{bmatrix}$$

j ... row index

i ... col index

$$\hookrightarrow c^H \cdot R_{xx} \cdot c = \overbrace{\sum_{j=0}^{N-1} c_j^* \cdot \underbrace{\sum_{i=0}^{N-1} r_{ji} \cdot c_i}_{\text{sum rows}}}^{\text{sum cols}} =$$

$$= \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} c_j^* \cdot r_{ji} \cdot c_i = \overbrace{\sum_{i=0}^{N-1} |c_i|^2 \cdot r_{ii}}^{\text{main diagonal}} + \overbrace{\sum_{i=0}^{N-1} \sum_{\substack{j=0 \\ j \neq i}}^{N-1} r_{ji} \cdot c_i \cdot c_j^*}^{\text{everything but main diagonal}}$$

$$\frac{\partial}{\partial c_k} [c^H \cdot R_{xx} \cdot c] = \frac{\partial}{\partial c_k} \Big(\sum_{i=0}^{N-1} |c_i|^2 \cdot r_{ii} + \sum_{i=0}^{N-1} \sum_{\substack{j=0 \\ i \neq j}}^{N-1} r_{ji} \cdot c_i \cdot c_j^*$$

$$\hookrightarrow |c_i| = (a_i + j \cdot b_i) \cdot (a_i - j b_i) =$$

$$= a_i^2 - j \cdot a_i b_i + j \cdot a_i \cdot b_i - j^2 \cdot b_i = \underline{\underline{a_i^2 + b_i^2}}$$

$$\hookrightarrow \frac{\partial}{\partial c_k} = \frac{\partial}{\partial a_k} + j \cdot \frac{\partial}{\partial b_k}$$

$$\hookrightarrow \frac{\partial}{\partial c_k} \Big(\sum_{i=0}^{N-1} |c_i|^2 \cdot r_{ii} \Big) = \frac{\partial \sum_i (a_i^2 + b_i^2) \cdot r_{ii}}{\partial a_k} + j \cdot \frac{\partial \sum_i (a_i^2 + b_i^2) \cdot r_{ii}}{\partial b_k}$$

$$\boxed{= 2 a_k \cdot r_{kk} + j \cdot 2 b_k \cdot r_{kk}}$$

$$\hookrightarrow \frac{\partial}{\partial a_k}\left(\sum_{i=0}^{N-1}\sum_{\substack{j=0\\ i\neq j}}^{N-1} r_{ji}\cdot c_i\cdot c_j^*\right) =$$

$$= \left(\frac{\partial}{\partial a_k} + j\cdot\frac{\partial}{\partial b_k}\right)\cdot\left(\sum_{i=0}^{N-1}\sum_{\substack{j=0\\ i\neq j}}^{N-1} r_{ji}\cdot(a_i + j\cdot b_i)\cdot(a_j - j\cdot b_j)\right) =$$ // first derivate over i, then j

$= \sqrt{-1}$ and not the index

$$= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} r_{jk}\cdot(1+0)\cdot(a_j - j\cdot b_j) + j\cdot r_{jk}\cdot(0 + j\cdot 1)\cdot(a_j - j\cdot b_j)$$

$$+ r_{ki}\cdot(a_i + j\cdot b_i)\cdot(1-0) + j\cdot r_{ki}\cdot(a_i + j\cdot b_i)\cdot(0 - j\cdot 1) =$$ // $r_{ki} = r_{ik}$

$$= \sum_{i=0}^{N-1} r_{ik}\cdot(a_i - \cancel{j\cdot b_i}) + r_{ik}\cdot j^2\cdot(\cancel{a_i} - j\cdot b_i) + r_{ik}\cdot(a_i + \cancel{j\cdot b_i}) - j^2\cdot(\cancel{a_i} + j\cdot b_i)\cdot r_{ik} =$$

$$= \sum_{i=0}^{N-1} r_{ik}\cdot a_i + r_{ik}\cdot j\cdot b_i + r_{ik}\cdot a_i + r_{ik}\cdot j\cdot b_i$$

$$= \sum_{i=0}^{N-1} \underline{2\cdot r_{ik}\cdot a_i} + 2j\cdot\underline{r_{ik}\cdot b_i}$$

$$= \sum_{i=0}^{N-1} 2\cdot r_{ik}\cdot(a_i + j\cdot b_i) = \sum_{i=0}^{N-1} 2\cdot r_{jk}\cdot c_i$$

$$= \sum_{i=0}^{N-1} 2\cdot r_{ik}\cdot c_i = \left[2\cdot\underline{R}_{xx}\cdot\underline{c}\right]_{i\text{-th row}}$$

accidentally switched row and col, should hold true nonetheless

$$\hookrightarrow \boxed{\underline{\nabla}_{\underline{c}}\left[\underline{c}^H\cdot\underline{R}_{xx}\cdot\underline{c}\right] = 2\cdot\underline{R}_{xx}\cdot\underline{c}}$$

$$\underline{\nabla}_{\underline{c}}\, J_{MSE}(\underline{c}) = 2\cdot\underline{R}_{xx}\cdot\underline{c} - 0 - 2\cdot\underline{p} + 0$$

$$= \cancel{2}\cdot\underline{R}_{xx}\cdot\underline{c} - \cancel{2}\underline{p} \overset{!}{=} 0$$ // to find minimum

$$\Rightarrow \boxed{\underline{c}_{MSE} = \underline{R}_{xx}^{-1}\cdot\underline{p}}$$ // Wiener Hopf solution for complex numbers

b) J_{min}

$$J_{min} = J_{MSE}(c_{MSE}) = c_{MSE}^H \cdot R_{xx} \cdot c_{MSE} - p^H c_{MSE} - p^T \cdot c_{MSE}^* + \sigma_d^2$$

$$= (R_{xx}^{-1} \cdot p)^H \cdot \underbrace{(R_{xx} \cdot R_{xx}^{-1})}_{I} p - p^H \cdot R_{xx}^{-1} \cdot p - \overbrace{\left(p^T \cdot (R_{xx}^{-1} \cdot p)^*\right)^T}^{\text{scalar}} + \sigma_d^2$$

$$= \cancel{(R_{xx}^{-1} \cdot p)^H} p - p^H \cdot R_{xx}^{-1} \cdot p - \cancel{(R_{xx}^{-1} \cdot p)^H} p + \sigma_d^2$$

$$= -p^H \cdot R_{xx}^{-1} \cdot p + E\{|d[n]|^2\} = \quad // \; d[n] = h^H \cdot x[n] + w[n]$$

$$= -p^H \cdot R_{xx}^{-1} \cdot p + E\{(h^H \cdot x[n] + w[n]) \cdot (h^H \cdot x[n] + w[n])^*\}$$

$$= -p^H \cdot R_{xx}^{-1} \cdot p + E\{h^H \cdot \overbrace{x[n] \cdot x[n]^H}^{E\{\} = R_{xx}} \cdot h + h^H \cdot \overbrace{x[n] \cdot w[n]^*}^{E\{\}=0} + h^T \cdot \overbrace{x^*[n] \cdot w[n]}^{E\{\}=0}$$

$$+ \overbrace{w[n] \cdot w[n]^*}^{E\{\} = \sigma_v^2}\} =$$

$$= -p^H \cdot R_{xx}^{-1} \cdot p + h^H \cdot R_{xx} \cdot h + \sigma_v^2 \quad // \; h = c_{MSE} = R_{xx}^{-1} \cdot p$$

$$= -p^H \cdot R_{xx}^{-1} \cdot p + (R_{xx}^{-1} \cdot p)^H \underbrace{R_{xx} \cdot R_{xx}^{-1}}_{I} p + \sigma_v^2$$

$$\hookrightarrow (R_{xx}^{-1} \cdot p)^H = p^H \cdot (R_{xx}^{-1})^H \overset{R_{xx}^H = R_{xx}}{=} p^H \cdot R_{xx}^{-1}$$

$$= \cancel{-p^H \cdot R_{xx}^{-1}} \cdot p + \cancel{p^H \cdot R_{xx}^{-1}} \cdot p + \sigma_v^2$$

$$\boxed{J_{min} = \sigma_v^2}$$