Lesson 4: Stationary stochastic processes

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Stationarity is a rather intuitive concept, it means that the **statistical properties of the process do not change over time**.

Stationary stochastic processes

There are two important forms of stationarity:

1. strong stationarity;
2. weak stationarity.

Stationary stochastic processes

Strong stationarity concerns the shift-invariance (in time) of its finite-dimensional distributions.

Weak stationarity only concerns the shift-invariance (in time) of first and second moments of a process.

Definition. The process $\{x_t; t \in \mathbb{Z}\}$ is **strongly stationary** if

$$F_{t_1+k, t_2+k, \cdots, t_s+k}(b_1, b_2, \cdots, b_s) = F_{t_1, t_2, \cdots, t_s}(b_1, b_2, \cdots, b_s)$$

for any finite set of indices $\{t_1, t_2, \cdots, t_s\} \subset \mathbb{Z}$ with $s \in \mathbb{Z}^+$, and any $k \in \mathbb{Z}$.

Thus the process $\{x_t; t \in \mathbb{Z}\}$ is strongly stationary if the joint distibution function of the vector $(x_{t_1+k}, x_{t_2+k}, ..., x_{t_s+k})$ is equal with the one of $(x_{t_1}, x_{t_2}, ..., x_{t_s})$ for for any finite set of indices $\{t_1, t_2, \cdots, t_s\} \subset \mathbb{Z}$ with $s \in \mathbb{Z}^+$, and any $k \in \mathbb{Z}$.

The meaning of the strongly stationarity is that the distribution of a number of random variables of the stochastic process is the same as we shift them along the time index axis.

Strongly stationary stochastic processes

If $\{x_t; t \in \mathbb{Z}\}$ is a strongly stationary process, then

$$x_1,\ x_2,\ x_3, \ldots$$

have the same distribution function.

$$(x_1, x_3),\ (x_5, x_7),\ (x_9, x_{11}), \ldots..$$

have the same joint distribution function and further

$$(x_1, x_3, x_5),\ (x_7, x_9, x_{11}),\ (x_{13}, x_{15}, x_{17}), \ldots$$

must have the same joint distribution function, and so on.

Strongly stationary stochastic processes

If the process is $\{x_t; t \in \mathbb{Z}\}$ is strongly stationary, then the joint probability distribution function of $(x_{t_1}, x_{t_2}, ..., x_{t_s})$ is invariant under translation.

iid process

An *iid* process is a strongly stationary process. This follows almost immediate from the definition.

Since the random variables $x_{t_1+k}, x_{t_2+k}, ..., x_{t_s+k}$ are *iid*, we have that

$$F_{t_1+k, t_2+k, \cdots, t_s+k}(b_1, b_2, \cdots, b_s) = F(b_1)F(b_2)\cdots F(b_s)$$

On the other hand, also the random variables $x_{t_1}, x_{t_2}, ..., x_{t_s}$ are *iid* and hence

$$F_{t_1, t_2, \cdots, t_s}(b_1, b_2, \cdots, b_s) = F(b_1)F(b_2)\cdots F(b_s).$$

We can conclude that

$$F_{t_1+k, t_2+k, \cdots, t_s+k}(b_1, b_2, \cdots, b_s) = F_{t_1, t_2, \cdots, t_s}(b_1, b_2, \cdots, b_s)$$

iid process

Remark. Let $\{x_t; t \in \mathbb{Z}\}$ be an *iid* process. We have that the conditional distribution of x_{T+h} given values of $(x_1, ..., x_T)$ is

$$P(x_{T+h} \leq b | x_1, ..., x_T) = P(x_{T+h} \leq b)$$

So the knowledge of the past has no value for predicting the future. An *iid* process is unpredictable.

Example. Under efficient capital market hypothesis, the stock price change is an *iid* process. This means that the stock price change is unpredictable from previous stock price changes.

Strongly stationary stochastic processes

Consider the discrete stochastic process

$$\{x_t; t \in \mathbb{N}\}$$

where $x_t = A$, with $A \sim U(3, 7)$ (A is uniformly distributed on the interval $[3, 7]$).

This process **is** of course strongly stationary.

Why?

Strongly stationary stochastic processes

Consider the discrete stochastic process

$$\{x_t; t \in \mathbb{N}\}$$

where $x_t = tA$, with $A \sim U(3,7)$

This process **is not** strongly stationary

Why?

Weakly stationary stochastic processes

If the second moment of x_t is finite for all t, then the mean $E(x_t)$, the variance $\mathrm{var}(x_t) = E[(x_t - E(x_t))^2] = E(x_t^2) - (E(x_t))^2$ and the covariance $\mathrm{cov}(x_{t_1}, x_{t_2}) = E[(x_{t_1} - E(x_{t_1}))(x_{t_2} - E(x_{t_2}))]$ are finite for all t, t_1 and t_2.

Why?

Hint: Use the Cauchy-Schwarz inequality
$|\mathrm{cov}(x, y)|^2 \leq \mathrm{var}(x)\mathrm{var}(y)$

Weakly stationary stochastic processes

Definition The process $\{x_t; t \in \mathbb{Z}\}$ is **weakly stationary**, or covariance-stationary if

1. the second moment of x_t is finite for all t, that is $E|x_t|^2 < \infty$ for all t
2. the first moment of x_t is independent of t, that is $E(x_t) = \mu \ \forall t$
3. the cross moment $E(x_{t_1} x_{t_2})$ depends only on $t_1 - t_2$, that is $\text{cov}(x_{t_1}, x_{t_2}) = \text{cov}(x_{t_1+h}, x_{t_2+h}) \ \forall t_1, t_2, h$

Weakly stationary stochastic processes

Thus a stochastic process is covariance-stationary if

1. it has the same mean value, μ, at all time points;
2. it has the same variance, γ_0, at all time points; and
3. the covariance between the values at any two time points, $t, t-k$, depend only on k, the difference between the two times, and not on the location of the points along the time axis.

Weakly stationary stochastic processes

An important example of covariance-stochastic process is the so-called white noise process.

Definition . A stochastic process $\{u_t; t \in \mathbb{Z}\}$ in which the random variables u_t, $t = 0 \pm 1, \pm 2...$ are such that

1. $E(u_t) = 0 \ \forall t$
2. $\text{Var}(u_t) = \sigma_u^2 < \infty \ \forall t$
3. $\text{Cov}(u_t, u_{t-k}) = 0 \ \forall t, \forall k$

is called **white noise** with mean 0 and variance σ_u^2, written $u_t \sim WN(0, \sigma_u^2)$.

First condition establishes that the expectation is always constant and equal to zero. Second condition establishes that variance is constant. Third condition establishes that the variables of the process are uncorrelated for all lags.

If the random variables u_t are independently and identically distributed with mean 0 and variance σ_u^2 then we will write

$$u_t \sim IID(0, \sigma_u^2)$$

White Noise process

Figure shows a possible realization of an $IID(0,1)$ process.

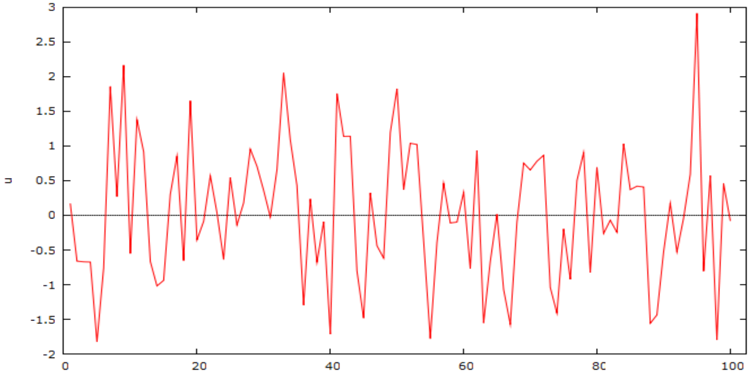

Figure : A realization of an $IID(0,1)$.

Random Walk process

An important example of weakly non-stationary stochastic processes is the following.
Let

$$\{y_t; t = 0, 1, 2, ...\}$$

be a stochastic processs where $y_0 = \delta < \infty$ and $y_t = y_{t-1} + u_t$ for $t = $ 1,2,..., with $u_t \sim WN(0, \sigma_u^2)$.

This process is called **random walk**.

Random Walk process

The mean of y_t is given by

$$E(y_t) = \delta$$

and its variance is

$$\mathrm{Var}(y_t) = t\sigma_u^2$$

Thus a **random walk** is not weakly stationary process.

Figure shows a possible realization of a random walk.

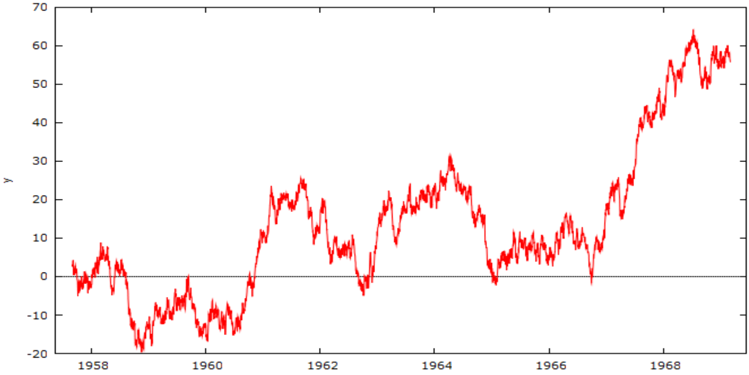

Figure : A realization of a random walk.

Relation between strong and weak Stationarity

First note that finite second moments are not assumed in the definition of strong stationarity, therefore, strong stationarity does not necessarily imply weak stationarity.

For example, an *iid* process with standard Cauchy distribution is strictly stationary but not weak stationary because the second moment of the process is not finite.

Relation between strong and weak Stationarity

If the process $\{x_t; t \in \mathbb{Z}\}$ is strongly stationary and has finite second moment, then $\{x_t; t \in \mathbb{Z}\}$ is weakly stationary.

PROOF. If the process $\{x_t; t \in \mathbb{Z}\}$ is strongly stationary, then

$$..., x_{-1},\ x_0,\ x_1, ...$$

have the same distribution function and

$$(x_{t_1}, x_{t_2}) \text{ and } (x_{t_1+h}, x_{t_2+h})$$

have the same joint distribution function for all t_1, and t_2 and h. Because, by hypothesis, the process $\{x_t; t \in \mathbb{Z}\}$ has finite second moment, this implies that

- $E(x_t) = \mu \ \ \forall t$
- $\text{cov}(x_{t_1}, x_{t_2}) = \text{cov}(x_{t_1+h}, x_{t_2+h}) \ \forall t_1, t_2, h$

Relation between strong and weak Stationarity

Of course a weakly stationary process is not necessarily strongly stationary.

weak stationarity $\not\Rightarrow$ strong stationarity

Relation between strong and weak Stationarity

Here we give an example of a weakly stationary stochastic process which is not strictly stationary.
Let $\{x_t; t \in \mathbb{Z}\}$ be a stochastic process defined by

$$x_t = \begin{cases} u_t & \text{if } t \text{ is even} \\ \frac{1}{\sqrt{2}}(u_t^2 - 1) & \text{if } t \text{ is odd} \end{cases}$$

where $u_t \sim iidN(0, 1)$.
This process is weakly stationary but it is not strictly stationary.

Relation between strong and weak Stationarity

We have

$$E(x_t) = \begin{cases} E(u_t) = 0 & \text{if } t \text{ is even} \\ \frac{1}{\sqrt{2}} E(u_t^2 - 1) = 0 & \text{if } t \text{ is odd} \end{cases}$$

and

$$\text{var}(x_t) = \begin{cases} \text{var}(u_t) = 1 & \text{if } t \text{ is even} \\ \frac{1}{2}\text{var}(u_{t-1}^2) = 1 & \text{if } t \text{ is odd} \end{cases}$$

Further, because x_t and x_{t-k} are independent random variables, we have

$$\text{cov}(x_t, x_{t-k}) = 0 \ \ \forall k$$

Thus, the process x_t is weakly stationary. In particular, $x_t \sim WN(0,1)$.

Relation between strong and weak Stationarity

Now, we note that

$$P(x_t \leq 0) = P(u_t \leq 0) = 0.5 \ \text{ for } \ t \ \text{ even}$$

and

$$\begin{aligned} P\left(x_t \leq 0\right) &= P\left(\frac{1}{\sqrt{2}}(u_t^2 - 1) \leq 0\right) \\ &= P\left(u_{t-1}^2 \leq 1\right) \\ &= P\left(|u_{t-1}| \leq 1\right) \\ &= P\left(-1 \leq u_{t-1} \leq 1\right) \\ &= 0.6826 \ \text{ for } \ t \ \text{ odd} \end{aligned}$$

Hence the random variables of the process are not identically distributed. This implies that the process is not strongly stationary

Relation between strong and weak Stationarity

There is one important case however in which weak stationarity implies strong stationarity.

If $\{x_t; t \in \mathbb{Z}\}$ is a weakly stationary Gaussian stochastic process, then $\{x_t; t \in \mathbb{Z}\}$ is strongly stationary.

Why?

Let $\{x_t; t \in \mathbb{Z}\}$ be a Gaussian stochastic process. Introducing the vector $\mathbf{b} = (b_1, b_2, ..., b_s)' \in \mathbb{R}^s$, the multidimensional density function of the vector $(x_{t_1}, x_{t_2}, ..., x_{t_s})$ is

$$f_{t_1,t_2,...,t_s}(\mathbf{b}) = \frac{1}{\sqrt{(2\pi)^s \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{b}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{b}-\boldsymbol{\mu})\right).$$

where $\boldsymbol{\mu} = (E(x_{t_1}), E(x_{t_2}), ..., E(t_s))'$ and $\boldsymbol{\Sigma} = \left[\mathrm{Cov}(x_{t_i}, x_{t_j})\right]$. We note that a multivariate Gaussian distribution is fully characterized by its first two moments.

Relation between strong and weak Stationarity

Let $\{x_t; t \in \mathbb{Z}\}$ be a Gaussian stochastic process. Assume that the process is **weakly** stationary. If the process i weakly stationary, then

1. $E(x_t) = \mu \;\; \forall t$
2. $\text{Var}(x_t) = \gamma_0 < \infty \;\; \forall t$
3. $\text{Cov}(x_{t_1+k}, x_{t_2+k}) = \text{Cov}(x_{t_1}, x_{t_2}) \;\; \forall t_1, t_2, \forall k$

and hence

$$f_{t_1+k, t_2+k, \dots, t_s+k}(\mathbf{b}) = f_{t_1, t_2, \dots, t_s}(\mathbf{b})$$

It follows that the joint distibution function of the vector $(x_{t_1+k}, x_{t_2+k}, \dots, x_{t_s+k})$ is equal with the one of $(x_{t_1}, x_{t_2}, \dots, x_{t_s})$ for for any finite set of indices $\{t_1, t_2, \cdots, t_s\} \subset \mathbb{Z}$ with $s \in \mathbb{Z}^+$, and any $k \in \mathbb{Z}$. This implies that the process $\{x_t; t \in \mathbb{Z}\}$ is **strongly** stationary.

Conversely, if $\{x_t; t \in \mathbb{Z}\}$ is a Gaussian strongly stationary stochastic process, then it is weakly stationary because it has finite variance.
Thus we can conclude that in the case of Gaussian stochastic process, the two definitions of stationarity are equivalent.

White Noise process

We note that a white noise process is not necessarily strongly stationary. Let w be a random variable uniformly distributed in the interval $(0; 2\pi)$. We consider the process $\{Z_t; t = 1, 2, ...\}$ defined by

$$Z_t = \cos(tw) \ \ t = 1, 2, ...$$

We have that

1. $E(Z_t) = 0 \ \ \forall t$
2. $\mathrm{Var}(Z_t) = \frac{1}{2} \ \ \forall t$
3. $\mathrm{Cov}(Z_t, Z_{t-k}) = 0 \ \ \forall t, \forall k$

Thus $Z_t \sim WN(0, .5)$. However, it can be shown that is not strongly stationary.