HOW TO SOLVE LQ PROBLEMS WITH TIME-VARYING TARGETS BY RICCATI'S THEORY

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In this short manual, we explain how to use Riccati's theory to solve an LQ control problem with targets. The related MATLAB code is downloadable freely. We start considering the case of finite time horizon to later address the case of infinite time horizon.

1. Finite time horizon

We minimize over $u \in L^2(0,T)$ the functional

$$J^T(u) = \frac{1}{2}\left[\int_0^T \|u(t)-q(t)\|^2 dt + \beta\int_0^T \|C(x(t)-z(t))\|^2 dt + \gamma\|D(x(T)-z(T))\|^2\right],$$

where:

$$\begin{cases} \frac{d}{dt}x(t) + Ax(t) = Bu(t) & t\in(0,T)\\ x(0)=x_0. \end{cases} \tag{1}$$

In the above control problem, $A \in \mathcal{M}_{n\times n}$, $B\in\mathcal{M}_{n\times m}$, $C\in\mathcal{M}_{r\times n}$ and $D\in\mathcal{M}_{r\times n}$. The control $u:[0,T]\longrightarrow\mathbb{R}^m$, while the state $x:[0,T]\longrightarrow\mathbb{R}^n$. The control target is $q\in C^1([0,T];\mathbb{R}^m)$ and the state target is $z\in C^1([0,T];\mathbb{R}^n)$. $\beta\geq 0$ and $\gamma\geq 0$ are positive parameters.

By the Direct Methods in the Calculus of Variations and strict convexity, the above problem admits an unique optimal control u^T. The corresponding optimal state is denoted by x^T.

1.1. Description of the algorithm.

We compute the optimal pair (u^T, x^T) by using the well-known Riccati's theory (see, e.g. [1, Lemma 2.6] and [2, section 4.3]).

Step 1 **Computation of the solution to the Riccati Differential Equation**
We determine the Riccati operator, satisfying the Riccati Differential Equation

$$\begin{cases} \mathcal{E}_t = \beta C^*C - (\mathcal{E}A + A^*\mathcal{E}) - \mathcal{E}BB^*\mathcal{E} & \forall t\in(0,+\infty)\\ \mathcal{E}(0) = \gamma D^*D. \end{cases}$$

We solve the above nonlinear ODE, by employing MATLAB solver `ode113`.

Step 2 **Computation of the remainder function**
We take into account the targets computing the remainder function h^T

$$\begin{cases} -\frac{d}{dt}h^T + (A^* + \mathcal{E}(T-t)BB^*)h^T + \mathcal{E}(T-t)\left(\frac{dz}{dt} + Az - Bq\right) = 0 & t\in(0,T)\\ h^T(T) = 0. \end{cases}$$

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Step 3 **Computation of the optimal state**
The optimal state solves the closed loop system

$$\begin{cases} \frac{d}{dt}x^T(t) + Ax^T(t) = -BB^*\left(\mathcal{E}(T-t)(x^T(t) - z(t)) + h^T(t)\right) + Bq(t) & t \in (0,T) \\ x^T(0) = x_0. \end{cases}$$

Step 4 **Computation of the optimal control**
We compute the optimal control in a feedback form

$$u^T(t) = -B^*\left(\mathcal{E}(T-t)(x^T(t) - z(t)) + h^T(t)\right) + q(t).$$

1.2. **Example.** Take

$$A := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad D := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Choose $\beta = 26$, $\gamma = 0$, $x_0 = [1.4; 1.4]$, $q \equiv 0$, $z(t) = [\sin(t); \sin(t)]$ and $T = 10$. We obtain figures 1, 2 and 3.

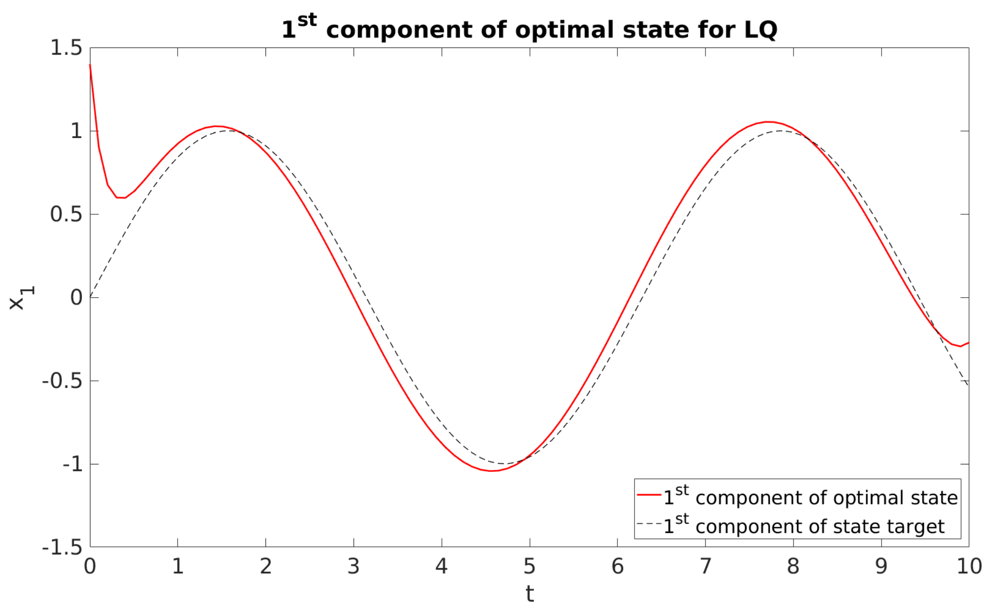

Figure 1. first component of the optimal state.

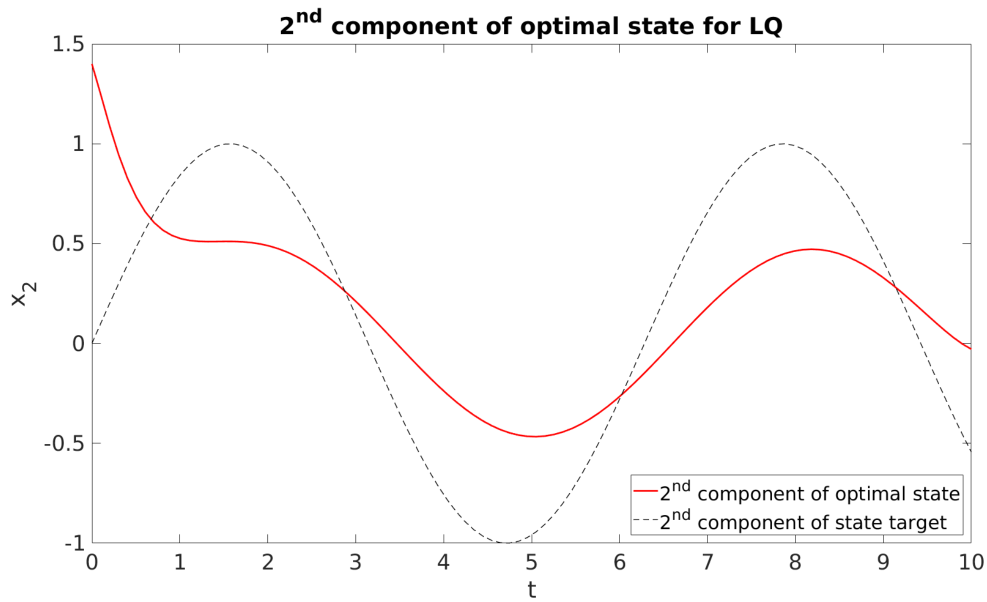

FIGURE 2. second component of the optimal state.

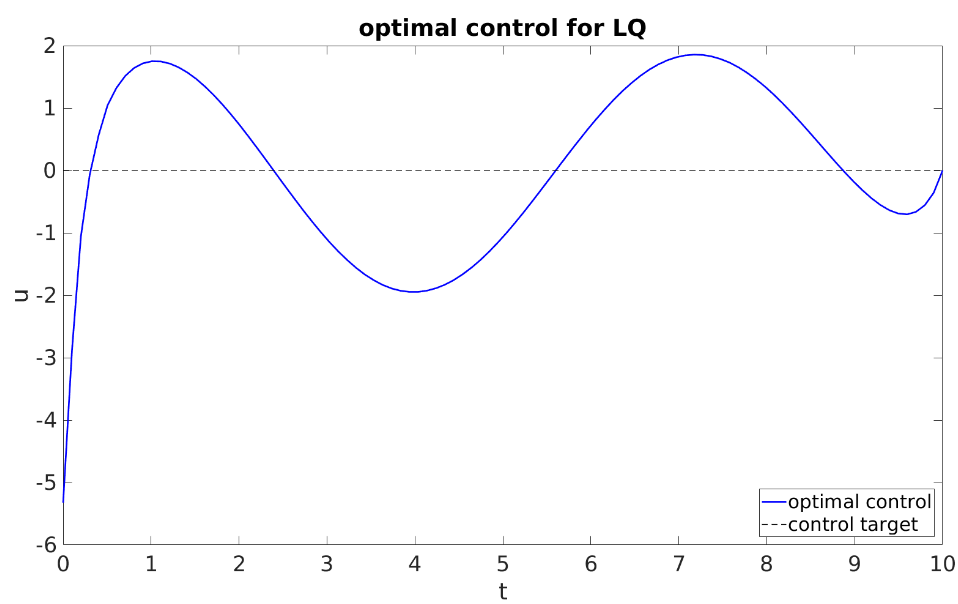

FIGURE 3. optimal control.

Since the parameter β is large enough and the control acts only on the first component of (1)

- the first component of the state is close to the target;
- the second component of the state is less close to the target;
- the control is far from its target.

The algorithm described in this guide can be employed to test the fulfillment of the turnpike property (see, for instance, [1] and [3]). In agreement with the theory, the turnpike effect is evident if:

- the targets are constants;
- (A, B) is controllable;
- (A, C) is observable, $\beta > 0$ and $\gamma = 0$;
- the time horizon T is large enough.

2. Infinite time horizon

We consider the optimal control problem

$$\min_{u\in L^2_{loc}(0,\infty)} J^{\infty}(u) = \frac{1}{2}\left[\int_0^{\infty} \|u(t) - q(t)\|^2 dt + \beta \int_0^{\infty} \|C(x(t) - z(t))\|^2 dt\right],$$

where:

$$\begin{cases} \frac{d}{dt}x(t) + Ax(t) = Bu(t) & t \in (0, +\infty) \\ x(0) = x_0. \end{cases}$$

In the above control problem, $A \in \mathcal{M}_{n\times n}$, $B \in \mathcal{M}_{n\times m}$ and $C \in \mathcal{M}_{r\times n}$. The control $u : (0, +\infty) \longrightarrow \mathbb{R}^m$, while the state $x : [0, +\infty) \longrightarrow \mathbb{R}^n$. The control target is $q \in L^2_{loc}((0, +\infty); \mathbb{R}^m)$ and the state target is $z \in W^{1,2}_{loc}((0, +\infty); \mathbb{R}^n)$. $\beta \geq 0$ and $\gamma \geq 0$ are positive parameters.

$\boxed{\text{Assumptions}}$

(1) the targets q and z satisfies the equation

$$\frac{d}{dt}z(t) + Az(t) = Bq(t) \qquad t \in (0, +\infty);$$

(2) (A, B) is controllable and (A, C) is observable.

The above assumptions guarantee the existence of a control $u \in L^2_{loc}((0, +\infty); \mathbb{R}^m)$ such that $J^{\infty}(u) < +\infty$. By the Direct Methods in the Calculus of Variations and strict convexity, the above problem admits an unique optimal control u^{∞}. The corresponding optimal state is denoted by x^{∞}.

2.1. Description of the algorithm.

We compute the optimal pair (u^{∞}, x^{∞}) by using the Algebraic Riccati Equation (see, for instance, [1, Lemma 2.6] and [2, section 4.3]).

Step 1 **Computation of the solution to the Algebraic Riccati Equation**
We determine $\widehat{E}$, the unique symmetric positive definite solution to the equation:

$$(\widehat{E}A + A^*\widehat{E}) + \widehat{E}BB^*\widehat{E} = C^*C \qquad \text{(ARE)}$$

We solve the above nonlinear algebraic equation, by employing MATLAB function `care`.

Step 2 **Computation of the optimal state**
First of all, we solve the Cauchy Problem

$$\begin{cases} \frac{d}{dt}\eta(t) + \left(A + BB^* \widehat{E}\right)\eta(t) = 0 & t \in (0,T) \\ \eta(0) = x_0 - z(0). \end{cases}$$

The optimal state $x^\infty = \eta + z$.
Step 3 **Computation of the optimal control**
We determine the optimal control in a feedback form

$$u^\infty(t) = -B^* \widehat{E} z(t) + q(t).$$

References

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