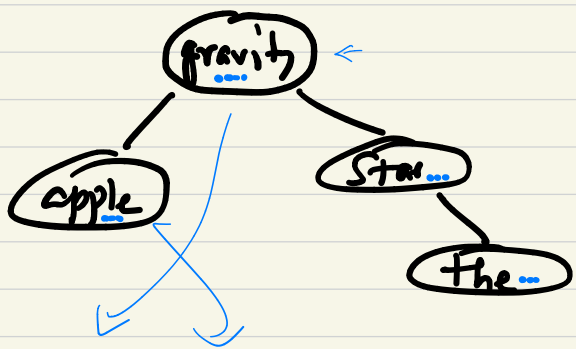
Lecture 04 Date : 11 Jan 24

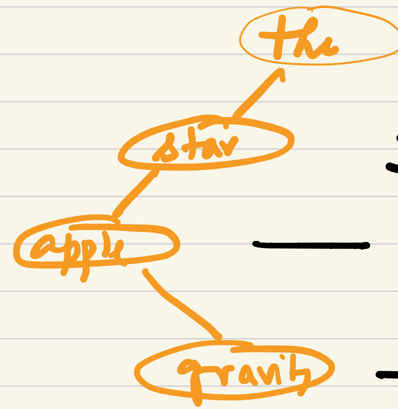
Optimal Binary Search Tree

keys: k[1... n] $\quad$ a_4 the, a_1 apple, a_3 star, a_2 gravity.

frequencies f[1... n] $\quad$ 1 2 3 6



$6 + 2\times 2 + 3\times 2 + 1\times 3 = 19$



1×1

3×2

2×3

6×4

37

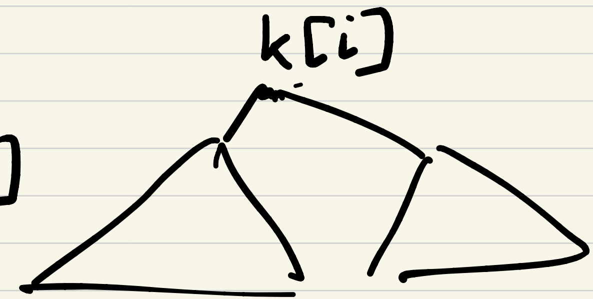
$f[1 \cdots n]$, $k[1 \cdots n]$ $k[1] < k[2] < \cdots < k[n]$

$f[i]$ is the freq of $k[i]$.

Let T be a BST for $k[1..n]$. with root $k[r]$

$\text{cost}(T, f[1 \ldots n])$

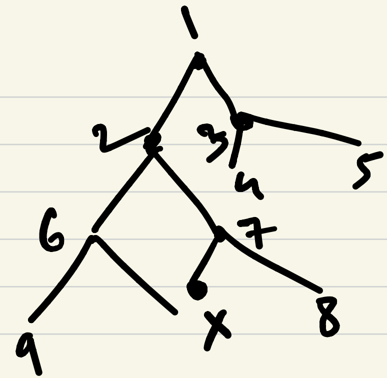
$$= \sum_{j=1}^{n} f[j] \cdot \# \text{ancestors of } k[j]$$



$$= \boxed{\sum_{j=1}^{r-1} f[j] \cdot \#\text{anc of } k[j]} \leftarrow \text{left subtree}$$

$$+ f[r] + \boxed{\sum_{j=r+1}^{n} f[j] \cdot \#\text{anc. of } k[j]}$$

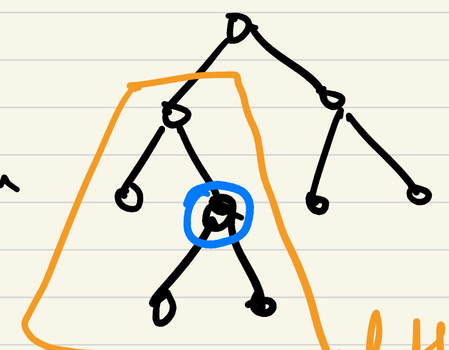
right subtree.



1, 2, 7, x

$$= f[r] + \sum_{j=1}^{r-1} f[j] \left(\#\text{anc of } k[j] \text{ in left}(T) + 1\right)$$

$$+ \sum_{j=r+1}^{n} f[j] \left(\#\text{anc. of } k[j] \text{ in right}(T) + 1\right)$$



left(T)

$$= \sum_{i=1}^{n} f[i] + \sum_{j=1}^{r-1} f[j]\,(\#\text{ anc. of } k[j] \text{ in } left(T)) + \sum_{j=r+1}^{n} f[j]\,(\#\text{ anc of } k[j] \text{ in } rt(T))$$

$$= \sum_{i=1}^{n} f[i] + cost(left(T), f[1 \ldots r-1]) + cost(right(T), f[r+1, \ldots, n])$$

$cost(i, j) :=$ cost of an optimal BST for $f[i \ldots j]$

$$\mathrm{cost}(i,j) = \begin{cases} 0 & i > j \\ \sum_{l=i}^{j} f[l] + \min_{i \le k \le j} \left(\mathrm{cost}(i,k-1) + \mathrm{cost}(k+1,j) \right) \end{cases}$$

$$\mathrm{cost}(i,i) = \sum_{l=i}^{i} f[l] + \underbrace{\mathrm{cost}(i,i-1)}_{0} + \underbrace{\mathrm{cost}(i+1,i)}_{0}$$

$$= f[i]$$

cost(i, j)

if $i > j$:

then return 0 $\leftarrow$

$\triangleright$ $q \leftarrow \sum_{l=i}^{j} f[l]$

$v \leftarrow \infty$ $\triangleright$ v is storing cost(i,j)

for $k \leftarrow i$ to j

do if $q + \text{cost}(i, k-1) + \text{cost}(k+1, j) < v$

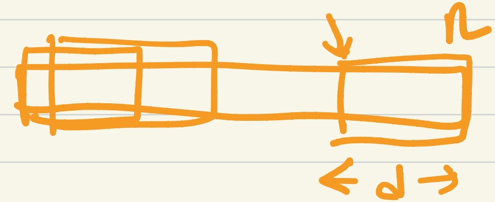
then $v \leftarrow q + \text{cost}(i, k-1) + \text{cost}(k+1, j)$

return v

cost(27, 5)

X

$O(n)$ | **for** $i \leftarrow 1$ **to** n.
do $cost[i, i-1] \leftarrow 0$.



① **for** $d \leftarrow 1$ **to** $n \longrightarrow n$

② **do for** $i \leftarrow 1$ **to** $n - d + 1 \rightarrow n$

do $j \leftarrow i + d - 1$

$\triangleright\ q \leftarrow \sum_{l=i}^{j} f[l] \longrightarrow n$

$v \leftarrow \infty$

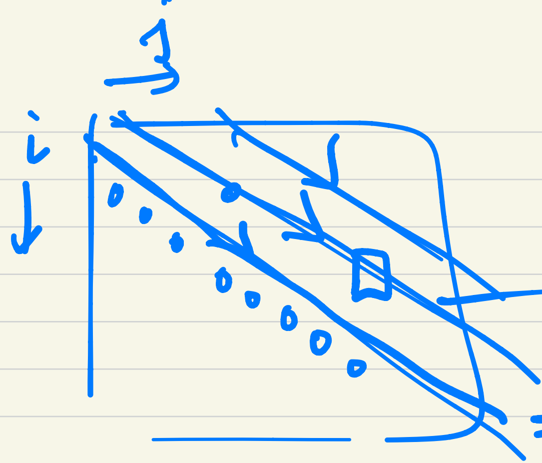
③ **for** $k \leftarrow i$ **to** $j \longrightarrow n$

do if $q + cost(i, k-1) + cost(k+1, j) < v$

then $v \leftarrow q + cost(i, k-1) + cost(k+1, v)$

$cost[i,j]$

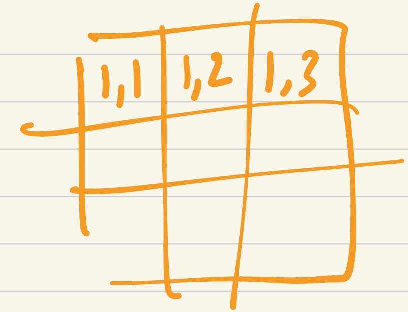
$O(n^3)$

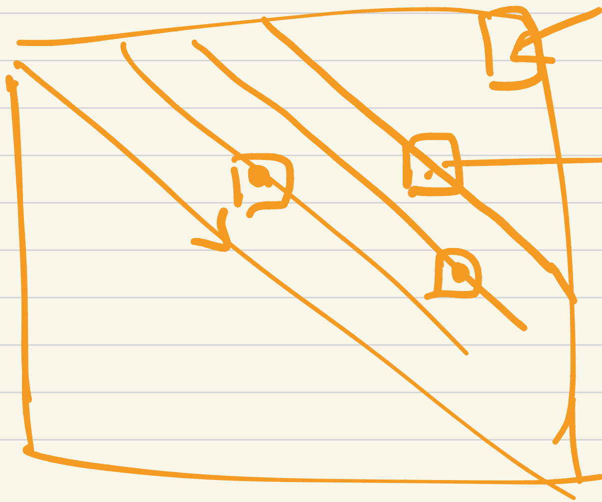


$i, i+1$

$d=2$

$=(k=i) \Leftrightarrow d=1$





$(i, i+d-1)$

$\boxed{(i, k-1)}$ $(k+1, i+d-1)$