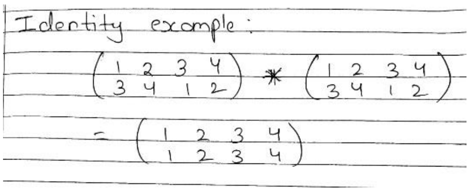
Group Theory :

A group is a set of operations which satisfies certain properties.

Group: G : set s and operation *, satisfies four axioms:

1. Identity : There is an identity element e ∈ s, such that ∀ g ∈ G, e * g = g *e = g.

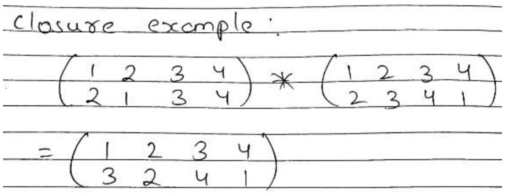


Identity example :

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

2. Closure : ∀ x, y ∈ s, x *y ∈ s.

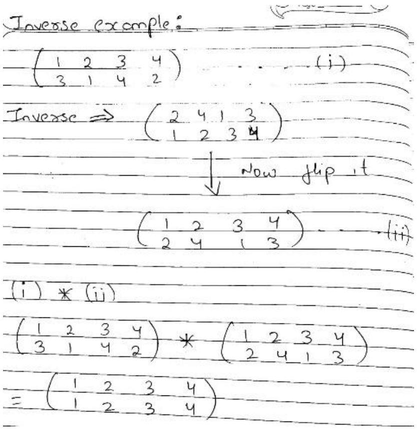


Closure example :

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

3. Inverse: ∀x ∈ ∃s, **∃y,** such that x*y = y*x = e



Inverse example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \ldots\ldots (i)$$

Inverse ⇒ $\begin{pmatrix} 2 & 4 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

$\downarrow$ Now flip it

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \ldots\ldots (ii)$$

(i) * (ii)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

4. Associativity: (x*y) * z= x*(y*z)

Note: if we take composition of any two permutation, the answer is again a permutation. In other words it is a bijective function.

➔ Here is an example of a multiplication table for a binary operation $\ast$ on the set G = {a, b, c, d}.

*	a	b	c	d
a	a	b	c	a
b	a	c	d	d
c	a	b	d	c
d	d	a	c	b

Note that $(a \ast b) \ast c = b \ast c = d$ but $a \ast (b \ast c) = a \ast d = a$.

Proof by contradiction

Suppose that we want to prove that the statement P is true. We begin by assuming that P is false. We then try to deduce a contradiction, i.e. some statement Q which we know is false. If we succeed, then our assumption that P is false must be wrong! So P is true, and our proof is finished.

Example 1: “For all rational numbers x and all irrational numbers y, the sum x + y is irrational”

Let us assume the negation of what we are trying to prove: namely that there exist a rational number x and an irrational number y such that x + y is rational. We observe that $y = (x + y) – x$

Now x+y and x are rational by assumption, and the difference of two rational numbers is rational. Thus y is rational. But that contradicts our assumptions. So our assumptions cannot be right. So if x is rational and y is irrational then x + y is irrational.

Example 2: If $a, b \in Z$, then $a^2 - 4b$ is not equal to 2.

Suppose this proposition is false. This conditional statement being false means there exist numbers a and b for which $a, b \in Z$ is true but $a^2 - 4b$ not equal to 2 is false. Thus there exist integers $a, b \in Z$ for which $a^2 - 4b = 2$. From this equation we get $a^2 = 4b + 2 = 2(2b + 1)$, so a^2 is even.

Since a^2 is even, it follows that a is even, so a = 2c for some integer c. Now plug a = 2c back into the equation $a^2 - 4b = 2$.

We get $(2c)^2 - 4b = 2$, so $4c^2 - 4b = 2$. Dividing by 2, we get $2c^2 - 2b = 1$.

Therefore $1 = 2(c^2 - b)$, and since $c^2 - b \in Z$, it follows that 1 is even. Since we know 1 is not even, something went wrong. But all the logic after the first line of the proof is correct, so it must be that the first line was incorrect. In other words, we were wrong to assume the proposition was false. Thus the proposition is true.

Proof by Uniqueness

"Exactly one element satisfies $P(x)$"

– Existence: $\exists x\ P(x)$

– Uniqueness: $\forall y\ P(y) \rightarrow y = x$

– $\exists x \forall y\ P(x) \wedge P(y) \rightarrow y = x$

– $\exists x \forall y\ P(y) \leftrightarrow (y = x)$

Example 1: $ar + b = 0$ has a unique solution when a, b are real and $a \neq 0$

Existence:

$ar + b = 0$

$ar = -b$

$r = -b/a$

Uniqueness:

Assume $\exists s\ as + b = 0$

Then, $ar + b = as + b$

$ar + b - b = as + b - b$

$ar/\mathrm{a} = as/a$

$\mathrm{r} = s$

$r = -b/a$ is the solution to $ar + b = 0$

Example 2: Let A be an n×n invertible matrix. Prove that the inverse matrix of A is unique.

To prove the uniqueness, suppose that you have two inverse matrices B and C and show that in fact B = C.

Recall that B is the inverse matrix if it satisfies

AB=BA=I, where I is the identity matrix.

Suppose that there are two inverse matrices B and C of the matrix A. Then they satisfy

AB=BA=I and AC=CA=I

To show that the uniqueness of the inverse matrix, we show that B=C as follows. Let I be the n×n identity matrix.

We have

B=BI

B=B(AC)

B=(BA)C (by the associativity)

B=IC=C

Thus, we must have B=C, and there is only one inverse matrix of A.

The concept of counter-examples :

A counterexample is an example that disproves a universal ("for all") statement.

Suppose you have a quantified statement: "All x's satisfy property P": ∀xP(x).

What is its negation?

$\neg\forall x P(x) \leftrightarrow \exists x \neg P(x)$.

Example:

Take any three distinct primes >2, then multiply them and add 2.

- 3,5,7
  (3*5*7)+2 = 107
  Here, 107 is a prime number.

  To disapprove:

- 11,7,5
  (11*7*5)+2 = 387
  Here, 387 is not a prime number.

Proof by Contraposition:

A proof by contraposition takes advantage of the mathematical equivalence **P → Q ⇔¬Q →¬P.** That is, a proof by contraposition begins by assuming that Q is false (i.e., ¬Q is true). It then produces a series of distinct implications leading to the conclusion that P is false (i.e., ¬P is true). This implies that Q cannot be false when P is true, so P->Q.

Bipartite Graph:

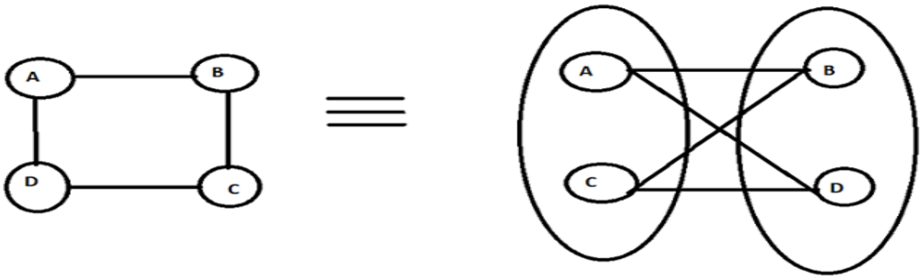
A bipartite graph is one whose vertices V can be divided into two independent sets V1 and V2, every edge of graph connects one vertex in V1 and other vertex in V2.

If V1 and V2 have same number of vertices then it is called **balanced bipartite graph.**

A bipartite graph contains some properties:

- It consists of 2 sets of vertices X and Y.
- The vertices of set X join only with vertices of set Y.
- The vertices with same set do not join.

Example:



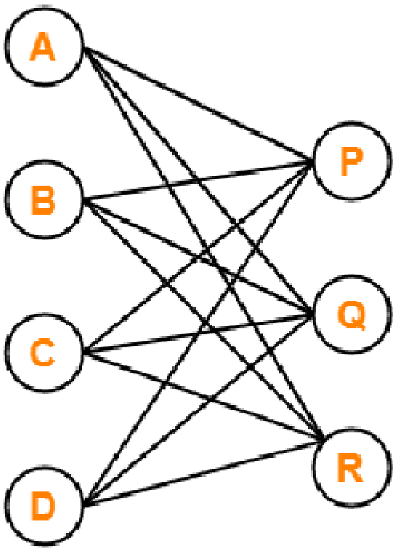
Here,

- The vertices of the graph can be decomposed into two sets.
- The two sets are X= {A, C} and Y= {B, D}.
- The vertices of set X join only with the vertices of set Y and vice-versa.
- The vertices within the same sets do not join
- Therefore, it is bipartite graph.

Complete Bipartite Graph:

A bipartite graph where every vertex of set X is joined with every vertex of set Y is called **complete bipartite graph.**

Example:



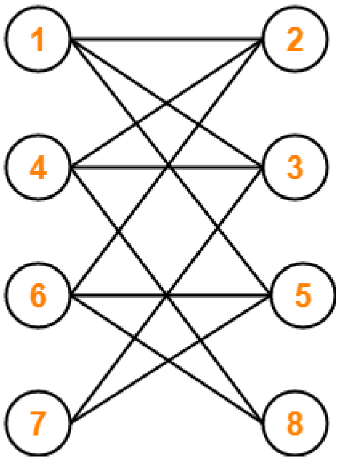
Here,

- This graph is a bipartite graph as well as complete graph.
- Therefore, it is a complete bipartite graph.
- This graph is called as $K_{4, 3.}$

Bipartite Graph Properties:

- Every sub graph of a bipartite graph is itself bipartite.
- Bipartite graphs contains no odd cycles (a cycle with an odd number of vertices)
- There does not exist a perfect matching for a bipartite graph with bipartition X and Y if |X|≠ |Y|.
- In any bipartite graph with bipartition X and Y ,
  Sum of degree of vertices of set X = Sum of degree of vertices of set Y.

Example:



Here,

- This graph consists of two sets of vertices.
- The two sets are X = {1,4,6,7} and Y = {2,3,5,8}.
- The vertices of set X are joined only with the vertices of set Y and vice- versa.
- No two vertices are joined with same sets.
- This satisfies the definition of bipartite graph.

Therefore, Graph is a **bipartite graph.**