Teleparallel Equivalent of General Relativity (TEGR)

In the **Teleparallel Equivalent of General Relativity (TEGR)**, gravity is described using *torsion* instead of curvature. The formulation employs *tetrad (vierbein) fields* and the *Weitzenböck connection*, which has torsion but no curvature.

Key Mathematical Concepts

1. **Tetrad (Vierbein) Fields:**

$$g_{\mu\nu} = \eta_{ab}\, e^a_\mu\, e^b_\nu, \tag{1}$$

   where:

   - e^a_μ are the tetrad fields (basis of the tangent space),
   - $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric.

2. **Weitzenböck Connection:**

$$\Gamma^\lambda_{\mu\nu} = e^\lambda_a\, \partial_\mu e^a_\nu. \tag{2}$$

   This connection has *zero curvature* but *non-zero torsion*. The torsion tensor is given by:

$$T^\lambda_{\mu\nu} = e^\lambda_a(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu). \tag{3}$$

3. **Torsion Tensor:**

$$T^\lambda_{\mu\nu} = e^\lambda_a(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu). \tag{4}$$

4. **Gravitational Action in TEGR:**

$$T = S_\lambda^{\ \mu\nu} T^\lambda_{\mu\nu}, \tag{5}$$

   where $S_\lambda^{\ \mu\nu}$ is the *superpotential*:

$$S_\lambda^{\ \mu\nu} = \frac{1}{2}\left(K^{\mu\nu}_{\ \ \ \lambda} + \delta^\mu_\lambda\, T^{\alpha\nu}_{\ \ \ \alpha} - \delta^\nu_\lambda\, T^{\alpha\mu}_{\ \ \ \alpha}\right), \tag{6}$$

   and $K^{\mu\nu}_{\ \ \ \lambda}$ is the *contorsion tensor*:

$$K^{\mu\nu}_{\ \ \ \lambda} = -\frac{1}{2}\left(T^{\mu\nu}_{\ \ \ \lambda} - T^{\nu\mu}_{\ \ \ \lambda} - T_\lambda^{\ \mu\nu}\right). \tag{7}$$

5. **TEGR Field Equations:**

$$e\, G^{\lambda\nu} = 2e\left(\nabla_\lambda T^\lambda_{\ \nu} - \frac{1}{2} g_{\lambda\nu} T\right), \tag{8}$$

   where $e = \det(e^a_\mu)$, and $G^{\lambda\nu}$ is the Einstein tensor.

Elimination of Curvature and the Role of Torsion

- In GR, gravity is described by curvature via the Riemann curvature tensor, using the Levi-Civita connection (torsion-free).
- In TEGR, curvature is eliminated in favor of a flat connection with torsion, encoded in the torsion scalar T.

General Relativity (GR)

In **General Relativity (GR)**, gravity is described by the curvature of spacetime, which is encoded in the *metric tensor* and the *Levi-Civita connection.*

Key Mathematical Concepts

1. **Metric Tensor:**

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu. \tag{9}$$

The metric tensor $g_{\mu\nu}$ describes the geometry of spacetime.

2. **Levi-Civita Connection (Christoffel Symbols):**

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{10}$$

This connection is torsion-free and metric-compatible.

3. **Riemann Curvature Tensor:**

$$R^\lambda{}_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda_{\mu\sigma} - \partial_\sigma \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\sigma} - \Gamma^\lambda_{\sigma\rho}\Gamma^\rho_{\mu\nu}. \tag{11}$$

4. **Ricci Tensor and Ricci Scalar:**

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \tag{12}$$

5. **Gravitational Action in GR:**

$$S_{GR} = \frac{1}{2\kappa} \int d^4x\, \sqrt{-g}\, R + S_{\text{matter}}, \tag{13}$$

where $\kappa = 8\pi G$.

6. **Einstein Field Equations:**

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{14}$$

Comparison: GR vs. TEGR

Concept	**GR**	**TEGR**
Fundamental Object	Metric tensor $g_{\mu\nu}$	Tetrad fields e_μ^a
Connection	Levi-Civita connection (torsion-free)	Weitzenböck connection (flat, torsion)
Torsion	Zero (torsion-free)	Non-zero
Curvature	Non-zero (Riemann tensor)	Zero (Weitzenböck connection)
Geometrical Property	Curved spacetime geometry	Flat spacetime with torsion
Action	Einstein-Hilbert action $\int \sqrt{-g}R$	Teleparallel action $\int eT$
Field Equations	Einstein field equations	Derived from torsion scalar T