Analytical Problem 2.1

↳ problem class problem 1.14

from problem class sheet page 8; 1.7

a) $J(\underline{c}[n]) = E\{e^2[n]\} + \alpha \cdot \| \underline{c}[n] \|^2$

$\underline{c}^T \cdot \underline{c}$

$$= E\{ |d[n]|^2 \} - 2(\underline{p}^H \cdot \underline{c})^T + \underline{c}^H \cdot \underline{R}_{xx} \cdot \underline{c} + \alpha \cdot \| \underline{c}[n] \|^2$$

$$\underline{\nabla}_c J_{MSE}(\underline{c}[n]) = 0 - 2 \cdot \underline{p}^* + 2 \cdot \underline{R}_{xx} \cdot \underline{c}[n] + \alpha \cdot 2 \cdot \underline{c}[n]$$ // assumption: $\underline{c} \in \mathbb{R}$, $\underline{p} \in \mathbb{R}$

$$= 2 \cdot (\underline{R}_{xx} \cdot \underline{c}[n] - \underline{p}) + 2 \cdot \alpha \cdot \underline{c}[n]$$

Gradient search: $-\underline{\nabla}_c J_{MSE}(\underline{c}) \Big|_{c = c[n-1]}$

$$-\underline{\nabla}_c J_{MSE}(\underline{c}) \Big|_{c = c[n-1]} = \underbrace{-2 \cdot (\underline{R}_{xx} \cdot \underline{c}[n-1] - \underline{p}) - 2 \cdot \alpha \cdot \underline{c}[n-1]}_{} = \Omega$$

Update rule: $\underline{c}[n] = \underline{c}[n-1] + \tilde{\mu} \cdot ((\underline{p} - \underline{R}_{xx} \cdot \underline{c}[n-1]) - \alpha \cdot \underline{c}[n-1]) \cdot 2$ // $\mu = \tilde{\mu} \cdot 2$

$$\underline{c}[n] = \underline{c}[n-1] - \mu \cdot \alpha \cdot c[n-1] + \mu \cdot (\underline{p} - \underline{R}_{xx} \cdot \underline{c}[n-1])$$

$$\boxed{\underline{c}[n] = \underline{c}[n-1] \cdot (1 - \mu \cdot \alpha) + \mu \cdot (\underline{p} - \underline{R}_{xx} \cdot \underline{c}[n-1])}$$

b) Where does it converge to?

$n \to \infty \Rightarrow c_\infty = ?$

$$\underline{c}[n] = \underline{c}[n-1] \cdot (1 - \mu \cdot \alpha) + \mu \cdot (\underline{p} - \underline{R}_{xx} \cdot \underline{c}[n-1])$$ // $n \to \infty$; $n-1 \to \infty$

$$\underline{c}_\infty = \underline{c}_\infty \cdot (1 - \mu \cdot \alpha) + \mu \cdot (\underline{p} - \underline{R}_{xx} \cdot \underline{c}_\infty)$$

$$\underline{c}_\infty \cdot (1 - 1 + \mu \cdot \alpha) = \mu \cdot (\underline{p} - \underline{R}_{xx} \cdot \underline{c}_\infty)$$

$$\underline{c}_\infty \cdot \cancel{\mu} \cdot \alpha = \cancel{\mu} \cdot (\underline{p} - \underline{R}_{xx} \cdot \underline{c}_\infty)$$

$$\underline{I} \cdot \underline{c}_\infty \cdot \alpha = \underline{p} - \underline{R}_{xx} \cdot \underline{c}_\infty$$

$$(\underline{R}_{xx} + \underline{I}\alpha) \cdot \underline{c}_\infty = p$$

$$\boxed{\underline{c}_\infty = (\underline{R}_{xx} + \underline{I}\alpha)^{-1} \cdot p}$$ // not quiet the Wiener-Hopf solution

c) misalignment $\underline{v}[n]$

$$\underline{v}[n] = \underline{c}[n] - \underline{c}_\infty$$

$$p = (\underline{R}_{xx} + a) \cdot \underline{c}_\infty$$

$$\underline{c}[n] = (1-\mu a) \cdot \underline{c}[n-1] + \mu (\underline{p} - \underline{R}_{xx} \cdot \underline{c}[n-1]) \quad | -\underline{c}_\infty$$

$$\underline{c}[n] - \underline{c}_\infty = (1-\mu \cdot a) \cdot \underline{c}[n-1] - \underline{c}_\infty - (1-\mu \cdot a) \cdot \underline{c}_\infty + (1-\mu \cdot a) \cdot \underline{c}_\infty + \mu (\underline{p} - \underline{R}_{xx} \cdot \underline{c}[n-1]) \quad | \underline{p} = \underline{R}_x$$

$$\underline{v}[n] = (1-\mu \cdot a) \cdot \underline{v}[n-1] - \cancel{\underline{c}_\infty} + \cancel{c_\infty} - \mu \cdot a \cdot c_\infty + \mu \cdot (\underline{R}_{xx} \cdot \underline{c}_\infty + a \cdot \underline{c}_\infty - \underline{R}_{xx} \cdot \underline{c}[n-1])$$

$$v[n] = (1-\mu a) \cdot v[n-1] - \mu \cancel{a} \underline{c}_\infty + \mu \cancel{a} \, \underline{c}_\infty + \mu \cdot \underline{R}_{xx} \cdot (-v[n-1])$$

$$\underline{v[n] = (1-\mu \cdot a) \cdot \underline{v}[n-1] - \mu \cdot \underline{R}_{xx} \, \underline{v}[n-1]}$$

$$\boxed{v[n] = \left((1-\mu \cdot a) \cdot \underline{\underline{I}} - \mu \cdot \underline{R}_{xx}\right) \cdot \underline{v}[n-1]} \quad // \, \underline{v}[n] = \left[(1-\mu \cdot a) \cdot I - \mu \cdot \underline{R}_{xx}\right]^n \cdot \underline{v}[0]$$

d) decoupling

$$\underline{v}[n] = \left[(1-\mu a) \cdot \underline{\underline{I}} - \mu \cdot \underline{R}_{xx}\right] \cdot \underline{v}[n-1] =$$

Eigendecomposition: $\underline{R}_{xx} = \underline{Q} \cdot \underline{\Lambda} \cdot \underline{Q}^H$

$\underline{Q}$ ... unitary matrix ; $\underline{Q}^{-1} = \underline{Q}^H$ // adjungierte matrix, NICHT adjunkte"?

$\underline{\Lambda}$ ... diagonal matrix consisting of eigenvalues ; $\underline{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_{m-1})$

$$\|\underline{Q}^H \cdot \underline{a}\|_2^2 = \underline{a}^H \cdot \underline{Q} \cdot \overset{\underline{Q}^{-1}}{\underline{Q}^H} \cdot \underline{a} = \underline{a}^H \cdot \underline{I} \cdot \underline{a} = \underline{\underline{\|\underline{a}\|_2^2}}$$ // Q matrix does not scale vector, it only rotates it

$$v[n] = \left[(1-\mu \cdot a) \cdot \underline{I} - \mu \cdot \underline{Q} \cdot \underline{\Lambda} \, \underline{Q}^H\right] \cdot \underline{v}[n-1] \quad | \, \underline{Q}^H \cdot \text{ from left}$$

$$\underbrace{\underline{Q}^H \underline{v}[n]}_{\tilde{\underline{v}}[n]} = (1-\mu \cdot a) \cdot \underline{Q}^H \cdot \underline{v}[n-1] - \underline{Q}^H \mu \cdot \underline{Q} \cdot \underline{\Lambda} \cdot \underline{Q}^H \cdot \underline{v}[n-1]$$

$$\tilde{\underline{v}}[n] = (1-\mu \cdot a) \cdot \tilde{\underline{v}}[n-1] - \mu \cdot \overbrace{\underline{Q}^H \underline{Q}}^{I} \cdot \underbrace{\underline{\Lambda}}_{\text{diagonal matrix}} \cdot \tilde{\underline{v}}[n-1]$$

$$\boxed{\tilde{\underline{v}}[n] = (1-\mu \cdot a) \cdot \tilde{\underline{v}}[n-1] - \mu \cdot \underline{\Lambda} \, \tilde{\underline{v}}[n-1] \quad // \text{decoupled}} \quad \tilde{v}_1[n] = (1-\mu \cdot a) \cdot \tilde{v}_1[n-1] - \mu \cdot \lambda_1 \cdot \tilde{v}_1[n-1]$$

e) $\underline{\tilde{v}}[n] = \left[(1-\mu\cdot a)\cdot\underline{\underline{I}} - \mu\cdot\underline{\underline{\Lambda}}\right]\cdot\underline{\tilde{v}}[n-1] = \cancel{[(1-\mu}$

$= \left[(1-\mu\cdot a)\,\underline{\underline{I}} - \mu\cdot\underline{\underline{\Lambda}}\right]\cdot\left[(1-\mu\cdot a)\cdot\underline{\underline{I}} - \mu\cdot\underline{\underline{\Lambda}}\right]\cdot\underline{\tilde{v}}[n-2]$

$$\boxed{\underline{\tilde{v}}[n] = \left[(1-\mu\cdot a)\cdot\underline{\underline{I}} - \mu\cdot\underline{\underline{\Lambda}}\right]^n\cdot\underline{\tilde{v}}[0]}$$

f) assumption: a influences convergence time // What is meant by: "find an expression of the time constants τ_i"

$\tilde{v}_i[n] = \left[(1-\mu\cdot a) - \mu\cdot\lambda_i\right]^n\cdot\tilde{v}_i[0]$

$|\tilde{v}_i[n]| = \underbrace{|(1-\mu\cdot a) - \mu\cdot\lambda_i|}_{\text{exponential decay: } |e^{-\frac{n}{\tau_i}}|}{}^n\cdot|\tilde{v}_i[0]|$ // $\tilde{v}_i[n] \overset{!}{=} 0$ for $n \to \infty$

$\Rightarrow |(1-\mu\cdot a) - \mu\cdot\lambda_i|^n = |e^{-\frac{n}{\tau_i}}|$ // $e > 0 \ \forall n$

$e^{-\frac{n}{\tau_i}} = |(1-\mu\cdot a) - \mu\cdot\lambda_i|^n$ // ln

$-\frac{n}{\tau_i}\cdot\ln(e) = \cancel{n}\cdot\ln|(1-\mu\cdot a) - \mu\cdot\lambda_i|$

$$\underline{\underline{\tau_i = -\frac{1}{\ln|(1-\mu\cdot a) - \mu\cdot\lambda_i|}}}$$

if λ_i are different $\Rightarrow \tau_i$ is different for each component

// convergence time can be ~~with~~ changed w/o changing the step size (or eigenvalues)