TEMA 6

SOLUCIÓN DE

LOS PROBLEMAS

PROBLEMA 1

$$\int \frac{x^3+2x^2-4x-3}{x^3+2x^2-3x}dx$$

Solución:

$$\int \frac{x^3+2x^2-4x-3}{x^3+2x^2-3x}dx = \int \left(1+\frac{-x-3}{x^3+2x^2-3x}\right)dx = \int \left(1-\frac{(x+3)}{x\,(x^2+2x-3)}\right)dx =$$

$$= \int \left(1-\frac{\cancel{(x+3)}}{x\,\cancel{(x+3)}\,(x-1)}\right)dx = \int \left(1-\frac{1}{x\,(x-1)}\right)dx = \int \left(1+\frac{1}{x}-\frac{1}{x-1}\right)dx =$$

$$= x + \mathrm{Ln}(|x|) - \mathrm{Ln}(|x-1|) + C$$

PROBLEMA 2

$$\int \frac{x+1}{x^4-4x^3+4x^2}dx$$

Solución:

$$\int \frac{x+1}{x^4-4x^3+4x^2}dx = \int \frac{x+1}{x^2(x^2-4x+4)}dx = \int \frac{x+1}{x^2(x-2)^2}dx$$

$$\frac{x+1}{x^2(x-2)^2} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2} = \frac{Ax(x-2)^2+B(x-2)^2+Cx^2(x-2)+Dx^2}{x^2(x-2)^2} =$$

$$= \frac{A(x^3-4x^2+4x)+B(x^2-4x+4)+C(x^3-2x^2)+Dx^2}{x^2(x-2)^2} =$$

$$= \frac{(A+C)x^3+(-4A+B-2C+D)x^2+(4A-4B)x+4B}{x^2(x-2)^2} \Longrightarrow$$

$$\Longrightarrow \left.\begin{array}{rcrcrcrcl} A & & & + & C & & & = & 0 \\ -4A & + & B & - & 2C & + & D & = & 0 \\ 4A & - & 4B & & & & & = & 1 \\ & & 4B & & & & & = & 1 \end{array}\right\} \Longrightarrow \begin{cases} A = \dfrac{1}{2} \\ B = \dfrac{1}{4} \\ C = -\dfrac{1}{2} \\ D = \dfrac{3}{4} \end{cases}$$

$$\int \frac{x+1}{x^4-4x^3+4x^2}dx = \int \left(\frac{1}{2}\frac{1}{x}+\frac{1}{4}\frac{1}{x^2}-\frac{1}{2}\frac{1}{x-2}+\frac{3}{4}\frac{1}{(x-2)^2}\right)dx =$$

$$= \frac{1}{2}\mathrm{Ln}(|x|) - \frac{1}{4}\frac{1}{x} + -\frac{1}{2}\mathrm{Ln}(|x-2|) - \frac{3}{4}\frac{1}{(x-2)} + C$$

PROBLEMA 3

$$\int (2x^3 - 5x + 3)e^{-x} dx$$

Solución:

Utilizaremos integración por partes, ya que la integral se ajusta al modelo $\int P(x)e^{ax+b} dx$.

$$\int x^2 e^x dx = \left\{ \begin{array}{lll} u = 2x^3 - 5x + 3 & \longrightarrow & du = (6x^2 - 5)dx \\ dv = e^{-x} dx & \longrightarrow & v = -e^{-x} \end{array} \right\} =$$

$$= -\left(2x^3 - 5x + 3\right) e^{-x} + \int \left(6x^2 - 5\right) e^{-x} dx = \left\{ \begin{array}{lll} u = 6x^2 - 5 & \longrightarrow & du = 12x dx \\ dv = e^{-x} dx & \longrightarrow & v = -e^{-x} \end{array} \right\} =$$

$$= -\left(2x^3 - 5x + 3\right) e^{-x} - \left(6x^2 - 5\right) e^{-x} + \int 12x e^{-x} dx = \left\{ \begin{array}{lll} u = 12x & \longrightarrow & du = 12 dx \\ dv = e^{-x} dx & \longrightarrow & v = -e^{-x} \end{array} \right\} =$$

$$= -\left(2x^3 - 5x + 3\right) e^{-x} - \left(6x^2 - 5\right) e^{-x} - 12x e^{-x} + \int 12 e^{-x} dx =$$

$$= -\left(2x^3 - 5x + 3\right) e^{-x} - \left(6x^2 - 5\right) e^{-x} - 12x e^{-x} - 12 e^{-x} + C =$$

$$= \left(-2x^3 - 6x^2 - 7x - 10\right) e^{-x} + C$$

PROBLEMA 4

$$\int \frac{3x^3+4x^2+3x-1}{2x^4+2x^2}dx$$

Solución:

$$\int \frac{3x^3+4x^2+3x-1}{2x^4+2x^2}dx = \frac{1}{2}\int \frac{3x^3+4x^2+3x-1}{x^4+x^2}dx = \frac{1}{2}\int \frac{3x^3+4x^2+3x-1}{x^2\left(x^2+1\right)}dx$$

$$\frac{3x^3+4x^2+3x-1}{x^2\left(x^2+1\right)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1} = \frac{Ax\left(x^2+1\right)+B\left(x^2+1\right)+(Cx+D)x^2}{x^2\left(x^2+1\right)} =$$

$$= \frac{Ax^3+Ax+Bx^2+B+Cx^3+Dx^2}{x^2\left(x^2+1\right)} = \frac{(A+C)x^3+(B+D)x^2+Ax+B}{x^2\left(x^2+1\right)}$$

$$\left.\begin{array}{rcr} A+C & = & 3 \\ B+D & = & 4 \\ A & = & 3 \\ B & = & -1 \end{array}\right\} \Longrightarrow \begin{cases} A= & 3 \\ B= & -1 \\ C= & 0 \\ D= & 5 \end{cases}$$

$$\int \frac{3x^3+4x^2+3x-1}{2x^4+2x^2}dx = \frac{1}{2}\int\left(\frac{3}{x}-\frac{1}{x^2}+\frac{5}{x^2+1}\right)dx =$$

$$= \frac{1}{2}\left(\int \frac{3}{x}dx - \int \frac{1}{x^2}dx + \int \frac{5}{x^2+1}dx\right) = \frac{1}{2}\left(3\mathrm{Ln}(|x|)+\frac{1}{x}+5\arctan(x)\right)+C$$

PROBLEMA 5

$$\int (x^2 - 3x + 2)e^x\,dx$$

Solución:

Utilizaremos integración por partes, ya que la integral se ajusta al modelo $\int P(x)e^{ax+b}\,dx$.

$$\int \left(x^2 - 3x + 2\right) e^x\,dx = \left\{ \begin{array}{lll} u = x^2 - 3x + 2 & \longrightarrow & du = (2x-3)\,dx \\ dv = e^x\,dx & \longrightarrow & v = e^x \end{array} \right\} =$$

$$= \left(x^2 - 3x + 2\right) e^x - \int (2x-3)e^x\,dx = \left\{ \begin{array}{lll} u = 2x - 3 & \longrightarrow & du = 2\,dx \\ dv = e^x\,dx & \longrightarrow & v = e^x \end{array} \right\} =$$

$$= \left(x^2 - 3x + 2\right) e^x - \left((2x-3)e^x - 2\int e^x\,dx\right) =$$

$$= \left(x^2 - 3x + 2\right) e^x - (2x-3)e^x + 2e^x + C = \left(x^2 - 5x + 7\right) e^x + C$$

PROBLEMA 6

$$\int (x+2)\mathrm{Ln}(x+3)\,dx$$

Solución:

Utilizaremos integración por partes, ya que la integral se ajusta al modelo $\int P(x)\mathrm{Ln}(ax+b)\,dx$.

$$\int (x+2)\mathrm{Ln}(x+3)\,dx = \left\{ \begin{array}{lcl} u = \mathrm{Ln}(x+3) & \longrightarrow & du = \dfrac{1}{x+3}dx \\ dv = (x+2)\,dx & \longrightarrow & v = \dfrac{x^2}{2} + 2x \end{array} \right\} =$$

$$= \left(\frac{x^2}{2}+2x\right)\mathrm{Ln}(x+3) - \int \frac{\frac{x^2}{2}+2x}{x+3}dx = \left(\frac{x^2}{2}+2x\right)\mathrm{Ln}(x+3) - \frac{1}{2}\int \frac{x^2+4x}{x+3}dx =$$

$$= \left(\frac{x^2}{2}+2x\right)\mathrm{Ln}(x+3) - \frac{1}{2}\int \left(x+1+\frac{-3}{x+3}\right)dx =$$

$$= \left(\frac{x^2}{2}+2x\right)\mathrm{Ln}(x+3) - \frac{1}{2}\left(\int x\,dx + \int dx - 3\int \frac{1}{x+3}dx\right) =$$

$$= \left(\frac{x^2}{2}+2x\right)\mathrm{Ln}(x+3) - \frac{1}{2}\left(\frac{x^2}{2}+x-3\mathrm{Ln}(|x+3|)\right) + C =$$

$$= \left(\frac{x^2}{2}+2x+\frac{3}{2}\right)\mathrm{Ln}(|x+3|) - \frac{1}{4}x^2 - \frac{1}{2}x + C$$

<u>Nota</u>: En el último paso podemos transformar $\mathrm{Ln}(x+3)$ en $\mathrm{Ln}(|x+3|)$ para ampliar el conjunto de valores $x \in \mathbb{R}$ que son solución de la integral.

PROBLEMA 7

$$\int \frac{x^2+4x+4}{x^2+2x-3}dx$$

Solución:

$$\int \frac{x^2+4x+4}{x^2+2x-3}dx = \int \left(1+\frac{2x+7}{x^2+2x-3}\right)dx = \int dx + \int \frac{2x+7}{(x-1)(x+3)}dx$$

$$\frac{2x+7}{(x-1)(x+3)} = \frac{A}{x-1}+\frac{B}{x+3} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)} = \frac{(A+B)x+(3A-B)}{(x-1)(x+3)} \Longrightarrow$$

$$\Longrightarrow \left.\begin{array}{rcl} A+B & = & 2 \\ 3A-B & = & 7 \end{array}\right\} \Longrightarrow \begin{cases} A = \dfrac{9}{4} \\ B = -\dfrac{1}{4} \end{cases}$$

$$\int \frac{x^2+4x+4}{x^2+2x-3}dx = \int dx + \int \frac{2x+7}{(x-1)(x+3)}dx =$$

$$= \int dx + \frac{9}{4}\int \frac{1}{x-1}dx - \frac{1}{4}\int \frac{1}{x+3}dx = x + \frac{9}{4}\mathrm{Ln}(|x-1|) - \frac{1}{4}\mathrm{Ln}(|x+3|) + C$$

PROBLEMA 8

$$\int \frac{1}{\sqrt[4]{x^3}(\sqrt{x}+\sqrt[3]{x})}dx$$

Solución:

Para resolver esta integral vamos a realizar el cambio de variable $x = t^{12}$, con lo que $t = \sqrt[12]{x}$ y $dx = 12t^{11}dt$:

$$\int \frac{1}{\sqrt[4]{x^3}(\sqrt{x}+\sqrt[3]{x})}dx = \int \frac{12t^{11}}{\sqrt[4]{(t^{12})^3}\left(\sqrt{t^{12}}+\sqrt[3]{t^{12}}\right)}dt = \int \frac{12t^{11}}{t^9\left(t^6+t^4\right)}dt = \int \frac{12}{t^2\left(t^2+1\right)}dt$$

$$\frac{12}{t^2\left(t^2+1\right)} = \frac{A}{t}+\frac{B}{t^2}+\frac{Ct+D}{t^2+1} = \frac{At(t^2+1)+B(t^2+1)+(Ct+D)t^2}{t^2(t^2+1)} =$$

$$= \frac{A(t^3+t)+B(t^2+1)+(Ct^3+Dt^2)}{t^2(t^2+1)} = \frac{(A+C)t^3+(B+D)t^2+At+B}{t^2(t^2+1)} \Longrightarrow$$

$$\left.\begin{array}{rcr} A+C &=& 0 \\ B+D &=& 0 \\ A &=& 0 \\ B &=& 12 \end{array}\right\} \Longrightarrow \begin{cases} A = & 0 \\ B = & 12 \\ C = & 0 \\ D = & -12 \end{cases}$$

$$\int \frac{1}{\sqrt[4]{x^3}(\sqrt{x}+\sqrt[3]{x})}dx = \int \frac{12}{t^2\left(t^2+1\right)}dt = \int \left(\frac{12}{t^2}+\frac{-12}{t^2+1}\right)dt =$$

$$= 12\int \frac{1}{t^2}dt - 12\int \frac{1}{t^2+1}dt = -\frac{12}{t} - 12\arctan(t) + C =$$

$$= \frac{-12}{\sqrt[12]{x}} - 12\arctan(\sqrt[12]{x}) + C$$

PROBLEMA 9

$$\int \frac{2x^2+9x-1}{x^2+2x-3}dx$$

Solución:

$$\int \frac{2x^2+9x-1}{x^2+2x-3}dx = \int \left(2+\frac{5x+5}{x^2+2x-3}\right)dx = \int \left(2+5\frac{x+1}{x^2+2x-3}\right)dx =$$

$$= \int \left(2+5\frac{2}{2}\frac{x+1}{x^2+2x-3}\right)dx = \int \left(2+\frac{5}{2}\frac{2x+2}{x^2+2x-3}\right)dx =$$

$$2\int dx + \frac{5}{2}\int \frac{2x+2}{x^2+2x-3}dx = 2x + \frac{5}{2}\mathrm{Ln}\left(\left|x^2+2x-3\right|\right) + C$$

PROBLEMA 10

$$\int \frac{3x-10}{x^2+2x+1}dx$$

Solución:

$$\int \frac{3x-10}{x^2+2x+1}dx = \int \frac{3x-10}{(x+1)^2}$$

$$\frac{3x-10}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2} = \frac{Ax+(A+B)}{(x+1)^2} \Longrightarrow$$

$$\Longrightarrow \left.\begin{array}{rcr} A & = & 3 \\ A+B & = & -10 \end{array}\right\} \Longrightarrow \begin{cases} A = & 3 \\ B = & -13 \end{cases}$$

$$\int \frac{3x-10}{x^2+2x+1}dx = \int \left(\frac{3}{x+1} - \frac{13}{(x+1)^2}\right)dx =$$

$$= 3\int \frac{1}{x+1} - 13\int (x+1)^{-2}dx = 3\mathrm{Ln}(|x+1|) + \frac{13}{x+1} + C$$

PROBLEMA 11

$$\int \frac{x^2+4x+4}{x^2+2x-3}dx$$

Solución:

$$\int \frac{x^2+4x+4}{x^2+2x-3}dx = x + \frac{9}{4}\mathrm{Ln}(|x-1|) - \frac{1}{4}\mathrm{Ln}(|x+3|) + C$$

PROBLEMA 12

$$\int \frac{-x^2+2x+1}{x^3+2x^2+x}dx$$

Solución:

$$\int \frac{-x^2+2x+1}{x^3+2x^2+x}dx = \mathrm{Ln}(|x|) - 2\mathrm{Ln}(|x+1|) - \frac{2}{x+1} + C$$

PROBLEMA 13

$$\int \frac{-x^3+3x^2-7x+1}{x^4-x^3-x^2+x}dx$$

Solución:

$$\int \frac{-x^3+3x^2-7x+1}{x^4-x^3-x^2+x}dx = \mathrm{Ln}(|x-1|) + \frac{2}{x-1} + \mathrm{Ln}(|x|) - 3\mathrm{Ln}(|x+1|) + C$$

PROBLEMA 14

$$\int \frac{3x^3+4x^2+3x-1}{2x^4+2x^2}dx$$

Solución:

$$\int \frac{3x^3+4x^2+3x-1}{2x^4+2x^2}dx = \frac{3}{2}\mathrm{Ln}(|x|) + \frac{1}{2x} + \frac{5}{2}\arctan(x) + C$$

PROBLEMA 15

$$\int (x^2 - 3x + 2)e^x \, dx$$

Solución:

$$\int \left(x^2 - 3x + 2\right) e^x \, dx = \left(x^2 - 5x + 7\right) e^x + C$$

PROBLEMA 16

$$\int (x + 2)\mathrm{Ln}(x + 3) \, dx$$

Solución:

$$\int (x + 2)\mathrm{Ln}(x + 3) \, dx = \left(\frac{x^2 + 4x + 3}{2}\right) \mathrm{Ln}(|x + 3|) - \frac{x^2}{4} - \frac{x}{2} + C$$

PROBLEMA 17

$$\int x^3 e^{2x} \, dx$$

Solución:

$$\int x^3 e^{2x} \, dx = \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8}\right) e^{2x} + C$$

PROBLEMA 18

$$\int x^3 e^{x^2} \, dx$$

Solución:

$$\int x^3 e^{x^2} \, dx = \left(\frac{x^2}{2} - \frac{1}{2}\right) e^{x^2} + C$$

PROBLEMA 19

$$\int \frac{x^2+4x+4}{x^2+2x+3}dx$$

Solución:

$$\int \frac{x^2+4x+4}{x^2+2x+3}dx = x + \mathrm{Ln}\left(\left|x^2+2x+3\right|\right) - \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C$$

PROBLEMA 20

$$\int \cos(5x)\,\mathrm{sen}(3x)\,dx$$

Solución:

$$\int \cos(5x)\,\mathrm{sen}(3x)\,dx = \frac{\cos(2x)}{4} - \frac{\cos(8x)}{16} + C$$

PROBLEMA 21

$$\int \frac{1}{(1+x^2)^2}dx$$

Solución:

Si intentamos utilizar el método habitual para integrales racionales comprobaremos que no llegamos a ninguna solución:

$$\frac{1}{(1+x^2)^2} = \frac{Ax+B}{(1+x^2)} + \frac{Cx+D}{(1+x^2)^2} = \frac{(Ax+B)(1+x^2)+(Cx+D)}{(1+x^2)^2} =$$

$$= \frac{Ax+Ax^3+B+Bx^2+Cx+D}{(1+x^2)^2} = \frac{Ax^3+Bx^2+(A+C)x+(B+D)}{(1+x^2)^2}$$

$$\Longrightarrow \left.\begin{array}{r} A=0 \\ B=0 \\ A+C=0 \\ B+D=1 \end{array}\right\} \Longrightarrow \begin{cases} A=0 \\ B=0 \\ C=0 \\ D=1 \end{cases} \Longrightarrow \frac{1}{(1+x^2)^2} = \frac{1}{(1+x^2)^2} \quad !!!$$

Al no conseguir ningún avance, claramente es necesario emplear otro método. En este caso, vamos a intentar el siguiente cambio:

$$t = \arctan(x) \Longrightarrow x = \tan(t) \Longrightarrow dx = \frac{1}{\cos^2(t)}dt$$

$$\int \frac{1}{(1+x^2)^2}dx = \int \frac{1}{\left(1+\frac{\text{sen}^2(t)}{\cos^2(t)}\right)^2}\frac{1}{\cos^2(t)}dt = \int \frac{1}{\left(\frac{\text{sen}^2(t)+\cos^2(t)}{\cos^2(t)}\right)^2}\frac{1}{\cos^2(t)}dt =$$

$$= \int \frac{1}{\frac{1}{\cos^4(t)}\cos^2(t)}dt = \int \cos^2(t)dt = \int \left(\frac{1+\cos(2t)}{2}\right)dt =$$

$$= \frac{t}{2} + \frac{1}{4}\text{sen}(2t) + C = \frac{1}{2}\arctan(x) + \frac{1}{4}\text{sen}(2\arctan(x)) + C = \frac{1}{2}\arctan(x) + \frac{1}{2}\frac{x}{1+x^2}$$

PROBLEMA 22

$$\int \sqrt{\frac{1+x}{1-x}} dx$$

Solución:

En este problema vamos a utilizar el siguiente cambio:

$$t^2 = \frac{1+x}{1-x} \implies t^2 - xt^2 = 1 + x \implies x\left(1+t^2\right) = t^2 - 1 \implies x = \frac{t^2-1}{t^2+1}$$

$$x = \frac{t^2-1}{t^2+1} \implies dx = \frac{2t(t^2+1) - 2t(t^2-1)}{(t^2+1)^2} dt \implies dx = \frac{4t}{(t^2+1)^2}$$

A continuación sustituimos en la integral:

$$\int \sqrt{\frac{1+x}{1-x}} dx = \int \sqrt{t^2} \frac{4t}{\left(t^2+1\right)^2} dt = \int 4 \frac{t^2}{\left(t^2+1\right)^2} dt = 4 \int \frac{t^2+1-1}{(t^2+1)^2} =$$

$$= 4 \int \left(\frac{1}{t^2+1} - \frac{1}{\left(t^2+1\right)^2} \right) dt = 4 \arctan(t) - 4\frac{1}{2} \left(\arctan(t) + \frac{t}{t^2+1} \right) + C =$$

$$= 2\arctan(t) - \frac{2t}{t^2+1} + C = 2\arctan\left(\sqrt{\frac{1+x}{1-x}}\right) - 2\frac{\sqrt{\frac{1+x}{1-x}}}{\frac{1+x}{1-x}+1} + C =$$

$$= 2\arctan\left(\sqrt{\frac{1+x}{1-x}}\right) - (1-x)\sqrt{\frac{1+x}{1-x}} + C = 2\arctan\left(\sqrt{\frac{1+x}{1-x}}\right) - \sqrt{1-x^2} + C$$

Otra forma de resolver la integral sería la siguiente:

$$\int \sqrt{\frac{1+x}{1-x}} dx = \int \sqrt{\frac{(1+x)(1+x)}{(1-x)(1+x)}} dx = \int \frac{(1+x)}{\sqrt{1-x^2}} dx =$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx + \int x \cdot \left(1-x^2\right)^{-1/2} dx = \operatorname{arc\,sen}(x) - \left(1-x^2\right)^{1/2} + C = \operatorname{arc\,sen}(x) - \sqrt{1-x^2} + C$$

PROBLEMA 23

$$\int \frac{e^{5x}+8}{e^{3x}-8}dx$$

Solución:

Puesto que el máximo común divisor de 3 y 5 es 1, vamos a utilizar el siguiente cambio:

$$t = e^x \implies dt = e^x dx \implies dx = \frac{dt}{t}$$

$$\int \frac{e^{5x}+8}{e^{3x}-8}dx = \int \frac{t^5+8}{(t^3-8)}\frac{dt}{t} = \int \left(\frac{t^5+8}{t^4-8t}\right)dt = \int \left(t+\frac{8t^2+8}{t^4-8t}\right)dt$$

$$\frac{8t^2+8}{t^4-8t} = \frac{8t^2+8}{t\,(t-2)\,(t^2+2t+4)} = \frac{A}{t}+\frac{B}{t-2}+\frac{Ct+D}{t^2+2t+4} =$$

$$= \frac{A(t-2)\,(t^2+2t+4)+Bt\,(t^2+2t+4)+(Ct+D)\,(t^2-2t)}{t^4-8t} =$$

$$= \frac{(A+B+C)t^3+(2B-2C+D)t^2+(4B-2D)t-8A}{t^4-8t}$$

$$\left.\begin{array}{rrrrrrrcr} A & + & B & + & C & & & = & 0 \\ & & 2B & - & 2C & + & D & = & 8 \\ & & 4B & & & - & 2D & = & 0 \\ -8A & & & & & & & = & 8 \end{array}\right\} \implies \begin{cases} A = & -1 \\ B = & \dfrac{5}{3} \\ C = & -\dfrac{2}{3} \\ D = & \dfrac{10}{3} \end{cases}$$

$$\int \left(t+\frac{8t^2+8}{t^4-8t}\right)dt = \int \left(t-\frac{1}{t}+\frac{5/3}{t-2}+\frac{(-2/3)t+(10/3)}{t^2+2t+4}\right)dt =$$

$$= \int \left(t-\frac{1}{t}+\frac{5}{3}\frac{1}{t-2}-\frac{1}{3}\frac{2t-10}{t^2+2t+4}\right)dt =$$

$$= \int \left(t-\frac{1}{t}+\frac{5}{3}\frac{1}{t-2}-\frac{1}{3}\frac{2t+2}{t^2+2t+4}+\frac{1}{3}\frac{12}{t^2+2t+4}\right)dt =$$

$$= \frac{t^2}{2}-\mathrm{Ln}(|t|)+\frac{5}{3}\mathrm{Ln}(|t-2|)-\frac{1}{3}\mathrm{Ln}(|t^2+2t+4|)+4\int \frac{1}{(t+1)^2+3}dt =$$

$$= \frac{t^2}{2}-\mathrm{Ln}(|t|)+\frac{5}{3}\mathrm{Ln}(|t-2|)-\frac{1}{3}\mathrm{Ln}(|t^2+2t+4|)+4\sqrt{3}\arctan\left(\frac{t+1}{\sqrt{3}}\right)+C =$$

$$= \frac{e^{2x}}{2}-x+\frac{5}{3}\mathrm{Ln}(|e^x-2|)-\frac{1}{3}\mathrm{Ln}(|e^{2x}+2e^x+4|)+4\sqrt{3}\arctan\left(\frac{e^x+1}{\sqrt{3}}\right)+C$$

PROBLEMA 24

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$$

Solución:

Una forma de resolver esta integral es haciendo el cambio $x = t^4$, con lo que $dx = 4t^3dt$:

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx = \int \frac{\sqrt[3]{1+t}}{t^2}4t^3dt = 4\int t\sqrt[3]{1+t}\,dt = 4\int t(1+t)^{1/3}dt =$$

$$= \left\{ \begin{array}{lcl} u = t & \longrightarrow & du = dt \\ dv = (1+t)^{1/3}dt & \longrightarrow & v = \frac{3}{4}(1+t)^{4/3} \end{array} \right\} =$$

$$= 4\left(\frac{3}{4}t(1+t)^{4/3} - \frac{3}{4}\int (1+t)^{4/3}dt\right) = 3t(1+t)^{4/3} - \frac{9}{7}(1+t)^{7/3} + C =$$

$$= 3\sqrt[4]{x}(1+\sqrt[4]{x})^{4/3} - \frac{9}{7}(1+\sqrt[4]{x})^{7/3} + C$$

Otra forma de resolver la integral en la variable t sería mediante el cambio de variable $1+t=\omega^3$, con lo que $dt = 3\omega^2 d\omega$:

$$4\int t\sqrt[3]{1+t}\,dt = \int 4(\omega^3-1)\omega 3\omega^2 d\omega =$$

$$= 12\int (\omega^6 - \omega^3)d\omega = 12\left(\frac{\omega^7}{7} - \frac{\omega^4}{4}\right) = 12\left(\frac{(1+t)^{7/3}}{7} - \frac{(1+t)^{4/3}}{4}\right) =$$

$$= 12\left(\frac{(1+\sqrt[4]{x})^{7/3}}{7} - \frac{(1+\sqrt[4]{x})^{4/3}}{4}\right) + C$$

PROBLEMA 25

$$\int \frac{1}{\cos^2(x)\,\text{sen}^6(x)}dx$$

Solución:

Puesto que la función es par a la vez en el seno y el coseno, vamos a utilizar el siguiente cambio de variable:

$$t = \tan(x) \qquad \text{sen}^2(x) = \frac{t^2}{1+t^2} \qquad \cos^2(x) = \frac{1}{1+t^2} \qquad dx = \frac{dt}{1+t^2}$$

Aplicamos el cambio de variable para resolver la integral:

$$\int \frac{1}{\cos^2(x)\,\text{sen}^6(x)}dx = \int \frac{1}{\left(\frac{1}{\cancel{1+t^2}}\right)\left(\frac{t^2}{1+t^2}\right)^3} \frac{dt}{(\cancel{1+t^2})} = \int \frac{\left(1+t^2\right)^3}{t^6}dt =$$

$$= \int \left(\frac{t^6+3t^4+3t^2+1}{t^6}\right)dt = \int \left(1+\frac{3t^4+3t^2+1}{t^6}\right)dt =$$

$$= \int \left(1+3t^{-2}+3t^{-4}+t^{-6}\right)dt = t-3t^{-1}-t^{-3}-\frac{1}{5}t^{-5}+C =$$

$$= \tan(x) - \frac{3}{\tan(x)} - \frac{1}{\left(\tan(x)\right)^3} - \frac{1}{5}\frac{1}{\left(\tan(x)\right)^5} + C$$

PROBLEMA 26

$$\int \frac{1}{7x^2-8}dx$$

Solución:

$$\int \frac{1}{7x^2-8}dx = \int \frac{1}{7\left(x^2-\frac{8}{7}\right)}dx = \frac{1}{7}\int \frac{1}{x^2-\frac{8}{7}}dx =$$

$$= \frac{1}{7}\int \frac{1}{\left(x+\sqrt{\frac{8}{7}}\right)\left(x-\sqrt{\frac{8}{7}}\right)}dx = \frac{1}{7}\int \left(\frac{-\frac{1}{2}\sqrt{\frac{7}{8}}}{x+\sqrt{\frac{8}{7}}} + \frac{\frac{1}{2}\sqrt{\frac{7}{8}}}{x-\sqrt{\frac{8}{7}}}\right)dx =$$

$$= -\frac{\sqrt{7}}{14\sqrt{8}}\mathrm{Ln}\left(\left|x+\sqrt{\frac{8}{7}}\right|\right) + \frac{\sqrt{7}}{14\sqrt{8}}\mathrm{Ln}\left(\left|x-\sqrt{\frac{8}{7}}\right|\right) =$$

$$= \frac{\cancel{\sqrt{7}}}{2\cancel{\sqrt{7}}\sqrt{7}2\sqrt{2}}\mathrm{Ln}\left(\left|\frac{x-\sqrt{\frac{8}{7}}}{x+\sqrt{\frac{8}{7}}}\right|\right) + C = \frac{1}{4\sqrt{14}}\mathrm{Ln}\left(\left|\frac{x-\sqrt{\frac{8}{7}}}{x+\sqrt{\frac{8}{7}}}\right|\right) + C$$

PROBLEMA 27

$$\int \frac{\arctan\left(\frac{x}{2}\right)}{4+x^2}dx$$

Solución:

En este problema vamos a utilizar el siguiente cambio de variable:

$$t = \arctan\left(\frac{x}{2}\right) \Longrightarrow dt = \frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2} = \frac{2}{x^2+4}dx \Longrightarrow dx = \frac{x^2+4}{2}dt$$

Realizamos ahora el cambio en la integral:

$$\int \frac{\arctan\left(\frac{x}{2}\right)}{4+x^2}dx = \int \frac{t}{\cancel{4+x^2}}\frac{\cancel{x^2+4}}{2}dt = \int \frac{t}{2}dt = \frac{t^2}{4} + C = \frac{\left(\arctan\left(\frac{x}{2}\right)\right)^2}{4} + C$$

PROBLEMA 28

$$\int \frac{2x-5}{3x^2-2}dx$$

Solución:

$$\int \frac{2x-5}{3x^2-2}dx = 3 \cdot \frac{1}{3}\int \frac{2x-5}{3x^2-2}dx = \frac{1}{3}\int \frac{6x-15}{3x^2-2}dx =$$

$$= \frac{1}{3}\int \left(\frac{6x}{3x^2-2} - \frac{15}{3x^2-4}\right) dx = \frac{1}{3}\mathrm{Ln}\left(\left|3x^2-2\right|\right) - 5\int \frac{1}{(3x^2-2)}dx =$$

$$= \frac{1}{3}\mathrm{Ln}\left(\left|3x^2-2\right|\right) - 5\int \frac{1}{3\left(x+\sqrt{\frac{2}{3}}\right)\left(x-\sqrt{\frac{2}{3}}\right)}dx =$$

$$= \frac{1}{3}\mathrm{Ln}\left(\left|3x^2-2\right|\right) - \frac{5}{3}\int \left(\frac{-\frac{1}{2}\sqrt{\frac{3}{2}}}{x+\sqrt{\frac{2}{3}}} + \frac{\frac{1}{2}\sqrt{\frac{3}{2}}}{x-\sqrt{2\frac{2}{3}}}\right)dx =$$

$$= \frac{1}{3}\mathrm{Ln}\left(\left|3x^2-2\right|\right) - \frac{5}{6}\sqrt{\frac{3}{2}}\mathrm{Ln}\left(\left|x+\sqrt{\frac{2}{3}}\right|\right) + \frac{5}{6}\sqrt{\frac{3}{2}}\mathrm{Ln}\left(\left|x-\sqrt{\frac{2}{3}}\right|\right) =$$

$$= \frac{1}{3}\mathrm{Ln}\left(\left|3x^2-2\right|\right) - \frac{5}{6}\sqrt{\frac{3}{2}}\mathrm{Ln}\left(\left|\frac{x-\sqrt{\frac{2}{3}}}{x+\sqrt{\frac{2}{3}}}\right|\right) + C$$

PROBLEMA 29

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}dx$$

Solución:

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}dx = \left\{\begin{array}{l} t = e^x+1 \\ dt = e^x dx \end{array}\right\} = \int \frac{(t-1)^{\cancel{2}}}{\sqrt{t}}\,\frac{dt}{\cancel{t-1}} = \int \frac{t-1}{t^{1/2}}dt =$$

$$= \int \left(t^{1/2} - t^{-1/2}\right)dt = \frac{2}{3}t^{3/2} - 2t^{1/2} + C = \frac{2}{3}\left(e^x+1\right)^{3/2} - 2\left(e^x+1\right)^{1/2} + C =$$

$$= \frac{2}{3}\left(e^x+1\right)\sqrt{e^x+1} - 2\sqrt{e^x+1} + C = \frac{2}{3}\left(e^x-2\right)\sqrt{e^x+1} + C$$

PROBLEMA 30

$$\int x(3x+1)^7 dx$$

Solución:

$$\int x(3x+1)^7 dx = \left\{ \begin{array}{l} t = 3x+1 \\ dt = 3dx \end{array} \right\} = \int \frac{t-1}{3} t^7 \frac{dt}{3} =$$

$$= \frac{1}{9}\int \left(t^8 - t^7\right) dt = \frac{t^9}{81} - \frac{t^8}{72} + C = \frac{(3x+1)^9}{81} - \frac{(3x+1)^8}{72} + C$$

PROBLEMA 31

$$\int \cos^3(x) dx$$

Solución:

$$\int \cos^3(x) dx = \left\{ \begin{array}{l} t = \operatorname{sen}(x) \\ dt = \cos(x) dx \end{array} \right\} = \int \cos^2(x)\cos(x) dx = \int \cos^2(x) dt =$$

$$= \int \left(1 - \operatorname{sen}^2(x)\right) dt = \int \left(1 - t^2\right) dt = t - \frac{t^3}{3} + C = \operatorname{sen}(x) - \frac{\operatorname{sen}^3(x)}{3} + C$$

Alternativamente, la integral también se puede resolver de la siguiente manera sin necesidad de realizar el cambio de variable:

$$\int \cos^3(x) dx = \int \cos(x)\cos^2(x) dx = \int \cos(x)\left(1 - \operatorname{sen}^2(x)\right) dx =$$

$$= \int \cos(x) - \int \cos(x)\operatorname{sen}^2(x) dx = \operatorname{sen}(x) - \frac{\operatorname{sen}^3(x)}{3} + C$$

PROBLEMA 32

$$\int e^x \cos 2x \, dx$$

Solución:

$$I = \int e^x \cos(2x) dx = \left\{ \begin{array}{lll} u = \cos(2x) & \longrightarrow & du = -2\,\text{sen}(2x) \\ dv = e^x dx & \longrightarrow & v = e^x \end{array} \right\} =$$

$$= e^x \cos(2x) + 2\int e^x \,\text{sen}(2x) dx = \left\{ \begin{array}{lll} u = \text{sen}(2x) & \longrightarrow & du = 2\cos(2x) dx \\ dv = e^x dx & \longrightarrow & v = e^x \end{array} \right\} =$$

$$= e^x \cos(2x) + 2\left(e^x \,\text{sen}(2x) - 2\int e^x \cos(2x) dx \right) =$$

$$= e^x \cos(2x) + 2e^x \,\text{sen}(2x) - 4\int e^x \cos(2x) dx =$$

$$= e^x \cos(2x) + 2e^x \,\text{sen}(2x) - 4I \implies$$

$$\implies 5I = e^x \cos(2x) + 2e^x \,\text{sen}(2x) \implies$$

$$\implies I = \frac{1}{5} e^x \big(\cos(2x) + 2\,\text{sen}(2x) \big) + C$$

PROBLEMA 33

$$\int x\sqrt{1+x}\, dx$$

Solución:

$$\int x\sqrt{1+x}\, dx = \left\{ \begin{array}{lll} u = x & \longrightarrow & du = dx \\ dv = (1+x)^{1/2} dx & \longrightarrow & v = \dfrac{2}{3}(1+x)^{3/2} \end{array} \right\} =$$

$$= \frac{2}{3} x(1+x)^{3/2} - \frac{2}{3}\int (1+x)^{3/2} dx = \frac{2}{3} x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + C$$

PROBLEMA 34

$$\int \frac{x^2}{\sqrt{1+x}} dx$$

Solución:

$$\int \frac{x^2}{\sqrt{1+x}} dx = \int x^2(1+x)^{-1/2} = \left\{ \begin{array}{lll} u = x^2 & \longrightarrow & du = 2x\,dx \\ dv = (1+x)^{-1/2}dx & \longrightarrow & v = 2(1+x)^{1/2} \end{array} \right\} =$$

$$= 2x^2(1+x)^{1/2} - 4\int x(1+x)^{1/2}dx = \left\{ \begin{array}{lll} u = x & \longrightarrow & du = dx \\ dv = (1+x)^{1/2}dx & \longrightarrow & v = \frac{2}{3}(1+x)^{3/2} \end{array} \right\} =$$

$$= 2x^2(1+x)^{1/2} - 4\left(\frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\int (1+x)^{3/2}dx\right) =$$

$$= 2x^2(1+x)^{1/2} - \frac{8}{3}x(1+x)^{3/2} + \frac{16}{15}(1+x)^{5/2} + C =$$

$$= 2x^2\sqrt{1+x} - \frac{8}{3}x(1+x)\sqrt{1+x} + \frac{16}{15}(1+x)^2\sqrt{1+x} + C =$$

$$= \sqrt{1+x}\left(2x^2 - \frac{8}{3}x - \frac{8}{3}x^2 + \frac{16}{15} + \frac{32}{15}x + \frac{16}{15}x^2\right) + C =$$

$$= \sqrt{1+x}\left(\frac{16}{15} - \frac{8}{15}x + \frac{6}{15}x^2\right) + C = \frac{2}{15}\sqrt{1+x}(8 - 4x + 3x^2) + C$$

PROBLEMA 35

$$\int x\,\mathrm{sen}^2(x)dx$$

Solución:

$$\int x\,\mathrm{sen}^2(x)dx = \int x\left(\frac{1-\cos(2x)}{2}\right)dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos(2x)dx =$$

$$= \left\{ \begin{array}{lll} u = x & \longrightarrow & du = dx \\ dv = \cos(2x)dx & \longrightarrow & v = \frac{1}{2}\mathrm{sen}(2x) \end{array} \right\} = \frac{1}{2}\frac{x^2}{2} - \frac{1}{2}\left(\frac{1}{2}x\,\mathrm{sen}(2x) - \frac{1}{2}\int \mathrm{sen}(2x)dx\right) =$$

$$= \frac{1}{2}\left(\frac{x^2}{2} - \frac{1}{2}x\,\mathrm{sen}(2x) - \frac{1}{2}\cos(2x)\right) + C = \frac{1}{4}\left(x^2 - x\,\mathrm{sen}(2x) - \cos(2x)\right) + C$$

PROBLEMA 36

$$\int \frac{1}{4x^2+4x+2}dx$$

Solución:

$$\int \frac{1}{4x^2+4x+2}dx = \frac{1}{2}\int \frac{1}{2x^2+2x+1}dx = \frac{1}{2}\int \frac{1}{2\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right)}dx =$$

$$= \frac{1}{2}\int \frac{1}{2\cdot\frac{1}{4}\left(4\left(x+\frac{1}{2}\right)^2+1\right)}dx = \frac{1}{2}\int \frac{2}{\left(2\left(x+\frac{1}{2}\right)\right)^2+1}dx = \frac{1}{2}\arctan(2x+1)+C$$

PROBLEMA 37

$$\int \frac{1}{\sqrt{(4-x^2)^3}}dx$$

Solución:

Vamos a resolver este problema mediante el siguiente cambio de variable trigonométrico:

$$x = 2\operatorname{sen}(t) \qquad \sqrt{4-x^2} = 2\cos(t) \qquad dx = 2\cos(t)dt$$

Aplicamos a continuación dicho cambio de variable:

$$\int \frac{1}{\sqrt{(4-x^2)^3}}dx = \int \frac{2\cos(t)}{\sqrt{(4-4\operatorname{sen}^2(t))^3}}dt = \int \frac{2\cos(t)}{\sqrt{(4\cos^2(t))^3}}dt =$$

$$= \int \frac{2\cos(t)}{8\cos^3(t)}dt = \frac{1}{4}\int \frac{1}{\cos^2(t)dt}dt$$

Realizamos a continuación el siguiente cambio:

$$\omega = \tan(t) \qquad d\omega = \frac{1}{\cos^2(t)}dt \quad \cos^2(t) = \frac{1}{1+\omega^2} \qquad \operatorname{sen}^2(t) = \frac{\omega^2}{1+\omega^2}$$

Continuamos ahora con la nueva variable:

$$\frac{1}{4}\int \frac{1}{\cos^2(t)dt}dt = \frac{1}{4}\int d\omega = \frac{1}{4}\omega \neq C = \frac{1}{4}\tan(t)+C =$$

$$\frac{1}{4}\tan\left(\operatorname{arc\,sen}\left(\frac{x}{2}\right)\right) = \frac{1}{4}\frac{x}{\sqrt{4-x^2}}+C$$

PROBLEMA 38

$$\int \frac{\operatorname{sen}(x)}{\cos(x)\left(1+\cos^2(x)\right)} dx$$

Solución:

$$\int \frac{\operatorname{sen}(x)}{\cos(x)\left(1+\cos^2(x)\right)} dx = -\mathrm{Ln}\left(|\cos(x)|\right) + \frac{1}{2}\mathrm{Ln}\left(\left|1+\cos^2(x)\right|\right) + C = \mathrm{Ln}\left(\frac{\sqrt{1+\cos^2(x)}}{|\cos(x)|}\right) + C$$

Sugerencia: Realiza el cambio de variable $t = \cos(x)$.

PROBLEMA 39

$$\int \frac{1}{2-\cos(x)} dx$$

Solución:

$$\int \frac{1}{2-\cos(x)} dx = \frac{2}{\sqrt{3}} \arctan\left(\sqrt{3}\tan\left(\frac{x}{2}\right)\right) + C$$

Sugerencia: Realiza el cambio de variable $t = \tan\left(\frac{x}{2}\right)$.

PROBLEMA 40

$$\int \frac{\cos(x)}{1+\cos(x)} dx$$

Solución:

$$\int \frac{\cos(x)}{1+\cos(x)} dx = x - \tan\left(\frac{x}{2}\right) + C$$

Sugerencia: Realiza el cambio de variable $t = \tan\left(\frac{x}{2}\right)$.

PROBLEMA 41

$$\int \frac{1}{\cos(x) + 2\,\mathrm{sen}(x) + 3} dx$$

Solución:

$$\int \frac{1}{\cos(x) + 2\,\mathrm{sen}(x) + 3} = \arctan\left(\tan\left(\frac{x}{2}\right) + 1\right) + C$$

Sugerencia: Realiza el cambio de variable $t = \tan\left(\frac{x}{2}\right)$.

PROBLEMA 42

$$\int \frac{1 - \sqrt{3x+2}}{1 + \sqrt{3x+2}} dx$$

Solución:

$$\int \frac{1 - \sqrt{3x+2}}{1 + \sqrt{3x+2}} dx = -\frac{4}{3}\mathrm{Ln}\left(\left|\sqrt{3x+2} + 1\right|\right) + C$$

Sugerencia: Realiza el cambio de variable $\sqrt{3x+2} = t$.

PROBLEMA 43

$$\int \frac{1}{\sqrt{x+1} + \sqrt[4]{x+1}} dx$$

Solución:

$$\int \frac{1}{\sqrt{x+1} + \sqrt[4]{x+1}} dx = 2\sqrt{x+1} - 4\sqrt[4]{x+1} + 4\mathrm{Ln}\left(\left|\sqrt[4]{x+1} + 1\right|\right) + C$$

Sugerencia: Realiza el cambio de variable $x + 1 = t^4$.

PROBLEMA 44

$$\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx$$

Solución:

$$\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx =$$

$$= 2\text{Ln}\left(\left|\sqrt{x+1}-1\right|\right) - \text{Ln}\left(\left|x+2+\sqrt{x+1}\right|\right) - \frac{2}{\sqrt{3}}\arctan\left(\frac{2\sqrt{x+1}+1}{\sqrt{3}}\right) + C$$

Sugerencia: Realiza el cambio de variable $\sqrt{x+1} = t$.

PROBLEMA 45

$$\int \frac{\sqrt[3]{x+1}}{x}dx$$

Solución:

$$\int \frac{\sqrt[3]{x+1}}{x}dx = 3\sqrt[3]{x+1} - \text{Ln}\left(\left|\sqrt[3]{x+1}-1\right|\right) +$$

$$+\frac{1}{2}\text{Ln}\left(\left|\sqrt[3]{(x+1)^2}+\sqrt[3]{(x+1)}+1\right|\right) + \sqrt{3}\arctan\left(\frac{2\sqrt[3]{x+1}+1}{\sqrt{3}}\right) + C$$

Sugerencia: Realiza el cambio de variable $x+1 = t^3$.