Lab Material for Probability Lecture:

1. Consider two medical tests, A and B, for a virus. Test A is 95% effective at recognizing the virus when it is present, but has a 10% false positive rate (indicating that the virus is present, when it is not). Test B is 90% effective at recognizing the virus, but has a 5% false positive rate. The two tests use independent methods of identifying the virus. The virus is carried by 1% of all people. Say that a person is tested for the virus using only one of the tests, and that test comes back positive for carrying the virus. Which test returning positive is more indicative of someone really carrying the virus? Justify your answer mathematically.

   P(Test_A= Positive | Virus= True)= 0.95
   P(Test_A= Positive | Virus= False)= 0.10
   P(Test_B= Positive | Virus= True)= 0.90
   P(Test_B= Positive | Virus= False)= 0.05
   P(Virus= True)= 0.01

$$P(\text{Virus} = \text{True} \mid \text{Test}_A = \text{Positive}) = \frac{P(\text{Test\_A} = P \mid \text{Virus} = T) \times P(\text{Virus} = T)}{P(\text{Test\_A} = P \mid \text{Virus} = T) \times P(\text{Virus} = T) + P(\text{Test\_A} = P \mid \text{Virus} = F) \times P(\text{Virus} = F)}$$

$$= \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.10 \times 0.99} = 0.0876$$

$$P(\text{Virus} = \text{True} \mid \text{Test}_B = \text{Positive}) = \frac{P(\text{Test\_B} = P \mid \text{Virus} = T) \times P(\text{Virus} = T)}{P(\text{Test\_B} = P \mid \text{Virus} = T) \times P(\text{Virus} = T) + P(\text{Test\_B} = P \mid \text{Virus} = F) \times P(\text{Virus} = F)}$$

$$= \frac{0.90 \times 0.01}{0.90 \times 0.01 + 0.05 \times 0.99} = 0.1538$$

Test B is more indicative because $P(\text{Virus} = \text{True} \mid \text{Test}_B = \text{Positive}) > P(\text{Virus} = \text{True} \mid \text{Test}_A = \text{Positive})$

2. After your yearly checkup, the doctor has bad news and good news. The bad news is that you tested positive for a serious disease and that the test is 99% accurate (i.e., the probability of testing positive when you do have the disease is 0.99, as is the probability of testing negative when you don't have the disease). The good news is that this is a rare disease, striking only 1 in 10,000 people of your age. Why is it good news that the disease is rare? What are the chances that you actually have the disease?

P(Test= Positive | Disease = True)= 0.99

P(Test= Negative | Disease = False)= 0.99

P(Disease = True)= 0.0001

$$P(\text{Disease} \ = \ \text{True} \mid \text{Test} = \text{Positive})$$
$$= \frac{P(\text{Test} = \ P \mid \text{Disease} \ = \ T) \times P(\text{Disease} \ = \ T)}{P(\text{Test} = \ P \mid \text{Disease} \ = \ T) \times P(\text{Disease} \ = \ T) + P(\text{Test} = \ P \mid \text{Disease} \ = \ F) \times P(\text{Disease} \ = \ F)}$$
$$= \frac{0.99 \times 0.0001}{0.99 \times 0.0001 + 0.01 \times 0.9999} = 0.0098$$

Its good news because now your probability of actually having it has decreased to 0.0098!!!

3. Would it be rational for an agent to hold the three beliefs P(A)=0.4, P(B)=0.3, and P(A∨B)=0.5? If so, what range of probabilities would be rational for the agent to hold for A∧B?

P(A∨B)=P(A)+P(B)-P(A∧B) → 0.5= 0.4+0.3- P(A∧B) → P(A∧B)= 0.7-0.5=0.2 which is possible since its between 0 and 1.

4. Would it be rational for an agent to hold the three beliefs P(A)=0.4, P(B)=0.3, and P(A∨B)=0.7? What type of relationship holds between A and B?

P(A∨B)=P(A)+P(B)-P(A∧B) → 0.7= 0.4+0.3- P(A∧B) → P(A∧B)= 0.7-0.7=0, indicating that A and B are mutually exclusive.

5. Given the full joint distribution shown bellow, calculate the following:

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

- P(*toothache*)
- P(*cavity*)
- P(*toothache* ∧ *cavity*)
- P(*toothache* ∨ *cavity*)
- P(*toothache* | *cavity*)
- P(*cavity* | *toothache* , *catch*)
- P(*toothache* | *cavity* , *catch*)
- Is *toothache* conditionally independent of *catch* given *cavity?* In other words: Does knowing that there is a *catch* have any effect on having *toothache,* when you that there is *cavity*? Justify your answer mathematically.

- P(*toothache*) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2
- P(*cavity*) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2
- P(*toothache* ∧ *cavity*) = 0.108 + 0.012 = 0.12
- P(*cavity* ∨ *toothache*)= P(*toothache*) + P(*cavity*) - P(toothache ∧ *cavity*) = 0.2 + 0.2 – 0.12= 0.28
- P(*toothache* | *cavity*)= P(*toothache* ∧ *cavity*)/ P(*cavity*) = 0.12/0.2 = 0.6
- P(*cavity* | *toothache, catch*) = P(*toothache, catch, cavity*) / P(*toothache, catch*)= 0.108/(0.108+0.016)= 0.871
- P(*toothache* | *cavity, catch*) = P(*toothache, catch, cavity*) / P(*catch, cavity*)= 0.108/(0.108+0.072)= 0.6
  So P(*toothache* | *catch, cavity*) = P(*toothache* | *cavity*) meaning that *toothache* is conditionally independent of *catch* given *cavity*.

Note: P(*toothache* | *catch*, *cavity*) = P(*toothache* | *catch*): *toothache* is conditionally independent *cavity* of given *catch.*

Hints:
A and B are independent: P(A| B)= P(A) ,
A and B are independent: P(A, B)= P(A)P(B),
A and B are independent given C: P(A| B, C)= P(A| C)