MA395

Essay

Structural Foundations in Topoi

2024-04-25

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1 Introduction

Suppose you were asked, "is $3 \in \mathbb{N}$?" Being a natural number, 3 is indeed a member of $\mathbb{N}$, so the answer is "yes". A little tricker is the question, "is $0 \in \mathbb{N}$?", but there is at least a meaningful and unambiguous answer as long as we are clear about the meaning of the symbol $\mathbb{N}$.* On the other hand, the question, "is $\pi \in \mathbb{Q}$?", would quickly receive an answer of "no".

Now, suppose you were then asked, "is $\pi \in \log$?"

You'd might pause for a moment, before again answering in the negative, but for a different reason than before. After all, π is a number, and log is a function, so π being a member of log – whatever that means – would be ridiculous! A better answer might be to declare the question as meaningless [Lei14].

This illustrates the intuitive notion of *type*, which may be particularly familiar to programmers. Many programming languages (called *strongly typed* languages) require you to declare the type of a variable before using it, with the idea being that strictly enforcing the type of every variable stops the programmer from performing nonsensical operations like adding an `int` to a `bool`, or trying to divide by a `string`.

However, in the standard foundational framework of ZFC, Zermelo–Fraenkel set theory with Choice – the "assembly language" of mathematics, if you will – *everything* is a set, so the question "is $\pi \in \log$?" should have a yes-or-no answer.

For instance, in ZFC, membership is a global relation, so it is always a valid question to ask whether any two arbitrary objects are members of each other, even if the answer is entirely meaningless. Because of this, the way ZFC uses the word "set" is very different from what mathematicians usually mean when they say "set". In ZFC, π is a set, as is log – but ask any mathematician to list the elements of π or log, and you will likely have difficulties in receiving an answer.

The benefit of this style of axiomatisation is simplicity – everything is a set, so we don't have to have extra rules to deal with every possible different kind of object. On the other hand, we lose this basic notion of type, because everything is of type `set`. (We say that ZFC is a *single-sorted* theory.)

As we saw above, this isn't always sensible. The axioms of ZFC allow even more nonsensical questions beyond asking whether any two random objects contain each other or are equal, and many of the axioms of ZFC themselves are similarly incomprehensible in ordinary mathematical usage. For instance, the axiom of regularity states that every non-empty set X contains a member $x \in X$ such that $x \cap X = \varnothing$. But, pick any ordinary set, say, $\mathbb{R}$, and the resulting statement is difficult to interpret. What does an expression like $3 \cap \mathbb{R}$ even mean?

One response to this problem might be to say that set theory offers not only a set of axioms, but also a collection of standard encodings of different mathematical objects. We can again compare this situation with computers: down in a hard drive, every file – text, image, audio, video, etc. – is ultimately encoded as bits: as pure combinations of 0s and 1s. So, one could argue that it doesn't matter that these questions don't have meaningful answers, because nobody is claiming that they should; just as how opening a text file with the wrong encoding results in garbled nonsense doesn't stop it from being useful when opened correctly.

But, even if we accept that the encodings are arbitrary, this problem of being able to create meaningless statements goes deeper than just posing questions about set memberships. One enlightening exercise, as posed by Benacerraf [Ben65], is to consider the question, "is $3 \in 17$?"

1.1 Is $3 \in 17$?

Benacerraf describes two children, Johnny and Ernie,† who have learnt mathematics from axiomatic set-theoretic foundations (as opposed to the more commonly preferred method of starting from "counting", which he calls the "vulgar way"), say for instance, ZFC. To introduce the notion of "counting", and other

*We will take the answer to be "yes" in this paper.

†Named in reference to *John von Neumann* and *Ernst Zermelo*.

common uses of natural numbers to these children is simple, as their teachers merely need to point out the common "vulgar" names of set-theoretic constructions they already know. However, there is some choice in the matter here.

Johnny is taught that there is a set, N, which ordinary people call the "(natural) numbers", that is equipped with a well-ordering called the *less-than* relation. Furthermore, this set contains an element that ordinary people refer to as the natural number 0; the empty set, and the *successor* $s(n)$ for any set n is the set $s(n) = n \cup \{n\}$ – so every number n is simply the collection of numbers less than it.*

The normal properties of natural numbers assumed by ordinary people can then be exhibited as concrete theorems for Johnny. While the common "vulgar" explanation of addition, multiplication, exponentiation, etc., are informal recursive definitions, Johnny can concretely define these procedures in terms of the successor operation, so these operations are derivable from this theory. Restricting our focus to finite sets, Johnny can also encode the common notion of counting with *cardinality* – a set has n elements if it can be put in bijective correspondence with the set of natural numbers less than n, and this definition is well-defined as Johnny's first order theory is sufficiently powerful to construct such a correspondence for any finite n.

At this point, Johnny can now communicate with the vulgar, with all the common constructions and usages of numbers fully encoded within his first order theory of sets. Note that all we have done is specify the set N and explain the notions of 0 and successor to Johnny. The laws of arithmetic can be derived from there, as can any other "extramathematical" uses of numbers. For instance, the notion of counting can be similarly encoded with the additional provision of a definition of cardinality. It can be reasonably agreed that this information is both necessary and sufficient to completely characterise the natural numbers for common usage.

1.2 The Isomorphism Problem

Now, Ernie is also provided with a set to be labelled N, a designated element $0 \in N$, and a definition of a successor function, so all the previous statements apply similarly to Ernie. The two are thus equally knowledgeable about the natural numbers and can both prove numerous theorems about them; and in discussion with ordinary people, they are in agreement.

The problems first arise when they consider the statement, "is $3 \in 17$?"

Johnny argues that the statement is true, while Ernie disagrees. Attempts to resolve this by consulting ordinary people are met with nothing but confusion – after all, to ordinary people, numbers are just that – *numbers* – and not sets.

Examining their given information reveals the origin of this discrepancy: by Johnny's definition of a successor function $s(n) = n \cup \{n\}$, every number is the set of numbers less than it, so $17 = \{0,1,2,3,\ldots,15,16\}$; clearly, $3 \in 17$. However, Ernie's successor function is instead defined by $s(n) = \{n\}$,† so $17 = \{16\}$ and 3 is nowhere to be found; clearly, $3 \notin 17$. This isn't the only disagreement between the two systems either.

Johnny claims that a set has n elements if and only if it can be placed in bijection with the set of numbers less than n – and here, Ernie agrees; then, Johnny claims further that a set has n elements if and only if it can be placed in bijection with the number n itself – but for Ernie, every number contains only a single element (apart from zero, which is empty), so their notions of cardinality also disagree.

The source of the disagreements between Johnny and Ernie is obvious – the difference between their successor functions, and by extension, the set N. But what is *not* obvious, is how these disagreements should be reconciled.

Each account of the naturals is equally valid and correct when considered in isolation, with neither one to be preferred over the other. In more modern language, both constructions yield valid models of the Peano axioms – that is, the resulting semirings are isomorphic. So, if we accept Johnny's construction,

*This is the standard von Neumann construction of the naturals.

†This is the historical Zermelo construction of the naturals.

there is no good reason why we shouldn't also accept Ernie's. Moreover, Johnny's and Ernie's accounts really are arbitrary, and there are infinitely many ways to assign sets to numbers – infinitely many choices of N, $0 \in N$, and $s : N \to N$ – that satisfy the Peano axioms.

Of course, we could choose to accept both accounts, and agree that $\big\{\varnothing,\{\varnothing\},\big\{\{\varnothing,\{\varnothing\}\}\big\}\big\} = 3 = \big\{\{\{\varnothing\}\}\big\}$, but this is clearly absurd, so we explore the alternative: at least one of the two accounts is false. We can actually make a stronger statement – at most one of the accounts (out of the infinite possibilities) can be "correct".

The belief that there *is* a true account is called set-theoretic Platonism – this is the idea that there is a particular set of sets somewhere in the universe which is the "real" set of natural numbers, regardless of whether there exists an argument to prove this or not, or even if we can ever find it.

Benacerraf rejects the possibility that there is no such argument, saying, "...if the number 3 is really one set rather than another, it must be possible to give some cogent reason for thinking so; for the position that this is an unknowable truth is hardly tenable. But there seems to be little to choose among the accounts. Relative to our purposes in giving an account of these matters, one will do as well as another, stylistic preferences aside."

This last sentence is at the heart of *structuralism.*

1.3 Structuralism

Mathematics, as mathematicians actually use it, does not demand of the natural numbers that they exist as some specific object, but only that they have the structure we require – when we work with the natural numbers, *we don't care* about the specific construction used; only that they have semiring structure; that they support recursive definitions and induction; that have a canonical embedding into the integers, etc.

In fact, this is how we usually describe and use objects in mathematics. Some basic examples of this are vectors and groups: a vector space is anything that satisfies the vector space axioms; and similarly, a group is anything that satisfies the group axioms. In neither definitions we do we prescribe what the vector space or group itself actually consists of, only requiring that whatever object or objects it is *behaves* in a certain way.

To the structuralist, mathematics is the study of structures independent of the things they are composed of. As seen above, this is the approach taken in many other mathematical contexts, so it is strange that the foundations of mathematics itself are commonly formulated in a way that is distinctly *not* structural in nature. But this does not have to be the case.

In a *structural set theory*, sets are objects that are characterised by their connections to other sets as prescribed by functions or relations – and this is essentially how sets are used in common practice of mathematics. Elements of sets themselves have no identity or internal structure beyond that which is given by functions and relations. In particular, this means that elements are not sets,* and cannot be members of other sets (not in the sense that it is false that they are, but in the sense that it is meaningless to ask whether they are [nLa23a]), so elements of different arbitrary sets are not comparable.

It is meaningless to ask whether $3 = \{\{\{\varnothing\}\}\}$ or not, because it is not asked in the context of the rest of the natural numbers – for instance, we know $2 \neq 3$, because, for example, 2 is strictly less than 3, which is a property of natural numbers; but it seems wrong to argue that $3 \neq \{\{\{\varnothing\}\}\}$ because, say, we know that 3 has three elements (or none, or seventeen, or infinitely many), while $\{\{\{\varnothing\}\}\}$ only has one, because *we don't know this.* The number of elements of 3 isn't a part of the structure of the natural numbers, so we cannot meaningfully deny that $3 = \{\{\{\varnothing\}\}\}$ on the grounds that 3 contains a different number of elements than $\{\{\{\varnothing\}\}\}$.

*Such an element is called a *urelement.* In classical material set theories, we usually have no urelements, as it is possible to embed material set theories with urelements into a version without urelements, simplifying the theory. In structural set theories, however, *every* element is a urelement.

What makes the number 3 the number 3 is exactly its relations to other natural numbers, so structuralism tells us that a more sensible question than "what is 3?" would be "what are *all* the natural numbers?" – or more precisely, "what *structure* is the natural numbers?"

We just saw that the question "is $3 \in 17$?" has a different answer depending on which construction is chosen – Johnny says "yes"; and Ernie, "no" – despite the resulting collections being isomorphic. We say that ZFC is not *isomorphism invariant.*

In contrast, structural definitions are always isomorphism invariant (with respect to the relevant structure). For instance, we structurally characterise a *natural numbers object* as a triple $(N,0,s)$ consisting of a set N, a distinguished element $0 \in N$, and a *successor function* $s : N \to N$. The natural numbers object then expresses natural arithmetic in terms of these components. Importantly, in this characterisation, it doesn't actually matter what the elements of N actually are, only that they carry this structure – the elements themselves are meaningless in isolation.

The structuralist then says that the number "3" is "the third place in a natural numbers object", rather than any particular set like in ZFC. We don't have to argue what set 3 is, because 3 *isn't a set* – it is a relation or structure that some particular objects may exhibit. The statement "is $3 \in 17$?" is then not a well-formed statement, because the $\in$ relation isn't compatible with members of this structure in this way.

In this way, structural set theories are not only isomorphism invariant, but are also free from much of the arbitrary constructions and additional baggage of ZFC that are never actually used in common mathematics.

Mathematicians generally do not appeal to any kind of axioms when doing mathematics – even when working with sets – without any loss of accuracy to their work. In practice, we never actually think of functions or numbers as sets, and that doesn't ever seem to pose a problem. So, it appears that we all subconsciously have a set of operating principles we use for manipulating mathematical constructions that are *good enough* for almost all purposes. The idea is that the axioms of structural foundations are much closer to these intuitions because we formulate our axioms in terms of how we want objects to behave, rather than as a list of rules about which objects exist.

1.4 Primitive Notions

In ZFC, and many other traditional *material* axiomatisations of set theory, the basic primitive notions are of *sets*, *elements*, and *membership*, and everything else is derived from there.

For instance, consider the notion of a *function*. Informally, a function is a special kind of correspondence between pairs of objects, where every given object is assigned exactly one corresponding object by the function. It is also helpful to view a function as an *operation* or as some kind of input-output process that is applied to an object to obtain its associated object (its *image*).

We can represent a function $f : A \to B$ as a *relation* – a subset of $A \times B$ – given by $\hat{f} = \big\{(x,y) : y \text{ is the } f\text{-image of } x\big\}$, where (a,b) is an ordered pair. Conversely, to distinguish which relations $R \subseteq A \times B$ represent functions, we use the property that functions assign exactly one image to each input: if a relation $\hat{f}$ satisfies the property that $(x,y) \in \hat{f}$ and $(x,z) \in \hat{f}$ imply that $y = z$, then $\hat{f}$ is a representation of some function.

This construction encodes our informal notion of a function as a set of ordered pairs satisfying a certain property. The next step is a trick commonly used in mathematics; we drop the distinction between the notion of a function and its abstraction as a set, and we say that this formal representation is itself the *definition* of a function [Gol84].

This definition works well on a technical level, and much theory can be developed with it, but there are several conceptual hurdles that come alongside it. One point of difficulty is with the *codomain* of the function: we can easily define the sets $\mathrm{dom}(f) = \big\{x : \exists y : (x,y) \in f\big\}$ and $\mathrm{im}(f) = \big\{y : \exists x : (x,y) \in f\big\}$, but there is no way to recover the codomain of a function from this definition.

This is not a problem in some branches of mathematics, such as analysis, or even much of set theory. However, in more algebraic or topological areas, this poses some difficulties.

Let $A \subset B$, and consider the functions $\mathrm{id}_A : A \to A$ and $\iota : A \hookrightarrow B$ both defined by $x \mapsto x$. The former is the identity function on A, while the latter is the inclusion of A into B, with the usage of the term "inclusion" indicating that we should view the function as including the elements of A into B. These functions are conceptually very distinct, but they are both the set $\{(x,x) : x \in A\}$.

This is not only a conceptual problem, but a practical one in some cases: if, for instance, we take $A = S^1$ and $B = \mathbb{C}$, then the identity and inclusion maps yield very different induced homomorphisms in first homology.

Even if we patch this definition to specify the information of the codomain separately, this definition still fails to faithfully capture the *dynamic* quality of a function; we often speak of a function *acting* on or being *applied* to an input, and the symbol between the domain and codomain of a function is even an *arrow*! Even further, more specialised functions like some transformations in linear algebra, geometry, group theory, etc. are explicitly described as motions of space. In contrast, the characterisation of a function as a set is inherently static.

Because functions are exactly how sets relate to one another, they are very important in a structural context. In fact, in our structural axiomatisation of set theory, we will instead take sets and *functions* to be our primitive notions, with elements and the membership relation now being derived. This choice of primitive notions lends itself well to be described with the language of category theory.

2 Categories

We briefly state some standard categorical definitions, generally adapting those from [Lei14] (though with some notable differences). For a more introductory and motivated treatment of these definitions, see [Kit22].

Loosely speaking, a *category* consists of a collection of *objects*, with *morphisms* or *arrows* pointing between objects, subject to a couple of axioms pertaining to how morphisms compose. These axioms derive from the properties of function composition, so in many ways, a category is a vast generalisation of sets and set functions.

One of the basic precepts of category theory is that objects have no internal identity; in an arbitrary category, objects are not (necessarily) sets, so it makes no sense to try "look inside" an object. Even if they do happen to be sets, it turns out that looking at the morphisms connecting to that object is sufficient to determine it (almost) uniquely – in this way, category theory is inherently structural, making it well-suited for discussing structural foundations.

Formally, a *category* $\mathscr{C}$ consists of:

- A class $\mathrm{ob}(\mathscr{C})$ of *objects* in $\mathscr{C}$. We often write $A \in \mathscr{C}$ to abbreviate $A \in \mathrm{ob}(\mathscr{C})$.
- For all (ordered) pairs of objects $A,B \in \mathrm{ob}(\mathscr{C})$, a class $\hom_{\mathscr{C}}(A,B)$ of *maps* or *arrows* called *morphisms* from A to B, called the *hom-set* or *hom-class* of morphisms from A to B, also sometimes written $\mathscr{C}(A,B)$ or $\hom(A,B)$ if the ambient category is clear. If $f \in \hom(A,B)$, we write $f : A \to B$ or $A \xrightarrow{f} B$. The union of all of these classes is the hom-set of $\mathscr{C}$, and is written $\hom(\mathscr{C})$.
- For any three objects $A,B,C \in \mathrm{ob}(\mathscr{C})$, a binary operation, $\circ : \hom(A,B) \times \hom(B,C) \to \hom(A,C)$, $(f,g) \mapsto g \circ f$, called *composition*, such that,
  - (*associativity*) if $f : A \to B$, $g : B \to C$, and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$;
  - (*identity*) for every object $X \in \mathrm{ob}(\mathscr{C})$, there exists a morphism $\mathrm{id}_X : X \to X$ called the *identity morphism* on X, such that every morphism $f : A \to X$ satisfies $\mathrm{id}_X \circ f = f$, and every morphism $g : X \to B$ satisfies $g \circ \mathrm{id}_X = g$.

Note that, despite the name, a hom-set is not necessarily a set (under ZFC), and may in fact be a proper class.* If the class of morphisms between any pair of objects does happen to be a set, then $\mathscr{C}$ is *locally small*; and if $\mathrm{ob}(\mathscr{C})$ is also a set, then $\mathscr{C}$ is additionally *small*.

We list a few illustrative examples of categories:

(i) The prototypical example of a category is the category of sets and set functions, **Set**. Identity morphisms are identity functions, and associativity follows from basic properties of set functions.

(ii) The category of groups and group homomorphisms, **Grp**. Every group is a set with extra structure, and every group homomorphism is a set function that happens to preserve this structure, so associativity and identity are inherited from **Set**.

(iii) Similarly, collections of sets with extra structure and maps that preserve that structure generally form categories called *concrete*† categories. For example, the category of:

- Monoids and monoid homomorphisms, **Mon**;
- Rings and ring homomorphisms, **Ring**;
- Metric spaces and non-expansive maps, **Met**;
- Topological spaces and continuous maps, **Top**;
- C^p-manifolds and p-times differentiable maps $\mathbf{Man}^p$;
- Measurable spaces and measurable functions, **Mea**;
- Vector spaces and linear maps over a fixed field K, $\mathbf{Vect}_K$; etc.

(iv) Let $(M, *)$ be a monoid, $\mathrm{ob}(\mathscr{C}) = \{\bullet\}$, and $\hom(\bullet,\bullet) = G$. For any two morphisms f and g, define the composition $f \circ g$ to be $f * g$. Identity and associativity follow from the monoid axioms, so $\mathscr{C}$ is a category. In this way, any monoid can be regarded as a category on a single object.

In particular, we see that the structure of this category is captured entirely within the morphisms, and the object itself is unimportant, so much so that we don't even assign it any characteristics beyond being a featureless point, $\bullet$. Note that this also means this category is *not* concrete, as the object $\bullet$ is not a set.

(v) Let X be any set equipped with a preorder (a reflexive and transitive relation), $\leq$. Let $\mathrm{ob}(\mathscr{C}) = X$, and for any two elements $x,y \in X$, define a unique morphism $f : x \to y$ if and only if $x \leq y$. Reflexivity gives identity morphisms, and transitivity guarantees that compositions exist, so any preordered set can be regarded as a category. More generally, a *thin* or *posetal* category is a category with at most one morphism in every hom-set, so up to isomorphism, every preordered set is a thin category.

(vi) For any topological space X, its fundamental groupoid $\Pi_1(X)$ is a category. Its objects are points in X, and morphisms are homotopy classes of paths, with composition given by path concatenation.

(vii) Any ordinal constructed in the von Neumann style, $n = \{m : m < n\}$, defines a category $\mathbb{n}$ on n objects with morphisms given by set inclusions. (This defines a poset, so this is a specific example of the above preordering categories.)

For instance, $\mathbb{0}$ is the category with zero objects and morphisms (the *empty category*); $\mathbb{1}$ is the category with one object (the *trivial* or *terminal category*); $\mathbb{2}$ is the category with two objects and a single non-identity morphism between them (the *arrow category*), often depicted as $0 \to 1$; etc.

*Given that we are attempting to axiomatise sets themselves, this statement may be somewhat confusing, but for now we will take the word "set" as a declaration of *intent* to form a collection that does not invoke paradoxes or contradictions – no universal or Russell sets. We can get surprisingly far with the intuitive idea of a set as a "bag of featureless dots".

†More properly, a concrete category is a category equipped with a faithful functor to **Set**, but informally, they are categories that "look like **Set** with extra structure".

More generally, $\mathbb{n}$ is the category freely generated by the graph

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$

in the sense that every non-identity morphism can be uniquely factored as a composite of morphisms in the displayed graph [Rie17]. For instance, $0 \subset 3$, so there should be a morphism $0 \to 3$ in this category, but this is just the composition of $0 \to 1$, $1 \to 2$, and $2 \to 3$, which are all displayed in the graph.

(viii) For any cardinal n, we can define a category with n objects and no morphisms aside from the required identities. Such a category is called a *discrete* category.

Note that an *indiscrete* or *codiscrete* category is *not* simply a category that is not discrete, but is instead a category where every hom-set is a singleton (i.e. the category forms a complete digraph).

A category can be interpreted as a *context* or a *universe* in which we perform some kind of mathematics. For instance, group theory is performed within **Grp**, topology within **Top**, differential geometry within $\mathbf{Man}^\infty$, etc. Category theory thus provides a unified language for dealing with all of these different contexts in a uniform manner, and for translating statements and methods between these different disciplines.

Category theory itself was developed from within algebraic topology, where it was used to apply the tools of abstract algebra to topological contexts [Gol84], but has since become a branch of pure mathematics in its own right. In particular, category theory has imparted the lesson of conceptually reframing existing theories in structural, arrow-theoretic terms, which has often proven to be a valuable way to obtain new insights and underlying connections.

The most general context for performing mathematics is the category of sets, **Set**: all mathematical objects can all be translated down into structures on sets (or in the case of material foundations, into sets themselves), so axiomatising the category of sets is one method of formalising alternative foundations of mathematics.

2.1 Diagrams

The structure of a collection of objects and morphisms in a category is often visually represented as a directed graph, called a *diagram*. We have already used $A \to B$ to denote a morphism from A to B, but we can also draw larger diagrams to represent more objects and morphisms. For instance, this diagram depicts 3 objects with morphisms between them:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Note that it is standard to omit identity morphisms from these diagrams to reduce clutter.

Because categories require compositions to exist, following a path through a diagram always gives a valid morphism between the endpoint objects. For instance, there is a path from X to Z that passes through morphisms f and g, so there is a morphism $g \circ f : X \to Z$ in this category. Furthermore, a diagram is *commutative* if for every pair of objects in the diagram, all routes between them are equal. For instance, the diagram above is commutative if and only if $h = g \circ f$. This also justifies the omission of identity morphisms in general diagrams; they don't meaningfully add any additional paths to the diagram.

2.2 Constructing Categories

Given two categories, $\mathscr{C}$ and $\mathscr{D}$, the *product category* $\mathscr{C} \times \mathscr{D}$ is the category with objects $\mathrm{ob}(\mathscr{C} \times \mathscr{D}) = \mathrm{ob}(\mathscr{C}) \times \mathrm{ob}(\mathscr{D})$ and morphisms $\hom_{\mathscr{C}\times\mathscr{D}}((A,B),(A',B')) = \hom_{\mathscr{C}}(A,A') \times \hom_{\mathscr{D}}(B,B')$, with compositions defined componentwise [Mac13]. That is, if $A \xrightarrow{f} A'$ and $B \xrightarrow{g} B'$ are objects and morphisms in categories $\mathscr{C}$ and $\mathscr{D}$ respectively, then we have the objects and morphism $(A,B) \xrightarrow{(f,g)} (A',B')$ in the product category $C \times D$.

Example. Take the arrow category, $\mathbb{2}$, with objects and single non-identity morphism $0 \xrightarrow{f} 1$. The product category $\mathbb{2} \times \mathbb{2}$ can then be represented as the diagram,

$$\begin{tikzcd}
(0,0) \arrow[r, "(\mathrm{id}_0,\, f)"] \arrow[d, "(f,\, \mathrm{id}_0)"'] \arrow[dr, "(f,\, f)" description] & (0,1) \arrow[d, "(f,\, \mathrm{id}_1)"] \\
(1,0) \arrow[r, "(\mathrm{id}_1,\, f)"'] & (1,1)
\end{tikzcd}$$

The diagonal morphism is often omitted from diagrams of this category, replaced by the requirement that the square commutes.

Another way to construct new categories from an existing category is to *dualise* the category. The *dual* or *opposite* category $\mathscr{C}^{\mathrm{op}}$ of a category $\mathscr{C}$ is the category with the same class of objects, but with the domains and codomains of all morphisms interchanged. That is, $\mathrm{ob}(\mathscr{C}) = \mathrm{ob}(\mathscr{C}^{\mathrm{op}})$, and $\hom_{\mathscr{C}}(A,B) = \hom_{\mathscr{C}^{\mathrm{op}}}(B,A)$ for all objects A and B.

More generally, the *principle of duality* states that every categorical definition and theorem has a *dual* definition and theorem, obtained by reversing the direction of all morphisms in the categories involved. Dual notions are often prefixed with *co-*, as in domains and *co*domains.

Theorem 2.1 (Conceptual Duality)**.** *Let Σ be a statement that holds in all categories. Then the dual statement Σ^* holds for all categories.*

Proof. [Bor+94, adapted][Kit22] If Σ holds in a category $\mathscr{C}$, then Σ^* holds in $\mathscr{C}^{\mathrm{op}}$. Every category is the dual of its dual, so Σ^* holds in all categories. ■

There is also a notion of a *subcategory*; given a category $\mathscr{C}$, a category $\mathscr{D}$ is a subcategory of $\mathscr{C}$ if $\mathrm{ob}(\mathscr{D})$ is a subcollection of $\mathrm{ob}(\mathscr{C})$ and $\hom_{\mathscr{D}}(A,B)$ is a subcollection of $\hom_{\mathscr{C}}(A,B)$ for any objects A and B in $\mathscr{D}$. A subcategory is furthermore said to be *full* if for every pair of objects A and B, every morphism $A \to B$ in $\mathscr{C}$ is also in $\mathscr{D}$. That is, a full subcategory $\mathscr{D}$ is a subcollection of objects of $\mathscr{C}$ with all possible morphisms included.

2.3 Morphisms

2.3.1 Isomorphisms

Suppose we have objects A and B and morphisms $f : A \to B$ and $g : B \to A$ such that the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[loop left, "\mathrm{id}_A"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow[loop right, "\mathrm{id}_B"]
\end{tikzcd}$$

That is, $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$, so f and g are *mutually inverse*. Then, we say that f and g are *isomorphisms*, and we alternatively label g by f^{-1}. If an isomorphism between a pair of objects A and B exists, we say that A and B are *isomorphic* and we write $A \cong B$.

Isomorphic objects are, as far as the ambient category is concerned, effectively identical – anything you can say about one object will apply just as well to any other isomorphic object.

In algebra, we often see phrases such as "*the* group S_3", or "*the* (semiring of) naturals", despite the fact that there exists many ostensibly distinct objects to which these names could refer – for instance, the set of isometries that preserve an equilateral triangle and the set of automorphisms on a set of cardinality 3 could both reasonably be labelled "S_3".

This reflects the idea that we often do not know or care about whether two objects are literally equal, but only that they are isomorphic with respect to whatever property we care about. In contrast, in set theory, we sometimes do care about whether two elements of a set are exactly equal or not, since, for example, sets are entirely determined by their elements.

This indicates that for different contexts and types of data, we care about different degrees of likeness. For elements of sets, this notion of likeness is equality. For objects in a category (such as, say, groups in **Grp**), this notion is isomorphism. In general, equality is too strong of a requirement in category theory in the sense that effectively all categorical results still hold if we weaken any requirements of equality to isomorphism – and further still, depending on the foundations used, arbitrary categories may not even admit a notion of equality between objects. Conversely, any purely categorical notion should also not refer to strict equality at all.

2.3.2 Monics and Epics

Consider a morphism $f : A \to B$ in some category $\mathscr{C}$. Suppose that for every pair of parallel morphisms into A

$$X \underset{g_1}{\overset{g_2}{\rightrightarrows}} A \xrightarrow{f} B$$

we have that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ (f is *left-cancellative*). Then, we say that f is a *monomorphism* (or is *monic*), and we write $f : A \rightarrowtail B$.

Monomorphisms generalise injective set functions, and in many categories where objects are structured sets and morphisms are structure preserving set functions, the two notions coincide.

Theorem 2.2. *The monomorphisms in* **Set** *are precisely the injections.*

Proof. Suppose $f : X \to A$ is injective, and let $g_1,g_2 : A \to B$ be functions such that $f \circ g_1 = f \circ g_2$. Then, for any $x \in X$, $f\big(g_1(x)\big) = f\big(g_2(x)\big)$, and by injectivity, $g_1(x) = g_2(x)$, so $g_1 = g_2$ and f is monic.

Now, suppose instead that $f : X \to Y$ is monic. Consider two elements $a,b \in X$ such that $f(a) = f(b)$ and define $g_1,g_2 : \{\bullet\} \to X$ by $g_1(\bullet) = a$ and $g_2(\bullet) = b$. Then, $f \circ g_1 = f \circ g_2$, and since f is monic, $g_1 = g_2$, giving $a = g_1(\bullet) = g_2(\bullet) = b$, and hence f is injective. ∎

Note that we have now characterised injectivity in terms of functions into and out of sets, making no mention of the elements within the set; thus describing injectivity in **Set** structurally.

(Non-empty) injective functions in **Set** are also always left-invertible, hinting at another connection between monics and invertibility.

If $f : A \to B$ is a morphism such that there exists a morphism $s : B \to A$ such that the composite $s \circ f$ is the identity on A, then f is a *split monomorphism*, and we say that s is the *section* of f, and that f is the *retraction* of s.

Note that being a split monomorphism is distinct from being a monomorphism; the former requires having a left-inverse, while the latter requires being left-cancellative, which, in general, are not the same thing. However, we do have:

Theorem 2.3. *Split monomorphisms are monomorphisms.*

Proof. Let $f : A \to B$ be a split monomorphism with left inverse $\ell : B \to A$, so $\ell \circ f = \mathrm{id}_A$, and let $g_1,g_2 : X \to A$ be morphisms such that $f \circ g_1 = f \circ g_2$. Then,

$$\begin{aligned} f \circ g_1 &= f \circ g_2 \\ \ell \circ f \circ g_1 &= \ell \circ f \circ g_2 \\ g_1 &= g_2 \end{aligned}$$

so f is monic. ∎

Note, however, that the converse does not hold in general; not all monomorphisms split, as demonstrated by the empty function in **Set**. For another example, let H be a subgroup of a group G in **Grp**, and consider the inclusion map $\iota : H \hookrightarrow G$. The inclusion map is injective as a set function, so it is monic in **Set**, which is inherited into **Grp**. But, ι has a left inverse if and only if $G \setminus H$ is normal in G, so this monomorphism does not split in general.

Dually, a morphism $f : A \to B$ is then an *epimorphism* (or is *epic*) if it is monic in $\mathscr{C}^{\mathrm{op}}$. That is, if for every pair of parallel morphisms from B

$$A \xrightarrow{f} B \overset{g_2}{\underset{g_1}{\rightrightarrows}} X$$

we have $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ (f is *right-cancellative*), and we write $f : A \twoheadrightarrow B$.

Epimorphisms, like monomorphisms and injections, generalise surjective set functions, and in **Set**, the two notions coincide.

Theorem 2.4. *The epimorphisms in* **Set** *are precisely the surjections.*

Proof. Suppose $f : X \to Y$ is surjective, and let $g_1,g_2 : Y \to A$ be functions such that $g_1 \circ f = g_2 \circ f$. By surjectivity, for every $y \in Y$ there is an $x \in X$ such that $y = f(x)$, so $g_1(y) = g_1\big(f(x)\big) = g_2\big(f(x)\big) = g_2(y)$, so $g_1 = g_2$ and hence f is epic.

Suppose otherwise that $f : X \to Y$ is epic, and define $g_1,g_2 : Y \to 2 = \{\top,\bot\}$ by $g_1(y) = \top$ and

$$g_2(y) = \begin{cases} \top & \exists x \in X : f(x) = y \\ \bot & \text{otherwise} \end{cases}$$

That is, g_2 maps the elements in the image of f to $\top$ and those outside to $\bot$. Then, $g_1 \circ f = g_2 \circ f$ by construction, and as f is epic, we have $g_1 = g_2$ so g_2 is the constant map at $\top$. So, the image of f is equal to Y, and f is surjective. ∎

However, in categories of structured sets, epimorphisms are generally *not* surjective (unlike monomorphisms, which generally *are* injective), so the analogy with surjectivity should be taken less literally with epimorphisms. However, we again have a notion of splitting for epimorphisms:

If $f : A \to B$ is a morphism such that there exists a morphism $s : B \to A$ such that the composite $f \circ s$ is the identity on B, then f is a *split epimorphism*, and we say that s is the *section* of f, and that f is the *retraction* of s.

Corollary 2.4.1. *Split epimorphisms are epimorphisms.*

Proof. Dual of previous theorem. ∎

Again, the converse does not hold in general; not all epimorphisms will split in an arbitrary category.

The previous two results together imply the following:

Theorem 2.5. *Isomorphisms are monic and epic.*

In certain categories, such as **Set**, every morphism that is simultaneously monic and epic (a *bimorphism*) is an isomorphism, and such a category is called *balanced*, but in general, the converse of this result does not hold. For instance, consider the inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ in **Ring**.

The inclusion map is injective, so it is monic in **Set**, which is inherited into **Ring**. Now, consider two maps f and g from $\mathbb{Q}$ to some ring R:

$$\mathbb{Z} \overset{\iota}{\hookrightarrow} \mathbb{Q} \overset{f}{\underset{g}{\rightrightarrows}} R$$

Because every rational $\frac{a}{b} \in \mathbb{Q}$ can be written as the product $a \cdot b^{-1}$ of an integer and a multiplicative inverse of an integer, the ring homomorphism f must map the rational $\frac{a}{b}$ to,

$$f\left(\frac{a}{b}\right) = f(a) \cdot f(b)^{-1}$$

and similarly for g, so if f and g agree over the integers – that is, if $f \circ \iota = g \circ \iota$ – then they are equal everywhere, and hence ι is epic. So, ι is a bimorphism, but is clearly not an isomorphism, so **Ring** is not a balanced category.

2.4 Functors

A *functor*, $F : \mathscr{C} \to \mathscr{D}$ between categories $\mathscr{C}$ and $\mathscr{D}$, consists of a mapping on objects and a mapping on morphisms, such that,

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object X in $\mathscr{C}$;
- $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in $\hom(\mathscr{C})$.

That is, the functor preserves identity morphisms and composition of morphisms. Equivalently, for every pair of objects $A,B \in \mathrm{ob}(\mathscr{C})$, the functor F induces a mapping $F_{A,B} : \hom_{\mathscr{C}}(A,B) \to \hom_{\mathscr{D}}(F(A),F(B))$ that respects the structure of the categories. If this induced function is surjective, then F is *full*; if it is injective, then F is *faithful*; and if it is bijective, then F is *fully faithful*. If a fully faithful functor $F : \mathscr{C} \to \mathscr{D}$ is additionally injective on isomorphism classes, it is called an *embedding*, and F is said to *embed* $\mathscr{C}$ into $\mathscr{D}$.

Functors encapsulate the idea that categorical constructions should also tell you what to do with mappings.

Example (Hom-Functor). Let $\mathscr{C}$ be locally small, and fix any object $A \in \mathrm{ob}(\mathscr{C})$. For any object $X \in \mathrm{ob}(\mathscr{C})$, we can form the set of morphisms $X \to A$, which is exactly $\hom_{\mathscr{C}}(X,A)$. Note that because $\mathscr{C}$ is locally small, this hom-set is a set and not a proper class, so it can be viewed as some element of **Set**.

This assignment of hom-sets to objects is functorial; there is a functor $\hom(A,-)$, also denoted h_A, that sends every object X to the hom-set $h_A(X) = \hom_{\mathscr{C}}(A,X)$. This functor sends every morphism $f : X \to Y$ to the function $\hom(A,f) : \hom_{\mathscr{C}}(A,X) \to \hom_{\mathscr{C}}(A,Y)$, also denoted $h_A(f)$, defined by mapping every $g \in \hom_{\mathscr{C}}(A,X)$ to its postcomposition by f to obtain $f \circ g \in \hom_{\mathscr{C}}(A,Y)$, thus defining the *covariant hom-functor*.

A *covariant* functor from $\mathscr{C}$ to $\mathscr{D}$ is simply a functor $\mathscr{C} \to \mathscr{D}$. In contrast, a *contravariant* functor from $\mathscr{C}$ to $\mathscr{D}$ is a functor $\mathscr{C} \to \mathscr{D}^{\mathrm{op}}$ (or equivalently, $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$). That is, a contravariant functor reverses the direction of morphisms. This happens naturally in some constructions (particularly in many topological constructions involving preimages), and in those cases, it is easier to say that a functor is contravariant than to start appending "$^{\mathrm{op}}$" to half the categories involved.

Contravariant functors with codomain **Set** are common enough that they have their own name: a *presheaf* on a category $\mathscr{C}$ is a functor $\mathscr{C}^{\mathrm{op}} \to$ **Set**, the name deriving from the notion of presheaves on topological spaces.

Example (Hom-Functor). Again, let $\mathscr{C}$ be locally small, and fix any object $A \in \mathrm{ob}(\mathscr{C})$. For any other object X, we can similarly form the set of morphisms $A \to X$, which is exactly $\hom_{\mathscr{C}}(A,X) \in \mathrm{ob}(\mathbf{Set})$.

This assignment of hom-sets to objects is again functorial; the functor $\hom(-,A)$, also denoted h^A, sends every object X to the hom-set $h^A(X) = \hom_{\mathscr{C}}(X,A)$. Let $f : X \to Y$ be a morphism in $\mathscr{C}$. Unlike in the covariant case, there is no natural way to construct an induced function $h^A(f) = \hom(f,A) : \hom_{\mathscr{C}}(X,A) \to \hom_{\mathscr{C}}(Y,A)$, but we can easily construct one in the opposite direction by mapping every morphism $g \in \hom_{\mathscr{C}}(Y,A)$ to its precomposition by f to obtain $g \circ f \in \hom_{\mathscr{C}}(X,A)$. Thus, this construction sends objects in $\mathscr{C}$ to **Set** while reversing all morphisms, defining the *contravariant* hom-functor $h^A : \mathscr{C}^{\mathrm{op}} \to$ **Set**.

Example (Hom-Bifunctor). The notation hom(A,−) and hom(−,B) in the previous examples suggests that there might be a functor hom(−,−) that sends (ordered) pairs of objects to the hom-set between them. In order for this construction to be functorial, we also need to map the pairs of morphisms between these pairs of objects to some function between the hom-sets. That is, if $f : X \to Y$ and $h : B \to A$ are morphisms, then there should be an induced function $\hom(h,f) : \hom(A,X) \to \hom(B,Y)$, noting that the first argument is reversed due to contravariance. By alternatively fixing each component of the functor, we can construct the following square:

$$\begin{tikzcd} \hom(A,X) \arrow[r, "\hom(h{,}X)"] \arrow[d, "\hom(A{,}f)"'] & \hom(B,X) \arrow[d, "\hom(B{,}f)"] \\ \hom(A,Y) \arrow[r, "\hom(h{,}Y)"'] & \hom(B,Y) \end{tikzcd}$$

We will follow how a morphism $g \in \hom(A,X)$ is mapped under this square along the two different paths, in a technique called *diagram chasing*. The vertical arrows are the covariant hom-functors that precompose their inputs by f, and the horizontal arrows are the contravariant hom-functors that postcompose their inputs by h.

So, along the upper path, we have $g \mapsto g \circ h \mapsto f \circ (g \circ h)$, and along the lower path, we have $g \mapsto f \circ g \mapsto (f \circ g) \circ h$. But by the associativity of composition, these paths are equal, so the diagram commutes for all f, g, and h; there is a unique morphism from $\hom(A,X)$ to $\hom(B,Y)$ induced by h and f:

$$\begin{tikzcd} \hom(A,X) \arrow[r, "\hom(h{,}X)"] \arrow[d, "\hom(A{,}f)"'] \arrow[dr, dashed, "\hom(h{,}f)" description] & \hom(B,X) \arrow[d, "\hom(B{,}f)"] \\ \hom(A,Y) \arrow[r, "\hom(h{,}Y)"'] & \hom(B,Y) \end{tikzcd}$$

which is exactly the statement that the function $\hom(h,f)$ is well-defined. So, hom(−,−) is indeed a valid functor $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$. Because this functor takes objects and morphisms from a product category, it is also called a *bifunctor*. The ordinary covariant and contravariant hom-functors are then just the partial applications of this bifunctor in the first and second arguments, respectively.

Theorem 2.6. *Functors preserve commutative diagrams.*

Proof. [Kit22] Because functors preserve compositions, for any paths $a_1 \circ a_2 \circ \cdots \circ a_n$ and $b_1 \circ b_2 \circ \cdots \circ b_m$ connecting a pair of objects in a commutative diagram, we have,

$$\begin{aligned} F(a_1) \circ F(a_2) \circ \cdots \circ F(a_n) &= F(a_1 \circ a_2 \circ \cdots \circ a_n) \\ &= F(b_1 \circ b_2 \circ \cdots \circ b_m) \\ &= F(b_1) \circ F(b_2) \circ \cdots \circ F(b_m) \end{aligned}$$

so the image of the two paths are also equal, so the image of the diagram remains commutative. ■

In particular, this implies that isomorphism diagrams are also preserved, so

- If f is an isomorphism, then $F(f)$ is also an isomorphism;
- If $A \cong B$ are isomorphic objects, then $F(A) \cong F(B)$.

A functor that satisfies the converse of the first statement is said to *reflect* isomorphisms, and a functor that satisfies the converse of the second is said to *create* isomorphisms [Rie17].

Theorem 2.7. *Fully faithful functors (i) reflect and (ii) create isomorphisms.*

That is, for a fully faithful functor $F : \mathscr{C} \to \mathscr{D}$*,* *(i) If a morphism f in $\mathscr{C}$ is such that $F(f)$ is an isomorphism in $\mathscr{D}$, then f is an isomorphism;*

(ii) If a pair of objects X and Y in $\mathscr{C}$ are such that $F(X) \cong F(Y)$, then $X \cong Y$.

Proof. Suppose $f : A \to B$ is a morphism such that $F(f) : F(A) \to F(B)$ is an isomorphism with inverse $\tilde{g} : F(B) \to F(A)$. As F is full, there exists a morphism $g : B \to A$ such that $F(g) = \tilde{g}$, so $F(g \circ f) = F(g) \circ F(f) = \tilde{g} \circ F(f) = \mathrm{id}_{F(A)} = F(\mathrm{id}_A)$, so by faithfulness, $g \circ f = \mathrm{id}_A$. Exchanging f and g in the previous yields $f \circ g = \mathrm{id}_B$, so f is an isomorphism.

Suppose $F(A) \cong F(B)$, so there is an isomorphism $\tilde{f} : F(A) \to F(B)$ with inverse $\tilde{g}$. As F is full, there exist morphisms $f : A \to B$ and $g : B \to A$ such that $F(f) = \tilde{f}$ and $F(g) = \tilde{g}$. Then, $F(g \circ f) = F(g) \circ F(f) = \tilde{g} \circ \tilde{f} = \mathrm{id}_{F(A)} = F(\mathrm{id}_A)$, so by faithfulness, $g \circ f = \mathrm{id}_A$. Again, exchanging f and g in the previous yields $f \circ g = \mathrm{id}_B$, so f and g are isomorphisms and hence $A \cong B$. ∎

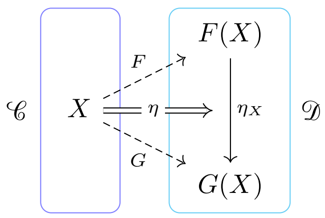
Note that the converse of this theorem does not hold in that functors that create or reflect isomorphisms are not necessarily full or faithful.

These two conditions may seem similar, but they do not imply each other in general. For instance, let $\mathscr{C}$ be a category in which every object is isomorphic, but there exist non-isomorphism morphisms, e.g. the category $\mathbf{Set}_n$ of sets of cardinality n. Because every object is isomorphic, any functor $F : \mathscr{C} \to \mathscr{D}$ to any category $\mathscr{D}$ trivially creates isomorphisms, but will not, in general, reflect isomorphisms.

2.5 Natural Transformations

Given categories and functors $\mathscr{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{D}$, a *natural transformation* is a mapping $\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{D} \end{tikzcd}$ or $\eta : F \Rightarrow G$ between functors.

The functors F and G map objects and morphisms in $\mathscr{C}$ to objects and morphisms in $\mathscr{D}$, so to define a mapping $F \Rightarrow G$, we want to associate the images of objects and morphisms under F to their images under G. For objects, this just means that if X is in $\mathscr{C}$, then $F(X)$ should be associated with $G(X)$ – this is just a morphism in $\hom_{\mathscr{D}}(F(X),G(X))$. So, the natural transformation η associates each object $X \in \mathrm{ob}(\mathscr{C})$ to a morphism $h_X : F(X) \to G(X)$ called the *component* of η at X.



[Kit22]

However, there could be many morphisms $F(X) \to G(X)$ we could choose. We need a way of selecting these components that is consistent throughout the whole category.

Consider a morphism $f : A \to B$ in $\mathscr{C}$. Under F and G, we have the images $F(f) : F(A) \to F(B)$ and $G(f) : G(A) \to G(B)$. Along with the components $\eta_A : F(A) \to G(A)$ and $\eta_B : F(B) \to G(B)$, this completes the square

$$\begin{tikzcd} & A \arrow[r, dashed, "F"] \arrow[d, "f"] \arrow[rrr, bend left, dashed, "G"] \arrow[rr, Rightarrow, bend left, "\eta"] & F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"] \arrow[dr, dashed, "(\exists!)?"] & G(A) \arrow[d, "G(f)"] & \\ \mathscr{C} & B \arrow[r, dashed, "F"] \arrow[rrr, bend right, dashed, "G"'] \arrow[rr, Rightarrow, bend right, "\eta"'] & F(B) \arrow[r, "\eta_B"] & G(B) & \mathscr{D} \end{tikzcd}$$ [Kit22]

In this diagram, there are two paths from $F(A)$ to $G(B)$, namely, $\eta_B \circ F(f)$, and $G(f) \circ \eta_A$, and without any further conditions on the components of η, these paths may be distinct. However, if we require that these paths are equal – that the diagram commutes – then this forces our selection of components to be consistent throughout the whole category. This coherency condition is called the *naturality* requirement.

So, overall, a natural transformation a natural transformation $\eta : F \Rightarrow G$ between functors $F,G : \mathscr{C} \to \mathscr{D}$ is a collection of morphisms $\left(F(X) \xrightarrow{\eta_X} G(X)\right)_{X \in \mathrm{ob}(\mathscr{C})}$ indexed by the objects of $\mathscr{C}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "f"] & F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\ B & F(B) \arrow[r, "\eta_B"] & G(B) \end{tikzcd}$$

That is, $\eta_B \circ F(f) = G(f) \circ \eta_A$ for all $f : A \to B$ in $\hom(\mathscr{C})$.

Natural transformations are collections of morphisms between the images of two functors that are canonical or consistent in some way in that they preserve some of the functoriality. Informally, a construction involving a collection of mappings between objects is said to be "natural" if those mappings can be extended to some natural transformation over the whole category, and "unnatural" otherwise. Often, unnatural constructions depend on some arbitrary choice e.g. of basis, generator, relations, etc. while natural constructions are independent of these choices.

Consider the following diagram of categories, functors, and natural transformations:

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=60, "H"'{name=H}] & \mathscr{D} \arrow[from=F, to=G, Rightarrow, "\alpha"] \arrow[from=G, to=H, Rightarrow, "\beta"] \end{tikzcd}$$

From the diagram, it would seem that we should be able to compose α and β to obtain a natural transformation $\beta \circ \alpha : F \Rightarrow H$. Such a composition is called a *vertical composition.*[*]

Consider an object X in $\mathscr{C}$. The components of α and β at X are the morphisms $\alpha_X : F(X) \to G(X)$ and $\beta_X : G(X) \to H(X)$ – these morphisms are compatible in that we can compose them, so we can define the component $(\beta \circ \alpha)_X$ to be $\beta_X \circ \alpha_X : F(X) \to H(X)$. For naturality, consider the following

[*]There is also a related notion of *horizontal composition* that combines natural transformations $\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "G"'{name=G}] & \mathscr{D} \arrow[r, bend left, "F'"{name=F2}] \arrow[r, bend right, "G'"'{name=G2}] & \mathscr{E} \arrow[from=F, to=G, Rightarrow, "\alpha"] \arrow[from=F2, to=G2, Rightarrow, "\beta"] \end{tikzcd}$ into a natural transformation $\alpha \diamond \beta : F' \circ F \Rightarrow G' \circ G$ that we will not use.

diagram:

$$\begin{array}{ccccccc}
A & & F(A) & \xrightarrow{\alpha_A} & G(A) & \xrightarrow{\beta_A} & H(A) \\
\Big\downarrow f & & \Big\downarrow F(f) & & \Big\downarrow G(f) & & \Big\downarrow H(f) \\
B & & F(B) & \xrightarrow{\alpha_B} & G(B) & \xrightarrow{\beta_B} & H(B)
\end{array}$$

α and β are natural transformations, so each square individually commutes, and hence the outer square also commutes, so $\beta \circ \alpha$ is natural, as required.

The collection of natural transformations between functors between two fixed categories $\mathscr{C}$ and $\mathscr{D}$, under vertical composition, forms the *functor category* $[\mathscr{C},\mathscr{D}]$ (also written as $\mathscr{D}^{\mathscr{C}}$) that has functors from $\mathscr{C}$ to $\mathscr{D}$ as objects, natural transformations as morphisms, and vertical composition as composition. Identities in this category are given by identity natural transformations that associate every object with the identity morphism on their image.

An isomorphism in a functor category is then called a *natural isomorphism*. That is, if $\eta : F \Rightarrow G$ and $\vartheta : G \Rightarrow F$ are natural transformations such that $\vartheta \circ \eta = \mathrm{id}_F$ and $\eta \circ \vartheta = \mathrm{id}_G$, then η and ϑ are natural isomorphisms, and we write η^{-1} for ϑ. If two functors F and G are naturally isomorphic, then we write $F \cong G$, or, we say that $F(X) \cong G(X)$ *naturally in* X whenever we need to bind a variable.

Note that the statement "$F(X) \cong G(X)$" is a statement about the objects $F(X)$ and $G(X)$ in $\mathscr{D}$, while "$F(X) \cong G(X)$ naturally in X" is a much stronger statement about the functors F and G in $[\mathscr{C},\mathscr{D}]$. In particular, $F(X) \cong G(X)$ naturally in X not only requires that there are isomorphisms $F(X) \cong G(X)$ for every X, but also that these individual isomorphisms can be selected in some consistent way such that all naturality diagrams commute; it is entirely possible for $F(X) \cong G(X)$ to hold for all X, but for no such selection of isomorphisms to exist and for $F \ncong G$.

Example (Dual Vector Spaces). In linear algebra, the *dual* V^* of a vector space V over a field K is the vector space of linear functionals $V \to K$ equipped with pointwise addition and scalar multiplication.

It is well known that any finite-dimensional vector space V is isomorphic to its dual V^*, and also to the dual of its dual, or the *double dual*, $V^{**} = (V^*)^*$, the space of linear functionals $V^* \to K$. These isomorphisms follow from a standard construction that, given a basis of V, yields a *dual basis* of V^* of the same dimension. However, in many ways, V^{**} has a lot more in common with V than the dual V^* does, and we can make this idea precise by showing that the collection of isomorphisms $V \cong V^{**}$ is natural in the sense that it can be extended to a natural transformation, while the isomorphisms $V \cong V^*$ cannot.

There is a contravariant functor $(-)^* : \mathbf{Vect}^{\mathrm{op}} \to \mathbf{Vect}$ that sends vector spaces V to their dual V^*, and linear maps $f : U \to V$ to their transpose $f^* : V^* \to U^*$, defined by precomposing linear functionals $\omega \in V^*$ by f to obtain $\omega \circ f \in U^*$. Applying this functor again yields the covariant double dual functor $(-)^{**} : \mathbf{Vect}_K \to \mathbf{Vect}_K$ that maps vector spaces V to their double dual V^{**}. Note that the elements of V^* are themselves functions $V \to K$, so the elements of V^{**} are functionals that send functions $V \to K$ to elements in K.

One obvious way to map these functionals to elements is just to supply the functions in V^* with an input $v \in V$ – that is, for any $\omega \in V^*$, we have $\omega(v) \in K$ by definition, so the evaluation mapping $\omega \mapsto \omega(v)$ is an element of V^{**}. So, for each vector space V, we have a linear map $\eta_V : V \to V^{**}$ that sends vectors to their associated evaluation mappings [Per21]. We show that these maps are natural in the formal sense.

Let $f : V \to W$ be a linear transformation, and consider the following diagram:

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "f"] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\eta_W"] & W^{**} \end{tikzcd}$$

Let $v \in V$ and $\omega \in W^*$. Along the lower path, we have,

$$\big[\eta_W\big(f(v)\big)\big](\omega) = \omega\big(f(v)\big)$$

and along the upper path, we have,

$$\begin{aligned} \big[f^{**}\big(\eta_V(v)\big)\big](\omega) &= \big[\eta_V(v) \circ f^*\big](\omega) \\ &= \big[\eta_V(v)\big]\big(f^*(\omega)\big) \\ &= \big[\eta_V(v)\big](\omega \circ f) \\ &= (\omega \circ f)(v) \\ &= \omega\big(f(v)\big) \end{aligned}$$

so the diagram commutes. If we view the objects on the left side of the commutative square as the images of objects under the identity functor, then we see that every map η_V is a morphism $\mathrm{id}_{\mathbf{Vect}_K}(V) \to (V)^{**}$, so the entire collection $(\eta_V)_{V \in \mathrm{ob}(\mathbf{Vect}_K)}$ defines a natural transformation $\mathrm{id}_{\mathbf{Vect}_K} \Rightarrow (-)^{**}$, with the above diagram verifying naturality.

For finite-dimensional spaces, η is furthermore a isomorphism; if we restrict our attention to the finite-dimensional case in the subcategory $\mathbf{FinVect}_K$, then η defines a natural isomorphism $\mathrm{id}_{\mathbf{Vect}_K} \cong (-)^{**}$. In contrast, the dual V^* is *not* naturally isomorphic to V, even in finite dimensions, simply because the single dual functor $(-)^*$ is contravariant and lives in a different functor category than the identity. Even if we extend the notion of naturality to cover contravariance, there is a deeper reason why the single dual is not natural or "canonical", unlike η:

Consider the case where f is an endomorphism, interpreted as a *change of basis*:

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "f"] & V^{**} \arrow[d, "f^{**}"] \\ V \arrow[r, "\eta_V"] & V^{**} \end{tikzcd}$$

What this diagram is saying is that if we change the basis of V, and also the basis of V^{**} with the induced function f^{**}, then the map η is completely unaffected: that is, η does not depend on the choice of basis [Per21]. In contrast, every isomorphism $V \to V^*$ that can be constructed (even in finite dimensions) will depend on some choice of basis of V, and moreover, changing the basis of V does not change the basis of V^* in a way that is compatible with these isomorphisms.

Natural transformations can also be composed with functors, in a sense. Let F,$G : \mathscr{C} \to \mathscr{D}$ and $H : \mathscr{D} \to \mathscr{E}$ be functors, and $\eta : F \Rightarrow G$ be a natural transformation. The *whiskering* $H \cdot \eta$ of H by η is the natural transformation $H \circ F \Rightarrow H \circ G$ defined by $(H \cdot \eta)_X = H(\eta_X)$.

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{D} \arrow[r, "H"] & \mathscr{X} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "H\circ G"'] \arrow[r, phantom, "\Downarrow H\cdot\eta"] & \mathscr{X} \end{tikzcd}$$

(This is a special case of the horizontal composition where one of the natural transformations is the identity natural transformation, so $H \cdot \eta = \eta \diamond \mathrm{id}_H$.)

2.6 Equivalence of Categories

We have seen numerous examples of structures and structure-preserving maps forming categories, and the same holds true for categories and functors themselves: the collection of small categories and functors forms a large category, **Cat**.*

If there is an isomorphism between two categories $\mathscr{C}$ and $\mathscr{D}$ in **Cat**, then $\mathscr{C}$ and $\mathscr{D}$ are *isomorphic categories* – categories which differ only in the labelling of their objects and morphisms.

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] & \mathscr{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

$$G \circ F = \mathrm{id}_{\mathscr{C}} \qquad \text{and} \qquad F \circ G = \mathrm{id}_{\mathscr{D}}$$

Again, we write $\mathscr{C} \cong \mathscr{D}$ if there exists an isomorphism between $\mathscr{C}$ and $\mathscr{D}$. Like with other isomorphic objects, results about one immediately gives identical results about the other, but unfortunately, isomorphism of categories tends to be too strong of a requirement in that very few useful categories are isomorphic to each other.

However, as mentioned earlier, we care about different degrees of likeness for different types of data. For elements of a set, this notion is strict equality, while for objects in a category, this notion is isomorphism. Applying this idea to a functor category, we see that natural isomorphism is the appropriate degree of likeness for functors.

Now, looking back at the definition of isomorphic categories above, we notice that the compositions of F and G are required to be *equal* to the identity functors, but as we have just seen, this degree of likeness is unreasonably strict. For functors, we really should be using natural isomorphisms, and indeed, if we only require the compositions to be naturally isomorphic to the identity functor, we obtain a weaker but much more useful notion of likeness called *equivalence of categories.*

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] & \mathscr{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

$$G \circ F \cong \mathrm{id}_{\mathscr{C}} \qquad \text{and} \qquad F \circ G \cong \mathrm{id}_{\mathscr{D}}$$

In this case, we say that $\mathscr{C}$ and $\mathscr{D}$ are *equivalent*, and we write $\mathscr{C} \simeq \mathscr{D}$. We also call the functors F and G *equivalences.*

Example. Consider the category $\mathbf{FinVect}_k$ of finite-dimensional vector spaces over a field k, and the category $\mathbf{Mat}(k)$, where morphisms are matrices with entries in k, and objects are the dimensions of those matrices (so, if n and m are objects, then $\hom(n,m)$ is the collection of $m \times n$ matrices). That $\mathbf{FinVect}_k$ and $\mathbf{Mat}(k)$ are not isomorphic categories can be deduced by observing that there are no isomorphic objects in $\mathbf{Mat}(k)$ – but, these categories are clearly related in some way as matrices are well known to represent linear transformations, and indeed, these categories are not isomorphic, but equivalent, with the equivalence $\mathbf{FinVect}_k \to \mathbf{Mat}_k$ sending each vector space to its dimension, and each linear transformation to its corresponding matrix. Each choice of basis for each vector space provides a different equivalence, also demonstrating that an equivalence is not unique.

Example. Up to isomorphism, every thin category is a preordered set, but up to equivalence, every thin category is a partially ordered set.

2.7 The Yoneda Lemma

Let $1 = \{\bullet\}$ be the set with one element. For any set X, a function $1 \to X$ amounts to selecting an element of X since the only data required to uniquely characterise such a function is just the image of $\bullet$ in X. Similarly, in any space X, the functions $1 \to X$ (where 1 is the one-point space) are essentially just points in X. In fact, in arbitrary categories with terminal objects 1, we will *define* a map $1 \to X$ to be an *element* of X.

*There is not a simple category of all categories for the same reason that there is no set of all sets, but given a choice of Grothendieck universe, a similar category can be constructed, and is denoted **CAT**.

We can extend this idea of "objects as functions" by picking different choices of domain spaces. For instance, the functions $[0,1] \to X$ are just paths in X; the functions $\mathbb{N} \to X$ are the sequences in X; and the functions $S^1 \to X$ are the topological loops in X. In some of these cases, these are even the *definitions* of these constructions.

More generally, given an object A in a category $\mathscr{C}$, a *generalised element* of A is any morphism with codomain A, and the domain of such a morphism is called the *domain of variation* of the element [Kos12]. We also alternatively call a morphism $S \to A$ a generalised element of A with *shape* S [Lei14]. Note that there really isn't any mathematical difference between a "generalised element" and a morphism, but the change in naming represents a change in perspective, as above.

Note, however, that this is more than just a semantic trick; this concept of treating objects as special cases of functions – or more generally, of morphisms – is a fundamental idea in category and structural set theory, and arguably the most important result in category theory [Rie17], which we display below, expands on this idea.

Now, we can exhibit basic objects like points or paths as certain types of maps, but how can we apply this idea to arbitrary objects X? That is, we have captured certain basic features within X with the maps $1 \to X$, $[0,1] \to X$, $S^1 \to X$, etc., but is it possible to characterise the entirety of X itself using these maps? A priori, there is no reason we should expect that the entire structure of X is contained within these maps. This is the content of the next result.

Lemma 2.8 (Yoneda)**.** *Let $\mathscr{C}$ be a locally small category. Then,*

$$\hom_{[\mathscr{C},\mathbf{Set}]}(h_A,F) \cong F(A)$$

naturally in $F \in \mathrm{ob}([\mathscr{C},\mathbf{Set}])$ and $A \in \mathrm{ob}(\mathscr{C})$.

Proof. [Kit22, abridged] We give a proof of the isomorphism only.

Let $f : A \to B$ be a morphism, $\eta : h_A \Rightarrow F$ be a natural transformation, and consider the naturality diagram of η:

$$\begin{tikzcd}
h_A(A) \arrow[r, "\eta_A"] \arrow[d, "h_A(f)"'] & F(A) \arrow[d, "F(f)"] \\
h_A(B) \arrow[r, "\eta_B"'] & F(B)
\end{tikzcd}$$

We chase the identity $\mathrm{id}_A \in \hom(A,A) = h_A(A)$ around the diagram:

$$\begin{tikzcd}
\mathrm{id}_A \arrow[r, mapsto, "\eta_A"] \arrow[d, mapsto, "h_A(f)"'] & \eta_A(\mathrm{id}_A) \arrow[d, mapsto, "F(f)"] \\
f \arrow[r, mapsto, "\eta_B"'] & \eta_B(f) = F(f)\big(\eta_A(\mathrm{id}_A)\big)
\end{tikzcd}$$

The input to the function on the right side is always $\eta_A(\mathrm{id}_A)$, implying that any natural transformation $h_A \Rightarrow F$ is completely determined by its value at id_A. This naturally induces a function $\phi : \hom_{[\mathscr{C},\mathbf{Set}]}(h_A,F) \to F(A)$ defined by $\eta \mapsto \eta_A(\mathrm{id}_A)$. Conversely, given an element $u \in F(A)$, we can define the components of a unique natural transformation $\eta : h_A \Rightarrow F$ by $\eta_B(f) = F(f)(u)$, where $f : A \to B$, defining a mapping $\psi : F(A) \to \hom_{[\mathscr{C},\mathbf{Set}]}(h_A,F)$. Then, $\psi\big(\phi(\eta)\big) = \psi\big(\eta_A(\mathrm{id}_A)\big) = \eta$, and $\phi\big(\psi(u)\big) = \phi(\eta) = \eta_A(\mathrm{id}_A) = u$, so the functions are mutually inverse, hence defining an isomorphism $F(A) \cong \hom_{[\mathscr{C},\mathbf{Set}]}(h_A,F)$.

For a proof of the naturality of this isomorphism, see [Kit22]. ■

If we take F to be another hom-functor in the Yoneda lemma, we obtain:

$$\hom_{[\mathscr{C},\mathbf{Set}]}(h_A,h_B) \cong h_B(A) = \hom_{\mathscr{C}}(B,A)$$

so the natural transformations $h_A \Rightarrow h_B$ are in bijection with the morphisms $B \to A$. This assignment of natural transformations from morphisms is the action of the contravariant functor $h_\bullet : \mathscr{C}^{\mathrm{op}} \to [\mathscr{C},\mathbf{Set}]$ defined on objects by $h_\bullet(A) = h_A$. In fact, we have already seen this functor – $h_\bullet$ is exactly the partial application of the hom-bifunctor $\hom(-,-)$ in the first argument (e.g., $h_\bullet(A) = \hom(-,-)(A,-) = \hom(A,-) = h_A$). Dually, we can also partially apply the hom-bifunctor in the second argument to obtain the covariant functor $h^\bullet : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$.

The Yoneda lemma then says that the functors $h_\bullet$ and $h^\bullet$ give embeddings of $\mathscr{C}^{\mathrm{op}}$ and $\mathscr{C}$ into $[\mathscr{C},\mathbf{Set}]$ and $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$, respectively – there is a copy of every (locally small) category contained within the collection of functors between its dual and **Set**.

These functors are called the covariant and contravariant *Yoneda embeddings*, and are often collectively denoted よ, with context disambiguating between the two.

Theorem 2.9 (Yoneda Embedding)**.** *Let $\mathscr{C}$ be a locally small category. Then, the Yoneda embeddings* よ $: \mathscr{C} \hookrightarrow [\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$ *and* よ $: \mathscr{C}^{\mathrm{op}} \hookrightarrow [\mathscr{C},\mathbf{Set}]$ *are embeddings – that is,* よ *is fully faithful, and injective on objects up to isomorphism.*

Proof. See [Kit22]. ■

From functoriality, the Yoneda embeddings imply that if $X \cong Y$, then $\hom(X,-) \cong \hom(Y,-)$ and $\hom(-,X) \cong \hom(-,Y)$. More notably, full faithfulness also implies the converse – that is, the Yoneda embeddings create isomorphisms:

Corollary 2.9.1. *If* $\hom(X,-) \cong \hom(Y,-)$ *or* $\hom(-,X) \cong \hom(-,Y)$*, then* $X \cong Y$*.*

That is, the maps in to and maps out from an object contain exactly as much information as the object itself; the collections of these maps are isomorphic if and only if the associated objects are, so objects are completely characterised by their generalised elements.

3 Universal Properties

3.1 Terminal and Initial Objects

An object $T \in \mathrm{ob}(\mathscr{C})$ is *terminal* if for every object $X \in \mathrm{ob}(\mathscr{C})$ there exists a unique morphism $X \to T$. Dually, an object $I \in \mathrm{ob}(\mathscr{C})$ is *initial* if for every object $X \in \mathrm{ob}(\mathscr{C})$, there exists a unique morphism $I \to X$ (or equivalently, if it is terminal in $\mathscr{C}^{\mathrm{op}}$). Terminal and initial objects are also sometimes collectively called *universal* objects, with context disambiguating between the two cases.

In **Set**, any singleton set $1 = \{\bullet\}$ is terminal as for any set X, a function $X \to 1$ exists, defined by mapping every element of X to $\bullet$, and is unique as there are is only one possible target for every input. Conversely, the empty set is initial as for any set X, the empty function $\varnothing \to X$ exists and is unique.

In many categories of structured sets such as **Set**, **Top**, **Ring**, and **Grp**, terminal objects are often singleton sets, so terminal objects are often denoted by 1. Initial objects are slightly less well-behaved, but are often the empty set, as is in **Set** or **Top**,* and are often denoted by 0.

Terminal and initial objects may not necessarily exist; for instance, in the preorder category $(\mathbb{N}, \leq)$, 0 is initial, but no terminal object exists; and in $(\mathbb{Z}, \leq)$, terminal and initial objects both fail to exist. However, if initial or terminal objects *do* exist, then they are unique up to unique isomorphism – that is, if C and C' are distinct and terminal (initial), then there is a unique isomorphism $C \to C'$ – and we say that they are *essentially unique*.

*Rings and groups require identities, so the empty set is not in **Ring** or **Grp**. Instead, the initial object in **Ring** is $\mathbb{Z}$; and in **Grp**, the trivial group.

Theorem 3.1. *Terminal (initial) objects are essentially unique.*

Proof. Suppose C and C' are distinct and terminal. Because C is terminal, the morphisms $f : C' \to C$ and $\mathrm{id}_C : C \to C$ are unique, and because C' is terminal, the morphisms $g : C \to C'$ and $\mathrm{id}_{C'} : C' \to C'$ are unique. Then, the compositions $g \circ f$ and $f \circ g$ must be identities, so they form a unique isomorphism. The essential uniqueness of initial objects follows from duality. ■

Terminal and initial objects can, however, coincide. For instance, in **Grp**, the trivial group is both terminal and initial. In these cases, the object is called a *null* or *zero* object.

3.2 Representability

By the Yoneda lemma, objects are determined entirely by the maps into or out from them; informally, a *universal property* is a description of these maps.

Terminal and initial objects are examples of objects characterised by universal properties – in this case, the universal property that the maps to or from them exist uniquely – and in fact, we can reverse this somewhat by saying that an object has a universal property, or is *universal*, if we can find some category in which it is initial or terminal.

Universal properties are a way of describing the "best", "largest", "smallest", "most ____" etc. object in a category – or in the case of terminal and initial objects, the "final" and "first" objects – and the exact notion of "best" will define different types of objects.

Through almost the exact same reasoning as for terminal and initial objects, objects characterised by universal properties are essentially unique (though we will prove this more formally soon). This also provides another way of proving that a collection of objects are isomorphic; just find a universal property they satisfy in common.

Once a universal property of some object has been identified, we often then ignore the specifics of how it was constructed in the first place, as the universal property alone is sufficient information to recover the object essentially uniquely. This allows us to more easily work with constructions that have unwieldy definitions but simple universal properties.

Now, the maps into or out from an object A are captured by the hom-functors h_A and h^A (or more concisely, by the Yoneda functors $\text{よ}(A)$), so a unversal property is a description of these functors. We formally give this description in terms of an isomorphism.

A covariant or contravariant functor F from a locally small category $\mathscr{C}$ to **Set** is *representable* if $F \cong \text{よ}(A)$ for some $A \in \mathrm{ob}(\mathscr{C})$ (where the Yoneda embedding necessarily matches the variance of F). The object A, along with the natural transformation $F \Rightarrow \text{よ}(A)$ is then called a *representation* of F. Representable functors then carry information about the hom-functors they are isomorphic to, hence encoding a universal property about their representing object.

We rephrase the universal property of initial objects using representations as an example. Let $\Delta 1 : \mathscr{C} \to$ **Set** be the *constant functor* that sends every object in $\mathscr{C}$ to a fixed singleton set 1, and every morphism to the identity function id_1. Then, an object A is initial if and only if $h_A \cong \Delta 1$. The natural isomorphism requires that $\mathrm{hom}(A,B)$ contains exactly a single morphism for every object B, which is precisely our previous definition of an object being initial. Equivalently, we could say that a category $\mathscr{C}$ has an initial object if and only if the constant functor $\Delta 1 : \mathscr{C} \to$ **Set** is representable.

Theorem 3.2. [Rie17] *Let X and Y be objects of a locally small category $\mathscr{C}$. If either the covariant or contravariant functors represented by X and Y are naturally isomorphic, then X and Y are isomorphic.*

Proof. The Yoneda embeddings are fully faithful, and hence create isomorphisms. It follows that an isomorphism between represented functors must be induced by a unique isomorphism between the representing objects. ■

In particular, if X and Y represent the same functor, then X and Y are isomorphic; and moreover, this isomorphism is unique. That is to say, the object representing a functor is essentially unique, thus extending the claim from terminal and inital objects to all objects characterised by universal properties.

We will now explore some of the many important constructions that can be characterised in this way.

3.3 Products

Let X and Y be objects in a category $\mathscr{C}$. We should intuitively expect that any notion of a *product* should consist of another object $P \in \mathrm{ob}(\mathscr{C})$ that is related to X and Y; that is, an object equipped with a pair of morphisms, called *projections*, to X and Y:

$$\begin{tikzcd} & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

This resulting diagram shape of one object pointing to two others is called a *span*.

But not just any object with maps to X and Y can be the product – we need the product to be universal; or, the "best" one possible. The notion of "best" here is that for all other spans

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X & & Y \end{tikzcd}$$

we require for there to exist a unique *factorisation* through P. That is, there exists a unique morphism $f : A \to P$ such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[d, dashed, "\exists! f"] \arrow[ddl, "g"'] \arrow[ddr, "h"] & \\ & P \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ X & & Y \end{tikzcd}$$

and we say that A *factors through* P. We then write $X \times Y$ for the product object P, and $\langle g,h \rangle$ for the unique morphism f, and we call g and h the *components* of the *pairing* $\langle g,h \rangle$. This latter notation is justified as f is uniquely determined by g and h, as prescribed by the universal property. Conversely, given any map into a product, we can reconstruct its components by postcomposing it by each projection map.

There are notable similarities to terminal objects with this definition, in that we require a unique morphism to exist; P appears to be "terminal" with respect to other objects that have maps to X and Y. (We formalise this observation later Section 4.3, with the notion of a category of cones.)

Just like with terminal objects, products do not always exist in any given category. For instance, for any pair of distinct objects in a discrete category, there is no way to form a span connecting the two objects, as every morphism in a discrete category is an identity, so these products do not exist. However, if the product *does* exist, then it is essentially unique due to the universal property.

Example (Products in **Set**). The Cartesian product $X \times Y$ is a categorical product.

The Cartesian product $X \times Y$ comes naturally equipped with two projection mappings to the component sets defined by $(x,y) \mapsto x$ and $(x,y) \mapsto y$. Now, suppose there is another set A with maps $X \xleftarrow{g} A \xrightarrow{h} Y$. Let $a \in A$, so $g(a) \in X$ and $h(a) \in Y$. To make the diagram commute, the obvious – and only – choice for $f : A \to X \times Y$ is to have $f(a) = \big(g(a),h(a)\big)$.

For uniqueness, suppose there exists another another mapping $f' : A \to X \times Y$ such that the product diagram commutes, and let $f'(a) = (x,y)$. Then, by commutativity, $g(a) = \big(\pi_1 \circ f'\big)(a) = \pi_1(x,y) = x$, and similarly, $h(a) = \big(\pi_2 \circ f'\big)(a) = \pi_2(x,y) = y$, so $f'(a) = \big(g(a),h(a)\big) = f(a)$, and the factorisation is unique.

Note that this characterisation of the Cartesian product as a categorical product makes no mention of the elements of $A \times B$ at all (or any other set), depending entirely on mappings between sets, and specifically on how the product *interacts* with other sets; thus formulating the Cartesian product structurally.

Example (Products in **Top**). If we take A to be the one-point space $\mathbf{1}$, then the maps g, h, and f are just elements of X, $P = X \times Y$, and Y, respectively. The product then says that there is a bijection

$$\hom_{\mathbf{Top}}(\mathbf{1},X \times Y) \cong \hom_{\mathbf{Top}}(\mathbf{1},X) \times \hom_{\mathbf{Top}}(\mathbf{1},Y)$$

so the points of $X \times Y$ correspond to the points in the Cartesian product of the underlying sets of X and Y. Then, if A is the set $X \times Y$ equipped with different topologies, the existence of f then requires that sets in A are open whenever they are open in P, so the topological product P must have the coarsest topology on $X \times Y$ possible such that the projection maps are still continuous, thus defining the product topology. This similarly extends to infinite products of topological spaces $(X_i)_{i \in \mathscr{I}}$, where the topology on the product space $\prod_{i \in \mathscr{I}} X_i$ is defined to be the coarsest topology such that each projection $\pi_j : \prod_{i \in \mathscr{I}} X_i \to X_j$ is continuous.

Example (Products in **Cat**). A product category $\mathscr{C} \times \mathscr{D}$ of two small categories $\mathscr{C}$ and $\mathscr{D}$ is a categorical product in the category **Cat** of small categories.

Consider a pair of morphisms $g : A \to X$ and $h : B \to Y$. We can construct the product $A \times B$ and $X \times Y$ to obtain:

$$\begin{tikzcd}
 & A \times B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, dashed, "g \times h"] & \\
A \arrow[dd, "g"'] & & B \arrow[dd, "h"] \\
 & X \times Y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X & & Y
\end{tikzcd}$$

Now, notice that the compositions $g \circ \pi_1$ and $h \circ \pi_2$ forms a span into X and Y, so by the universal property of the product, there exists a unique map $A \times B \to X \times Y$. Again, this map is entirely determined by g and h, so we write $g \times h$ for this *product morphism.* Note that this is distinct from the pairing $\langle g,h \rangle$, which is a map from a single object into a product, while a morphism of the form $g \times h$ is a map from a product into another product.

It may be helpful to consider these morphisms in **Set**: as before, a pairing function $\langle f,g \rangle : A \to X \times Y$ acts on elements $a \in A$ by $\langle f,g \rangle(a) = (f(a),g(a))$. On the other hand, a product function $s \times t : A \times B \to X \times Y$ acts on pairs $(a,b) \in A \times B$ pointwise, $(s \times t)(a,b) = \big(s(a),t(b)\big)$.

Another special case of the product is given by taking the product of an object X with itself, forming the span with two other copies of X using the projection maps:

$$\begin{tikzcd} & X \times X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

Another span can be given by three copies of X equipped with a pair of identity maps:

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "\mathrm{id}_X"] & \\ X & & X \end{tikzcd}$$

By the universal property of the product, there must exist a unique morphism $X \to X \times X$ such that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[d, dashed, "\Delta_X"] \arrow[ddl, "\mathrm{id}_X"'] \arrow[ddr, "\mathrm{id}_X"] & \\ & X \times X \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ X & & X \end{tikzcd}$$

This morphism is called the *diagonal morphism* of X, denoted by $\langle \mathrm{id}_X, \mathrm{id}_X \rangle$, Δ, or Δ_X.

For example, in **Set**, the diagonal function is given by the function $x \mapsto (x,x)$ that sends every element to the corresponding diagonal subset of the Cartesian square.

The diagonal morphism also provides a link between the product morphism and the pairing morphism: suppose we have a product $A \times B$, and a pair of maps $f : X \to A$ and $g : X \to B$. Then, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[dr, "\langle f,g \rangle"'] & X \times X \arrow[d, "f \times g"] \\ & A \times B \end{tikzcd}$$

3.3.1 Coproducts

We can dualise the notion of a product into a *coproduct* (or *categorical sum*) by reversing the direction of all morphisms in the previous definition. Given two objects X and Y, the coproduct is an object P equipped with *insertion* maps,

$$\begin{tikzcd} & P & \\ X \arrow[ur, "\iota_1"] & & Y \arrow[ul, "\iota_2"'] \end{tikzcd}$$

with the universal property that for all objects and maps

$$\begin{tikzcd} & A & \\ X \arrow[ur, "g"] & & Y \arrow[ul, "h"'] \end{tikzcd}$$

of the same shape (a *cospan*), there exists a unique factorisation of P through A. That is, there exists a unique map $f : P \to A$ such that the following diagram commutes:

$$\begin{tikzcd} & A & \\ & P \arrow[u, dashed, "\exists! f"] & \\ X \arrow[uur, "g"] \arrow[ur, "\iota_1"'] & & Y \arrow[uul, "h"'] \arrow[ul, "\iota_2"] \end{tikzcd}$$

Then, we write $X \amalg Y$, or less commonly $X + Y$, for P, and $[g,h]$ for the *copairing* map f.

Example (Coproducts in **Set**). The disjoint union $X \sqcup Y$ is a categorical coproduct.

The obvious choice for the insertion maps is just to send every element of X and Y to their embedded copy in $X \sqcup Y$. For the map f, consider an element $x \in X$, and its image $g(x) \in A$. The insertion map ι_1 sends x to its copy in $X \sqcup Y$, so for the left triangle to commute, this copy needs to be mapped to $g(x)$ by f, and similarly for every copy of $y \in Y$ in $X \sqcup Y$. So, we define $f : X \sqcup Y \to A$ by,

$$f(u) = \begin{cases} g(u) & u \in X \\ h(u) & u \in Y \end{cases}$$

Again, this is the only possible map we could define that makes the diagram commute, so the disjoint union of sets is a categorical coproduct.

3.4 Pullbacks

Let $\mathscr{C}$ be a category, and consider the following span:

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

The *pullback* or *fibred product* of this diagram is an object $P \in \mathrm{ob}(\mathscr{C})$ equipped with a pair of projection maps $\pi_1 : P \to X$ and $\pi_2 : P \to Y$ such that the *pullback square* below commutes:

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

But again, not just any object P with maps to X and Y will suffice; we want the "best" such square amongst all similar diagrams. The notion of "best" here is the same as for products; every other similar square should uniquely factorise through P. That is, for any commutative square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "h"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

in $\mathscr{C}$, there must exist a unique map $f : A \to P$ such that

$$\begin{tikzcd} A \arrow[dr, dashed, "\exists! f"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\ & P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd}$$

commutes. The symbol $\lrcorner$ marks the square as a pullback and not as a simple commutative square – that is, that P, π_1, and π_2 satisfy a universal property. We then say that π_1 is the pullback of t along s, and similarly, that π_2 is the pullback of s along t, and the object P is then also written as $X \times_Z Y$. The pullback of a morphism along itself is also called the *kernel pair* of that morphism.

Note that if Z is terminal, then the entire diagram commutes trivially as every map from any given object into the terminal object must be equal, so the pullback in this case is exactly the ordinary product. Or, put another way, the product $X \times Y$ is just the pullback of $X \to 1$ and $Y \to 1$.

Example (Pullbacks in **Set**). We need to pick a set for P and a pair of projection maps such that the lower right square of the diagram commutes in the best possible way.

If we ignore the maps to Z for a moment, we could simply pick P to be the product $X \times Y$. The maps to Z then add the extra constraints in that when mapping elements from X to Z and Y to Z, they must be equal. So, intuitively, we'd might think that the pullback of a diagram $X \xrightarrow{s} Z \xleftarrow{t} Y$ in **Set** should be:

$$P = \big\{(x,y) \in X \times Y : s(x) = t(y)\big\}$$

with π_1 and π_2 given by the standard projection maps, as they were for products.

To verify universality of this construction, suppose there is another set A with maps $X \xleftarrow{g} A \xrightarrow{h} Y$ such that the outer square commutes. Let $a \in A$, so $g(a) \in X$ and $h(a) \in Y$. To make the two triangles commute, this forces f to be defined by $f(a) = \big(g(a),h(a)\big)$. We then need to check that the image of f is within P; that is, that $s\big(g(a)\big) = t\big(h(b)\big)$. But this is just the commutativity requirement of the outer square, which holds by assumption.

Example (Preimages). The preimage of a set under a function can be exhibited as a type of pullback:

Given a function $f : X \to Y$, and a subset $B \subseteq Y$, we can use the inclusion map $\iota : B \hookrightarrow Y$ to form the span

$$\begin{tikzcd} & B \arrow[d, hook, "\iota"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The pullback of this diagram is then given by the preimage of B, with the projection π_1 given by the inclusion mapping $j : f^{-1}[B] \hookrightarrow X$, and the projection π_2 given by the restriction of f to $f^{-1}[B]$:

$$\begin{tikzcd} f^{-1}[B] \arrow[r, "f|_{f^{-1}[B]}"] \arrow[d, hook, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook, "\iota"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Theorem 3.3. *Monomorphisms are stable under pullback. That is, if s is a monomorphism, then π_2 is also a monomorphism. Similarly, if t is a monomorphism, then so is π_1.*

Proof. Let $s : X \rightarrowtail Z$ be a monomorphism, and suppose there is an object A with two maps $p,q : A \to P$ such that $\pi_2 \circ p = \pi_2 \circ q$. Then, $t \circ \pi_2 \circ p = t \circ \pi_2 \circ q$, and the commutativity of the pullback square yields $s \circ \pi_1 \circ p = s \circ \pi_1 \circ q$. Since s is monic, we then have $\pi_1 \circ p = \pi_1 \circ q$.

So, A has a pair of morphisms to X and Y, given by $(\pi_1 \circ p = \pi_1 \circ q) : A \to X$ and $(\pi_1 \circ p = \pi_1 \circ q) : A \to Y$, and hence forms a commutative square with Z. But, by the universal property of the pullback, this square must factor uniquely through P, so the map $A \to P$ is unique, and hence $p = q$, so π_2 is monic. The proof for t and π_1 is similar. ■

3.4.1 Pushouts

Dually, the *pushout* of a cospan diagram

$$\begin{tikzcd} & Y \\ X & Z \arrow[l, "s"] \arrow[u, "t"'] \end{tikzcd}$$

is an object $P \in \mathrm{ob}(\mathscr{C})$ with morphisms $\iota_1 : X \to P$ and $\iota_2 : Y \to P$ such that the pushout square below commutes and is universal:

$$\begin{tikzcd} P \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[l, "\iota_2"'] \\ X \arrow[u, "\iota_1"] & Z \arrow[l, "s"] \arrow[u, "t"'] \end{tikzcd}$$

That is, for any other commutative square

$$\begin{tikzcd} A & Y \arrow[l, "h"'] \\ X \arrow[u, "g"] & Z \arrow[l, "s"] \arrow[u, "t"'] \end{tikzcd}$$

there exists a unique map $f : P \to A$ such that

$$
\begin{tikzcd}
A & & \\
& P \arrow[ul, dotted, "\exists! f"'] & Y \arrow[l, "\iota_2"'] \arrow[ull, bend right, "h"'] \\
& X \arrow[u, "\iota_1"] \arrow[uul, bend left, "g"] & Z \arrow[l, "s"] \arrow[u, "t"'] \arrow[ul, phantom, "\lrcorner", very near end]
\end{tikzcd}
$$

commutes. Again, the *cokernel pair* of a morphism is the pushout of the morphism against itself.

Corollary 3.3.1. *Epimorphisms are stable under pushout.*

Proof. Dual of previous theorem. ■

Example (Pushouts in **Set**). As for pullbacks, ignoring Z yields $P = X \sqcup Y$, with ι_1 and ι_2 the usual insertion maps. The commutativity of the pushout square then requires that for any $z \in Z$, $s(z)$ is equal to $t(z)$ in $X \sqcup Y$; However, by construction, these elements are never equal in a disjoint union.

To force this equality, we quotient out by the equivalence relation $\sim$ generated by $s(z) \sim t(z)$ for all $z \in Z$, so the pushout is given by ${A \sqcup B}\big/{\sim}$.

Example (Unions and Intersections). Consider two sets X and Y, and their intersection $X \cap Y$ and (ordinary) union $X \cup Y$. The intersection is naturally equipped with inclusions into X and Y, and similarly, X and Y have inclusions into $X \cup Y$. The induced square is then simultaneously a pullback and a pushout:

$$
\begin{tikzcd}
X \cap Y \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook] \\
X \arrow[r, hook] & X \cup Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

3.5 Equalisers

A *fork* is a collection of objects and morphisms such that the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[r, "g"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y
\end{tikzcd}
$$

That is, this diagram is a fork if and only if $s \circ g = t \circ g$, and we say that g *equalises* s and t.

Let X and Y be objects in a category $\mathscr{C}$ with parallel morphisms $s,t : X \to Y$:

$$
\begin{tikzcd}
X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y
\end{tikzcd}
$$

The *equaliser* of s and t is an object E equipped with a map $\iota : E \to X$ such that the following diagram is a fork, and is universal:

$$
\begin{tikzcd}
E \arrow[r, "\iota"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y
\end{tikzcd}
$$

That is, every other fork through X and Y beginning at an object A factors uniquely through E:

$$\begin{tikzcd} A \arrow[d, dashed, "\exists! f"'] \arrow[dr, "g"] & & \\ E \arrow[r, "\iota"'] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \end{tikzcd}$$

and we write eq(s,t) for E.

Example (Equalisers in **Set**). In set theory, the notion of an equaliser is often defined to be the set of values upon which two functions agree. That is, given two functions f,$g : X \to Y$, the equaliser is the subset $\{x \in X : f(x) = g(x)\}$ of X. This function equaliser is a categorical equaliser, with ι given by the obvious inclusion map.

Example (Kernels). Let G and H be groups, and let $\varepsilon : G \to H$ be the trivial homomorphism defined by $g \mapsto \mathrm{id}_H$ for all $g \in G$. The equaliser of ε and any group homomorphism $\phi : G \to H$ is exactly the kernel of ϕ equipped with the inclusion mapping $\ker(\phi) \hookrightarrow G$:

$$\begin{tikzcd} \ker(\phi) \arrow[r, hook, "\iota"] & G \arrow[r, shift left, "\varepsilon"] \arrow[r, shift right, "\phi"'] & H \end{tikzcd}$$

Theorem 3.4. *The equaliser of two morphisms is monic.*

Proof. [Gol84, adapted] Let $\iota : E \to X$ equalise s,$t : X \to Y$, and let g,$h : A \to E$ be morphisms such that $\iota \circ g = \iota \circ h$, so the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & E \arrow[r, "\iota"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \end{tikzcd}$$

Then,

$$\begin{aligned} s \circ (\iota \circ g) &= (s \circ \iota) \circ g \\ &= (t \circ \iota) \circ g \\ &= t \circ (\iota \circ g) \end{aligned}$$

so $\iota \circ g$ equalises s and t and defines a fork $A \to X \rightrightarrows Y$. By the universal property of the equaliser, this fork must then factorise uniquely through E, so the map $A \to E$ that makes the diagram

$$\begin{tikzcd} A \arrow[d, dashed, "g=h"'] \arrow[dr, "\iota \circ g"] & & \\ E \arrow[r, "\iota"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \end{tikzcd}$$

commute is unique and hence ι is monic. ∎

A morphism that is the equaliser of some pair of parallel morphisms is called a *regular monomorphism*. As shown above, regular monomorphisms are monomorphisms, but the converse does not always hold. For instance, in **Top**, the monomorphisms are injective continuous functions, while the regular monomorphisms are topological embeddings (injective continuous functions that are homeomorphic on their image equipped with the subspace topology).

3.6 Coequalisers

Dualising, a *cofork* is a collection of objects and morphisms such that the following diagram commutes:

$$X \underset{t}{\overset{s}{\rightrightarrows}} Y \xrightarrow{g} A$$

and we say that g *coequalises* s and t.

The *coequaliser* of a pair of parallel morphisms $s,t : X \rightrightarrows Y$ is an object C equipped with a map $\pi : Y \to C$ such that the following diagram is a universal cofork:

$$X \underset{t}{\overset{s}{\rightrightarrows}} Y \xrightarrow{\pi} C$$

and we write $\mathrm{coeq}(s,t)$ for C.

Dual to regular monomorphisms, a morphism that is the coequaliser of some pair of parallel morphisms is called a *regular epimorphism.* By the dual of the previous theorem, every regular epimorphism is an epimorphism, but again, the converse does not necessarily hold.

4 Limits

In the previous constructions, we have started with some initial data – for products; a discrete pair of objects; for pullouts, a span; for equalisers, a parallel pair of morphisms – and we construct a new object equipped with maps to the given data in the most "general" way possible. That is, any other similar object will factor through our universal construction.

The notion of a *limit* unifies these three constructions. But first, we more precisely formulate how this initial data is specified.

4.1 Diagrams

So far, we have been frequently representing collections of objects and morphisms with the use of directed graphs, which we called diagrams. We formalise this notion in more detail now.

Recall that elements of a set X is can be viewed as functions $1 \to X$. Analogously, objects in a category $\mathscr{C}$ can be viewed as a functor $\mathbb{1} \to \mathscr{C}$ from the trivial category $\mathbb{1}$ – the functor just sends the unique object of $\mathbb{1}$ to some object of $\mathscr{C}$. A morphism in $\mathscr{C}$ can then similarly be viewed as a functor $\mathbb{2} \to \mathscr{C}$, where $\mathbb{2}$ is the arrow category $[\bullet \to \bullet]$ – such a functor then picks out two objects, and a morphism between them. More generally, this suggests that we can view any collection of objects and morphisms to be a functor from some indexing category to the target category:

A *diagram* (*of shape* $\mathscr{I}$) in a category $\mathscr{C}$ is a functor $D : \mathscr{I} \to \mathscr{C}$, where $\mathscr{I}$ is called the *indexing category.* If the indexing category $\mathscr{I}$ is small, then the diagram D is said to be small.

The directed graph representations used previously is then obtained by drawing the images of these functors, and functoriality requires that any compositions that exist in $\mathscr{I}$ also exist in this image graph, so the resulting graph is always commutative.

Theorem 4.1. *Functors preserve commutative diagrams.*

Proof. A commutative diagram in $\mathscr{C}$ is given by a functor $D : \mathscr{I} \to \mathscr{C}$. Given any functor $F : \mathscr{C} \to \mathscr{D}$, the composition $F \circ D : \mathscr{I} \to \mathscr{D}$ is by definition a commutative diagram in $\mathscr{D}$. ∎

We proved this result earlier by applying the functor to pairs of paths between objects and moving composition operations about with functoriality, but with this characterisation of diagrams, the result is easily apparent.

One important type of diagram is as follows: let $\mathscr{C}$ and $\mathscr{J}$ be categories, and let $\mathscr{J}$ be small; then, for any object $C \in \text{ob}(\mathscr{C})$, the *constant functor* (or *constant diagram*) $\Delta C : \mathscr{J} \to \mathscr{C}$ maps every object in $\mathscr{J}$ to C, and every morphism in $\mathscr{J}$ to id_C, similarly to an ordinary constant set function.

The assignment of objects to their constant functors is itself functorial; the *diagonal functor* $\Delta : \mathscr{C} \to [\mathscr{J},\mathscr{C}]$ sends every object $C \in \text{ob}(C)$ to the constant functor $\Delta C : \mathscr{J} \to \mathscr{C}$, and every morphism $f : A \to B$ to a natural transformation $\Delta f : \Delta A \to \Delta B$ that takes the same value f at every object in $\mathscr{J}$.

For an explanation to the diagonal functor's name, consider the case where $\mathscr{J} = \mathbf{2}$ is the discrete category on two objects. First, note that any functor $\mathbf{2} \to \mathscr{C}$ simply picks out pairs of objects of $\mathscr{C}$, so $[\mathbf{2},\mathscr{C}] \cong \mathscr{C} \times \mathscr{C}$. This gives the *binary* diagonal functor, $\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$, defined by $\Delta(X) = (X,X)$ and $\Delta(f) = \langle f,f \rangle$ for objects X and morphisms f.

For objects X, the pair (X,X) is really just the image of a constant functor from $\mathbf{2}$ to $\mathscr{C}$, consistent with the above, and for morphisms f, the pairing $\langle f,f \rangle$ is similarly just the components of a natural transformation $\Delta X \Rightarrow \Delta X$. This may seem similar to diagonal morphisms as defined earlier (Section 3.3), and in fact, diagonal functors are exactly the diagonal morphisms in **Cat**.

4.2 Cones

A *cone* over a diagram $F : \mathscr{J} \to \mathscr{C}$ with *summit* $X \in \text{ob}(\mathscr{C})$ is a natural transformation $\phi : \Delta X \to F$, and the components of this natural transformation are called the *legs* of the cone.

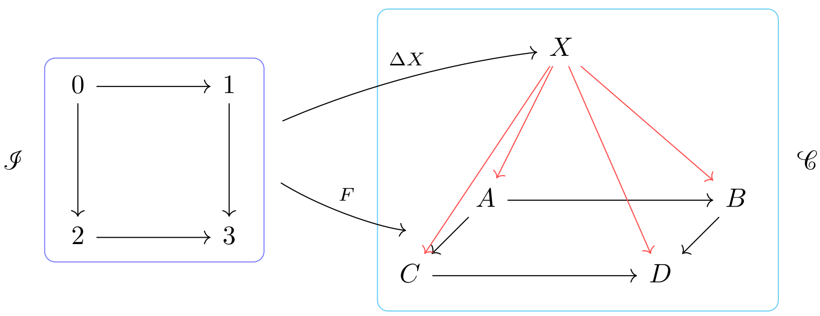
It may be helpful here to think of a natural transformation purely in terms of its components. That is, a cone over a diagram $F : \mathscr{J} \to \mathscr{C}$ is an assignment of a morphism $\phi_J : X \to F(J)$ for each object $J \in \text{ob}(\mathscr{J})$ in the diagram. Because one of the functors sends everything to a single object, the naturality square is contracted into a triangle; so, naturality is just the requirement that for every morphism $f : J \to K$ in $\mathscr{J}$, the following triangle in $\mathscr{C}$ commutes:

$$\begin{tikzcd} & X \arrow[dl, "\phi_J"'] \arrow[dr, "\phi_K"] & \\ F(J) \arrow[rr, "F(f)"'] & & F(K) \end{tikzcd}$$

As an example, suppose $\mathscr{J} = \mathbb{2} \times \mathbb{2}$, so the diagram in $\mathscr{C}$ is some commutative square:

$$\mathscr{J} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 2 \arrow[r] & 3 \end{tikzcd} \xrightarrow{F} \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \quad \mathscr{C}$$

(Here, $F(0) = A$, $F(1) = B$, etc.) A cone over this diagram with summit X is a collection of morphisms from X to each object of the diagram:



subject to the constraint that every triangle in the diagram involving two legs of the cone commutes. (The visual appearance of the resulting diagram also lends this construction its name.)

As with every other categorical construction, we can dualise the notion of a cone. A *cocone* under a diagram $F : \mathscr{I} \to \mathscr{C}$ with *nadir* $X \in \mathrm{ob}(\mathscr{C})$ is a natural transformation $\psi : F \to \Delta X$. The standard visualisation here is to place the object X below the diagram, drawing morphisms down to it from above.

$$\begin{tikzcd}
& A \arrow[rr] \arrow[dl] \arrow[dd] & & B \arrow[dl] \arrow[ddll] \\
C \arrow[rr] \arrow[dr] & & D \arrow[dl] & \\
& X & &
\end{tikzcd}$$

For a slight technicality, we've been saying that a cone with summit X over a diagram $F : \mathscr{I} \to \mathscr{C}$ is a natural transformation $\Delta X \Rightarrow F$, which is perfectly fine for almost all use cases, because X is completely determined by ϕ whenever $\mathscr{I}$ is non-empty. However, if this is the case, then ΔX and F are both the empty functor, which has nothing to do with X, so the natural transformation doesn't actually contain enough information to recover X.

For this reason, a cone is more properly a *pair* (X,ϕ), so that the summit is specified separately. This also necessitates a small modification to ΔX – to account for the case of the empty diagram, we use not the constant functor, but instead the unique functor from $\mathscr{I}$ to the terminal category $\mathbb{1} = \{\bullet\}$ composed with the inclusion $\iota_X : \mathbb{1} \to \mathscr{C}$ defined by $\bullet \mapsto X$ – but for non-empty diagrams, they function exactly as we have described previously.

We will interchangably refer to a (co)cone both as a natural transformation ϕ, and as a pair (X,ϕ) that includes the summit/nadir separately, depending on context.

4.3 Universal Cones

Let $F : \mathscr{I} \to \mathscr{C}$ be a diagram. The *limit* of F is the universal cone $\phi : \Delta X \Rightarrow F$ over the diagram F, and we denote the summit X of this cone by $\lim F$. Dually, the *colimit* is the universal cocone under F, with nadir denoted $\operatorname{colim} F$. By a slight abuse of language, sometimes, we call the summit X of this universal cone "the limit" by itself, but the limit is properly the entire data of the cone. To distinguish the two notions, we call this universal cone the *limit cone*, and the summit the *limit object*, or just *limit*.

The notion of universality for cones is again of unique factorisation; a cone over a diagram D with summit X is universal if every other cone over D with summit X' factors through it uniquely. That is, there exists a unique morphism $f : X' \to X$

$$\begin{tikzcd}
& & X' \arrow[d, dashed, "\exists! f"] \arrow[ddl] \arrow[ddr] \arrow[dddll] \arrow[ddd, bend left] & \\
& & X \arrow[dl] \arrow[dr] \arrow[ddll] \arrow[dd] & \\
& A \arrow[rr] \arrow[dl] & & B \arrow[dl] \\
C \arrow[rr] & & D &
\end{tikzcd}$$

such that every leg of the non-universal cone factors through the corresponding leg of the universal cone, so the triangle

$$\begin{tikzcd}
X' \arrow[d, dashed, "f"'] \arrow[dr, bend left, "\phi'_J"] & \\
X \arrow[r, "\phi_J"'] & F(J)
\end{tikzcd}$$

commutes for all $J \in \mathrm{ob}(\mathscr{I})$. Comparing X' and X with the constant functors $\Delta X'$ and ΔX, we see that f must be given by the single component of the constant natural transformation Δf.

In this case, we also call f a *cone morphism*, as it defines a way of mapping between cones. In fact, this allows us to define a *category of cones* – if $F : \mathscr{I} \to \mathscr{C}$ is a diagram, then the collection of cones and cone morphisms over that diagram forms a category. Associativity is inherited from $\mathscr{C}$, and identities are given by identity morphisms on the summits.

This allows us to characterise limits as terminal objects in the category of cones over a diagram. This justifies the use of the phrasing "*the* limit", rather than "*a* limit", as any two ostensibly distinct limits over the same diagram will be isomorphic up to unique isomorphism:

Theorem 4.2. *(Co)limits are essentially unique.*

That is, given any two universal cones $\phi : \Delta X \Rightarrow F$ and $\psi : \Delta Y \Rightarrow F$ over the same diagram $F : \mathscr{I} \to \mathscr{C}$, there is a unique cone isomorphism $\Delta X \cong \Delta Y$ (a cone morphism in the sense described above that additionally has an inverse).

Proof. A limit of F is a terminal object in the category of cones over F. But by Theorem 3.1, terminal objects are essentially unique. Essential uniqueness of colimits follows from duality. ∎

Conversely, if we begin with a map $f : X' \to X$, where X is the summit of a limit cone $\phi : \Delta X \Rightarrow F$, then the universal property says that the induced natural transformation $\psi : \Delta X' \Rightarrow F$ with components defined by precomposing every component of ϕ with f is also a cone, and furthermore, the uniqueness of the factorisation through a limit cone implies that this association between maps into X and cones over F is a bijection.* More concisely,

Theorem 4.3. *The maps into a limit object* $\lim F$ *are precisely the cones over the diagram* F*.*

4.4 Examples

Consider the most trivial possible case of $\mathscr{I}$ being the empty category, so all diagrams $F : \mathscr{I} \to \mathscr{C}$ are empty. A cone over the empty diagram with summit X is just the object X with no other data, so the limit of this diagram is then just an object such that any cone – which is just another object – factors through it uniquely. That is, there is a unique morphism to the limit object from every other object. This is exactly the terminal object of $\mathscr{C}$, so the limit of the empty diagram is the terminal object. Dually, a cocone under the empty diagram with nadir X is again just the object X with no other data, so the colimit of the empty diagram is the initial object.

If we take $\mathscr{I}$ to be the trivial category, then a diagram $F : \mathscr{I} \to \mathscr{C}$ consists of a single object, say, X_1. A cone over F with summit X is then just the object X equipped with a morphism to X_1 that every other object uniquely factors through it. But, we can just take $X = X_1$, as every object factors through the identity on X_1, so the limit is just the object itself. The colimit is also just the object X_1, for the same reason.

*The precomposition here may be reminiscent of the Yoneda embeddings; in fact, this assignment is functorial, and limits can be expressed as a type of representation, though we will not be doing this here.

If $\mathscr{I}$ is the discrete category on two objects, then a cone over $F : \mathscr{I} \to \mathscr{C}$ with summit X is the object X equipped with a pair of morphisms to the diagram:

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
X_1 & & X_2
\end{tikzcd}$$

The limit is then the object through which every other similar span factors – which is exactly the product of the two objects in the diagram, so a product is a special case of a limit. Similarly, the colimit of the two object diagram is a universal cospan, or, the coproduct.

We can generalise the binary (co)product to take n input objects by taking $\mathscr{I}$ to be the discrete category on n objects. The limit of the resulting diagram with objects $(X_i)_{i\in\mathscr{I}}$ is then the n-ary product, written as $\prod_{i\in\mathscr{I}} X_i$, and similarly, the colimit is the n-ary coproduct, written as $\coprod_{i\in\mathscr{I}} X_i$.

$$\begin{tikzcd}
 & & & \prod\limits_{i\in\mathscr{I}} X_i \arrow[dlll] \arrow[dll, "\pi_{-2}"'] \arrow[dl, "\pi_{-1}"'] \arrow[d, "\pi_0"] \arrow[dr, "\pi_1"] \arrow[drr, "\pi_2"] \arrow[drrr] & & & \\
\cdots \arrow[drrr] & X_{-2} \arrow[drr, "\iota_{-2}"'] & X_{-1} \arrow[dr, "\iota_{-1}"'] & X_0 \arrow[d, "\iota_0"] & X_1 \arrow[dl, "\iota_1"] & X_2 \arrow[dll, "\iota_2"] & \cdots \arrow[dlll] \\
 & & & \coprod\limits_{i\in\mathscr{I}} X_i & & &
\end{tikzcd}$$

This generalisation encompasses the previous two examples as well; the empty and trivial categories can be viewed as discrete categories on zero and one objects, respectively, so the *nullary* product of the empty family is exactly the terminal object (which is one of the reasons why the terminal object is denoted 1), and the *unary* product is just the original input object.

If we instead take the limit of a span $\bullet \to \bullet \leftarrow \bullet$, we obtain the diagram:

$$\begin{tikzcd}
\lim F \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_0"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

The projection π_0 can be omitted as it is implied by commutativity, so the limit cone is just a commutative square. The universal property then says that every similar square factors through $\lim F$, so this diagram is exactly the pullback of the two morphisms in the original span. Similarly, the colimit of a span $\bullet \leftarrow \bullet \to \bullet$ is then the pushout of the two morphisms.

We can also take the limit over a parallel pair of morphisms $\bullet \rightrightarrows \bullet$ to obtain the equaliser:

$$\begin{tikzcd}
\lim F \arrow[d, "\iota"'] \arrow[dr, "f"] & \\
\bullet \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \bullet
\end{tikzcd}$$

Commutativity implies that ι forms a fork with s and t, and that f can be omitted to obtain the usual equaliser diagram. Colimits over parallel pairs then similarly yield coequalisers.

4.5 Completeness

Diagrams can be of any shape, but limits over arbitrary diagrams do not always exist in arbitrary categories. For example, we have already seen that products of distinct objects do not exist in discrete categories due to a lack of morphisms to use as projections.

A category $\mathscr{C}$ *has limits of shape* $\mathscr{I}$ if every diagram F of shape $\mathscr{I}$ in $\mathscr{C}$ admits a limit in $\mathscr{C}$. Similar variations on this wording can be applied to special classes of limits like products ("$\mathscr{C}$ has products") [Lei14]. One important case is of *small limits*, where the indexing category of the diagram is small.

A category $\mathscr{C}$ is *complete* if it has all small limits. That is, if it has limits of shape $\mathscr{I}$ for every small category $\mathscr{I}$. Dually, $\mathscr{C}$ is *cocomplete* if it has all small colimits. If a category is both complete and cocomplete, it is *bicomplete*. The condition of having *all* (large) limits is too strong of a condition for any useful categories to satisfy, so we often do not consider it.

A weaker form of completeness is of *finite completeness* – a category is *finite* if it has finitely many *morphisms* (which also implies that there are finitely many objects), and a *finite limit* is a limit of shape $\mathscr{I}$ for a finite indexing category $\mathscr{I}$. A category $\mathscr{C}$ is then *finitely complete* if it has all finite limits, *finitely cocomplete* if it has all finite colimits, and *finitely bicomplete* if it is finitely complete and cocomplete.

Theorem (Existence Theorem for Limits)**.**

- *If a category $\mathscr{C}$ has all products and binary equalisers, then $\mathscr{C}$ is complete.*
- *If $\mathscr{C}$ has binary products, a terminal object and binary equalisers, then $\mathscr{C}$ is finitely complete.*

Proof. Let $\mathscr{C}$ be a category with all products and equalisers, and let $F : \mathscr{I} \to \mathscr{C}$ be a diagram. If we ignore all morphisms in $\mathscr{I}$, then the limit of F would just be the product $\prod_{I \in \mathscr{I}} F(I)$. The morphisms then add the additional constraints that for each morphism $f : A \to B$ in $\mathscr{I}$, the triangle

$$\begin{tikzcd}
\prod\limits_{I \in \mathscr{I}} F(I) \arrow[d, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
F(A) \arrow[r, "F(f)"'] & F(B)
\end{tikzcd}$$

must commute.

We construct another product $\prod_{(f:A\to B) \text{ in } \mathscr{I}} F(B)$ indexed over the morphisms in $\mathscr{I}$. Now, recall that a map from an object X into a product $\prod X_i$ is determined entirely by the component maps $X \to X_i$. The two routes $\prod_{I \in \mathscr{I}} F(I) \to F(B)$ through the triangle above, given by π_B and $F(f) \circ \pi_A$, can be reindexed by morphisms in $\mathscr{I}$ to obtain:

$$\begin{aligned} s_{(f:A\to B)} &= \pi_B \\ t_{(f:A\to B)} &= F(f) \circ \pi_A \end{aligned}$$

and hence these components define a pair of maps into the product indexed by morphisms:

$$\begin{tikzcd}
\prod\limits_{I \in \mathscr{I}} F(I) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \prod\limits_{(f:A\to B) \text{ in } \mathscr{I}} F(B)
\end{tikzcd}$$

Because we want the triangles of the form given above to commute, these paths should be equal, so we construct the equaliser $L \xrightarrow{p} \prod_I F(I)$ of s and t. We claim that the components of p form a limit cone on F.

First, note that $s \circ p = t \circ p$ are morphisms into a product, so their components also agree by the universal property of the product. Writing π' for the projections of $\prod_f F(B)$, we have,

$$s \circ p = t \circ p$$

$$\begin{aligned}
\pi'_f \circ s \circ p &= \pi'_f \circ t \circ p \\
s_f \circ p &= t_f \circ p \\
\pi_B \circ p &= F(f) \circ \pi_A \circ p \\
p_B &= F(f) \circ p_A
\end{aligned}$$

which is exactly the commutativity requirement of a cone, so the components of p form a cone $\phi : \Delta L \Rightarrow F$. Any other cone $\psi : \Delta L' \Rightarrow F$ must induce a map that equalises s and t, so L' factors through L by the universal property of the equaliser, and hence ϕ is a limit cone.

$$\begin{tikzcd}
L' \arrow[d, dashed, "\exists!"'] \arrow[dr, "p'"] & & \\
L \arrow[r, "p"'] & \prod\limits_{I \in \mathscr{I}} F(I) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \prod\limits_{(f:A \to B) \text{ in } \mathscr{I}} F(B)
\end{tikzcd}$$

We have now expressed an arbitrary limit in terms of products and equalisers, so $\mathscr{C}$ is complete.

Now suppose that $\mathscr{C}$ only admits products that are binary. By induction, $\mathscr{C}$ also has n-ary products for finite $n \geq 1$, and since $\mathscr{C}$ has a terminal object (the nullary product), it has all finite products. The previous argument then applies, but only for finite indexing categories, and hence $\mathscr{C}$ is finitely complete. ■

The converse of the propositions are also clearly true, so the implications are biconditional; (finite) completeness is equivalent to having (finite) products and binary equalisers.

4.6 Limits and Functors

Let $D : \mathscr{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$ with limit object X and limit cone ϕ. A functor $F : \mathscr{C} \to \mathscr{D}$ *preserves* this limit if $F(X)$ is the limit object of the composite diagram $F \circ D : \mathscr{I} \to \mathscr{D}$, and the whiskering $F \cdot \phi$ (that is, the natural transformation with components defined by $(F \cdot \phi)_A = F(\phi_A)$) is a limit cone on $F \circ D$.

Theorem 4.4. *The hom-bifunctor* $\hom(-,-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ *preserves limits in both arguments.*

This means that, for example, $\hom(X,A) \times \hom(X,B) \cong \hom(X,A \times B)$ and $\hom(A,X) \times \hom(B,X) \cong \hom(A \amalg B,X)$, with the product being transformed into a coproduct in the first argument by contravariance.

Proof. Recall that the morphisms into a limit object $\lim D$ correspond to the cones over the diagram D (Theorem 4.3). More specifically, the morphisms $X \to \lim D$ are the cones over D with apex X, so we have

$$\hom_{\mathscr{C}}(X, \lim D) \cong \hom_{[\mathscr{I},\mathscr{C}]}(\Delta X,D) \tag{1}$$

naturally in X.

If we have $\mathscr{C} = \mathbf{Set}$ and $X = 1$ in the previous, then

$$\begin{aligned}
\hom_{\mathbf{Set}}(1, \lim D) &\cong \hom_{[\mathscr{I},\mathbf{Set}]}(\Delta 1,D) \\
\lim D &\cong \hom_{[\mathscr{I},\mathbf{Set}]}(\Delta 1,D) \qquad (2)
\end{aligned}$$

so the set of cones over a diagram with apex 1 is precisely the limit of that diagram.

Now, consider a cone ϕ over the composite diagram $\hom(X,-) \circ D = \hom\big(X,D(-)\big) : \mathscr{I} \to \mathbf{Set}$ with apex 1. Every morphism $f : I \to J$ between objects I,J in $\mathscr{I}$ induces a function $\hom(\mathrm{id}_X,f) = f \circ (-)$

from $\hom\big(X,D(I)\big)$ to $\hom\big(X,D(J)\big)$, so such a cone is a collection of components such that the triangle

$$\begin{tikzcd} & 1 \arrow[dl, "\phi_I"'] \arrow[dr, "\phi_J"] & \\ \hom\big(X,D(I)\big) \arrow[rr, "f\circ(-)"'] & & \hom\big(X,D(J)\big) \end{tikzcd}$$

commutes for all $I,J \in \mathrm{ob}(\mathscr{I})$ and $f \in \hom_{\mathscr{I}}(I,J)$.

Because 1 is a singleton set, the component ϕ_I effectively just selects a single function from the hom-set $\hom\big(X,D(I)\big)$. We can alteratively interpret this as ϕ assigning a function $\psi_I : X \to D(I)$ to each object I such that $f \circ \psi_I = \psi_J$ for all objects I,J in $\mathscr{I}$ i.e., such that

$$\begin{tikzcd} & X \arrow[dl, "\psi_I"'] \arrow[dr, "\psi_J"] & \\ D(I) \arrow[rr, "f"'] & & D(J) \end{tikzcd}$$

commutes, which is exactly a cone over D. Thus, the set of cones over F with apex X is isomorphic to the set of cones over the hom-sets $\hom\big(X,D(-)\big)$ with apex 1:

$$\hom_{[\mathscr{I},\mathscr{C}]}(\Delta X,D) \cong \hom_{[\mathscr{I},\mathbf{Set}]}\big(\Delta 1, \hom_{\mathscr{C}}\big(X,D(-)\big)\big) \tag{3}$$

Combining the above, we deduce that $\hom_{\mathscr{C}}(X, \lim D)$ is the limit of

So combining the previous isomorphisms, we have,

$$\begin{aligned} \hom_{\mathscr{C}}(X, \lim D) &\cong \hom_{[\mathscr{I},\mathscr{C}]}(\Delta X,D) && \text{[by (1)]} \\ &\cong \hom_{[\mathscr{I},\mathbf{Set}]}\big(\Delta 1, \hom_{\mathscr{C}}\big(X,D(-)\big)\big) && \text{[by (3)]} \\ &\cong \lim\big(\hom\big(X,D(-)\big)\big) && \text{[by (2) in reverse]} \end{aligned}$$

Preservation of limits in the first argument follows from duality. ■

5 Adjunctions

Let $\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{D}$ be categories and opposing functors. We say that L is *left adjoint* to R, and that L is *right adjoint* to R, if,

$$\hom_{\mathscr{C}}(L(A),B) \cong \hom_{\mathscr{D}}(A,R(B))$$

naturally in $A \in \mathrm{ob}(\mathscr{C})$ and $B \in \mathrm{ob}(\mathscr{D})$, and we write $L \dashv R$ to denote this relationship.

The image of a morphism under this isomorphism is called its *adjoint transposition*. For a morphism $f : A \to R(B)$ in $\mathscr{C}$, its adjoint transposition $L(A) \to B$ in $\mathscr{D}$ is called its *left adjunct*, and is denoted $f^\sharp$, and similarly, the adjoint transposition of a morphism $g : L(A) \to B$ in $\mathscr{D}$ is called its *right adjunct*, and is denoted $g^\flat$.

Note that the functors L and R are *adjoint*, while the morphisms f and $f^\sharp$ or $g^\flat$ and g are *adjunct* [Mac13].

Adjoint functors correspond to a weak form of equivalence between categories, in that every equivalence functor and its inverse are adjoint, and conversely, if L and R are both fully faithful, then they form an equivalence of categories.

Example.

- The functor $\mathscr{C} \to \mathbb{1}$ to the trivial category has a right adjoint if and only if $\mathscr{C}$ has a terminal object, and a left adjoint if and only if $\mathscr{C}$ has an initial object.
- If $\mathscr{C}$ has binary products, then the *cartesian product bifunctor* $- \times - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is right adjoint to the diagonal functor $\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$.
- A so-called *forgetful functor* is a functor that does nothing to objects and morphisms apart from "forgetting" some of the structure of the original category. For instance, the forgetful functor U : **Grp** $\to$ **Set** sends groups to their underlying sets – "forgetting" the group structure – and group homomorphisms to themselves, but considered as functions. The left adjoint to a forgetful functor from a category of algebraic objects to **Set** is given by the functor that constructs the free object of the appropriate type on any given set.

Theorem 5.1. *A left or right adjoint, if it exists, is unique up to natural isomorphism.*

Proof. We give the proof for the uniqueness of a right adjoint. The uniqueness of left adjoints follows similarly.

Suppose $R_1,R_2 : \mathscr{C} \to \mathscr{D}$ are right adjoint to $L : \mathscr{D} \to \mathscr{C}$, so for each object D in $\mathscr{D}$, we have the natural isomorphisms of functors

$$\hom\big(-,R_1(D)\big) \cong \hom\big(L(-),D\big) \cong \hom\big(-,R_2(D)\big)$$

also natural in D. By the Yoneda lemma, this isomorphism is induced by an isomorphism $R_1(D) \cong R_2(D)$. As the Yoneda embedding is fully faithful, $R_1(D) \cong R_2(D)$ is also natural in D, and we have $R_1 \cong R_2$. ■

Theorem 5.2. *Left adjoints preserve colimits and right adjoints preserve limits.*

Proof. Let $(L \dashv R) : \mathscr{C} \to \mathscr{D}$ be a pair of adjoint functors, and let $F : \mathscr{I} \to \mathscr{D}$ be a diagram in $\mathscr{D}$ whose limit $\lim F$ exists. Then, for every C in $\mathscr{C}$,

$$\begin{aligned} \hom_{\mathscr{C}}\big(C,R(\lim F)\big) &\cong \hom_{\mathscr{D}}\big(L(C),\lim F\big) \\ &\cong \lim \hom_{\mathscr{D}}\big(L(C),F(-)\big) \\ &\cong \lim \hom_{\mathscr{C}}\big(C,(R \circ F)(-)\big) \\ &\cong \hom_{\mathscr{C}}\big(C,\lim(R \circ F)\big) \end{aligned}$$

with all isomorphisms natural in C, so by the Yoneda lemma, $R(\lim F) \cong \lim(R \circ F)$. That left adjoints preserve colimits follows from duality. ■

6 Subobjects

Recall that an element $x \in X$ is simply a map $x : 1 \to X$, where 1 is terminal. We have also seen the notion of a generalised element of shape S in which we identify certain features of a space with maps $S \to X$ into the space, but can also abstract the notion of a *subset* into categories other than **Set** in a similar way.

Note that the notion of a subset in material set theories is not isomorphism invariant: for instance, consider the sets $\{1\}$, $\{2\}$, $\{\text{cat}\}$, and $X = \{1,2,3\}$. We have that $\{1\} \subseteq X$ and $\{2\} \subseteq X$ are distinct subsets of X, and $\{\text{cat}\} \nsubseteq X$, but these singleton sets are all isomorphic! Categories don't care about how we label elements (and in arbitrary categories, it currently doesn't make sense to ask about elements of objects anyway).

What we *do* care about is how these sets *embed* into X or not – just as elements of sets are not themselves sets, subsets of sets are also not sets.

Consider the class of monomorphisms into an object, X. We can define a preorder on this class as follows. Let $f : A \rightarrowtail X$ and $g : B \rightarrowtail X$ be monomorphisms into X. Then, let $f \leq g$ if f factors through g – that is, there exists a morphism $h : A \to B$ such that $f = g \circ h$ (h is necessarily unique if it exists, since g is monic, and must also be a monomorphism, as f is monic). If both $f \leq g$ and $g \leq f$, or equivalently, if h is an isomorphism, then f and g are *isomorphic* morphisms, and we write $f \cong g$.

A *subobject* of an object X is then an isomorphism class of monomorphisms into X. If $S : A \rightarrowtail X$ is a monomorphism into X, then we write $[S] \subseteq X$ for the isomorphism class/subset represented by S. As a small abuse of notation, we sometimes pick a representative monomorphism S and just write $S \subseteq X$.

If we interpret this definition in **Set**, a monomorphism $f : A \rightarrowtail X$ describes (or rather, represents) the subset $f(A)$, which, as a set, is isomorphic to A via f. Note that the object A is entirely irrelevant here; a different function $g : A \rightarrowtail X$ with image distinct from f represents a different subset from f; and conversely, a function $h : B \rightarrowtail X$ that agrees with f will witness the same subset as f. (In fact, they will be isomorphic, and will reside in the same isomorphism class, as we'd might hope.)

This is what distinguishes, say, $\{1\}$ from $\{2\}$ in the context of being a subset of $X = \{1,2,3\}$; these subsets are distinct because the ways they embed into X do not admit a factorisation in both directions – being a subobject is not really a relation on objects, but rather on morphisms.

This definition also helps to resolve some problems in material set theories. For instance, one question that inevitably arises in material set theory, is to ask whether $\mathbb{Z}$ is a *really* subset of $\mathbb{R}$ or not. The former is a set of equivalence classes of von Neumann ordinals (or Zermelo ordinals, etc.), while the latter is a set of Dedekind cuts (or equivalence classes of Cauchy sequences, or elements of a complete ordered field, etc.), so it seems that $\mathbb{Z}$ cannot be a subset of $\mathbb{R}$ – but when integers and real numbers are interpreted in the conventional non-set-theoretic way, every integer is clearly a real number. (This is also very similar in flavour to our earlier question of whether $3 \in 17$ or not.)

The structural viewpoint says that it doesn't make sense to ask whether $\mathbb{Z} \subseteq \mathbb{R}$ or not because their *elements* are the same or not, but instead to ask if there is some *map* $\mathbb{Z} \to \mathbb{R}$ that *witnesses* that $\mathbb{Z} \subseteq \mathbb{R}$. Note that the maps $\mathbb{Z} \to \mathbb{R}$ also depend on the structure being considered, or rather, the ambient category – if $\mathbb{Z}$ and $\mathbb{R}$ are being considered as, say, groups or rings (i.e. objects in **Grp** or **Ring**, respectively), then we can only ask if there are any group or ring homomorphisms (i.e. morphisms in the respective category) that provide a reason for us to write $\mathbb{Z} \subseteq \mathbb{R}$, in the context of being a subgroup or subring.

Furthermore, this definition includes information about *how* an object embeds into another, rather than just *that it does* in some unspecified way. This definition of a subobject in more general categories of structured sets is also often more natural than the set-theoretic notion of having the underlying set of a structure be a subset of some larger structure (as well as generalising to non-concrete categories).

This does, however, have the somewhat odd side effect that, for example, $\{0,1\} \subseteq \mathbb{Z}$ is not the same as the subset $\{0,1\} \subseteq \mathbb{R}$ because one is an isomorphism class of monomorphisms into $\mathbb{Z}$, and the other of monomorphisms into $\mathbb{R}$. However, they are related in that we can lift the former isomorphism class to the latter with a suitable function $\mathbb{Z} \to \mathbb{R}$. This further reflects the structural idea that objects should be characterised by their connections to other objects – the ambient containing set is different, so these subsets hold different structures and are hence distinct entities.

Now, the $\leq$ relation on monomorphisms is only a preorder as it is not antisymmetric: $f \leq g$ and $g \leq f$ only imply that f and g are isomorphic and not strictly equal. However, since subobjects are isomorphism classes of monomorphisms, isomorphic morphisms do induce strictly equal subobjects, so the preorder $\leq$ induces a partial order $\subseteq_X$ on the subobjects of X. Again, we often pick representative monomorphisms A and B and abbreviate $[A] \subseteq_X [B]$ to just $A \subseteq_X B$. This notion of containment of subsets is inherently local to subsets of an ambient set X, unlike in a material set theory, where the subset relation, being defined in terms of the global membership relation, is compatible with any two arbitrary sets.

Importantly, the subset relation $\subseteq_X$ between *two subobjects* of X is distinct from the symbol $\subseteq$ indicating that a subobject belongs to a true *object* X. The former is a relation, local to the collection of subobjects of some given object, while the latter is just notation for a class of morphisms.

For the notion of membership on subsets, we say that x *is a member of* a and write $x \in a$ if $x \in X$, $a : S \rightarrowtail X$ is a subset of X, and there exists an element $\bar{x} \in S$ such that $a(\bar{x}) = x$:

$$\begin{tikzcd} & S \arrow[d, "a"] \\ 1 \arrow[ur, "\bar{x}"] \arrow[r, "x"'] & X \end{tikzcd}$$

That is, x is a member of a if x lifts through a. Again, membership for subsets is only defined locally within a containing set S, unlike in a material set theory, so it doesn't make sense to ask whether $x \in y$ or not for arbitrary sets x and y.

We can also dualise the notion of a subobject. The collection of epimorphisms from an object X are similarly preordered by factorisation – that is, we write $f \leq g$ for epimorphisms $f : X \twoheadrightarrow A$ and $g : X \twoheadrightarrow B$ if there exists a (necessarily unique and epic) morphism $h : A \to B$ such that $f = h \circ g$, and two epimorphisms are isomorphic if they factor through each other, or equivalently, if h is an isomorphism. A *quotient object* of X is then an isomorphism class of epimorphisms from X.

One important kind of subobject is given by the notion of an *image*. In **Set**, we can identify the image of a function $f : A \to B$ with a particular subset of B, namely, the subset consisting of all the elements of B of the form $f(a)$ for some $a \in A$. We can describe this situation more generally, without reference to elements, as follows.

The *image* of a morphism $f : A \to B$ is the minimal subobject of B through which f factorises universally into a composition $A \xrightarrow{e} \mathrm{im}(f) \overset{m}{\rightarrowtail} B$. That is, $f = m \circ e$, and for every other factorisation $A \xrightarrow{e'} S \overset{m'}{\rightarrowtail} B$, we have $\mathrm{im}(f) \subseteq_B S$. Then, the morphism $e : A \to \mathrm{im}(f)$ is called the *corestriction* of f.

Dually, the *coimage* of a morphism is the image of the corresponding morphism in the opposite category, or equivalently, the maximal quotient object of A through which f factors through universally.

6.1 The Subobject Classifier

For **Set** in particular, another way to characterise a subset A of a given set X is as a function $\chi_A : X \to 2$, where $2 = \{\top, \bot\}$, by taking χ_A to be the indicator function of A defined by

$$\chi_A(x) = \begin{cases} \top & x \in A \\ \bot & x \notin A \end{cases}$$

So, there is a bijection between the subobjects $A \rightarrowtail X$ and the functions $\chi_A : X \to 2$ given by $\chi_A \mapsto A = \chi_A^{-1}\big[\{\top\}\big]$. Now, recall that preimages are a special case of pullbacks, so this bijection says that for every subset $A \subseteq X$, there is a unique function $\chi : X \to 2$ such that

$$\begin{tikzcd} A = f^{-1}\big[\{\top\}\big] \arrow[r, "!"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"] \\ X \arrow[r, "\chi"'] & 2 \end{tikzcd}$$

is a pullback [Lei11]. Nothing here is really specific to **Set**, so we can abstract this diagram into any arbitrary category that admits pullbacks and has a terminal object to obtain the following definition:

A *subobject classifier* in a category $\mathscr{C}$ is an object Ω and map $\top : 1 \to \Omega$ such that every monomorphism $m : A \rightarrowtail X$ is the pullback of $\top$ along a unique morphism $\chi_m : X \to \Omega$ called the *characteristic morphism*.

That is, for every monomorphism $m : A \rightarrowtail X$, there exists a unique morphism $\chi_m : X \to \Omega$ such that

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

is a pullback square.

The object Ω is then called the *object of truth values*, a morphism $X \to \Omega$ a *truth value*, and the morphism $\top : 1 \to \Omega$ the truth value *true*.

In a concrete category, commutativity of the square, $\chi_m \circ m = \top \circ !$, intuitively means that χ_m is "true" everywhere over the image of m. The diagram being a pullback then means that A is the "largest" subobject of X with this property, so χ_m is true exactly in the image of A, and if we have any other object with a map into X that makes χ_m similarly true, then it will factor uniquely through m.

We give another characterisation of a subobject classifier [Lei11].

For any object $X \in \mathrm{ob}(\mathscr{C})$, we write $\mathrm{Sub}_{\mathscr{C}}(X)$, or just $\mathrm{Sub}(X)$, to denote the collection of subobjects of X. If this collection is a set (as opposed to a proper class) for every object in a category, then the category is called *well-powered*.

Suppose $\mathscr{C}$ has finite limits and is locally small. Then, every map $f : X \to Y$ in $\mathscr{C}$ induces a map $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ between the subobject posets in the reverse direction by pullback. That is, if $g : B \rightarrowtail Y$ is a subobject in $\mathrm{Sub}(Y)$, then it is a monomorphism, and because pullbacks preserve monomorphisms, the pullback

$$\begin{tikzcd} X \times_Y B \arrow[r, "\pi_2"] \arrow[d, tail, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

of g along f is another monomorphism into X, which is just a subobject in $\mathrm{Sub}(X)$. This defines a functor $\mathrm{Sub} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$.

A subobject classifier is then exactly a representation of this functor.

Recall that for Sub to be representable, there must exist an object Ω such that $\mathrm{Sub}(X) \cong \hom_{\mathscr{C}}(X,\Omega)$ naturally in X. This intuitively corresponds to the previous idea that the subsets of X correspond to maps $X \to 2$ (this also implies that $\mathscr{C}$ is well-powered), and furthermore, that this correspondence is canonical. This holds similarly in arbitrary categories, in that the subobjects of an object X naturally correspond to morphisms $X \to \Omega$; hence the name subobject *classifier*.

Subobject posets also allow us to abstract various other familiar operations on subsets.

The pullback of (two representative monomorphisms of) two subobjects $A \rightarrowtail X$ and $B \rightarrowtail X$ is denoted $A \cap_X B$ or $A \wedge_X B$, and is called the *intersection* or *meet* of the two subobjects, and dually, the pushout of the two subobjects is denoted $A \cup_X B$ or $A \vee_X B$, and is called the *union* or *join* of the two subobjects. (When the object X is clear, the subscripts are often suppressed.)

These definitions are based on operations in the ambient category, $\mathscr{C}$; namely that of pullbacks and pushouts, but it turns out that these intersections and unions can be expressed entirely internally to the subobject poset:

Theorem 6.1. *The pullback of two subobjects of X in $\mathscr{C}$ is their product in $\mathrm{Sub}_{\mathscr{C}}(X)$ interpreted as a thin category, and similarly, their pushout in $\mathscr{C}$ is their coproduct in $\mathrm{Sub}_{\mathscr{C}}(X)$.*

Proof. Monomorphisms are stable under pullback, so the pullback of two objects is also a subobject in $\mathrm{Sub}(X)$. The universal property of the pullback then says that it factors through every other pair of monomorphisms into X, which is exactly a product in $\mathrm{Sub}(X)$. The proof for unions is similar.* ∎

Intuitively, the intersection of two subobjects with representing monomorphisms f and g in $\mathrm{Sub}(X)$ should be the maximal subobject that factors through both f and g, which is exactly their meet, $f \wedge g$, in the order-theoretic sense – but meets are exactly the products in a thin category. Similarly, the union of the representatives f and g should intuitively be the minimal subobject that f and g both factor through, which is exactly their order-theoretic join, $f \vee g$.

The above correspondence between pullbacks in $\mathscr{C}$ and products in $\mathrm{Sub}(X)$ allow us to transport some properties of $\mathscr{C}$ into this thin subobject poset category: if $\mathscr{C}$ is finitely (co-,bi-)complete, then the collection of subobjects is not just a poset, but is furthermore a meet-semilattice (join-semilattice, lattice, respectively). Given some favourable conditions† which we will assume, these meets and joins also distribute over each other, so the collection of subobjects is additionally a distributive lattice.

A *Boolean category* is a category in which every subobject $A \rightarrowtail X$ has a *complement* subobject $B \rightarrowtail X$ such that $A \wedge B \cong \varnothing$ is initial in the subobject lattice, and $A \vee B \cong X$ – that is, the subobject lattice $\mathrm{Sub}(X)$ of any object X is a *Boolean* lattice.

6.2 Power Objects

We can also abstract the notion of a power set into categories other than **Set**. In set theory, the power set $\mathscr{P}(A)$, also written perhaps more suggestively as 2^A, of a set A is the set of all subsets of A. This is a definition reliant on the set-theoretic subset relation, which is not a categorical notion, so we want to find a way to characterise 2^A with the maps to or from it.

Also note that the notion of a power object is far more specific to **Set** than subobjects are. For instance, there isn't really a notion of a "power group", in that the collection of all subgroups of a group does not have group structure itself – collections of subobjects in this sense generally do not inherit the structure required to be an object in their own right.

The idea here is that the functions $B \to 2^A$ are naturally isomorphic to subobjects of $A \times B$:

$$\hom(B, 2^A) \cong \mathrm{Sub}(A \times B)$$

Or equivalently, that the power set of a set A is exactly a representation of the functor $\mathrm{Sub}(A \times -)$. This is already a complete description of power sets that generalises to arbitrary categories, but we can again give a more concrete definition in terms of pullbacks.

Consider a set B, along with a function $f : B \to 2^A$ that maps elements of B to subsets of A. This induces a relation $R \subseteq A \times B$ that identifies which elements of A belong to the images of elements in B:

$$R = \big\{(a,b) \in A \times B : a \in f(b)\big\}$$

That is, aRb if and only if $a \in f(b)$. There is also the canonical relation $\in_A \subseteq A \times 2^A$ that identifies which elements of A belong to which subsets S of A:

$$\in_A = \big\{(a,S) \in A \times 2^A : a \in S\big\}$$

So $a \in_A S$ if and only if... $a \in S$.

Now, we can define a function $R \to \in_A$ by applying f to the second component of elements in R, so $(a,b) \in R$ if and only if $\big(a, f(b)\big) \in (\in_A)$. Then, we see that R is the preimage of $\in_A$ by $\mathrm{id}_A \times f$ (with the

*But *not* dual. Dualising this proof yields a result about unions in the poset of *epi*morphisms of X.

†The category must be at least a *coherent category*, which we have not discussed, but all the categories we will see later will be coherent. In particular, topoi are always coherent.

appropriate restrictions), so we have the following pullback:

$$\begin{tikzcd}
R = f^{-1}[\in_A] \arrow[r, "\mathrm{id}_A \times f\big|_R"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \in_A \arrow[d, hook] \\
A \times B \arrow[r, "\mathrm{id}_A \times f"'] & A \times 2^A
\end{tikzcd}$$

Again, we can abstract this diagram into other categories.

A *power object* of an object A consists of an object Ω^A and a monomorphism $\in_A \rightarrowtail A \times \Omega^A$ such that for every other object B and monomorphism $m : R \rightarrowtail A \times B$, there exists a unique morphism $\chi_m : B \to \Omega^A$ such that

$$\begin{tikzcd}
R \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \in_A \arrow[d, tail] \\
A \times B \arrow[r, "\mathrm{id}_A \times \chi_m"'] & A \times \Omega^A
\end{tikzcd}$$

is a pullback square.

Now, the notation 2^A suggests a connection between how we classified subsets of A with maps $A \to 2$ before, and in fact, this definition is compatible with the notion of a subobject classifier in that if $A \cong 1$ is terminal, then $1 \times \Omega^1 \cong \Omega$ and $\mathrm{id}_A \times \chi_m \cong \chi_m$, so the pullback reduces to the subobject classifier pullback square from before, and the power object of a terminal object is exactly the subobject classifier.

7 Monoidal Categories

A monoidal category is a category that has properties similar to an algebraic monoid; it is equipped with a binary endofunctor that satisfies the monoid axioms in a certain sense.

A *monoidal category* $(\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of:

- A category $\mathscr{C}$;
- A bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the *tensor product*, written in infix notation;
- A designated object I in $\mathscr{C}$ called the *unit*;
- A natural isomorphism $\alpha : \big((-) \otimes (-)\big) \otimes (-) \Rightarrow (-) \otimes \big((-) \otimes (-)\big)$ with components of the form $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ called the *associator*;
- A natural isomorphism $\lambda : I \otimes (-) \Rightarrow (-)$ with components of the form $\lambda_A : (I \otimes A) \to A$ called the *left unitor*;
- A natural isomorphism $\rho : (-) \otimes I \Rightarrow (-)$ with components of the form $\rho_A : (A \otimes I) \to A$ called the *right unitor*;

subject to the *coherence conditions* that the following diagrams commute:

- the *triangle identity*:

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_A"] \\
& A \otimes B &
\end{tikzcd}$$

- the *pentagon identity*:

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C \otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A \otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes \mathrm{id}_D"'] & & A \otimes (B \otimes (C \otimes D)) \arrow[d, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B \otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D)
\end{tikzcd}$$

The tensor product being a bifunctor means that the category is closed with respect to the tensor product, while the left and right unitors say that $I \otimes A \cong A$ and $A \otimes I \cong A$ for any object A, so I acts as the identity of the tensor product. The associator then says that $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ for all objects A, B, and C, so the tensor product is also associative. This is all the structure that a monoid demands, so, why do we have the additional coherence conditions?

The analogue of this expression in a monoidal category is an object of the form $(A \otimes I) \otimes B$; the right unitor guarantees that the unit I acts like the identity, giving $A \otimes I \cong A$, so $(A \otimes I) \otimes B \cong A \otimes B$, but again, we could use the associator to first rebracket $(A \otimes I) \otimes B \cong A \otimes (I \otimes B)$ before reducing with the left unitor. However, there's no reason why we should expect that these two orderings will produce exactly equal objects. The triangle identity is exactly the condition that this equality *does* hold, and similarly, the pentagon identity guarantees that every way we rebracket an expression yields isomorphic objects.

For similar reasons, monoidal categories also admit several notions of commutativity. A *braided monoidal category* is a monoidal category equipped with an additional natural isomorphism with components of the form $\beta_{A,B} : A \otimes B \to B \otimes A$ called the *braiding*, subject to two additional coherence conditions called the *hexagon identities* that ensure compatibility with the associator.

The tensor product in a braided monoidal category is then commutative in the sense that reversing the order of a tensor product yields isomorphic objects, as given by the braiding. However, applying the braiding twice may yield objects that are not strictly equal, but only isomorphic. If these objects *are* strictly equal – that is, $\beta_{A,B} \circ \beta_{B,A} = \mathrm{id}_{A \otimes B}$ – then the category is furthermore a *symmetric monoidal category*. The tensor product in a symmetric monoidal category is then "as commutative as possible".

One special case of a monoidal category is if the monoidal structure is given by the categorical product; such a category is called a *cartesian monoidal category*. Because categorical products are essentially unique, every cartesian monoidal category is necessarily symmetric monoidal.

Example. **Set** is monoidal, with the tensor product given by the categorical product, so **Set** is cartesian monoidal. Note that $(A \times B) \times C \neq A \times (B \times C)$, but there is an obvious isomorphism between them that we can use as the associator. Similarly, the unitors are given by the isomorphisms $1 \times A \cong A$ and $A \times 1 \cong A$.

Categories can often be monoidal in multiple ways; for instance, **Set** is also monoidal if we take the tensor product to be the categorical coproduct (we say that **Set** is *cocartesian monoidal*). Again, we don't have strict equality here, with $(A \sqcup B) \sqcup C \neq A \sqcup (B \sqcup C)$, but there is again a canonical isomorphism between the two objects. The unitors are then given by the isomorphisms $\varnothing \sqcup A \cong A$ and $A \sqcup \varnothing \cong A$.

For another example, the category $\mathbf{Vect}_K$ of vector spaces over a field K is also monoidal in multiple ways, with the tensor product given by either the traditional tensor product, or the direct sum of vector spaces. In this case, these two structures are actually compatible in that the tensor product distributes over the direct sum, giving the category an additional semiring structure.

8 Internalisation

Recall the standard definition of a group:

> A *group* $(G,*)$ is a set G equipped with a binary operation $* : G \times G \to G$ that is associative, admits an identity element $e \in G$ (is *unitary*), and every element $g \in G$ has an inverse $g^{-1} \in G$ under $*$.

At this point, we should be used to viewing various mathematical constructions as morphisms, and we might be tempted to do the same here. The binary operation is already a function, and the identity element can, as usual, also be viewed as a function $e : 1 \to G$. Similarly, we have the function $(-)^{-1} : G \to G$ that sends an element to its inverse.

Now, because G is a set and e, $*$, and $(-)^{-1}$ are set functions, the associativity, identity, and inverse axioms can be entirely encoded by the requirement that certain diagrams in **Set** relating the three functions commute [nLa23b]:

- Unitality:

$$\begin{tikzcd} G \times 1 \arrow[r, "\mathrm{id} \times e"] \arrow[dr, "\cong"'] & G \times G \arrow[d, "*"] & 1 \times G \arrow[l, "e \times \mathrm{id}"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd}$$

- Associativity:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{id}\times *"] \arrow[d, "*\times\mathrm{id}"'] & G \times G \arrow[d, "*"] \\
G \times G \arrow[r, "*"'] & G
\end{tikzcd}$$

- Inverses:

$$\begin{tikzcd}
 & & G \arrow[dll, "\exists!"'] \arrow[drr, "\Delta"] & & \\
1 \arrow[dr, "e"'] & & & & G \times G \arrow[dl, "(-)^{-1}\times\mathrm{id}"] \\
 & G & & G \times G \arrow[ll, "*"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & G \arrow[dll, "\exists!"'] \arrow[drr, "\Delta"] & & \\
1 \arrow[dr, "e"'] & & & & G \times G \arrow[dl, "\mathrm{id}\times(-)^{-1}"] \\
 & G & & G \times G \arrow[ll, "*"] &
\end{tikzcd}$$

For instance, in **Set**, we can interpret the two paths in the left inverse diagram as the chains of functions $g \mapsto (g,g) \mapsto (g^{-1},g) \mapsto g^{-1} * g$ and $g \mapsto 1 \mapsto e$, so commutativity says that $g^{-1} * g = e$.

However, we should notice that this characterisation of groups does not explicitly refer to the elements within the group – all the requirements are now to do with how the group interacts with these three functions. Furthermore, the only structure of **Set** that is used in the above characterisation is the existence of finite products. This is not specific to **Set**, and indeed, there is no reason why this definition needs to be tied to **Set** at all; all the previous diagrams make sense in any arbitrary category that admits these limits, even if we cannot necessarily interpret G to be a set in that category.

An *internal group* in a category $\mathscr{C}$ that admits finite products is an object G equipped with morphisms $e : 1 \to G$ (where 1 is terminal in $\mathscr{C}$), $* : G \times G \to G$, and $(-)^{-1} : G \to G$ such that the diagrams above commute.

If $\mathscr{C}$ is **Set**, then we just have the definition of an ordinary group; if $\mathscr{C} = \mathbf{Top}$, we obtain topological groups; if $\mathscr{C} = \mathbf{Man}^{\infty}$, we obtain Lie groups; if $\mathscr{C} = \mathbf{Grp}$, we obtain abelian groups, etc.

This process of abstracting a structure like a group into an object or objects within a general category is called *internalisation* – and we can do this with many other constructions, creating internal monoids,* rings, lattices, etc. For instance, the subobject classifier in a Boolean category is exactly an internal Boolean algebra.

These diagrams can also be dualised to obtain so-called cogroups, comonoids, corings, etc. but these dual objects often do not correspond to any standard algebraic structures [For02], though cogroups do arise naturally in algebraic topology. For example, the n-sphere S^n is precisely a cogroup object in the homotopy category of pointed topological spaces, and is related to why the higher homotopy groups are in fact groups. Categories themselves can also be internalised within categories with sufficient pullbacks. For instance, small categories are precisely the categories internal to **Set**. In general, the more structure the ambient category has, the more that is able to be internalised.

We can abstract further and replace the products with tensor products to produce internal objects in general monoidal categories. For instance, an internal monoid in **Ab** with monoidal structure given by the tensor product $\otimes_{\mathbb{Z}}$, is precisely a ring(!); and an internal monoid in $\mathbf{Vect}_K$, with monoidal structure given by the tensor product $\otimes_K$ of vector spaces, is exactly a K-algebra, etc.

*Dropping the commutative diagram for inverses in the above definition of an internal group yields this particular construction.

8.1 Internal Homs

Clearly, internalisation is very useful, as it unifies many seemingly distinct constructions. But for now, we are interested in the internalisation of a categorical hom-set. We begin by abstracting the similar notion of a *function set*.

Given two sets A and B, the collection of functions from A to B form a set $[A,B]$, called the function set. We consider the functions into $[A,B]$ from another set X.

Such a function takes an argument from X, and returns a function $A \to B$. We can alternatively interpret this as a function that takes an argument from *both* X and A, and returns an element in B, so there is a bijection between functions $X \to [A,B]$ and $X \times A \to B$.*

This allows us to easily abstract a function set into any arbitrary category that admits products as follows: given a pair of objects A and B, the *internal hom-object*, or just *internal hom*, is an object $[A,B]$ such that

$$\hom(X,[A,B]) \cong \hom(X \times A,B)$$

naturally in X. This assignment of objects to internal hom objects is also functorial, defining the *internal hom-functor* $[-,-] : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ that sends pairs of objects to their internal homs, much like the ordinary hom-bifunctor.

Replacing the product in the above with a tensor product, internal hom-functors further generalise to categories that may not admit products.

If a monoidal category admits all internal hom objects, it is called a *closed monoidal category*. More precisely, a monoidal category is closed monoidal if for every object A, the functor $(-) \otimes A : \mathscr{C} \to \mathscr{C}$ that sends objects to their right tensor product by A has a right adjoint, $[A,-] : \mathscr{C} \to \mathscr{C}$, that sends objects to the internal hom out from A. That is,

$$\hom(X,[A,B]) \cong \hom(X \otimes A,B)$$

naturally in all three variables. These categories are "closed" in the sense that taking hom-sets leaves you within the category.

If the category is further cartesian monoidal, then it is called a *cartesian closed category*.

Example. Any locally small category has at most a set of morphisms between any pair of objects – which is an object of **Set**. **Set** itself is locally small, so the hom-set between every pair of objects is just another object of **Set** (specifically, the function set between them), so **Set** has all internal homs, and is hence closed monoidal. **Set** is also cartesian monoidal, and so is furthermore cartesian closed.

In cartesian closed categories, the internal hom $[A,B]$ is also written as B^A and is called an *exponential object*. This notation is compatible with the notation Ω^A for power objects (when they exist), as the power object of A is precisely the exponential object of A into the subobject classifier Ω, and this can be taken as an alternative definition of the power object. More generally, this notation is also compatible with the categorical product in that we have $A^1 \cong A$, $A^2 \cong A \times A$ (where $2 \coloneqq 1 \amalg 1$), etc. In more detail,

$$\begin{aligned}
\hom(X,A^2) &\cong \hom(X \times (1 \amalg 1),A) \\
&\cong \hom((X \times 1) \amalg (X \times 1),A) \\
&\cong \hom(X \times 1,A) \times \hom(X \times 1,A) \\
&\cong \hom(X,A) \times \hom(X,A) \\
&\cong \hom(X,A \times A)
\end{aligned}$$

so by the Yoneda lemma, $A^2 \cong A \times A$ (and so on, by induction).

This compatibility with categorical products allows for a more concise characterisation of cartesian closed categories: a category is cartesian closed if and only if it has finite products, and the right cartesian

*Particularly in computer science and formal logic, the reverse direction of this bijection is called *currying*, and is related to the notion of partial application.

product functor $(-) \times A$ admits a right adjoint $(-)^A$ for every object A. That is, for every pair of objects A and B, there is an object B^A such that

$$\hom(X, B^A) \cong \hom(X \times A, B)$$

naturally in all three variables. The left adjunct $f^\flat : X \to B^A$ of a morphism $f : X \times A \to B$ is also called the *exponential transpose* of f, and similarly, the right adjunct is called the *exponential cotranspose.*

In the special case where $X \cong 1$ is terminal, this isomorphism becomes

$$\hom(1, B^A) \cong \hom(1 \times A, B) \cong \hom(A, B)$$

In other words, the elements of B^A (that is, the morphisms $1 \to B^A$) are naturally isomorphic to the morphisms $A \to B$, so the exponential object B^A can be thought of as the "object of morphisms $A \to B$", much like a function set in **Set**. Given a morphism $f : A \to B$, we write $[f]$ for its isomorphic copy or "label" in B^A.

Another interesting case is given by $X = B^A$, with the isomorphism then being:

$$\hom(B^A, B^A) \cong \hom(B^A \times A, B)$$

The image of the identity map on B^A is called the *evaluation map*, denoted by $\mathrm{ev} : B^A \times A \to B$. Concretely, in **Set**, or more generally on generalised elements, the evaluation map is given by evaluating a function $[f] \in B^A$ at a value $a \in A$; $([f], a) \mapsto f(a)$, hence the name.

The evaluation map also satisfies the universal property that given any object X and map $e : X \times A \to B$, there is a unique morphism $u : X \to B^A$ such that $\mathrm{ev} \circ (u \times \mathrm{id}_A) = e$:

$$\begin{tikzcd}
X \times A \arrow[d, dashed, "u \times \mathrm{id}_A"'] \arrow[dr, "e"] & \\
B^A \times A \arrow[r, "\mathrm{ev}"'] & B
\end{tikzcd}$$

9 ETCS

9.1 Topoi

The notion of a *topos* (plural *topoi*) was first introduced in algebraic geometry by Grothendieck in the early 1960s as a generalisation of sheaves of sets in topology. Every topological space induces a topos, and conversely, every topos, as defined by Grothendieck, behaves much like a generalised topological space. These topoi are called *Grothendieck topoi*, and are now prevalent in modern algebraic geometry.

A more general notion of a topos was soon developed by Lawvere and Tierny over the same decade, which we will now introduce.

An (*elementary* or *Lawvere–Tierny*) *topos* is a category that:

- is finitely complete;
- is cartesian closed;
- has a subobject classifier.

This definition seems rather short for a structure we claim to be so important – and it is, deceptively so; a topos carries a vast amount of additional rich structure that just happens to follow from these few axioms. We give a few basic properties of topoi.

Lemma 9.1. *Every monomorphism in a topos is regular. That is, every monomorphism occurs as the equaliser of some pair of parallel morphisms.*

Proof. Let $m : X \to Y$ be a monomorphism. Then, it is a subobject of Y, so it is classified by the unique map $\chi_m : X \to \Omega$ that makes the following diagram a pullback square:

$$\begin{tikzcd}
X \arrow[r, "!_A"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
Y \arrow[r, "\chi_m"'] & \Omega
\end{tikzcd}$$

We claim that m is the equaliser of χ_m and $\top \circ !_Y$ (where $!_Y$ is the unique map $Y \to 1$): Let $f : A \to Y$ also equalise χ_m and $\top \circ !_Y$, i.e. $\chi_m \circ f = \top \circ !_Y \circ f$. Our maps are now:

$$\begin{tikzcd}
A \arrow[dr, dashed, "g"] \arrow[drr, bend left, "!_A"] \arrow[ddr, bend right, "f"'] & & \\
& X \arrow[r, "!_X"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
& Y \arrow[r, "\chi_m"'] \arrow[ur, "!_Y"] & \Omega
\end{tikzcd}$$

Because 1 is terminal, $!_Y \circ f = !_A$, so we have $\chi_m \circ f = \top \circ !_A$ and the outer square commutes, so the universal property of the pullback yields a unique map $g : A \to X$ making the whole diagram commute.

By commutativity of the left triangle, $f = m \circ g$, and hence

$$\begin{tikzcd}
A \arrow[d, dashed, "g"] \arrow[dr, "f"] & & \\
X \arrow[r, tail, "m"'] & Y \arrow[r, shift left, "\chi_m"] \arrow[r, shift right, "\top \circ !_Y"'] & \Omega
\end{tikzcd}$$

commutes, so m is an equaliser, as required. ■

Corollary 9.1.1. *Topoi are balanced. That is, every bimorphism is an isomorphism.*

Proof. From the previous lemma, every bimorphism in a topos is an epic regular monomorphism.

Let $f : E \to A$ be an epic regular monomorphism. As f is a regular monomorphism, it is the equaliser of a pair of parallel morphisms $g,h : A \to B$, so we have $g \circ f = h \circ f$. Since f is epic, we have $g = h$.

The equaliser of $g = h$ is the identity map, and by the universal property of the equaliser, E must factor through A essentially uniquely, so $f : E \to A$ must be this isomorphism. ■

Despite starting with only a finitely complete category, the subobject classifier and cartesian closed structure together also imply that a topos also has all finite colimits:

Theorem 9.2. *[Par74] Every topos is finitely cocomplete.*

We also have a result stating that morphisms in a topos satisfy a epi-mono factorisation structure:

Theorem 9.3 (Image Factorisation)**.** *In a topos, every arrow factors essentially uniquely through its image into the composition of an epimorphism with a monomorphism.*

Proof. [MM12] Let $f : A \to B$ be a morphism. We construct the following diagram in stages.

$$\begin{tikzcd}
A \arrow[r, dashed, "e"'] \arrow[rr, bend left, "f"] \arrow[d, equal] & M \arrow[r, dashed, tail, "m"'] & B \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] \arrow[d, equal] & X \arrow[d, dashed, "u"] \\
A \arrow[r, "g"'] & N \arrow[r, tail, "h"'] & B \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y
\end{tikzcd}$$

First, construct the cokernel pair $x,y : B \to X$ of f, and let $m : M \to B$ be the equaliser of x and y. By the universal property of the equaliser, f factors uniquely through the equaliser m, so $f = m \circ e$ for some $e : A \to M$. Note that, as an equaliser, m is monic.

Now, suppose f also factorises as $f = h \circ g$ with h monic. As every monomorphism in a topos is regular, h is the equaliser of some pair of morphisms $s,t : B \to Y$, so we have $s \circ h = t \circ h$, and precomposing by g yields $s \circ f = t \circ f$. Then, because x,y is the cokernel pair of f, X factors through Y via a unique map $u : X \to Y$, giving

$$\begin{aligned} s \circ m &= u \circ x \circ m \\ &= u \circ y \circ m \\ &= t \circ m \end{aligned}$$

so m also equalises s and t and therefore factors uniquely through h. As h is arbitrary, we have that m is the minimal subobject of B and hence $M = \operatorname{im}(f)$. It remains to show that e is epic.

Perform this construction again on e to obtain the chain

$$A \xrightarrow{e'} \operatorname{im}(e) \rightarrowtail^{m'} \operatorname{im}(f) \rightarrowtail^{m} B$$

equal to f. In particular, f factors through the monomorphism (subobject) $m \circ m'$, so the image also factors through $m \circ m'$, as it is the minimal subobject, so $m = (m \circ m') \circ v$ for some $v : M \to \operatorname{im}(e)$. It follows that $m' \circ v = \mathrm{id}_M$, so m' is an isomorphism.

As before, m' is the equaliser of the cokernel pair x', y' of e. But, because m' is an isomorphism, we have $x' = y'$, so the cokernel pair of e is x', x', and hence e is epic, as required. ■

The prototypical example of a topos is **Set**, but **Set** also has a couple of special properties it does not share with most other topoi, which we will explore soon. On the other hand, the existence of terminal objects allow us to consider elements of objects in arbitrary topoi; the subobject classifier Ω allows us to consider subobjects; and exponentials allow us to consider objects of morphisms from one object to another, as well as power objects in the form of exponentials of the subobject classifier; so, along with finite completeness and the cartesian closed structure, arbitrary topoi behave in many ways like **Set**.

In particular, this means that almost all categorical constructions in **Set** can readily be internalised in an arbitrary topos, and many theorems about these constructions similarly apply to their internalised variants. In this sense, topoi are just a kind of well-behaved generalised space in which objects behave "like sets".

9.2 Set

We give some characteristics of **Set** that distinguish it from other topoi [Lei11], appealing only to "obvious" properties that sets and functions should satisfy.

Firstly, **Set** is non-trivial: that is, **Set** $\not\simeq \mathbb{1}$. Another way to characterise this property is that the terminal and inital objects of **Set** both exist, and are not isomorphic:

(i) $0 \ncong 1$.

In **Set**, the terminal object 1 also has another special property: if the parallel morphisms $f,g : X \to Y$ are such that every map $x : 1 \to X$ equalises f and g, then $f = g$.

More generally, an object S is called a *separator* or is said to *separate morphisms* if for every pair of parallel morphisms $f,g : X \to Y$, if $f \circ s = g \circ s$ for every $s : S \to X$, then $f = g$. So, if a category admits a separator, then just by looking at (compositions with) the generalised elements of shape S, we can distinguish all morphisms in that category.

The next property of **Set** is then:

(ii) The terminal object 1 is a separator.

In **Set**, however, this has an additional important interpretation: recall that maps $1 \to X$ are elements of X, so, given a function $f : X \to Y$, the composition $f \circ x : 1 \to Y$ is an element of Y, which we might choose to write as $f(x)$. Thus, not only are elements a special case of functions, but evaluation of functions is a special case of composition. The property above then says that if $f(x) = g(x)$ for all x, then f and g are the same function − this is saying that functions have no internal identity, and are completely defined by their effects on elements (and implicitly, the data of their (co)domains). This property is similar to the axiom of extensionality in ZFC, but for functions instead of sets.

A topos that satisfies properties (i) and (ii) is called a *well-pointed* topos.

The next property quite specific to **Set** is, roughly speaking, the existence of the natural numbers. To state this more formally, we need to characterise the natural numbers categorically. One feature of the natural numbers that we often use, particularly with induction, is that they support recursive definitions.

Given a set X, and an element $x \in X$, every function $r : X \to X$ generates a unique sequence of elements $(x_i)_{i=1}^{\infty} \subseteq X$ such that $x_0 = x$ and $x_{n+1} = r(x_n)$. Note that such a sequence is indexed by the natural numbers, so this yields a correspondence between applications of r to x, and applications of the successor function to 0 in the subscripts. Moreover, a sequence in X is just a generalised element of shape $\mathbb{N}$, or a morphism $f : \mathbb{N} \to X$, so this is really a statement about the natural numbers. If we write $s : \mathbb{N} \to \mathbb{N}$ for the successor function, then the previous just says that the following diagram commutes:

$$\begin{tikzcd}
 & \mathbb{N} \arrow[r, "s"] \arrow[dd, "f"] & \mathbb{N} \arrow[dd, "f"] \\
1 \arrow[ur, "0"] \arrow[dr, "x"'] & & \\
 & X \arrow[r, "r"'] & X
\end{tikzcd}$$

where 1 is terminal.

A *natural numbers object* in a category $\mathscr{C}$ is a triple $(\mathbb{N},0,s)$ consisting of an object $\mathbb{N} \in \mathrm{ob}(\mathscr{C})$, a morphism $0 : 1 \to \mathbb{N}$ from the terminal object 1, and a *successor morphism* $s : \mathbb{N} \to \mathbb{N}$ with the universal property that all other similar triples (X,x,r) factor through $(\mathbb{N},0,s)$ uniquely. That is, there exists a unique morphism $f : \mathbb{N} \to X$ such that the previous diagram commutes. This universal property also means that natural numbers objects are essentially unique, so we are safe to speak about *the* natural numbers.

The sequence f given by this axiom is said to be defined by *simple recursion* with *starting value* x and *transition rule* r.

Arithmetic operations $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such as addition, multiplication, exponentiation, etc. can then be defined in terms of their exponential transpositions $\mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ by simple recursion. For instance, the following diagram defines addition of natural numbers:

$$\begin{tikzcd}
 & \mathbb{N} \arrow[r, "s"] \arrow[dd, "+^\flat"] & \mathbb{N} \arrow[dd, "+^\flat"] \\
1 \arrow[ur, "0"] \arrow[dr, "\mathrm{id}_{\mathbb{N}}^\flat"'] & & \\
 & \mathbb{N}^{\mathbb{N}} \arrow[r, "s^{\mathbb{N}}"'] & \mathbb{N}^{\mathbb{N}}
\end{tikzcd}$$

[Kos12, adapted]

Commutativity of the left triangle says that adding zero is the identity function, so $0 + n = n$, and commutativity of the square on the right says $(s \circ n) + m = s \circ (n + m)$, which is precisely the Peano definition of addition.

The third distinguishing property of **Set** is then:

(iii) **Set** has a natural numbers object.

The last special property of **Set** that we will need is that every surjective function $f : A \twoheadrightarrow B$ has a right section – a function $s : B \to A$ such that $f \circ s = \mathrm{id}_B$. This can be stated in categorical terms as:

(iv) Epimorphisms split.

The function s is defined by assigning each element $b \in B$ an element from $f^{-1}[b]$, (which is non-empty as f is surjective). However, this implies the existence of a choice function for any arbitrary f and thus, the statement that every epimorphism splits is precisely the Axiom of Choice. More generally, a category is said to *satisfy the Axiom of Choice*, or to *have Choice*, if every epimorphism splits.

So, the content of this section can be stated concisely as:

Sets and set functions form a well-pointed topos
with natural numbers object and Choice.

The category of sets of course has more properties than this; for instance, power objects always exist; the category is balanced; the subobject classifier has two objects, $\Omega \cong 2 := 1 \amalg 1$; and the topos is Boolean, etc. but these properties all follow from the previous conditions.

The question is now, what conditions do we need to enforce on sets and set functions to ensure that they *do* form such a topos?

One answer is of course, ZFC – and indeed, any model of ZFC will satisfy these properties, so the category of ZFC sets will satisfy the above.

This is the answer many mathematicians recognise and know in the back of their mind, but often do not like to concern themselves with, because the axioms of ZFC are generally quite far removed from the study of mathematics – the specific axioms seem unimportant compared to the need for the end result to satisfy these requirements.

This answer is thus rather unsatisfying, or even irrelevant, because all of the above requirements were derived from "obvious" properties of sets that we often use – extensionality of functions, existence of natural numbers, etc. At no point did we have to consult with a list of axioms to decide these properties, because they all follow from our informal idea of what sets should be and how set functions should behave.

In particular, this means that anything that satisfies the above requirements will behave *like a set.* This is the idea behind the alternative answer given by Lawvere in his *Elementary Theory of the Category of Sets*, or ETCS: we take these properties *as our axioms.* That is, we do not require that sets satisfy the axioms of ZFC, but instead require that sets and set functions form a well-pointed topos with natural numbers object and Choice.

At this point, one may think that there is some circularity here: that ETCS depends on the notion of a category, which itself depends on the notion of "collections" of objects and morphisms, which seem quite similar to "sets".

The straightforward formalist response is that category theory (and specialisations thereof, like ETCS) and ZFC are *first-order theories*, so they are all just collections of sentences in the first-order language over some signature – at a fully formalised level, none of these theories mention or depend on any prior notion of sets, because they are just alphabets of symbols, together with lists of axioms.

However, outside of formal logic, we usually don't think of theories in this way – as manipulations of strings of abstract symbols – but instead as descriptions of some universe of interest. This answer may thus be somewhat unsatisfactory in that it doesn't answer the question intuitively, so an alternative explanation for ETCS in particular is that, although motivated by category-theoretic ideas, ETCS does not intrinsically depend on the notion of a category – category theory is just a convenient language with which we can express the axioms of ETCS concisely. It is certainly possible to state the axioms of ETCS without mentioning categories at all.

For reference, the axioms stated without mentioning categories are, informally,

1. Function composition is associative and has identities
2. There exists an empty set
3. There exists a set with one element
4. Functions are completely characterised by their actions on elements
5. Given sets X and Y, we may form the Cartesian product $X \times Y$
6. Given sets X and Y, we may form the set Y^X of functions from X to Y
7. Given a function $f : X \to Y$ and $y \in Y$, we may form the fibre $f^{-1}[y]$
8. The subsets of a set X correspond to the functions $X \to \{0,1\}$
9. The natural numbers form a set
10. Every surjection admits a section

Stated in this way, the comparison with ZFC is now more obvious: ZFC says "there are things called sets; there is a binary relation $\in$ defined on sets; and some axioms hold.", and ETCS says "there are things called sets; for every pair of sets there are things called functions; there is a (partial) binary operation $\circ$ on functions called composition; and some axioms hold." In neither case do we specify what these "things" are, nor do we suppose that these "things" form any structure like a set or category beyond what the axioms demand. The point is, circularity is no more of a problem for ETCS than it is for ZFC.

As noted in [Lei12], the axioms of ETCS as stated above also appear to be more *fundamental* in some way than ZFC. Suppose that one day, we find that ZFC had been proved to be inconsistent: that some logician had started with the axioms of ZFC, and had irrefutably derived a logical contradiction from them. It is likely that most mathematicians, being generally detached from ZFC in the first place, would not be deeply bothered by this fact, and could continue on, generally confident that their theorems and results still hold true in the sense that their negations do not.

In contrast, the axioms of ETCS are modelled on core properties of sets and functions that we often use – a proof that ETCS were inconsistent would be devastating. We would no longer be able to safely assume that function composition is associative, that products or function sets exist, etc.

As an aside, note that in this paper, we are describing ETCS as a *two-sorted* first-order theory, roughly meaning that we have two distinct "kinds" of things – namely, objects (sets) and morphisms (functions).

Many introductions to logic, however, only discuss single-sorted theories (like ZFC, or fragments thereof), so some may object to this usage of a two-sorted theory. Fortunately, it is in fact possible to express ETCS as a single-sorted theory, where objects in the sense of the two-sorted theory are just a special type of morphism in the single-sorted theory. Specifically, objects are in bijection with identity morphisms, so they can be treated as a special case of morphisms. We then just add a source, target, and composition predicate to our underlying logic to obtain the desired single-sorted theory.

While we will not be discussing this style of axiomatisation, it is interesting that the primitive notion of this theory is not of sets, as in ZFC, but of functions – this is yet another illustration of the structural idea that connections are more important than objects.

9.3 Constructing the Universe

We prove some standard set theory machinery used for constructing common mathematical objects and the rest of the set-theoretic universe. Most of the proofs in this section have been roughly adapted from [LC05], with most modifications due to differences in definitions and conventions. (In particular, the ETCS axioms are very different.)

We have already constructed (local) intersections and unions as meets and joins in the subobject lattice, but we would like to extend this to indexed families of sets. In fact, we can prove (a structural version of) ZFC's axiom of the union in ETCS.

First, recall that a subobject $m : A \rightarrowtail X$ is classified by a characteristic morphism $\chi_m : X \to \Omega$ such that

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

is a pullback square. Then, given a subset $\alpha : I \rightarrowtail \Omega^X$ of the power object of X, i.e., an indexed family of subsets of X, the union of the α_i is then a subset $a : \bigcup_\alpha \to X$ of X such that for any $x \in X$, there exists an index $i \in I$ such that the function labelled by $\alpha(i)$ sends x to $\top \in \Omega$. That is,

$$x \in a \leftrightarrow \exists i \in I : \mathrm{ev}_{\Omega^X}(\alpha(i),x) = \top$$

where ev_{Ω^X} is the evaluation map on Ω^X, and $\top : 1 \to \Omega$ is the truth value true.

Theorem 9.4 (Indexed Unions)**.** *Given $\alpha : I \rightarrowtail \Omega^X$, the union $\bigcup_\alpha$ as defined above exists.*

Proof. Taking the exponential transpose, $\alpha^\flat : I \times X \to \Omega$, the desired property of $\bigcup_\alpha$ simplifies to:

$$x \in a \leftrightarrow \exists i \in I : \alpha^\flat(i,x) = \top$$

First, construct the equaliser of $\alpha^\flat$ and $\top \circ !_{I\times X}$, and consider the projection $\pi_2 : I \times X \to X$:

$$\begin{tikzcd} E \arrow[r, "k"] \arrow[d, dashed, two heads, "q"'] & I \times X \arrow[r, shift left, "\alpha^\flat"] \arrow[r, shift right, "\top\circ !_{I\times X}"'] \arrow[d, "\pi_2"] & \Omega \\ \bigcup_\alpha \arrow[r, dashed, tail, "a"'] & X & \end{tikzcd}$$

By image factorisation (Theorem 9.3), the composition $\pi_2 \circ k : E \to X$ factors through its image (essentially uniquely) into the epimorphism q and monomorphism a.

Let $x \in a$, i.e., x is a function $1 \to X$ and there exists a lift $\bar{x} \in \bigcup_\alpha$ of x. Then, since q is an epimorphism, there exists $\hat{x} \in E$ such that $q(\hat{x}) = \bar{x}$, so we have $x = a \circ q \circ \hat{x} = \pi_2 \circ k \circ \hat{x}$.

$$\begin{tikzcd}
 & & I & & \\
 & E \arrow[r, "k"] \arrow[d, dashed, two heads, "q"'] & I \times X \arrow[u, "\pi_1"] \arrow[r, "!_{I\times X}"'] \arrow[d, "\pi_2"] \arrow[rr, bend left, "\alpha^\flat"] & 1 \arrow[r, "\top"'] & \Omega \\
 & \bigcup\alpha \arrow[r, dashed, tail, "a"'] & X & & \\
1 \arrow[uur, bend left, "\hat{x}"] \arrow[ur, "\bar{x}"] \arrow[urr, bend right, "x"'] & & & &
\end{tikzcd}$$

Now, define $i \coloneqq \pi_1 \circ k \circ \hat{x}$. Then,

$$\begin{aligned}
\alpha^\flat(i,x) &= \alpha^\flat(\pi_1 \circ k \circ \hat{x}, \pi_2 \circ k \circ \hat{x}) \\
&= \alpha^\flat \circ k \circ \hat{x} \\
&= \top \circ !_{I\times X} \circ k \circ \hat{x} \\
&= \top
\end{aligned}$$

since $!_{I\times X} \circ k \circ \hat{x} : 1 \to 1$ is necesssarily the identity. So, if $x \in a$, then there exists $i \coloneqq \pi_1 \circ k \circ \hat{x} \in I$ such that $\alpha^\flat(i,x) = \top$, as required.

For the reverse implication, suppose there exists an index $i \in I$ such that $\alpha^\flat(i,x) = \top$. Then, by the universal property of the equaliser, there is some $\hat{x} \in E$ such that $k(\hat{x}) = (i,x)$. Then, applying q to $\hat{x}$, we have $x \in a$, as required. ■

Next, we prove that primitive recursion can be performed in ETCS, allowing the construction of a vast class of important functions. For instance, multiplication, division, the factorial function, the exponential function, and the function that returns the nth prime are all primitive recursive; in fact, most of the computable functions encountered in mathematics are primitive recursive.

Theorem 9.5 (Primitive Recursion). *Given a pair of morphisms $f_0 : A \to B$ and $u : \mathbb{N} \times A \times B \to B$, there is a unique morphism $f : \mathbb{N} \times A \to B$ such that for all $n \in \mathbb{N}$ and $a \in A$,*

- $f(0,a) = f_0(a)$;
- $f\big(s(n),a\big) = u\big(n,a,f(n,a)\big)$.

Proof. Primitive recursion is more complicated than the simple recursion given by the natural numbers object in two main ways.

Firstly, the values of f depend not only on $n \in \mathbb{N}$, but also on the members a of the parameter object A. To simplify, we instead find the exponential transpose of f. That is, the function $f^\flat : \mathbb{N} \to B^A$ such that

- $f^\flat(0) = [f_0]$;
- For all $n \in \mathbb{N}$ and $a \in A$, $\mathrm{ev}\big(y(s(n)),a\big) = u\big(n,a,\mathrm{ev}(f^\flat(n),a)\big)$, where ev is the evaluation map for B^A.

The next complication is that the transition rule now depends not only on the value of the previous step, but also on the number n of previous steps that have already been taken. For this, we instead construct a sequence $F : \mathbb{N} \to \mathbb{N} \times B^A$ of ordered pairs where the first coordinate is just used to track the number of previous steps.

In other words, we define by simple recursion the graph of $f^\flat$, and from there, we can then recover $f^\flat$ by composing with a projection map. Explicitly, we require

- $F(0) = \langle 0, f^\flat(0) \rangle = \langle 0, [f_0] \rangle$;
- For all $n \in \mathbb{N}$ and $a \in A$,
  - *(i)* $(\pi_1 \circ F)\big(s(n)\big) = s(n)$;
  - *(ii)* $\mathrm{ev}\big((\pi_2 \circ F)(s(n)), a\big) = u\big(n, a, \mathrm{ev}\big((\pi_2 \circ F)(n), a\big)\big)$.

By the universal property of the natural numbers object, the existence of F can be given by finding a map $r : \mathbb{N} \times B^A \to \mathbb{N} \times B^A$ such that for any $n \in \mathbb{N}$, $[h] \in B^A$, and $a \in A$,

(i) $(\pi_1 \circ r)(n, [h]) = s(n)$;

(ii) $\mathrm{ev}\big((\pi_2 \circ r)(n, [h]), a\big) = u\big(n, a, \mathrm{ev}([h], a)\big)$.

We claim that such a map is given by $r := \langle s \circ \pi_1, G^\flat \rangle$, where $G^\flat$ is the exponential transpose of the map $G : A \times \mathbb{N} \times B^A \to B$ defined by the chain:

$$A \times \mathbb{N} \times B^A \xrightarrow{\Delta_A \times \mathrm{id}_{\mathbb{N}} \times \mathrm{id}_{B^A}} (A \times A) \times \mathbb{N} \times B^A \overset{\substack{\text{associator } \alpha, \\ \text{braiding } \beta}}{\cong} \mathbb{N} \times A \times (B^A \times A) \xrightarrow{\mathrm{id}_{\mathbb{N}} \times \mathrm{id}_A \times \mathrm{ev}} \mathbb{N} \times A \times B \xrightarrow{u} B$$

(Multiple associator components have been omitted to significantly simplify the chain.)

We verify that r satisfies the two desired properties:

(i)
$$\begin{aligned} (\pi_1 \circ r)(n, [h]) &= (s \circ \pi_1)(n, h) \\ &= s\big(\pi_1(n, h)\big) \\ &= s(n) \end{aligned}$$

(ii)
$$\begin{aligned} \mathrm{ev}\big((\pi_2 \circ r)(n, [h]), a\big) &= \mathrm{ev}(G^\flat(n, [h]), a) \\ &= G(a, n, [h]) \\ &= \big(u \circ (\mathrm{id}_{\mathbb{N}} \times \mathrm{id}_A \times \mathrm{ev}) \circ (\overset{\alpha;\beta}{\because}) \circ (\Delta_A \times \mathrm{id}_{\mathbb{N}} \times \mathrm{id}_{B^A})\big)(a, n, [h]) \\ &= \big(u \circ (\mathrm{id}_{\mathbb{N}} \times \mathrm{id}_A \times \mathrm{ev}) \circ (\overset{\alpha;\beta}{\because})\big)\big((a, a), n, [h]\big) \\ &= \big(u \circ (\mathrm{id}_{\mathbb{N}} \times \mathrm{id}_A \times \mathrm{ev})\big)\big(n, a, ([h], a)\big) \\ &= u\big(n, a, \mathrm{ev}([h], a)\big) \end{aligned}$$

as required.

Finally, uniqueness is given by function extensionality. ∎

Note that existence only depends on finite completeness, exponentials, and the existence of the natural numbers object, and uniqueness on well-pointedness, so primitive recursion can actually be performed in much more general categories than just those that are models of ETCS; for instance, the functor category $[\mathscr{C}, \mathbf{Set}]$ for any small category $\mathscr{C}$.

Next, we verify that the natural numbers object in fact behaves as we would like:

Theorem 9.6 (Peano Postulates). *The natural numbers object* $(\mathbb{N}, 0, s)$ *satisfies the Peano postulates. That is,*

(i) The successor function $s : \mathbb{N} \to \mathbb{N}$ *is monic;*

(ii) If $m = s(n)$ *for some* $n \in \mathbb{N}$*, then* $m \neq 0$*;*

(*iii*) *If* $S \subseteq \mathbb{N}$ *and for all* $n \in \mathbb{N} : n \in S \to s(n) \in S$*, then* $S = \mathrm{id}_{\mathbb{N}}$.

Proof.

(*i*) The predecessor function p can be defined by primitive recursion with $f_0(-) = 0$ and $u(n, -, -) = n$. The parameter object A isn't actually used here, so suppressing it from the arguments, primitive recursion gives

- $p(0) = 0$;
- $p(s(n)) = n$.

Since s has a left-inverse, it is injective and hence monic.

(*ii*) Suppose $0 = s(n)$ for some $n \in \mathbb{N}$. Then,

$$\begin{aligned} n &= p\big(s(n)\big) \\ &= p(0) \\ &= 0 \end{aligned}$$

so $s(0) = 0$.

Let X be some set, $x \in X$ an element, and $r : X \to X$ an endofunction on X, and let $f : \mathbb{N} \to X$ be the unique morphism given by the universal property of the natural numbers object. As $\mathbb{N}$ factors through X, we have $f(0) = x$ and $r \circ f = f \circ s$, so,

$$\begin{aligned} r(x) &= (r \circ f)(0) \\ &= (f \circ s)(0) \\ &= f(0) \\ &= x \end{aligned}$$

Since $x \in X$ is arbitrary, $r = \mathrm{id}_X$ (as $x : 1 \to X$ equalises r and id_X and 1 is a separator), and since X is arbitrary, this implies that every endofunction is an identity function, which is a contradiction.

(*iii*) Let $S : A \rightarrowtail \mathbb{N}$ be a (representing monomorphism of a) subset of $\mathbb{N}$ with $0 \in S$, i.e., there exists $\bar{0} \in A$ such that $S(\bar{0}) = 0$; and such that $\forall n \in \mathbb{N} : n \in S \to s(n) \in S$.

The latter requirement implies that S is contained in its preimage $s^{-1}[S]$, so the map sending the lift to n to the lift of $s(n)$ is total, and extends to a morphism $t : A \to A$, satisfying $S \circ t = s \circ S$.

By simple recursion, $\bar{0}$ and t define a unique sequence $f : \mathbb{N} \to A$ such that $f(0) = \bar{0}$ and $(f \circ s)(n) = (t \circ f)(n)$ for all $n \in \mathbb{N}$.

Then, we have $(S \circ f)(0) = S(\bar{0}) = 0$ and

$$\begin{aligned} (S \circ f) \circ s &= S \circ (f \circ s) \\ &= S \circ (t \circ f) \\ &= (S \circ t) \circ f \\ &= (s \circ S) \circ f \\ &= s \circ (S \circ f) \end{aligned}$$

The identity $\mathrm{id}_{\mathbb{N}}$ also satisfies these equations, so $S \circ f = \mathrm{id}_{\mathbb{N}}$ by uniqueness of the map given by recursion. Then, for any $n \in \mathbb{N}$, we have $n = (S \circ f)(n) = S\big(f(n)\big)$, so $f(n)$ is a lift of n and hence $n \in S$.

■

In particular, this third point allows us to perform induction in any model of ETCS.

At this point, we have now developed sufficient machinery to construct much of modern set theory – Cantor's theorem, the Cantor–Schröder–Bernstein theorem, Zorn's lemma, etc. – as well as embedding the rest of mathematics into sets. Most of the set-theoretic universe at this point is constructed similarly to ZFC, constructing new sets by taking products and quotients of existing sets.

We end with a metatheorem that quantifies how strongly the axioms of ETCS characterise its models:

Theorem 9.7. *[New14] If $\mathscr{C}$ and $\mathscr{S}$ are models of* ETCS*, then $\mathscr{C} \simeq \mathscr{S}$, with the equivalence given by the adjoint functors $T \dashv \hom_{\mathscr{S}}(1,-)$, where*

$$T(X) := \coprod_{x \in X} 1$$

10 Discussion

10.1 Relative Strength

ETCS is slightly weaker than ZFC, in the sense that there are statements provable in ZFC that are not provable in ETCS, but only to a slight extent, as these statements are generally beyond the interest of even most researching mathematicians (outside of those studying set theory/model theory/foundations). Undergraduate mathematics in particular (again, outside of a course on ZFC) also assumes no properties of sets beyond ETCS, so it would seem that ETCS is more than sufficient for most practical applications – in exchange for a very slightly weaker ontology, we obtain a great deal of conceptual clarity.

However, if one still needs these extra statements, then ETCS can be extended to encompass them. This is not unusual for a set of axioms; for instance, the (generalised) continuum hypothesis has been famously proven to be independent from ZFC, and is often taken as an additional axiom on top of ZFC when working with large cardinals in set theory.

The relationship between ETCS and ZFC has been well-studied, and it is known that ETCS is equivalent to the fragment of ZFC called *Restricted Zermelo with Choice* [Lei11][MM12], and it is also known what extra conditions need to be added on top of ETCS to obtain an axiom system with strength equivalent to ZFC (in the formal sense that a proposition is provable in this extended ETCS if and only if it is provable in ZFC).

This condition missing from ETCS is some form of an axiom of *collection* – axiom schemata that permit the construction of certain new sets from existing sets. These axiom schemata hence contribute to the size of the universe of constructible sets substantially, but conversely, this expansiveness is often not of particular importance in the practice of "ordinary" mathematics, so the omission of collection in ETCS is not damaging for most applications [nLa23c].

In ZFC, a form of a collection axiom is given by the axiom schema of replacement – informally, given a first-order formula φ, and a set x, we are permitted to form the set $\{\varphi(y) : y \in x\}$.

ETCS can similarly be extended with a collection axiom, which is exhibited as a form of cocompleteness [Osi74]. Informally, the axiom of collection in ETCS+C states that the category of sets has all coproducts $\coprod_{i \in I} X_i$ of families $(X_i)_{i \in I}$ of sets specified by first-order formulae. With this additional axiom, ETCS+C is then equivalent to the entirety of ZFC.

For another example, we can also augment ETCS with the Continuum Hypothesis, just like with ZFC. An elementary topos with collection is said to *satisfy the Continuum Hypothesis* (CH) if for all objects X, if there exists monomorphisms $\mathbb{N} \rightarrowtail X \rightarrowtail \Omega^{\mathbb{N}}$, then there exists either a monomorphism $X \rightarrowtail N$, or a monomorphism $\Omega^N \rightarrowtail X$, where $\mathbb{N}$ is the natural numbers object and Ω is the subobject classifier.

If the topos is Boolean, as in the case of any topos that satisfies ETCS, then (a categorical version of) the Cantor–Schröder–Bernstein theorem holds, so the existence of these latter two monomorphisms imply that there exists either an isomorphism $X \cong \mathbb{N}$, or an isomorphism $X \cong \Omega^{\mathbb{N}}$ (not necessarily equal to either monomorphism in either case), thus recovering the ordinary set-theoretic Continuum Hypothesis.

10.2 Material and Structural Sets

ETCS and ZFC both deal with "sets", but these notions are so distinct that it seems unhelpful to call them both by the same name. We will call a set in the style of ZFC a *material-set* and a set in the style of ETCS a *structural-set*.

In ZFC, we have the axiom of extensionality, which says that two material-sets are equal if and only if they have exactly the same elements. That is, material-sets are determined entirely by their elements. However, in ETCS, a weak extensionality principle is given by the Yoneda lemma: structural-sets are characterised only up to isomorphism by their generalised elements. However, we often only ask if two sets contained within a larger ambient set are equal. In that case, the strong extensionality principle for functions given by well-pointedness characterises structural-(sub)sets up to true equality.

Because elements of structural-sets are functions, this means that they themselves are never structural-sets, unlike in ZFC, where elements of material-sets are always themselves material-sets. This is perhaps closer to how we often use sets in ordinary mathematics; we never actually treat, say, the (real, natural, etc.) number "3" as a set itself.

In the introduction, we saw an argument of Benacerraf's that numbers cannot be sets, since numbers have no properties beyond arithmetic relations amongst themselves, and sets *do* have properties other than that. In this view, the natural numbers are envisioned as elements of an *abstract structure*, where elements have no properties beyond what is endowed upon them by that structure.

In ZFC, we define $\mathbb{N}$ to be some particular material-set, say $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\},\ldots\}$, with all the arithmetic relations constructed on top of the chosen encoding, but this yields unwanted additional properties, like $3 \in 17$ (or not, given a different encoding) that we have to ignore.

In contrast, the natural numbers object in ETCS is a structural-set $\mathbb{N}$, equipped with an element 0 and a successor function s. Natural arithmetic is expressed in terms of the zero element 0 and the successor function s, so natural numbers, or elements of this abstract structure, have arithmetic relations between each other, but no additional properties beyond that.

More generally, *any* structural-set is precisely an abstract structure in this sense. An element $x \in X$ – a function $x : 1 \to X$ – has no identity or internal content except that it is an element of X, and is distinct from other elements of X.

Another effect of this is that, elements of structural sets, being functions, must also always be inherently attached to a set, unlike elements of material-sets, which are themselves objects that may exist in isolation; in ETCS, we can never refer to any element as existing by itself as some kind of free-floating Platonic essence, void of any connecting structure.

Given a material-set X, then for any other set A, we can ask whether $A \in X$ or not, regardless of any prior relations between A and X. This statement, "$A \in X$", is then a proposition in the formal sense: that is, it has a truth value, can be proven, can be combined with logical connectives, quantified over, etc.

In contrast, if X is a structural-set, then there are some things which are *intrinsically* elements of X, namely, the functions $1 \to X$. If a thing is not *given* as an element of X, then it *is not* an element of X, and similarly, an element $1 \to X$ cannot also be an element $1 \to Y$ of a different structural-set Y.

Thus, the statement $A \in X$ is not something one would ever *prove* about two pre-existing objects A and X. Consequently, the statement $x \in X$ is *not a proposition* in a structural set theory.

As an illustration of this difference, consider the statement "for all $x \in \mathbb{R}$, $x^2 \geq 0$" [Shu13]. If $\mathbb{R}$ is a material-set, then this statement could be read as "for all things x, if $x \in \mathbb{R}$, then $x^2 \geq 0$". Formally, the corresponding sentence in first order logic is $\forall x : x \in \mathbb{R} \to x^2 \geq 0$.

However, if $\mathbb{R}$ is a structural-set, then $x \in \mathbb{R}$ is a logical atom and cannot be the premise of an implication. Thus, the statement should be read as "it is a property of every real number that its square is non-negative."

Arguably, this is closer to how quantification is used in practice: we generally don't mean, "it is a property of any and all things in mathematics that *if* it happens to be a real number, *then* its square is non-negative." For instance, under the material interpretation, one particular instance of "for all $x \in \mathbb{R}$, $x^2 \geq 0$" is, "if the ring $\mathbb{Q}[x]$ happens to be a real number, then its square is non-negative", which could be reasonably agreed is a statement that most mathematicians would not naturally regard part of the content of "for all $x \in \mathbb{R}$, $x^2 \geq 0$".

Conversely, sometimes, we *do* want to regard $A \in X$ as a proposition. For instance [Shu13], if L is the set of complex numbers with real part $\frac{1}{2}$, then we are very interested in proving that "for all $z \in \mathbb{C}$, if $\zeta(z) = 0$ and z is not an even negative integer, then $z \in L$". In this statement, the first $\in$ is read structurally – z is *given* to be a complex number – while the second $\in$ is read materially; being the consequent of an implication, $z \in L$ should certainly be read as a proposition.

The observation here is that L is a subset of $\mathbb{C}$: z is already given to be an element of the structural-set $\mathbb{C}$ (i.e. it is a function $1 \to \mathbb{C}$), so it is possible to ask whether it happens to belong to this subset L (does it lift through the function witnessing $L \subseteq \mathbb{C}$?). As seen earlier, function extensionality characterises subsets up to true equality, so ETCS supports this use of material-subsets and propositional membership.

All this previous discussion also applies similarly to the subset relation: like elements, subobjects are (classes of) morphisms, so the statement $A \subseteq X$ for structural-sets X is similarly not a proposition – it is just notation for a class of monomorphisms with codomain X. However, the relation $\subseteq_X$ between subsets of a fixed set X *can* be used propositionally since it is a proper relation, so ETCS also supports propositional containment.

There are, however, some constructions that are somewhat less natural as structural-sets – in particular, function sets and power sets. In ETCS, function sets are given by exponential objects. We have a natural isomorphism

$$\hom(1,B^A) \cong \hom(A,B)$$

characterising the elements of B^A as being isomorphic to functions $A \to B$; but, this is only an isomorphism, and not true equality – elements of B^A are not literally functions $A \to B$, instead only being "labels" for them. To access the functions they reference, we need to invoke the evaluation map $\mathrm{ev} : B^A \times A \to B$.

Power sets have a similar problem to function sets in ETCS: the elements of a power set are characterised by the isomorphism

$$\hom(1,\Omega^A) \cong \mathrm{Sub}(A)$$

but this is again only up to isomorphism and not true equality, so elements of Ω^A are also only "labels" for subsets of A. This is one place where material-sets really are more conceptually clear: the elements of a material-power set $\mathscr{P}(X)$ are genuine subsets of X.

On the other hand, function sets are also rather unnatural in ZFC: we first have to pick an arbitrary encoding of an ordered pair, then define a function to be a special type of set of ordered pairs. This chain of encodings also results in lots of undesirable side-effects. At least in ETCS, the set of labels is given by a universal property and is hence isomorphism invariant.

10.3 Types

The problem here is that ZFC concerns itself exclusively with material-sets, and ETCS with structural-sets, when in mathematics, we often need to work with both. The awkwardness in these constructions, comes predominantly from forcing us to interpret structural-sets within a framework that only supports material-sets, or the reverse.

There are various solutions to this. For one, we could just keep adding more and more primitive notions to our systems until we have everything we need, i.e., add atoms to ZFC, and define the naturals to be a set of atoms; add functions to avoid encoding ordered pairs, etc., or, add a membership predicate to ETCS. However, a more systematic approach is to be desired.

It turns out that such a foundational system already exists, namely, *type theory*. In particular, variants such as *Martin–Löf dependent type theory* (MLTT) or *Homotopy Type Theory* (HoTT).

Type theories generally work like structural set theories. For instance, we have *types* which behave very much like structural-sets, and "elements" of types are called *terms*, and we write the *type declaration* $x : X$ for a term x of type X.

Just like with the structural usage of the $\in$ relation, a type declaration is not a proposition, as terms are intrinsically of some given immutable type, just like elements are intrinsically attached to structural-sets. Instead, these kinds of statements are called *judgements* – meta-assertions that cannot be proven within the theory.

Terms may also have *some* internal structure, unlike elements of structural-sets, though they do not *have* to, also unlike elements of material-sets, and the kind of internal structure they may have is controlled by their type. Then, *type constructors* can be used similarly to "adding primitive notions" to a set theory, i.e., a type constructor for ordered pairs, etc.

It turns out that category theory is also the natural language for the semantics of type theory; and conversely, type theory is a natural language for the syntax of category theory. This is beyond the scope of this discussion, but informally, we may interpret the objects of a category as types, and a subobject $\phi \rightarrowtail A$ can be regarded as a proposition by interpreting ϕ as the collection of terms of type A for which ϕ is true. Logical operations are then given by various limits in the subobject lattice.

This type system associated to each category is called that category's *internal logic*, and different kinds of categories induce different kinds of internal logics. For instance, Boolean categories induce type theories equivalent to classical first-order logics, while elementary topoi generate constructive higher-order logics. In particular, the internal logic of an ∞-topos is a model of a variant of Homotopy Type Theory.

Conversely, any type theory can be converted into a category by constructing objects from types, subobjects from relations, morphisms from functions, etc. It turns out that set theories can also be embedded within type theories, and type theories can also be embedded within sets: sets, categories, and types, can all be implemented within each other; one explicit construction is given in [Awo11].

10.4 Final Remarks

Once we have membership, functions, unions, quotients, products, etc. in whichever choice of set-theoretic foundations, the following development of mathematics is mostly the same: at a certain point, once basic mathematical structures have been constructed and encoded, for all practical purposes, it matters not if one begins with ZFC or ETCS.

After all, asking "is $3 \in 17$?" is not really a problem of practical concern in ZFC. However, it is still pedagogically fruitful to ask such questions. One advantage of teaching ETCS as a foundation is that it introduces the notions of isomorphisms and universal properties to students early on. It can also clarify why some material constructions are constructed in the way they are (i.e. "they are arbitrary choices of models of a (co)limit", "they satisfy the relevant structural property", etc.), even if we do not choose to use ETCS in practice.

Beyond this, the significance of ETCS is not from its use (or non-use) as a foundation of mathematics, but moreso from the research into topos theory that followed. ETCS was one of the first attempts of a categorical analysis of logic, and though it did not see much use as a foundation itself, the more general theory of topoi that followed is now the main language of categorical logic.

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