Signal Processing and Speech Communication Lab.
Graz University of Technology

AS

Adaptive Systems—Homework Assignment 3

v1.0

Name(s) Matr.No(s).

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Your solutions to the problems (your calculations, the answers to each task as well as the OCTAVE/MATLAB plots) have to be uploaded to the TeachCenter as a single `⋆.pdf` file, no later than **2020/2/20**. Use **this page** as the title page and fill in your **name(s) and matriculation number(s)**. Submitting your homework as a LaTeX document can earn you **up to 3 bonus points!**
All scripts needed for your OCTAVE/MATLAB solutions (all `⋆.m` files) have to be uploaded to the TeachCenter as a single `⋆.zip` archive, no later than **2020/2/20**.
All filenames consist of the assignment number and your matriculation number(s) such as `Assignment3_MatrNo1_MatrNo2.⋆`, for example,

Problem solutions:	`Assignment3_01312345_01312346.pdf`
OCTAVE/MATLAB files:	`Assignment3_01312345_01312346.zip`

Please make sure that your approaches, procedures, and results are clearly presented. Justify your answers! A single upload of the files per group is sufficient.

Analytical Problem 3.1 (12 Points)—Linear Prediction of an AR Process

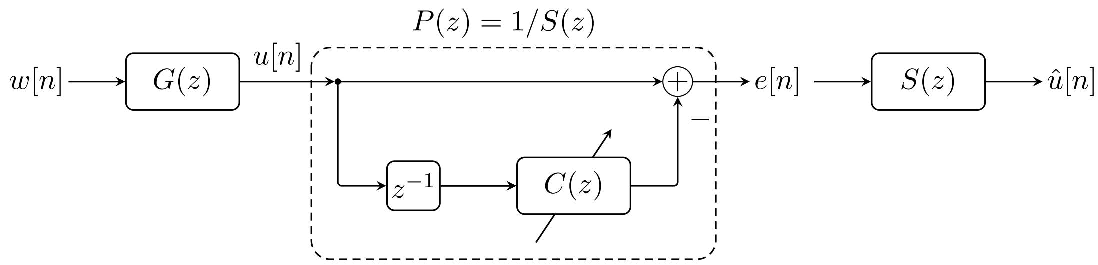


Figure 1: Adaptive linear predictor.

Let $u[n]$ be samples of an AR process with process generator difference equation

$$u[n] = w[n] + u[n-1] - \frac{1}{8}u[n-2]$$

where $w[n]$ are samples of zero-mean, white Gaussian noise. The variance of the AR process is known to be $\sigma_u^2 = 1$.

(a) (4 points) Derive the first three samples of the autocorrelation sequence $r_{uu}[k]$, $k = 0, 1, 2$. Also, compute the variance of the white-noise input, σ_w^2, and the noise gain of the recursive process generator filter, $G_G = \frac{\sigma_u^2}{\sigma_w^2}$. Explicitly write down $G(z)$.

(b) (3 points) For the above AR process, compute the MSE-optimal linear predictor of zeroth order[1], i.e., $C(z) = c_0$. Also, compute the prediction gain $G_P = \frac{\sigma_u^2}{\sigma_e^2}$. Is the used predictor order optimal for the given source?

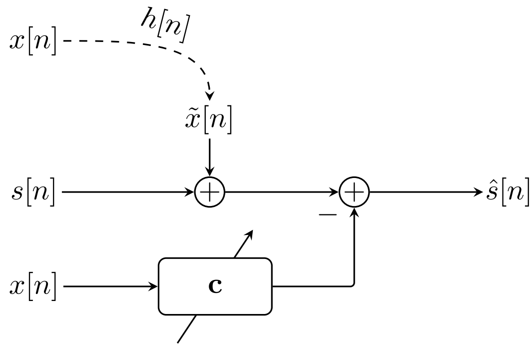
(c) (3 points) For the above AR process, compute the MSE-optimal linear predictor of first order, i.e., $C(z) = c_0 + c_1 z^{-1}$. Again compute the prediction gain $G_P = \frac{\sigma_u^2}{\sigma_e^2}$ and explain whether the used predictor order optimal for the given source.

(d) (2 points) Specify the transfer function of the synthesis filter $S(z)$ (i.e., the inverse of the prediction-error filter $P(z)$) for both predictor orders $N = 1$ and $N = 2$ and compare it to $G(z)$. Calculate the noise gain G_S of $S(z)$. Can you use the noise gain of $S(z)$ to compute the variance of $\hat{u}[n]$?

[1] Note that the overall prediction filter – including the delay element and the direct path depicted in Fig. 1 – is a first-order filter.

Analytical Problem 3.2 (11 Points)—Interference Cancelation

Consider the following interference-cancelation problem:



A signal $s[n]$ (assume that $s[n]$ is a white-noise signal with unit variance) is superimposed with a periodic interference $\tilde{x}[n]$, which is itself a filtered version of a periodic signal

$$x[n] = \cos\left(\frac{\pi}{4}n + \varphi_1\right) + \cos\left(\frac{3\pi}{4}n + \varphi_2\right)$$

where φ_1 and φ_2 are independent and uniformly distributed as $\mathcal{U}[0, 2\pi)$. The autocorrelation function of $x[n]$ is given as

$$r_{xx}[k] = \frac{1}{2}\cos\left(\frac{\pi}{4}k\right) + \frac{1}{2}\cos\left(\frac{3\pi}{4}k\right).$$

Assume further that the filter between $x[n]$ and $\tilde{x}[n]$ is a first-order IIR system with impulse response

$$h[n] = \left(\frac{1}{3}\right)^n u[n].$$

You should now design an adaptive filter which has access to $x[n]$ in order to reconstruct $s[n]$ as accurately as possible, i.e., you should minimize the *mean-squared error* (MSE)

$$\mathsf{E}\big[e^2[n]\big] = \mathsf{E}\big[(\hat{s}[n] - s[n])^2\big].$$

(a) (1 points) What is the maximum filter length N you can choose for the adaptive filter $\mathbf{c}$? Justify your answer!

(b) (2 points) Note that the signal $\tilde{x}[n]$ can be written as

$$\tilde{x}[n] = A_1 \cos\left(\frac{\pi}{4}n + \varphi_1 + \phi_1\right) + A_2 \cos\left(\frac{3\pi}{4}n + \varphi_2 + \phi_2\right)$$

where A and ϕ are amplitude and phase changes introduced by the filter $h[n]$. Compute A_1, A_2, ϕ_1, and ϕ_2 for the impulse response given above. You can of course use numerical tools to answer this question.

(c) (3 points) For $N = 4$, determine the autocorrelation matrix $\mathbf{R}_{xx} = \mathsf{E}\big[\mathbf{x}[n]\mathbf{x}[n]^T\big]$ of the signal $x[n]$ and the crosscorrelation vector $\mathbf{p} = \mathsf{E}\big[\tilde{x}[n]\mathbf{x}[n]\big]$. **Hint:** You can simplify the last calculation by computing just the k-th element of $\mathbf{p}$, $p_k = \mathsf{E}\big[\tilde{x}[n]x[n-k]\big]$ and evaluate it for $k = 0, \ldots, N-1$. With this you will get

$$p_k = \frac{A_1}{2}\cos\left(\frac{\pi}{4}k + \phi_1\right) + \frac{A_2}{2}\cos\left(\frac{3\pi}{4}k + \phi_2\right).$$

(d) (1 points) Determine the Wiener solution $\mathbf{c}_{\mathrm{MSE}}$ for the third-order filter $\mathbf{c}$, i.e., for $N = 4$. You can use numerical software to compute the result.

(e) (2 points) Show that for the Wiener solution the mean-squared error vanishes, i.e., that $\mathsf{E}\big[e^2[n]\big] = 0$.

(f) (2 points) Plot the frequency response of the IIR filter $h[n]$ and of your FIR interference cancelation filter with coefficient vector $\mathbf{c}_{\mathrm{MSE}}$ using `freqz`. What do you observe?

OCTAVE/MATLAB Problem 3.3 (10 Points)—BONUS: Frequency Estimation Design Challenge

The file `soundfile.mat` contains a sinusoid with varying frequency, superimposed by white noise with constant variance σ_w^2, i.e., it contains the signal

$$x[n] = \sin(\theta[n] \cdot n) + w[n]$$

where $\theta[n]$ is the frequency of the sinusoid at time n and a sampling frequency of $f_s = 8000\,\text{Hz}$. It is your goal to estimate the frequency of this sinusoid; specifically, the goal is to deliver a frequency vector

$$\hat{\boldsymbol{\theta}} = [\hat{\theta}[1], \hat{\theta}[2], \ldots, \hat{\theta}[K]]^\mathsf{T}$$

where K is the length of the signal $x[n]$. You have to obey the following design rules:

- The signal contained in the .mat-file is your only input. No other information is available to you.
- You must not use any frequency estimation scripts available in Matlab toolboxes/Octave packages.
- Your solution must include one of your adaptive filter implementations (with possible modifications made by yourself).
- The allowed order of your adaptive filter is at most 100 (i.e., $N \leq 100$).

To participate in the challenge and to get the points for this example, you have to deliver the following material within your submission:

- All your Matlab/Octave script(s) needed for your solution to the problem.
- A .mat-file containing your estimate of the frequency vector.
- A description of your approach in your protocol.

The team with the <u>least absolute error</u> defined as $||\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}||_1 = \sum_k |\hat{\theta}[k] - \theta[k]|$ is the winning team and is awarded with a certificate and TU Graz merchandizing material!

Good luck!

Analytical Problem 3.1

Linear Prediction of AR process

a)

$$u[n] = w[n] + u[n-1] - \frac{1}{8}\cdot u[n-2]$$

$$r_{uu}[k] = E\{u^*[n]\cdot u[n+k]\}$$

$$= E\{u^*[n]\cdot w[n+k] + \overbrace{u^*[n]\cdot u[n-1+k]}^{r_{uu}[k-1]} - \overbrace{u^*[n]\cdot u[n-2+k]}^{r_{uu}[k-2]}\cdot\frac{1}{8}\}$$

$$r_{uu}[k] = E\{u^*[n]\cdot w[n+k]\} + r_{uu}[k-1] - \frac{1}{8}\cdot r_{uu}[k-2]$$

NR = $$= E\{\overbrace{w^*[n]\cdot w[n+k]}^{\sigma_v^2\delta[k]} + \underbrace{u^*[n-1]\cdot w[n+k]}_{=0 \text{ for } k\geq 0} - \frac{1}{8}\cdot \underbrace{u[n-2]\cdot w[n+k]}_{=0 \text{ for } k\geq -1}\} =$$

$$= \sigma_w^2\,\delta[k] \quad \text{for } k\geq 0$$

$$\boxed{\underset{k\geq 0}{r_{uu}[k]} = \sigma_v^2\,\delta[k] + r_{uu}[k-1] - \frac{1}{8}\cdot r_{uu}[k-2]}$$

k = 0:

$$r_{uu}[0] = \sigma_v^2 + r_{uu}[-1] - \frac{1}{8}\cdot r_{uu}[-2] = \sigma_v^2 = 1$$

k = 1:

$r_{uu}[1] = r_{uu}[0] - \frac{1}{8}\cdot r_{uu}[-1]$ // R_{uu} has to be symmetric $\rightarrow r_{uu}[-1] = r_{uu}[+1]$

$(r_{uu}[-k] = r_{uu}[k]$

$$\frac{9}{8}\cdot r_{uu}[1] = r_{uu}[0] \Rightarrow r_{uu}[1] = \frac{8}{9}\cdot\sigma_v^2 = \frac{8}{9}$$

$$\cancel{\frac{9}{8}\cdot r_{uu}[1] = \sigma_v^2 + r_{uu}[-1] - \frac{1}{8}\cdot r_{uu}[-2]}$$

$$\cancel{\frac{1}{8}\cdot r_{uu}[1] = \sigma_v^2 - \frac{1}{8}\cdot r_{uu}[2]}$$

$$\cancel{r_{uu}[2] = 8\cdot\sigma_w^2 - r_{uu}[1] = \left(8\cdot\sigma_v^2\cdot\frac{1}{r_{uu}[1]} - 1\right)\cdot r_{uu}[1]}$$

~~k = 2:~~

$$\cancel{r_{uu}[2] = r_{uu}[1] - \frac{1}{8}\cdot r_{uu}[0]}$$

$$\cancel{= r_{uu}[1] - \frac{1}{8}\cdot\frac{9}{8}\cdot r_{uu}[1]}$$

$$\cancel{r_{uu}[2] = \left(1 - \frac{9}{64}\right)\cdot r_{uu}[1]}$$

$\underline{k=2}$:

$$r_{uu}[2] = r_{uu}[1] - \frac{1}{8} \cdot r_{uu}[0]$$
$$= \frac{8}{9} - \frac{1}{8} \cdot 1$$
$$r_{uu}[2] = \frac{64}{72} - \frac{9}{72} = \underline{\underline{\frac{55}{72}}}$$

$$r_{uu}[0] = \sigma_u^2 = \sigma_w^2 + r_{uu}[-1] - \frac{1}{8} \cdot r_{uu}[-2] \quad // r_{uu}[-k] = r_{uu}[k],$$

$$\sigma_w^2 = \sigma_u^2 - r_{uu}[1] + \frac{1}{8} \cdot r_{uu}[2]$$
$$= 1 - \frac{8}{9} + \frac{1}{8} \cdot \frac{55}{72} \approx \underline{\underline{\frac{1}{64}}} = \underline{\underline{\frac{119}{576}}}$$

$$G_G = \frac{\sigma_u^2}{\sigma_w^2} = \frac{1}{\frac{119}{576}} = \underline{\underline{\frac{576}{119}}}$$

$$G(z) = \frac{U(z)}{W(z)}$$

$$u[n] = w[n] + u[n-1] - \frac{1}{8} \cdot u[n-2]$$

$$U(z) = W(z) + U(z) \cdot z^{-1} - \frac{1}{8} \cdot U(z) \cdot z^{-2}$$

$$U(z) \cdot \left(1 - z^{-1} + \frac{z^{-2}}{8}\right) = W(z)$$

$$G(z) = \frac{U(z)}{W(z)} = \frac{1}{1 - z^{-1} + \frac{z^{-2}}{8}} = \underline{\underline{\frac{z^2}{z^2 - z + \frac{1}{8}}}}$$

b) $\underline{c}_{MSE} = \underline{R}_{uu}^{-1} \cdot \underline{p}$

0-th order: $R_{uu} = r_{uu}[0] = \sigma_u^2$

$$\underline{p} = E\{d[n] \cdot \underline{x}[n]\} = \underbrace{E\{u[n] \cdot u[n-1]\}}_{r_{uu}[-1] = r_{uu}[1]}$$

$$c_0 = r_{uu}[0] \cdot r_{uu}[1] = \sigma_u^2 \cdot \frac{8}{9} = \underline{\underline{\frac{8}{9}}}$$

b) ~~ff~~

$$G_p = \frac{\sigma_u^2}{\sigma_e^2}$$

$$\sigma_e^2 = E\{e[n]\cdot e^*[n]\} \qquad e[n] = u[n] - \underset{\in\mathbb{R}}{c_0}\cdot u[n-1]$$

$$= E\{(u[n] - c_0\cdot u[n-1])\cdot(u[n] - c_0\cdot u[n-1])^*\}$$

$$= E\{\underbrace{u^2[n]}_{r_{uu}[0]} - \underbrace{2u[n]\cdot c_0\cdot u[n-1]}_{2\cdot c_0\cdot r_{uu}[-1]} + \underbrace{c_0^2\cdot u^2[n-1]}_{c_0^2\cdot r_{uu}[0]}\}$$

$$= r_{uu}[0] - 2c_0\cdot r_{uu}[-1] + c_0^2\cdot r_{uu}[0]$$

$$= \sigma_u^2 - 2c_0\cdot\frac{8}{9} + c_0^2\cdot\sigma_u^2$$

$$= \sigma_u^2 - 2\cdot\left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^2\cdot 1$$

$$= \cancel{9 - 2\cdot\frac{8}{81}} \quad \sigma_u^2 - \left(\frac{8}{9}\right)^2$$

$$= \underline{\underline{\frac{17}{81}}}$$

Optimal?: No because the system is of 2^{nd} order ($u[n-2] \to z^{-2}$) and the predictor is of first order (with delay element)

$$G_p = \frac{\sigma_u^2}{\sigma_e^2} = \frac{1}{\frac{17}{81}} = \underline{\underline{\frac{81}{17}}} \approx 4{,}76$$

c) $C(z) = c_0 + c_1 z^{-1}$

$$\underline{R}_{uu} = \begin{bmatrix} r_{uu}[0] & r_{uu}[1] \\ r_{uu}[-1] & r_{uu}[0] \end{bmatrix} = \begin{bmatrix} 1 & \frac{8}{9} \\ \frac{8}{9} & 1 \end{bmatrix}$$

$$\underline{p} = E\{d[n]\cdot \underline{x}[n]\} \qquad d[n] = u[n]$$

$$\underline{x}[n] = \underline{u}[n-1]$$

$$\underline{p} = \begin{pmatrix} E\{u[n]\cdot u[n-1]\} \\ E\{u[n]\cdot u[n-2]\} \end{pmatrix} = \begin{pmatrix} r_{uu}[-1] \\ r_{uu}[-2] \end{pmatrix} = \begin{pmatrix} 8/9 \\ 55/72 \end{pmatrix}$$

$$\underline{c}_{MSE} = \underline{R}_{uu}^{-1}\cdot\underline{p} = \frac{1}{1-\left(\frac{8}{9}\right)^2}\cdot\begin{pmatrix} 1 & -\frac{8}{9} \\ -\frac{8}{9} & 1 \end{pmatrix}\cdot\begin{bmatrix} 8/9 \\ 55/72 \end{bmatrix}$$

$$= \frac{81}{17}\cdot\begin{pmatrix} 17/81 \\ -\frac{17}{648} \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ -\frac{1}{8} \end{pmatrix} = c_{MSE}} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

$\sigma_e^2 = E\{e^*[n] \cdot e[n]\}$ $\qquad e[n] = u[n] - \underline{c}_{MSE}^T \cdot \underline{u}[n-1]$

$$= u[n] - c_0 \cdot u[n-1] - c_1 \cdot u[n-2]$$

$$\sigma_e^2 = E\{(u[n] - \underset{\in\mathbb{R}}{c_0} \cdot u[n-1] - c_1 \cdot u[n-2]) \cdot (u[n] - c_0 \cdot u[n-1]$$

$$\sigma_e^2 = E\{(u[n] - \underline{c}_{MSE}^T \cdot \underline{u}[n-1])^* \cdot (u[n] - \underline{c}_{MSE}^T \cdot \underline{u}[n-1])\}$$

$$= E\{\underbrace{u^2[n]}_{E\{\}=\sigma_u^2} - 2 \cdot \underline{c}_{MSE}^T \cdot \underline{u}[n-1] \cdot u[n] + \underline{c}_{MSE}^T \cdot \underline{u}[n-1] \cdot \overbrace{(\underline{u}^T[n-1] \cdot \underline{c}_{MSE})}^{\text{scalar}}{}^T\}$$

$$= \sigma_u^2 - 2 \cdot \underline{c}_{MSE}^T \cdot E\{\underline{u}[n-1] \cdot u[n]\} + \underline{c}_{MSE}^T \cdot \underline{R}_{uu} \cdot \underline{c}_{MSE}$$

$$= \sigma_u^2 - 2 \cdot \underline{c}_{MSE}^T \cdot \begin{bmatrix} r_{uu}[-1] \\ r_{uu}[-2] \end{bmatrix}^* + \underline{c}_{MSE}^T \cdot \underline{R}_{uu} \cdot c_{MSE}$$

$\downarrow$ MATLAB

$\sigma_e^2 = \underline{0{,}2066}$ $\qquad G_p = \frac{\sigma_u^2}{\sigma_e^2} = \underline{4{,}84}$ Optimal?: Yes, because system and predictor do match.

d) $P(z) = \frac{E(z)}{U(z)}$

N=2:

$e[n] = u[n] - \underline{c}_{MSE}^T \cdot \underline{u}[n-1]$

$e[n] = u[n] - c_0 \cdot u[n-1] - c_1 \cdot u[n-2]$

$\downarrow$

$E(z) = U(z) - c_0 \cdot U(z) \cdot z^{-1} - c_1 \cdot U(z) \cdot z^{-2}$

$E(z) = U(z) \cdot (1 - c_0 \cdot z^{-1} - c_1 \cdot z^{-2})$

$P(z) = \frac{E(z)}{U(z)} = 1 - c_0 \cdot z^{-1} - c_1 \cdot z^{-2}$

$$S(z) = \frac{1}{P(z)} = \frac{1}{1 - c_0 \cdot z^{-1} - c_1 \cdot z^{-2}} = \underline{\underline{\frac{z^2}{z^2 - c_0 \cdot z - c_1}}} \overset{c_0 = 1,\ c_1 = [illegible]}{=} \underline{\frac{z^2}{z^2 - z + \frac{1}{[illegible]}}}$$

N=1:

$e[n] = u[n] - c_0 \cdot u[n-1]$

$\downarrow$

$$E(z) = U(z) - c_0 \cdot U(z) \cdot z^{-1} \Rightarrow P(z) = \frac{E(z)}{U(z)} = 1 - c_0 \cdot z^{-1} \Rightarrow S(z) = \frac{1}{1 - \frac{8}{9} \cdot z^{-1}} = \underline{\underline{\frac{z}{z - \frac{8}{9}}}}$$

$\underline{N=1}$: $S(z) = \dfrac{z}{z - \frac{8}{9}}$; $c_0 = \frac{8}{9}$; $G_s = \dfrac{1}{G_p} = \underline{\underline{\dfrac{17}{81}}}$

$\underline{N=2}$: $S(z) = \dfrac{z^2}{z^2 - z + \frac{1}{8}}$; $c_0 = 1$, $c_1 = -\frac{1}{8}$; $G_s = \dfrac{1}{G_p} = \dfrac{1}{4{,}84} = \underline{\underline{0{,}2066}}$

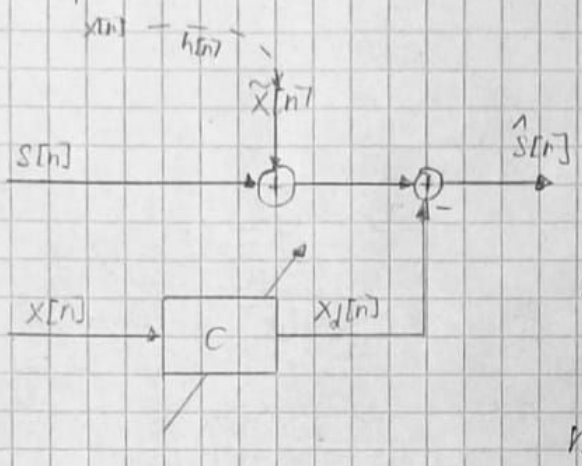
same

$G(z) = \dfrac{z^2}{z^2 - z + \frac{1}{8}}$

Noise Gain: $G_s = \dfrac{1}{G_p}$!!!

Analytical Problem 3.2

Interference Cancellation



$$s[n] \sim \mathcal{N}(0, \sigma_s^2)$$

$$x[n] = \cos\left(\frac{\pi}{4} n + \varphi_1\right) + \cos\left(\frac{3\pi}{4} n + \varphi_2\right)$$

$$\varphi_1, \varphi_2 \sim U[0, 2\pi]$$

$$r_{xx}[k] = \frac{1}{2} \cdot \cos\left(\frac{\pi}{4} \cdot k\right) + \frac{1}{2} \cdot \cos\left(\frac{3\pi}{4} \cdot k\right)$$

$h[n] = \left(\frac{1}{3}\right)^n \cdot u[n]$ // first order IIR system

Minimize $E\{e^2[n]\} = E\{(\hat{s}[n] - s[n])^2\}$

a) Maximum filter length of $\underline{c}$

$\hookrightarrow$ $x[n]$ consists of 2 cos, which cause a line spectrum of 4 lines ($L = 4$)

R_{xx} is invertible for $N \leq L \Rightarrow N_{max} = 4$

b) $\tilde{x}[n] = A_1 \cos\left(\frac{\pi}{4} n + \varphi_1 + \phi_1\right) + A_2 \cos\left(\frac{3\pi}{4} n + \varphi_2 + \phi_2\right)$

$\tilde{x}[n] = \cancel{x[n] * h[n]} \quad (x * h)[n]$

unit step

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} h[m] \cdot x[n-m] = \sum_{m=-\infty}^{\infty} \left(\frac{1}{3}\right)^m \cdot u[m] \cdot x[n-m] =$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^m \cdot \left[\cos\left(\frac{\pi}{4}(n-m) + \varphi_1\right) + \cos\left(\frac{3\pi}{4}(n-m) + \varphi_2\right)\right]$$

Set $n = 0$

$$\tilde{x}[0] = \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^m \left[\cos\left(\frac{\pi}{4} \cdot (-m) + \varphi_1\right) + \cos\left(\frac{3\pi}{4} \cdot (-m) + \varphi_2\right)\right] \quad // \cos(-x) = \cos(x)$$

$$= \sum_{m=0}^{\infty} \frac{\cos\left(\frac{\pi}{4} \cdot m - \varphi_1\right)}{3^m} + \frac{\cos\left(\frac{3\pi}{4} \cdot m - \varphi_2\right)}{3^m}$$

$$\tilde{x}[0] = A_1 \cdot \cos(\varphi_1 + \phi_1) + A_2 \cdot \cos(\varphi_2 + \phi_2)$$

$\hookrightarrow$ $A_1 = \sum_{m=0}^{\infty} \frac{1}{3^m} = \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^m$ geometrische Reihe $= \frac{1}{1 - \frac{1}{3}} = \frac{3}{3-1} = \underline{\underline{\frac{3}{2}}}$

$$= \sum_{m=0}^{\infty} \frac{\cos(\frac{\pi}{4} m - \varphi_1)}{3^m} + \sum_{m=0}^{\infty} \frac{\cos(\frac{3\pi}{4} m - \varphi_2)}{3^m} \quad // \cos(x \mp y) = \cos(x) \cdot \cos(y) \pm \sin(x) \cdot \sin(y)$$

$$= \sum_{m=0}^{\infty} \frac{\cos(\frac{\pi}{4} m) \cdot \cos(\varphi_1) + \sin(\frac{\pi}{4} m) \cdot \sin(\varphi_1)}{3^m} + \sum_{m=0}^{\infty} \frac{\cos(\frac{3\pi}{4} m) \cdot \cos(\varphi_2) + \sin(\frac{3\pi}{4} m) \cdot \sin(\varphi_2)}{3^m}$$

$$\tilde{x}[0] = A_1 \cos(\varphi_1 + \phi_1) + A_2 \cdot \cos(\varphi_2 + \phi_2)$$

$$= A_1 [\cos(\varphi_1) \cdot \cos(\phi_1) - \sin(\varphi_1) \cdot \sin(\phi_1)] + A_2 \cdot [\cos(\varphi_2) \cdot \cos(\phi_2) - \sin(\varphi_2) \cdot \sin(\phi_2)]$$

$$= A_1 \cos(\phi_1) \cdot \cos(\varphi_1) - A_1 \sin(\phi_1) \cdot \sin(\varphi_1) + A_2 \cos(\phi_2) \cdot \cos(\varphi_2) - A_2 \cdot \sin(\phi_2) \cdot \sin(\varphi_2)$$

$$\tilde{x}[0] = \sum_{m=0}^{\infty} \frac{\cos(\frac{\pi}{4} m)}{3^m} \cdot \cos(\varphi_1) + \sum_{m=0}^{\infty} \frac{\sin(\frac{\pi}{4} m)}{3^m} \cdot \sin(\varphi_1) + \sum_{m=0}^{\infty} \frac{\cos(\frac{3\pi}{4} m)}{3^m} \cdot \cos(\varphi_2) + \sum_{m=0}^{\infty} \frac{\sin(\frac{3\pi}{4} m)}{3^m} \cdot \sin(\varphi_2)$$

$$\hookrightarrow A_1 \cdot \cos(\phi_1) = \sum_{m=0}^{\infty} \frac{\cos(\frac{\pi}{4} m)}{3^m} \xrightarrow{\text{MATLAB}} = 1{,}1948 = A$$

$$\hookrightarrow -A_1 \cdot \sin(\phi_1) = \sum_{m=0}^{\infty} \frac{\sin(\frac{\pi}{4} m)}{3^m} = 0{,}3683 = B$$

$$\hookrightarrow A_2 \cdot \cos(\phi_2) = \sum_{m=0}^{\infty} \frac{\cos(\frac{3\pi}{4} m)}{3^m} = 0{,}7808 = C$$

$$\hookrightarrow -A_2 \cdot \cancel{\cos}\sin(\phi_2) = \sum_{m=0}^{\infty} \frac{\sin(\frac{3\pi}{4} m)}{3^m} = 0{,}1489 = D$$

$$\left(NR: \cos(\tfrac{\pi}{4} m) = \{1, \cos(0), \cos(\tfrac{\pi}{4}), \cos(\tfrac{\pi}{2}), \cos(\tfrac{3\pi}{4}), \cos(\pi), \ldots\} \right.$$
$$\left. = \{1, \tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}, -1, -\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}, 1, \ldots\} \right)$$

$$A_1 \cdot \cos(\phi_1) = A$$

$$-A_1 \cdot \sin(\phi_1) = B \Rightarrow A_1 = -\frac{B}{\sin(\phi_1)}$$

$$-\frac{B}{\sin(\phi_1)} \cdot \cos(\phi_1) = A \quad // \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$-\frac{B}{A} = \frac{\sin(\phi_1)}{\cos(\phi_1)} = \tan(\phi_1) \Rightarrow \phi_1 = \arctan\left(-\frac{B}{A}\right) = \underline{-0{,}2992} \text{ rad}$$

$$\Rightarrow \phi_2 = \arctan\left(-\frac{D}{C}\right) = \underline{-0{,}18844} \text{ rad}$$

$$A_1 = \frac{A}{\cos(\phi_1)} = \underline{1{,}2503} \qquad A_2 = \frac{C}{\cos(\phi_2)} = \underline{0{,}7947}$$

c) $N=4$

$$R_{xx} = E\{\underline{x}[n] \cdot \underline{x}^T[n]\} = \begin{bmatrix} r_{xx}[0] & r_{xx}[1] & r_{xx}[2] & r_{xx}[3] \\ r_{xx}[-1] & r_{xx}[0] & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \end{bmatrix}$$

$$r_{xx}[k] = \frac{1}{2} \cdot \cos\left(\frac{\pi}{4} k\right) + \frac{1}{2} \cdot \cos\left(\frac{3\pi}{4} k\right)$$

$$r_{xx}[0] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = \underline{1}$$

$$r_{xx}[1] = \frac{1}{2} \cdot \overset{\frac{1}{\sqrt{2}}}{\cos\left(\frac{\pi}{4}\right)} + \frac{1}{2} \cdot \overset{-\frac{1}{\sqrt{2}}}{\cos\left(\frac{3\pi}{4}\right)} = \underline{\underline{0}} = r_{xx}[-1]$$

$$r_{xx}[2] = \frac{1}{2} \cdot \overset{0}{\cos\left(\frac{\pi}{2}\right)} + \frac{1}{2} \cdot \overset{0}{\cos\left(\frac{6\pi}{4}\right)} = \underline{\underline{0}}$$

$$r_{xx}[3] = \frac{1}{2} \cdot \overset{-\frac{1}{\sqrt{2}}}{\cos\left(\frac{3}{4}\pi\right)} + \frac{1}{2} \cdot \overset{\frac{1}{\sqrt{2}}}{\cos\left(\frac{9}{4}\pi\right)} = \underline{\underline{0}}$$

$$R_{xx} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p = E\{\tilde{x}[n] \cdot \underline{x}[n]\} = \begin{pmatrix} E\{\tilde{x}[n] \cdot x[n]\} \\ E\{\tilde{x}[n] \cdot x[n-1]\} \\ E\{\tilde{x}[n] \cdot x[n-2]\} \\ E\{\tilde{x}[n] \cdot x[n-3]\} \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$p_k = E\{\tilde{x}[n] \cdot x[n-k]\} = E\{\frac{A_1}{2} \cdot \cos\left(\frac{\pi}{4} k + \phi_1\right) + \frac{A_2}{2} \cdot \cos\left(\frac{3\pi}{4} k + \phi_2\right)$$

$$p_0 = \frac{A_1}{2} \cdot \cos\left(\frac{\pi}{4} 0 + \phi_1\right) + \frac{A_2}{2} \cdot \cos\left(\frac{3\pi}{4} k + \phi_2\right)$$

$\downarrow$ MATLAB

$$\underline{p} = \begin{bmatrix} 0{,}7878 \\ 0{,}3293 \\ 0{,}1078 \\ 0{,}0366 \end{bmatrix}$$

d) $$c_{MSE} = \underline{R}_{xx}^{-1} \cdot \underline{p} \overset{\text{Matlab}}{=} \underline{I} \cdot \underline{p} = \underline{\underline{p}}$$

e) Wiener Hopf solution: $\underline{c}_{MMSE} = \underline{h}$

$$E\{e^2[n]\} = E\{(\hat{s}[n] - s[n])^2\} = E\{\hat{s}^2[n] - 2\cdot\hat{s}[n]\cdot s[n] + s^2[n]\} =$$

NR: $$E\{\hat{s}^2[n]\} = E\{(s[n] + \tilde{x}[n] - \cancel{\underline{x}[n]} \cdot \cancel{s[n]}\ \underline{c}^T \cdot \underline{x}[n])^2\} =$$

$$= E\{(s[n] + \underline{h}^T \cdot \underline{x}[n] - \underline{h}^T \cdot \underline{x}[n])^2\} =$$

$$= E\{(s[n])^2\}$$

NR: $$E\{2\cdot\hat{s}[n]\cdot s[n]\} = E\{2\cdot(s[n] + \cancel{x}\underline{h}^T \cdot \underline{x}[n] - \overset{\underline{h}^T}{\underline{c}^T} \cdot \underline{x}[n]) \cdot s[n]\}$$

$$= 2\cdot E\{s[n]\cdot s[n]\}$$

$$= 2\cdot E\{s^2[n]\}$$

$$E\{e^2[n]\} = E\{s^2[n]\} - 2\cdot E\{s^2[n]\} + E\{s[n]\} = \underline{0}\ ✓$$

f) $$H(z) = \mathcal{Z}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]\cdot z^{-n} =$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n \cdot u[n] \cdot z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \cdot \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3z}\right)^n = \text{// geometrische Reihe}$$

$$= \frac{1}{1-\frac{1}{3z}} = \underline{\underline{\frac{3z + 0\cdot z^0}{3z-1}}}$$

$$C(z) = \frac{c_0 + c_1\cdot z^{-1} + c_2\cdot z^{-2} + c_3\cdot z^{-3}}{1}$$

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