

# Teleparallel Gravity: Relativistic Gravity in Flat Spacetime

## Abstract

This presentation reviews the mathematical and conceptual framework of Teleparallel Gravity, a theory of gravity that is equivalent to General Relativity. In Teleparallelism, gravity is encoded in torsion instead of curvature, allowing for a description in terms of flat, Minkowski spacetime. This, in principle, allows for an approach to relativistic gravity that more closely parallels other familiar field theories, such as electromagnetism. We will explore conceptual foundations, highlighting the distinctions between changes of reference frame versus changes of coordinates, while also providing a physically transparent way to understand the theory. We illustrate these ideas with a visualizable model involving the stereographic projection of two-dimensional spherical geometry to a two-dimensional plane.

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# 1 Introduction

Teleparallel Gravity (TG) offers an alternative formulation to Einstein's General Relativity (GR), where gravity is not described by the curvature of spacetime but by torsion in a flat, Minkowski background.

## 1.1 Teleparallel Gravity vs. General Relativity

While GR uses the curvature of spacetime to describe gravity, TEG utilizes torsion in a flat spacetime. The key mathematical distinction lies in the connection used: GR relies on the Levi-Civita connection, which is torsion-free, while TG uses the Weitzenböck connection, which has zero curvature but non-zero torsion.

In GR, the Riemann curvature tensor  $R^\lambda_{\mu\nu\kappa}$  encodes the gravitational field:

$$R^\lambda_{\mu\nu\kappa} = \partial_\nu \Gamma^\lambda_{\mu\kappa} - \partial_\kappa \Gamma^\lambda_{\mu\nu} + \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta}$$

In contrast, TG replaces curvature with torsion  $T^\lambda_{\mu\nu}$ :

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu}$$

where  $\Gamma^\lambda_{\mu\nu}$  is the Weitzenböck connection. This torsion is directly related to the gravitational field in TG.

## 1.2 Significance of Flat Spacetime

Flat spacetime simplifies the conceptual framework for understanding gravity, allowing for a direct comparison with other field theories like electromagnetism. This section discusses how the lack of curvature in TG leads to simpler models and easier calculations in certain contexts, particularly when considering global properties of spacetime.

# 2 Foundations of Relativistic Mechanics and Inertial Frames

This section establishes the groundwork in relativistic mechanics, starting with the first law of motion and the concept of inertial frames.

## 2.1 Inertial and Non-Inertial Frames

An inertial frame is one where objects remain at rest or in uniform motion unless acted upon by a force. Non-inertial frames involve accelerations. Mathematically, we express the transformation between inertial frames using the Lorentz transformation:

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

where  $\Lambda^\mu_\nu$  is the Lorentz transformation matrix satisfying  $\Lambda^\mu_\alpha \Lambda^\nu_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$ , with  $\eta_{\mu\nu}$  being the Minkowski metric.

In non-inertial frames, additional fictitious forces, such as centrifugal force or Coriolis force, appear. These forces can be incorporated into the equations of motion via a time-dependent transformation or by introducing a general connection  $\Gamma^\lambda_{\mu\nu}$ .

## 2.2 Covariant Derivatives and Tensors

To describe physical laws in a covariant manner, we introduce the concept of covariant derivatives, which allow us to differentiate tensors while respecting the underlying geometric structure of spacetime. The covariant derivative of a vector  $V^\mu$  is given by:

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda$$

In TG, the connection  $\Gamma^\mu_{\nu\lambda}$  is the Weitzenböck connection, which preserves torsion but not curvature.

## 2.3 First Law of Motion and Covariant Description

Expanding upon the first law, we will explore its implications in relativistic mechanics, particularly how acceleration in a given frame can be described in a coordinate-independent way using covariant derivatives. The key equation here is the geodesic equation in TG:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

This equation governs the motion of particles in TG and is analogous to the geodesic equation in GR but with the torsion-based connection.

# 3 Covariant Formalism and Reference Frames

In this section, the covariant formalism is applied to the study of reference frames, distinguishing between changes of reference frame and coordinate transformations.

## 3.1 Reference Frames and Coordinates

While a change in reference frame involves transitioning between different observers' perspectives, a coordinate transformation merely re-labels points in spacetime. For instance, a boost in the  $x$ -direction is a change of reference frame:

$$t' = \gamma(t - \frac{vx}{c^2}), \quad x' = \gamma(x - vt)$$

where  $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$  is the Lorentz factor.

In contrast, a rotation about the  $z$ -axis is a coordinate transformation:

$$x' = x \cos \theta - y \sin \theta, \quad y' = x \sin \theta + y \cos \theta$$

These transformations affect physical observables differently, especially in the context of TG, where torsion behaves distinctly under changes of frame versus changes of coordinates.

### 3.2 Lorentz Transformations and Frame Changes

We discuss how Lorentz transformations serve as the core mathematical tool for changing reference frames in special relativity. We will also consider generalizations to non-inertial frames and discuss how these transformations are interpreted differently in TG compared to GR.

For example, under a Lorentz boost:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where  $\beta = \frac{v}{c}$ , the torsion tensor transforms covariantly:

$$T'^\lambda{}_{\mu\nu} = \Lambda^\lambda{}_\alpha \Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu T^\alpha{}_{\sigma\rho}$$

This property ensures that the physical content of TG remains invariant under changes of inertial frames.

### 3.3 Covariant Formalism Applied to Teleparallel Gravity

Here, we will show how TG's formalism allows for a covariant description of gravity in terms of torsion, contrasting this with the curvature-based description in GR. The equations governing the dynamics of torsion will be derived and discussed. The fundamental equation of motion in TG is:

$$\partial_\nu (e T^{\nu\mu\lambda}) - \frac{1}{e} (\partial_\nu e) T^{\nu\mu\lambda} + e (\Gamma^\mu_{\sigma\nu} T^{\nu\sigma\lambda} - \Gamma^\lambda_{\sigma\nu} T^{\nu\sigma\mu}) = 0$$

where  $e$  is the determinant of the tetrad  $e^a_\mu$ , and  $T^{\nu\mu\lambda}$  is the torsion tensor.

## 4 Teleparallel Gravity and the Analogy with Electromagnetism

One of the advantages of TG is its closer analogy with electromagnetism, allowing for a more unified field theory approach.

### 4.1 Field Equations in TG

Deriving the field equations for TG, we will compare them with the Maxwell equations of electromagnetism, showing the formal similarities and discussing the physical differences. The field equations in TG are derived from the Lagrangian density:

$$\mathcal{L}_{TG} = \frac{e}{2\kappa} T^\lambda{}_{\mu\nu} S_\lambda{}^{\mu\nu}$$

where  $S_\lambda{}^{\mu\nu}$  is the superpotential defined as:

$$S_{\lambda}{}^{\mu\nu} = \frac{1}{2} (K^{\mu\nu}{}_{\lambda} + \delta_{\lambda}^{\mu} T^{\alpha\nu}{}_{\alpha} - \delta_{\lambda}^{\nu} T^{\alpha\mu}{}_{\alpha})$$

The Euler-Lagrange equations yield:

$$\partial_{\nu} (e S_{\lambda}{}^{\mu\nu}) = -\frac{e}{2\kappa} T^{\mu}{}_{\nu\lambda}$$

This equation is analogous to the Maxwell equations in electromagnetism:

$$\partial_{\nu} F^{\mu\nu} = J^{\mu}$$

where  $F^{\mu\nu}$  is the electromagnetic field tensor, and  $J^{\mu}$  is the current density.

## 4.2 Force-Free Frames in Gravity and Electromagnetism

We will explore the conditions under which a frame can be found where the effects of a gravitational field vanish, analogous to how an electromagnetic field can be canceled out in a particular reference frame. In TG, such frames are characterized by the vanishing of the torsion tensor in that frame:

$$T^{\lambda}{}_{\mu\nu} = 0$$

In electromagnetism, a similar condition applies to the electromagnetic field tensor:

$$F^{\mu\nu} = 0$$

These conditions are not global but can be achieved locally in regions of spacetime where the field strengths are sufficiently weak.

## 4.3 Implications for Particle Motion

In this subsection, we will discuss how the motion of particles is affected in TG, drawing parallels to how particles move in electromagnetic fields. The equation of motion for a test particle in TG is:

$$\frac{d^2 x^{\mu}}{d\tau^2} + T^{\mu}{}_{\nu\lambda} \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} = 0$$

This is analogous to the Lorentz force law in electromagnetism:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \frac{q}{m} F^{\mu}{}_{\nu} \frac{dx^{\nu}}{d\tau} = 0$$

where  $q$  is the charge of the particle,  $m$  is its mass, and  $F^{\mu}{}_{\nu}$  is the electromagnetic field tensor.

# 5 Spherical Geometry and Stereographic Projection

A two-dimensional spherical model provides an intuitive way to visualize concepts in TG.

## 5.1 Stereographic Projection: A Mathematical Tool

We will introduce stereographic projection, a method for mapping a sphere onto a plane, and explore its mathematical properties. The stereographic projection is given by:

$$x = \frac{2R\xi}{\xi^2 + \eta^2 + 1}, \quad y = \frac{2R\eta}{\xi^2 + \eta^2 + 1}, \quad z = R \frac{\xi^2 + \eta^2 - 1}{\xi^2 + \eta^2 + 1}$$

where  $(\xi, \eta)$  are coordinates on the plane, and  $(x, y, z)$  are coordinates on the sphere of radius  $R$ .

## 5.2 Visualizing Teleparallelism on the Sphere

Using stereographic projection, we will develop a model for understanding torsion in TG, illustrating how gravity can be visualized in flat spacetime. The curvature and torsion of the sphere can be mapped onto the plane, allowing us to visualize the effects of gravity in TG.

## 5.3 Connecting Spherical and Hyperbolic Geometries to Space-time

We will explore how concepts from spherical and hyperbolic geometry can be extended to four-dimensional spacetime, providing deeper insights into the structure of TG. The relationship between the curvature of the sphere and the torsion in TG will be analyzed using the Gauss-Bonnet theorem.

# 6 Weitzenböck Connection and Teleparallelism

In TG, the Weitzenböck connection replaces the Levi-Civita connection used in GR. This section will delve into the mathematical structure and physical interpretation of the Weitzenböck connection.

## 6.1 Tetrads and the Weitzenböck Connection

We will derive the Weitzenböck connection from the tetrads and discuss its role in TG. A tetrad  $e_\mu^a$  is a set of four linearly independent vector fields that relate the spacetime coordinates to the flat Minkowski coordinates:

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$$

The Weitzenböck connection is defined by:

$$\Gamma_{\mu\nu}^\lambda = e_a^\lambda \partial_\nu e_\mu^a$$

This connection has zero curvature but non-zero torsion, making it ideal for the TG framework.

## 6.2 Torsion as the Fundamental Quantity

Torsion, rather than curvature, plays the central role in TG. The torsion tensor can be written as:

$$T^\lambda_{\mu\nu} = e^\lambda_a (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu)$$

We will develop the equations governing torsion and discuss how they lead to the field equations of TG. The dynamics of torsion in TG are described by:

$$\mathcal{D}_\nu T^{\mu\nu\lambda} = 0$$

where  $\mathcal{D}_\nu$  is the covariant derivative with respect to the Weitzenböck connection.

## 6.3 Physical Interpretation and Applications

The Weitzenböck connection provides a different way of interpreting gravitational phenomena. This section will explore applications of TG, such as in cosmology and the study of gravitational waves. We will discuss how the torsion tensor influences the propagation of gravitational waves and how TG provides alternative explanations for cosmological phenomena like dark energy and dark matter.

# 7 Conclusion and Future Directions

In concluding, we will summarize the key points and suggest directions for future research.

## 7.1 Summary of Key Ideas

We will recap the distinctions between TG and GR, the significance of flat spacetime, and the role of torsion in describing gravity. The equivalence between TG and GR will be emphasized, particularly in how both theories predict the same observational phenomena, but with different underlying geometric interpretations.

## 7.2 Potential for Unification

TG offers possibilities for unifying gravity with other forces, a potential that we will explore here. We will discuss how TG might be integrated into a broader framework that includes electromagnetism and the weak and strong nuclear forces, potentially leading to a unified field theory.

## 7.3 Open Questions and Future Research

Finally, we will discuss the open questions in TG, such as the need for empirical tests, and suggest avenues for further theoretical development. Open issues like the quantization of TG, its implications for black hole physics, and its potential to explain dark energy will be highlighted as key areas for future research.