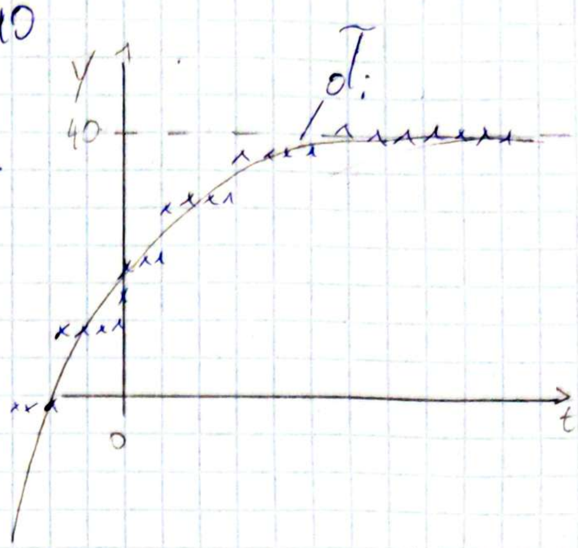
Fit Exponential Function

- $y = a - b \cdot e^{-\frac{t}{\tau}}$ e.g. $a = 40$, $b = 1$, $\tau = 5$



Find a, b, τ using Gradient Descent.

$\frac{dy}{da} = 1$

$\frac{dy}{db} = 0 - e^{-\frac{t}{\tau}}$

- $\frac{dy}{d\tau} = 0 - b \cdot e^{-\frac{t}{\tau}} \cdot \left(\frac{t}{\tau^2}\right)$ // $\frac{d\left(\frac{t}{\tau}\right)}{d\tau} = \frac{d}{d\tau}\left(t \cdot \tau^{-1}\right) = -t \cdot \tau^{-2} \cdot (-1)$

Cost function

$$f(a,b,\tau) = \left[\tilde{d} - \left(a - b \cdot e^{-\frac{t}{\tau}}\right)\right]^2 =$$

$$= \left[\tilde{d}^2 - 2\left(a - b \cdot e^{-\frac{t}{\tau}}\right) \cdot \tilde{d} + \left(a - b \cdot e^{-\frac{t}{\tau}}\right)^2\right]$$

Easy start: only a is unknown:

- $\frac{df}{da} =$

Bsp e.g. $\tau = 1$, $b = 1$ measured at time t_i

$y = a - e^{-t}$ $\quad \tilde{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix}$ $\quad \underline{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{bmatrix}$

Cost fct.

$$f(a) = \sum_{i=1}^{n} \left(d_i - \left(a - e^{-t_i}\right)\right)^2$$

- $$f(a) = \sum_{i=1}^{n} \left(d_i^2 - 2\left(a - e^{-t_i}\right) d_i + \left(a - e^{-t_i}\right)^2\right)$$

$$f(a) = \sum_{i=1}^{n} d_i^2 - 2ad_i + 2\cdot e^{t_i}\cdot d_i + a^2 - 2a\cdot e^{-t_i} + e^{-t_i\cdot 2}$$

$$\frac{df}{da} = \sum_{i=1}^{n} 0 - 2d_i + 0 + 2a - 2\cdot e^{-t_i} + 0$$

∇f

$$= \sum_{i=1}^{n} -2d_i + 2a - 2\cdot e^{-t_i}$$

$$= \underline{\underline{\sum_{i=1}^{n} 2\cdot(a - d_i - e^{-t_i})}}$$

$$a_{n+1} = a_n - \gamma\cdot\nabla f(a_n)$$

$$J_n = \underline{e}_n^T\cdot\underline{e}_n$$

$$\underline{J} = \begin{bmatrix} J_1 \\ J_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \underline{e}_1^T\cdot\underline{e}_1 \\ \underline{e}_2^T\cdot\underline{e}_2 \\ \vdots \end{bmatrix} = \left[\underline{d} - a_n\cdot\underline{e}^t\right]^T$$

$$J = \sum_{i=1}^{N} (d_i - (a - e^{-t_i}))^2 \quad // \quad [d_0, d_2, \dots, d_N]$$

$$= \sum_{i=1}^{N} e_i^2 = \underline{e}^T\cdot\underline{e} \quad // \quad [e_1, e_2, e_3, \dots]\cdot\begin{bmatrix} e_1 \\ e_2 \\ \vdots \end{bmatrix} = e_1^2 + e_2^2 + \dots = \sum_i e_i^2$$

$$\underline{e} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \end{bmatrix} - a\cdot\begin{bmatrix} e^{t_1} \\ e^{t_2} \\ \vdots \end{bmatrix} = \underline{d} - a\cdot\underline{e}^t$$

$$\frac{de}{da} = \underline{d} - e^t$$

$$\nabla_a J = \frac{d}{da}\sum_i (d_i - (a - e^{-t_i}))^2 =$$

$$= \sum_i \frac{d}{da}(d_i - (a - e^{-t_i}))^2 =$$

$$= \sum_i 2\cdot(d_i - (a - e^{-t_i}))^1\cdot\left(\frac{d}{da}(d_i - (a - e^{-t_i}))^a\right)$$

$$= \sum_i 2\cdot(d_i - (a - e^{-t_i}))\cdot(0 - 2\cdot(a - e^{-t_i})\cdot(1))$$

$$= \sum_i 2\cdot(d_i - (a - e^{-t_i}))\cdot(-2)\cdot(a - e^{-t_i})$$

$$\cancel{J = \sum_i \left[d_i - (a - \exp(-t))\right]}$$

$$\cancel{\nabla_a J = \frac{d}{da} \sum_i \left[d_i - (a - \exp(-t))\right]^2}$$

$$\cancel{= \sum_i 2 \cdot [d_i -}$$

$$J = \sum_{i=1}^{N} \left[d_i - (a - e^{-t_i})\right]^2$$

$$= \sum_i \left[d_i + e^{-t_i} - a\right]^2$$

$$\frac{dJ}{da} = \frac{d}{da}\left(\sum_i \left[d_i + e^{-t_i} - a\right]^2\right)$$ Summenregel

$$= \sum_i \frac{d}{da}\left[\quad - \text{''} - \quad\right]^2$$

$$= \sum_i 2 \cdot \left[d_i + e^{-t_i} - a\right]^1 \cdot (0 + 0 - 1)$$

$$= \underline{\underline{\sum_i -2 \cdot \left[d_i + e^{-t_i} - a\right]}} \quad ?$$

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1 measurement

d scalar d, t

$$J = (d - (a - b \cdot e^{-\frac{t}{\tau}}))^2 ; \quad J = (d + b \cdot e^{-\frac{t}{\tau}} - a)^2$$

$$\cancel{\frac{dJ}{da} = 2(d}$$

$$\frac{dJ}{da} = 2 \cdot (d + b \cdot e^{-\frac{t}{\tau}} - a) \cdot (0 + 0 - 1)$$

$$\underline{\underline{\frac{dJ}{da} = -2 \cdot (d + b \cdot e^{-\frac{t}{\tau}} - a)}} \overset{!}{=} 0$$

$$a = d + b \cdot e^{-\frac{t}{\tau}}$$

$$= 37{,}1813 + \underbrace{1 \cdot e^{-\overset{2}{\cancel{10}}}}_{0{,}8187}{}^{0{,}67} = 40 \checkmark$$

$\Rightarrow$

$$J = \sum_{i=1}^{n} \left(d_i - (a - e^{-t_i})\right)^2$$

$$\cancel{\ln(J) = K = \sum_{i=1}^{n} \ln\left(d_i - (a - e^{-t_i})\right)^2}$$ // this altered the function but it should not have an impact → IT DOES IMPACT IT HUGELY

$$\ln(J) = L = \ln\left[\left\{\sum_{i=1}^{n} \left(d_i - (e^{-t_i} + a)\right)^2\right\}\right]$$

$$\frac{dL}{da} = \frac{d}{da}\left(-\text{''}- \right)$$

$$= \frac{1}{\sum_i \left(d_i - (e^{-t_i} + a)\right)^2} \cdot \left[\frac{d}{da}\left(\sum_i \left(d_i - (e^{-t_i} + a)\right)^2\right)\right]$$

$$= -\text{''}- \cdot \left[\sum_i 2 \cdot \left(d_i + e^{-t_i} + a\right) \cdot (-1)\right]$$

$$= -\frac{2\sum_i(-a + e^{-t_i} + d_i)}{(-a + e^{-t} + d)^2}$$

↳ Since J can get to 0 $\ln(J) \to \infty$ ↳ bad for numerical stability → Solution add constant to J s.t. it can not be 0

$$\cancel{J_{mod} = \sum_{i=1}^{N}\left(d_i - (a - e^{-t_i}\right.}$$

$$J = \left[\sum_{i=1}^{N} \left(d_i - (a - b \cdot e^{-\frac{t_i}{\tau}})\right)^2\right] + 10$$

$L = \ln(J)$ → Ableitungsrechner.net

$$\cancel{\frac{dL}{da} = -2 \cdot \sum_i \left(-a + \frac{b \cdot e^{-t}}{\tau} + d\right)}$$

$$\frac{dL}{da} = -\frac{\sum 2\left(-a + b \cdot e^{-\frac{t_i}{\tau}} + d_i\right)}{\sum_i \left(-a + b \cdot e^{-\frac{t_i}{\tau}} + d_i\right)^2 + 10}$$

Extension to more variables

$$L = \ln\left[\sum_{i=1}^{N} \left(d_i - a + b \cdot e^{-\frac{t}{\tau}}\right)^2 + 10\right]$$

$$\vec{\nabla}_c L = \begin{bmatrix} \frac{\partial f}{\partial a} \\ \frac{\partial f}{\partial b} \\ \frac{\partial f}{\partial \tau} \end{bmatrix} \qquad \underline{c} = \begin{bmatrix} a \\ b \\ \tau \end{bmatrix}$$

$$\underline{\underline{\frac{\partial f}{\partial a} = -\frac{\sum_i 2 \cdot \left(-a + b \cdot e^{-\frac{t}{\tau}} + d\right)}{\sum \left(-a + b \cdot e^{-\frac{t}{\tau}} + d\right)^2 + 10}}}$$ //Ableitungsrechner.net

$$\frac{\partial f}{\partial b} = + \frac{\sum_i 2 \cdot e^{-\frac{t}{\tau}} \cdot \left(e^{-\frac{t}{\tau}} \cdot b + d - a\right)}{\sum_i \left(e^{-\frac{t}{\tau}} \cdot b + d - a\right)^2 + 10}$$

$$\frac{\partial f}{\partial \tau} = \frac{\sum_i 2b \cdot t \cdot e^{-\frac{t}{\tau}} \cdot \left(b \cdot e^{-\frac{t}{\tau}} + d - a\right)}{\left[\sum_i \tau \cdot \left(b \cdot e^{-\frac{t}{\tau}} + d - a\right)^2\right] + \tau^2 \cdot 10}$$