A semi-discretization of the Isothermal Euler Equations

Michael Schuster

Friedrich-Alexander Universität Erlangen-Nürnberg

DeustoTech, Universidad de Deusto, Bilbao

July 2, 2018

Introduction and mathematical model

The aim is to model the gas transport through a pipeline network with compressor stations as controllable elements. Consider a connected, directed graph $G = (\mathcal{V}, \mathcal{E})$ with a set of nodes $\mathcal{V}$ ($|\mathcal{V}| = n$) and a set of edges $\mathcal{E}$ ($|\mathcal{E}| = m$). An edge e can either be a flow edge ($e \in \mathcal{E}_F$, $|\mathcal{E}_F| = m_1$) (pressure drop along the pipe caused by friction) or a compressor edge ($e \in \mathcal{E}_C$, $|\mathcal{E}_C| = m_2$) (pressure rise along the pipe caused by control). For notation, we define $f(e)$ as the feet of an edge and $h(e)$ as the head of an edge for all edges $e \in \mathcal{E}$. For one flow edge $e \in \mathcal{E}_F$ consider the isothermal Euler equations (see [3], [2])

$$\textbf{(ISO)} \quad \begin{cases} \dfrac{\partial}{\partial t} p_e + a^2 \dfrac{\partial}{\partial x} q_e = 0 \\ \dfrac{\partial}{\partial t} q_e + \dfrac{\partial}{\partial x}\left(a^2 \dfrac{(q_e)^2}{p_e} + p_e\right) = -\dfrac{\lambda_e^F a^2}{2D_e} \dfrac{|q_e| q_e}{p_e} \end{cases} \tag{1}$$

which model the gas flow through a single pipe. Here, $p_e = p_e(t, x)$ and $q_e = q_e(t, x)$ are pressure and flow, D_e is the (constant) diameter of pipe e, a is the sound speed in the gas and λ_e^F denotes the (constant) pipe friction coefficient. The Mach number M_e is given by the quotient of velocity and sound speed, so it holds

$$M_e = \frac{v_e}{a} = a\frac{q_e}{p_e}$$

The gas flow is called subsonic, if $0 < |v_e| < a$ (i.e. $|M_e| < 1$) and it is called supersonic, if $|v_e| > a$ (i.e. $|M_e| > 1$). We are only interested in subsonic solutions because this is the only relevant case for gas transportation in real world.
For a compressor station, it must hold (see [1])

$$\frac{c}{\gamma}\left(\left(\frac{p_e(t, L_e)}{p_e(t, 0)}\right)^{\gamma} - 1\right) = \tilde{u}_e(t).$$

Here we have constants c, γ and pressures $p_e(t, L_e)$ and $p_e(t, 0)$ at the end and at the beginning of the edge. We assume that the flow through this edge is constant and we define $u(t) := \left(\frac{\gamma}{c}\tilde{u}_e(t) + 1\right)^{\frac{1}{\gamma}}$, so we can write the compressor properties for an edge $e \in \mathcal{E}_C$ as

$$\begin{aligned} p_e(t, L_e) &= u(t)\ p_e(t, 0), \\ q_e(t, L_e) &= q_e(t, 0). \end{aligned} \tag{2}$$

Also we formulate some node conditions (boundary and coupling conditions) for the network. Motivated by real gas networks, we assume that for all input nodes $v \in \mathcal{V}_{\text{in}}$, a time-dependent pressure function $\underline{p}_e(t)$ ($e \in \mathcal{E}$ s.t. $f(e) \in \mathcal{V}_{\text{in}}$) is given and for all output nodes $v \in \mathcal{V}_{\text{out}}$, a flow function $\overline{b}_e(t)$ ($e \in \mathcal{E}$ s.t. $h(e) \in \mathcal{V}_{\text{out}}$) is given. For all inner nodes, we demand continuity in pressure and conservation of mass. We define the set $E_0(v)$ as all edges, that are connected to node v and $E_0^+(v)$ resp. $E_0^-(v)$ as all edges, that arrive at node v resp. that leave node v. So it is

$$\begin{aligned} p_e(t, L_e) &= p_f(t, 0) \qquad \forall e \in E_0^+(v), f \in E_0^-(v), \\ \sum_{e \in E_0^+(v)} q_e(t, L_e) &= \sum_{e \in E_0^-(v)} q_e(t, 0). \end{aligned} \tag{3}$$

So our aim is to find time- and space-dependent functions p_e and q_e (for all $e \in \mathcal{E}$) such that for some space-dependent initial conditions $p_{e,0}$ and $q_{e,0}$ the equations (1), (2) and (3) are fulfilled.

The system in Riemann invariants

Now, we reformulate the system using its Riemann invariants and we talk about existence and uniqueness of a solution. For every flow edge $e \in \mathcal{E}_F$ the eigenvalues of A_e are given by

$$\begin{aligned} \lambda_{e,1} &= a^2 \frac{q_e}{p_e} + a = a(M_e + 1), \\ \lambda_{e,2} &= a^2 \frac{q_e}{p_e} - a = a(M_e - 1) \end{aligned}$$

and the corresponding left eigenvectors are

$$\begin{aligned} l_{e,1} &= \frac{1}{p_e}\left[+a,\ (1 - a\frac{q_e}{p_e}\right] = \frac{1}{p_e}\left[+a,\ (1 - M_e)\right], \\ l_{e,2} &= \frac{1}{p_e}\left[-a,\ (1 + a\frac{q_e}{p_e}\right] = \frac{1}{p_e}\left[-a,\ (1 + M_e)\right] \end{aligned}.$$

Multiplying the PDE in **(SYS)** with the left eigenvectors, on get the Riemann invariants

$$\begin{aligned} r_{e,1}(t, x, p_e(t, x), q_e(t, x)) &= \ln(p_e) + a\frac{q_e}{p_e} = \ln(p_e) + M_e \\ \text{and} \quad r_{e,2}(t, x, p_e(t, x), q_e(t, x)) &= \ln(p_e) - a\frac{q_e}{p_e} = \ln(p_e) - M_e. \end{aligned}$$

We set $R_e(t, x, \Psi_e(t, x)) := [r_{e,1}(t, x, p_e(t, x), q_e(t, x)),\ r_{e,2}(t, x, p_e(t, x), q_e(t, x))]^T$. So (1) is equivalent to

$$\frac{\partial}{\partial t} R_e(t, x) + \begin{bmatrix} \lambda_{e,1}(t, x, R(t, x)) & 0 \\ 0 & \lambda_{e,2}(t, x, R(t, x)) \end{bmatrix} \frac{\partial}{\partial x} R_e(t, x) = F_e(t, x, R_e(t, x)). \tag{4}$$

For further analysis, we will omit writing the explicit dependence on $(p_e(t, x), q_e(t, x))$. Next we write the compressor property and the node conditions in terms of Riemann invariants. Therefor we assume that every compressor edge $e_C \in \mathcal{E}_C$ is locally linear in the graph, that means there exists exactly one edge $e \in \mathcal{E}$ with $h(e) = f(e_C)$ and there also exists exactly one edge $\tilde{e} \in \mathcal{E}$ with $f(\tilde{e}) = h(e_C)$. This is a weak assumption on the graph and it implies that $f(e_C) \notin \mathcal{V}_{\text{in}}$ and $h(e_C) \notin \mathcal{V}_{\text{out}}$. So we can write the compressor properties as

$$\begin{aligned} p_{e+1}(t, 0) &= u_e(t) p_{e-1}(t, L_e) \\ \text{and} \quad q_{e+1}(t, 0) &= q_{e-1}(t, L_e), \end{aligned}$$

where $e - 1$ and $e + 1$ are the unique edges connected to $f(e)$ and $h(e)$ for all $e \in \mathcal{E}_C$. One can interpret this step as a removal of the compressor edges from the graph, but its properties must still hold. So we can only look for a solution on the flow edges meeting the compressor properties. Now, we write the compressor properties in terms of Riemann invariants:

$$\left.\begin{aligned} (r_{e+1,1} + r_{e+1,2})(t, 0) &= 2\ln(u_e(t)) + (r_{e-1,1} + r_{e-1,2})(t, L_{e-1}), \\ u_e(t)(r_{e+1,1} - r_{e+1,2})(t, 0) &= (r_{e-1,1} - r_{e-1,2})(t, L_{e-1}). \end{aligned}\right. \quad \forall e \in \mathcal{E}_C \tag{5}$$

We also write the coupling conditions in terms of Riemann invariants:

$$\begin{aligned}(r_{e,1}+r_{e,2})(t,L_e) &= (r_{\tilde{e},1}+r_{\tilde{e},2})(t,0) \qquad \forall e,\tilde{e}\in\mathcal{E} \text{ s.t. } h(e)=f(\tilde{e}),\\ \sum_{\{e\in\mathcal{E}|h(e)=v\}}(r_{e,1}-r_{e,2})(t,L_e) &= \sum_{\{e\in\mathcal{E}|f(e)=v\}}(r_{e,1}-r_{e,2})(t,0) \quad \forall v\in\mathcal{V}_{\text{in}}.\end{aligned} \tag{6}$$

For the boundary conditions we have

$$\begin{aligned}(r_{e,1}+r_{e,2})(t,0) &= 2\ln(\underline{p}_e(t)) \qquad \forall e\in\mathcal{E} \text{ s.t. } f(e)\in\mathcal{V}_{\text{in}},\\ (r_{e,1}-r_{e,2})(t,L_e)\exp\left(\frac{(r_{e,1}+r_{e,2})(t,L_e)}{2}\right) &= 2\bar{b}(t) \qquad \forall e\in\mathcal{E} \text{ s.t. } h(e)\in\mathcal{V}_{\text{out}}.\end{aligned} \tag{7}$$

So our model is now

$$(\mathbf{RSYS}) \quad \begin{cases}\frac{\partial}{\partial t}R_e(t,x)+\Lambda_e(t,x,R(t,x))\frac{\partial}{\partial x}R_e(t,x) = F_e(t,x,R_e(t,x)) \quad \forall e\in\mathcal{E}_F\\ \text{s.t. } (5),(6),(7) \text{ are fulfilled}\\ \text{and } R_e(0,x)=R_{e,0}(x)\end{cases} \tag{8}$$

So our aim now is to find Riemann invariants $r_{e,1}(t,x)$, $r_{e,2}(t,x)$ for all $e\in\mathcal{E}_F$, s.t. (**RSYS**) is fulfilled. In [3], *Theorem 5.1*, the authors give a general well posedness result for systems like ours. With this, the pressure and the flow can be computed the following:

$$\begin{aligned}p_e(t,x) &= \exp\left(\frac{(r_{e,1}+r_{e,2})(t,x)}{2}\right),\\ q_e(t,x) &= \frac{(r_{e,1}-r_{e,2})(t,x)}{2a}\exp\left(\frac{(r_{e,1}+r_{e,2})(t,x)}{2}\right).\end{aligned}$$

Semidiscretization of the system

In a next step, we discretize (**RSYS**) in space using a finite differences scheme. We can assume w.l.o.g. that every flow edge in the graph has the same length L. We separate the space interval $[0,L]$ in $n+1$ uniformly distributed points x_i with $x_1=0$, $x_{n+1}=L$ and $\Delta x = x_{k+1}-x_k = L/n$ for all $k=1,\cdots,n$.
For all $i=2,\cdots,n$ we approximate the space derivative at x_i using a central difference quotient. For the upper and lower point, we use a left and a right difference quotient. So the finite differences matrix $D\in\mathbb{R}^{2(n+1)\times 2(n+1)}$ is a block diagonal matrix with block diagonal $\frac{1}{2\Delta x}(\tilde{D},\tilde{D})$ and

$$\tilde{D} = \begin{bmatrix}-2 & 2 & 0 & 0 & \cdots & & 0\\ -1 & 0 & 1 & 0 & & & \\ 0 & -1 & 0 & 1 & & & \\ \vdots & & \ddots & \ddots & \ddots & & \vdots\\ & & & -1 & 0 & 1 & 0\\ & & & 0 & -1 & 0 & 1\\ 0 & & \cdots & 0 & 0 & -2 & 2\end{bmatrix}.$$

For notation, we introduce a third index to the Riemann invariants, such that $r_{e,i,j}(t) = r_{e,i}(t,x_j)$ for $e\in\mathcal{E}_F$, $i\in\{1,2\}$ and $j\in\{1,\cdots,n+1\}$. For all $t\in[0,R]$ we define time-dependent vectors $r^D_{e,i}(t)\in\mathbb{R}^{n+1}$ as vectors of all space discretized Riemann invariants, evaluated at the grid points, and we set $R^D_e(t) := [r^D_{e,1}(t), r^D_{e,2}(t)]^T$. We also add the third index to the function on the right, such that $F_{e,i,j}(t,R^D_e(t)) = F_{e,i}(t,x_j,R(t,x_j))$, we define time dependent vectors $F^D_{e,i}(t,R^D_e(t))\in\mathbb{R}^{n+1}$ for the right hand side corresponding to the time-dependent Riemann vectors $r^D_{e,i}(t)$ and we set $F^D_e(t,R^D_e(t)) = [F^D_{e,1}(t,R^D_e(t)), F^D_{e,2}(t,R^D_e(t))]^T$. Let Λ^D_e be the discretized matrix of eigenvalues of edge e. The next step is to insert the boundary- and node conditions. If the initial conditions meet these boundary- and node conditions, it is sufficient inserting the time-derivatives of them.

Because of this step, the mass matrix is not the identity anymore. We'll give an example for that later.
For solving this semi-discretized problem for a given finite time T, we choose a equidistant partition of the time interval. We use a *Crank-Nicolson-Method* to solve the time-dependent problem. For a general time-dependent problem $\dot{y} = f(y)$ that means we do not directly compute y^{n+1} using

$$y^{n+1} = y^n + \Delta t f(y^n) \quad \text{or} \quad y^{n+1} = y^n + \Delta t f(y^{n+1}),$$

which is the explicit resp. implicit Euler method, we use a auxiliary variable $y^\theta = \theta y^{n+1} + (1-\theta) y^n$ for a $\theta \in [0,1]$ which we use for evaluating f. We improve this value as long as possible to get a solution for the next time step. If we choose $\theta = 0$ this is equivalent to the explicit Euler method and for $\theta = 1$ it is equivalent to the implicit Euler method. In our problem, for better stability of the scheme, we insert the formula for y^θ in the linear parts and we use y^θ for evaluating the nonlinear parts. This leads to

$$M \frac{(R_e^D)^{n+1} - (R_e^D)^n}{\Delta t} + \Lambda_e^D D \left(\theta (R_e^D)^{n+1} + (1-\theta)(R_e^D)^n \right) = F_e^D((R_e^D)^\theta),$$

which is equivalent solving

$$(M + \Delta t \theta \Lambda_e^D D)(R_e^D)^{n+1} = (M - \Delta t (1-\theta) \Lambda_e^D D)(R_e^D)^n + \Delta t F_e^D((R_e^D)^\theta).$$

We formulate an algorithm for the computation. The exponential index is for the time step.

Algorithm

- `set initial data` $(R_e^D)^1$
- `for all time steps` $1, \cdots, n$ `do`
  - `set` $(\tilde{R}_e^D)^{n+1} := y^n$
  - `while true`
    - `set` $(R_e^D)^\theta := \theta (\tilde{R}_e^D)^{n+1} + (1-\theta)(R_e^D)^n$
    - `compute` $M_e^D, \Lambda_e^D, D, F_e^D$ `using` $(R_e^D)^\theta$
    - `compute` $(R_e^D)^{n+1}_{\mathtt{Tmp}}$ `using`
      $(M_e^D + \Delta t \theta \Lambda_e^D D)(R_e^D)^{n+1}_{\mathtt{Tmp}} = (M_e^D - \Delta t (1-\theta) \Lambda_e^D D)(R_e^D)^n + \Delta t F_e^D$
    - `if` $\|(\tilde{R}_e^D)^{n+1} - (R_e^D)^{n+1}_{\mathtt{Tmp}}\| \leq tol$ `then set` $(R_e^D)^{n+1} := (R_e^D)^{n+1}_{\mathtt{Tmp}}$
    - `else set` $(\tilde{R}_e^D)^{n+1} = (R_e^D)^{n+1}_{\mathtt{Tmp}}$
  - `end while`
- `end for`

`d`

A minimal example

Now we want to illustrate the results using a minimal example. Consider therefor the minimal linear graph $G = (\mathcal{V}, \mathcal{E})$ with four nodes, two flow edges and one compressor edge:

$$\begin{aligned} \mathcal{V} &= \{0,1,2,3\}, \\ \mathcal{E}_F &= \{e_1, e_2\} = \{(0,1),(2,3)\}, \\ \text{and} \quad \mathcal{E}_C &= \{e_C\} = \{(1,2)\}. \end{aligned}$$

With the boundary conditions described before, the graph is the following:

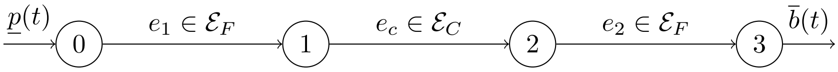


Figure 1: Schemeatic representation of the minimal graph

The Riemann invariants for both edges are given by

$$\begin{aligned}
r_{1,1}(t,x) &= \ln(p_1) + a\frac{q_1}{p_1} &&= \ln(p_1) + M_1,\\
r_{1,2}(t,x) &= \ln(p_1) - a\frac{q_1}{p_1} &&= \ln(p_1) - M_1,\\
r_{2,1}(t,x) &= \ln(p_2) + a\frac{q_2}{p_2} &&= \ln(p_2) + M_2,\\
r_{2,2}(t,x) &= \ln(p_2) - a\frac{q_2}{p_2} &&= \ln(p_2) - M_2.
\end{aligned}$$

Removing the compressor edges, like described before, the node conditions get redundant here. The system written in Riemann invariants, using $R_1(t,x) = [r_{1,1}(t,x), r_{1,2}(t,x)]^T$, $R_2(t,x) = [r_{2,1}(t,x), r_{2,2}(t,x)]^T$, $R(t,x) = [R_1(t,x), R_2(t,x)]^T$ and $\tilde{\Lambda}(R(t,x)) \in \mathbb{R}^{4\times 4}$ with $\operatorname{diag}\tilde{\Lambda}(R(t,x)) = (\Lambda_1(R_1(t,x)), \Lambda_2(R_2(t,x)))$, is then

$$(\mathbf{RSYS}) \quad \begin{cases}
\frac{\partial}{\partial t}R(t,x) + \tilde{\Lambda}(R(t,x))\frac{\partial}{\partial x}R(t,x) = F(R(t,x)) & \\
\left.\begin{array}{l}(r_{2,1}+r_{2,2})(t,0) = 2\ln(u_{e_C}) + (r_{1,1}+r_{1,2})(t,L)\\ u(t)(r_{2,1}-r_{2,2})(t,0) = (r_{1,1}-r_{1,2})(t,L)\end{array}\right\} & \text{compressor properties}\\
\left.\begin{array}{l}(r_{1,1}+r_{1,2})(t,0) = 2\ln(\underline{p}(t))\\ (r_{2,1}+r_{2,2})(t,L) + 2\ln((r_{2,1}-r_{2,2})(t,L)) = 2\ln(2a\overline{b}(t))\end{array}\right\} & \text{boundary conditions}\\
\left.\begin{array}{l}R_1(0,x) = \underline{R}_1(x)\\ R_2(0,x) = \underline{R}_2(x)\end{array}\right\} & \text{initial conditions}
\end{cases}$$

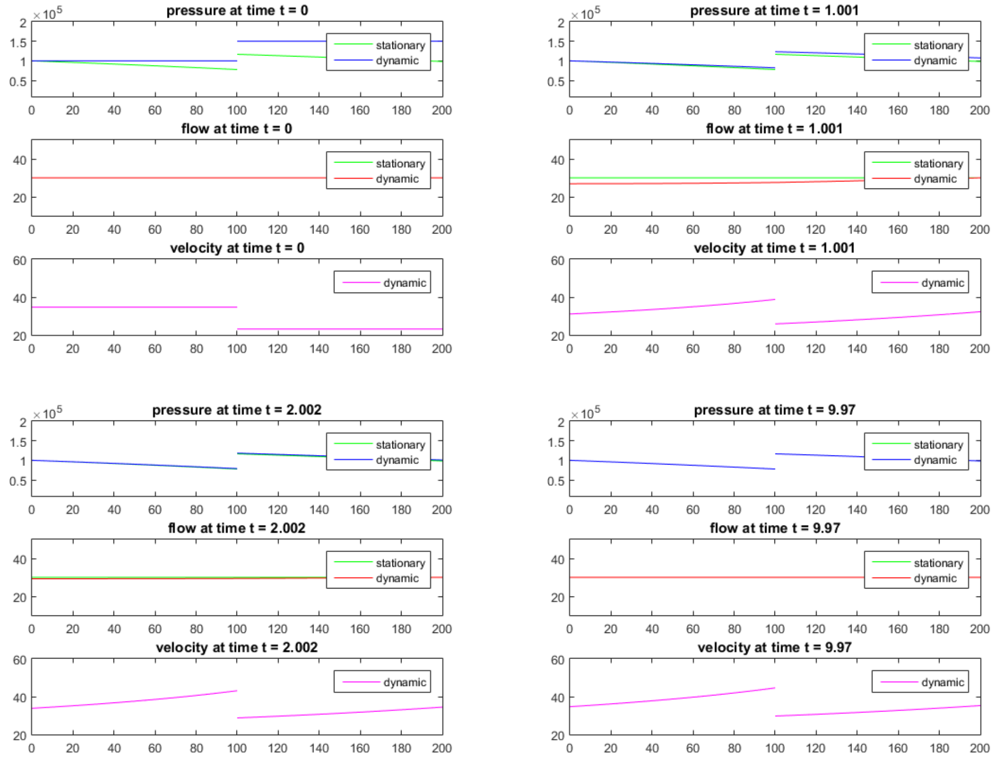
with initial conditions $\underline{R}_1(x)$, $\underline{R}_2(x)$. We remove the compressor edge from the graph and use the space-discretization described before. For inserting the boundary conditions we use the representation

$$\begin{aligned}
r_{1,1,1}(t) &= 2\ln(\underline{p}(t)) - r_{1,2,1}(t) && \text{lower boundary}\\
r_{2,2,n+1}(t) &= 2\mathrm{W}\left(-a\overline{b}(t)\exp(-r_{2,1,n+1}(t))\right) + r_{2,1,n+1}(t) && \text{upper boundary}
\end{aligned}$$

and for inserting the compressor properties, we use the representation

$$\begin{aligned}
r_{1,2,n+1}(t) &= \frac{(1-u(t))r_{1,1,n+1}(t) + 2u(t)r_{2,2,1}(t) - 2u(t)\ln(u(t))}{(u(t)+1)},\\
r_{2,1,1}(t) &= \frac{(1-u(t))r_{1,1,n+1}(t) + 2u(t)r_{2,2,1}(t) - 2u(t)\ln(u(t))}{(u(t)+1)}.
\end{aligned}$$

The following pictures show the results of implementation for constant boundary data.



Dynamic solution for time 0, 1, 2, 10

So what one can observe in this results is, that the dynamic solution converges to the stationary solution, if a C^0 compatibility condition for the initial and the boundary condition is fulfilled. Also one can see that there is a discontinuity in pressure, which is because of the compressor station. As demanded, the flow is continuous. The stationary solution was computed with a classical Newton method.

Outlook

This work should be a basis of analyzing a long time behavior of controlling the isothermal Euler equations. The aim is to observe turnpike properties in optimal control problems related to the isothermal Euler equations.

References

[1] Domschke, Pia ; Hiller, Benjamin ; Lang, Jens ; Tischendorf, Caren: Modellierung von Gasnetzwerken: Eine Übersicht / Technische Universität Darmstadt. Version: 2017. `http://www3.mathematik.tu-darmstadt.de/fb/mathe/preprints.html`. 2017 (2717). – Forschungsbericht

[2] Gugat, M. ; Ulbrich, S.: The isothermal Euler equations for ideal gas with source term: Product solutions, flow reversal and no blow up. In: *Journal of Mathematical Analysis and Applications* (2017). `http://dx.doi.org/10.1016/j.jmaa.2017.04.064`. – DOI 10.1016/j.jmaa.2017.04.064

[3] Gugat, M. ; Ulbrich, S.: Lipschitz solutions of initial boundary value problems for balance laws. In: *Mathematical Models and Methods in Applied Scineces* (2018). `http://dx.doi.org/10.1142/S0218202518500240`. – DOI 10.1142/S0218202518500240