

UNIFORM EXPONENTIAL AND POLYNOMIAL STABILITY AND APPROXIMATION IN CONTROL OF A THERMOELASTIC MODEL

Model Problem

We consider the numerical approximation of the following coupled thermoelastic wave models

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \gamma \theta_x(x,t) = 0 & \text{in } \Omega \times (0, \infty) \\ \theta_t(x,t) - \Delta \theta(x,t) - \gamma u_{tx}(x,t) = 0 & \text{in } \Omega \times (0, \infty) \\ u(x,t) = 0 = \theta(x,t) & \text{on } \partial\Omega \times (0, \infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x) & \text{on } \Omega \end{cases} \quad (1)$$

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \gamma \theta(x,t) = 0 & \text{in } \Omega \times (0, \infty) \\ \theta_t(x,t) - \Delta \theta(x,t) - \gamma u_t(x,t) = 0 & \text{in } \Omega \times (0, \infty) \\ u(x,t) = 0 = \theta(x,t) & \text{on } \partial\Omega \times (0, \infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x) & \text{on } \Omega \end{cases} \quad (2)$$

where $u(x,t)$ is the displacement (longitudinal or transverse, depending upon the application) at position x along a bounded smooth domain $\Omega \subset \mathbb{R}$ and time t , and $\theta(x,t)$ is the temperature deviation from the reference temperature at position x and time t , $u_0(x)$, $v_0(x)$ and $\theta_0(x)$ are initial data in a suitable space. The small positive constant γ is a thermo-mechanical coupling parameter and is generally small in comparison to 1. System (1) differs from system (2) at the coupling terms, where we have replaced the strong coupling ($\gamma \theta_x$ and γu_{tx}) by a weak coupling ($\gamma \theta$ and γu_t). It is well known from literature that system (1) and (2) are respectively exponentially and polynomially stable, see [3, 7] and [5, 6, 9].

In this post, we will show by numerical experiments, how the coupling terms affect quantitative and qualitative properties of thermoelastic systems (1) and (2). These results could be found in [8, 10].

To do this, we consider a semi discretization version of both systems (1) and (2), obtained with finite element method, which has the following form

$$(S_i) \begin{cases} z'_n(t) = A_{i,n} z_n(t), & t \geq 0, n \in \mathbb{N}, i = 1, 2 \\ z_n(0) = z_{n0}, & n \in \mathbb{N} \end{cases} \quad (3)$$

where for all $n \in \mathbb{N}$, $z_n = (u_n, v_n, \theta_n)^T$ is the semi discrete solution, z_{n0} is the discretized initial data, $A_{i,n}$ the discretized dynamic and the subscript \cdot_i refers to system (1) and (2) with

$$A_{i,n} = \begin{pmatrix} 0 & D_n & 0 \\ -D_n & 0 & -\gamma F_{i,n} \\ 0 & \gamma F_{i,n} & -D_n^2 \end{pmatrix}, \quad i = 1, 2$$

and $F_{2,n} = I_n$,

$$D_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & n \end{pmatrix}, \quad (F_{1,n})_{ij} = \begin{cases} -\frac{4}{\pi} \frac{ij}{i^2 - j^2}, & |i - j| = \text{odd}, \\ 0, & \text{otherwise.} \end{cases}$$

Spectral properties of thermoelastic systems

Figure 1 and Figure 2 show how the coupling terms affect the placement of eigenvalues of the dynamic A_n . In Figure 1, we see that a uniform distance between the eigenvalues and the imaginary axis is preserved, see Table 1. Another observation is that for fixed n , the eigenvalues of higher frequency modes, in particular, the one of the n^{th} mode, are closer to the imaginary axis. Moreover, as the number of modes increases, these eigenvalues bend back towards the vertical line $\lambda = -\frac{\gamma^2}{2}$, a fact which has been already shown in [4]. Therefore, the corresponding spectral element approximation scheme preserves the property of exponential stability.

TABLE 1. Distance between $\sigma(A_{1,n})$ and the imaginary axis for the spectral element method

n	$\min\{-\text{Re}\lambda, \lambda \in \sigma(A_{1,n})\}$
8	8.9227×10^{-4}
16	8.9383×10^{-4}
24	8.9402×10^{-4}
32	8.9407×10^{-4}

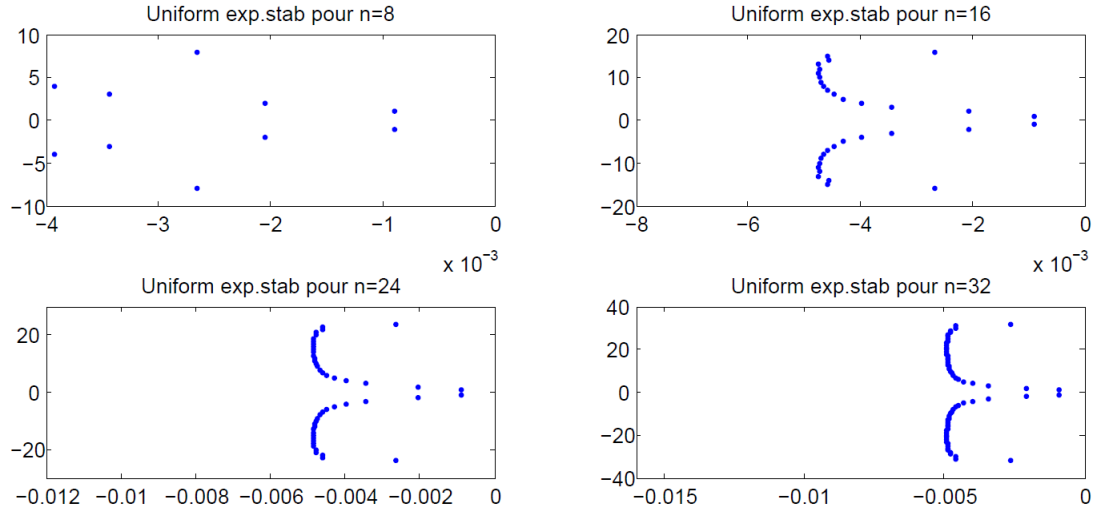


FIGURE 1. Location of the complex eigenvalues of the matrix $A_{1,n}$ with the finite element method

In Figure 2, conversely to Figure 1 where a uniform distance between the eigenvalues and the imaginary axis is preserved, we observe that, as the number of modes increases, an asymptotic behaviour appears in the neighborhood of the imaginary axis at $\pm\infty$. This property is mainly related to systems with polynomial decay, see [1].

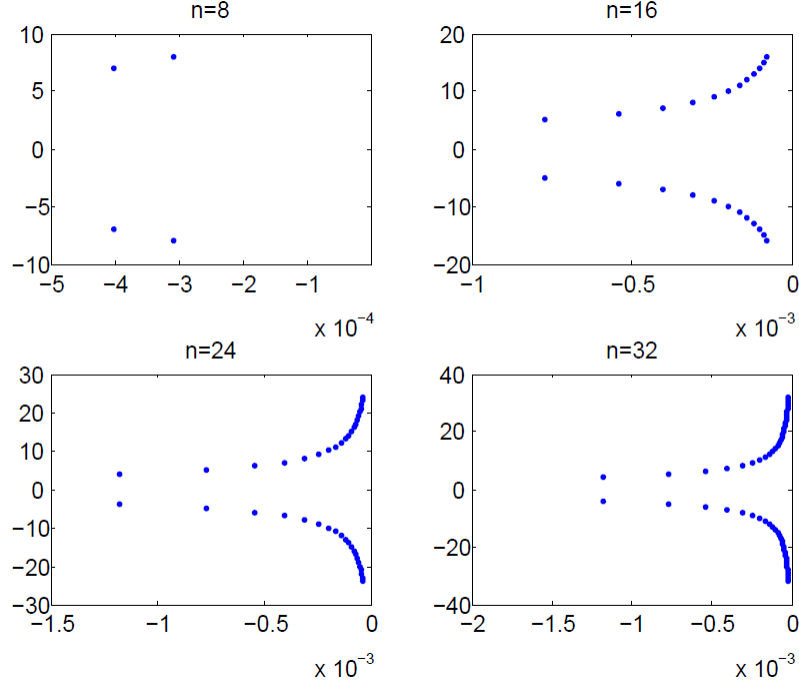


FIGURE 2. Location of the complex eigenvalues of the matrix $A_{2,n}$ with the finite element method.

Uniform and polynomial decay of the energy

The discrete energy associated to system (3) is given by

$$E_{i,n}(t) = \frac{1}{2} \sum_{j=1}^n \left\{ |u_j(t)|^2 + |v_j(t)|^2 + |\theta_j(t)|^2 \right\}, \quad i = 1, 2. \quad (4)$$

The discrete energy $E_{1,n}$ associated to system (1) decays exponentially to zero, see Figure 3, in the following sense: $\exists M, \alpha$ positive constants such that

$$E_{1,n}(t) \leq M e^{-\alpha t} E_{1,n}(0), \quad n \in \mathbb{N}, \quad t > 0.$$

However, the introduction of the weak coupling term in system (1) has changed the dynamic and consequently the behavior of energy (4). In this case, we say that system (2) decays polynomially to zero, see Figure 4, in the following sense: $\exists M, \alpha$ positive constants such that

$$E_{2,n}(t) \leq \frac{M}{t} \|A_{2,n} z_{n0}\|^2, \quad n \in \mathbb{N}, \quad t > 0.$$

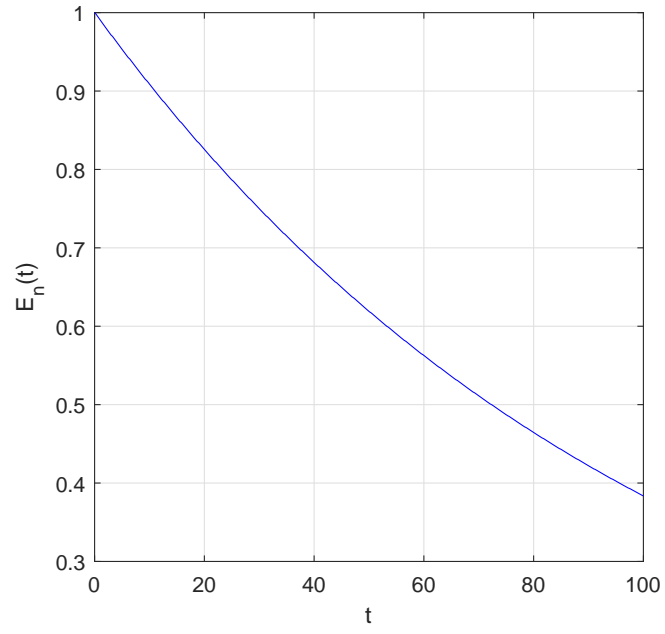
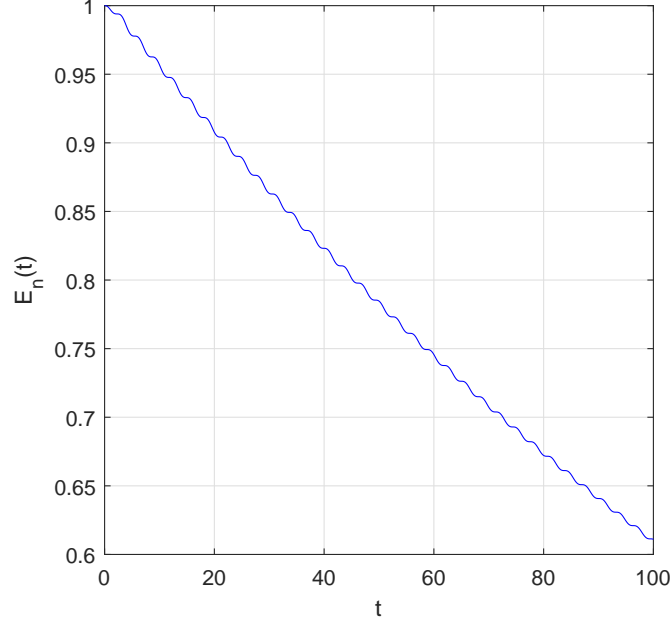


FIGURE 3. Exponential decay of $E_{1,n}(t)$

FIGURE 4. Polynomial decay of $E_{2,n}(t)$

Effect of smoothness of the initial data on the rate of decay of energy

It has been shown theoretically, see [1, 2], that the energy associated to system (2) is very sensitive to the smoothness of its initial data. This fact, has been also observed numerically, see Figure 6. we use a uniform mesh with $n = 100$ elements, fix the final time at $T = 100$, use $dt = 0.1$ and consider the following initial conditions for u , u_t and θ

$$u(x, 0) = 0, \quad \theta(x, 0) = 0, \quad u_t(x, 0) = \sin(jx), \quad j = 1, 2, 3.$$

Through Figure 6, we notice that for $j = 1$, the approximate energy $E_{2,n}(t)$ decays to zero as the time t increases. Moreover, we observe that the decay rate depends strongly to j . That is, when j increases, initial data are very oscillating. We say in this case that the rate of decay of the discrete energy $E_{2,n}(t)$ is very sensitive to the choice of the initial data. However, the behavior of the energy associated to system (1) remains indifferent to the smoothness of initial data when $n \rightarrow \infty$, see Figure 5.

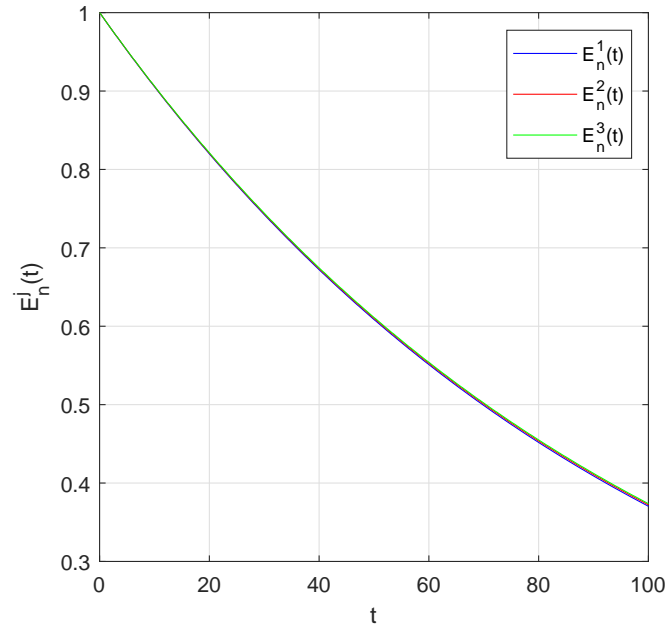
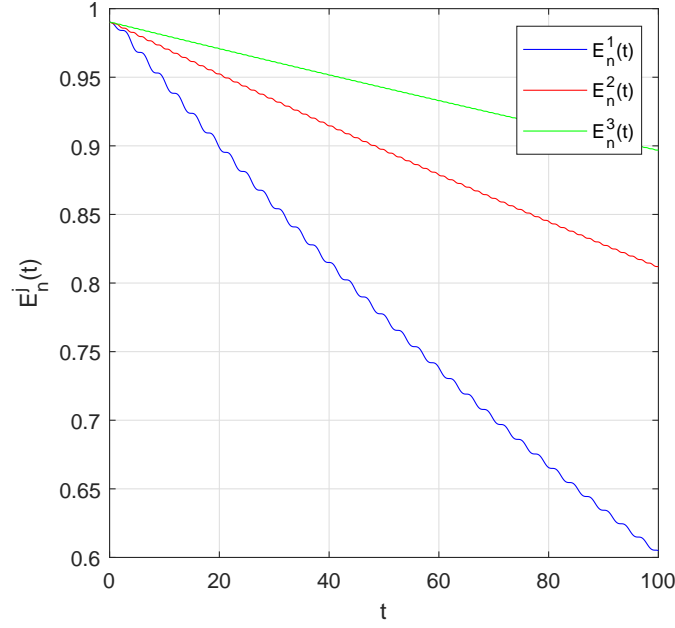


FIGURE 5. No effect of smoothness on exponential decay of $E_{1,n}(t)$

FIGURE 6. Effect of smoothness on polynomial decay of $E_{2,n}(t)$

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