Inverse design for the one-dimensional Burgers equation

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1 The problem

We consider the following one-dimensional Burgers equation

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where u is the state, u_0 is the initial state and the flux function f is defined by $f(u) = \frac{u^2}{2}$. Kruzkov's theory [18] provides existence and uniqueness of a solution of (1) with initial datum $u_0 \in L^\infty(\mathbb{R})$. This solution is called a weak-entropy solution, denoted by $(t,x) \to S_t^+(u_0)(x)$. For a given function u^T, we introduce the backward entropy solution $(t,x) \to S_t^-(u^T)(x)$ as follows: for every $t \in [0,T]$, for a.e $x \in \mathbb{R}$,

$$S_t^-(u^T)(x) = S_t^+(x \to u^T(-x))(-x).$$

We study the problem of inverse design for (1). This problem consists in identifying the set of initial data evolving to a given target at a final time.

Due to the time-irreversibility of the Burgers equation, some target functions are unattainable from weak-entropy solutions of this equation, making the inverse problem under consideration ill-posed. To get around this issue, we introduce the following optimal control problem

$$\inf_{u_0 \in \mathcal{U}_{\mathrm{ad}}^0} J_0(u_0) := \|u^T(\cdot) - S_T^+(u_0)(\cdot)\|_{L^2(\mathbb{R})}, \tag{$\mathcal{O}_T$}$$

where u^T is a given target function and the class of admissible initial data $\mathcal{U}_{\mathrm{ad}}^0$ in $(\mathcal{O}_T)$ is defined by

$$\mathcal{U}_{\mathrm{ad}}^0 = \{u_0 \in BV(\mathbb{R}) / \|u_0\|_{BV(\mathbb{R})} < C \text{ and } \mathrm{supp}(u_0) \subset K_0\}. \tag{2}$$

Above, BV stands for functions of bounded variation and $C > 0$ is a constant large enough. The study of $(\mathcal{O}_T)$ is motivated by the minimization of the sonic boom effects generated by supersonic aircrafts [11, 3, 2].

To solve the optimal control problem $(\mathcal{O}_T)$, some difficulties arise from a theoretical and numerical point of view.

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- Since the entropy solution u of (1) may contain shocks even if the initial datum is a smooth function, this generates important added difficulties that have been the object of intensive study in the past, see [7, 8, 4, 5] and the references therein. In particular, the authors make sense of the derivative of J_0 in $(\mathcal{O}_T)$ in a weak way by requiring strong conditions on the set of initial data. This leads to require that entropy solutions of (1) have a finite number of non-interacting jumps.

- When J_0 is weakly differentiable, gradient descent methods have been implemented in [9, 10, 1] to solve numerically the optimal problem $(\mathcal{O}_T)$. In the cases where it was applied successfully, only one possible initial datum emerges, namely the backward entropy solution $S_T^-(u^T)$.This is mainly due to the numerical viscosity that numerical schemes introduce to gain stability. To find some multiple minimizers, the authors in [17] use a filtering step in the backward adjoint solution.

Dans [19], we fully characterize the set of minimizers of the optimal control problem $(\mathcal{O}_T)$.

Theorem 1.1 *Let $u^T \in BV(\mathbb{R})$. The optimal control problem $(\mathcal{O}_T)$ admits multiple optimal solutions. Moreover, for a.e $T > 0$, the initial datum $u_0 \in BV(\mathbb{R})$ is an optimal solution of $(\mathcal{O}_T)$ if and only if $u_0 \in BV(\mathbb{R})$ verifies $S_T^+(u_0) = S_T^+(S_T^-(u^T))$.*

A characterisation of the set $\{u_0 \in BV(\mathbb{R}), S_T^+(u_0) = S_T^+(S_T^-(u^T))\}$ is given in [12]. An illustration of Theorem 1.1 is given in Figure 1.

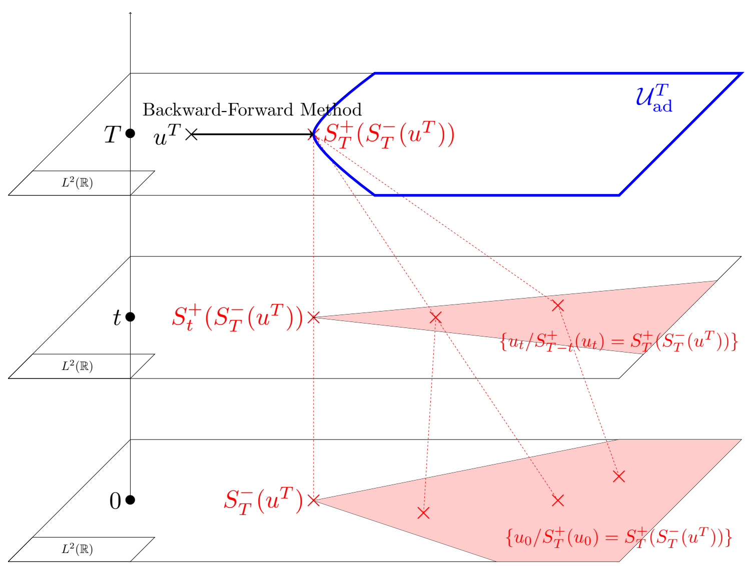


Figure 1: The backward-forward solution $S_T^+(S_T^-(u^T))$ is the projection of u^T onto the set of attainable target functions. The shaded area in red at time $t = 0$ represents the set of minimizers of $(\mathcal{O}_T)$.

The proof of Theorem 1.1 is structured as follows. From [12, Theorem 3.1, Corollary 3.2], [17, Corollary 1] or [15], there exists $u_0 \in BV(\mathbb{R})$ such that $S_T^+(u_0) = q$ if and only if q satisfies the one-sided Lipschitz condition [6, 16, 20, 14], i.e $\partial_x q \leq \frac{1}{T}$ in $\mathcal{D}'(\mathbb{R})$. Thus, the optimal problem $(\mathcal{O}_T)$ can be rewritten as follows.

$$\min_{q \in \mathcal{U}^T_{\text{ad}}} J_1(q) := \|u^T - q\|_{L^2(\mathbb{R})}, \tag{3}$$

where the admissible set $\mathcal{U}_{\rm ad}^T$ is defined by

$$\mathcal{U}_{\rm ad}^T = \{q \in BV(\mathbb{R})/\ \partial_x q \leq \frac{1}{T} \text{ and } \|q\|_{BV(\mathbb{R})} \leq C \text{ and } \mathrm{Supp}(q) \subset K_1\}.$$

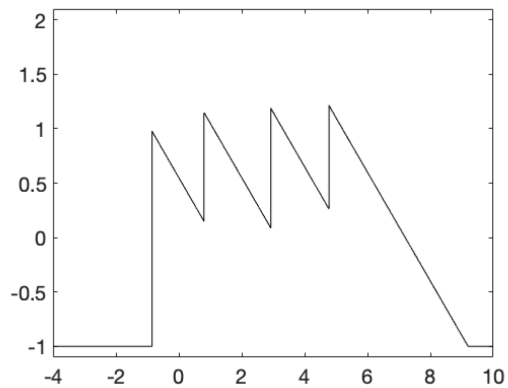
Above, K_1 an open bounded interval large enough. Note that the optimal problem (3) is not related to the PDE model (1). We prove that $q = S_T^+(S_T^-(u^T))$ is a critical point of (3) using the first-order optimality conditions applied to (3) and the full characterization of the set $\{u_0 \in BV(\mathbb{R})/\ S_T^-(u_0) = S_T^-(u^T)\}$ given in [19, Theorem A.2].

2 Numerical simulations

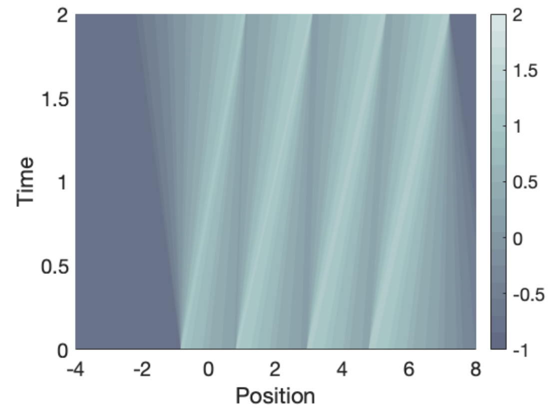
In [19, Section 3], we implement a wave-front tracking algorithm to construct numerically the set of minimizers of $(\mathcal{O}_T)$. We consider for instance, a target function u^T defined by

$$u^T(x) = \begin{cases} 2 & \text{if } x \in (-0.2, 1.1) \bigcup (2, 3.1) \bigcup (4.1, 5.3) \bigcup (6.1, 7.2), \\ -1 & \text{otherwise.} \end{cases} \tag{4}$$

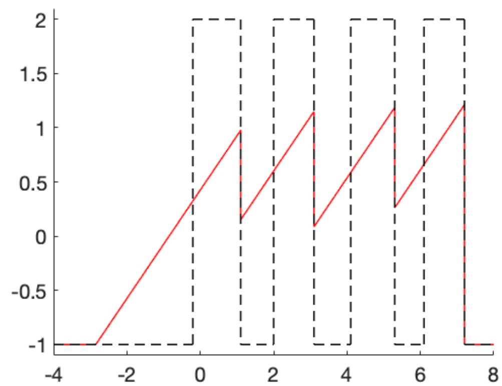
From Theorem 1.1, the backward solution $S_T^-(u^T)$ is an optimal solution of $(\mathcal{O}_T)$ and u_0 is an optimal solution of $(\mathcal{O}_T)$ if and only if $S_T^+(u_0) = S_T^+(S_T^-(u^T))$. In Figure 2, the target function u^T, the backward solution $S_T^-(u^T)$ and the backward-forward solution $S_T^+(S_T^-(u^T))$ are plotted.



$x \quad \to \quad S_T^-(u^T)(x)$



$(t,x) \quad \to \quad S_t^+(S_T^-(u^T))(x)$



u^T and $x \to S_T^+(S_T^-(u^T))(x)$

Figure 2: Plotting of the target function u^T defined in (4), the optimal solution $S_T^-(u^T)$ and the backward-forward solution $S_T^+(S_T^-(u^T))$

Note that $S_T^+(S_T^-(u^T))$ has four different shocks located at $x = 1.1$, $x = 3.1$, $x = 5.3$ and $x = 7.2$. If we use a conservative numerical method as Godunov scheme, the approximate solution of $S_T^+(S_T^-(u^T))$ doesn't have shocks because of numerical viscosity that numerical schemes introduced, see Figure 3. This implies that only one minimizer of $(\mathcal{O}_T)$ can be constructed using a Godunov scheme, which is the backward entropy solution $S_T^-(u^T)$. When a wave-front tracking algorithm is implemented, the approximate solution of $S_T^+(S_T^-(u^T))$ has shocks since we track the discontinuities from u^T to $S_T^+(S_T^-(u^T))$. This implies that all initial data u_0 that coincide with the approximate solution of $S_T^+(S_T^-(u^T))$ can be recovered, see [19, Section 3].

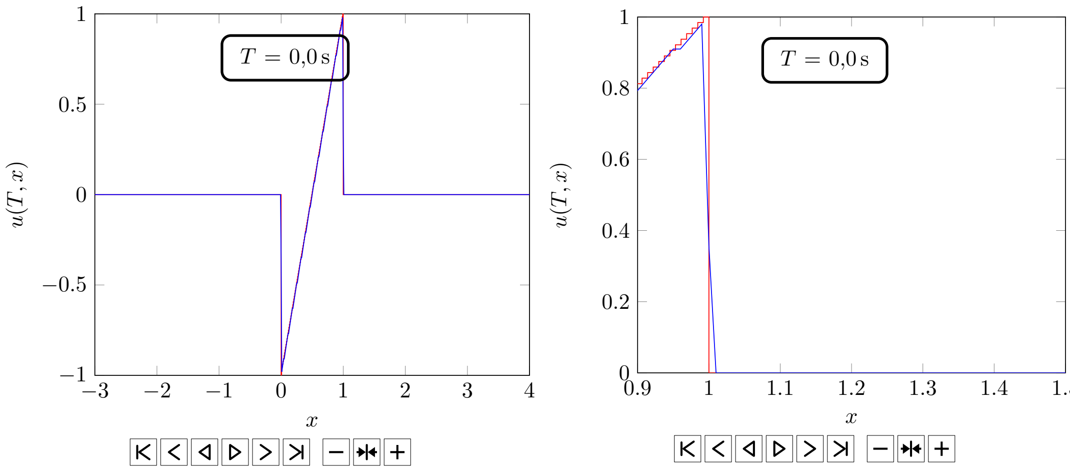


Figure 3: Approximation solution of $S_T^+(u_0)$ with u_0 an N-wave constructed with a wave-front tracking algorithm (in red) and a Godunov scheme (in blue)

At the left top of Figure 4, the weak-entropy solution of (1) with initial data $S_T^-(u^T)$ coincides with $S_T^+(S_T^-(u^T))$ at time T. In Figure 4, three other optimal solutions of $(\mathcal{O}_T)$ u_0 are plotted. In particular, we have $S_T^+(u_0) = S_T^+(S_T^-(u^T))$.

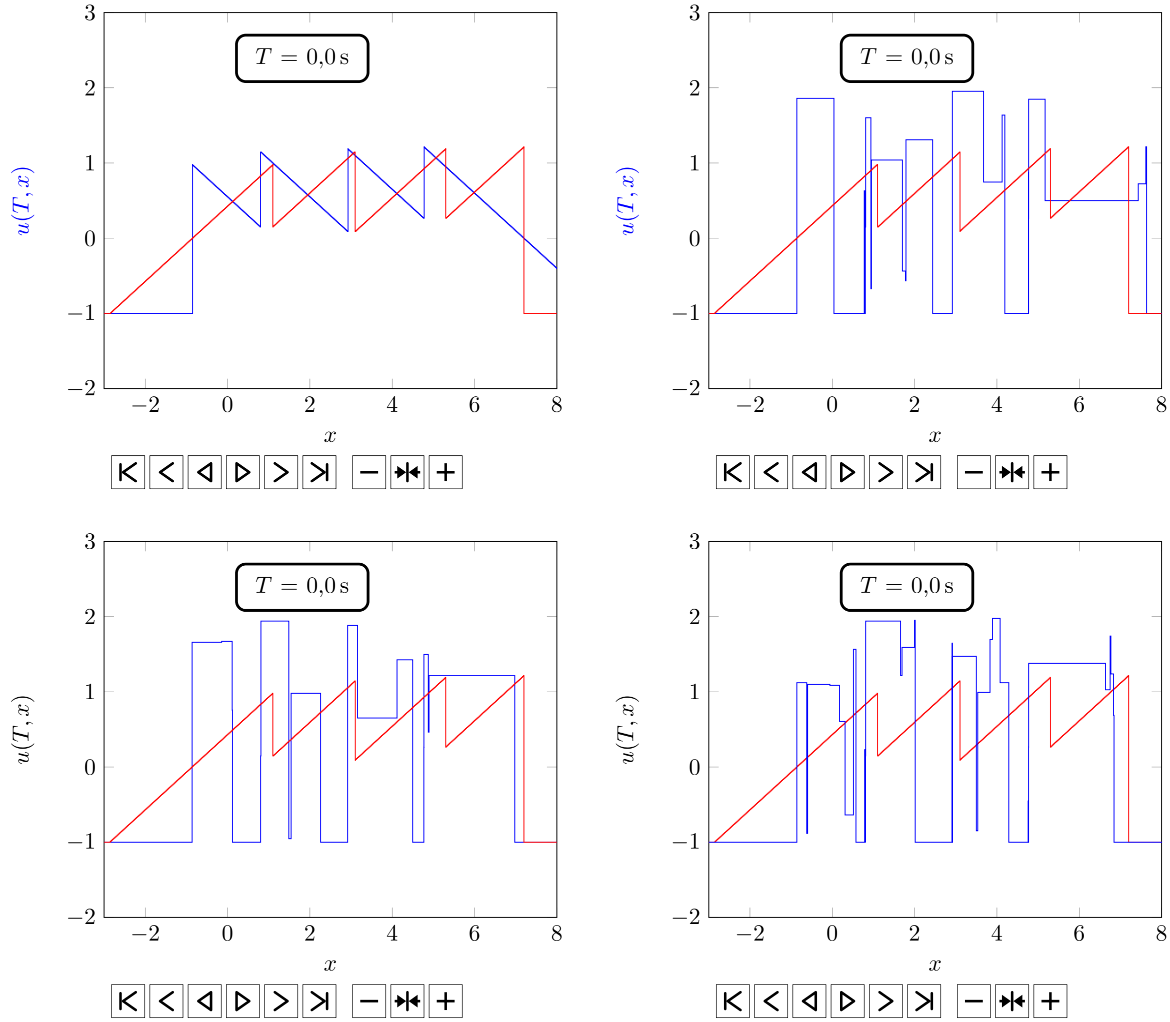


Figure 4: Plotting of four optimal solutions of $(\mathcal{O}_T)$. The weak-entropy solutions of (1) associated to the four optimal solutions u_0 evolve to the backward-forward solution $S_T^+(S_T^-(u^T))$ at time $T = 2$, i.e $S_T^+(u_0) = S_T^+(S_T^-(u^T))$

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