1 Problem 2.6 - Bonus Wiener-Hopf Solution

(a)

$$\begin{aligned}
J_{MSE(\underline{c})} &= E[|e[n]|^2] = E[e[n]\ e[n]^*] \\
J_{MSE(\underline{c})} &= E[(y[n]-d[n])\ (y[n]^*-d[n]^*)] \\
J_{MSE(\underline{c})} &= E[y[n]\ y[n]^* - d[n]\ y[n]^* - d[n]^*\ y[n] + d[n]\ d[n]^*] \\
J_{MSE(\underline{c})} &= E[\underline{c}^H\ \underline{x}[n]\ (\underline{c}^T\ \underline{x}[n]^*)^T - d[n]\ (\underline{c}^T\ \underline{x}[n]^*)^T - d[n]^*\ (\underline{c}^H\ \underline{x}[n])^T + d[n]\ d[n]^*\ ] \\
J_{MSE(\underline{c})} &= E[\underline{c}^H\ \underbrace{\underline{x}[n]\ \underline{x}[n]^H}_{R_{xx}}\ \underline{c}] - E[\underbrace{d[n]\ \underline{x}[n]^H}_{p^H}\ \underline{c}] - E[\underbrace{d[n]^*\ \underline{x}[n]^T}_{p^T}\ \underline{c}^*] + E[\underbrace{d[n]\ d[n]^*}_{\sigma_d^2}] \\
J_{MSE(\underline{c})} &= \underline{c}^H\ \underline{R}_{xx}\underline{c} - \underline{p}^H\ \underline{c} - \underline{p}^T\ \underline{c}^* + \sigma_d^2
\end{aligned}$$

Minimize $\rightarrow$ $derivate(\underline{\bigtriangledown}_c)$

$$\underline{\bigtriangledown}_c = \begin{pmatrix} \frac{\partial}{\partial\ c_1} \\ \vdots \\ \frac{\partial}{\partial\ c_k} \\ \vdots \\ \frac{\partial}{\partial\ c_N} \end{pmatrix}\ with\ \frac{\partial}{\partial\ c_k} = \frac{\partial}{\partial\ a_k} + j\ \frac{\partial}{\partial\ b_k} \text{ for complex } c_k$$

$$\begin{aligned}
\underline{\bigtriangledown}_c\ J_{MSE(\underline{c})} &= \underline{\bigtriangledown}_c[\underline{c}^H\ \underline{R}_{xx}\underline{c} - \underline{p}^H\ \underline{c} - \underline{p}^T\ \underline{c}^* + \underbrace{\sigma_d^2}_{=0}\ ] \\
\underline{\bigtriangledown}_c[\underline{p}^H\ \underline{c}]^T &= \underline{\bigtriangledown}_c[\underline{c}^T\ \underline{p}^*] = \underline{\bigtriangledown}_c[\sum_{k=0}^{N-1} c_k\ p_k^*] \\
\underline{\bigtriangledown}_c[\underline{p}^H\ \underline{c}]^T &= \underline{\bigtriangledown}_c \sum_k (a_k + j\ b_k)\ p_k* \\
\underline{\bigtriangledown}_c[\underline{p}^H\ \underline{c}]^T &= \begin{pmatrix} \frac{\partial\ \sum_k (a_k+j\ b_k)\ p_k*}{\partial\ a_0} + j\ \frac{\partial\ \sum_k (a_k+j\ b_k)\ p_k*}{\partial\ b_0} \\ \vdots \\ \vdots \\ \frac{\partial\ \sum_k (a_k+j\ b_k)\ p_k*}{\partial\ a_{N-1}} + j\ \frac{\partial\ \sum_k (a_k+j\ b_k)\ p_k*}{\partial\ b_{N-1}} \end{pmatrix} \\
\underline{\bigtriangledown}_c[\underline{p}^H\ \underline{c}]^T &= \begin{pmatrix} p_0^* + j\ (j\ p_0^*) \\ \vdots \\ \vdots \\ p_{N-1}^* + j\ (j\ p_{N-1}^*) \end{pmatrix} \\
\underline{\bigtriangledown}_c[\underline{p}^H\ \underline{c}]^T &= \begin{pmatrix} p_0^* -\ p_0^* \\ \vdots \\ \vdots \\ p_{N-1}^* -\ p_{N-1}^* \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \underline{0}
\end{aligned}$$

$$\underline{\nabla}_{\underline{c}}[\underline{p}^T\ \underline{c}*]^T = \underline{\nabla}_{\underline{c}}[\underline{c}^H\ \underline{p}] = \underline{\nabla}_{\underline{c}}[\sum_{k=0}^{N-1} c_k^*\ p_k]$$

$$\underline{\nabla}_{\underline{c}}[\underline{p}^T\ \underline{c}*]^T = \underline{\nabla}_{\underline{c}}[\sum_k (a_k - j\ b_k)\ p_k]$$

$$\underline{\nabla}_{\underline{c}}[\underline{p}^T\ \underline{c}*]^T = \begin{pmatrix} \frac{\partial\ \sum_k (a_k - j\ b_k)\ p_k}{\partial\ a_0} + j\ \frac{\partial\ \sum_k (a_k - j\ b_k)\ p_k}{\partial\ b_0} \\ \vdots \\ \vdots \\ \frac{\partial\ \sum_k (a_k - j\ b_k)\ p_k}{\partial\ a_{N-1}} + j\ \frac{\partial\ \sum_k (a_k - j\ b_k)\ p_k}{\partial\ b_{N-1}} \end{pmatrix}$$

$$\underline{\nabla}_{\underline{c}}[\underline{p}^T\ \underline{c}*]^T = \begin{pmatrix} p_0 - j\ (j\ p_0) \\ \vdots \\ \vdots \\ p_{N-1} - j\ (j\ p_{N-1}) \end{pmatrix}$$

$$\underline{\nabla}_{\underline{c}}[\underline{p}^T\ \underline{c}*]^T = \begin{pmatrix} p_0 + \ p_0 \\ \vdots \\ \vdots \\ p_{N-1} + \ p_{N-1} \end{pmatrix} = 2\ \underline{p}$$

$$\underline{\nabla}_{\underline{c}}[\underline{c}^H\ \underline{R}_{xx}\ \underline{c}]$$

$$\underline{R}_{xx}^T = \underline{R}_{xx} = \begin{pmatrix} r_{00} & r_{01} & \dots & r_{0i} \\ r_{10} & r_{11} & & \\ \vdots & & \ddots & \\ r_{j0} & & & r_{ji} \end{pmatrix}$$

$$\begin{aligned}
\underline{c}^H\ \underline{R}_{xx}\ \underline{c} &= \sum_{j=0}^{N-1} c_j^* \sum_{i=0}^{N-1} r_{ji}\ c_i \\
&= \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} c_j^*\ r_{ji}\ c_i \\
&= \sum_{i=0}^{N-1} |c_i|^2\ r_{ii} + \sum_{j=0}^{N-1} \sum_{i=0;i\neq j}^{N-1} c_j^*\ r_{ji}\ c_i
\end{aligned}$$

$$\frac{\partial}{\partial\ c_k}\ [\underline{c}^H\ \underline{R}_{xx}\ \underline{c}] = \frac{\partial}{\partial\ c_k}\ (\sum_{i=0}^{N-1} |c_i|^2\ r_{ii} + \sum_{j=0}^{N-1} \sum_{i=0;i\neq j}^{N-1} c_j^*\ r_{ji}\ c_i)$$

$$\begin{aligned}
|c_i| &= (a_i + j\ b_i)\ (a_i - j\ b_i) = a_i^2 + b_i^2 \\
\frac{\partial}{\partial\ c_k} &= \frac{\partial}{\partial\ a_k} + j\ \frac{\partial}{\partial\ b_k} \text{ for complex } c_k \\
\frac{\partial}{\partial\ c_k}(\sum_{i=0}^{N-1} |c_i|^2\ r_{ii}) &= \frac{\partial \sum_i (a_i^2 + b_i^2)\ r_{ii}}{\partial\ a_k} + j\ \frac{\partial \sum_i (a_i^2 + b_i^2)\ r_{ii}}{\partial\ b_k} \\
&= 2\ a_k\ r_{kk} + j\ 2\ b_k\ r_{kk}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\ c_k}(\sum_{j=0}^{N-1} \sum_{i=0;i\neq j}^{N-1} c_j^*\ r_{ji}\ c_i) &= (\frac{\partial}{\partial\ a_k} + j\ \frac{\partial}{\partial\ b_k})(\sum_{j=0}^{N-1} \sum_{i=0;i\neq j}^{N-1} r_{ji}\ (a_i + j\ b_i)\ (a_j - j\ b_j)) \\
&= \sum_{j=0}^{N-1} \sum_{i=0;i\neq j}^{N-1} r_{jk}\ (1+0)\ (a_j - j\ b_j) + j\ r_{jk}\ (0+j)(a_j - j\ b_j) \\
&\qquad + r_{ki}\ (1-0)\ (a_i + j\ b_i) + j\ r_{ki}\ (0-j)(a_i + j\ b_i) \\
&= \sum_{i=0}^{N-1} r_{ik}\ (a_i - \cancel{j\ b_i}) + r_{ik}\ j^2\ (\cancel{a_i} - j\ b_i) + r_{ik}\ (a_i + \cancel{j\ b_i}) - j^2\ (\cancel{a_i} + j\ b_i) r_{ik} \\
&= \sum_{i=0}^{N-1} r_{ik}\ a_i + r_{ik}\ j\ b_i + r_{ik}\ a_i + r_{ik}\ j\ b_i \\
&= \sum_{i=0}^{N-1} 2\ r_{ik}\ a_i + 2\ j\ r_{ik}\ b_i \\
&= \sum_{i=0}^{N-1} 2\ r_{ik}(a_i + j\ b_i) = \sum_{i=0}^{N-1} 2\ r_{ik} c_i = 2\ \underline{R}_{xx}\ \underline{c}
\end{aligned}$$

$$\underline{\nabla}_{\underline{c}}[\underline{c}^H\ \underline{R}_{xx}\ \underline{c}] = 2\ \underline{R}_{xx}\ \underline{c}$$

$$\begin{aligned}
\underline{\nabla}_{\underline{c}} J_{MSE(\underline{c})} &= 2\ \underline{R}_{xx}\ \underline{c} - 0 - 2\ \underline{p} + 0 \\
&= \cancel{2}\ \underline{R}_{xx}\ \underline{c} - \cancel{2}\underline{p} \overset{!}{=} 0 \\
\Rightarrow\ \underline{c}_{MSE} &= \underline{R}_{xx}^{-1}\ \underline{p}
\end{aligned}$$

(b) J_{min}

$$
\begin{aligned}
J_{min} = J_{MSE}(\underline{c}_{MSE}) &= \underline{c}_{MSE}^H \, \underline{R}_{xx} \, \underline{c}_{MSE} - \underline{p}^H \, \underline{c}_{MSE} - \underline{p}^T \, \underline{c}_{MSE}^* + \sigma_d^2 \\
&= (\underline{R}_{xx}^{-1} \, \underline{p})^H (\underbrace{\underline{R}_{xx}\underline{R}_{xx}^{-1})}_{\underline{I}}) \, \underline{p} - \underline{p}^H \, \underline{R}_{xx}^{-1} \, \underline{p} - \underline{p}^T \, \underline{R}_{xx}^{-1} \, \underline{p} + \sigma_d^2 \\
&= \cancel{(\underline{R}_{xx}^{-1} \, \underline{p})^H} \, \underline{p} - \underline{p}^H \, \underline{R}_{xx}^{-1} \, \underline{p} - \cancel{(\underline{R}_{xx}^{-1} \, \underline{p})^H} \, \underline{p} + \sigma_d^2 \\
&= -\underline{p}^H \, \underline{R}_{xx}^{-1} \, \underline{p} + E[|d[n]|2] \\
&= -\underline{p}^H \, \underline{R}_{xx}^{-1} \, \underline{p} + E[(\underline{h}^H \, \underline{x}[n] + w[n]) \, (\underline{h}^H \, \underline{x}[n] + w[n])^*] \\
&= -\underline{p}^H \, \underline{R}_{xx}^{-1} \, \underline{p} + E[\underline{h}^H \, \underline{x}[n] \, \underline{x}[n]^H \, \underline{h} + \underline{h}^H \, \underline{x}[n] \, w[n]^* + \underline{h}^T \, \underline{x}[n]^* \, w[n] + w[n] \, w[n]^*] \\
&= -\underline{p}^H \, \underline{R}_{xx}^{-1} \underline{p} + \underline{h}^H \, \underline{R}_{xx} \, \underline{h} + \sigma_w^2 \\
&= -\underline{p}^H \, \underline{R}_{xx}^{-1} \underline{p} + (\underline{R}_{xx}^{-1} \underline{p})^H \, \underbrace{\underline{R}_{xx} \, \underline{R}_{xx}^{-1}}_{\underline{I}} \underline{p} + \sigma_w^2 \\
&= \cancel{-\underline{p}^H \, \underline{R}_{xx}^{-1} \, \underline{p}} + \cancel{\underline{p}^H \, \underline{R}_{xx}^{-1} \, \underline{p}} + \sigma_w^2
\end{aligned}
$$

$$J_{min} = \sigma_w^2$$