1 Analytical Problem 2.1 - Deterministic - Gradient Algorithm

(a)

$$J(\underline{c}[n]) = E[e[n]^2] + \alpha\ ||\underline{c}[n]||^2$$
$$J(\underline{c}[n]) = E[|d[n]|^2] - 2\ (\underline{p}^H\ \underline{c})^T + \underline{c}^H\ \underline{R}_{xx}\ \underline{c} + alpha\ ||\underline{c}[n]||^2$$

$$\underline{\nabla} J_{MSE}(\underline{c}[n]) = 0 - 2\ \underbrace{\underline{p}^*}_{\in\mathbb{R}} + 2\ \underline{R}_{xx}\ \underline{c}[n] + \alpha\ 2\ \underline{c}[n]$$
$$\underline{\nabla} J_{MSE}(\underline{c}[n]) = 2\ (\underline{R}_{xx}\ \underline{c}[n] - \underline{p}) + 2\ \alpha\ \underline{c}[n]$$

Gradient search: $-\underline{\nabla} J_{MSE}(\underline{c})|_{c=c[n-1]}$

$$-\underline{\nabla} J_{MSE}(\underline{c})|_{c=c[n-1]} = 2\ (\underline{R}_{xx}\ \underline{c}[n-1] - \underline{p}) + 2\ \alpha\ \underline{c}[n-1]$$
$$\text{Update rule: } \underline{c}[n] = \underline{c}[n-1] + \underbrace{\widetilde{\mu}}_{\mu=\widetilde{\mu}\ 2}\ ((\underline{p} - \underline{R}_{xx}\ \underline{c}[n-1]) - \alpha\ c[n-1])\ 2$$
$$\underline{c}[n] = \underline{c}[n-1] - \mu\ \alpha\ c[n-1]\ + \mu(\underline{p} - \underline{R}_{xx}\ \underline{c}[n-1])$$
$$\underline{c}[n] = \underline{c}[n-1]\ (1 - \mu\ \alpha) + \mu(\underline{p} - \underline{R}_{xx}\ \underline{c}[n-1])$$

(b) Where does it converge to?

$$n \to \infty\ \Rightarrow\ c_\infty = ?$$

$$\underline{c}[n] = \underline{c}[n-1]\ (1 - \mu\ \alpha) + \mu(\underline{p} - \underline{R}_{xx}\ \underline{c}[n-1])$$
$$\underline{c}_\infty = \underline{c}_\infty\ (1 - \mu\ \alpha) + \mu(\underline{p} - \underline{R}_{xx}\ \underline{c}_\infty)$$
$$\underline{c}_\infty\ (1 - 1 + \mu\ \alpha) = \mu\ (\underline{p} - \underline{R}_{xx}\ \underline{c}_\infty)$$
$$\underline{c}_\infty\ \cancel{\mu}\ \alpha = \cancel{\mu}\ (\underline{p} - \underline{R}_{xx}\ \underline{c}_\infty)$$
$$(\underline{R}_{xx} + \underline{I}\ \alpha)\ \underline{c}_\infty = \underline{p}$$
$$\underline{c}_\infty = (\underline{R}_{xx} + \underline{I}\ \alpha)^{-1}\ \underline{p}$$

(c) Misalignment vector $\underline{v}[n] = \underline{c}[n] - \underline{c}_\infty$

$$\underline{c}[n] = \underline{c}[n-1]\ (1 - \mu\ \alpha) + \mu(\underline{p} - \underline{R}_{xx}\ \underline{c}[n-1])$$
$$\underline{c}[n] - \underline{c}_\infty = \underline{c}[n-1]\ (1 - \mu\ \alpha) - \underline{c}_\infty + \mu(\underbrace{\underline{p}}_{=(\underline{R}_{xx}+\alpha)\ \underline{c}_\infty} - \underline{R}_{xx}\ \underline{c}[n-1]) - (1 - \mu\ \alpha)\ \underline{c}_\infty + (1 - \mu\ \alpha)\ \underline{c}_\infty$$
$$\underline{v}[n] = (1 - \mu\ \alpha)\ \underline{v}[n-1] - \cancel{\underline{c}_\infty} + \cancel{\underline{c}_\infty} - \mu\ \alpha\ \underline{c}_\infty + \mu(\underline{R}_{xx}\ \underline{c}_\infty + \alpha\underline{c}_\infty - \underline{R}_{xx}\ \underline{c}[n-1])$$
$$\underline{v}[n] = (1 - \mu\ \alpha)\ \underline{v}[n-1] - \cancel{\mu\ \alpha\ \underline{c}_\infty} + \cancel{\mu\ \alpha\ \underline{c}_\infty} + \mu\ \underline{R}_{xx}\ (-\underline{v}[n-1])$$
$$\underline{v}[n] = (1 - \mu\ \alpha)\ \underline{v}[n-1] + \mu\ \underline{R}_{xx}\ (-\underline{v}[n-1])$$
$$\underline{v}[n] = ((1 - \mu\ \alpha)\ \underline{I} - \mu\ \underline{R}_{xx})\ \underline{v}[n-1]$$

(d) decoupled

$$\underline{v}[n] = ((1-\mu\ \alpha)\ \underline{I} - \mu\ \underline{R}_{xx})\ \underline{v}[n-1]$$

Eigendecomposition: $\underline{R}_{xx} = \underline{Q}\ \underline{\Lambda}\ \underline{Q}^H$

$\underline{Q}$... unitary matrix; $\underline{Q}^{-1} = \underline{Q}^H$

$\underline{\Lambda}$... diagonal matrix consisting of eigenvalues; $\underline{\Lambda} = diag(\lambda_1, \lambda_2, ..., \lambda_{n-1})$

$||\underline{Q}^H\ \underline{a}||_2^2 = \underline{a}^H\ \underline{Q}\ \underline{Q}^H\ \underline{a} = \underline{a}^H\ \underline{I}\ \underline{a} = ||\underline{a}||_2^2$

Q matrix does not scale vector, it only rotates it

$$\begin{aligned}
\underline{v}[n] &= ((1-\mu\ \alpha)\ \underline{I} - \mu\ \underline{Q}\ \underline{\Lambda}\ \underline{Q}^H)\ \underline{v}[n-1] \\
\underbrace{\underline{Q}^H\ \underline{v}[n]}_{=\underline{\widetilde{v}}[n]} &= (1-\mu\ \alpha)\ \underline{Q}^H\ \underline{v}[n-1] - \underline{Q}^H\ \mu\ \underline{Q}\ \underline{\Lambda}\ \underline{Q}^H\ \underline{v}[n-1] \\
\underline{\widetilde{v}}[n] &= (1-\mu\ \alpha)\ \underline{\widetilde{v}}[n-1] - \mu \underbrace{\underline{Q}^H\ \underline{Q}}_{=\underline{I}}\ \underline{\Lambda}\ \underline{\widetilde{v}}[n-1] \\
\underline{\widetilde{v}}[n] &= (1-\mu\ \alpha)\ \underline{\widetilde{v}}[n-1] - \mu\ \underline{\Lambda}\ \underline{\widetilde{v}}[n-1]
\end{aligned}$$

(e)

$$\begin{aligned}
\underline{\widetilde{v}}[n] &= ((1-\mu\ \alpha)\ - \mu\ \underline{\Lambda})\ \underline{\widetilde{v}}[n-1] \\
\underline{\widetilde{v}}[n] &= ((1-\mu\ \alpha)\ - \mu\ \underline{\Lambda})\ ((1-\mu\ \alpha)\ - \mu\ \underline{\Lambda})\ \underline{\widetilde{v}}[n-2] \\
\underline{\widetilde{v}}[n] &= ((1-\mu\ \alpha)\ - \mu\ \underline{\Lambda})^n\ \underline{\widetilde{v}}[0]
\end{aligned}$$

(f)

$$\begin{aligned}
\underline{\widetilde{v}_i}[n] &= ((1-\mu\ \alpha)\ - \mu\ \underline{\lambda}_i)^n\ \underline{\widetilde{v}_i}[0] \\
\underbrace{|\underline{\widetilde{v}_i}|[n]}_{\underline{\widetilde{v}_i}[n]=0\ for\ n\to\infty} &= |\underbrace{(1-\mu\ \alpha)\ - \mu\ \underline{\lambda}_i|^n}_{exponential\ decoy:\ |e^{-\frac{n}{t_i}}|}\ |\underline{\widetilde{v}_i}[0]| \\
|(1-\mu\ \alpha)\ - \mu\ \underline{\lambda}_i|^n &= |e^{-\frac{n}{t_i}}| \\
e^{-\frac{n}{t_i}} &= |(1-\mu\ \alpha)\ - \mu\ \underline{\lambda}_i|^n \\
-\frac{\cancel{n}}{t_i}\ ln(e) &= \cancel{n}\ ln(|(1-\mu\ \alpha)\ - \mu\ \underline{\lambda}_i|) \\
t_i &= -\frac{1}{ln(|(1-\mu\ \alpha)\ - \mu\ \underline{\lambda}_i|)}
\end{aligned}$$

convergence time can be changed with changing the step size (or eigenvalue)