Binary Representation of Rational Numbers

If x is a rational number with $0 \leq x \leq 1$, then $x = 0.P\overline{R}$ where P and Q are binary integers.

In the following, rem $= x - \lfloor x \rfloor$, so $x = \lfloor x \rfloor +$ rem.

Example 1: $x = 5/7$

a/b	$\lfloor a/b \rfloor$	rem	
5/7	0	5/7	
decimal part			
10/7	1	3/7	
6/7	0	6/7	
12/7	1	5/7	repeat - go to 10/7

$5/7 = 0.\overline{101}$

Working backwards,

$$\begin{aligned}
0.\overline{101} &= 0.101101101\overline{101} \\
&= 0.101 + 0.000101 + 0.000000101 + \dots \\
&= 0.101 + \tfrac{1}{8}0.101 + \left(\tfrac{1}{8}\right)^2 0.101 + \dots \\
&= \tfrac{5}{8}\left(1 + \tfrac{1}{8} + \left(\tfrac{1}{8}\right)^2 + \cdots\right) \\
&= \tfrac{5}{8}\left(\frac{1}{1-\frac{1}{8}}\right) \\
&= \tfrac{5}{8}\tfrac{8}{7} \\
&= \tfrac{5}{7}
\end{aligned}$$

Example 2: $x = 11/14$

a/b	$\lfloor a/b \rfloor$	rem	
11/14	0	11/14	
decimal part			
11/7	1	4/7	
8/7	1	1/7	
2/7	0	2/7	
4/7	0	4/7	repeat - back to 8/7

$11/14 = 0.1\overline{101}$

Example 3: $x = 7/22 = \frac{1}{2}(7/11)$

a/b	$\lfloor a/b \rfloor$	rem	
7/22	0	7/22	
decimal part			
7/11	0	7/11	
14/11	1	3/11	
6/11	0	6/11	
12/11	1	1/11	
2/11	0	2/11	
4/11	0	4/11	
8/11	0	8/11	
16/11	1	5/11	
10/11	0	10/11	
20/11	1	9/11	
18/11	1	7/11	repeat - back to 14/11

$7/22 = 0.0\overline{1010001011}$

Example 4: $x = 72.1$

$72 = 1001000$ (7 bits)

$0.1 = 1/10$

a/b	$\lfloor a/b \rfloor$	rem	
1/10	0	1/10	
decimal part			
1/5	0	1/5	
2/5	0	2/5	
4/5	0	4/5	
8/5	1	3/5	
6/5	1	1/5	repeat - back to 2/5

$1/10 = 0.0\overline{0011}$

$72.1 = 1001000 + 0.0\overline{0011} = 10010000\overline{0011}$

72.1 as a single number

reduce 72.1 to a 24 bit number:

$72.1 = 1001000.00011001100110011001100110011$

Round off gives last digit as 1 and introduces an error term (-0.00000152587890625).

Use scientific notation (base 2) to write with a leading 1 followed by a decimal part:

$72.1 = 1.00100000011001100110011001100110011 \times 2^6$

IEEE representation:

sign (1 bit) = 0

exponent (8 bits) = 6+127 = 133 = 128 + 5 = 10000101

mantissa (23 bits - leading 1 omitted) = 00100000011001100110011001100110011

sign | exponent | mantissa = 0 | 10000101 | 00100000011001100110011001100110011

single(72) = 0100001010010000001100110011001100110011

In hexadecimal (this is not the hexadecimal representaion of 72.1).

Write in blocks of 4 starting from the right:

0100 0010 1001 0000 0011 0011 0011 0011 0011 0011

Convert to hexadecimal:

single(72.1) = 429033333_{16}

Using java.lang.floatToIntBits

```
>> value = java.lang.Float.floatToRawIntBits(72.1)

value =

   1.1167e+09

>> java.lang.Integer.toBinaryString(value)

ans =

1000010100100000011001100110011
```

Converting a string representation of a positive integer into binary

Matlab Implementation

```
function y = int2bin(str, y)
% input: str is a string representing a positive integer N
% output: the binary representation of N

keySet = {'0','1','2','3', '4', '5','6','7','8', '9'};
valueSet = uint64(0:9);
M = containers.Map(keySet,valueSet) % lookup table

n = length(str);

y = M(str(1));

for i = 2:n
    a = bitshift(y, 1) + bitshift(y, 3);
    b = M(str(i));
    y = a + b;
end

end
```