Taller #3 Metodos Mat.

Seccion 3.14

6) En el caso 3D tenemos que si $\{e_i\}$ define un sistema de coordenadas (dextrogiro) no necesariamente ortogonal, entonces demuestre que

(a)

$$e^i = \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)}, \quad i, j, k = 1, 2, 3 \text{ y sus permutaciones cíclicas}$$

$$e^i \cdot e_i = 1$$

$$e^i = \alpha (e_j \times e_k) \rightarrow e^i \text{ es ortonormal a } e_j \text{ y } e_k$$

entonces

$$\alpha = \frac{e^i}{e_j \times e_k}$$

$$e^i \cdot e_i = \alpha (e_j \times e_k) e_i$$

$$1 = \alpha e_i (e_j \times e_k)$$

$$1 = \frac{e^i}{e_j \times e_k} e_i (e_j \times e_k)$$

$$e^i = \frac{(e_j \times e_k)}{e_i (e_j \times e_k)}$$

(b) Si los volumenes : $V = e_i \cdot (e_2 \times e_3)$ y $\tilde{V} = e^1 \cdot (e^2 \times e^3)$, entonces $V\tilde{V} = 1$

Sabemos que $e^1 = e_2 \times e_3$

reemplazando en $\tilde{V}$

$$\tilde{V} = (e_2 \times e_3)(e^2 \times e^3) = 1$$

$$\tilde{V} = e_1 \cdot e^1 \quad , \quad V = e^1 \cdot e_1$$

$$\tilde{V} = 1 \quad , \quad V = 1$$

$$V\tilde{V} = 1$$

ahora en $V = e_1 (e_2 \times e_3)$

$$V = (e^2 \times e^3)(e_2 \times e_3)$$

c) ¿Qué vector satisface $a \cdot e^i = 1$? Demuestre que a es único

Podemos partir de lo demostrado anteriormente

$$e^i = \frac{e_j \times e_K}{e_i \cdot (e_j \times e_K)}$$

$$a \cdot e^i = 1$$

$$a \cdot \left(\frac{e_j \times e_K}{e_i \cdot (e_j \times e_K)} \right) = 1$$

$$a \cdot (e_j \times e_K) = e_i \cdot (e_j \times e_K)$$

$$a \cdot (e_j \times e_K) - e_i (e_j \times e_K) = 0$$

$$(e_j \times e_K)(a - e_i) = 0$$

Esto tiene que ser $\neq 0$

$$a - e_i = 0$$

$$\boxed{a = e_i}$$

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d) Encuentre el producto vectorial de dos vectores a y b que estan representadas en un sistema de coordenadas oblicuo

Dada la base $w_1 = 4i + 2j + k,$

$w_2 = 3i + 3j,$

$w_3 = 2k$

Entonces encuentre:

I. las bases recíprocas $\{e^i\}$

$$w^1 = \frac{(w_2 \times w_3)}{w_1 (w_2 \times w_3)} \longrightarrow w_1 (w_2 \times w_3)$$

$$(4\hat{i} + 2\hat{j} + k)(6i - 6j)$$

$$= 12$$

$$w_2 \times w_3 = \begin{vmatrix} i & j & k \\ 3 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6i - 6j$$

Entonces $w^1 = \frac{(6i - 6j)}{12} = \frac{1}{2}i - \frac{1}{2}j$

$$w^2 = \frac{(w_1 \times w_3)}{w_1 (w_2 \times w_3)} \qquad w_1 (w_2 \times w_3) = 12$$

$w_1 \times w_3$ $\cancel{(4i+2j+2k)(4i-8j)}$

$$\begin{vmatrix} i & j & k \\ 4 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 4i - 8j \qquad w^2 = \frac{4i - 8j}{12} \qquad w^2 = \frac{1}{3}i - \frac{2}{3}j$$

$$w^3 = \frac{w_1 \times w_2}{w_1 (w_2 \times w_3)}$$

$w_1 \times w_2$

$$w^3 = \frac{-3i + 3j + 6\hat{k}}{12}$$

$$\begin{vmatrix} \hat{i} & j & k \\ 4 & 2 & 1 \\ 3 & 3 & 0 \end{vmatrix} = -3i + 3j + 6\hat{k}$$

$$w^3 = -\frac{1}{4}i + \frac{1}{4}j + \frac{1}{2}k$$

II) las componentes covariantes y contravariantes del vector

$a = i + 2j + 3k$

Sea el contravariante

usando la base $w_1 = 4i + 2j + k$

$w_2 = 3i + 3j$

$i + 2j + 3k = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad w_3 = 2k$

$$\alpha \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{bmatrix} 4\alpha + 3\beta = 1 \\ 2\alpha + 3\beta = 2 \\ \alpha + 2\gamma = 3 \end{bmatrix} \quad \left[\begin{array}{ccc|c} 4 & 3 & 0 & 1 \\ 2 & 3 & 0 & 2 \\ 1 & 0 & 2 & 3 \end{array}\right]$$

Resolviendo entonces

$\alpha = -1/2$

$\beta = 1$

$\gamma = \frac{5}{4}$

Entonces

$$a = -1/2 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \frac{5}{4} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Sea el covariante

Usando la base reciproca o dual

$w^1 = \frac{1}{2}i - \frac{1}{2}j$

$w^2 = 1/3 i - \frac{2}{3}j$

$w^3 = -1/4 i + 1/4 j + 1/2 k$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 1/3 \\ -2/3 \\ 0 \end{pmatrix} + \gamma_1 \begin{pmatrix} -1/4 \\ 1/4 \\ 1/2 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1/2 & 1/3 & -1/4 & 1 \\ -1/2 & -2/3 & 1/4 & 2 \\ 0 & 0 & 1/2 & 3 \end{array}\right] \quad \begin{pmatrix} 1/2\,\alpha_1 + 1/3\,\beta_1 + (-1/4\,\gamma_1) = 1 \\ -1/2\,\alpha_1 + (-2/3\,\beta_1) + 1/4\,\gamma_1 = 2 \\ 0 + 0 + 1/2\,\gamma_1 = 3 \end{pmatrix}$$

Resolviendo

$\alpha_1 = 11/3$ entonces

$\beta_1 = 2$

$\gamma_1 = 6$

$$a = 11/3 \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1/3 \\ -2/3 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

7)

Considere una vez más el espacio vectorial de matrices hermíticas 2×2 y la definición de producto interno $\langle a|b\rangle \rightleftharpoons Tr(\mathbb{A}^{+}\mathbb{B})$ vista en 2.2.4

Hemos comprobado que las matrices de Pauli $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ vistas en 2.2.4 que forman base para ese espacio

- Encuentre entonces la base dual asociada a las base de Pauli y, adicional
- dado un vector genérico en este espacio vectorial, encuentre también su 1-forma asociada, osea el $\langle\,|$

Sean las bases de pauli

$$\sigma_0 = I\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\sigma_2^{+} = \begin{pmatrix}0 & i\\ -1 & 0\end{pmatrix}\quad \sigma_1^{+} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\sigma_3^{+} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\quad \sigma_0^{+} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$\sigma_0 = e_1 \quad \sigma_1 = e_2 \quad \sigma_2 = e_3$

$\sigma_3 = e_4$

Ortonormalizando las bases

$$\|\sigma_0\| = \sqrt{\langle\sigma_0|\sigma_0\rangle} = \sqrt{Tr\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}$$

$$\|\sigma_0\| = \sqrt{2}$$

$$\|\sigma_1\| = \sqrt{\langle\sigma_1|\sigma_1\rangle} = \sqrt{Tr\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}$$

$$\|\sigma_1\| = \sqrt{2}$$

$$\|\sigma_2\| = \sqrt{\langle\sigma_2|\sigma_2\rangle} = \sqrt{Tr\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}$$

$$\|\sigma_2\| = i\sqrt{2}$$

$$\|\sigma_3\| = \sqrt{\langle\sigma_3|\sigma_3\rangle} = \sqrt{Tr\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$$

$$\|\sigma_3\| = \sqrt{2}$$

Donde la base dual

$\langle e^1\|e_1\rangle = 1$	$\langle e^2\|e_1\rangle = 0$	$\langle e^3\|e_1\rangle = 0$	$\langle e^4\|e_1\rangle = 0$
$\langle e^1\|e_2\rangle = 0$	$\langle e^2\|e_2\rangle = 1$	$\langle e^3\|e_2\rangle = 0$	$\langle e^4\|e_2\rangle = 0$
$\langle e^1\|e_3\rangle = 0$	$\langle e^2\|e_3\rangle = 0$	$\langle e^3\|e_3\rangle = 1$	$\langle e^4\|e_3\rangle = 0$
$\langle e^1\|e_4\rangle = 0$	$\langle e^2\|e_4\rangle = 0$	$\langle e^3\|e_4\rangle = 0$	$\langle e^4\|e_4\rangle = 1$

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$$\langle a^1|\sigma_1\rangle = Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix}^\dagger \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^3 & a^1 \\ a^4 & a^2 \end{pmatrix} = \frac{1}{\sqrt{2}}(a^3+a^2) = 1 = a^3+a^2 = \sqrt{2}$$

$$\langle a^1|\sigma_2\rangle = Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix}^\dagger \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{1}{i\sqrt{2}} Tr\begin{pmatrix} ia^3 & -ia^1 \\ ia^4 & -ia^2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}i}(ia^3 - ia^2) = 0$$

$$\langle a^1|\sigma_3\rangle = Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix}^\dagger \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(a^1+a^4) = 0$$

$$\langle a^1|\sigma_0\rangle = Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} Tr\begin{pmatrix} a^1 & a^3 \\ a^2 & a^4 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}}(a^1+a^4) = 0$$

$$\langle b|\sigma_1\rangle = Tr\begin{pmatrix} b^1 & b^3 \\ b^2 & b^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}} Tr\begin{pmatrix} b^3 & b^1 \\ b^4 & b^2 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}}(b^3+b^2) = 0$$

$$\langle b|\sigma_2\rangle = Tr\begin{pmatrix} b^1 & b^3 \\ b^2 & b^4 \end{pmatrix} \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}i} Tr\begin{pmatrix} ib^3 & -ib^1 \\ ib^4 & -ib^2 \end{pmatrix}$$

$$= \frac{1}{i\sqrt{2}}(ib^3 - ib^2) = 1$$

$$\langle b|\sigma_3\rangle = Tr\begin{pmatrix} b^1 & b^3 \\ b^2 & b^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} Tr\begin{pmatrix} b^1 & -b^3 \\ b^2 & b^4 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(b^1+b^2) = 0$$

$$\langle b|\sigma_0\rangle = Tr\left(\begin{pmatrix} b^1 & b^3 \\ b^2 & b^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}\left(Tr\begin{pmatrix} b^1 & b^3 \\ b^2 & b^4 \end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}(b^1+b^4) = 0$$

$$\langle c|\sigma_1\rangle = Tr\begin{pmatrix} c^1 & c^3 \\ c^2 & c^4 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} Tr\begin{pmatrix} c^3 & c^1 \\ c^4 & c^2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(c^3+c^2) = 0$$

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$$\langle c|\sigma_2\rangle = Tr\begin{pmatrix} c^1 & c^3 \\ c^2 & c^4 \end{pmatrix}\frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{1}{i\sqrt{2}} Tr\begin{pmatrix} ic^3 & -ic^1 \\ ic^4 & -ic^2 \end{pmatrix}$$

$$= \frac{1}{i\sqrt{2}}(ic^3 - ic^2) = 0$$

$$\langle c|\sigma_3\rangle = Tr\begin{pmatrix} c^1 & c^3 \\ c^2 & c^4 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} Tr\begin{pmatrix} c^1 & -c^3 \\ c^2 & c^4 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(c^1 + c^4) = 1$$

$$\langle c|\sigma_0\rangle = Tr\begin{pmatrix} c^1 & c^3 \\ c^2 & c^4 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} Tr\begin{pmatrix} c^1 & c^3 \\ c^2 & c^4 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(c^1 + c^4)$$

$$\langle d|\sigma_1\rangle = Tr\begin{pmatrix} d^1 & d^3 \\ d^2 & d^4 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} Tr\begin{pmatrix} d^3 & d^1 \\ d^4 & d^2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(d^3 + d^2) = 0$$

$$\langle d|\sigma_2\rangle = \frac{1}{i\sqrt{2}}(id^3 - id^2) = 0 \qquad \langle d|\sigma_3\rangle = \frac{1}{\sqrt{2}}(d^1 + d^2) = 0$$

$$\langle d|\sigma_0\rangle = \frac{1}{\sqrt{2}}(d^1 + d^4) = 1$$

$$\langle\sigma_0| = \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 \end{pmatrix} \qquad \langle\sigma_3| = \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 \end{pmatrix}$$

Vector dual

$$\langle\sigma_1| = \begin{pmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix}$$

$$v = \alpha\begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & \sqrt{2}/2 \end{pmatrix} + \beta\begin{pmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix} + \dots +$$

$$\langle\sigma_2| = \begin{pmatrix} 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix}$$

$$c\begin{pmatrix} 0 & -\sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix} + y\begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 \end{pmatrix}$$

$v = ($