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DYCON BLOG POST

Consider a rotor, rotating about a fixed axis z. Assume that, because of wear and damage, the mass distribution is not homogeneous. This leads to dangerous vibrations in the rotation. A prototypical example can be a wind turbine, which is often affected by misalignment of the blades and/or mass imbalance of the hub and blades [3].

Two balancing heads are mounted at the endpoints of the axle, as in figure 1. Each balancing head is made of two masses free to rotate to compensate the imbalance. Our goal is to employ control theoretical techniques to to minimize the vibrations, moving the balancing masses.

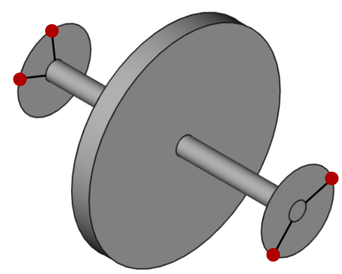

FIGURE 1. The rotor and the balancing device are represented. In the special case represented, the balancing heads are located at the endpoints of the spindle. The four balancing masses (two for each balancing head) are drawn in red.

Suppose the rotor is a rigid body rotating about a fixed axis. The angular velocity ω is assumed constant.

Consider $(O; (x, y, z))$ rotor-fixed reference frame.

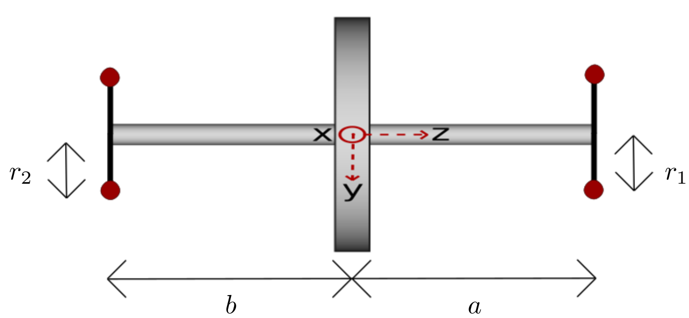

FIGURE 2. Front view of the system made of rotor and balancing device.

(A) *intermediate* angle (B) *gap* angle

FIGURE 3. One balancing head is considered. The balancing masses $(m_i, P_{i,1})$ and $(m_i, P_{i,2})$ are drawn in red. The bisector of the angle generated by $\overrightarrow{OP}_{i,1}$ and $\overrightarrow{OP}_{i,2}$ is the dashed line. The *intermediate* angle α_i is represented in 3a, while the *gap* angle γ_i is depicted in 3b. The angles α_i and γ_i give the position of the balancing masses in each balancing head.

We consider two planes π_1 and π_2 orthogonal to the rotation axis z. The balancing device (see figures 1 and 2) is made up two heads lying in each of these planes. The heads are integral with the rotor, whence they rotate together with the rotor. Each head is made of a pair of balancing masses, which are free to rotate orthogonally to the rotation axis z. Namely, we have

- two planes $\pi_1 := \{z = -a\}$ and $\pi_2 := \{z = b\}$, with $a,\, b \geq 0$;
- two mass-points $(m_1, P_{1,1})$ and $(m_1, P_{1,2})$ lying on π_1 at distance r_1 from the axis z, i.e., in the reference frame $(O; (x, y, z))$

$$\begin{cases} P_{1,1;x} = & r_1 \cos(\alpha_1 - \gamma_1) \\ P_{1,1;y} = & r_1 \sin(\alpha_1 - \gamma_1) \\ P_{1,1;z} = & -a, \end{cases} \quad \text{and} \quad \begin{cases} P_{1,2;x} = & r_1 \cos(\alpha_1 + \gamma_1) \\ P_{1,2;y} = & r_1 \sin(\alpha_1 + \gamma_1) \\ P_{1,2;z} = & -a; \end{cases} \tag{1}$$

- two mass-points $(m_2, P_{2,1})$ and $(m_2, P_{2,2})$ lying on π_2 at distance r_2 from the axis z, namely, in the reference frame $(O; (x, y, z))$

$$\begin{cases} P_{2,1;x} = & r_2 \cos(\alpha_2 - \gamma_2) \\ P_{2,1;y} = & r_2 \sin(\alpha_2 - \gamma_2) \\ P_{2,1;z} = & b, \end{cases} \quad \text{and} \quad \begin{cases} P_{2,2;x} = & r_2 \cos(\alpha_2 + \gamma_2) \\ P_{2,2;y} = & r_2 \sin(\alpha_2 + \gamma_2) \\ P_{2,2;z} = & b. \end{cases} \tag{2}$$

For any $i = 1, 2$, let b_i be the bisector of the angle generated by $\overrightarrow{OP}_{i,1}$ and $\overrightarrow{OP}_{i,2}$ (see figure 3). For any $i = 1, 2$, the *intermediate* angle α_i is the angle between the x-axis and the bisector b_i, while the *gap* angle γ_i is the angle between $\overrightarrow{OP}_{i,1}$ and the bisector b_i.

Note that the angles α_i and γ_i are defined with respect to the rotor-fixed reference frame $(O; (x, y, z))$. Indeed, the balancing device described above is integral with the rotor.

The imbalance is modelled by a resulting force F and a momentum N orthogonal to the rotation axis. In the rotor-fixed reference frame $(O; (x, y, z))$, set $P_1 := (0, 0, -a)$, $P_2 := (0, 0, b)$, $F := (F_x, F_y, 0)$ and $N := (N_x, N_y, 0)$. By imposing the equilibrium condition on forces and momenta, the force F and the momentum N

can be decomposed into a force F_1 exerted at P_1 contained in plane π_1 and a force F_2 exerted at P_2 contained in π_2.

In each plane, we generate a force to balance the system, by moving the balancing masses:

- in plane π_1, we compensate force F_1 by the centrifugal force:

$$B_1 = m_1 r_1 \omega^2 \left(\cos(\alpha_1 - \gamma_1) + \cos(\alpha_1 + \gamma_1), \sin(\alpha_1 - \gamma_1) + \sin(\alpha_1 + \gamma_1)\right); \tag{3}$$

- in plane π_2, we compensate force F_2 by the centrifugal force:

$$B_2 = m_2 r_2 \omega^2 \left(\cos(\alpha_2 - \gamma_2) + \cos(\alpha_2 + \gamma_2), \sin(\alpha_2 - \gamma_2) + \sin(\alpha_2 + \gamma_2)\right); \tag{4}$$

The overall imbalance of the system is then given by the resulting force in π_1

$$F_{ris,1} = B_1 + F_1$$

and the resulting force in π_2

$$F_{ris,2} = B_2 + F_2.$$

The overall imbalance on the system made of rotor and balancing device is measured by imbalance indicator

$$G = \|B_1 + F_1\|^2 + \|B_2 + F_2\|^2.$$

By trigonometric formulas, $G : \mathbb{R}^4 \longrightarrow \mathbb{R}$

$$\begin{aligned} G(\alpha_1, \gamma_1, \alpha_2, \gamma_2) \quad = \quad & \Big[\left|2m_1 r_1 \omega^2 \cos(\gamma_1)\cos(\alpha_1) + F_{1,x}\right|^2 \\ & + \left|2m_1 r_1 \omega^2 \cos(\gamma_1)\sin(\alpha_1) + F_{1,y}\right|^2 \\ & + \left|2m_2 r_2 \omega^2 \cos(\gamma_2)\cos(\alpha_2) + F_{2,x}\right|^2 \\ & + \left|2m_2 r_2 \omega^2 \cos(\gamma_2)\sin(\alpha_2) + F_{2,y}\right|^2 \Big]. \end{aligned} \tag{5}$$

Our task is to find a control strategy such that

- the balancing masses move from their initial configuration Φ_0 to a final configuration $\overline{\Phi}$, where the imbalance is compensated;
- the imbalance and velocities should be kept small during the correction process.

We address the minimization problem

$$\min_{\Phi \in \mathscr{A}} \frac{1}{2} \int_0^\infty \left[\|\dot{\Phi}\|^2 + \beta \hat{G}(\Phi) \right] dt, \tag{6}$$

where

$$\mathscr{A} := \left\{ \Phi \in H^1_{loc}((0, +\infty); \mathbb{T}^4) \quad \middle| \quad \Phi(0) = \Phi_0, \quad \text{and} \quad L(\Phi, \dot{\Phi}) \in L^1(0, +\infty) \right\}, \tag{7}$$

with $H^1_{loc}((0, +\infty); \mathbb{T}^4) := \cup_{T>0} H^1(0, T; \mathbb{T}^4)$. In [1], we obtained the following results:

1. there exists $\Phi \in \mathscr{A}$ minimizer of J;
2. $\Phi = (\alpha_1, \gamma_1; \alpha_2, \gamma_2)$ is C^∞ smooth and, for $i = 1, 2$, the following Euler-Lagrange equations are satisfied

$$\begin{cases} -\ddot{\alpha}_i = \beta \cos(\gamma_i) \left[-c_1^i \sin(\alpha_i) + c_2^i \cos(\alpha_i) \right] & \text{in } (0, \infty) \\ -\ddot{\gamma}_i = -\beta \sin(\gamma_i) \left[c_1^i \cos(\alpha_i) + c_2^i \sin(\alpha_i) - \cos(\gamma_i) \right] & \text{in } (0, \infty) \\ \alpha_i(0) = \alpha_{0,i}, \;\; \gamma_i(0) = \gamma_{0,i}, \;\; \dot{\Phi}(T) \underset{T \to +\infty}{\longrightarrow} 0. \end{cases} \tag{8}$$

3. for any optimal trajectory Φ for (6), there exists $\overline{\Phi} \in \mathscr{S}$ such that
$$\Phi(t) \underset{t\to+\infty}{\longrightarrow} \overline{\Phi}, \tag{9}$$
$$\dot{\Phi}(t) \underset{t\to+\infty}{\longrightarrow} 0. \tag{10}$$
and
$$\left|\hat{G}\left(\Phi(t)\right)\right| \underset{t\to+\infty}{\longrightarrow} 0. \tag{11}$$
If, in addition
$$m_1 r_1 > \frac{\sqrt{F_{1,x}^2 + F_{1,y}^2}}{2\omega^2} \quad \text{and} \quad m_2 r_2 > \frac{\sqrt{F_{2,x}^2 + F_{2,y}^2}}{2\omega^2}, \tag{12}$$
we have the exponential estimate
$$\|\Phi(t) - \overline{\Phi}\| + \|\dot{\Phi}(t)\| + |G\left(\Phi(t)\right)| \leq C \exp\left(-\mu t\right), \quad \forall\ t \geq 0, \tag{13}$$
with $\mu > 0$.

We performed some numerical simulations, employing the expert interior-point optimization routine `IPOpt` (see [4] and [5]), the modelling language being `AMPL` (see [2]). The related code is available in the Appendix.

In figures 4, 5, 6 and 7, we plot a simulation, with initial datum $\Phi_0 = (\alpha_{0,1}, \gamma_{0,1}; \alpha_{0,2}, \gamma_{0,2}) \coloneqq (1, 0.3, 0.6, 0.3)$. Condition (12) is verified. In agreement with our result, the exponential stabilization emerges. In figure 8, we depict the imbalance indicator versus time along the computed trajectories. As expected, it decays to zero exponentially.

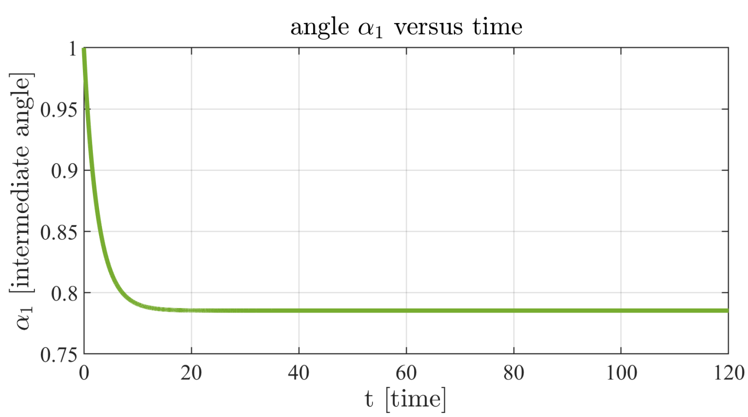

FIGURE 4. Intermediate angle α_1 versus time

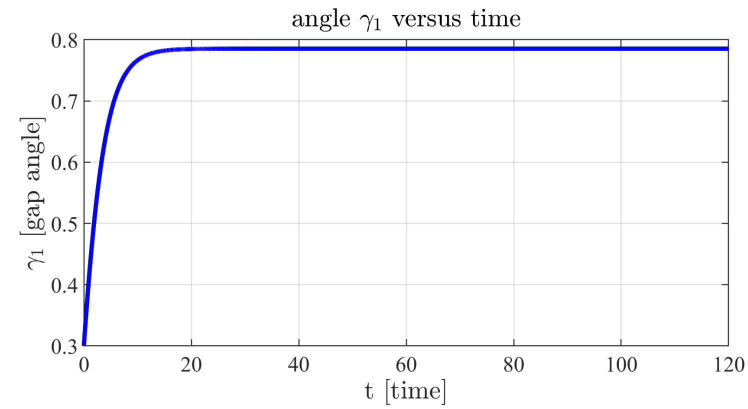

FIGURE 5. Gap angle γ_1 versus time

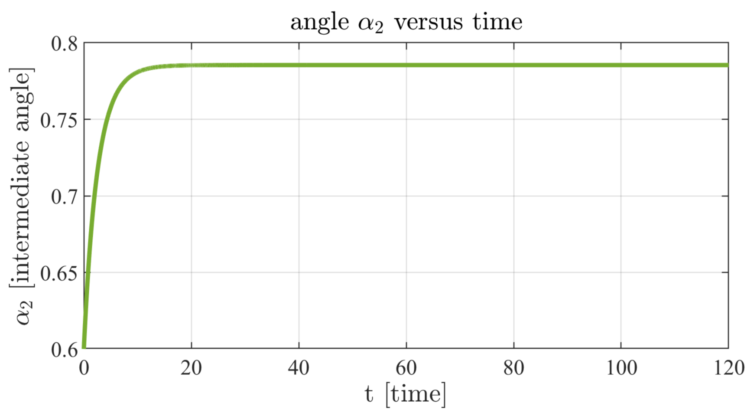

FIGURE 6. Intermediate angle α_2 versus time

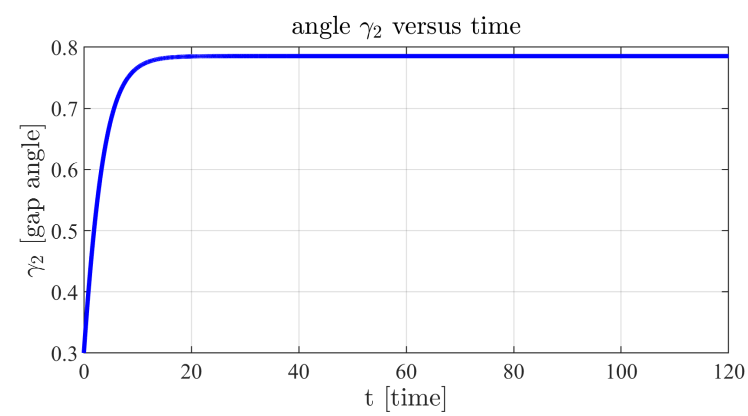

FIGURE 7. Gap angle γ_2 versus time

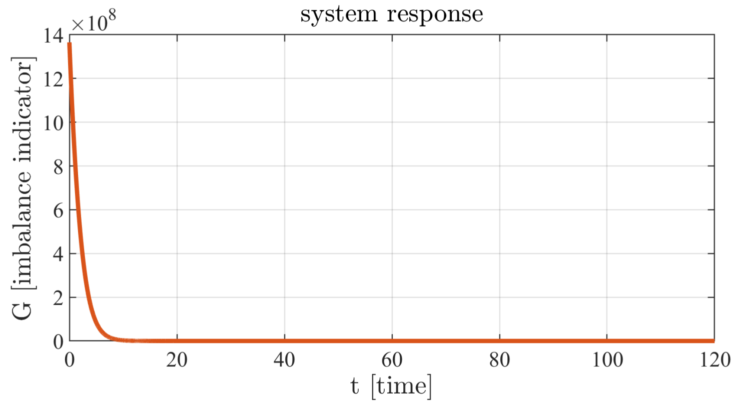

FIGURE 8. The imbalance indicator G along the computed trajectory versus time.

Appendix.

0.1. **IPOpt-AMPL code.** As announced, we provide the `IPOpt-AMPL` code related to the minimization problem (6).

```
#parameters for the algorithm execution.
param pi = 4*atan(1);
param Nt = 10000;
param T = 120;
param dt = T/(Nt);

#physical parameters.
param m1 = 1;
param m2 = 1;
param a = 1;
param b = 1;
param r1 = 1;
param r2 = 1;
param omega = 2000*(2*pi)/60;
param Fx = -2*m1*r1*omega^2;
param Fy = -2*m2*r2*omega^2;
param Nx = 0;
param Ny = 0;
param F1x = (b*Fx-Ny)/(a+b);
param F1y = (b*Fy+Nx)/(a+b);
param F2x = (a*Fx+Ny)/(a+b);
param F2y = (a*Fy-Nx)/(a+b);
#param ss = [0;pi;0;pi];
#WARNING:
#The condition below must be fulfilled.
#m_1r_1 > \frac{\sqrt{F_{1,x}^2+F_{1,y}^2}}{2\omega^2}\hspace{0.3 cm}\mbox{and}\hspace{0.3 cm

#weighting parameters.
```

```
param beta = 0.25*(2*omega^2)^(-2); #omega^4*(1000/120);

var Phi {i in 0..Nt, j in 1..4};  # Phi(t), state
var psi {i in 0..Nt-1, j in 1..4}; # psi(t), control, default 0;

minimize cost: sum {i in 0..Nt-1} (0.5*dt*(psi[i,1]^2+psi[i,2]^2+psi[i,3]^2+psi[i,4]^2)+(beta

subject to Phi_dyn_one {i in 1..Nt}:
((Phi[i,1]-Phi[i-1,1])/(dt)) = psi[i-1,1];
subject to Phi_dyn_two {i in 1..Nt}:
((Phi[i,2]-Phi[i-1,2])/(dt)) = psi[i-1,2];
subject to Phi_dyn_three {i in 1..Nt}:
((Phi[i,3]-Phi[i-1,3])/(dt)) = psi[i-1,3];
subject to Phi_dyn_four {i in 1..Nt}:
((Phi[i,4]-Phi[i-1,4])/(dt)) = psi[i-1,4];

subject to initial_condition_one: Phi[0,1] = 1;
subject to initial_condition_two: Phi[0,2] = 0.3;
subject to initial_condition_three: Phi[0,3] = 0.6;
subject to initial_condition_four: Phi[0,4] = 0.3;

option solver ipopt;
option ipopt_options "max_iter=2000 linear_solver=mumps halt_on_ampl_error yes";
solve;


printf: " # cost = %24.16e\n", cost; # > out_MMarposs1.txt;
printf: " # T = %24.16e\n", T; # > out_MMarposs1.txt;
printf: " # Nt = %d\n", Nt; # >> out_MMarposs1.txt;
printf: " # Data\n"; # >> out_MMarposs1.txt;
printf {i in 0..Nt}: " %24.16e\n", Phi[i,1]; # >> out_MMarposs1.txt;
printf {i in 0..Nt}: " %24.16e\n", Phi[i,2]; # >> out_MMarposs1.txt;
printf {i in 0..Nt}: " %24.16e\n", Phi[i,3]; # >> out_MMarposs1.txt;
printf {i in 0..Nt}: " %24.16e\n", Phi[i,4]; # >> out_MMarposs1.txt;
printf: " # Imbalance indicator evaluated at (Phi[Nt,1],Phi[Nt,2],Phi[Nt,3],Phi[Nt,4])\n"; #
printf: " %24.16e\n", ((2*m1*r1*omega^2*cos(Phi[Nt,2])*cos(Phi[Nt,1])+F1x)^2+(2*m1*r1*omega^2



end;
```

`IpOpt-AMPL` codes can be run for free in NEOS solvers
https://neos-server.org/neos/solvers/nco:Ipopt/AMPL.html

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