I can't believe you just excluded the middle

On Constructivism in Mathematics

by

Ryan Jun Kit Liu

MA4K9 Dissertation

Submitted to The University of Warwick

Mathematics Institute

April, 2025

Contents

Foreword

The characters of the dialogues* are purely fictional, and do not represent the positions of any specific persons living nor dead, and are intended only to be vehicles through which common objections and misconceptions are introduced, and more generally as foils to highlight differences in viewpoints/interpretations between different philosophies of mathematics. In particular, this paper is only intended to cover (varieties of) constructivism, though generally from a pluralistic viewpoint; other positions are touched upon only so far as to introduce objections against constructivism, and are intentionally not explored in significant depth for the sake of brevity and focus.

While the dialogues focus on philosophical discourse, this essay is not intended to be predominantly philosophical in nature. However, the origins of constructivism are highly philosophical in nature, and such discussion is useful to fully contextualise the motivations behind various definitions and axiomatisations.

On the other hand, the philosophical underpinnings of constructivism are inessential to its use, in so far as any other branch of mathematics. After all, a mathematician may not have any more understanding of the philosophy underpinning their work any more than a pigeon does of aerodynamics. But a pigeon flies nonetheless.

Thus, for the mathematically-minded reader uninterested in philosophy or history, we recommend just skimming over the dialogues, or even skipping through to either: ACT II for a showcase of counterintuitive constructive results; ACT III for a sample of some classical results being constructively recreated; or ACT IV for a more modern application of constructive mathematics in topos theory.

Constructivism and Intuitionism

These terms are discussed in more detail in the paper, but in the existing literature, these are often conflated, particularly in the context of logic and type theory (e.g. the internal *higher-order intuitionistic logic* of a topos, *intuitionistic type theory*, etc.).

For our purposes, *intuitionism* is reserved explicitly for the philosophical position and programme associated with Brouwer. Thus, we would rather describe, for example, the internal logic of a topos as *constructive*, rather than intuitionistic.

In light of this, the character INT would perhaps be more aptly named CONS; although she discusses Brouwer's historical constructions in ACT II, she does not share his philosophy and is instead more pluralistic in viewpoint (e.g., see the discussion and following dialogue in § 2). However, we find that INT is more readable and easily distinguishable from CLASS than CONS is, so the misnomer has been retained.

Weak Counterexamples

One of the common arguments employed in constructive mathematics is that of a *weak counterexample*, which depends intrinsically on the existence of unresolved conjectures. For the purposes of this paper, we have chosen:

- the *twin prime conjecture* – "there are infinitely many primes p such that $p - 2$ is also a prime.";
- and the (*strong*) *Goldbach conjecture* – "every even natural number greater than 2 is expressible as the sum of two primes."

At the time of writing, both of these conjectures remain open. However, if either of these have been resolved at the time of reading, weak counterexample arguments should be straightforward to rephrase in terms of other unresolved conjectures; the reader should feel free to pick their favourite unresolved conjecture(s) for this purpose, and do so as an exercise. (In fact, several of the sources in the existing literature referenced in this paper have weak counterexamples based on conjectures that have been resolved since their publishing.)

*Format heavily inspired by [Lak63], [Hey66], and [Lin72].

I

Discourse

Class. Ah, good to see you, Ms. Int.

Int. Likewise, Mr. Class. Your talk earlier was fascinating – though, I hope my questions didn't bother you?

Class. No, of course not – I found your comments equally as intriguing. Still, I have yet to understand your fixation on rejecting the *Axiom of Choice* and the *excluded middle*, and such. Are they not obviously true in reality?

Int. I believe we have spoken on this topic before. You take these axioms to be "obvious", but isn't it also obvious that rigid Euclidean motions should preserve volume? But of course, Choice says otherwise.

Humans are notoriously bad at understanding their own internal thought processes, often assuming that the mental models of reality they hold are "obviously" right, despite the fact that others find quite contradictory beliefs equally obvious.

This is especially true in the realm of the infinite: our ordinary logic was developed for reasoning about finite collections, being modelled on our informal intuitions about how such collections behave, so I am of the belief that extending logical principles like Choice to the infinite without justification is unwarranted.

Class. I accept the point about our instincts misleading us, but I think your objection is more about finiteness than these non-constructive principles. In particular, your argument does not extend to the excluded middle; in reality, it must surely be the case that it is either raining, or it is not – there is no third possibility – and there are no issues of finiteness to be observed here.

Int. This is a side effect of how we access empirical truths in the physical world; if we can, at any moment, look outside and verify whether or not it is raining, then we can constructively establish a proof of one disjunct or another – this, I agree. However, this just means that this particular instance of LEM holds in practice, and does not serve as justification for settling all similar matters – especially abstract mathematical instances of LEM – as a logical axiom.

For instance, consider the similar statement "Either it will rain *tomorrow* or it will not." Unlike your previous statement, there is no way to *presently* confirm the truth value of this proposition. Since we have no means of determining, at this moment, whether it will rain tomorrow, we lack a constructive proof of either disjunct. You might insist that, when tomorrow arrives, we will be able to confirm one way or the other – but that only establishes that the law of excluded middle may hold *retrospectively*, not that it is valid as a principle of reasoning before we have the necessary evidence.*

To assume that every statement is either true or false, even when we have no means of determining which, is to appeal to a kind of metaphysical realism about mathematics; to the idea that mathematical objects and truths exist in some objective reality, independent of our capacity to prove or verify them. But this is a philosophical assumption in metaphysics, not a logical necessity.

* * *

1 Introduction

Much of the historical (and indeed current) objection to the *Axiom of Choice* in set theory is due to the fact that it is not *constructive*: it declares the existence of a choice function for any family of non-empty sets (a function $f : X \to \bigcup X$ with $f(A) \in A$ for all $A \in X$), but does not indicate what this choice function might be.

For instance, the following result, often considered to be one of the most important results in general topology, is dependent on Choice:

*If you really wanted to take this example further, we could replace the question of whether it is raining or not with the spin of a quantum particle, in which case both disjuncts really *are* meaningless until a measurement is made.

Theorem 1.1 (Tychonoff)**.** *Given any family $\{X_\alpha\}_{\alpha\in\Lambda}$ of compact spaces, their product $X \coloneqq \prod_{\alpha\in\Lambda} X_\alpha$ is compact.*

Proof sketch. [Loo13] By complementation, a space is compact if and only if for any collection $\mathcal{F}$ of closed sets such that every finite intersection is non-empty, the intersection $\bigcap \mathcal{F}$ over the entire family is also non-empty.

Let $\mathcal{F}$ be such a family of closed sets in X with the finite intersection property. By Zorn's lemma (equivalent to Choice), we may extend $\mathcal{F}$ to a maximal family $\mathcal{F}_0$. By projection on to each X_α, we obtain a similar family in each space. By the compactness of the X_α, we pass to the finite and find a point that is in the intersection of the closure of every set of $\mathcal{F}_0$, and is therefore in $\mathcal{F}$. ■

There are many proofs of Tychonoff's theorem in the literature; some use well-ordering instead of Zorn's lemma, and some use Kelley's theorem on universal subnets, etc., all of which equivalent to Choice. In any case, the use of Choice is essential; Tychonoff's theorem is not just dependent on Choice, but is in fact *equivalent* to Choice [Nor].

Here is another intuitively "obvious" result:

Theorem 1.2. *The cartesian product of any family of non-empty sets is non-empty.*

However, an element of the cartesian product is, by definition, a choice function on the original family, so this statement is also equivalent to Choice. Clearly, Choice yields many important results.

Beyond this, however, there is another non-constructive principle embedded within classical logic itself that is commonly accepted, even by those who reject Choice; namely, the *Law of the Excluded Middle* (LEM).

Axiom (Excluded Middle)**.** *For every proposition p, either p or not p.*

$$\vdash p \vee \neg p$$

This law is also known as the (*principium*) *tertium non datur*, literally *third not given* – every statement is true or false; there is no third possibility.

Clearly, the information in a statement $p \vee \neg p$ is quite limited. For instance, consider the statement "*there exist* 1000 *consecutive* 0*s somewhere in the decimal representation of the real number* π". It is very possible that we might never be able to decide the truth value of this statement, yet we are forced to accept that one of the cases must be true.

However, this principle is so deeply embedded into classical mathematics that most mathematicians cannot easily point out when it is being used in a proof. To illustrate its usage, one of the most commonly quoted examples is as follows:

Theorem 1.3. *There exist a,$b \in \mathbb{R} \setminus \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.*

Classical proof. Consider $\sqrt{2}^{\sqrt{2}}$. By the law of the excluded middle, this is either rational or irrational.

If it is irrational, then we are done, with $a = b = \sqrt{2}$.

Otherwise,

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$$

so taking $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ suffices. ■

This proof is non-constructive in the sense that we have produced two candidates that potentially validate the statement, but have not shown *which one* is true, instead appealing to LEM to complete the proof. In particular, if, for some reason, we needed a concrete example of such irrationals a and b, then this proof is of no use to us.

However, the additional provision of a proof of the (ir)rationality of $\sqrt{2}^{\sqrt{2}}$ would then remove the need for LEM; thus, this classic example is poorly chosen, in that the use of LEM in its proof is inessential.

In contrast, here is a theorem whose proof depends on LEM intrinsically:

Theorem 1.4 (Cantor–Bernstein)**.** *If there exist injections $f : A \to B$ and $g : B \to A$, then there exists a bijection $h : A \to B$.*

Proof. Because unions and intersections of subsets are still subsets, the powerset $\mathcal{P}(A)$ ordered by inclusion is a complete lattice. Now, consider the map $F : \mathcal{P}(A) \to \mathcal{P}(A)$ defined on subsets $S \subseteq A$ by

$$F(S) = A \setminus g\big(B \setminus f(S)\big)$$

Since f and g are injective, this map is weakly monotone increasing. Thus by the Tarski-Knaster theorem, the set of fixed points is non-empty, so let $X \subseteq A$ be a fixed point of F.

Since $A \setminus X = g\big(B \setminus f(X)\big)$, we can define a bijection $h : A \to B$ by

$$h(x) = \begin{cases} f(x) & x \in X \\ g^{-1}(x) & x \in A \setminus X \end{cases}$$

By LEM, for each $x \in A$, either $x \in X$ or $x \notin X$, so this function is well-defined. ■

There are many proofs of the Cantor–Bernstein theorem; the proof given above is perhaps not the most common – that being the proof commonly attributed to König in which the orbits of elements under f and g are considered, where LEM is (repeatedly) used to find preimages of elements, or to state that there is none – but this proof sharply isolates the usage of LEM to a single place; in showing the well-definedness of h. One might think that LEM may be removed with a different approach or with some rearrangement, but as before, it turns out that this theorem is equivalent to LEM [PB22]; the usage is essential.

2 Constructivism

Many mathematicians will have at least heard of constructive mathematics in passing; or seen the occasional proof or result mentioning it in an aside; or even heard the local department set theorist/logician ramble about Choice or LEM in some form or another.

Unfortunately, many mathematicians dismiss such ideas without much thought, marking them as irrelevant to their own work or even as pointless and detached from reality. However, much of this stems from a general misunderstanding of what constructive mathematics actually says, which we hope to elucidate here.

2.1 Constructive mathematics is, broadly speaking, mathematics done without the Law of the Excluded Middle and other non-constructive principles.

Constructivism is, more generally, the philosophy that:

- Constructive mathematics is useful;
- Or, in more extreme cases, that non-constructive mathematics is "wrong".

The suffix "-ism" certainly carries the ideological connotations of this latter point, and historically, constructive mathematics was initially developed by mathematicians who believed in this.

We hasten to clarify that we disagree with this idea – and strongly so. However, we also believe the first point is underappreciated by many who could benefit from seeing this other approach to mathematics.

Class. Yet, mathematics seems to proceed quite well even with such assumptions. Surely, you would not claim that theorems proved by classical means are meaningless?

Int. Of course not – such a claim is blatantly absurd. After all, Newton and Laplace were able to predict celestial motions with remarkable accuracy long before the real numbers were rigourously defined. Whether or not we accept non-constructive principles or not, their work would remain equally as accurate and uncaring of our philosophical convictions.

To me, the problem is not that classical results are wrong or unreliable in any way, but that the idealistic assumptions of classical logic unnecessarily mask or sidestep questions about how things are constructed or how they can be concretely realised.

CLASS. That's a reasonable thought, but isn't there value in generality? Classical methods let us prove broad results without always getting bogged down in specific constructions.

INT. Certainly. Classical reasoning excels at powerful, sweeping results. But sometimes, constructive methods give additional insight – they tell us why something exists in a way that can be directly interpreted in explicit computational or algorithmic terms. And even classically, constructive proofs can be more illuminating, as they tend to be more explicit.

Even most classical mathematicians would agree that a constructive proof is more valuable than one that is not. Yet, many mathematicians brashly dismiss constructivism without the faintest attempt at engaging with its ideas.

These non-constructive principles, while powerful, are not always transparent in their application, and their unrestrained use can lead to conclusions that are not constructive or computationally meaningful. Even if one doesn't agree with their philosophy, simply being aware of constructivist concerns would help mathematicians recognise when they are relying on such principles unnecessarily, and possibly avoid using them when they aren't essential to the problem at hand.

After all, why would you spend time carefully constructing a specific witness to a theorem when you could just bash it with Choice or LEM and obtain a proof of no computational value?

I have heard many a mathematician reject constructivism, saying that they "believe in" Choice or LEM, but I feel that this is akin to "believing" in commutativity or not in the theory of groups. Each structure is interesting in its own right, and each is useful for different purposes.

Choiced sets are better-behaved for the purposes of functional analysis, cardinal arithmetic, category theory, etc. while Choiceless/LEMless sets contain better constructive content, are better behaved under computability relations, etc.

There are many modern mathematicians who work constructively not because of any philosophical leanings about the "wrongness" of non-constructive mathematics, but because constructive mathematics is interesting, and furthermore *useful*, even in a classical setting.

To further this, we hope to introduce some of the main ideas of constructivism and clarify some common misconceptions here.

2.1 Varieties of Constructivism

Constructivism is a wide spectrum of perspectives on mathematics, and is by no means homogeneous – there are many varieties of "constructive" mathematics, and the existing literature is not in agreement over the scope of constructivism. Even within a particular "branch" of constructivism, views and interpretations will vary widely.

We quickly discuss some terms commonly associated with constructivism, before outlining the three most influential schools of constructivism.

2.2 Finitism: the basic ideas of finitism are:

- Only finitely representable structures are objects of mathematics. In particular, operations on such structures consist only of combinatorial data, and are hence always *effective* or *computable*.
- Abstract notions such as (arbitrary) sets, quantifiers, operations, constructions, etc. are not finitistically acceptable.

For instance, while finitists accept the existence of each natural numbers individually, the set $\mathbb{N}$ of all natural numbers is rejected.

Skolem was the first mathematician to contribute substantially to the theory of finitist mathematics [Tro+11]. Skolem demonstrated that a fair part of arithmetic could be developed through finitist means.

Curry and Goodstein later formulated a purely equational calculus, *Primitive Recursive Arithemetic* (**PRA**), free from quantifiers and bound variables in which Skolem's work could be formalised, and later showed that classical propositional logic is a conservative extension of **PRA** [Tro+11].

A related viewpoint was endorsed by Hilbert: finite mathematical objects are concrete and existing, while infinite mathematical objects are "ideal" objects, and accepting these ideal objects does not cause any problems with finite objects. This is akin to the treatment of complex numbers in the solutions of the depressed cubics in the 16th Century; these quantities were not regarded to be concrete mathematical objects, but only useful abstractions that vanished at the end of a calculation, leaving the only the concrete "actual" result.

In the same way, Hilbert believed that it was possible to show that any result about finite objects that involved ideal infinite objects could be rewritten to exclude them. That is, the addition of ideal infinite objects is conservative over the finitist part.

This directly lead to Hilbert's programme of proving the consistency and completeness of set theory via finitist means. This idea of encoding everything in mathematics and philosophy into a universal formal language goes back at least to Leibniz; if there was a mathematical disagreement, you would not have to engage in tedious debate and argue with one another, but sit down and *calculate*. However, not long after Hilbert had formulated his programme, Gödel's incompleteness theorems dealt a fatal blow to this goal.

With the modern development of (seemingly) consistent theories involving infinities, along with further progress in formal logic, the mathematical landscape has since shifted away from this goal, and most modern mathematicians now reject finitism as a philosophy and/or mathematical foundation. However, finitist methods have since found applications in proof theory (i.e. in bounded arithmetic), finite model theory, and formal program verification.

2.3 Predicativism: This can be seen as a form of "constructivism with respect to definitions": in predicativism, objects cannot be defined *impredicatively*; that is, defining an object x by invoking a totality X of which x is an element. In particular, quantification over X when defining x is not permissible.

For instance, the definition of a real number x as the least upper bound of a set (c.f. Dedekind cuts) is impredicative because x is defined to be a particular element of a set (the set of upper bounds) that includes x. Because of this, predicativism naturally entails the introduction of *stratified* theories, where quantification over constructions at one level then yields a construction at a new higher level.

Note that predicativism is also compatible with classical logic, as it only restricts what is definitionally permissible, but is still often combined with the demands of other constructive schools of thought. We will not explicitly require predicativism in our constructivism, but it will make the occasional appearance later on.

2.4 Intuitionism is a constructive school of thought, originally conceived of by Brouwer (1881–1966), and later developed by Heyting (1898–1980).

The name is due to Brouwer's emphasis that mathematics, as written, is informal, and *mathematical intuition* is what allows us to recognise that a proof is convincing [Tro+11; MML18; VS17]. This was in opposition to the formalist idea that mathematics can be reduced to the pure syntactic manipulation of meaningless strings, independent from semantics [Mad92; Dav05]; in contrast, intuitionism holds that these strings are inherently attempts at capturing and conveying *mental constructions*.

The main ideas of intuitionism are as follows:

(a) The objects of mathematics are mental constructions, grasped only in the mind of the (idealised) mathematician.

The use of all mathematical language is thus inherently informal, induced by our limitations (when compared to an ideal mathematician with unlimited memory and perfect communication), and the desire to communicate our mental constructions to others.

(b) Mathematics is a matter of creation, not discovery. Mathematicians do not mentally reconstruct or discover preexisting mathematical objects existing independently of our thoughts. That is, intuitionism is anti-realist with regards to the ontology of mathematical objects.

In particular, mathematics is independent of the external experience of reality; with regards to mathematics, only the thought constructions of the ideal mathematician are exact.

(c) It does not make sense to ask the truth or falsity of a mathematical statement independently from our knowledge concerning the statement; a statement is *true* if we have a proof of it, and *false* if we can show that any supposed proof of it leads to a contradiction. Therefore, for an arbitrary statement, the excluded middle does not hold *a priori*.

It is this last point that forces a different interpretation of statements of the form "p or q holds", and "there exists x such that $P(x)$ holds"; in particular, $p \vee \neg p$ does not generally hold in the constructive interpretation of $\vee$ and $\neg$. In fact, these requirements are common to all three schools of constructivism listed here; we return to this discussion later. Specific to intuitionism, however, is the the principle of *Weak Continuity for Numbers*, WC-$\mathbb{N}$.

First, a *choice sequence* α is an indefinite process of building up a sequence of values, one at a time, by the ideal mathematician. At any particular step, the ideal mathematician has determined only finitely many values, and for the next step, they are constrained by a collection of restrictions (which may be empty) on future choices that define the choice sequence. At any particular instant, we can only access a finite completed initial segment, so constructively meaningful claims about choice sequences must be provable from some initial segment. The WC-$\mathbb{N}$ extends this idea as follows:

Axiom (WC-$\mathbb{N}$)**.** *For any formula $\varphi(\alpha,n)$ depending on sequences α and naturals n,*

$$\forall\alpha\exists n\big(\varphi(\alpha,n)\big) \to \forall\alpha\exists m\exists n\forall\beta \sqsupseteq \alpha_m\big(\varphi(\beta,n)\big)$$

That is, roughly speaking, if φ is a natural-valued function on choice sequences sending α to n, then there is a point m (depending on α) such if β agrees with α for at least the first m components, then β is also sent to n. In other words, on each choice sequence, φ is determined entirely by one of its finite initial segments.

This reflects the constructive idea that functions on infinite sequences must be constructible in a way that does not require knowledge of the entire sequence, but only some finite initial segment of the sequence, since we can never observe the entire infinite sequence. This roughly corresponds to the idea that functions defined on choice sequences are topologically continuous in the sense that the value of n does not change if we slightly perturb α in a way that does not alter its first few terms (whence the name).

2.5 Markov's Constructive Recursive Mathematics (CRM)**.**

(a) The objects of mathematics are *algorithms*, in the precise sense of strings formed from a formal grammar called a *Markov-algorithm.* Issues to do with memory are disregarded, and strings may be arbitrarily long (though still always finite).

(b) The abstraction of potential realisability is permissible, but not the abstraction of actual infinity.

For instance, we know how to compute the sum of any two arbitrarily large natural numbers, so we may regard natural addition as a well-defined operation in a potential sense, even though we cannot deal with an infinite collection of completed computations all at once.

(c) If it is impossible that an algorithmic computation does not terminate, then for some input, it does terminate. Logically, this is the statement of *Markov's Principle*:

$$\mathsf{MP}: \qquad \neg\neg\exists x\big(f(x)=0\big) \to \exists x\big(f(x)=0\big),\ (f:\mathbb{N}\to\mathbb{N}\text{ recursive})$$

This last point of MP also makes CRM incompatible with intuitionism, as Markov's principle is an instance of double negation elimination/excluded middle that does not provide a witness for which input the algorithm terminates on.

2.6 Bishop's Constructive Mathematics (BCM)**.** Bishop's approach is generally pragmatic in nature, focussing instead on the *practice* of constructive mathematics instead of its philosophy:

(*a*) Mathematical statements should have numerical meaning. In particular, existential quantifiers and disjunctions must, in principle, be capable of being made explicit. Furthermore, one can only show that an object exists by giving a finite routine for finding it.

Unlike Markov's programme, Bishop does not require that every object is given by an algorithm in the precise sense of recursion theory, and descriptions in terms of informal "rules" or "operations" are fully admissible.

According to Bishop, Brouwer had successfully criticised classical mathematics, but had failed to introduce a satisfactory alternative, instead wasting time on splitting various classically-equivalent concepts into multiple constructively-distinct concepts – each also of dubious utility – rather than concentrating on the distillation of singular mathematically-useful translations of classical definitions.

> The movement [Brouwer] had founded has long been dead, killed [...] chiefly by the failure of Brouwer and his followers to convince the mathematical public that abandonment of the idealistic viewpoint would not sterilize or cripple the development of mathematics. Brouwer and other constructivists were much more successful in their criticisms of classical mathematics than in their efforts to replace it with something better. [Bis67]

To this end, Bishop's programme contains the following guiding principles:

- Define concepts *affirmatively* (the significance of this will be apparent later);
- Avoid defining irrelevant concepts; if several definitions are classically equivalent but constructively distinct, use only the ones that yield useful results.
- Avoid *pseudo-generality*; if an extra assumption simplifies the theory and the examples that one is interested in satisfy the assumption, then the assumption should be made.

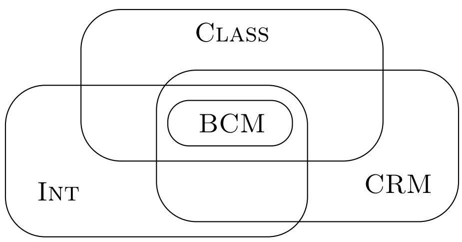
Because the aim of Bishop's programme was to reconstruct much of classical mathematics – though with more numerical content – without regards to any specific philosophical commitments, statements in Bishop's constructive mathematics are amenable to a wide variety of interpretations.

A statement of BCM may be interpreted intuitionistically without much change: Bishop's sequences have a natural interpretation as choice sequences in Brouwer's intuitionism; and the lack of a formal definition of algorithm in BCM also enables BCM to be interpreted in CRM similarly. In fact, BCM can be interpreted in computable mathematic under any reasonable model of computation for the same reason. And finally, since the only logical restrictions on BCM are on LEM, every theorem of BCM is a classical theorem too.

From a formal viewpoint, Class, Int, and CRM can be seen as BCM extended with additional axioms:

$$\begin{aligned} \textsc{Class} &= \mathrm{BCM} + \mathsf{LEM} \\ \textsc{Int} &= \mathrm{BCM} + \mathsf{WC\text{-}}\mathbb{N} \\ \mathrm{CRM} &= \mathrm{BCM} + \mathsf{MP} \end{aligned}$$

The situation is thus, informally, as follows:



([Tro+11])

In this paper, for the purposes of philosophy, we will generally follow Brouwer's programme, but for practical mathmatical purposes, we will follow in the footsteps of Bishop and aim to be as pragmatic as possible (apart from when intentionally showcasing certain counterintuitive constructive results).

3 Constructivity

Int. Mathematics is not the exercise of pushing formal strings around according to arbitrary syntactic rules – this is akin to regarding grammar as being the basis of human conversation, as opposed to ideas – nor is it the act of drawing empty conclusions from arbitrary axioms and inference rules – of playing with purely abstract concepts devoid of practical significance. However, many mathematical statements do seem to carry a rather idiosyncratic form of practical content.

Take, for instance, the conjecture that "*every even integer greater than 2 is the sum of two primes.*" The practical content of this statement is not that, if we go to the number line and observe it closely, we will see a certain thing happen; after all, we have already verified the conjecture for every integer that could possibly occur in nature or conceivably be of any practical significance; yet the perceived content of such a statement would be in its *proof*. Rather, its practical utility – if any – may be found in its use in deriving other theorems, each possibly of the same kind of practical content.

It seems then, that there is a divide in mathematics, between these statements that are merely theoretic – those making assertions about an *ideal* mathematical universe without any empirical demonstrability – and those that are pragmatic in nature, with immediate *constructive* validity, asserting that a certain sequence of realisable operations will yield concrete, observable outcomes.

We now expand on the notion of constructivity via some simple examples. Which objects are valid as (mental) constructions? The natural numbers are constructively unproblematic overall, but can we select from them an element at will?

Consider the following descriptions [Hey66] of two natural numbers a and b:

(1) a is the greatest prime such that $a - 1$ is also prime, or $a = 0$ if such a number does not exist.

(2) b is the greatest prime such that $b - 2$ is also prime, or $b = 0$ if such a number does not exist.

Classically, these descriptions are both perfectly valid as definitions, despite the qualitative differences between the two – the first one, we can prove its value ($a = 3$), while the second, we (currently) have no method to compute. Thus, this second definition is not constructively permissible.

Class. It seems odd that the validity of a definition, and by extension, the ontology of a natural number, should be contingent on human understanding – surely either infinitely many such pairs exist, or they do not. In the former case, we have $b = 0$, and in the latter, we have that b is the greatest such twin prime. In every possible case, b is well-defined; why should it matter whether we can compute this value or not?

What if, one day, it is proved that infinitely many twin primes exist, so we would have $b = 0$ from that point onwards. Then, was $b = 0$ or not before the proof was completed? [Menger, 1930]

Int. For your first point, this depends on what you mean by "exist". If "to exist" does not mean "to be constructed", then it must be purely metaphysical in meaning and thus outside the domain of mathematics. In the study of mental constructions, "to exist" can only mean "to be constructed", in which case this statement does not single out a natural number and thus fails to define anything at all.

For your second, we believe that b was (is?) *not* 0 before the proof was completed, in the sense that there is no computational value in saying so before then. This is perhaps not the same sense that you mean – that $b = 0$ in some absolute manner – but this is the only *mathematically meaningful* sense we can offer, devoid of metaphysical argumentation.

We should note that there is already some degree of idealisation present here. We think of 3, 500, $10^{10^{10}}$ as the same kind of object, despite our mental constructions of each one being very different. We can immediately grasp "three" as a collection of units, or as the short repetition of successorship, but mentally representing $10^{10^{10}}$ in the same way is absurd. Even the case of 500 is already stretching our capabilities.

Class. If what you care about is numerical value, then does the string $10^{10^{10}}$ define a natural number? There is no practical way to write down or compute such a value, so surely this, too, is not valid constructively?

Int. While there is no physical process by which we might write down the digits of this number, the crucial distinction between these two definitions is that we have a definite and unambiguous method of constructing $10^{10^{10}}$ in a finite number of steps, through its partial recursive definition. In contrast, we have no such guarantee for the supposed number b. The computational value in question is not one of practicalities, but possibilities.

This idea was first developed formally by A.S. Esenin-Vol'pin [Tro+11], whose programme questioned the view that there is (up to isomorphism) a unique sequence of natural numbers, since, in a strict realistic interpretation, the notion of indefinitely iterating the successor operation is unjustified. In doing so, he identified multiple universes of natural numbers, consisting of the "reasonably" small and "unreasonably" large numbers, separated according to the principles needed to generate them.

As an extreme form of constructivism, this approach was originally called *ultraintuitionism*, but is now more commonly referred as *ultrafinitism*. However, most constructivists view ultrafinitism as impractically restrictive:

> There are considerable obstacles to be overcome for a coherent and systematic development of ultra-finitism, and in our opinion no satisfactory development exists at present. [TD88]

However, ultrafinitism has since found modern applications in computability and complexity theory – in particular, it has strong correspondences with resource-bounded complexity classes: if intuitionism is the mathematics of the computable, then ultrafinitism is the mathematics of the *feasibly* computable. However, for the purposes of mathematics, we will not discuss ultrafinitism further.

Here is another example of a non-constructive argument [TD88]:

Lemma 3.1 (König). *Let G be connected locally finite graph. Then, G contains a ray. That is, an infinite simple path.*

Proof. We can "construct" the ray as follows. Select some vertex v_0. As G is connected, each of the infinitely-many vertices of G can be reached by a simple path that starts from v_0.

However, G is locally finite, so v_0 only has finitely many neighbours. By the pigeonhole principle, there is at least one neighbours through which infinitely-many of these paths route through. So connect the path through this neighbour and repeat. ■

Despite the construction apparently given, this proof is not constructively valid because, in general, we do not know how to decide whether a vertex has infinitely many paths through it or not.

These examples show that the natural constructive interpretations of "p or q" and "$\exists x\, P(x)$" as "we can *decide* between p and q", and "we can *construct* a particular x such that $P(x)$", respectively, are not compatible with all of the reasoning principles of classical logic.

For instance, reading "$\exists$" as explicit construction, we can already see that the classical equivalence of $\neg\forall$ and $\exists\neg$ cannot hold constructively, since knowing that something cannot hold everywhere does not provide a method of actually constructing the point at which it fails.

4 The Brouwer-Heyting-Kolmogorov Interpretation

Class. Sometimes, I think our disagreements could be resolved if we just agreed on what logical symbols *mean*. After all, logic is universal – is it not?

Int. Universal, perhaps, but not independent of interpretation. To you, each proposition is either true or false, regardless of whether we can determine which. But constructively, logic is not about truth in the abstract – it is about proof.

Class. And what, exactly, does that change? Surely, "A or B" means that either A is true, or B is true (or both, of course).

Int. Not quite. You treat disjunction as a statement about truth values: if $A \vee B$ is true, then you are happy that at least one of A or B is true, even if you have no means of deciding which is the case. Constructively, we require this additional justification: we may only assert $A \vee B$, if we have an explicit proof of A, or an explicit proof of B.

Class. So it comes back to the excluded middle then, I suppose?

Int. Not just that – it is a broader shift in what logical operators mean; consider implication. Classically, $A \to B$ is true simply whenever either A is false, or B is true; $A \to B \equiv \neg A \vee B$. But constructively, an implication is better viewed as a procedure that transform proofs of A into proofs of B; it asserts not just a relationship between truth values, but a way to transform evidence of one statement into evidence of another.

In classical logic, the inference rules and operations we use have been carefully designed to preserve *truth values* with respect to proof. In constructive logic, the inference rules instead preserve *justification* with respect to evidence and construction.

Classically, we can prove $p \vee q$ without necessarily proving either of p and q. We have already seen such an example earlier, when finding irrational a,b with a^b rational; we don't know which of $p : \left(a = b = \sqrt{2}\right)$ and $q : \left(a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}\right)$ is true (and in fact, we never gave a proof of either one), but are still able to prove that $p \vee q$ holds.

We should note that the original statement of this theorem is existential in nature; but again, we have classically proved that there exist a and b satisfying some condition, without actually producing any explicit concrete values for a and b.

Constructively, a disjunction $p \vee q$ is a constructive choice between p and q. To prove such a disjunction, we must explicitly prove (at least) one of p or q. Similarly, an existential quantifier $\exists x\, P(x)$ asserts the existence of a particular x such that $P(x)$ holds; constructively, the specific x that validates $P(x)$ must be given explicitly.

4.1 Because of this change in objective, the meanings of the logical connectives and quantifiers also change. The *proof-interpretation* or *Brouwer-Heyting-Kolmogorov interpretation* (BHK-interpretation) of constructive logic assigns the following meanings to the symbols:

- A proof of $A \wedge B$ consists of a proof of A and a proof of B.
- A proof of $A \vee B$ consists of a proof of A or a proof of B, along with information about which is the case.
- A proof of $A \rightarrow B$ consists of an algorithm that transforms a proof of A into a proof of B.
- A proof of $\forall x\, A(x)$ consists of an algorithm that transforms a proof that t is an element of the intended domain D into a proof of $A(t)$.
- A proof of $\exists x\, A(x)$ consists of an element $t \in D$ and a proof of $A(t)$.
- $\perp$ is not provable; a proof of $\neg A$ is a proof that A is not provable; or equivalently, an algorithm that derives a proof of $\perp$ from a supposed proof of A; that is, $A \rightarrow \perp$.

This interpretation predates the work of Church, Turing, and Kleene on formal computability, and as such, the usage of "algorithm" here – and hence the interpretation as a whole – is informal.* For instance, if we interpret "algorithm" in the sense of Markov's recursive functions, then we obtain a working interpretation much closer to that of CRM, while leaving the notion informal leans closer towards Bishop's pragmatic constructivism.

In any case, even on this informal level, we can show that a variety of logical principles are constructively (in)valid. For now, we will take algorithm to just mean any definable function, operating on proofs in an informal way that will hopefully become clear soon.

Example.

- The statement $p \rightarrow p$ is constructively valid for any p: we just take the identity function $\lambda x.x$ as our algorithm transforming a proof of p into a proof of p.
- The statement $p \rightarrow p \vee q$ is constructively valid for any p and q: such a proof is an algorithm that transforms a proof of p into a proof of $p \vee q$. A proof of $p \vee q$ is a proof of p, or a proof of q, tagged

*It should be noted that the development of a precise formal definition of what constitutes an "algorithm" did not have much impact on intuitionism, at least philosophically. This should not really be surprising, as most of these definitions describe algorithms by carving out a specific formal language in which they can be expressed, which is diametrically opposed to Brouwer's view of mathematics as the languageless mental activity of the ideal mathematician.

to indicated which is the case – i.e. a pair $\langle a,\ell\rangle$, where a proves p and $\ell = 0$, or a proves q and $\ell = 1$ – so if a proves p, then the function $\lambda a.\langle a,0\rangle$ provides the required transformation.

- Consider the statement $A \to (B \to A)$ for arbitrary propositions A and B. A proof of this statement is an algorithm that transforms a proof of A into an algorithm that transforms a proof of B into a proof of A. Under this reading, we can see that the proof of B is extraneous, so the second algorithm should just be the function that returns the proof of A given by the first. Explicitly; suppose a proves A. Then, the constant function $\lambda b.a$ is indeed an algorithm that transforms a proof of B into a proof of A, by simply returning a. So the whole proof is given by the function $\lambda a.(\lambda b.a)$ that takes a proof a of A and returns the constant function that returns a. $\triangle$

4.2 Note that, while the equivalence $\neg p \equiv p \to \bot$ is a classical *theorem*, constructively, this is the *definition* of negation. Thus, there is a certain ambiguity when we say statements such as "not p" or "we do not have p". This could be interpreted as the *mathematical* proposition $\neg p$, or $p \to \bot$. However, we could also mean the *factual* proposition "we have not yet decided p".

INT. Intuitionistically, every mathematical assertion is of the form "I have effected the construction A in my mind".

The *mathematical* negation would be "I have effected in a construction that deduces a contradiction from the construction of A", which, again, is a positive statement about a construction.

Conversely, the *factual* negation would be "I have not (yet) effected the the construction A in my mind"; this is not of the form of a mathematical assertion [Hey66].

To disambiguate between the two, we will use wording such as "p is *impossible*" or "p is *contradictory*" to mean that $p \to \bot$; and "we do not (yet) *know* that p" or "we cannot *assert* p" for the latter meaning.

4.3 Under the BHK interpretation, we can also prove that the *Principle of Explosion* $\bot \to P$ holds for any P; since $\bot$ has no proof, any algorithm vacuously transforms every proof of $\bot$ into a proof of P, as required.

Conversely, we can see that LEM is constructively problematic. If we were to accept LEM as a general principle, then we would have a universal proof strategy of obtaining, for any proposition P, either a proof of P, or a proof of $\neg P$. But, if such a method existed, we could also decide every currently-unresolved conjecture, which is clearly not the case. Thus, it cannot be the case that LEM holds in general.

4.4 This is an example of what Brouwer calls a *weak counterexample*; we have not formally *refuted* LEM, in that we have not derived a contradiction from universally assuming LEM, but have instead shown that universally assuming LEM implies that we should have access to information that we in fact do not have (i.e. a decision procedure for every proposition p).

In fact, we cannot refute even a single instance of LEM. That is, it is impossible to find a proposition p such that $(p \vee \neg p) \to \bot$, or $\neg(p \vee \neg p)$. This is because $\neg\neg(p \vee \neg p)$ holds constructively:

Suppose a proves $\neg(p \vee \neg p)$. If b proves $p \vee \neg p$, then $a(b)$ proves $\bot$. If c proves p, then $\langle c,0\rangle$ proves $p \vee \neg p$, and similarly, if d proves $\neg p$, then $\langle d,1\rangle$ proves $p \vee \neg p$. Then, the map $f := \lambda c.a\langle c,0\rangle$ witnesses $p \to \bot$, or equivalently, $\neg p$; similarly, the map $g := \lambda d.a\langle d,1\rangle$ that witnesses $\neg p \to \bot$. But then, $g(f) = \lambda c.a\big\langle c\langle a,0\rangle,1\big\rangle$ constructs $\bot$, so the map $\lambda c.g(f) = \lambda a.\lambda c.a\big\langle c\langle a,0\rangle,1\big\rangle$ witnesses $\neg\neg(p \vee \neg p)$.

4.5 The BHK interpretation is not *bivalent*, in the sense that it is constructively consistent to have a statement that is neither true nor false (read: provable nor refutable). But as shown above, we also cannot prove the existence of such statements. On the other hand, the *Law of Noncontradiction* still holds constructively:

Theorem (Noncontradiction)**.** *There is no proposition p such that p and $\neg p$.*

$$\vdash \neg(p \wedge \neg p)$$

$\neg(p \wedge \neg p)$ expands to $\big(p \wedge (p \to \bot)\big) \to \bot$. Suppose a proves p and b proves $p \to \bot$, so $\langle a,b\rangle$ proves $p \wedge (p \to \bot)$. Then, the application mapping $\lambda\langle a,b\rangle.b(a)$ transforms this proof into a proof of $\bot$, so the law of non-contradiction holds constructively.

As a more-standard logical proof, this would be rendered as:

Proof. Suppose there is a proposition p such that both p and $\neg p$ hold. Since $\neg p \equiv p \to \bot$, we have both p and $p \to \bot$, so by modus ponens, we have $\bot$. Therefore, there is no such p. ■

Incidentally, this is *not* a proof by contradiction, but rather, a proof of *negation*.

4.6 Proof and Refutation by Contradiction. *Proof by contradiction*, or *reduction ad absurdium* is a form of proof that establishes the validity of a proposition p by showing that assuming $\neg p$ leads to a contradiction:

$$\frac{\begin{array}{c}\neg p \\ \vdots \\ \bot\end{array}}{p}$$

Unfortunately, there is another similar logical pattern that is often conflated with proof by contradiction:

$$\frac{\begin{array}{c}p \\ \vdots \\ \bot\end{array}}{\neg p}$$

This second pattern is *refutation by contradiction*, or *proof of negation* [Bau17].

Both patterns assume some statement believed to be false and derive a contradiction from there, before concluding the opposite of the original statement. They are superficially similar in form, but one form adds a negation to the conclusion, while the other removes one. Classically, both are both valid so the distinction is immaterial, but constructively, the situation is more nuanced.

Because $\neg p \equiv p \to \bot$, proof of negation is constructively valid as an instance of implication introduction $((p \vdash q) \vdash p \to q)$. In contrast, the proof by contradiction proceeds by proving that assuming $\neg p$ yields a contradiction, i.e. that $\neg\neg p$. But then, proof by contradiction derives p from this.

That is, proof by contradiction requires the logical principle that $\neg\neg p \to p$ for any p. This is, for obvious reasons, called *double negation elimination* (DNE):

Axiom (Double Negation Elimination)**.** *For every proposition p, $\neg\neg p$ implies p.*

$$\vdash \neg\neg p \to p$$

A proposition p for which $\neg\neg p \to p$ holds is called ($\neg\neg$-)*stable*. Unfortunately:

Theorem 4.1. *Double negation elimination implies the law of the excluded middle.*

Proof. Given any proposition p, suppose $\neg(p \vee \neg p)$. If we had p, then we would also have $p \vee \neg p$, which together with $\neg(p \vee \neg p)$ yields $\bot$. So $\neg p$. But again, this gives $p \vee \neg p$, so $\bot$. So $\neg\neg(p \vee \neg p)$. By double negation elimination, we have $p \vee \neg p$. ■

Evidently, proof by contradiction is not constructively valid; this proof schema only works when the original statement is *un-negated*, or *affirmative*. Because of this, there is an inherent asymmetry between positive and negative statements. This point is perhaps more subtle than it seems. Compare the following situations:

- We have a complicated set of equations that has a solution in the real numbers, and we wish to show the solution is *irrational*. To do so, we assume the solution is rational and derive a contradiction. Therefore, the solution is irrational.
- We have a complicated set of equations that has a solution in the real numbers, and we wish to show the solution is *rational*. To do so, we assume the solution is irrational and derive a contradiction. Therefore, the solution is rational.

Despite the strategy being identical in both cases, the former is constructively valid, while the latter is *not*. The problem is that irrationality is a *negative definition*; to be *irrational* is (defined) to be *not rational*.

Let $R(x)$ be the proposition that x is rational. The first proof shows $R(x) \to \bot$, so we have $\neg R(x)$; that is, x is irrational, by definition of irrational. Conversely, the second proof shows $\neg R(x) \to \bot$, so we have $\neg\neg R(x)$; the proof only establishes that x is *not not* rational, and we cannot constructively eliminate this double negation.

Class. I know you reject the excluded middle, but what else could x possibly be be!

Int. The following example will be illustrative [Hey66, adapted]. I write the decimal fraction $\varrho = 0.333\ldots$, and I keep writing down 3s until the pth digit, where p is the least prime such that $p-2$ is also prime. So, if such a p exists, we would have $\varrho = \frac{10^p-1}{3\cdot 10^p}$.

Now suppose that ϱ cannot be rational. Then, $\varrho = \frac{10^p-1}{3\cdot 10^p}$ is impossible, and such a prime does not exist. But then, $\varrho = \frac{1}{3}$. So, it is impossible that ϱ cannot be rational. However, from this, we would not be justified in asserting that ϱ is rational, for this would be the claim that we have a method of constructing integers p,q such that $\varrho = p/q$. Such a construction would require that we either find the least twin prime, or prove that no such prime exists.

Class. And you reject the claim that that ϱ is equal to one of 1/3, 0.3, 0.33, 0.333, etc. even though we do not know which one?

Int. Yes: constructively, the value of such a proof is in the knowing. Our position would be better expressed by saying that ϱ could not be *different* from each of these numbers.

A similar problem arises when talking about *non-empty sets*. A set X is *inhabited* if it has an element:

$$\exists x(x \in X)$$

Or rather, X is inhabited if we have *explicitly constructed* an element $x \in X$. In contrast, a set X is non-empty if:

$$\neg\neg\exists x(x \in X)$$

Constructively, this does not say that X actually has an element, as showing that it is impossible for X to be empty does not generally provide a method to explicitly construct an element of X; in other words, inhabitedness is not stable.

Fortunately, many famous theorems usually thought of as being proofs by contradiction are really proofs by negation. For instance, Euclid's theorem (there are infinitely many primes), the irrationality of $\sqrt{2}$ (and more generally, other proofs by *infinite descent*), Cantor's theorem, the non-existence of a Halting oracle, and Russell's paradox are all proofs by negation, and are hence constructively valid.

4.1 Logic without LEM

As this is such a common misconception, we reiterate this point: constructively, we do not accept LEM, but this does *not* mean that we accept the *negation* of LEM. Constructivism only entails that we are ambivalent about the excluded middle; that is, we do not say that every instance of $p \vee \neg p$ automatically fails, only that it does not hold *a priori*. However, this seemingly innocuous change radically modifies our interpretation of "truth". For instance, simply by inspecting a truth table, we can see that LEM is classically a tautology:

p	$\neg p$	$p \vee \neg p$
0	1	1
1	0	1

Classically, a proposition is true if it is true in all possible models, regardless of our ability to construct a proof for it. Thus, classically, truth is a metalogical notion. Truth tables work under this assumption, taking a truth-functional approach to logic; verifying a tautology with a truth table is then a proof by *model checking*.

Conversely, since the constructive interpretation of logical symbols conflates truth with provability, they are no longer truth-functional. So, if we reject LEM, then simple tables no longer adequately capture the nuances of constructive truth.

4.7 Constructively, we also cannot allow the use of the Axiom of Choice, as it implies LEM. Many forms of constructivism do, however, accept weaker versions of Choice, such as *countable choice* AC_ω, where choice holds for families of inhabited sets indexed by the naturals; or *dependent choice* DC, in which, roughly speaking, we may choose depending on previous choices. In practice, these weaker variants are more than sufficient for general mathematics (outside of set theory).

Before we continue, we also address another common misconception. Suppose we have proved that there exists an some value such that φ holds. It is then common to say something along the lines of "choose x satisfying $\varphi(x)$..." to obtain such an instance of x. This is *not* an application of Choice, but merely the elimination of an existential quantifier ($\exists x\, \varphi(x) \vdash \varphi(t)$, where t is a fresh variable).

Now, note that Choice is formulated constructively (i.e. affirmatively) as:

Axiom (Choice)**.** *For every set X of* ***inhabited*** *sets, there is a choice function $X \to \bigcup X$.*

The strategy of our proof will be to associate the given proposition φ to a set X whose elements may coincide depending on the truth value of φ, and thus a choice function on X would decide φ.

Theorem 4.2 (Diaconescu)**.** *Axiom of Choice implies Law of the Excluded Middle.*[*]

Proof. For any proposition φ, define the sets

$$\begin{aligned} A &= \big\{x \in \{0,1\} : \varphi \vee x = 0\big\} \\ B &= \big\{x \in \{0,1\} : \varphi \vee x = 1\big\} \end{aligned}$$

Note that if P is true, then $A = B = \{0,1\}$, so $\varphi \to A = B$, and in particular, for any function g, we have $\varphi \to g(A) = g(B)$. Hence,

$$g(A) \neq g(B) \to \neg\varphi$$

Now, construct the pairing $X = \{A,B\}$. Both A and B are inhabited, witnessed by $0 \in A$ and $1 \in B$, so by Choice, there exists a choice function f on X. Thus,

$$\begin{aligned} f(A) \in A \wedge f(B) \in B \quad &\longleftrightarrow \quad \big(\varphi \vee f(A) = 0\big) \wedge \big(\varphi \vee f(B) = 1\big) \\ &\longleftrightarrow \quad \varphi \vee \big(f(A) = 0 \wedge f(B) = 1\big) \\ &\longrightarrow \quad \varphi \vee \big(f(A) \neq f(B)\big) \\ &\longrightarrow \quad \varphi \vee \neg\varphi \end{aligned}$$

■

Note that we only used Choice on a set of (at most) 2 elements, so more sharply, we have proved that Choice for all inhabited sets $X \leq^* 2$ implies LEM. Further still, classically, Choice is valid in all finite sets, and all the sets in the previous proof are all classically finite (with cardinality 1 or 2), so in fact, we have:

Corollary 4.2.1. Choice *for all inhabited sets $X \leq^* 2$ is equivalent to* LEM.

[*] In the presence of Separation, Pairing, and Extensionality.

II

Collapse

Class. It seems to me that the allure of classical mathematics is in its universality; we have such a rich collection of theorems and results, many of which have stood the test of time. I can't help but feel uneasy about the consequences of rejecting classical principles.

Int. Constructivism certainly doesn't allow us to keep everything from the classical tradition intact. However, we should remember that constructivism (purely in the sense of rejecting LEM) is strictly a generalisation of classical logic: a proof avoiding LEM is still a classical proof. It should be expected that a more general theory will prove fewer things.

That said, what I believe you are alluding to is that if we, as some constructivists do, accept some additional axioms à la Brouwer or Markov, we can prove some rather... strange results. While these new results are mathematically consistent, they don't always align with the classical intuition of what we expect from a theory.

Class. Yes, precisely: it seems to me that constructivism does more than just weaken some classical theorems – it fundamentally alters the mathematical landscape. Many familiar results become unprovable, while others transform into something unrecognizable. In these systems, what we're left with doesn't quite feel like the same mathematics at all.

Int. But this doesn't mean we lose mathematics altogether; it simply means that we must be more precise about what we claim. For instance, the failure of the full Hahn-Banach theorem constructively isn't just an unfortunate loss, but also an illustration that certain analytic properties require non-constructive reasoning.

Class. That's a generous interpretation. You could just as well say that constructivism fails to capture mathematical reality as we understand it.

* * *

5 Dissent

One can hardly speak about constructivism without also mentioning Hilbert's now-famous riposte:

> Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists. To prohibit existence statements and the principle of excluded middle is tantamount to relinquishing the science of mathematics altogether. For compared with the immense expanse of modern mathematics, what would the wretched remnants mean, the few isolated results, incomplete and unrelated, that the intuitionists have obtained without the use of logical ϵ-axiom? [HB34]

Hilbert's critique came at a time when constructivism had not yet demonstrated its potential. Rather, the early results of intuitionism were far from impressive. As Hilbert had accurately assessed, proponents of constructivism – in particular, Brouwer's programme – had indeed only produced a "few isolated results", and had, even further, devoted considerable effort into the fragmentation of classically-equivalent notions into numerous obscure and constructively-inequivalent definitions, and to developing theories revolving around unclear esoteric concepts like choice sequences. Even worse, some varieties of constructivism had started to develop theorems that not only directly contradicted classical results, but in many cases defied even common sense.

For instance, in Brouwer's framework, trichotomy on the real numbers fails in general, and not every non-empty bounded-above subset of the real numbers has a supremum. Followers of CRM had then muddied the waters further, proving that there exists a continuous real-valued map on [0,1] that is not uniformly continuous [BV07], and more dramatically, there exists an unbounded continuous real-valued map on [0,1]. Any single one of these results would have shattered our understanding of real analysis.

It was clear that constructivism had no purpose in mathematics, except, perhaps, as an idle curiosity.

Before we move on, we exhibit some of these constructive oddities.

6 All Functions are Continuous?

For instance:

Theorem 6.1. *Every function $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

This is exclusively a Brouwerian result, so before we may prove this result, we must first introduce some of his theory.

7 Choice Sequences

Int. We begin with the basic notion of the natural numbers. These are typically seen as the foundation of arithmetic and analysis, starting with 0 – or 1, depending on your taste (though this is not mine) – and building up through the process of counting.

In intuitionism, any mathematical object is accepted if it can be explicitly constructed, and the construction of the natural numbers is thus straightforward: we start with an initial number 0 and generate successive numbers by a clearly defined process (e.g. the successor operation). We also acknowledge the possibility of an indefinite repetition of this process. From this, we can determine the natural numbers.

Note that in saying that the natural numbers are *determined*, we do not mean that it forms an entire completed infinity, but only that there is never any choice in how to extend any given initial segment of the natural numbers, and that, given any mathematical object, we can always decide whether or not it can be reached by repeated application of the successor operation to 0, and hence whether or not it belongs to the natural numbers.

* * *

Apart from the use of LEM, Brouwer's main objection to classical logic was "its introduction and description of the continuum." [Van90]. According to Kant, the notion of the continuum appearing in the real plane, the real line, etc., is a primitive notion, formed from our intuitive understanding of *space*, and did not require axiomatisation in terms of more elementary concepts [MV20]. However, the development of non-Euclidean geometries had shown that our geometric intuitions about space were not unique.

The convention since then was to develop the continuum in Cantor's set theory. The continuum was then a collection of distinct objects – the real numbers – and was defined via Cauchy sequences, Dedekind cuts, sequences of nested rational intervals, etc. In any case, these methods of construction treated the real numbers as an actual, complete infinity. To Brouwer, this was not constructively acceptable.

The importance of infinite sequences in the arithmetic description of the continuum was clear. Borel had remarked that it was necessary to adopt accept sequences of arbitrary choices to describe the continuum, and, as a constructivist, he was uncomfortable with accepting such sequences, but ultimately did not completely reject the idea [MV20].

However, Brouwer had found a way to describe these infinite sequences constructively by dividing them up temporally into two parts: at any particular instance, a sequence, as Brouwer envisioned it, consists of a finite constructed part, which provides the usable content of the construction; and an indefinite not-yet determined part, which provides the flexibility that makes it possible to capture the rest of the continuum.

7.1 A *choice sequence* α is an unfinished and indefinitely ongoing process of selecting values one by one, by the ideal mathematician, [Ves79; Tro+96; TD14; BR87]. At any particular step, the ideal mathematician has determined only finitely many values, and for the next step, they are constrained by a collection of restrictions on future choices that define the choice sequence. This collection of restrictions may be entirely empty, or may fully restrict the sequence to only a single possibility. A choice sequence defined by the former is called *free* or *lawless*, while a deterministic choice sequence is called *lawlike*. For instance, in the above, the natural numbers are intuitionistically conceived as a kind of lawlike choice sequence.

It is critical to understand that choice sequences in general are to be thought of as an *ongoing* process, forever incomplete; if any reasoning about a choice sequence is to be constructively valid, it must only depend on some initial segment of the sequence.

As an aside, this notion of choice sequence is very similar to how real numbers are treated in computability theory, since an algorithm may only access a real number to finite precision at any instant, and in fact, an identical result about real valued functions holds in modern computable analysis [Wei95; Bra16].

7.1 Spreads

Unlike in classical mathematics, where a set of real numbers is understood as a completed totality, the constructive approach requires that any collection of choice sequences be defined in terms of the finite data available at any given stage. To formalize this, Brouwer introduced the notion of a *spread*. Rather than being an arbitrary collection of sequences, a spread is constructed by a rule that determines how sequences can be extended at each finite stage. This ensures that our mathematical objects remain finitely presented and constructively meaningful.

Let us fix some notation first. We denote the concatenation of two sequences $a = \langle a_0, \ldots, a_n \rangle$ and $b = \langle b_0, \ldots, b_m \rangle$ by $a * b = \langle a_0, \ldots, a_n, b_0, \ldots, b_m \rangle$. If $b = \langle k \rangle$ is a singleton list, then we may also write $a \frown k$ for $a * \langle k \rangle$. We denote by $\bar{\alpha}_n \coloneqq \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ the initial n-segment of a sequence α.

Given finite sequences a and b, if there exists a number k such that $a = b \frown k$, then we say that b is an *immediate descendant* of a, or that a is an *immediate ascendant* of b. More generally, if there is a sequence c such that $a = b * c$, then b is a *descendant* of a, or that a is an *ascendant* of b.

Given a finite sequence a and any sequence α, if there exists a natural number n such that $\bar{\alpha}_n = a$, then we say that α *extends* a, denoted by $\alpha \sqsupseteq a$.

7.2 A *spread* M consists of a *spread law* Λ_M and a *complementary law* Γ_M, [Van04; Coq04; Dum00].

A *spread law* is a function Λ that divides the space of finite sequences of natural numbers into the *admissible* and *inadmissible* as follows:

- Λ must be decidable. That is, we should be able to determine in finite time whether any given sequence $\langle a_0, \ldots, a_n \rangle$ is admissible or not.
- If a sequence a is admissible, then every initial segment of a must be admissible. Note that this means that the empty sequence $\langle\rangle$ is admissible for every spread law.
- If $\langle a_0, \ldots, a_n \rangle$ is admissible, then Λ decides for every natural number k whether $\langle a_0, \ldots, a_n, k \rangle$ is admissible or not. Moreover, there must exist at least one k for which $\langle a_0, \ldots, a_n, k \rangle$ is admissible.

We can interpret a spread as an infinite rooted directed tree labelled by finite sequences of natural numbers, with edges indicating the initial segment relation; the root of the tree labelled by the empty sequence, and every node having at least one child. This tree, necessarily unique up to isomorphism, is called the *underlying tree* of the spread, though we will choose to conflate the two notions, as is common in the literature. A choice sequence is then precisely a ray (i.e. unbounded path) in this tree.

A topology can be naturally defined on the underlying tree of every spread [MV20]: we consider the choice sequences as points, and take as a basis for the topology the sets

$$V_a = \{\beta \in M : \beta \sqsupseteq a\}$$

where a is a finite sequence. That is, two sequences are close together under this topology if they agree on some initial segment. This topology captures the intuitionistic idea of functions as mappings that operate on finite data and produce progressively refined results rather than acting on completed infinite sequences.

Example. [MV20]

1. Let Λ_M admit every sequence. Then, the spread M contains all natural choice sequences. This spread is called the *universal spread*, and the underlying tree T_U is called the *universal tree*. Equipped with the initial segment topology, this becomes the familiar Baire space ω^ω; this can be taken as the constructive definition of the Baire space.

2. Let Λ_M be defined as: given an admissible sequence a, $a \frown 0$ and $a \frown 1$ are admissible. Then, M contains all binary sequences, and the underlying tree is denoted by T_{01}. Again, with the initial segment topology, this constructively defines the Cantor space 2^ω. △

7.3 The *complementary law* Γ_M of a spread M assigns previously-constructed mathematical objects to finite Λ_M-admissible sequences [TD88]. This allows us to construct choice sequences of arbitrary objects. If a complementary law is not explicitly given, we assume that the law is the identity function, and the spread is called a *naked spread*.

A sequence $\langle a_n \rangle$ with the property that the initial segment $\langle a_0, \ldots, a_k \rangle$ is Λ_M-admissible for every k is called a Λ_M-*admissible* sequence. Then, the sequence of objects assigned by Γ_M to the initial segments of an admissible sequence $\langle a_n \rangle$ is called an *element* of the spread M. That is, the sequence $\langle \Gamma_M(\bar{a}_0), \Gamma_M(\bar{a}_1), \Gamma_M(\bar{a}_2), \ldots \rangle$, where the $\bar{a}_n = \langle a_1, a_2, \ldots, a_n \rangle$ are initial segments of $\langle a_n \rangle$.

We are now in a position to construct the real numbers.

7.2 Real Number Generators

Class. Let us accept that you have access to the natural numbers. From your description, though you do not accept them as a singular entity, you have a version of membership that suffices for our purposes. Do you also, then, accept Peano's axioms?

Int. Yes, though intuitionistically, they are not axioms, but theorems. Since the natural numbers arise as an explicit mental construction, they come equipped with some basic properties that include the Peano axioms. Let us abbreviate "natural number" by N, and the successor of n by n^+.

The first two properties of Peano (0 is an N, and if n is an N, then n^+ is an N) hold immediately from our construction; the next two (n^+ is not 0, and if x and y are N and $x^+ = y^+$, then $x = y$) can be deduced from the determinism of our construction. As for the so-called axiom of induction, suppose that P is a predicate of natural numbers such that $P(0)$ holds, and $P(n)$ implies that $P(n^+)$. Now, let m be any N. Running over the construction of m as the iteration of successorship to 0, we see that P holds at every step of the construction, and hence $P(m)$ holds.

Inequalities are also decidable in the same way; if m, distinct from n, appears in the construction of n, then we say $n > m$, and otherwise $n < m$. Now, $(m \neq n) \wedge (n < m) \to (n > m)$ is a theorem; since if m does not occur during the construction of n, then we may proceed until we reach m, in which case, we have found n during the construction of m. The usual recursive definitions of basic arithmetic operations can be constructed in the same way.

Class. Then, apart from some philosophical considerations, we seem to at least agree on the notion of a natural number. From here, I presume you similarly accept the construction of ordered pairs, and hence the integers, then rationals as unproblematic?

Int. Yes. In doing so, (in)equality on the integers and rationals are also decidable, and we construct the recursive arithmetic operations as normal. It is at the next step – that of the real numbers – where we diverge.

* * *

As is common in constructivism, various classically equivalent constructions of the real numbers diverge constructively. For the purposes of proving the continuity theorem, we will use the *Cauchy construction*. We abbreviate choice sequence to just sequences in this section. We closely follow [BV07; Hey66; TD88] in the following definitions, though any text on computable/constructive analysis would agree with these.

7.4 A sequence $\langle a_n \rangle$ of rational numbers is called a *Cauchy sequence* if for every natural number k there exists a natural number $n = n(k)$ such that

$$|a_{n+p} - a_n| < \frac{1}{k}$$

for every natural number p. As per usual, the existence is to be interpreted constructively. That is, $n(k)$ must be effectively computable.* For this reason, we will often say "*we can find* n such that..." to emphasise this point.

Example. The sequence $a \equiv \langle 2^{-n} \rangle$ is a Cauchy sequence. Given a natural number k, we have $n = \lceil \log_2 k \rceil$; this is computable as we can find this value using only elementary arithmetic operations (multiplication by 2, comparison, and successor) and bounded minimisation. △

Example. Define the sequence $b \equiv \langle b_n \rangle$ by:

$$b_n = \begin{cases} 1 & n \text{ is the least prime such that } p-2 \text{ is prime} \\ 2^{-n} & \text{otherwise} \end{cases}$$

b differs from a in at most one term, so b is classically a Cauchy sequence, but as long as the twin prime conjecture is unresolved, the n corresponding to $k = 2$ is not computable. Thus, b is not a Cauchy sequence by our definition. △

7.5 A *real number generator* (RNG) is a Cauchy sequence of rationals.

Two RNGs $a \equiv \langle a_n \rangle$ and $b \equiv \langle b_n \rangle$ are *identical* if $a_n = b_n$ for every n, and we denote this relation by $a \equiv b$. This is too strong to be of much use outside of definitions, so:

Two RNGs $a \equiv \langle a_n \rangle$ and $b \equiv \langle b_n \rangle$ *coincide* if for every k, we can find $n = n(k)$ such that

$$|a_{n+p} - b_{n+p}| < \frac{1}{k}$$

for all p, and we denote this relation by $a = b$. If $\neg(a = b)$, we write $a \neq b$.

From now on, if we refer to an RNG by a single letter, a, then we implicitly mean that $a \equiv \langle a_n \rangle$, so that a_n is the nth component of a.

Lemma 7.1. *Coincidence is an equivalence relation.*

Proof. Clearly, coincidence is reflexive since $|a_{n+p} - a_{n+p}| = 0 < 1/k$. Similarly, the absolute value function is symmetric, so coincidence is too.

Now, suppose $a = b$ and $b = c$, so for every k, we can find $n_1' = n_1(2k)$ and $n_2' = n_2(2k)$ such that $|a_{n_1'+p} - b_{n_1'+p}| < 1/2k$ and $|b_{n_2'+p} - c_{n_2'+p}| < 1/2k$. Then, if $n = \max\big(n_1(2k), n_2(2k)\big)$, we have by the triangle inequality $|a_{n+p} - c_{n+p}| < \frac{1}{k}$ for all p. ■

Note that, given any RNG a, there exists an RNG b such that $a = b$ and b converges arbitrarily fast. For instance, if we wish for b to satisfy $|b_{n+p} - b_n| < 1/n$ for every n and p, then we can define b by $b_k = a_{n(k)}$.

Theorem 7.2. *If $a \neq b$ is impossible, then $a = b$.*

CLASS. Is this not obvious?

INT. Classically perhaps, but remember; constructively, $a \neq b$ means $\neg(a = b)$, so the theorem is an double negation elimination instance which is not *a priori* admissible.

Proof. Fix k, and by taking maxima as in the previous proof, construct $n = n(k)$ such that both $|a_{n+p} - a_n| < 1/4k$ and $|b_{n+p} - b_n| < 1/4k$ for all p.

Now, suppose $|a_n - b_n| \geq 1/k$. Then, by the triangle inequality we have $|a_{n+p} - b_{n+p}| > \frac{1}{2k}$ for all p, so $a \neq b$, which is impossible. Thus, $|a_n - b_n| < 1/k$. But then, $|a_{n+p} - b_{n+p}| < 2/k$ for all p, so $a = b$. ■

*Computable, not in the formal recursion-theoretic sense, but constructible in finite time to the ideal mathematician. Still, if something is recursively computable, then it is clearly also constructible in this sense.

As is usual in constructive mathematics, we will prefer a *positive* notion of inequality:

7.6 For two RNGs a and b, a *lies apart from* b if we can find n and k such that

$$|a_{n+p} - b_{n+p}| > \frac{1}{k}$$

for every p, and we denote this relation by $a \# b$. Unlike the negative statement $a \neq b$, the statement $a \# b$ means we can explicitly demonstrate how separated a and b are; in general, $a \# b$ entails $a \neq b$, but not the converse. Clearly, apartness is also commutative and is also compatible with coincidence.

Theorem 7.3. *If $a \# b$ is impossible, then $a = b$.*

Proof. The previous proof of the implication for $a \neq b$ is also valid for this stronger theorem. ∎

7.7 We can lift operations on the rationals to RNGs by applying them pointwise:

- $a + b :\equiv \langle a_n + b_n \rangle$;
- $ab :\equiv \langle a_n \cdot b_n \rangle$;
- $-a :\equiv \langle -a_n \rangle$;
- If $a \# 0$, then $a^{-1} :\equiv \langle c_n \rangle$ where $c_n = a^{-1}$ if $a \neq 0$, and 0 otherwise.

By standard results in elementary analysis, these operations preserve the constructive Cauchy property and are compatible with coincidence.

7.8 We also define the ordering on RNGs as follows. For two RNGs a and b, a is *strictly less than* b if we can find n and k such that

$$b_{n+p} - a_{n+p} > \frac{1}{k}$$

for every p, and we denote this relation, as usual, by $a < b$. We also write $a > b$ for $a < b$. It is clear from the definition of this ordering that $a < b$ entails $a \# b$. We also have a converse:

Theorem 7.4. *If $a \# b$, then $a < b$ or $b < a$.*

Proof. Sinc $a \# b$, we can find n and k such that $|a_{n+p} - b_{n+p}| > 1/k$ for every p. By taking maxima, find $m > n$ such that $|a_m - a_{m+p}| < 1/4k$ and $|b_m - b_{m+p}| < 1/4k$, so by the triangle inequality and expanding the absolute value, we have either $a_m - a_b > 1/k$, or $b_m - a_m > 1/k$. In the first case, $a_{m+p} - b_{m+p} > 1/2k$ for every p, so $b < a$, and in the second, we similarly obtain $a < b$. ∎

We write $a \not< b$ or $a \geq b$ if $a < b$ is impossible, and similarly $a \not> b$ or $a \leq b$ if $a > b$ is impossible.

Note that $a \leq b$ is not the same as $(a < b \vee a = b)$, however. For instance, we earlier constructed a rational ϱ by terminating the decimal fraction 0.333... in a way that is dependent on the twin prime conjecture. Clearly $\varrho \not> 1/3$, but until twin primes is resolved, we cannot determine if $\varrho < 1/3$ or $\varrho = 1/3$.

7.3 Species

Intuitionistically, there are two ways to constructively define a collection of objects:

- By providing a common method of construction of its members;
- By providing a characteristic property or predicate that its members satisfy.

Spreads are a realisation of the former method. For the latter, we introduce the notion of a *species*. Previously, whenever we have been referring to a "collection" or a "set" of objects, we have already been implicitly using this idea.

7.9 A *species* is any property which mathematical objects can be supposed to possess [Hey66; MV20]. After a species S has been defined, any previously-defined mathematical object which satisfies S is said to be a *member* of S.

Example. Every spread M corresponds to a species: the property of being equal to an element of a spread M defines the corresponding *spread-species* S_M. △

Note that species are inherently predicative in their construction: they only collect together preexisting objects, so those objects must be definable without referring to some species that contains them.

7.10 We freely use the language of classical set theory ($\in$, $\cup$, $\cap$, $\subseteq$, etc.) to describe species, interpreting the symbols in the obvious ways, and with negations interpreted, as usual, as deriving contradictions.

As we would expect, this constructive reading introduces slight differences from classical set theory. For instance, if $T \subseteq S$, then the species $T \cup (S \setminus T)$ is not necessarily equal to S, while their equality is a theorem for sets. If S is, say, the species of real numbers, and T of rational numbers, then $T \cup (S \setminus T)$ is the species of real numbers whose (ir)rationality is known, which is not equal to S.

7.4 Real Numbers

A *real number* is a species of RNGs coinciding with a given RNG. If x is a real number, and ξ is one of its members, then we say that ξ *coincides* with x. The property of being a real number then forms another species called the (one-dimensional) *continuum* (of real numbers).

7.11 If a and b are real numbers, the *closed interval* $[a,b]$ is the species of real numbers x such that $x > a$ and $x > b$ are impossible, and $x < a$ and $x < b$ are impossible:

$$\neg(x > a \wedge x > b) \wedge \neg(x < a \wedge x < b)$$

This definition is complicated because we may not know which of a or b is larger, so both cases must be handled. Note that with this definition, $[a,b] = [b,a]$.

7.5 Canonical Number Generators

It is convenient to pick a representative for each real number of a particularly simple form.

Let a real number x be given by the RNG r (i.e. x is the species of RNGs coincident with r). As r is an RNG, for any n, we can find k such that x is well-approximated by r_k to within 2^{-n-3}:

$$|x - r_k| < 2^{-n-3}$$

From there, we can approximate r_k by a multiple x_n of 2^{-n} to within 2^{-n-1}, by defining x_n to be the nearest integer to $r_k 2^n$:

$$|r_k - x_n 2^{-n}| = \left| r_k - \frac{\operatorname{round}(r_k 2^n)}{2^n} \right| \leq 2^{-n-1}$$

Then by the triangle inequality,

$$\begin{aligned} |x - x_n 2^{-n}| &\leq |x - r_k| + |r_k - x_n 2^{-n}| \\ &< 2^{-n-3} + 2^{-n-1} \\ &= \frac{5}{8} 2^{-n} \end{aligned}$$

Repeating this for each n, we construct a sequence $\langle x_n 2^{-n} \rangle$ that, by construction, coincides with x, and which has the property that

$$|x_n 2^{-n} - x_{n+1} 2^{-n-1}| < \frac{5}{8} 2^{-n} + \frac{5}{16} 2^{-n} = \frac{15}{16} 2^{-n}$$

Since the x_n are integers, this implies

$$|x_n 2^{-n} - x_{n+1} 2^{-n-1}| \leq 2^{-n-1}$$

7.12 A *canonical number generator* (CNG) is an RNG of the form $\langle x_n 2^{-n} \rangle$ where each x_n is an integer, satisfying

$$|x_n 2^{-n} - x_{n+1} 2^{-n-1}| \leq 2^{-n-1}$$

The above then proves:

Theorem 7.5. *Every real number x coincides with a canonical number generator $\langle x_n 2^{-n} \rangle$ with*

$$|x - x_n 2^{-n}| < \frac{5}{8} 2^{-n}$$

It should be clear from our construction that the coefficient of $\frac{5}{8}$ can be improved to $\frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$ by tightening the bounds on the original approximations.

8 Weak Continuity for Numbers

We have seen that species are generally well behaved (ignoring constructive peculiarities common to all intuitionistic constructions), and in particular, the notion of a *subspecies* is unproblematic. However, given a *spread*, how can we extract subcollections of choice sequences?

More generally, what constraints on properties of choice sequences yield well-defined spreads? This is theoretically problematic because choice sequences are not fully given static objects, but ongoing processes whose values are, at any particular instance, incomplete. Consequently, if we attempt to classify or select choice sequences based on properties that depend on their entire structure, we encounter an inherent obstacle, in that such properties cannot be decided at any stage of the process.

For instance, suppose we have two choice sequences α and β, and wish to determine if α is greater than β pointwise. This cannot be determined, because at any given instance, we only have access to finite initial segments of α and β. Similarly, if we attempt to define a collection of sequences based on such a global property, we cannot verify whether a given sequence belongs to this collection.

8.1 In general, if a claim about a choice sequence as a whole is to be constructively meaningful, then it must be provable using only an initial segment of the sequence. Similarly, any statement about a collection of choice sequences must be provable using only finite sequences.

Suppose we have a functional $\Phi : \omega^\omega \to \mathbb{N}$ that is defined for all choice sequences. Because Φ is total and we cannot assume anything more than an initial segment of any choice sequence α, for every sequence α, there must exist some natural number m for which only the initial segment $\bar{\alpha}_m$ determines the value of Φ:

$$\forall \alpha \exists m \forall \beta \big(\bar{\beta}_m = \bar{\alpha}_m \to \Phi(\beta) = \Phi(\alpha) \big)$$

Or equivalently, since $\bar{\beta}_m = \bar{\alpha}_m \leftrightarrow \beta \sqsupseteq \bar{\alpha}_m$,

$$\forall \alpha \exists m \forall \beta \sqsupseteq \bar{\alpha}_m \big(\Phi(\beta) = \Phi(\alpha) \big)$$

8.2 Here is a seemingly unrelated question: when is a function $\Phi : \omega^\omega \to \mathbb{N}$ continuous?

As previously mentioned, there is a natural topology on the Baire space (and by extension, on every spread), given by the initial segment topology. Because $\mathbb{N}$ is discrete, the preimage of a singleton $\{k\}$ under a continuous map Φ – that is, the set of sequences α with $\Phi(\alpha) = k$ – must be open in ω^ω. In other words, the set of sequences with $\Phi(\alpha) = k$ all agree on some initial segment. So, a function $\Phi : \omega^\omega \to \mathbb{N}$ is continuous if and only if:

$$\forall \alpha \exists m \forall \beta \sqsupseteq \bar{\alpha}_m \big(\Phi(\beta) = \Phi(\alpha) \big)$$

That is, a functional $\Phi : \omega^\omega \to \mathbb{N}$ is constructively total if and only if it is continuous.

Brouwer takes this one step further, and introduces a form of dependent choice for choice sequences:

$$\forall \alpha \exists n \big(\varphi(\alpha, n) \big) \to \exists (\Phi \in \omega^\omega \to \mathbb{N}) \forall \alpha : \varphi\big(\alpha, \Phi(\alpha)\big)$$

That is, for any relation $\varphi(\alpha,n)$ associating each choice sequence α to at least one natural number n, we can define a function Φ that chooses one of the n for each α.

8.3 Combining this with the previous observation, we obtain the principle of *local* or *weak continuity for numbers* [TD88; Dum00]:

Axiom (WC-$\mathbb{N}$)**.** *For any formula $\varphi(\alpha,n)$ depending on sequences α and naturals n,*

$$\forall\alpha\exists n\big(\varphi(\alpha,n)\big) \to \forall\alpha\exists m\exists n\forall\beta \sqsupseteq \alpha_m\big(\varphi(\beta,n)\big)$$

Equivalently, the principle states that formulae on choice sequences are topologically stable, in the sense that if $\varphi(\alpha,n)$ holds for a sequence α, then it also holds for every other sequence in some basic open set in ω^ω containing α. This principle is hence "local" in the sense that we are only guaranteed this stability on open patches around each α individually, with no global coherence conditions.

As an aside, we note that WC-$\mathbb{N}$ is incompatible with classical logic:

Theorem 8.1 ([TD88])**.** WC-$\mathbb{N}$ *refutes* LEM*.*

Proof. Consider the following instance of LEM:

$$\forall\alpha\big(\forall k(\alpha_k = 0) \vee \neg\forall k(\alpha_k = 0)\big)$$

Introducing an extraneous variable, we can rewrite this as:

$$\forall\alpha\exists n\Big(\big(n = 0 \wedge \forall k(\alpha_k = 0)\big) \vee \big(n \neq 0 \wedge \neg\forall k(\alpha_k = 0)\big)\Big)$$

This is the form required by WC-$\mathbb{N}$, so by WC-$\mathbb{N}$, we have:

$$\forall\alpha\exists m\exists n\forall\beta \sqsupseteq \bar{\alpha}_m\Big(\big(n = 0 \wedge \forall k(\beta_k = 0)\big) \vee \big(n \neq 0 \wedge \neg\forall k(\beta_k = 0)\big)\Big)$$

Eliminating n, this simplifies to:

$$\forall\alpha\exists m\forall\beta \sqsupseteq \bar{\alpha}_m\big(\forall k(\beta_k = 0) \vee \neg\forall k(\beta_k = 0)\big)$$

Now specialise α to the constant zero function. Then, for some m,

$$\forall\beta \sqsupseteq \bar{\alpha}_m\forall k(\beta_k = 0) \vee \forall\beta \sqsupseteq \bar{\alpha}_m\neg\forall k(\beta_k = 0)$$

The left disjunct is false, since

$$\beta = \langle\underbrace{0,0,0,\dots,0}_{m},1,\dots\rangle$$

extends $\bar{\alpha}_m$ but contradicts $\forall k(\beta_k = 0)$; and the right disjunct is also false, since $\beta = \alpha$ also extends $\bar{\alpha}_m$, but $\forall k(\beta_k = 0)$ holds. ∎

8.1 Bars

Let M be a spread and F be the species of Λ_M-admissible sequences. Then, a subspecies $B \subseteq F$ is a *bar* on M if every choice sequence α in M has an initial segment in B:

$$\forall\alpha \in M\exists n(\bar{\alpha}_n \in B)$$

Given a finite sequence a, we say that B *bars* a *(in* M*)* if for every choice sequence α in M extending a, there exists a natural n such that $\bar{\alpha}_n \in B$. So, for instance, B being a bar on M is equivalent to B barring $\langle\rangle$ in M.

Informally, we can think of a bar B on a spread M as a kind of "covering" of the underlying tree U, in that every choice sequence passes through a finite sequence in B.

8.4 The motivation behind this definition is that, if we could constructively describe such a bar B, then we would have a collection of finite objects that, in a sense, describes all the infinite choice sequences in M. Of course, the set containing only the empty sequence trivially bars every spread, but our goal is to construct bars that capture useful properties of the collection.

A species S of finite sequences in M is *inductive* if whenever every Λ_M-admissible immediate descendent of a is in S, then a is also in S.

Theorem 8.2 (Bar Induction)**.** *Let P be a decidable bar on S and let Q be an inductive subspecies of F containing P. Then Q is a bar.*

Proof. [Hey66; Vel06] Suppose we have a proof $\mathfrak{R}$ that P is a bar on S. This proof can only depend on the following data:

- The species P (and F);
- The spread-law Λ_S.

Thus, $\mathfrak{R}$ must be a (by definition, finite) proof tree consisting of inferences of the following forms:

- $n \in P$, so n is barred (immediate inference);
- n is barred by P, so $n \frown k$ is barred by P (upwards inference);
- For all k, $n \frown k$ is barred by P, so n is barred by P (downward inference).

The last inference of $\mathfrak{R}$ must assert that $\langle\rangle$ is barred by P in S. If this inference is an immediate inference, then $Q \supseteq P$ is also immediately a bar by the same inference. Otherwise, suppose that $\langle\rangle$ is not in P.

Since $\langle\rangle$ has no ascendants, the final inference must then be a downwards inference. So, the barred condition of every singleton sequence must be proved in $\mathfrak{R}$ before that of $\langle\rangle$. Thus, any upwards inferences for 1-sequences are extraneous. Moreover, since the proof of $\langle\rangle$ being barred must follow that of every 1-sequence, the barred condition for every 1-sequence must be proven via a downwards inference. Thus, any upwards inferences for 2-sequences are also extraneous.

By induction, we can remove all instances of upwards inference from $\mathfrak{R}$ to obtain a proof $\mathfrak{R}'$. We can then further remove every instance of downwards inference that proves the barred condition for any sequence that is in P (for which immediate inference will suffice instead) or has an ascendant in P (unnecessary by the definition of a bar), or that has been otherwise previously proven in $\mathfrak{R}'$ to be barred by P. Call this simplified proof $\mathfrak{R}''$.

Now, replace in $\mathfrak{R}''$ every immediate inference "$a \in P$, so a is barred by P" in $\mathfrak{R}''$ by the claim "a belongs to Q"; and every downward inference "for all k, $a \frown k$ is barred, so a is barred" by the inference "every immediate descendant of a belongs to Q, so a belongs to Q". Because $P \subseteq Q$, the former holds, and because Q is inductive, the latter is sound, so this yields a valid proof $\mathfrak{R}^*$. Because $\mathfrak{R}''$ proves that $\langle\rangle$ is barred by P, $\mathfrak{R}^*$ proves that $\langle\rangle$ is in Q, and hence Q is a bar. ■

Class. I must admit, this line of argumentation is fascinating; it is certainly not the typical constructivist argument I expected. When I think of a constructive proof, I imagine explicit constructions – things we can compute or verify via calculation. But here, we proved an implication $p \to q$, not by assuming P is a bar and establishing directly that Q is a bar, but instead by reasoning and inducting over proofs themselves!

Int. Indeed. This proof takes seriously the BHK-interpretation of an implication as an algorithm transforming a proof of the antecedent into a proof of the consequent.

Class. But even so, this feels like quite the shift in methodology from what we have seen so far. The approach here feels almost "second order", so to speak. In classical mathematics, we typically work *within* a system to prove theorems; here, however, constructivism appears to be adopting an *external*, proof-theoretic perspective, treating proofs themselves as objects that can be manipulated. This seems to stray a bit from the usual constructive spirit of *direct* construction.

Int. The key distinction here is that we are not assuming an arbitrary proof of p exists, and then deducing q as you would do in a classical setting, but instead, we are systematically transforming proofs of p into proofs of q;

each step in our proof corresponds to an explicit constructive transformation. You might think of this, rather as a *proof schema* of sorts.

This might be surprising, since, in classical logic, the notion of a proof as an object in itself doesn't really have any weight; truth is truth, and the notion of a proof is auxiliary. But for intuitionists, the proofs themselves are really the main object of study.

Class. But isn't that a curious way to reason, especially for a constructivist? Constructivism tends to favor direct constructions over indirect arguments. Here, however, the argument hinges on an abstract manipulation of proofs themselves. Doesn't this resemble the sort of formalist metamathematical reasoning that constructivists often criticise?

Int. That depends on how one looks at it. If we interpret mathematics to be purely about constructing proofs, then studying how proofs interact and how they can be transformed becomes foundational. This is not merely a formalist metamathematical abstraction. The proof-theoretic aspects of intuitionism are not a departure from its philosophy, but rather a natural extension of it. Constructivism is not simply about computation; it is about maintaining a meaningful connection between proof and construction.

8.2 Fans

A spread whose underlying tree is locally finite is called a *fan*.

Example. The spread defining the Cantor space is a fan, because every node a has precisely two immediate descendants: $a \frown 0$ and $a \frown 1$. △

Lemma 8.3. *Every closed interval* $[a,b]$ *of the continuum coincides with a fan.*

Proof. Without loss of generality, suppose $a < b$. Let $\langle a_n 2^{-n}\rangle$ and $\langle b_n 2^{-n}\rangle$ be CNGs for a and b, respectively, and consider the spread S of the CNGs $\langle x_n 2^{-n}\rangle$ where x_n satisfies $a_n \leq x_n \leq b_n$.

Once x_n has been chosen in any choice sequence, at least one and at most three values are admissible for x_{n+1}, so S is a fan. By construction, every element of S coincides with some element of $[a,b]$, and conversely, any CNG for a real number in $[a,b]$ will satisfy the above inequalities. ■

Previously, we have seen the principle of weak continuity for numbers (WC-$\mathbb{N}$), which states that formulae that are defined on all choice sequences (and hence more generally, on spreads) are topologically stable in that if a formula φ holds for a choice sequence α, then there is a basic open set containing α in the Baire space over which φ holds. That is, φ must be determined entirely by some initial segment of α.

This stability is "weak" in that these stable open sets exist separately and locally for each choice sequence without any global coherence. That is, there is no uniform bound on the length on the initial segment that determines φ.

Because fans are spreads satisfying an additional finiteness condition, we'd might expect that they are better behaved, constructively. This intuition turns out to be correct: for fans, such a uniform bound exists:

Theorem 8.4 (Fan Theorem). *Let S be a fan, and Φ be a natural-valued function defined on every element δ of S. Then,*

$$\exists m \forall \alpha,\beta \big(\bar{\alpha}_m = \bar{\beta}_m \to \Phi(\alpha) = \Phi(\beta)\big)$$

Proof. [Hey66], Let K be the naked fan of Λ_S-admissible choice sequences, and F be the corresponding spread-species. We define a function ϕ on K as follows: if the element δ of S is associated by Γ_S to the element $d \in K$, define $\phi(d) = \Phi(\delta)$. By WC-$\mathbb{N}$, for a fixed d, $f(d)$ is determined entirely by some initial segment $\bar{d}_{m(d)}$ of d. Let P be the subspecies $\{\bar{d}_{m(d)} : d \in K\} \subseteq F$ of these initial segments. By construction, P is a bar on K.

We define another subspecies $Q \subseteq F$ as follows: a is in Q if we can find a natural number $m(a)$ such that for every K-continuation d of a, $\phi(d)$ is completely determined by $\bar{d}_{m(a)}$. Clearly, $P \subseteq Q$, since if $a \in P$, then the value of m given by P also suffices for Q. Q is also inductive, since if every immediate descendant of a belongs to Q, then we can compute $m(a)$ by taking the maximum m over all the immediate descendants of a; this is computable as K is a fan, so there are only finitely many such descendants.

Thus, by bar induction, we have that Q is a bar on K, so every sequence $d \in K$ passes through Q. Thus, we can find a uniform bound m such that for every $d \in K$, $\phi(d)$ is determined by the first m components of d; and hence $\Phi(\delta)$ is determined by the first m components of δ. ∎

9 All Functions are Continuous

We are now in a position to prove Brouwer's continuity theorem:

Theorem 9.1. *Every function $f : [a,b] \to \mathbb{R}$ is uniformly continuous.*

Proof. [Hey66] By Theorem 8.3, the interval $[a,b]$ coincides with a fan S. To every element ξ of S, we can associate a real number $y := f(\xi)$. Let $\eta = \langle \eta_n 2^{-n} \rangle$ be a CNG for y. Then, for any fixed n, we can associate η_n to ξ, thus defining a natural-valued function ϕ_n on S. By the fan theorem, there is a number $m = m(n)$ such that for each $\xi \in S$, $\phi_n(\xi) = \eta_n$ is determined entirely by $\bar{\xi}_m$.

Now, let x_1 and x_2 be real numbers in $[a,b]$ such that $|x_1 - x_2| < 2^{-m-2}$. Then, x_1 and x_2 have CNGs ξ_1 and ξ_2, respectively, whose initial segments up to m are equal. It follows that η_n is the same for both ξ_1 and ξ_2, and thus, $\left|f(\xi_1) - f(\xi_2)\right| < \frac{5}{4}2^{-n}$. That is, f is uniformly continuous on $[a,b]$. ∎

Corollary 9.1.1. *Every function $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

Proof. Every real number c is contained within some closed interval, for instance, $[c-1,c+1]$. By the previous theorem, f is (uniformly) continuous on this closed interval and is, in particular, continuous at c. ∎

III

Refutation

INT. Think of it this way: when Weierstrass and others first formalised the real numbers, they were horrified to discover that their definition implied the existence of continuous functions that were nowhere differentiable. At the time, this upended our understanding of differentiability and forced us to reconsider our entire geometric framework of calulus. However, we eventually decided to accept this definition, and instead modified our intuitions about geometry and smoothness.

Now with the power of foresight and our modern understanding of analysis, it might be difficult to fully understand their discomfort. In the end, the horrors of nowhere differentiable functions ultimately paved the way to the paradise of self-similar sets, fractals, and measure theory [Dav05].

And again, the introduction of Choice into classical mathematics implied, for instance, that there existed subsets of the real line that were not measurable. Once again, we had to adjust our geometric intuitions about lengths and measures.

CLASS. And so, you suggest that the oddities of constructivism are the same?

* * *

10 Impasse

In 1967, the year after Brouwer's death, Erret Bishop struck back against Hilbert's attack, publishing a textbook [Bis67] in which he rebuilds analysis from constructive foundations. In this book, he provides a thorough constructive treatment of calculus, measure theory, complex analysis, and functional analysis, and more, as well reconstructing several major modern theorems in analysis, including the Stone-Weierstrass theorem, the Hahn-Banach Theorem, the spectral theorem, the Lebesgue convergence theorems, and the beginnings of the theory of C*-algebras. It seemed that the "gaps" in constructivism mathematics were not a problem in practice: discontinuous functions are not computable and carry no meaningful numerical or computational information, and are thus constructively invisible, their absence hardly felt.

In developing his theory of analysis, Bishop showed that, despite the apparent veridical paradoxes of constructivism, it was not necessary to abandon major portions of modern mathematics:

> The thrust of Bishop's work was that both Hilbert and Brouwer had been wrong about an important point on which they had agreed. Namely both of them thought that if one took constructive mathematics seriously, it would be necessary to "give up" the most important of modern mathematics (such as, for example, measure theory or complex analysis). Bishop showed that this was simply false, and in addition that it is not necessary to introduce unusual assumptions that appear contradictory to the uninitiated. [Bee12]

Bishop's book is also remarkably *readable*. It does not invoke unfamiliar concepts such as choice sequences or spreads, and reads just like any other textbook; every page is instantly understandable to any classical analyst. Bishop had also shown that constructive mathematics did not require delving into logical symbol spaghetti to produce useful results constructively; it is sufficient to merely be mindful of where you are using LEM, along with some redefinitions.

With the development of recursion theory, the (non-uniformly) continuous map from [0,1] to $\mathbb{R}$ of CRM could also be reconciled with classical mathematics by the following reinterpretation of the statement to: "there is a *recursively* continuous *recursive* function from the closed interval [0,1] of the *recursive* real line to the *recursive* real line that is not *recursively* uniformly continuous." Rephrased this way, this result is no longer in conflict with CLASS, and indeed, is in fact a theorem of CLASS.

11 The Constructive Intermediate Value Theorem

It is well-known that IVT is not constructively valid. However, we will show some constructive alternatives in the style of Bishop.

Firstly, the continuity result from the previous act depends on WC-ℕ, which is incompatible with classical logic, and is hence not acceptable in BCM. Thus, unlike Brouwer, we cannot prove that our functions are all continuous. Bishop also treats sequences as static, fully determined (i.e. computable) functions defined on the naturals, unlike the dynamic, incomplete choice sequences of Brouwer's intuitionism; which aligns more closely with the classical interpretation of sequences. Apart from this, we largely take the same definition of a real number as an equivalence class of Cauchy sequence as before, and all of the previous theory not dependent on WC-ℕ still holds here. Firstly:

Lemma 11.1. *Equality is not decidable for real numbers.*

Proof. Let GC(n) be the proposition that n can be expressed as the sum of two primes. Now, consider the RNG with $a_n = \frac{1}{m}$ if there exists $m < n$ with m the least counterexample of Goldbach, and $\frac{1}{n}$ otherwise. This RNG is well-defined as for every n, GC(n) is decidable, since primes are computable and we can exhaustively compute the sums of all pairs of primes under n. But then, $a = 0$ if and only if the Goldbach conjecture is proved, and $a \# 0$ if and only if the Goldbach conjecture is refuted. ■

Theorem 11.2. *Trichotomy does not hold for real numbers.*

Proof. We split the $\# 0$ case in the previous example: define $a_n = \frac{1}{m}$ if there exists $m < n$ with m the least counterexample of Goldbach *and* $m \equiv 0 \pmod 4$, and $\frac{1}{n}$ otherwise, and define b_n in the same way, except $m \equiv 2 \pmod 4$. Then, $a = b$ if Goldbach holds; $a < b$ if the least counterexample is a multiple of 4, and $a > b$ otherwise; none of which we can determine. ■

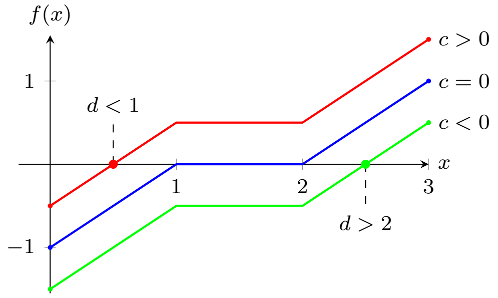
Firstly, we observe that IVT is not constructively valid. Define the function $g : [0,3] \to \mathbb{R}$ by:

$$g(x) = \begin{cases} x-1 & 0 \le x \le 1 \\ 0 & 1 \le x \le 2 \\ x-2 & 2 \le x \le \end{cases} \qquad \text{([TD88; Bab19])}$$

Strictly speaking, this is a slight abuse of notation, because which interval the *real number* x lies in may not be decidable, but applying this definition of g to *Cauchy sequences* resolves this problem. Now, let $c = a - b$, where a and b are the real numbers defined in the previous proof, and define $f(x) = g(x) + c$.

Now, $f(0) = c - 1 < 0$ and $f(3) = c + 1 > 0$, so by IVT, there exists $d \in (0,3)$ such that $f(d) = 0$. Since $\langle d_n \rangle$ is Cauchy, there exists n such that $|d_{n+p} - d_n| < 1/4$ for all p.

Since equality is decidable on the rationals, we have $d_n \le 3/2$ or $d_n > 3/2$. In the first case, this implies that $d_{n+p} < 1/4 + d_n \le 7/4 < 2$ for all p, so $d < 2$; and in the second case, this implies $d_{n+p} > d_n - 1/4 > 5/4 > 1$ for all p, so $d > 1$.



But if $d < 2$, then $\neg(c < 0)$, and if $d > 1$ then $\neg(c > 0)$. So we have $\neg(c < 0)$ or $\neg(c > 0)$. In either case, this determines more information about the Goldbach conjecture than we currently have. So IVT is constructively invalid.

Theorem 11.3 (Approximate Intermediate Value Theorem)**.** *If $f : [0,1] \to \mathbb{R}$ is continuous and $f(0) \leq 0 \leq f(1)$, then for every $\varepsilon > 0$, there exists $c \in [1,1]$ such that $|f(c)| < \varepsilon$.*

This approximate restatement faithfully captures the computational aspects of constructivism: while we may not be able to assert the existence of an exact root, we can still always approximate one to any degree of accuracy; given any desired level of precision ε, we can construct a point where f is arbitrarily close to zero. Our proof is by the means of an algorithm that computes these roots to arbitrary precision.

Proof. [Fra20] Fix $\varepsilon > 0$ and define the following sequences inductively:

$$\begin{aligned}
a_1 &= a \\
b_1 &= b \\
c_n &= (a_n + b_n)/2 \\
d_n &= \max\left(0, \min\left(1, \tfrac{1}{2} + \tfrac{f(c_n)}{\varepsilon}\right)\right) \\
a_{n+1} &= c_n - d_n(b-a)2^{-n} \\
b_{n+1} &= b_n - d_n(b-a)2^{-n}
\end{aligned}$$

The d_n are clamped to be between 0 and 1, and each interval $[a_{n+1},b_{n+1}]$ is of length $b_{n+1} - a_{n+1} = (b-a)2^{-n}$ and is contained in the previous $[a_n,b_n]$. The sequence c_n thus forms a Cauchy sequence.

We claim that for every $m \in \mathbb{N}$, either

- (i) $\exists j \leq m\big(\left|f(c_j)\right| < \varepsilon\big)$;
- (ii) or, $f(a_m) < 0$ and $f(b_m) > 0$.

We induct on m. The base case holds as, by assumption, $f(a) < 0$ and $f(b) > 0$. If case (i) holds, this is trivial. Otherwise, at least one of the following holds: $f(c_m) < -\varepsilon/2$, $\left|f(c_m)\right| < \varepsilon$, or $f(c_m) > \varepsilon/2$. If $\left|f(c_m)\right| < \varepsilon$, then the inductive step is trivial. Otherwise:

If $f(c_m) < -\varepsilon/2$, then $d_m = 0$, so $a_{m+1} = c_m$ and $b_{m+1} = b_m$, and similarly, if $f(c_m) < -\varepsilon/2$, then $d_m = 1$, so $a_{m+1} = a_m$ and $b_{m+1} = c_m$. In either case, $f(a_{m+1}) < 0$ and $f(b_{m+1}) > 0$ by the induction hypothesis.

This establishes the claim. Now, if (i) holds, the theorem follows. Otherwise,

$$|c - a_m| \leq |c - c_m| + |c_m - a_m| < \delta \to \left|f(c) - f(a_m)\right| < \varepsilon$$

and

$$|c - b_m| \leq |c - c_m| + |c_m - b_m| < \delta \to \left|f(c) - f(b_m)\right| < \varepsilon$$

So $f(c)$ is within ε of both a negative and positive number, so $\left|f(c)\right| < \varepsilon$. ∎

IV

The Ascent

Int. In many ways, a topos is a generalised category of sets. Within a topos, we have terminals, so can talk about global elements; we have exponential objects, so we can talk about function objects and power objects; and we have limits and colimits, so we can construct products and unions. We can even axiomatise the category of sets as a special kind of topos (see [Liu23] and ETCS).

However, in doing so, we need to enforce the Axiom of Choice as separate condition; Choice is not inherent to the framework of topoi, and in fact, topoi by default only support constructive reasoning. While this may seem restrictive, we gain generality in that every constructive result automatically transfers into every topos: from topoi of sheaves over a space in geometry, to the effective and realisability topoi in computability and proof theory.

* * *

Many of Hilbert's objections, though weakened, still hold today. We have since proved that a multitude of critical and fundamental results are incompatible with or independent from constructive logic. For instance, the well-ordering of cardinals, existence of maximal ideals, Hilbert's Nullstellensatz, Tychonoff's theorem, even the intermediate value theorem, and many more.

Though it is possible to rebuild much of analysis from the ashes, should we do so with the rest of mathematics? Though much would persist, much would also be lost in the reconstruction, with all of the previous critical theorems – along with many others – being unavoidable casulties. Even the way we morally justified the previous result of CRM was to pack it up into the box of computability and embed it back into Class!

11.1 And so, the question remains: why should we care? While some may take interest in the philosophy of mathematics, many do not. Most mathematicians are not interested in the philosophical, or even foundational, underpinnings of their work, so arguing over various interpretations of logic and over whether Choice or LEM should hold seems like an irrelevant exercise in logic, detached from mathematics as a whole.

Firstly, computer scientists and complexity theorists should care deeply about these results: *computability* is by its very nature constructive, and in fact, is a restricted version thereof: not only do we require that our proofs provide us a method of constructing a witness, we furthermore require that this method be *effective* with respect to some model of computation.

Secondly, logical theories and related systems are not just used in the foundations of mathematics; there are many instances of structures in mathematics that can *interpret* logical calculi in a formal sense, and deductions within these internal interpreted systems can yield external results about the external structure itself. We will see various examples of this, particularly in topos theory.

The catch is that, in many cases, these structures will only be able to interpret restricted fragments of classical logic: translations of Choice and LEM are simply false (in the model-theoretic sense) in a vast variety of structures that we are interested in. Thus, there is reason to study non-classical logic, even outside of a foundational setting: only non-classical reasoning will be of any use for proving results within these structures. We demonstrate this now.

12 Topos-Theoretic Background

We assume basic familiarity with category theory, and only recall some basic topos-theoretic notions. For a more detailed exposition, see my previous paper [Liu, 2024].

For our purposes, a *topos* will be defined as a finitely complete Cartesian closed category with a *subobject classifier*.

12.1 Notation and Terminology

We say that $x : T \to X$ is a *generalised element* of X at *stage of definition* T, or just a T*-element* of X. If T is terminal, then we recover the ordinary notion of elementhood, and x is said to be a *global element* of X. When viewing morphisms through this interpretation, we write $x \in_A X$ for an element $x : A \to X$.

A *subobject* of an object X is an equivalence class of monomorphisms into X, where two monos are equivalent if they factor through each other. For simplicity, we will often pick a representative and call the mono itself, or even the domain, a subobject. We write $A \subseteq B$ if the mono representing A factors through the mono representing B. Given an element x of X, we write $x \in_X a$ or just $x \in a$ if x factors through a.

12.2 The Subobject Classifier

A *subobject classifier* in a category $\mathscr{C}$ is an object Ω and a map $\mathtt{t} : 1 \to \Omega$ such that every monomorphism $m : A \rightarrowtail X$ is the pullback of $\mathtt{t}$ along a unique morphism $\chi_m : X \to \Omega$ called the *characteristic* or *classifying morphism* (of m).

That is, for every mono $m : A \rightarrowtail X$, there exists a unique morphism $\chi_m : X \to \Omega$ such that

$$\begin{tikzcd}
A \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathtt{t}"] \\
X \arrow[r, "\chi_m"'] & \Omega
\end{tikzcd}$$

is a pullback square. The object Ω is then called the *object of truth values*, any element of Ω is a *truth value*, and $\mathtt{t}$ is the truth value *true*. We also call any morphism $X \to \Omega$ a *predicate* on X. Because this map will occur frequently, we also abbreviate $\mathtt{t} \circ !_X$ to $\mathtt{t}_X$, or $\mathtt{t}$ when the object X is clear from context.

Note that, while every mono is required to have a unique classifying morphism, any particular morphism may classify many monos. However, $\chi_a = \chi_b$ if and only if $a \cong b$, since pullbacks are unique up to isomorphism. Conversely, any predicate ϕ is fully determined by the part of its domain that it takes to *true*, namely, the subobject that it classifies (i.e. the pullback of $\mathtt{t}$ along ϕ).

Lemma 12.1. *Let $\chi_m : X \to \Omega$ and $a : A \to X$. Then, $\chi_m a = \mathtt{t}_A$ if and only if a factors through the mono $m : M \to X$ classified by χ_m.*

Proof. If $\chi_m a = \mathtt{t}_A$, then there exists a (unique) morphism $f : A \to M$ such that $mf = a$:

$$\begin{tikzcd}
A \arrow[dr, dashed, "f"] \arrow[drr, bend left, "!_A"] \arrow[ddr, bend right, "a"'] & & \\
& M \arrow[r, "!_M"] \arrow[d, tail, "m"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathtt{t}"] \\
& X \arrow[r, "\chi_m"] & \Omega
\end{tikzcd}$$

Since $mf = a$, we have $\chi_m a = \chi_m m f = \mathtt{t}_M f = t_A$. ■

We can now see the utility of the $\in$ notation and the naming of the *true* morphism: using this terminology, the previous lemma simply says that $\chi_m(a)$ is *true* if and only if $a \in m$.

13 Internal Connectives

A subobject of a X has three equivalent descriptions as a mono $m : A \rightarrowtail X$ into X, a predicate $\phi : X \to \Omega$ on X, and as a global element $c : 1 \to \Omega^X$ of the power object Ω^X. We can translate between these descriptions with the following notation:

Mono		Predicate		Element
$m : A \rightarrowtail X$	$\longrightarrow$	$\chi_m : X \to \Omega$		
$\phi^* : A \rightarrowtail X$	$\longleftarrow$	$\phi : X \to \Omega$	$\longrightarrow$	$\ulcorner\phi\urcorner : 1 \to \Omega^X$
		$c^\sharp : X \to \Omega$	$\longleftarrow$	$c : 1 \to \Omega^X$

Often in category theory, we like to work with the first description since this approach is amenable to being described in terms of the subobject poset Sub(X) or more generally, the slice category $\mathscr{E}/X$, e.g. constructing intersections as (fibred) products of two monos.

However, we will show that these constructions can be *internalised* into the other two characterisations.

For the remainder of this section, let $a : A \to X$ and $b : B \to X$ be general subobjects, and $\phi,\psi : X \to \Omega$ be two predicates on X.

13.1 Conjunction

We define the *conjunction* morphism $\wedge$ as the classifying morphism of the pairing $\langle \mathsf{t},\mathsf{t}\rangle : 1 \rightarrowtail \Omega \times \Omega$:

$$\wedge \coloneqq \chi_{\langle \mathsf{t},\mathsf{t}\rangle} : \Omega \times \Omega \to \Omega$$

We will suggestively write $\phi \wedge \psi$ in infix notation for this composite.

Theorem 13.1. *$\phi \wedge \psi = \mathsf{t}_X$ if and only if $\phi = \mathsf{t}_X$ and $\psi = \mathsf{t}_X$.*

Proof. By pairing, $\phi = \mathsf{t}_X$ and $\psi = \mathsf{t}_X$ if and only if $\langle\phi,\psi\rangle$ factors through $\langle \mathsf{t},\mathsf{t}\rangle$ (or equivalently $\langle\phi,\psi\rangle \in \langle \mathsf{t},\mathsf{t}\rangle$), which is equivalent by the previous lemma to $\chi_{\langle \mathsf{t},\mathsf{t}\rangle}\langle\phi,\psi\rangle = \mathsf{t}_X$, which is precisely $\wedge\langle\phi,\psi\rangle = \phi \wedge \psi = \mathsf{t}_X$. ∎

More generally,

Corollary 13.1.1. *The intersection $a \cap b$ is the subobject classified by $\chi_a \wedge \chi_b$.*

Interpreting this statement in **Set** and accepting that this map $\chi_{\langle \mathsf{t},\mathsf{t}\rangle}$ indeed encodes conjunction, this is intuitively obvious, because an element is in $a \cap b$ if and only if it is in a (it is *true* under χ_a) *and* it is in b (it is *true* under χ_b). In fact, our notation of $\in$ for factoring makes this reasoning formal:

Proof. By the definition of an intersection, $c \in a \cap b$ if and only if $c \in a$ and $c \in b$, for c an arbitrary element of X. That is, $\chi_{a\cap b}c = \mathsf{t}_C$ if and only if $\chi_a c = \mathsf{t}_C$ and $\chi_b c = \mathsf{t}_C$. By previous theorem, this holds if and only if $\wedge\langle\chi_a,\chi_b\rangle c = \mathsf{t}$. ∎

From this, we can also deduce properties of $\wedge$ by transporting external properties of $\cap$ along this equivalence. For instance, we have $\phi \wedge \phi = \phi$, since $\phi^* \cap \phi^* = \phi^*$; and $\phi \wedge \mathsf{t} = \phi$ for any predicate ϕ, since both sides classify ϕ^*; etc.

13.2 Implication

Let $\leq_1$ be the equaliser of $\wedge,\pi_1 : \Omega \times \Omega \to \Omega$:

$$\Omega_1 \xrightarrow{\leq_1} \Omega \times \Omega \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} \Omega$$

Given a pair of predicates $\phi,\psi : X \to \Omega$, what does it mean to have $\langle\phi,\psi\rangle \in \leq_1$ (or in infix notation $\phi \leq_1 \psi$)? This would mean that $\langle\phi,\psi\rangle$ factors through $\leq_1$, so by the definition of an equaliser, $\phi = \phi \wedge \psi$.

Theorem 13.2. *$a \subseteq b$ if and only if $\chi_a \leq_1 \chi_b$*

Proof. $a \subseteq b$ is equivalent to $a \equiv a \cap b$, which is precisely $\chi_a = \chi_a \wedge \chi_b$, or, $\chi_a \leq_1 \chi_b$. ■

In other words, $\phi \leq_1 \psi$ means that ψ is "at least as true" as ϕ; it sends at least the same elements of X to true (and possibly some more); if $\phi x = \mathtt{t}$, then $x \in \phi^*$, so $x \in \psi^*$, which implies $\psi x = \mathtt{t}$.

We define the *material conditional* or *implication* morphism $\rightarrow$ as the classifying morphism of $\leq_1$:

$$\rightarrow := \chi_{\leq_1} : \Omega \times \Omega \rightarrow \Omega$$

As usual, we write $\phi \rightarrow \psi$ for $\rightarrow\langle\phi,\psi\rangle$.

Theorem 13.3. *$(\phi \rightarrow \psi) = \mathtt{t}$ if and only if $\phi \leq_1 \psi$.*

Proof. Expanding definitions, $(\phi \rightarrow \psi) = \mathtt{t}$ if and only if $\chi_{\leq_1}\langle\phi,\psi\rangle = \mathtt{t}$ if and only if $\phi \leq_1 \psi$. ■

Given subobjects $a : A \rightarrow X$ and $b : B \rightarrow X$, we define the (*material*) *implicate* $a \Rightarrow b$ to be the subobject classified by $\chi_a \rightarrow \chi_b$.

Theorem 13.4. *$a \cap b \subseteq c$ if and only if $a \subseteq b \Rightarrow c$.*

Proof. $a \subseteq b \Rightarrow c$ is equivalent to $\rightarrow \langle\chi_b,\chi_c\rangle a = \mathtt{t}_A$, equivalent to $\rightarrow \langle\chi_b a,\chi_c a\rangle = \mathtt{t}_A$ which holds if and only if $\chi_b a \leq_1 \chi_c a$, if and only if $a \cap b \subseteq a \cap c$, which is equivalent to $(a \cap b) \subseteq c$. ■

13.3 Universal Quantification

For each object X, the label $\ulcorner \mathtt{t}_X \urcorner : 1 \rightarrow \Omega^X$ represents the maximal subobject $\mathrm{id}_X : X \rightarrowtail X$. Since $\ulcorner \mathtt{t}_X \urcorner$ is from a terminal, it is necessarily monic, and hence has a classifying morphism.

For each object X, we define the *universal quantifier* morphism $\forall_X$ as the classifying morphism of $\ulcorner \mathtt{t}_A \urcorner : 1 \rightarrow \Omega^A$:

$$\forall_X := \chi_{\ulcorner \mathtt{t}_X \urcorner} : \Omega^X \rightarrow \Omega$$

Intuitively, $\forall_X$ sends a subobject of X to *true* if and only if it is the maximal subobject; or, equivalently, $\forall_X$ sends a predicate on X to *true* if and only if it is true universally over all of X.

Let $r : R \rightarrowtail Y \times X$ be a relation classified by $\chi_r : Y \times X \rightarrow \Omega$, equivalent via exponential tranposition (currying) to $\chi_r^\flat : Y \rightarrow \Omega^X$. The *universal quantification* $\forall x.R$ of r over X is the pullback of $\ulcorner \mathtt{t}_X \urcorner$ along $\chi_r^\flat$:

$$\begin{tikzcd}
\forall x.R \arrow[r, "!"] \arrow[d, tail, "\forall x.r"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\ulcorner \mathtt{t}_X \urcorner"] \\
Y \arrow[r, "\chi_r^\flat"] & \Omega^X
\end{tikzcd}$$

Or equivalently, $\forall a.R$ is the subobject classified by $\forall_X \chi_r^\flat$.

Theorem 13.5. *For any $s : S \rightarrowtail Y$, $S \subseteq \forall x.R$ if and only if $S \times X \subseteq R$.*

That is, $\forall x.R$ is the largest subobject of Y that is related to all of X under r.

Proof. $S \subseteq \forall x.R$ if and only if $\chi_r s = \ulcorner \mathtt{t}_X \urcorner \circ !_S$. Taking the exponential transposition of both sides yields $\chi_r(s \times \mathrm{id}_X) = t_{S\times X}$, so $s \times \mathrm{id}_X$ factors through r; that is, $S \times X \subseteq R$. ■

14 The Internal Language

As previously mentioned, every topos can interpret an internal logical calculus called an *internal language*. In fact, most structured categories can do this, with the strength of the interpreted logical theories depending on the structure of the category, but a topos has a particularly simple and expressive internal language – however, this language is in general, constructive. The internal language is easily extendable depending on the particular topos we are interested in; for instance, in the *effective topos* of computable presentations, Markov's principle is valid, and the internal language is much closer to that of CRM. More generally, there exist topoi that model all of the results we have discussed previously, and many more. There are even topoi in which $\mathbb{N}^{\mathbb{N}}$ injects into $\mathbb{N}$ [Bau11], or where the reals are countable [BH24]!

Beyond these model-theoretic curiosities, the internal language has concrete applications in mathematics. The basic unmodified internal language allows us to reason about the objects of a topos as if they were sets, applying set-theoretic results to the much more complicated structures of any arbitrary topos.

For instance, in a *sheaf topos*, sheaves of rings appear to just be ordinary rings from the perspective of the internal language, and thus, any constructively valid theorem about rings yields, by transport into the internal language, a corresponding theorem about sheaves of rings. Similarly, any constructively valid theorem about an object can be transported into the effective topos to obtain a version that holds for a computable presentation of that object. These techniques have very recently been applied to great success in algebraic geometry [Ble21].

The internal language is often called a *type theory* in the literature, since every term of the language is associated to a *type* that restricts what other terms it can interact with. However, this is not a type theory in the modern sense, as its signature contains separate function symbols not created by lambda abstractions. This is because a topos has two kinds of "functions"; actual morphisms, and elements of exponential objects. Because of this, we instead prefer the term *typed predicate calculus* to describe the internal language.

For this section, we fix an elementary topos $\mathscr{E}$. Given an expression τ in our formal language, we write $[\![\tau]\!]$ for its interpretation in $\mathscr{E}$.

14.1 Types

The types of our language are defined inductively from a collection of *base types* via the following *type constructors*:

(T1) Every base type is a type.

(T2) If A and B are types, then B^A is a type.

(T3) If A and B are types, then $A \times B$ is a type.

In our interpretation, every object of $\mathscr{E}$ is a base type, and function/product types are interpreted in the obvious way, e.g. $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$ and $[\![B^A]\!] = [\![B]\!]^{[\![A]\!]}$.

14.2 Terms

Terms are not only attached to a type, but are defined relative to a list of free variables. A (*variable*) *context* is a list $x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$, where A_i are types and x_i are distinct variables. We use capital Greek letters $\Gamma, \Delta, \ldots$ as metavariables for contexts. We write

$$\Gamma \vdash t : A$$

for the statement that *t is a term of type A in context Γ*. Note that this is a metalogical claim (a *judgement*), and not a provable proposition, like the (co)domain declaration of a morphism $f : A \to B$.

Fix a context Γ. We define the various *term formation* rules and their interpretations as follows [Sou21a; McL92; Str03; MM12]:

(t1) All of the variables declared in Γ are terms:

$$\frac{}{\Gamma \vdash x_i : A_i}$$

(I1) Each variable $x_i : A_i$ is interpreted as the ith projection:

$$[\![\Gamma \vdash x_i : A_i]\!] = \pi_i : \prod_j A_j \to A_i$$

(t2) For every morphism $c : 1 \to B$, there is a *constant* term $c : B$. We write ! for the unique constant of type 1.

(I2) This constant is interpreted composing with by the unique morphism into 1:

$$[\![\Gamma \vdash c : B]\!] = c \circ !_{A_1 \times \cdots \times A_n} : \prod_i A_i \xrightarrow{!} 1 \xrightarrow{c} B$$

(t3) Given a variable $y : A$ and a term $s : B$, there is a term $\lambda x{:}A.s$ of type B^A:

$$\frac{\Gamma, y : A \vdash s : B}{\Gamma \vdash \lambda y{:}A.s : B^A}$$

(I3) For any term $s : B$, if the variable $y : A$ is not in Γ, then the interpretation of $\lambda y.s$ is the exponential transposition of $[\![\Gamma, y : A \vdash s : B]\!] : \prod_i A_i \times A \to B$:

$$[\![\Gamma \vdash \lambda y.s : B^A]\!] = [\![\Gamma, y : A \vdash s : B]\!]^\flat : \prod_i A_i \to B^A$$

If the variable $y : A$ is in Γ, then we can simply rename the bound instances of y in s by a fresh variable (i.e. α-conversion in the lambda calculus) and proceed as above.

(t4) For every morphism $f : A \to B$ and term $a : A$, there is a term $fa : B$:

$$\frac{\Gamma \vdash f : A \to B, \quad \Gamma \vdash a : A}{\Gamma \vdash fa : B}$$

(I4) The interpretation of fa is given by composing f with the interpretation of $[\![\Gamma \vdash a : A]\!]$:

$$[\![\Gamma \vdash fa : B]\!] = f[\![\Gamma \vdash a : A]\!] : \prod_i A_i \xrightarrow{[\![\Gamma \vdash a]\!]} A \xrightarrow{f} B$$

(t5) Given terms $a : A$ and $b : B$, there is a term $\langle a,b \rangle : A \times B$:

$$\frac{\Gamma \vdash a : A, \quad \Gamma \vdash b : B}{\Gamma \vdash \langle a,b \rangle : A \times B}$$

(I5) The interpretation of the pair $\langle a,b \rangle$ is given by the pairing of the components' interpretations:

$$\big[\!\big[\Gamma \vdash \langle a,b \rangle\big]\!\big] = \big\langle [\![\Gamma \vdash a]\!], [\![\Gamma \vdash b]\!] \big\rangle : \prod_i A_i \to A \times B$$

Note that on the left, the brackets in $\langle a,b \rangle$ are just formal symbols, while on the right, they denote pairing from the categorical product.

A variable $x : X$ is *free* unless bound by a lambda operator $\lambda x{:}X$. A term with no free variables is called *closed*. For instance, every constant term is closed. Given a variable $v : X$ and a term $t : X$, for any term s, we write $s[t/v]$ for the syntactic substitution of every *free* occurence of v by the term t. If no free variables of t then become bound in $s[t/v]$, we say that t is *free for v in s*.

Combining formation rules 4 and 5, we have that for any terms $f : B^A$ and $a : A$, there is a term $\mathrm{ev}_A\big(\langle f,a \rangle\big) : B$, which we will abbreviate to $f(a)$:

$$[\![\Gamma \vdash f(a) : B]\!] : \prod_i A_i \xrightarrow{\langle f,a \rangle} B^A \times A \xrightarrow{\mathrm{ev}_A} B$$

We also omit angle brackets when using this notation, writing $f(x,y)$ for $f\big(\langle x,y \rangle\big)$.

Example. Let $g : Y \to A$ be a morphism, $y : Y$ be a variable, and $\tau : Y$ be a term with free variable $x : X$, and consider the term $(\lambda y{:}Y.gy)(\tau)$. This term is interpreted by:

$$\begin{aligned}
[\![x : X \vdash (\lambda y{:}Y.gy)(\tau)]\!] &= [\![x : X \vdash \mathrm{ev}(\lambda y{:}Y.gy,\tau)]\!] \\
&= \mathrm{ev}\langle [\![x : X \vdash \lambda y{:}Y.gy]\!],[\![x : X \vdash \tau]\!]\rangle \\
&= \mathrm{ev}\langle [\![x : X,y : Y \vdash gy]\!]^\flat,[\![x : X \vdash \tau]\!]\rangle \\
&= \mathrm{ev}\langle (g[\![x : X,y : Y \vdash y]\!])^\flat,[\![x : X \vdash \tau]\!]\rangle \\
&= \mathrm{ev}\langle (g\pi_2)^\flat,[\![x : X \vdash \tau]\!]\rangle \\
&= g\pi_2\langle \mathrm{id}_H ,[\![x : X \vdash \tau]\!]\rangle \\
&= g[\![x : X \vdash \tau]\!]
\end{aligned}$$

These rewriting rules match what we would expect, since applying a function $y \mapsto gy$ to a term τ should just yield $g(\tau)$, which is precisely what we see here. △

We will also use set-builder notation for lambda abstraction over Ω. That is, if $x : X$ and $\phi : \Omega$, then

$$\{x : X \mid \phi\} \coloneqq \lambda x{:}X.(\phi : \Omega)$$

This matches our reading of maps $\phi : X \to \Omega$ as subobjects of X.

Terms $\varphi,\psi,\ldots$ of type Ω are called *formulae*. Again by formation rules 4 and 5, we can apply the logical connectives $\wedge$ and $\to$ to formulae to obtain composite terms also of type Ω.

Given any formula ϕ and variable $x : X$, we define the formula $\forall x{:}X.\phi$ as an abbreviation for $\forall_X\{x : X \mid \phi\} = \forall_X \lambda_x{:}X.\phi$. So, $\forall x{:}X.\phi = \mathsf{t}$ if and only if $\{x : X \mid \phi\} = X$. This formula is interpreted by:

$$\prod_i A_i \xrightarrow{[\![\Gamma,y:X\vdash\phi]\!]} \Omega^X \xrightarrow{\forall_X} \Omega$$

We have not constructed $\bot$, $\neg\phi$, $\phi \vee \psi$, and $\exists x : X$, but it turns out that because we can quantity over Ω, we can construct these operations from $\wedge$, $\to$, and $\forall$ via the *Russell–Prawitz embeddings*:

$$\begin{aligned}
\bot &\equiv \forall p{:}\Omega.p \\
\neg\varphi &\equiv \varphi \to \bot \\
\varphi \vee \psi &\equiv \forall\varphi{:}\Omega.(\varphi \to p) \wedge (\tau \to p) \to p \\
\exists x{:}A.\varphi(x) &\equiv \forall p{:}\Omega.(\forall x{:}A.\varphi(x) \to p) \to p
\end{aligned}$$

14.3 Extensions

Given a formula ϕ and a context $\Gamma \equiv x_1 : A_1,\ldots,x_n : A_n$, the *extension* $[\Gamma \mid \phi]$ of ϕ over Γ is the subobject of $[\![\Gamma]\!]$ classified by $[\![\Gamma \vdash \phi]\!]$. We think of the extension as the largest subcontext of Γ in which ϕ is universally valid.

For instance:

$$\begin{aligned}
[\Gamma \mid \mathsf{t}] &= [\![\Gamma \vdash \mathsf{t}]\!]^* \\
&= (\mathsf{t} \circ !_{[\![\Gamma]\!]})^* \\
&= [\![\Gamma]\!]
\end{aligned}$$

is the entire context; *true* holds everywhere. We also have:

$$\begin{aligned}
[\Gamma \mid \phi \wedge \psi] &= (\wedge \circ \langle [\![\Gamma \vdash \phi]\!],[\![\Gamma \vdash \psi]\!]\rangle)^* \\
&= [\![\Gamma \vdash \phi]\!]^* \cap [\![\Gamma \vdash \psi]\!]^* \\
&= [\Gamma \mid \phi] \cap [\Gamma \mid \psi]
\end{aligned}$$

and through similar reasoning,

$$[\Gamma \mid \phi \to \psi] \equiv [\Gamma \mid \phi] \Rightarrow [\Gamma \mid \phi]$$

Example. [Sou21a] Suppose φ is a formula with free variables $a : A$ and $b : B$. When is $c : B$ in

$$\begin{aligned}[b : B \mid \forall a{:}A.\varphi] &= [\![b : B \vdash \forall a{:}A.\varphi]\!]^* \\ &= [\![b : B \vdash \forall_A \lambda a{:}A.\varphi]\!] \\ &= \left(\forall_A [\![b : B \vdash \lambda a{:}A.\varphi]\!]\right)^* \\ &= \left(\forall_A [\![b : B, a : A \vdash \varphi]\!]^\flat\right)^*\end{aligned}$$

Let $r : R \to B \times A$ be the subobject classified by $\chi_r \coloneqq [\![b : B, a : A \vdash \varphi]\!] : B \times A \to \Omega$, so:

$$\begin{aligned}&= \forall_A \chi_r^\flat \\ &= \forall a.R\end{aligned}$$

When is a subobject $x : X \rightarrowtail B$ contained in this extension? By Theorem 13.5, this is precisely when $X \times A \subseteq R$. That is, $X \subseteq [b : B \mid \forall a{:}A.\varphi]$ if and only if X and A satisfy the relation R. △

A formula ϕ is *true* if $[\Gamma \mid \phi] = [\![\Gamma]\!]$, or equivalently, if $[\![\Gamma \vdash \phi]\!] = \mathsf{t}_{[\![\Gamma]\!]}$, where Γ contains precisely the free variables of ϕ.

Example. Let $y : Y$ and $\tau : Y$ be as in the example in the previous section, and replace $g : Y \to A$ by a classifying morphism $\chi_r : Y \to \Omega$. When is $\tau \in \{y : Y \mid \chi_r y\}$ *true*?

$$\begin{aligned}\left[x : X \mid \tau \in \{y : Y \mid \chi_r y\}\right] &= X \\ [\![x : X \mid \tau \in \{y : Y \mid \chi_r y\}]\!] &= \mathsf{t}_X \\ \chi_r [\![x : X \vdash \tau]\!] &= \mathsf{t}_X \\ [\![x : X \vdash \tau]\!] &\in r\end{aligned}$$

That is, $\tau \in \{y : Y \mid \chi_r y\}$ if and only if $\tau \in r$. △

More generally, for any finite set of formulas $\Phi \equiv \phi_1, \ldots, \phi_n$ called a *propositional context*, we write $[\Gamma \mid \Phi]$ for the intersection of extensions of each formula in Φ:

$$[\Gamma \mid \Phi] \coloneqq \bigcap_{\phi \in \Phi} [\Gamma \mid \phi]$$

Then, Φ *implies* φ, written $\Gamma \mid \Phi \vdash \varphi$ if $[\Gamma \mid \Phi] \subseteq [\Gamma \mid \varphi]$, where Γ contains precisely the free variables of φ.

15 Topos Logic

We have been directly manipulating terms and extensions above. We package some of these together as a *sequent*; an expression of the form

$$\Gamma \mid \Phi \vdash \varphi$$

where Γ is a variable context (often omitted), Φ is a propositional context, and φ is a formula, representing the claim that the formulae in Φ imply φ in context Γ.

There is a system of sound inference rules for these sequents that make them easier to manipulate than working with terms and extensions directly [Str03; McL92]. This system of rules is called *topos logic*. The exact rules are given later, but the point is that essentially, they are almost identical to that of ordinary inference rules for predicate logic: under these rules, extensions behave almost identically to subsets, and sequents behave almost identically to ordinary predicate sequent calculus.

For instance [McL92], this topos-logic proof tree shows $\neg(\varphi \vee \psi) \vdash \neg\varphi$:

$$\dfrac{\dfrac{\dfrac{\varnothing}{\neg(\varphi \vee \psi), (\varphi \vee \psi), \varphi \vdash \bot} \qquad \dfrac{\dfrac{\varnothing}{\varphi \vdash \varphi \vee \psi}}{\neg(\varphi \vee \psi), \varphi \vdash \varphi \vee \psi}}{\neg(\varphi \vee \psi), \varphi \vdash \bot}}{\neg(\varphi \vee \psi) \vdash \neg\varphi}$$

from which (by repeating the argument mutatis mutandis for ψ) we can derive the de Morgan law $\vdash \neg(\varphi \vee \psi) \to (\neg\varphi \wedge \neg\psi)$; the proof is effectively identical to that of an ordinary proof tree.

The main differences in topos logic from ordinary set theory are that:

- Sets are *local*; variables are tied to a single sets, as are quantifiers. (On the other hand, we gain the ability to quantify over truth values.)
- In general, $\varphi \vee \neg\varphi$ and $\neg\neg\varphi \to \varphi$ are not provable – topos logic is, by default, constructive.

15.1 Structural Rules

$$\frac{}{\Gamma \mid \varnothing \vdash \top}\,(\text{True}) \qquad \frac{}{\Gamma \mid \bot \vdash \varphi}\,(\text{False})$$

$$\frac{\Gamma \vdash \varphi : \Omega}{\Gamma \mid \varphi \vdash \varphi}\,(\text{Axiom}) \qquad \frac{\Gamma \mid \Phi \vdash \varphi}{\Gamma \mid \Phi,\psi \vdash \varphi}\,(\text{Weak})$$

$$\frac{\Gamma \mid \Phi,\varphi,\varphi \vdash \psi}{\Gamma \mid \Phi,\varphi \vdash \psi}\,(\text{Contr}) \qquad \frac{\Gamma \mid \Phi_1,\varphi_1,\varphi_2,\Phi_2 \vdash \psi}{\Gamma \mid \Phi_1,\varphi_2,\varphi_1,\Phi_2 \vdash \psi}\,(\text{Perm})$$

$$\frac{\Gamma \mid \Phi \vdash \varphi}{\Gamma[x/s] \mid \Phi[x/s] \vdash \varphi[x/s]}\,(\text{Subst})$$

for any term s free for x in all the formulae of Φ

$$\frac{\Gamma \mid \Phi \vdash \varphi \qquad \Delta \mid \Phi,\varphi \vdash \psi}{\Gamma,\Delta \mid \Phi \vdash \psi}\,(\text{Cut})$$

given that every variable free in φ is also free in Φ or ψ

15.2 Logical Rules

$$\frac{}{\Gamma \mid \Phi \vdash \top}\,(\top) \qquad \frac{\Gamma \mid \Phi \vdash \varphi, \qquad \Gamma \mid \Phi \vdash \psi}{\Gamma \mid \Phi \vdash \varphi \wedge \psi}\,(\wedge\text{I})$$

$$\frac{\Gamma \mid \Phi \vdash \varphi_1 \wedge \varphi_2}{\Gamma \mid \Phi \vdash \varphi_1}\,(\wedge_1\text{E}) \qquad \frac{\Gamma \mid \Phi \vdash \varphi_1 \wedge \varphi_2}{\Gamma \mid \Phi \vdash \varphi_2}\,(\wedge_2\text{E})$$

$$\frac{\Gamma \mid \Phi,\varphi \vdash \psi}{\Gamma \mid \Phi \vdash \varphi \to \psi}\,(\to\text{I}) \qquad \frac{\Gamma \mid \Phi \vdash \varphi \to \psi \qquad \Gamma \mid \Phi \vdash \varphi}{\Gamma \mid \Phi \vdash \psi}\,(\to\text{E})$$

$$\frac{\Gamma,x : X \mid \Phi \vdash \varphi(x)}{\Gamma \mid \Phi \vdash \forall x{:}X.\varphi(x)}\,(\forall\text{I}) \qquad \frac{\Gamma \mid \Phi \vdash \forall x{:}X.\varphi(x)}{\Gamma,x : X \mid \Phi \vdash \varphi(x)}\,(\forall\text{E})$$

Derived rules for $\bot$, $\vee$, $\exists$:

$$\frac{\Gamma \mid \Phi,\varphi \vdash \bot}{\Gamma \mid \Phi \vdash \neg\varphi}\,(\neg\text{I}) \qquad \frac{\Gamma \mid \Phi \vdash \neg\varphi}{\Gamma \mid \Phi,\varphi \vdash \bot}\,(\neg\text{E})$$

$$\frac{\Gamma \mid \Phi \vdash \varphi_1}{\Gamma \mid \Phi \vdash \varphi_1 \vee \varphi_2}\,(\vee_1\text{I}) \qquad \frac{\Gamma \mid \Phi \vdash \varphi_2}{\Gamma \mid \Phi \vdash \varphi_1 \vee \varphi_2}\,(\vee_2\text{I})$$

$$\frac{\Gamma \mid \Phi \vdash \varphi_1 \vee \varphi_2 \qquad \Gamma \mid \Phi,\varphi_1 \vdash \psi \qquad \Gamma \mid \Phi,\varphi_2 \vdash \psi}{\Gamma \mid \Phi \vdash \psi}\,(\vee\text{E})$$

$$\frac{\Gamma \mid \Phi \vdash \varphi(t)}{\Gamma \mid \Phi \vdash \exists x{:}X.\varphi(x)}\,(\exists\text{I}), \qquad \frac{\Gamma \mid \Phi \vdash \exists x{:}X.\varphi(x) \qquad \Gamma,x{:}X \mid \Phi,\varphi(x) \vdash \psi}{\Gamma \mid \Phi \vdash \psi}\,(\exists\text{E})$$

15.3 Soundness

Here, we prove some lemmata that imply the soundness of topos logic, following [McL92].

Firstly, variables that are not free in a term do not affect its interpretation, apart from possibly modifying the domain:

Lemma 15.1 (Superfluous Variables). *Let $s : X$ be any term, and let $\Gamma \equiv a_1 : A_1 \ldots, a_n : A_n$ be a context including all the variables free in s. Let $\Delta \equiv b_1 : B_1, \ldots, b_k : B_k$ be another context including all the variables in Γ. Then, $[\![\Delta \vdash s]\!]$ is given by:*

$$\prod_i B_i \xrightarrow{p} \prod_j A_j \xrightarrow{[\![\Gamma \vdash s]\!]} X$$

where p is the product pairing of projections of $[\![\Delta]\!]$ onto the relevant factors that are in $[\![\Gamma]\!]$.

Proof. This is clear from all of our term formation rules, excepts perhaps lambda abstraction. For terms $\lambda x.s$, the projection of $\prod_i B_i \times X \to \prod_j A_j \times X$ is given by $p \times \mathrm{id}_X$, and the transpose of $[\![s]\!](p \times \mathrm{id}_X)$ is given by $[\![s]\!]^\flat p$, which is of the required form. ■

So, in a larger context Δ, the interpretation of φ may be, at first, appear to be more complex due to the presence of extra variables in Δ, but this projection ensures that the interpretation in Δ in fact restricts to the interpretation of Γ along p. That is, only the relevant variables from Γ affect the interpretation.

Corollary 15.1.1. *For any formula φ and contexts Γ and Δ as above, the extension $[\Delta \mid \varphi]$ is the pullback of $[\Gamma \mid \varphi]$ along p. Thus, if $\Gamma \mid \Phi \vdash \varphi$ holds, then $[\Delta \mid \Phi] \subseteq [\Delta \mid \varphi]$ for every context Δ including all the variables free in both Φ and φ.*

Recall that $s[t/v]$ is the syntactic substitution of a term $t : X$ in place of every free occurence of a variable $v : X$ in the term s, and that t is free for v in s if no free variables of t become bound in the resulting term. We write $s[t_i/v_i]_{i=1}^n$ or $s[t_1/v_1, \ldots, t_n/v_n]$ for the (simultaneous) substitution of each of the t_i for v_i.

Lemma 15.2 (Substitution Lemma). *Let $s : X$ be any term, and let $\Gamma \equiv a_1 : A_1 \ldots, a_n : A_n$ be a context including all the variables free in s. For each variable $v_i \in \Gamma$, let t_i be a term free for v_i in s. Let $\Delta = b_1 : B_1, \ldots, b_k : B_k$ be another context containing all the variables free in $s[t_i/v_i]_{i=1}^n$. Then, $\left[\!\left[\Delta \vdash s[t_i/v_i]_{i=1}^n\right]\!\right]$ is given by:*

$$\prod_i B_i \xrightarrow{[\![\Delta \vdash \langle t_1, \ldots, t_n \rangle]\!]} \prod_j A_j \xrightarrow{[\![\Gamma \vdash s]\!]} X$$

Proof. For s a variable, this is trivial. Otherwise, this holds through identical reasoning to that of the superfluous variable lemma. ■

Corollary 15.2.1. *The extension $\left[\Delta \mid \varphi[t_i/v_i]_{i=1}^n\right]$ is the pullback of $[\Gamma \mid \varphi]$ along the product pairing $[\![\Delta \vdash \langle t_1, \ldots, t_n \rangle]\!]$. Because this also holds for any finite set of formulae, if $[\Gamma \mid \Phi] \subseteq [\Gamma \mid \varphi]$, then $\left[\Delta \mid \Phi[t_i/v_i]_{i=1}^n\right] \subseteq \left[\Delta \mid \varphi[t_i/v_i]_{i=1}^n\right]$.*

Having transferred to extensions, the soundness of structural rules now follow by basic properties of inclusion and intersection. Contraction, Permutation, and Weakening follow quickly from the construction of extensions, while the previous corollary validates Substitution, etc. We also have that $[\Gamma \mid \Phi] \subseteq [\Gamma \mid \varphi]$ implies that $[\Gamma \mid \Phi, \varphi] = [\Gamma \mid \Phi]$, which validates Cut elimination.

The rules for $\wedge$ are sound since $[\Gamma \mid \Phi] \subseteq [\Gamma \mid \varphi]$ and $[\Gamma \mid \Phi] \subseteq [\Gamma \mid \psi]$ if and only if $[\Gamma \mid \Phi] \subseteq [\Gamma \mid \varphi] \cap [\Gamma \mid \varphi]$; implication is sound by Theorem 13.4; and the $\forall$ rules are sound by Theorem 13.5, since by the superfluous variable theorem, $[\Gamma, x : X \mid \Phi] = [\Gamma \mid \Phi] \times X$ if and only if $x : X$ is not free in Γ. Finally, the rules for $\neg$, $\vee$, and $\exists$ are derived from sound rules, and are thus sound.

V

Resolution

Beyond its role in topos theory, constructive logic is also fundamental in computable mathematics, where it provides a rigourous framework for reasoning about computable functions, real numbers, differential equations, etc. In computable analysis for instance, our earlier implementation of real numbers as Cauchy sequences with specified moduli of convergence closely resembles how real numbers are handled in numerical computation: a real number is not an single fixed object, but rather an incomplete process that yields approximations to arbitrary precision. This constructive perspective has been particularly influential in fields like optimisation and numerical methods, where ensuring the correctness of approximations is critical.

Constructive logic has also become central in *proof assistants* and *formal verification*, which are increasingly important in both pure mathematics and computer science. Systems like Coq, Agda, and Lean rely on constructive type theory as their foundation, allowing for the mechanical formalisation of mathematical proofs in a way that ensures computational realisability. This connection between constructive logic and type theory has also led to the development of *homotopy type theory* (HoTT), which is rapidly becoming one of the most active areas of research in mathematics and theoretical computer science.

Constructive logic is not merely a restriction of classical reasoning, nor a vague philosophical position, but rather, a powerful tool that reveals new structures, enables computational interpretations of mathematical theorems, and even provides alternative foundations for modern mathematics in the form of type theory.

Whether in the study of algebraic geometry and topos theory, computability theory and numerical functions, or proof assistants and formal program verification, constructive methods continue to influence mathematics in a variety of directions, both theoretical and applied.

* * *

Class. Well, I must admit, this journey through constructive mathematics has been both more enlightening – and more extensive – than I had expected – from the early philosophical concerns, all the way to modern applications in topos theory.

Int. I think the real value of constructive mathematics lies not in replacing classical results as the original constructivists aimed to do, but in providing a different lens through which we can understand them. And even for those who do not subscribe to a constructive philosophy, working constructively is a practical matter in some of these areas, like topos theory.

Class. That does leave me with one lingering question: if constructivism offers all this insight – even within classical mathematics – why is it still so often met with resistance?

Int. It is early days yet; the internal language is still a relatively new concept, and only recently have more powerful results been proved using this technique. With the development of type theory, constructive reasoning is slowly gaining ground, but it remains far from being a widespread area of study in mathematics. Perhaps we are merely waiting for another Bishop to complete the next great constructive work.

But most of all, constructivism demands a shift in perspective – and shifts in perspective are rarely easy. Mathematics has never been static. Each new idea – negative numbers, complex numbers, non-Euclidean geometry – was once seen as radical before becoming indispensable. Perhaps one day, constructivism will not be seen as an alternative approach, but simply as a natural part of mathematics.

Class. Perhaps. At the very least, I can no longer dismiss it outright. There is much to reconsider.

Int. Good. That is, after all, where all mathematics begins.

References

[Lak63] Lakatos, Imre. *Proofs and Refutations*. Nelson London, 1963.

[Hey66] Heyting, Arend. *Intuitionism: An Introduction*. Vol. 41. Elsevier, 1966.

[Lin72] Linderholm, Carl E. *Mathematics made Difficult*. World Publishing, 1972.

[Loo13] Loomis, Lynn H. *Introduction to Abstract Harmonic Analysis*. Courier Corporation, 2013.

[Nor] Norwood, Zach. *Tychonoff's Theorem Implies AC*.

[PB22] Pradic, Cécilia and Brown, Chad E. *Cantor-Bernstein implies Excluded Middle*. 2022. arXiv: 1904.09193 [math.LO]. URL: https://arxiv.org/abs/1904.09193.

[Tro+11] Troelstra, Anne Sjerp et al. "History of constructivism in the 20th century". In: *Set Theory, Arithmetic, and Foundations of Mathematics* (2011), pp. 150–179.

[MML18] MARTINO., Enrico, Martino, Enrico, and Lue. *Intuitionistic Proof versus Classical Truth*. Springer, 2018.

[VS17] Van Atten, Mark and Sundholm, Göran. "LEJ Brouwer's 'Unreliability of the Logical Principles': A New Translation, with an Introduction". In: *History and Philosophy of Logic* 38.1 (2017), pp. 24–47.

[Dav05] Davies, E. Brian. "A Defence of Mathematical Pluralism". In: *Philosophia Mathematica* 13.3 (2005), pp. 252–276. DOI: 10.1093/philmat/nki017.

[Mad92] Maddy, Penelope. *Realism in Mathematics*. Oxford University Press, 1992.

[Bis67] Bishop, Errett. *Foundations of Constructive Analysis*. 1967.

[TD88] Troelstra, A.S. and Dalen, D. van. *Constructivism in Mathematics, Vol 1*. Studies in Logic and the Foundations of Mathematics. Elsevier Science, 1988. ISBN: 9780080570884.

[Bau17] Bauer, Andrej. "Five stages of accepting constructive mathematics". In: *Bulletin of the American Mathematical Society* 54.3 (2017), pp. 481–498.

[HB34] Hilbert, David and Bernays, Paul. *Foundations of Mathematics Vol. 1*. 1934.

[BV07] Bridges, Douglas S and Vita, Luminita Simona. *Techniques of Constructive Analysis*. Springer Science & Business Media, 2007.

[Van90] Van Stigt, Walter P. *Brouwer's Intuitionism*. North-Holland Amsterdam, 1990.

[MV20] Moschovakis, Joan R. and Vafeiadou, Garyfallia. *Intuitionistic Mathematics and Logic*. 2020. arXiv: 2003.01935 [math.LO]. URL: https://arxiv.org/abs/2003.01935.

[Ves79] Vesley, Richard. "Choice sequences. A chapter of intuitionistic mathematics. Oxford logic guides. Clarendon Press, Oxford1977, ix+ 170 pp." In: *The Journal of Symbolic Logic* 44.2 (1979), pp. 275–276.

[Tro+96] Troelstra, Anne Sjerp et al. "Choice sequences: a retrospect". In: *CWI Quarterly* 9 (1996), pp. 143–149.

[TD14] Troelstra, A.S. and Dalen, D. van. *Constructivism in Mathematics, Vol 2*. Studies in Logic and the Foundations of Mathematics. Elsevier Science, 2014. ISBN: 9780080955100.

[BR87] Bridges, Douglas and Richman, Fred. *Varieties of Constructive Mathematics*. Vol. 97. Cambridge University Press, 1987.

[Wei95] Weihrauch, Klaus. *A Simple Introduction to Computable Analysis*. Fernuniv., Fachbereich Informatik, 1995.

[Bra16] Brattka, Vasco. "Computability and Analysis, a Historical Approach". In: *Pursuit of the Universal*. Springer International Publishing, 2016, pp. 45–57. ISBN: 9783319401898. DOI: 10.1007/978-3-319-40189-8_5.

[Coq04] Coquand, Thierry. "About Brouwer's fan theorem". In: *Revue internationale de philosophie* 230 (2004), pp. 483–489.

[Dum00] Dummett, Michael. *Elements of Intuitionism.* Vol. 39. Oxford University Press, 2000.

[Van04] Van Atten, Mark. *On Brouwer.* Wadsworth, 2004.

[Vel06] Veldman, Wim. *Brouwer's Real Thesis on Bars.* CS 6. Université Nancy 2, 2006.

[Bee12] Beeson, Michael J. *Foundations of Constructive Mathematics: Metamathematical Studies.* Vol. 6. Springer Science & Business Media, 2012.

[Bab19] Babb, Ryan Acosta. *All Functions are Continuous! A provocative introduction to constructive analysis.* University of Warwick. 2019.

[Fra20] Frank, Matthew. "Interpolating between choices for the approximate intermediate value theorem". In: *Logical Methods in Computer Science* 16 (2020).

[Liu23] Liu, Kit. *Structural Foundations in Topoi.* University of Warwick. 2023.

[Bau11] Bauer, Andrej. *An injection from N^N to N.* 2011.

[BH24] Bauer, Andrej and Hanson, James E. *The Countable Reals.* 2024. arXiv: `2404.01256 [math.LO]`. URL: `https://arxiv.org/abs/2404.01256`.

[Ble21] Blechschmidt, Ingo. *Using the internal language of toposes in algebraic geometry.* 2021. arXiv: `2111.03685 [math.AG]`. URL: `https://arxiv.org/abs/2111.03685`.

[Sou21a] Southwell, Richard. *Internal Language of a Topos (Mitchell-Bénabou Language).* 2021. URL: `https://www.youtube.com/watch?v=66glvnHX-a8`.

[McL92] McLarty, Colin. *Elementary Categories, Elementary Toposes.* Clarendon Press, 1992.

[Str03] Streicher, Thomas. "Category Theory and Categorical Logic". In: *Lecture Notes, Technische Universität Darmstadt.* (2003).

[MM12] MacLane, S. and Moerdijk, I. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Springer New York, 2012. ISBN: 9781461209270.

[Sou21b] Southwell, Richard. *Topos Theory Essentials.* 2021. URL: `https://www.youtube.com/watch?v=t7tLRUI1E-E`.

[Iem24] Iemhoff, Rosalie. "Intuitionism in the Philosophy of Mathematics". In: *The Stanford Encyclopedia of Philosophy.* Ed. by Zalta, Edward N. and Nodelman, Uri. Summer 2024. Metaphysics Research Lab, Stanford University, 2024.

[Mil20] Mileti, Joseph R. *A Mathematical Introduction to Mathematical Logic.* 2020.

[FH06] Fourman, Michael P and Hyland, J Martin E. "Sheaf models for analysis". In: *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9–21, 1977.* Springer. 2006, pp. 280–301.

[FS06] Fourman, Michael P and Scott, Dana S. "Sheaves and logic". In: *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9–21, 1977.* Springer. 2006, pp. 302–401.

[Gra06] Grayson, Robin J. "Heyting-valued models for intuitionistic set theory". In: *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9–21, 1977.* Springer. 2006, pp. 402–414.

[Lei24] Leinster, Tom. *Axiomatic Set Theory.* University of Edinburgh. 2024.

[Hin97] Hindley, J Roger. *Basic Simple Type Theory.* 42. Cambridge University Press, 1997.

[Mit72] Mitchell, William. "Boolean topoi and the theory of sets". In: *Journal of Pure and Applied Algebra* 2.3 (1972), pp. 261–274.

[Ber21] Berger, Josef. *Brouwer's fan theorem.* 2021. arXiv: `2001.00064 [math.LO]`. URL: `https://arxiv.org/abs/2001.00064`.

[Jac99] Jacobs, Bart. *Categorical Logic and Type Theory.* Vol. 141. Elsevier, 1999.

[Shu16] Shulman, Michael. "Categorical logic from a categorical point of view". In: *Available on the web* 19 (2016), pp. 20–22.

[Bri99] Bridges, Douglas S. "Constructive mathematics: a foundation for computable analysis". In: *Theoretical computer science* 219.1-2 (1999), pp. 95–109.

[Som11] Sommaruga, Giovanni. *Foundational Theories of Classical and Constructive Mathematics*. Vol. 76. Springer Science & Business Media, 2011.

[Hen12] Hendtlass, Matthew. "The intermediate value theorem in constructive mathematics without choice". In: *Annals of Pure and Applied Logic* 163.8 (2012). Continuity, Computability, Constructivity: From Logic to Algorithms, pp. 1050–1056. ISSN: 0168-0072. DOI: `https://doi.org/10.1016/j.apal.2011.12.026`.

[LS88] Lambek, Joachim and Scott, Philip J. *Introduction to Higher-order Categorical Logic*. Vol. 7. Cambridge University Press, 1988.

[Bel14] Bell, John Lane. *Intuitionistic Set Theory*. Vol. 50. College Publications London, 2014.

[Acz11] Aczel, Peter. *Local Constructive Set Theory and Inductive Definitions*. Springer, 2011.

[SU06] Sørensen, Morten Heine and Urzyczyn, Pawel. *Lectures on the Curry-Howard Isomorphism*. Vol. 149. Elsevier, 2006.

[Avi22] Avigad, Jeremy. *Mathematical Logic and Computation*. Cambridge University Press, 2022.

[VV94] Van Dalen, Dirk and Van Dalen, Dirk. *Logic and structure*. Vol. 3. Springer, 1994.

[Tro73] Troelstra, Anne Sjerp. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*. Springer, 1973.

[Van05] Van Dalen, Dirk. *Mystic, Geometer, and Intuitionist: The Life of LEJ Brouwer*. Vol. 2. Oxford University Press, 2005.

[Bel] Bell, John L. *Notes on toposes and local set theories*.

[Coq14] Coquand, Thierry. "Recursive Functions and Constructive Mathematics". In: *Constructivity and computability in historical and philosophical perspective* (2014), pp. 159–167.

[Hyl82] Hyland, J Martin E. "The effective topos". In: *Studies in Logic and the Foundations of Mathematics*. Vol. 110. Elsevier, 1982, pp. 165–216.

[Car14] Caramello, Olivia. "Topos-theoretic background". In: (2014). URL: `http://www.oliviacaramello.com/Unification/ToposTheoreticPreliminariesOliviaCaramello.pdf`.

[Bel08] Bell, John L. *Toposes and Local Set Theories: An Introduction*. Courier Corporation, 2008.

[BW00] Barr, Michael and Wells, Charles. *Toposes, Triples, and Theories*. Springer-Verlag, 2000.

[BMW85] Barr, Michael, Mclarty, Colin, and Wells, Charles. "Variable set theory". In: *Unpublished manuscript* (1985).

[Gol84] Goldblatt, Robert. *Topoi: The Categorial Analysis of Logic*. Elsevier, 1984. ISBN: 9780444867117.

[Kos12] Kostecki, Ryszard. *An Introduction to Topos Theory*. 2012.

[Lei11] Leinster, Tom. *An informal introduction to topos theory*. 2011. arXiv: `1012.5647 [math.CT]`.

[Str04] Streicher, Thomas. *Universes in Toposes*. Clarendon Press, 2004. ISBN: 9780198566519.