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Analytical derivatives of the orbit response matrix and dispersion

Abstract

This internal report presents the formulas for the response matrix and dispersion derivatives with respect to the quadrupole strengths. The equations validity is extended to the thick quadrupole case. The formulas hold only in the constant momentum case, the discrepancies with the constant path case are assessed for the ALBA lattice. These formulas make the optics correction, in particular the LOCO analysis, several times faster.

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Contents

1 Introduction

In the case of the ALBA lattice, a LOCO [1] fit takes several minutes. Routinely a LOCO measurement and analysis is performed every week. Usually, this analysis runs iteratively on the measurement data. The first analysis uses only the available correction knobs, that is, the 112 quadrupoles, while the second analysis also uses the quadrupole component of the 32 combined function bending (CFD) magnets as quadrupole correctors as well. The most time-consuming part of that analysis is the calculation of the orbit response matrix (ORM) and the change of dispersion as a function of the fit parameters which are performed numerically using AT 2.0 [2].

In this internal report, we show how we can perform some of such calculations in a faster way. For example, in the case of the uncoupled response matrix, where the exact formula is well known, the derivatives of the formula can be calculated instead of numerical differentiation.

In [3], following a different formulation, equivalent results are presented. In that case, the formulas are designed for the ORM fitting algorithm used at the ESRF. The formulas that we are presenting here can be implemented in LOCO and apply to the thick quadrupole and dipole case which is essential in the case of the ALBA storage ring lattice as well as for the low emittance lattice present designs.

The next sections are dedicated to show the adequate formulas for the ORM and dispersion derivatives calculation. In Appendix A, the ORM derivative is compared to the numerical calculation for the ALBA case. In Appendix B, the dispersion derivative is compared with the numerical calculation for the ALBA case. In Appendix C, the performance of LOCO making use of the above mentioned analytical formulas is compared with the usual LOCO using numerical calculations for the ALBA case.

2 Constant energy uncoupled response matrix quadrupole derivative

In this case the derivation is based in the closed orbit formula [4]:

$$R_{i,j} = \frac{\sqrt{\beta_i \beta_j}}{2sin(\pi\nu)} cos(|\psi_i - \psi_j| - \pi\nu), \tag{1}$$

where $R_{i,j}$ represents the orbit response at the $i-th$ beam position monitor (BPM) for the $j-th$ corrector in each plane, β_i is the corresponding plane beta function at the BPM, β_j is the corresponding plane beta function at the corrector, ν is the betatron tune in the corresponding plane, ψ_i is the corresponding plane betatron phase at the $i-th$ BPM and ψ_j is the corresponding plane betatron phase at the $j-th$ corrector. Using the chain rule, its derivative with respect to the $k-th$ quadrupole reads:

$$\frac{dR_{i,j}}{dq_k} = \frac{\partial R_{i,j}}{\partial \beta_i}\frac{d\beta_i}{dq_k} + \frac{\partial R_{i,j}}{\partial \beta_j}\frac{d\beta_j}{dq_k} + \frac{\partial R_{i,j}}{\partial \nu}\frac{d\nu}{dq_k} + \frac{\partial R_{i,j}}{\partial \psi_i}\frac{d\psi_i}{dq_k} + \frac{\partial R_{i,j}}{\partial \psi_j}\frac{d\psi_j}{dq_k}, \tag{2}$$

Each of the derivatives with respect to the optical functions β, ψ and the tune ν are calculated from equation 1 and are expressed as follows:

$$\begin{aligned}
\frac{\partial R_{i,j}}{\partial \beta_i} &= \frac{\sqrt{\beta_j}}{4\sqrt{\beta_i} sin(\pi\nu)} C_{i,j,1} \\
\frac{\partial R_{i,j}}{\partial \beta_j} &= \frac{\sqrt{\beta_i}}{4\sqrt{\beta_j} sin(\pi\nu)} C_{i,j,1} \\
\frac{\partial R_{i,j}}{\partial \nu} &= -\pi \frac{\sqrt{\beta_i \beta_j}}{2 sin^2(\pi\nu)} \left[C_{i,j,1} cos(\pi\nu) - s(\psi_i - \psi_j) S_{i,j,1} sin(\pi\nu)\right] \\
\frac{\partial R_{i,j}}{\partial \psi_i} &= -\frac{\sqrt{\beta_i \beta_j}}{2 sin(\pi\nu)} S_{i,j,1} \\
\frac{\partial R_{i,j}}{\partial \psi_j} &= \frac{\sqrt{\beta_i \beta_j}}{2 sin(\pi\nu)} S_{i,j,1}
\end{aligned} \tag{3}$$

In the previous formula, the following definitions have been used:

$$\begin{aligned}
C_{i,j,n} &= cos(n|\psi_i - \psi_j| - n\pi\nu) \\
S_{i,j,n} &= s(\psi_i - \psi_j) sin(n|\psi_i - \psi_j| - n\pi\nu),
\end{aligned} \tag{4}$$

where $s()$ represents the sign function. Each of the derivatives with respect to the quadrupole strength q_k is calculated in the next subsections. A comparison with numerical calculations for the ALBA case is presented in appendix A.

2.1 Tune change with the quadrupole strength

In this case the relation is well known [4]:

$$\frac{d\nu}{dq_k} = \pm \frac{\beta_k L_k}{4\pi}, \tag{5}$$

The sign is positive for the horizontal plane and negative in the vertical plane.

2.2 Beta change with the quadrupole strength

Also in this case, the well known beta beating formula at any lattice location i is used:

$$\frac{d\beta_i}{dq_k} = \mp \frac{\beta_i \beta_k L_k}{2 sin(2\pi\nu)} C_{i,k,2}, \tag{6}$$

The sign is negative for the horizontal plane and positive in the vertical plane.

2.3 Phase change with the quadrupole strength

Also in this case, there is an explicit formula that can be found in the literature. However, it can also be directly obtained using equation 6. Here, an small demonstration follows. The phase advance can be calculated from the beta function as:

$$\psi_i = \int_0^{z_i} \frac{dt}{\beta(t)}, \tag{7}$$

hence, its derivative with respect to the quadrupoles value:

$$\frac{d\psi_i}{dq_k} = -\int_0^{z_i} \frac{d\beta(t)}{dq_k} \frac{dt}{\beta(t)^2}, \tag{8}$$

and now, using equation 6, it can be written as:

$$\frac{d\psi_i}{dq_k} = \pm \frac{\beta_k L_k}{2sin(2\pi\nu)} \int_0^{z_i} cos(2|\psi(t) - \psi_k| - 2\pi\nu) \frac{dt}{\beta(t)}. \tag{9}$$

Using again equation 7 we can change the integration variable:

$$\frac{d\psi_i}{dq_k} = \pm \frac{\beta_k L_k}{2sin(2\pi\nu)} \int_0^{\psi_i} cos(2|\psi - \psi_k| - 2\pi\nu) d\psi. \tag{10}$$

This integral can be solved. First we should notice that:

$$\frac{d\left[s(\psi - \psi_k) sin(2|\psi - \psi_k| - 2\pi\nu)\right]}{d\psi} = 2\delta(\psi - \psi_k) sin(2|\psi - \psi_k| - 2\pi\nu) + 2cos(2|\psi - \psi_k| - 2\pi\nu), \tag{11}$$

where $\delta()$ represents the Dirac's delta function and we have used $s^2(\psi - \psi_k) = 1$. Integrating the previous equation and isolating the term that also appears in equation 10 we obtain:

$$\int_0^{\psi_i} cos(2|\psi - \psi_k| - 2\pi\nu) d\psi = \left[\frac{s(\psi - \psi_k)}{2} sin(2|\psi - \psi_k| - 2\pi\nu)\right]_0^{\psi_i} + \theta(\psi_i - \psi_k) sin(2\pi\nu), \tag{12}$$

where $\theta()$ is the Heaviside's step function. Then equation 10 also reads:

$$\frac{d\psi_i}{dq_k} = \pm \frac{\beta_k L_k}{2sin(2\pi\nu)} \left[\left[\frac{s(\psi - \psi_k)}{2} sin(2|\psi - \psi_k| - 2\pi\nu)\right]_0^{\psi_i} + \theta(\psi_i - \psi_k) sin(2\pi\nu)\right]. \tag{13}$$

Finally, we obtain:

$$\frac{d\psi_i}{dq_k} = \pm \frac{\beta_k L_k}{4sin(2\pi\nu)} \quad \left[S_{i,k,2} + sin(2\psi_k - 2\pi\nu) + 2\theta(\psi_i - \psi_k) sin(2\pi\nu)\right], \tag{14}$$

which, as usual, changes sign in the vertical plane. Notice that by substituting ψ_i by $2\pi\nu$ and ψ_k by 0 in equations 14, one can recover equation 5. Also, notice that the second term not containing ψ_i terms will be canceled out once added up in equation 2 with the similar term from $\frac{d\psi_j}{dq_k}$.

2.4 Complete formula

We can include the above equations in a single formula, which results in the following expression:

$$\begin{aligned} \frac{dR_{i,j}}{dq_k} = \mp \frac{\sqrt{\beta_i\beta_j}\beta_k L_k}{8sin(\pi\nu)sin(2\pi\nu)} & \left[C_{i,j,1}\left[C_{i,k,2} + C_{j,k,2} + 2cos^2(\pi\nu)\right]\right. \\ & \left. + S_{i,j,1}\left[S_{i,k,2} - S_{j,k,2} + sin(2\pi\nu)(2\theta(\psi_i - \psi_k) - 2\theta(\psi_j - \psi_k)) - s(\psi_i - \psi_j)\right]\right], \end{aligned} \tag{15}$$

where the sing is negative for the horizontal plane and negative for the vertical plane.

2.5 Thick quadrupole equations

Equation 15 is only valid for thin quadrupoles. In this section we modify the formula to make it valid for tick quadrupoles as well. Regarding the variation of the Twiss functions inside the quadrupole, four types of terms in equation 15 have to be considered:

1. No phase variation terms: $\beta_k L_k$
2. Sin like terms: $\beta_k L_k\, sin(2\psi_k)$
3. Cos like terms: $\beta_k L_k\, cos(2\psi_k)$
4. Other terms: $\beta_k L_k\, S_{k,i,2}$ or $\beta_k L_k\, C_{k,i,2}$

In the following subsections, the different terms modifications will be solved for a thick focusing quadrupole. The generalization to defocussing quadrupoles and combined function bending magnets can be done a posteriori. In the case of the defocussing magnet the quadrupole strength q_k must be substituted by $-q_k$ and hence $sin(\sqrt{q_k})/\sqrt{q_k}$ is substituted by $sinh(\sqrt{q_k})/\sqrt{q_k}$. sin and cos terms do not appear explicitly in equation 15, but they are very useful to calculate the other more convoluted terms.

2.5.1 No phase variation terms

This term appears in equation 5 and 14. In this simplest case, the effective beta function should be used. The following substitution should be done:

$$\beta_k L_k \longmapsto I_{k,0} \equiv \int_0^{L_k} \beta_k(z)dz \tag{16}$$

For a thick focusing quadrupole, the transfer matrix along the quadrupole is the following:

$$A(q_k, s|0) = \begin{pmatrix} cos(\sqrt{q_k}z) & sin(\sqrt{q_k}z)/\sqrt{q_k} \\ -\sqrt{q_k}zin(\sqrt{q_k}z) & cos(\sqrt{q_k}z) \end{pmatrix}, \tag{17}$$

The Twiss transfer matrix can be obtained from the transfer matrix and allows to express analytically the beta function variation inside the quadrupole:

$$\begin{pmatrix} \beta_k(z) \\ \alpha_k(z) \\ \gamma_k(z) \end{pmatrix} = \begin{pmatrix} cos^2(\sqrt{q_k}z) & -\frac{sin(2\sqrt{q_k}z)}{\sqrt{q_k}} & sin^2(\sqrt{q_k}z)/q_k \\ \frac{\sqrt{q_k}}{2}sin(2\sqrt{q_k}z) & cos(2\sqrt{q_k}z) & -\frac{1}{2\sqrt{q_k}}sin(2\sqrt{q_k}z) \\ q_k sin^2(\sqrt{q_k}z) & \sqrt{q_k}sin(2\sqrt{q_k}z) & cos^2(\sqrt{q_k}z) \end{pmatrix} \begin{pmatrix} \beta_k \\ \alpha_k \\ \gamma_k \end{pmatrix}. \tag{18}$$

Hereafter, we choose the following convention: When optical functions β_k, α_k, γ_k or ψ_k have no explicit dependency with position, they represent the value at the beginning of the $k-th$ element. In particular, the beta function variation along the quadrupole reads:

$$\beta_k(z) = \frac{\beta_k}{2} + \frac{\gamma_k}{2q_k} + \left[\frac{\beta_k}{2} - \frac{\gamma_k}{2q_k}\right] cos(2\sqrt{q_k}z) - \frac{\alpha_k}{\sqrt{q_k}} sin(2\sqrt{q_k}z) \tag{19}$$

Notice that it is consistent with the aforementioned convention, since from the previous equation, we find $\beta_k(0) = \beta_k$ and $\beta'_k(0) = -2\alpha_k$. Using the previous equation, the integral $I_{k,0}$ can be calculated:

$$I_{k,0} = \left[\frac{\beta_k}{2} + \frac{\gamma_k}{2q_k}\right] L_k + \left[\frac{\beta_k}{2} - \frac{\gamma_k}{2q_k}\right] \frac{sin(2\sqrt{q_k}L_k)}{2\sqrt{q_k}} + \frac{\alpha_k}{2q_k}\left[cos(2\sqrt{q_k}L_k) - 1\right] \tag{20}$$

2.5.2 Sin like term

This term does not appear explicitly in the equation 15, but it is useful for some of them, for example, when integrating the term $\beta_k(z)S_{i,k,2}$. The following substitution should be made:

$$\beta_k L_k\, sin(2(\psi_k(z)-\psi_k)) \longmapsto I_{k,s,2} \equiv \int_0^{L_k} \beta_k(z)\, sin(2(\psi_k(z)-\psi_k))dz, \tag{21}$$

where again, when the z dependency is not explicit, it indicates the value at the beginning of the quadupole. In order to solve this integral, the general transfer matrix expression can be used:

$$A(k,s|0) = \begin{pmatrix} \sqrt{\frac{\beta_k(z)}{\beta_k}}(cos(\psi_k(z)-\psi_k)+\alpha_k sin(\psi_k(z)-\psi_k)) & \sqrt{\beta_k(z)\beta_k}sin(\psi_k(z)-\psi_k) \\ \frac{\alpha_k-\alpha_k(z)}{\sqrt{\beta_k(z)\beta_k}}cos(\psi_k(z)-\psi_k) - \frac{1-\alpha_k\alpha_k(z)}{\sqrt{\beta_k(z)\beta_k}}sin(\psi_k(z)-\psi_k) & \sqrt{\frac{\beta_k}{\beta_k(z)}}(cos(\psi_k(z)-\psi_k)-\alpha_k(z)sin(\psi_k(z)-\psi_k)) \end{pmatrix} \tag{22}$$

which matching the $A(k,s|0)_{1,1}$ and $A(k,s|0)_{1,2}$ terms with equation 17 gives the following result:

$$\begin{aligned} \sqrt{\beta_k(z)}sin(\psi_k(z)-\psi_k) &= \frac{1}{\sqrt{q_k\beta_k}}sin(\sqrt{q_k}z) \\ \sqrt{\beta_k(z)}cos(\psi_k(z)-\psi_k) &= \sqrt{\beta_k}cos(\sqrt{q_k}z) - \frac{\alpha_k}{\sqrt{q_k\beta_k}}sin(\sqrt{q_k}z), \end{aligned} \tag{23}$$

The previous equations are useful to calculate integral $\Sigma_{k,1}$ since:

$$I_{k,s,2} = \int_0^{L_k} 2\beta_k(z)\, sin((\psi_k(z)-\psi_k))cos((\psi_k(z)-\psi_k))dz. \tag{24}$$

After few algebraic manipulations, the previous integral can be expressed as:

$$I_{k,s,2} = \frac{1}{2q_k}\left[1-cos(2\sqrt{q_k}L_k)+\frac{\alpha_k}{\beta_k}(\frac{sin(2\sqrt{q_k}L_k)}{\sqrt{q_k}}-2L_k)\right]. \tag{25}$$

2.5.3 Cos like term

This term does not appear explicitly in the equation 15, but it is useful for some of them, for example when integrating the $\beta_k(z)C_{i,k,2}$ term. The following substitution should be done:

$$\beta_k L_k\, cos(2(\psi_k(z)-\psi_k)) \longmapsto I_{k,c,2} \equiv \int_0^{L_k} \beta_k(z)\, cos(2(\psi_k(z)-\psi_k))dz, \tag{26}$$

where again, when the z dependency is not explicit it indicates the value at the beginning of the quadupole. We can rewrite the previous equation as:

$$\begin{aligned} I_{k,c,2} &= \int_0^{L_k} \beta_k(z)\left[1-2sin^2(\psi_k(z)-\psi_k)\right]dz \\ &= I_{k,0} - 2\int_0^{L_k} \beta_k(z)sin^2(\psi_k(z)-\psi_k)dz. \end{aligned} \tag{27}$$

Making use of equation 23, the following equation can be solved:

$$I_{k,c,2} = I_{k,0} + \frac{1}{q_k\beta_k}\left[\frac{sin(2\sqrt{q_k}L_k)}{2\sqrt{q_k}} - L_k\right]. \tag{28}$$

2.5.4 Other terms

In equation 15, the following terms appear:

$$\begin{aligned}
\beta_k L_k\, S_{i,k,2} \longmapsto \Sigma_{i,k,2} &\equiv \int_0^{L_k} \beta_k(z)\, s(\psi_i - \psi_k(z)) sin(2|\psi_i - \psi_k(z)| - 2\pi\nu) dz \\
\beta_k L_k\, C_{i,k,2} \longmapsto \Gamma_{i,k,2} &\equiv \int_0^{L_k} \beta_k(z)\, cos(2|\psi_i - \psi_k(z)| - 2\pi\nu) dz \\
\beta_k L_k\, \theta(\psi_i - \psi_k) \longmapsto \Delta_{i,k,2} &\equiv \int_0^{L_k} \beta_k(z) \theta(\psi_i - \psi_k(z)) dz.
\end{aligned} \tag{29}$$

The integrals in the previous equations can be solved using equations 25 and 28. However, we should treat the case when $i = k$ separately. Although this may seem an unlikely case, as we will see further in the text, it needs to be considered.

2.5.4.1 Case $i \neq k$

First we should notice that:

$$s(\psi_i - \psi_k(z)) sin(2|\psi_i - \psi_k(z)| - 2\pi\nu) = \begin{cases} \psi_i > \psi_k(z), \psi_k & sin(2(\psi_i - \psi_k) - 2\pi\nu) cos(2(\psi_k(z) - \psi_k)) \\ & -cos(2(\psi_i - \psi_k) - 2\pi\nu) sin(2(\psi_k(z) - \psi_k)) \\ \psi_i < \psi_k(z), \psi_k & -sin(2(\psi_k - \psi_i) - 2\pi\nu) cos(2(\psi_k(z) - \psi_k)) \\ & -cos(2(\psi_k - \psi_i) - 2\pi\nu) sin(2(\psi_k(z) - \psi_k)) \end{cases}, \tag{30}$$

where ψ_k has been included as the phase at the beginning of element k. That definition allows us to combine the two cases as follows:

$$\begin{aligned}
s(\psi_i - \psi_k(z)) sin(2|\psi_i - \psi_k(z)| - 2\pi\nu) =& s(\psi_i - \psi_k) sin(2|\psi_i - \psi_k| - 2\pi\nu) cos(2(\psi_k(z) - \psi_k)) \\
&- cos(2|\psi_i - \psi_k| - 2\pi\nu) sin(2(\psi_k(z) - \psi_k),
\end{aligned} \tag{31}$$

which in our previous notation reads:

$$S_{i,z_k,2} = S_{i,k,2} cos(2(\psi_k(z) - \psi_k)) - C_{i,k,2} sin(2(\psi_k(z) - \psi_k)), \tag{32}$$

Similarly, with the second term in equation 29, we have:

$$cos(2|\psi_i - \psi_k(z)| - 2\pi\nu) = \begin{cases} \psi_i > \psi_k(z), \psi_k & cos(2(\psi_i - \psi_k) - 2\pi\nu) cos(2(\psi_k(z) - \psi_k)) \\ & +sin(2(\psi_i - \psi_k) - 2\pi\nu) sin(2(\psi_k(z) - \psi_k)) \\ \psi_i < \psi_k(z), \psi_k & cos(2(\psi_k - \psi_i) - 2\pi\nu) cos(2(\psi_k(z) - \psi_k)) \\ & -sin(2(\psi_k - \psi_i) - 2\pi\nu) sin(2(\psi_k(z) - \psi_k)) \end{cases}, \tag{33}$$

which again can be simplified as follows:

$$C_{i,z_k,2} = C_{i,k,2} cos(2(\psi_k(z) - \psi_k)) + S_{i,k,2} sin(2(\psi_k(z) - \psi_k)), \tag{34}$$

After these algebraic manipulations, the integrals in equation 29 can be rewritten in terms of $I_{k,0}$, $I_{k,s}$ and $I_{k,c}$:

$$\begin{aligned}
\Sigma_{i,k,2} &= I_{k,c,2} S_{i,k,2} - I_{k,s,2} C_{i,k,2} \\
\Gamma_{i,k,2} &= I_{k,c,2} C_{i,k,2} + I_{k,s,2} S_{i,k,2} \\
\Delta_{i,k,2} &= I_{k0} \theta(\psi_i - \psi_k),
\end{aligned} \tag{35}$$

2.5.4.2 Case $i = k$

In this special case, integrals in equation 29 become:

$$\begin{aligned}
\Sigma_{k,k,2} &= \int_0^{L_k} \beta_k(z)\, s(\psi_k - \psi_k(z)) sin(2|\psi_k - \psi_k(z)| - 2\pi\nu) dz \\
\Gamma_{k,k,2} &= \int_0^{L_k} \beta_k(z)\, cos(2|\psi_k - \psi_k(z)| - 2\pi\nu) dz \\
\Delta_{k,k,2} &= \int_0^{L_k} \beta_k(z) \theta(\psi_k - \psi_k(z)) dz.
\end{aligned} \tag{36}$$

Since by definition $\psi_k(z) > \psi_k$, the previous equation becomes:

$$\begin{aligned}
\Sigma_{k,k,2} &= -\int_0^{L_k} \beta_k(z)\, sin(2(\psi_k(z) - \psi_k) - 2\pi\nu) dz \\
\Gamma_{k,k,2} &= \int_0^{L_k} \beta_k(z)\, cos(2(\psi_k(z) - \psi_k) - 2\pi\nu) dz \\
\Delta_{k,k,2} &= 0.
\end{aligned} \tag{37}$$

Expanding the *sin* and *cos* terms in the previous equations we get:

$$\begin{aligned}
\Sigma_{k,k,2} &= I_{k,c,2} sin(2\pi\nu) - I_{k,s,2} cos(2\pi\nu) \\
\Gamma_{k,k,2} &= I_{k,c,2} cos(2\pi\nu) + I_{k,s,2} sin(2\pi\nu) \\
\Delta_{k,k,2} &= 0.
\end{aligned} \tag{38}$$

2.5.4.3 All cases

Notice that the result from equation 38 does not correspond with equation 35 evaluated at $i = k$. While $C_{k,k,2} = cos(2\pi\nu)$, the other terms do not match because $S_{k,k,2} = -s(0) sin(2\pi\nu)$, $s(0) = 0$ and $\theta(0) = 1$. A compact formula including both cases can be achieved if we consider the following modified sign and step functions:

$$\tilde{s}(x) = \begin{cases} x > 0 & 1 \\ x = 0 & -1 \\ x < 0 & -1 \end{cases}, \tag{39}$$

and

$$\tilde{\theta}(x) = \begin{cases} x > 0 & 1 \\ x = 0 & 0 \\ x < 0 & 0 \end{cases}. \tag{40}$$

With the previous definitions, the solution for equation 29 valid for all cases is:

$$\begin{aligned}
\Sigma_{i,k,2} &= I_{k,c,2} \tilde{S}_{i,k,2} - I_{k,s,2} C_{i,k,2} \\
\Gamma_{i,k,2} &= I_{k,c,2} C_{i,k,2} + I_{k,s,2} \tilde{S}_{i,k,2} \\
\Delta_{i,k,2} &= I_{k0} \tilde{\theta}(\psi_i - \psi_k),
\end{aligned} \tag{41}$$

where $\tilde{S}_{i,k,n} = \tilde{s}(\psi_i - \psi_k) sin(n|\psi_i - \psi_k| - n\pi\nu)$.

2.6 Thick quadrupole response matrix derivative formula

Starting from equation 15, using the definitions from the previous subsection, we obtain:

$$\begin{aligned}\frac{dR_{i,j}}{dq_k} = \mp \frac{\sqrt{\beta_i\beta_j}}{8sin(\pi\nu)sin(2\pi\nu)}&\left[C_{i,j,1}\left[\Gamma_{i,k,2}+\Gamma_{j,k,2}+2I_{k,0}cos^2(\pi\nu)\right]\right.\\ &\left.+S_{i,j,1}\left[\Sigma_{i,k,2}-\Sigma_{j,k,2}+I_{k,0}sin(2\pi\nu)(2\tilde{\theta}(\psi_i-\psi_k)-2\tilde{\theta}(\psi_j-\psi_k))-s(\psi_i-\psi_j)\right]\right],\end{aligned} \tag{42}$$

where the negative sing corresponds to the horizontal plane ORM and the positive sing applies in the case of the vertical ORM.

2.7 thick quadrupole Phase derivative formula

It is also useful to know the phase derivative w.r.t a thick quadrupole change. In this case we start with 14. Most terms have been already discussed above except the term with $\beta_k L_k sin(2\psi_k - 2\pi\nu)$. This term can be expressed in terms of previously solved terms since:

$$\begin{aligned} sin(2\psi_k(z)-2\pi\nu) =& sin(2\psi_k(z)-2\psi_k)cos(2\psi_k-2\pi\nu)\\ &+cos(2\psi_k(z)-2\psi_k)sin(2\psi_k-2\pi\nu),\end{aligned} \tag{43}$$

With this and the previous sections results we get:

$$\begin{aligned}\frac{d\psi_i}{dq_k} = \pm\frac{1}{4sin(2\pi\nu)}&\left[\Sigma_{i,k,2}+cos(2\psi_k-2\pi\nu)I_{k,s,2}\right.\\ &\left.+sin(2\psi_k-2\pi\nu)I_{k,c,2}+2sin(2\pi\nu)\Delta_{i,k}\right],\end{aligned} \tag{44}$$

2.8 Edge focusing effect

For combined function dipoles the previous formulas can be applied to the hard edge model with some precautions. Fist of all, in the horizontal plane the previous formulas have to be modified in the following way:

$$q_{k,x} \longmapsto q_k + \frac{1}{\rho^2}, \tag{45}$$

where ρ is the dipole radius of curvature. Additionally, according to [5], since the thin lens version of the dipole's edge element transport matrix reads:

$$\begin{aligned}\begin{pmatrix} x_{end}\\ x'_{end}\end{pmatrix} &= \begin{pmatrix} 1 & 0\\ C'_x & 1\end{pmatrix}\begin{pmatrix} x_{start}\\ x'_{start}\end{pmatrix}\\ \begin{pmatrix} y_{end}\\ y'_{end}\end{pmatrix} &= \begin{pmatrix} 1 & 0\\ C'_y & 1\end{pmatrix}\begin{pmatrix} y_{start}\\ y'_{start}\end{pmatrix},\end{aligned} \tag{46}$$

where:

$$\begin{aligned} C'_x =& \frac{tan(E_1)}{\rho}\\ C'_y =& -\frac{tan(E_1 - \frac{I_{fint}g(1+sin^2(E_1))}{\rho cos(E_1)})}{\rho},\end{aligned} \tag{47}$$

where E_1 is the entrance edge angle, I_{fint} is the so called fringe field integral and g is the dipole gap. Hence, the dipole's edge element Twiss functions transport matrix reads:

$$\begin{aligned}\begin{pmatrix}\beta_{x,end}\\ \alpha_{x,end}\\ \gamma_{x,end}\end{pmatrix} &= \begin{pmatrix}1 & 0 & 0\\ -C'_x & 1 & 0\\ C'^2_x & -2C'_x & 1\end{pmatrix}\begin{pmatrix}\beta_{x,start}\\ \alpha_{x,start}\\ \gamma_{x,start}\end{pmatrix}\\ \begin{pmatrix}\beta_{y,end}\\ \alpha_{y,end}\\ \gamma_{y,end}\end{pmatrix} &= \begin{pmatrix}1 & 0 & 0\\ -C'_y & 1 & 0\\ C'^2_y & -2C'_y & 1\end{pmatrix}\begin{pmatrix}\beta_{y,start}\\ \alpha_{y,start}\\ \gamma_{y,start}\end{pmatrix}.\end{aligned} \tag{48}$$

Therefore for our calculations, the Twiss functions at the beginning of a bending magnet have to be substituted according to the following rule:

$$\begin{aligned}\gamma_{x,k} &\longmapsto \gamma_{x,k} + \beta_{x,k}C'^2_x - 2\alpha_{x,k}C'_x\\ \alpha_{x,k} &\longmapsto \alpha_{x,k} - \beta_{x,k}C'_x\\ \gamma_{y,k} &\longmapsto \gamma_{y,k} + \beta_{y,k}C'^2_y - 2\alpha_{y,k}C'_y\\ \alpha_{y,k} &\longmapsto \alpha_{y,k} - \beta_{y,k}C'_y,\end{aligned} \tag{49}$$

3 Horizontal plane dispersion quadrupole derivative

The horizontal dispersion is originated in the dipoles (here with index j) and can be expressed analytically as follows:

$$\eta_{x,i} = \frac{\sqrt{\beta_{x,i}}}{2sin(\pi\nu_x)}\sum_j \int_0^{L_j} h_j\sqrt{\beta_{x,j}(z)}cos(|\psi_{x,i} - \psi_{x,j}(z)| - \pi\nu)dz, \tag{50}$$

where h_j is the dipole's curvature. Within this section all functions will refer to the horizontal plane, the x subindex will not be made explicit. This time the integral along the dipole's field has been left explicit since, as shown in the previous section, the Twiss functions have a pronounced variation inside the ALBA dipoles. Notice that equation 50 can be integrated similarly to the thick quadrupole formula in section 2.5:

$$\int_0^{L_j} \sqrt{\beta_j(z)}cos(|\psi_i - \psi_j(z)| - \pi\nu)dz = I_{j,c,1}C_{i,j,1} + I_{j,s,1}\tilde{S}_{i,j,1}, \tag{51}$$

where the following definitions, have been used:

$$\begin{aligned}I_{j,c,1} &\equiv \int_0^{L_j} \sqrt{\beta_j(z)}\, cos((\psi_j(z) - \psi_j))dz = \frac{\sqrt{\beta_j}}{\sqrt{q_j}}sin(\sqrt{q_j}L_j) + \frac{\alpha_j\left[cos(\sqrt{q_j}L_j) - 1\right]}{q_j\sqrt{\beta_j}}\\ I_{j,s,1} &\equiv \int_0^{L_k} \sqrt{\beta_j(z)}\, sin((\psi_j(z) - \psi_j))dz = -\frac{cos(\sqrt{q_j}L_j) - 1}{q_j\sqrt{\beta_j}}.\end{aligned} \tag{52}$$

These equations are integrated thanks to equation 23. Notice that here since we are considering dipoles in the horizontal plane: $q_j \longmapsto q_j + h_j^2$. Also we must keep in mind that β_j corresponds to the horizontal beta function at the beginning of the bending magnet but after having applied the edge effect. Equation 50 can be rewritten as:

$$\eta_{x,i} = \sum_j h_j\left(\hat{I}_{j,c,1}R^{thin}_{i,j} + \hat{I}_{j,s,1}T^{thin}_{i,j}\right), \tag{53}$$

where $\hat{I}_{j,c,1} \equiv \frac{I_{j,c,1}}{\sqrt{\beta_j}}$, $\hat{I}_{j,s,1} \equiv \frac{I_{j,s,1}}{\sqrt{\beta_j}}$, $R_{i,j}^{thin}$ is the thin magnet response matrix as in equation 1 and $T_{i,j}^{thin}$ is the sinus version of it (let us call it sinus response matrix):

$$T_{i,j}^{thin} = \frac{\sqrt{\beta_i \beta_j}}{2sin(\pi\nu)} \tilde{S}_{i,j,1}. \tag{54}$$

Note that, in this section, the subindex j, unlike equation 1, refers to each one of the bending magnets starting longitudinal position. Now equation 53 can be derived respect to the quadrupole strengths:

$$\frac{d\eta_{x,i}}{dq_k} = \sum_j h_j \left(\frac{d\hat{I}_{j,c,1}}{dq_k} R_{i,j} + \frac{d\hat{I}_{j,s,1}}{dq_k} T_{i,j} + \hat{I}_{j,c,1} \frac{dR_{i,j}}{dq_k} + \hat{I}_{j,s,1} \frac{dT_{i,j}}{dq_k} \right). \tag{55}$$

The term $\frac{dR_{i,j}}{dq_k}$ in the previous equation was already derived in section 2. The $\frac{dT_{i,j}}{dq_k}$ term can be derived in a similar way, this will be done in the next subsection. The other terms pending to be derived are $\frac{d\hat{I}_{j,s,1}}{dq_k}$ and $\frac{d\hat{I}_{j,c,1}}{dq_k}$, those will be addressed in the second subsection. A comparison with the numerical calculations is presented in appendix B.

3.1 The sinus response matrix derivative

Equation 54 derivative can be taken similarly to equation 2:

$$\frac{dT_{i,j}}{dq_k} = \frac{\partial T_{i,j}}{\partial \beta_i} \frac{d\beta_i}{dq_k} + \frac{\partial T_{i,j}}{\partial \beta_j} \frac{d\beta_j}{dq_k} + \frac{\partial T_{i,j}}{\partial \nu} \frac{d\nu}{dq_k} + \frac{\partial T_{i,j}}{\partial \psi_i} \frac{d\psi_i}{dq_k} + \frac{\partial T_{i,j}}{\partial \psi_j} \frac{d\psi_j}{dq_k}, \tag{56}$$

Each of the derivatives with respect to the optical functions β, ψ and the tune ν are calculated from equation 54 and are expressed as follows:

$$\begin{aligned}
\frac{\partial T_{i,j}}{\partial \beta_i} =& \frac{\sqrt{\beta_j}}{4\sqrt{\beta_i} sin(\pi\nu)} \tilde{S}_{i,j,1} \\
\frac{\partial T_{i,j}}{\partial \beta_j} =& \frac{\sqrt{\beta_i}}{4\sqrt{\beta_j} sin(\pi\nu)} \tilde{S}_{i,j,1} \\
\frac{\partial T_{i,j}}{\partial \nu} =& -\pi \frac{\sqrt{\beta_i \beta_j}}{2sin^2(\pi\nu)} \left[\tilde{S}_{i,j,1} cos(\pi\nu) + \tilde{s}(\psi_i - \psi_j) C_{i,j,1} sin(\pi\nu) \right] \\
\frac{\partial T_{i,j}}{\partial \psi_i} =& -\delta(\psi_i - \psi_j) \frac{\sqrt{\beta_i \beta_j}}{2sin(\pi\nu)} \tilde{S}_{i,j,1} + \frac{\sqrt{\beta_i \beta_j}}{2sin(\pi\nu)} C_{i,j,1} \\
\frac{\partial T_{i,j}}{\partial \psi_j} =& \delta(\psi_i - \psi_j) \frac{\sqrt{\beta_i \beta_j}}{2sin(\pi\nu)} \tilde{S}_{i,j,1} - \frac{\sqrt{\beta_i \beta_j}}{2sin(\pi\nu)} C_{i,j,1}
\end{aligned} \tag{57}$$

The last two lines in the previous equations contain the Dirac´s delta function which takes infinite value at $\psi_i = \psi_j$. However, the contribution of the two terms from the two last lines cancels out. Combining equation 57 with equations 5, 6 and 14 we obtain the derivative of the sinus response matrix respect to thin quadrupole strengths.

$$\begin{aligned}
\frac{dT_{i,j}}{dq_k} =& -\frac{\sqrt{\beta_i \beta_j} \beta_k L_k}{8sin(\pi\nu) sin(2\pi\nu)} \Big[\tilde{S}_{i,j,1} \left[C_{i,k,2} + C_{j,k,2} + 2cos^2(\pi\nu) \right] \\
& + C_{i,j,1} \left[S_{j,k,2} - S_{i,k,2} + sin(2\pi\nu)(2\theta(\psi_j - \psi_k) - 2\theta(\psi_i - \psi_k)) + \tilde{s}(\psi_i - \psi_j) \right] \Big],
\end{aligned} \tag{58}$$

The thick quadrupole version reads:

$$\frac{dT_{i,j}}{dq_k} = -\frac{\sqrt{\beta_i\beta_j}}{8sin(\pi\nu)sin(2\pi\nu)}\left[\tilde{S}_{i,j,1}\left[\Gamma_{i,k,2}+\Gamma_{j,k,2}+2I_{k,0}cos^2(\pi\nu)\right]\right.$$
$$\left.+C_{i,j,1}\left[\Sigma_{j,k,2}-\Sigma_{i,k,2}+I_{k,0}sin(2\pi\nu)(2\tilde{\theta}(\psi_j-\psi_k)-2\tilde{\theta}(\psi_i-\psi_k))+\tilde{s}(\psi_i-\psi_j)\right]\right], \tag{59}$$

3.2 The other terms

We still need to take the derivatives $\frac{d\hat{I}_{j,s,1}}{dq_k}$ and $\frac{d\hat{I}_{j,c,1}}{dq_k}$. The term with $\hat{I}_{j,s,1}$ can be calculated using the chain rule again:

$$\frac{d\hat{I}_{j,s,1}}{dq_k} = \frac{\partial\hat{I}_{j,s,1}}{\partial\beta_j}\frac{d\beta_j}{dq_k}+\frac{\partial\hat{I}_{j,s,1}}{\partial q_j}\frac{dq_j}{dq_k} = -\frac{cos(\sqrt{q_j}L_j)-1}{q_j\beta_j^2}\frac{\beta_j\beta_k L_k}{2sin(2\pi\nu)}C_{j,k,2}+\frac{\partial\hat{I}_{j,s,1}}{\partial q_j}\delta_{j,k}, \tag{60}$$

where, the horizontal plane sign has been taken and $\delta_{j,k}$ is the Kronecker's delta which will only contribute when the quadrupole index corresponds to a bending magnet. The thick quadrupole form is:

$$\frac{d\hat{I}_{j,s,1}}{dq_k} = \hat{I}_{j,s,1}\frac{\Gamma_{j,k,2}}{2sin(2\pi\nu)}+\frac{\partial\hat{I}_{j,s,1}}{\partial q_j}\delta_{j,k}. \tag{61}$$

The term with $\hat{I}_{j,c,1}$ is slightly more complicated since it implies taking the derivative of the α_j function:

$$\frac{d\hat{I}_{j,c,1}}{dq_k} = \frac{\partial\hat{I}_{j,c,1}}{\partial\beta_j}\frac{d\beta_j}{dq_k}+\frac{\partial\hat{I}_{j,c,1}}{\partial\alpha_j}\frac{d\alpha_j}{dq_k}+\frac{\partial\hat{I}_{j,c,1}}{\partial q_j}\frac{dq_j}{dq_k}. \tag{62}$$

The α_j term can be solved since:

$$\frac{d\alpha_j}{dq_k} = -\frac{1}{2}\frac{d}{ds_j}\frac{d\beta_j}{dq_k}. \tag{63}$$

Combining the previous equation with the horizontal plane case (minus sign) of equation 6 we obtain:

$$\frac{d\alpha_j}{dq_k} = \frac{1}{2}\frac{d}{ds_j}\left[\frac{\beta_j\beta_k L_k}{2sin(2\pi\nu)}C_{j,k,2}\right], \tag{64}$$

which after some algebra becomes:

$$\frac{d\alpha_j}{dq_k} = -\frac{\beta_k L_k}{2sin(2\pi\nu)}\left[\alpha_j C_{j,k,2}+S_{j,k,2}\right]. \tag{65}$$

Now we can express analytically the contribution of the $\hat{I}_{j,c,1}$ derivative:

$$\begin{aligned}\frac{d\hat{I}_{j,c,1}}{dq_k} =& \frac{\alpha_j\left[cos(\sqrt{q_j}L_j)-1\right]}{q_j\beta_j}\frac{\beta_k L_k}{2sin(2\pi\nu)}C_{j,k,2}\\ &-\frac{cos(\sqrt{q_j}L_j)-1}{q_j\beta_j}\frac{\beta_k L_k}{2sin(2\pi\nu)}\left[\alpha_j C_{j,k,2}+S_{j,k,2}\right]+\frac{\partial\hat{I}_{j,c,1}}{\partial q_j}\delta_{j,k},\end{aligned} \tag{66}$$

which can be simplified and for the thin quadrupole case becomes:

$$\frac{d\hat{I}_{j,c,1}}{dq_k} = \frac{\beta_k L_k}{2sin(2\pi\nu)}\hat{I}_{j,s,1}S_{j,k,2}+\frac{\partial\hat{I}_{j,c,1}}{\partial q_j}\delta_{j,k}, \tag{67}$$

while in the thick quadrupole case it is written as:

$$\frac{d\hat{I}_{j,c,1}}{dq_k} = \frac{\hat{I}_{j,s,1}\Sigma_{j,k,2}}{2sin(2\pi\nu)}+\frac{\partial\hat{I}_{j,c,1}}{\partial q_j}\delta_{j,k}, \tag{68}$$

3.3 Complete dispersion derivative formula

Grouping all results together in the thick quadrupole case, equation 55 it becomes:

$$
\begin{aligned}
\frac{d\eta_{x,i}}{dq_k} = &\left[\frac{\partial \hat{I}_{k,c,1}}{\partial q_k}C_{i,k,1} + \frac{\partial \hat{I}_{k,s,1}}{\partial q_k}S_{i,k,1}\right]\frac{h_k\sqrt{\beta_i\beta_k}}{2sin(\pi\nu)} \\
&-\sum_j \frac{h_j\sqrt{\beta_i\beta_j}}{8sin(\pi\nu)sin(2\pi\nu)}\left[\hat{I}_{j,c,1}\left[C_{i,j,1}\left[\Gamma_{i,k,2}+\Gamma_{j,k,2}+2I_{k,0}cos^2(\pi\nu)\right]\right.\right. \\
&\left.+S_{i,j,1}\left[\Sigma_{i,k,2}-\Sigma_{j,k,2}+I_{k,0}sin(2\pi\nu)(2\tilde{\theta}(\psi_i-\psi_k)-2\tilde{\theta}(\psi_j-\psi_k)-s(\psi_i-\psi_j))\right]\right] \\
&+\hat{I}_{j,s,1}\left[\tilde{S}_{i,j,1}\left[\Gamma_{i,k,2}-\Gamma_{j,k,2}+2I_{k,0}cos^2(\pi\nu)\right]\right. \\
&\left.\left.+C_{i,j,1}\left[-\Sigma_{j,k,2}-\Sigma_{i,k,2}+I_{k,0}sin(2\pi\nu)(2\tilde{\theta}(\psi_j-\psi_k)-2\tilde{\theta}(\psi_i-\psi_k)+\tilde{s}(\psi_i-\psi_j))\right]\right]\right].
\end{aligned}
\tag{69}
$$

Where the first two terms only contribute when the quadrupole field number k is also a bending magnet and $h_k \neq 0$. Next, we make those two terms explicit:

$$
\begin{aligned}
\frac{\partial \hat{I}_{k,s,1}}{\partial q_k} &= \frac{1}{q_k\beta_k}\left[\frac{sin(\sqrt{q_k}L_k)}{2\sqrt{q_k}}L_k + \frac{cos(\sqrt{q_k}L_k)-1}{q_k}\right] \\
\frac{\partial \hat{I}_{k,c,1}}{\partial q_k} &= \frac{cos(\sqrt{q_k}L_k)}{2q_k}L_k - \frac{sin(\sqrt{q_k}L_k)}{2q_k^{3/2}} - \alpha_k\frac{\partial \hat{I}_{k,s,1}}{\partial q_k}
\end{aligned}
\tag{70}
$$

4 Complete thick quadrupole and thick corrector response matrix derivative

A thick corrector version of the response matrix in Eq.1 can be obtained integrating along the corrector length:

$$
R_{i,j} = \frac{\sqrt{\beta_i}}{2sin(\pi\nu)}\frac{1}{L_j}\int_0^{L_j}\sqrt{\beta_j(z)}cos(|\psi_i-\psi_j(z)|-\pi\nu)dz, \tag{71}
$$

This expression is equivalent to the dispersion formula in Eq.50 provided that dipoles are exchanged by correctors, without summation, without the curvature coefficient, and adding the length as a denominator instead. Hence, the response matrix derivative Eq.42 which was valid for thick quadrupoles and thin correctors can be extended to the thick corrector case using the result in Eq.69. The final complete formula valid for thick correctors and quadrupoles is:

$$
\begin{aligned}
\frac{dR_{i,j}}{dq_k} = &\mp\frac{\sqrt{\beta_i\beta_j}}{8sin(\pi\nu)sin(2\pi\nu)}\left[\frac{\hat{I}_{j,c,1}}{L_j}\left[C_{i,j,1}\left[\Gamma_{i,k,2}+\Gamma_{j,k,2}+2I_{k,0}cos^2(\pi\nu)\right]\right.\right. \\
&\left.+S_{i,j,1}\left[\Sigma_{i,k,2}-\Sigma_{j,k,2}+I_{k,0}sin(2\pi\nu)(2\tilde{\theta}(\psi_i-\psi_k)-2\tilde{\theta}(\psi_j-\psi_k)-s(\psi_i-\psi_j))\right]\right] \\
&+\frac{\hat{I}_{j,s,1}}{L_j}\left[\tilde{S}_{i,j,1}\left[\Gamma_{i,k,2}-\Gamma_{j,k,2}+2I_{k,0}cos^2(\pi\nu)\right]\right. \\
&\left.\left.+C_{i,j,1}\left[-\Sigma_{j,k,2}-\Sigma_{i,k,2}+I_{k,0}sin(2\pi\nu)(2\tilde{\theta}(\psi_j-\psi_k)-2\tilde{\theta}(\psi_i-\psi_k)+\tilde{s}(\psi_i-\psi_j))\right]\right]\right].
\end{aligned}
\tag{72}
$$

In most cases the corrector magnets do not have any quadrupolar component($q_j = 0$) so

$$\begin{aligned}\frac{\hat{I}_{j,c,1}}{L_j} &= \lim_{q_j->0} \frac{1}{L_j\sqrt{q_j}} sin(\sqrt{q_j}L_j) + \frac{\alpha_j\left[cos(\sqrt{q_j}L_j)-1\right]}{q_j\beta_jL_j} = 1-\frac{\alpha_jL_j}{2\beta_j}\\ \frac{\hat{I}_{j,s,1}}{L_j} &= \lim_{q_j->0} \frac{cos(\sqrt{q_j}L_j)-1}{q_j\beta_jL_j} = \frac{L_j}{2\beta_j}.\end{aligned} \tag{73}$$

In the thin corrector case $L_j << \beta_j$, then the previous equations become $\frac{\hat{I}_{j,c,1}}{L_j} = 1$ and $\frac{\hat{I}_{j,s,1}}{L_j} = 0$ so that Eq73 becomes its thin corrector version as in Eq.42.

5 Horizontal plane dispersion dipole derivative

LOCO does not considers the dipole curvature as a fitting parameter for the dispersion. However, as becomes clear inspecting equations 50 and 53, there is a close relation between the dipole's curvature and the dispersion function. In this section the analytical derivative of the dispersion with respect to the dipole's curvature is compared to the numerical results. Again, we will consider only the thick quadrupole case. Once more we use the chain rule to evaluate the derivative of equation 53:

$$\frac{d\eta_{x,i}}{dh_j} = \frac{\partial\eta_{x,i}}{\partial h_j} + \frac{\partial\eta_{x,i}}{\partial\alpha_j}\frac{d\alpha_j}{dh_j} + \frac{\partial\eta_{x,i}}{\partial q_j}\frac{dq_j}{dh_j}, \tag{74}$$

where the first term is much more important than the others. Using equations 45 and 49 the following indirect dependencies are derived:

$$\begin{aligned}\frac{d\alpha_j}{dh_j} &= -\,\beta_j tan(E_1)\\ \frac{dq_j}{dh_j} &= 2h_j.\end{aligned} \tag{75}$$

Hence all three terms in equation 74 can be written in terms of already known expressions:

$$\begin{aligned}\frac{\partial\eta_{x,i}}{\partial h_j} &= \hat{I}_{j,c,1}R_{i,j} + \hat{I}_{j,s,1}T_{i,j}\\ \frac{\partial\eta_{x,i}}{\partial\alpha_j}\frac{d\alpha_j}{dh_j} &= -\,h_jR_{i,j}tan(E_1)\frac{cos(\sqrt{q_j}L_j)-1}{q_j}\\ \frac{\partial\eta_{x,i}}{\partial q_j}\frac{dq_j}{dh_j} &= 2h_j^2\left[\frac{\partial\hat{I}_{j,c,1}}{\partial q_j}R_{i,j} + \frac{\partial\hat{I}_{j,s,1}}{\partial q_j}T_{i,j}\right].\end{aligned} \tag{76}$$

6 Combined function dipole

A change in the combined function dipole (CFD) has an impact on the orbit response matrix through two distinct terms. The first depends on the conventional quadrupole component change which has been detailed in the previous sections. An additional term arises from the orbit and energy variations resulting from the dipole component change.

The orbit variation introduces a quadupole-like effect that depends on the orbit at the sextupoles while the enegy variation introcudes a change on all the quadrupoles. In this section we make the these terms explicit for the horizontal plane case, the extrapolation to the vertical plane case is straight forward.

6.1 Complete Response matrix thick corrector formula

Usually CFD (here with index k) are long and the beam optics changes a lot along them. In those cases Eq. 1 are not valid, instead one needs to calculate:

$$R_{n,k} = \frac{\sqrt{\beta_n}}{2L_k sin(\pi\nu)} \int_0^{L_k} \sqrt{\beta_k(z)}\, cos(|\psi_n - \psi_k(z)| - \pi\nu)dz, \tag{77}$$

where n is a summation index over each BPM. When the energy variation produced by the kick is included one has:

$$R_{n,k} = \frac{\sqrt{\beta_n}}{2L_k sin(\pi\nu)} \int_0^{L_k} \sqrt{\beta_k(z)}\, cos(|\psi_n - \psi_k(z)| - \pi\nu)dz - \int_0^{L_k} \frac{\eta_n\eta_k}{\alpha_p C L_k}dz, \tag{78}$$

Where α_p is the ring momentum compaction factor, C is the ring circumference, η_n the dispersion function at evey n-th BPM and η_k the dispersion function at every CFD.

Notice that Eq.77 is equivalent to the dispersion contribution in Eq. 50 so the same calculation applies (except for the curvature h_j):

$$R_{n,k} = \left(\hat{I}_{k,c,1} R_{n,k}^{thin} + \hat{I}_{k,s,1} T_{n,k}^{thin}\right) - \frac{\eta_n \langle\eta_k\rangle}{\alpha_p C}, \tag{79}$$

where the term $-\frac{\langle\eta_k\rangle}{\alpha_p C}$ corresponds to the energy change induced by the kick.

6.2 Complete Response matrix thick corrector thick quadrupole derivative formula

In this case Eq. 15 needs to be updated with Eq. 79 result. Usually response matrices like $R_{n,k}$ in Eq. 79 are not measured, we do not need to fit it so there is no need to explicitly calculate that derivative.

6.3 BPM Orbit correction and energy changes

A given change in the diopole quadrupole component dq_k produces a poportional dipolar kick given by $db_k = \frac{b_k}{q_k}dq_k$, where b_k is the reference angle and q_k the reference quadrupole strength. The orbit change dx_n generated at the $n-th$ BPM is:

$$dx_n = R_{n,k}\frac{b_k}{q_k}dq_k + \eta_n d\delta_{non\,CFD}, \tag{80}$$

Where the energy change $d\delta_{non\,CFD}$ represents any energy change not due to the $k-th$ CFD since that is already included in the $R_{n,k}$ term. $d\delta_{non\,CFD}$ will be changed by the RF energy feedback (RFEF) so here it is left as a variable. This BPM orbit change should be corrected by each of the $m-th$ horizontal orbit corrector magnets (HCM). The HCM kick change $d\theta_m$ to perform the correction can be approximated by the inverse of their response matrix $R_{n,m}$:

$$d\theta_m = -\sum_n R_{n,m}^{-1} dx_n = -\sum_n R_{n,m}^{-1} R_{n,k}\frac{b_k}{q_k}dq_k - d\delta_{non\,CFD}\sum_n R_{n,m}^{-1}\eta_n. \tag{81}$$

Depending on how the RFEF is set different outcomes are possible, typically the orbit correctors equivalent energy change is zeroed:

$$d\delta_{HCM} = -\frac{1}{\alpha_c C}\sum_m \eta_m d\theta_m = 0. \tag{82}$$

To accomplish that goal, the RF frequency f_{RF} is then adequately changed, and the precise value is not of relevance here. Combining eq.81 and eq.82 we obtain:

$$d\delta_{non\,CFD} = -\frac{1}{\sum_{n,m}\eta_m R^{-1}_{n,m}\eta_n}\sum_{n,m}\eta_m R^{-1}_{n,m} R_{n,k}\frac{b_k}{q_k}dq_k. \tag{83}$$

Then, the total change of energy due to the orbit feedback and the RFEF is:

$$d\delta = d\delta_{HCM} + d\delta_{CFD} + d\delta_{non\,CFD} = -\left[\frac{\eta_n\langle\eta_k\rangle}{\alpha_p C} + \frac{1}{\sum_{n,m}\eta_m R^{-1}_{n,m}\eta_n}\sum_{n,m}\eta_m R^{-1}_{n,m}R_{n,k}\right]\frac{b_k}{q_k}dq_k. \tag{84}$$

6.4 Orbit effect at the sextupoles

Now both the HCM and CFD effects shold be combined to obtain the orbit at the $l-th$ sextupole x_l:

$$\begin{aligned} dx_{l,k} &= \sum_m R_{l,m}d\theta_j + R_{l,k}db_k + \eta_l d\delta_{non\,CDF} \\ &= \left[R_{l,k} - \sum_{m,n} R_{l,m}R^{-1}_{n,m}R_{n,k} - \frac{\eta_l + \sum_{n,m}R_{l,m}R^{-1}_{n,m}\eta_n}{\sum_{n,m}\eta_m R^{-1}_{n,m}\eta_n}\sum_{n,m}\eta_m R^{-1}_{n,m}R_{n,k}\right]\frac{b_k}{q_k}dq_k. \end{aligned} \tag{85}$$

Due to the extra orbit, every sextupole in the machine will have an optics effect dq_l that is proportional to that displacement and the sextupole polynomial strength s_l:

$$\frac{dq_l}{dq_k} = 2s_l\frac{dx_{l,k}}{dq_k} = 2s_l\left[R_{l,k} - \sum_{m,n}R_{l,m}R^{-1}_{n,m}R_{n,k} - \frac{\eta_l + \sum_{n,m}R_{l,m}R^{-1}_{n,m}\eta_n}{\sum_{n,m}\eta_m R^{-1}_{n,m}\eta_n}\sum_{n,m}\eta_m R^{-1}_{n,m}R_{n,k}\right]\frac{b_k}{q_k}. \tag{86}$$

The orbit resulting from the CFD change and the HCM correction is quite localized and not smoooth at all. Hence There can be large orbit changes along each sextupole. Besides, some sextupoles also act as HCM which may further contribute to a large orbit variation there. This forces us to recalculate the jacobian taking that orbit variation into account. The orbit inside the sextupole with a θ_l dipolar kick can be approximated as:

$$\begin{pmatrix} x_{l,k}(z) \\ x'_{l,k}(z)\end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\begin{pmatrix} x_{l,k} \\ x'_{l,k}\end{pmatrix} + \theta_l\begin{pmatrix}\frac{z^2}{2L_l} \\ \frac{z}{L_l}\end{pmatrix}. \tag{87}$$

Hence the quadrupolar variation effect inside the sextupole can be written as:

$$\frac{dq_l(z)}{dq_k} = 2s_l\left(\frac{dx_{l,k}}{dq_k} + \frac{dx'_{l,k}}{dq_k}z + \frac{d\theta_{l,k}}{dq_k}\frac{z^2}{2L_l}\right). \tag{88}$$

Where the orbit angle $\frac{dx'_{l,k}}{dq_k}$ can be obtained calculating Eq.86 at both the beginning and the end of the drift before each sextupole. The additional m-th HCM angle inside the l-th sextupole is calculated as:

$$\frac{d\theta_{l,k}}{dq_k} = R^{-1}_{n,l=m} R_{n,k} \frac{b_k}{q_k}. \tag{89}$$

Also, making use of the expressions detailed from Eq.17 to -Eq.22, we can express the optics variation within the sextupole as follows:

$$\begin{aligned} \beta_l(z) &= \beta_k - 2\alpha_l z + \gamma_l z^2 \\ \sqrt{\beta_l(z)} sin(\psi_l(z) - \psi_l) &= \frac{z}{\sqrt{\beta_l}} \\ \sqrt{\beta_l(z)} cos(\psi_l(z) - \psi_l) &= \frac{1}{\sqrt{\beta_l}} \left(\beta_l - z\alpha_l\right), \end{aligned} \tag{90}$$

To obtain the additional quadrupolar contribution at the sextupoles the thin quadrupole formula from Eq.15 needs to be integrated considering the previous orbit and optics variations inside the sextupoles.

The no phase term this time becomes an integral taking into account the orbit variation at the sextupoles:

$$\beta_l L_l \frac{dq_l}{dq_k} \longmapsto I_{l,k,0} \equiv \int_0^{L_l} \frac{dq_l(z)}{dq_k} \beta_l(z) dz, \tag{91}$$

which is integrated as follows:

$$\begin{aligned} I_{l,k,0} =& 2s_l \frac{dx_{l,k}}{dq_k} \beta_l L_l + s_l \left(\frac{dx'_{l,k}}{dq_k} \beta_l - 2\frac{dx_{l,k}}{dq_k} \alpha_l \right) L_l^2 \\ &+ 2s_l \left(\frac{dx_{l,k}}{dq_k} \gamma_l + \frac{d\theta_{l,k}}{dq_k} \frac{\beta_l}{2L_l} - 2\frac{dx'_{l,k}}{dq_k} \alpha_l \right) \frac{L_l^3}{3} + s_l \left(\frac{dx'_{l,k}}{dq_k} \gamma_l - \frac{d\theta_{l,k}}{dq_k} \frac{\alpha_l}{L_l} \right) \frac{L_l^4}{2} + s_l \frac{d\theta_{l,k}}{dq_k} \frac{\gamma_l}{5} L_l^4 \\ =& \beta_l L_l s_l \left(2\frac{dx_{l,k}}{dq_k} + L_l \left(\frac{dx'_{l,k}}{dq_k} + \frac{1}{3}\frac{d\theta_{l,k}}{dq_k} \right) \right) - \alpha_l L_l^2 s_l \left(2\frac{dx_{l,k}}{dq_k} + L_l \left(\frac{4}{3}\frac{dx'_{l,k}}{dq_k} + \frac{1}{2}\frac{d\theta_{l,k}}{dq_k} \right) \right) \\ &+ \gamma_l L_l^3 s_l \left(\frac{2}{3}\frac{dx_{l,k}}{dq_k} + L_l \left(\frac{1}{2}\frac{dx'_{l,k}}{dq_k} + \frac{1}{5}\frac{d\theta_{l,k}}{dq_k} \right) \right) \end{aligned} \tag{92}$$

The sin term taking into account the orbit variation at the sextupole:

$$\beta_l L_l \, sin(2(\psi_l(z) - \psi_l)) \frac{dq_l}{dq_k} \longmapsto I_{l,k,s,2} \equiv \int_0^{L_l} \frac{dq_l(z)}{dq_k} \beta_l(z) \, sin(2(\psi_l(z) - \psi_l)) dz, \tag{93}$$

which is integrated as follows:

$$\begin{aligned} I_{l,k,s,2} &= \int_0^{L_l} \frac{dq_l(z)}{dq_k} 2\beta_l(z) \, sin((\psi_l(z) - \psi_l)) cos((\psi_l(z) - \psi_l)) dz \\ &= 2s_l \frac{dx_{l,k}}{dq_k} L_l^2 + \frac{4s_l}{3} \left(\frac{dx'_{l,k}}{dq_k} - \frac{dx_{l,k}}{dq_k} \frac{\alpha_l}{\beta_l} \right) L_l^3 + s_l \left(\frac{1}{2L_l} \frac{d\theta_{l,k}}{dq_k} - \frac{dx'_{l,k}}{dq_k} \frac{\alpha_l}{\beta_l} \right) L_l^4 - 2s_l \frac{d\theta_{l,k}}{dq_k} \frac{\alpha_l}{5\beta_l} L_l^4 \\ &= L_l^2 s_l \left(2\frac{dx_{l,k}}{dq_k} + L_l \left(\frac{4}{3}\frac{dx'_{l,k}}{dq_k} + \frac{1}{2}\frac{d\theta_{l,k}}{dq_k} \right) \right) - 2\frac{\alpha_l}{\beta_l} L_l^3 s_l \left(\frac{2}{3}\frac{dx_{l,k}}{dq_k} + L_l \left(\frac{1}{2}\frac{dx'_{l,k}}{dq_k} + \frac{1}{5}\frac{d\theta_{l,k}}{dq_k} \right) \right). \end{aligned} \tag{94}$$

The cos term taking into account the orbit variation at the sextupole:

$$\beta_l L_l \, cos(2(\psi_l(z) - \psi_l)) \frac{dq_l}{dq_k} \longmapsto I_{l,k,c,2} \equiv \int_0^{L_l} \frac{dq_l(z)}{dq_k} \beta_l(z) \, cos(2(\psi_l(z) - \psi_l)) dz, \tag{95}$$

which is integrated as in the case of Eq27:

$$\begin{aligned} I_{l,k,c,2} &= I_{l,0} - 2\int_0^{L_l} \frac{dq_l(z)}{dq_k}\beta_l(z)\, sin^2((\psi_l(z) - \psi_l))dz \\ &= I_{l,k,0} - 2\int_0^{L_l} \frac{dq_l(z)}{dq_k}\frac{z^2}{\beta_l} \\ &= I_{l,k,0} - \frac{L_l^3 s_l}{\beta_l}\left(\frac{2}{3}\frac{dx_{l,k}}{dq_k} + L_l\left(\frac{1}{2}\frac{dx'_{l,k}}{dq_k} + \frac{1}{5}\frac{d\theta_{l,k}}{dq_k}\right)\right). \end{aligned} \tag{96}$$

The terms that appear in the thin aproximation of the response matrix derivative are the following:

$$\begin{aligned} \beta_l\frac{dq_l}{dq_k}L_l\, S_{i,l,2} &\longmapsto \Sigma_{i,l,k,2} \equiv \int_0^{L_l}\beta_l(z)\frac{dq_l(z)}{dq_k}\, s(\psi_i - \psi_l(z))sin(2|\psi_i - \psi_l(z)| - 2\pi\nu)dz \\ \beta_l\frac{dq_l}{dq_k}L_l\, C_{i,l,2} &\longmapsto \Gamma_{i,l,k,2} \equiv \int_0^{L_l}\beta_l(z)\frac{dq_l(z)}{dq_k}\, cos(2|\psi_i - \psi_l(z)| - 2\pi\nu)dz \\ \beta_l\frac{dq_l}{dq_k}L_l\, \theta(\psi_i - \psi_l) &\longmapsto \Delta_{i,l,k,2} \equiv \int_0^{L_l}\beta_l(z)\frac{dq_l(z)}{dq_k}\theta(\psi_i - \psi_l(z))dz. \end{aligned} \tag{97}$$

Here the term $\beta_l(z)\frac{dq_l(z)}{dq_k}$ plays the same role as $\beta_k(z)$ in Eq.29. It can be can be solved in an equivalent using the $\tilde{s}$ and $\tilde{\theta}$ functions defined in Eq.39 and Eq.40:

$$\begin{aligned} \Sigma_{i,l,k,2} &= I_{l,k,c,2}\tilde{S}_{i,l,2} - I_{l,k,s,2}C_{i,l,2} \\ \Gamma_{i,l,k,2} &= I_{l,k,c,2}C_{i,l,2} + I_{l,k,s,2}\tilde{S}_{i,l,2} \\ \Delta_{i,l,k,2} &= I_{l,k,0}\tilde{\theta}(\psi_i - \psi_l), \end{aligned} \tag{98}$$

As in the Eq.42 case, with the above described modification the response matrix change due to the orbit change at the l-th sextupole produced by the k-th CFD can be written as:

$$\begin{aligned} \frac{\partial R_{i,j}}{\partial q_l}\frac{dq_l}{dq_k} = \mp \frac{\sqrt{\beta_i\beta_j}}{8sin(\pi\nu)sin(2\pi\nu)}&\left[C_{i,j,1}\left[\Gamma_{i,l,k,2} + \Gamma_{j,l,k,2} + 2I_{l,k,0}cos^2(\pi\nu)\right]\right. \\ &\left.+S_{i,j,1}\left[\Sigma_{i,l,k,2} - \Sigma_{j,l,k,2} + 2\Delta_{i,l,k,2}sin(2\pi\nu) - 2\Delta_{j,l,k,2}sin(2\pi\nu) - I_{l,k,0}sin(2\pi\nu)s(\psi_i - \psi_j)\right]\right], \end{aligned} \tag{99}$$

This equation can be slightly simplified so that the response matrix change produced by the orbit at the l-th sextupole produced by the k-th CFD can be written as:

$$\begin{aligned} \frac{\partial R_{i,j}}{\partial q_l}\frac{dq_l}{dq_k} = \mp \frac{\sqrt{\beta_i\beta_j}}{8sin(\pi\nu)sin(2\pi\nu)}&\left[C_{i,j,1}\left(\Gamma_{i,l,k,2} + \Gamma_{j,l,k,2}\right) + S_{i,j,1}\left(\Sigma_{i,l,k,2} - \Sigma_{j,l,k,2}\right)\right. \\ &\left.+2S_{i,j,1}sin(2\pi\nu)\left(\Delta_{i,l,k,2} - \Delta_{j,l,k,2}\right) + 2I_{l,k,0}cos(\pi\nu)cos(\psi_i - \psi_j)\right], \end{aligned} \tag{100}$$

For every CFD the effect of each sextupole has to be summed up, so the previous equation will com into play as a sum for all the sextupoles l:

$$\begin{aligned} \sum_l \frac{\partial R_{i,j}}{\partial q_l}\frac{dq_l}{dq_k} = \mp \frac{\sqrt{\beta_i\beta_j}}{8sin(\pi\nu)sin(2\pi\nu)}&\left[C_{i,j,1}\left(\Gamma^{CFD}_{i,k,2} + \Gamma^{CFD}_{j,k,2}\right) + S_{i,j,1}\left(\Sigma^{CFD}_{i,k,2} - \Sigma^{CFD}_{j,k,2}\right)\right. \\ &\left.+2S_{i,j,1}sin(2\pi\nu)\left(\Delta^{CFD}_{i,k,2} - \Delta^{CFD}_{j,k,2}\right) + 2I^{CFD}_{k,0}cos(\pi\nu)cos(\psi_i - \psi_j)\right], \end{aligned} \tag{101}$$

where the following definitions are used:

$$
\begin{aligned}
I^{CFD}_{k,0} &= \sum_l I_{l,k,0} \\
\Sigma^{CFD}_{i,k,2} &= \sum_l I_{l,k,c,2}\tilde{S}_{i,l,2} - I_{l,k,s,2}C_{i,l,2} \\
\Gamma^{CFD}_{i,k,2} &= \sum_l I_{l,k,c,2}C_{i,l,2} + I_{l,k,s,2}\tilde{S}_{i,l,2} \\
\Delta^{CFD}_{i,k,2} &= \sum_l I_{l,k,0}\tilde{\theta}(\psi_i - \psi_l),
\end{aligned}
\tag{102}
$$

6.5 Total CFD effect on the response matrix

The effect of the CFD Jacobian $\frac{dR_{i,j}}{dq_k}|_{CFD}$ is the addition of the direct quadrupole term plus a term coming from the orbit induced quadrupolar term at the sextupoles made explicitly in Eq. 101 and the energy term in Eq.**??**:

$$
\left.\frac{dR_{i,j}}{dq_k}\right|_{CFD} = \frac{dR_{i,j}}{dq_k} + \sum_l \frac{dR_{i,j}}{dq_l}\frac{dq_l}{dq_k} + \frac{dR_{i,j}}{d\delta}\frac{d\delta}{dq_k} \tag{103}
$$

The response matrix energy derivative $\frac{dR_{i,j}}{d\delta}$ in the previous equation can be analytically obtained in two different ways:

$$
\begin{aligned}
&Method\,1 \qquad \frac{dR_{i,j}}{d\delta} = \frac{R_{i,j}(d\delta) - R_{i,j}(0)}{d\delta} \\
&Method\,2 \qquad \frac{dR_{i,j}}{d\delta} = \sum_r \frac{dR_{i,j}}{dq_r}q_r
\end{aligned}
\tag{104}
$$

The first method is slower since it implies calculating the all the optics functions again for an slightly different energy. The results are identical within numerical noise.

6.6 Horizontal plane dispersion CFD derivative

In this case, the pure quadrupole formula in Eq.69 is complemented with an additional term proportional to the additional changes in energy similar to Eq.104, a term similar to Eq.101 and also a term given by the curvature changes when changing the the CFD.

$$
\left.\frac{d\eta_{x,i}}{dq_k}\right|_{CFD} = \frac{d\eta_{x,i}}{dq_k} + \sum_l \frac{d\eta_i}{dq_l}\frac{dq_l}{dq_k} + \frac{\partial\eta_{x,i}}{\partial\delta}\frac{d\delta}{dq_k} + \sum_j \frac{\partial\eta_{x,i}}{\partial h_j}\frac{dh_j}{dq_k} \tag{105}
$$

Where the index j here includes all elements that generate curvature: all the correctors and the bending magnets. We will deal first with the energy term which is simpler and later with the curvature term.

6.6.1 Energy term

The energy variation can be obtained numerically calculating the optics for an slightly different energy or analytically combining the energy effect of the ring dipoles and quadrupoles and the orbit effect at quadrupoles and sextupoles:

$$
\frac{\partial\eta_{x,i}}{\partial\delta} = \sum_k \frac{\partial\eta_{x,i}}{\partial q_k}\frac{dq_k}{d\delta} + \sum_j \frac{\partial\eta_{x,i}}{\partial h_j}\frac{dh_j}{d\delta} + 2\sum_l \frac{\partial\eta_{x,i}}{\partial q_l}\eta_{x,l}s_l + \sum_k \frac{\partial\eta_{x,i}}{\partial h_k}\eta_{x,k}q_k \tag{106}
$$

The quadrupolar component $\frac{\partial \eta_{x,i}}{\partial q_j}$ can obtained using Eq.69. The dipolar component $\frac{\partial \eta_{x,i}}{\partial h_j}$ is described by Eq.74. The dependency on the energy of the quadrupolar and dipolar terms can be assumed to be linear:

$$\begin{aligned} \frac{dq_j}{d\delta} &= -q_j \\ \frac{dh_j}{d\delta} &= -h_j \end{aligned} \tag{107}$$

6.6.2 Curvature term

The third term in the Eq.105 r.h.s. should contain al curvature sources which after a CFD change are the HCM correctors, the quadrupoles and the dipoles. In the case of correctors, the curvature change can be calculated using Eq.81

$$\left.\frac{dh_j}{dq_k}\right|_{HCM} = \frac{1}{L_j}\frac{dH_j}{dq_k} = -\frac{1}{L_j}\sum_n R^{-1}_{n,m} R_{n,k}\frac{b_k}{q_k} \tag{108}$$

The contribution from the quadrupoles can be calculated with the angle change at the quadrupole, for that we can use the position and angle change (equivalent to Eq.85 but at the quadrupoles instead of the sextupoles) and the transfer matrix Eq.17:

$$\left.\frac{dh_j}{dq_k}\right|_{quad} = -\frac{1}{L_j}\frac{dx_{j,k}}{dq_k}\sqrt{q_j}sin(\sqrt{q_j}L_j) + \frac{1}{L_j}\frac{dx'_{j,k}}{dq_k}(cos(\sqrt{(q_j)}L_j) - 1) \tag{109}$$

Finally the contribution from the dipoles is given by the change in the dipolar component of the same CFD that is varied:

$$\left.\frac{dh_j}{dq_k}\right|_{CFD} = \frac{h_j}{q_k}\delta_{j,k} \tag{110}$$

Using Eq.53, each term contributes proportionally to:

$$\frac{\partial \eta_{x,i}}{\partial h_j} = \hat{I}_{j,c,1}R^{thin}_{i,j} + \hat{I}_{j,s,1}T^{thin}_{i,j} \tag{111}$$

6.7 Orbit effect at the sextupoles

Similarly to subsection 6.4, the orbit at every sextupole produces a quadrupolar component, which produces an effect that should be integrated along each $l-th$ sextupole's length. This time we start with Eq.69 and repeat the same approximations of second order polynomial variation of the orbit inside the sextupoles as in the case of the response matrix derivative.

$$\begin{aligned} \sum_l \frac{d\eta_{x,i}}{dq_l}\frac{dq_l}{dq_k} = -\sum_j \frac{h_j\sqrt{\beta_i\beta_j}}{8sin(\pi\nu)sin(2\pi\nu)} & \left[\hat{I}_{j,c,1}\left[C_{i,j,1}\left[\Gamma^{CFD}_{i,k,2} + \Gamma^{CFD}_{j,k,2} + 2I^{CFD}_{k,0}cos^2(\pi\nu)\right]\right.\right. \\ & \left. + S_{i,j,1}\left[\Sigma^{CFD}_{i,k,2} - \Sigma^{CFD}_{j,k,2} + sin(2\pi\nu)(2\Delta^{CFD}_{i,k,2} - 2\Delta^{CFD}_{j,k,2} - I^{CFD}_{k,0}s(\psi_i - \psi_j))\right]\right] \\ & + \hat{I}_{j,s,1}\left[\tilde{S}_{i,j,1}\left[\Gamma^{CFD}_{i,k,2} - \Gamma^{CFD}_{j,k,2} + 2I^{CFD}_{k,0}cos^2(\pi\nu)\right]\right. \\ & \left.\left. + C_{i,j,1}\left[-\Sigma^{CFD}_{j,k,2} - \Sigma^{CFD}_{i,k,2} + sin(2\pi\nu)(2\Delta^{CFD}_{j,k,2} - 2\Delta^{CFD}_{i,k,2} + I^{CFD}_{k,0}\tilde{s}(\psi_i - \psi_j))\right]\right]\right]. \end{aligned} \tag{112}$$

Notice that the first two terms of Eq.69 do not appear in the previous equation. Those would correspond to a quadrupolar term in the sextupole proportional to the generated extra curvature and that would be a second order term.

7 Off-diagonal response matrix and vertical plane dispersion

A.Franchi [3] found an expression for the off-diagonal response matrix and vertical plane dispersion change as a function of the skew quadrupole strengths. In the next sections we will make use of that same formulas which we repeat here just for completeness. First, the off diagonal response matrix $R_{ij}^{(xy)}$ and $R_{ij}^{(yx)}$ derivatives are:

$$\begin{aligned}\frac{dR_{ij}^{(xy)}}{ds_k} \simeq & \frac{L_k}{8}\sqrt{\beta_{i,x}\beta_{j,y}\beta_{k,x}\beta_{k,y}}\left[\frac{1}{sin[\pi(Q_x-Q_y)]}\left[\frac{cos(\tau_{x,ki}-\tau_{y,ki}+\tau_{y,ji})}{sin(\pi Q_y)}-\frac{cos(\tau_{x,kj}-\tau_{y,kj}+\tau_{x,ji})}{sin(\pi Q_x)}\right]\right. \\ & \left. +\,[\frac{1}{sin[\pi(Q_x+Q_y)]}\left[\frac{cos(\tau_{x,ki}+\tau_{y,ki}-\tau_{y,ji})}{sin(\pi Q_y)}+\frac{cos(\tau_{x,kj}+\tau_{y,kj}+\tau_{x,ji})}{sin(\pi Q_x)}\right]\right] \\ \frac{dR_{ij}^{(yx)}}{ds_k} \simeq & \frac{L_k}{8}\sqrt{\beta_{i,y}\beta_{j,x}\beta_{k,x}\beta_{k,y}}\left[\frac{1}{sin[\pi(Q_x-Q_y)]}\left[-\frac{cos(\tau_{x,ki}-\tau_{y,ki}-\tau_{x,ji})}{sin(\pi Q_x)}+\frac{cos(\tau_{x,kj}-\tau_{y,kj}-\tau_{y,ji})}{sin(\pi Q_y)}\right]\right. \\ & \left. +\,[\frac{1}{sin[\pi(Q_x+Q_y)]}\left[\frac{cos(\tau_{x,ki}+\tau_{y,ki}-\tau_{x,ji})}{sin(\pi Q_x)}+\frac{cos(\tau_{x,kj}+\tau_{y,kj}+\tau_{y,ji})}{sin(\pi Q_y)}\right]\right],\end{aligned} \tag{113}$$

where s_k represents the $k-th$ skew quadrupole strength and $\tau_{x,ij}$ and $\tau_{y,ij}$ are phase advance differences defined as follows:

$$\tau_{w,ab} = \begin{cases} \psi_{w,b}-\psi_{w,a}-\pi Q_w & if \psi_{w,b} > \psi_{w,a} \\ \psi_{w,b}-\psi_{w,a}+\pi Q_w & if \psi_{w,a} > \psi_{w,b} \end{cases}, \quad w = x, y \tag{114}$$

Also, according to reference [3], the vertical plane dispersion derivative is expressed:

$$\frac{d\eta_{y,i}}{ds_k} = \frac{L_k\sqrt{\beta_{y,i}\beta_{y,k}}}{2sin(\pi\nu_y)}\eta_{x,k}cos(\tau_{y,ik}), \tag{115}$$

In the case of the skew magnets, the thick magnet formula has not been used. At ALBA those magnets are quite thin and are located at places where the optical functions vary quite linearly. Hence, it is enough to use the average of the optics functions.

7.1 Thick skew magnet case

First, notice that according to equation 4 definitions, we have the following relations between Andrea's notation and ours:

$$\begin{aligned} cos(\tau_{w,ab}) =& C_{w,b,a,1} = cos(n|\psi_{w,b}-\psi_{w,a}|-\pi\nu_x) \\ sin(\tau_{w,ab}) =& S_{w,b,a,1} = s(\psi_{w,b}-\psi_{w,a})sin(|\psi_{w,b}-\psi_{w,a}|-\pi\nu), \end{aligned} \tag{116}$$

Also, notice that, regarding the magnet index k, all terms in equation 113 are variations of the following expression:

$$\sqrt{\beta_{x,k}\beta_{y,k}}L_k\,cos(\tau_{x,ka} \pm_1 \tau_{y,ka} \pm_2 \tau_{w,ji}), \quad a = i, j, \quad w = x, y \tag{117}$$

The thick magnet equivalent of those terms will ve called $\Omega_{k,a,w}$. It is defined as:

$$\Omega_{k,a,w} \equiv \int_0^{L_k} \sqrt{\beta_{x,k}(z)\beta_{y,k}(z)}\,cos(\tau_{x,ka}(z) \pm_1 \tau_{y,ka}(z) \pm_2 \tau_{w,ji})dz \tag{118}$$

The previous equation can be expanded and it has the following general expression:

$$\begin{aligned}\Omega_{k,a,w} \equiv \int_0^{L_k} dz\sqrt{\beta_{x,k}(z)\beta_{y,k}(z)}\ [&C_{x,k,a,1}(z)C_{y,k,a,1}(z)C_{w,j,i,1}\\ &\mp_1 S_{x,k,a,1}(z)S_{y,k,a,1}(z)C_{w,j,i,1}\\ &\mp_2 S_{x,k,a,1}(z)C_{y,k,a,1}(z)S_{w,j,i,1}\\ &\mp_2 \pm_1 C_{x,k,a,1}(z)S_{y,k,a,1}(z)S_{w,j,i,1}]\end{aligned} \tag{119}$$

7.1.1 Sin × sin like term

This term does not appear explicitly in the equation 113, but it is useful for some of them, for example when integrating the $\sqrt{\beta_{x,k}\beta_{y,k}}S_{x,i,k,1}S_{y,i,k,1}$ term. The following substitution should be done:

$$\begin{aligned}&\sqrt{\beta_{x,k}\beta_{y,k}}L_k\, sin(\psi_{x,k}(z)-\psi_{x,k})\, sin(\psi_{y,k}(z)-\psi_{y,k}) \longmapsto I_{k,ss,1}\\ &I_{k,ss,1} \equiv \int_0^{L_k} \sqrt{\beta_{x,k}(z)\beta_{y,k}(z)}\, sin(\psi_{x,k}(z)-\psi_{x,k})\, sin(\psi_{y,k}(z)-\psi_{y,k})dz,\end{aligned} \tag{120}$$

Again we need to make use of equation 23 wich

$$I_{k,ss,1} = \frac{1}{q_k\sqrt{\beta_{x,k}\beta_{x,k}}}\int_0^{L_k} sin(\sqrt{q_k}z)\, sinh(\sqrt{q_k}z)dz, \tag{121}$$

The previous integral is also solved in Andrea's paper, see appendix C in [3].

$$I_{k,ss,1} = \frac{1}{q_k\sqrt{\beta_{x,k}\beta_{x,k}}}\int_0^{L_k} sin(\sqrt{q_k}z)\, sinh(\sqrt{q_k}z)dz, \tag{122}$$

8 Conclusions

An explicit and analytical form of the ORM and dispersion function derivatives has been calculated. This allows to implement it in the LOCO code that performs the fit, replacing the numerical assessment used by the standard version of LOCO. The LOCO fit procedure has been made faster by a factor 4 by replacing the numerical coupled constant path (NCCP) simulations by the analytical uncoupled constant energy (AUCE) expressions. In particular, the LOCO fit included in the ALBA weekly startup procedure will take 2 min instead of 8min. The simplifications in the formulas have an impact below at the 10^{-4} level in beta beat and quadrupole strength corrections. For our purposes, the present level of agreement is satisfactory and the new LOCO script will be used in the ALBA storage ring weekly setup procedure.

A Appendix A: Numerical comparison for ALBA ORM

For the numerical comparison we will use the Matlab based tracking code AT [6] . The analytical calculation of the response matrix derivative with respect each to each one of the 112 quadrupoles and the 32 combined function bending magnets using equation 2 takes 0.7 seconds. On the other hand, calculating the numerical difference of two response matrices having changed one quadrupole takes 0.4 seconds. Hence the analytical method is potentially 32 times faster. As stated previously, the case described so far includes only constant energy calculations. The error associated to that simplification has been numerically evaluated for the ALBA case, and corresponds to a 0.28% rms error in the horizontal plane while in the vertical plane there is no significant effect. Next the previously described formulas will be compared with numerical constant energy simulations. Since, at ALBA, the beta functions in the combined function dipoles have a pronounced hyperbolic variation, those cases will be treated separately from the rest of normal quadupoles.

ORMS error	Hor.Plane				Vert.Plane			
	thin		thick		thin		thick	
	quads	dipoles	quads	dipoles	quads	dipoles	quads	dipoles
whitout coupling	1.18%	10.72%	0.00%	0.00%	1.67%	1.21%	0.00%	0.00%
0.5% coupling	1.18%	11.03%	0.04%	1.14%	1.74%	1.45%	0.40%	0.28%

Table 1: ORMS error of equation 15 and 42 respect to the numerical constant energy response matrix derivative.

B Appendix B: Numerical comparison for the ALBA dispersion derivative

This time, only the thick quadrupole case (equation 69) is considered. The results have been compared to the constant energy 4D simulations. The ORMS difference of the 4D simulations respect to the constant frequency 6D simulations is 0.28% both with and without coupling. Since, at ALBA, the beta function in the combined function dipoles has a pronounced minimum, those cases will be treated separately from the rest of normal quadupoles.

ORMS error	quads	dipoles
whitout coupling	0.04%	0.05%
0.5% coupling	0.22%	2.00%

Table 2: ORMS error of equation 69 respect to the numerical constant energy dispersion derivative.

C Appendix C: Analytical uncoupled constant energy LOCO versus numerical coupled constant path LOCO results

The thick quadrupole analytical formulas described so far have been used to speed up the LOCO response matrix and dispersion derivatives calculation.

The analytical uncoupled constant energy (AUCE) formulas assume neither there is any change of the off-diagonal response matrix with respect to the quadrupoles strengths, nor that there is any change of the diagonal response matrix with respect to the skew quadrupole strengths. Also, the AUCE formulas do not take into account the constant path corrections that are due in electron machines as ALBA.

In this section the fitting results using such formulas are compared to the numerical coupled constant path (NCCP) LOCO method. In the next subsections the LOCO fit is compared on

60 simulated lattices and also on 20 real measurement data sets. In the case of the simulated lattices, the fitted machine functions can be compared to the simulated ones. In the case of the measurement data sets, the NCCP LOCO fits are compared with the AUCE LOCO fit.

C.1 Random Simulated latices

For the comparison 60 simulated LOCO measurements have been used and the fit result has been compared to the modeled machine functions. Several LOCO fitting schemes have been used: the quadrupole part is fitted with the 112 quadrupoles (112Q) or also using as a fit parameter the quadrupole strengths of the combined function dipoles (112Q+32D). The skew part has been fitted using the 32 available skew magnets (32S) or using the 120 sextupoles as coupling sources (120S). In each case, 5 fit iterations are used. The fitting algorithm used was the scaled Levenberg Marquardt with $\lambda = 0.05$.

Dispite the AUCE approximation, as table 3 shows, the LOCO fit agrees remarkably well with the NCCP LOCO fit. This is not very surprising since a small error in the derivatives as shown in tables 2 and 1 is washed away in the iterative LOCO process. Regarding the total LOCO evaluation time, using the analytical formulas only reduces a factor 3 or 4 improvement. The improvement is small compared to the calculation time difference between the formulas and the numerical simulations. However, the total LOCO fit time includes other tasks like the the LOCO matrix SVD calculation or the fitting structure construction.

ORMS error wrt simulation	112Q				112Q+32D			
	32S		120S		32S		120S	
	AUCE	NCCP	AUCE	NCCP	AUCE	NCCP	AUCE	NCCP
$\Delta\beta_x/\beta_x[\%]$	1.55	1.54	1.55	1.54	0.92	0.91	0.92	0.92
$\Delta\beta_y/\beta_y[\%]$	2.47	2.47	2.47	2.47	1.00	0.99	1.00	1.00
$\Delta\eta_x/\eta_x[\%]$	1.03	1.02	1.03	1.02	0.66	0.66	0.66	0.66
$\Delta\eta_y[mm]$	0.91	0.92	0.30	0.35	0.91	0.91	0.29	0.29
$\Delta\epsilon_y/\epsilon_x[\%]$	0.02	0.02	0.01	0.01	0.02	0.19	0.01	0.01
$\Delta\theta[mrad]$	5.67	5.70	2.90	2.97	5.62	5.64	2.81	2.80
$\Delta k_{quad}/k_{quad}[\%]$	0.03		0.03		0.01		0.01	
Elapsed time $[min]$	3.69	10.09	4.60	16.48	4.13	11.77	5.07	18.11

Table 3: LOCO fit ORMS error for various quantities, for the AUCE and NCCP cases. In both cases the fit result is compared with the model machine functions at every lattice element. Also the quadrupole fit parameters and the total LOCO analysis time are compared.

C.2 Measured data Loco fit differences

In this chapter the differences between the AUCE and NCCP LOCO fits to 20 measured data sets are listed. The data sets where acquired from 09/05/2016 to 31/10/2016 as part of the ALBA routine startup procedure. For each case two fitting parameters schemes are used: 112Q+32S and 112Q+32D+120S as during the startup procedure. Table 4 shows a very good agreement for all the machine functions of the two LOCO fits. The typical CPU time for the NCCP LOCO fits is 8min while for the AUCE and NCCP LOCO fits is 2min.

ORMS difference	112Q+32S	112Q+32D+120S
$\Delta\beta_x/\beta_x[\%]$	0.03	0.05
$\Delta\beta_y/\beta_y[\%]$	0.02	0.04
$\Delta\eta_x/\eta_x[\%]$	0.44	0.78
$\Delta\eta_y[mm]$	0.03	0.06
$\Delta\epsilon_y/\epsilon_x[\%]$	0.00	0.00
$\Delta\theta[mrad]$	0.26	0.33
$\Delta k_{quad}/k_{quad}[\%]$	0.00	0.01

Table 4: LOCO fit ORMS difference.