1. Let A and B be two finite sets. Suppose A has an element x such that $x \notin B$. Then:

   (a) $|A| > |B|$
   (b) $|A| < |B|$
   (c) $|A| = |B|$
   (d) We cannot determine the relative values of $|A|$ and $|B|$ from the information given (correct answer)

   (a), (b) and (c) scenarios are all possible. For (a) take the sets as $A = \{1, 2, 3\}$ and $B = \emptyset$ and $x = 1$. For scenario (b) take $A = \{1\}$ and $B = \{2, 3, 4\}$ and $x = 1$. For scenario (c) take $A = \{1\}$ and $B = \{2\}$ and $x = 1$. Thus the correct answer is (d) that the conclusion cannot be made on the basis of the information provided.

2. Consider the powerset partial order of the four element set $S = \{1, 2, 3, 4\}$. The number of elements in a lattice with $\{2\}$ as source and $\{2, 3, 4\}$ as sink is:

   (a) 2
   (b) 4 (correct answer)
   (c) 5
   (d) 6

   Since it is a lattice, there has to be only one minimal element and one maximal element. They are both given. So the number of elements that are present in the sink and not in the source is 2. So one can generate all elements of the lattice by taking the source union a subset of the extra elements. The number of subsets of the extra elements is 2^2 which is 4.

3. The number of elements in a largest upper lattice (but not a lattice) contained within the power set partial order of the four element set $S = \{1, 2, 3, 4\}$ is:

   (a) 8
   (b) 14
   (c) 15 (correct answer)
   (d) 16

   The entire power set partial order is a lattice. Here we want an upper lattice that is not a lattice. Thus there must be more than one source element. Removing the empty set achieces this. Thus, the number of elements is the power set without the empty set which yields 15.

4. The number of elements in a largest partial order that is neither an upper lattice, nor a lower lattice nor a lattice, contained within the power set partial order of the four element set $S = \{1, 2, 3, 4\}$ is:

   (a) 8
   (b) 14 (correct answer)
   (c) 15
   (d) 16

   Here, in a similar manner to the previous question, we leave out both the empty set and the whole set which ensures t is neither an upper lattice nor a lower lattice.

5. The number of ordered pairs (including self loops) in the power-set partial order (not the hasse diagram) of a four element set is:

   (a) 16
   (b) 32
   (c) 64
   (d) 81 (correct answer)

This is a generalization of the concept of number of subsets. Note that we have here a set of S of 4 elements. We have an ordered pair in the relation when there are two subsets $A \subseteq S$ and $B \subseteq S$ where in addition we have $A \subseteq B$. This can be thought of as a three level hierarchy. Presence in A guarantees presence in B and S. Absence from A can be further classified as present in B or absent in B. However presence in B implies present in S but not vice versa. Thus there is a three level hierarchy for each element. Thus, every element can be labelled as 0,1 or 2 to indicate its level. Since each element has three possible labels and there are 4 elements the answer is $3^4 = 81$. This generalizes the technique by which we establisth that the number of subsets of an n element set is 2^n by labelling using 0 or 1.

6. Consider an eight element set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and a relation R on this set. You are told that the relation when restricted to the subset $S_1 = \{1, 2, 3, 4\}$ is an equivalence relation, with exactly one equivalence class. You are also told that the relation when restricted to the subset $S_2 = \{5, 6, 7, 8\}$ is a partial order with the same structure as the power-set partial order of a two element set. The largest possible number of elements in an equivalence relation on S is:

   (a) 4
   (b) 5
   (c) 6 (correct answer)
   (d) 8

   The provided correct answer is constructed by taing the reflexive, symmetric and transitive nature of the first four elements and restricting the reflexive antisymmetric and transitive nature of the last 4 elements in such a way that the restriction is reflexive symmetric and transitive. The reflexive and transitive attributes are already met. So we need a restricted portion which is both anti symmetric and symmetric. This happens when you take the two singleton sets in the power set partial order of a two element set. Neither is related to the other. And thus the elements are in an equivalence relation. These two can augment the first four elements to give an equivalence relation on 6 elements.

7. Consider the power set partial order of a four element set S. The largest equivalence relation that may be obtained by deleting some ordered pairs from this relation is:

   (a) 1
   (b) 2
   (c) 6 (correct answer)
   (d) 8

   This is also what is known as an **anti-chain** in the theory of partial orders. It is a set of elements no distinct pair of which is related to each other. This means the anti-symmetry becomes symmetry because neither direction is there an edge. It can be verified that the collection of all two element subsets of a 4 element set are such that neither is a subset of the other and hence they form an equivalence relation. There are six such sets.

8. Suppose there is a relation R defined on an eight element set S. Suppose R is both an equivalence relation and a partial order. Then the number of equivalence classes of R is:

   (a) 1
   (b) 2
   (c) 4
   (d) 8 (correct answer)

   Again, ths is like the previous example. The point of difference between an equivalence relation and a partial order is the **symmetrc** and **antisymmetric** clauses. These are blurred when the equivalnece classes of the equivalence relation are allwith just one element. In this special case the relation also becomes a partial order. Thus the only case is when each of the eight elements form single element equivalence classes.

9. Consider the ordered pairs $\{(1,20),(2,19),(3,18),(4,17),(5,16)\}$. The number of ways of arranging these pairs in a linear list, such that a pair x which has a bigger number on each coordinate than a pair y must appear after y in the linear list is:

   (a) 1

   (b) 4

   (c) 6

   (d) 120 (correct answer)

   In the given list every two ordered pairs is such that one of them is bigger on one coordinate and the other is bigger on the other coordinate. Thus there is no restriction on the relative order of any two elements. Thus all possible permutations of the five ordered pairs is valid leading to 5!=120.

10. Consider the ordered pairs $\{(1,16),(2,17),(3,18),(4,19),(5,20)\}$. The number of ways of arranging these pairs in a linear list, such that a pair x which has a bigger number on each coordinate than a pair y must appear after y in the linear list is:

    (a) 1 (correct answer)

    (b) 4

    (c) 6

    (d) 120

    Here, all ordered pairs are synchronised with other ordered pairs meaning if a pair si smaller on one coordinate it is also smaller on the other coordinate. Thus they form a total order and there is no flexibility. Hence the answer is just 1 order.

11. Consider the ordered pairs $\{(1,16),(2,20),(3,17),(4,18),(5,19)\}$. The number of ways of arranging these pairs in a linear list, such that a pair x which has a bigger number on each coordinate than a pair y must appear after y in the linear list is:

    (a) 1

    (b) 4 (correct answer)

    (c) 6

    (d) 120

    Here the first third fourth and fifth pairs are to be placed in thar order alone. The second element has to come after the first but has no other restrictions. Thus there are four orders.

12. Consider the ordered pairs $\{(1,16),(2,19),(3,20),(4,17),(5,18)\}$. The number of ways of arranging these pairs in a linear list, such that a pair x which has a bigger number on each coordinate than a pair y must appear after y in the linear list is:

    (a) 1

    (b) 4

    (c) 6 (correct answer)

    (d) 120

    Here the first three pairs must come in that order. The last fourth and fifth must come in that order. The fourth and fifth must appear after thr first but no other restrictions. Thus it is like a **shuffle function** between the second and third in that order and the fourth and fifth in that order. We know this is $\binom{4}{2} = 6$.

13. Consider a relation $R_{intersect}$ defined on the power-set of a set minus the empty set. Two subsets are related to each other in $R_{intersect}$ if and only if they have non-empty intersection. Which is the smallest iterative version of the relation $R_{intersect}$ so that every pair of sets is related to each other?

(a) $R_{intersect}$

(b) $R^2_{intersect}$ (correct answer)

(c) $R^3_{intersect}$

(d) $R^4_{intersect}$

Two sets which are not intersecting each other are both intersecting their unions. Thus it takes at most two steps to go from any set to any other set in this relation.

14. Consider the power set of a 13 element set where we can move from any subset to another subset in one step by one of three operations:

   - add an element to the subset
   - remove an element from the subset
   - take the complement of the subset

   The maximum number of steps needed to move from any subset to any other subset is:

   (a) 1
   (b) 6
   (c) 7 (correct answer)
   (d) 12

   To go from $\{1, 2, 3, 4\}$ to $\{5, 6, 7\}$ without using the complement operation takes 7 steps (4 deletes and 3 inserts). If we complement we will get a 9 element superset of the target set from which we need to delete 6 elements yielding 7 steps. To argue that we will never need more steps than 7, we observe that the intersection of the two sets need not be touched if we are not complementing the initial set. The total numberof operations is the **symmetric difference** of the two sets. If this is at most 7 then we are done. If the symmetric defence is more than 7, then we can complement the source set when the symmetric difference will necessarily be less than or equal to 6. This shows that we never need more than 7 steps.

15. Suppose you wish to construct 5 sets $A_1, \ldots, A_5$ such that for any subset of this set of sets, there is a unique element present in exactly those sets. Example $|A_1 \cap A_2 \cap \overline{A_3} \cap A_4 \cap \overline{A_5}| = 1$ and so on, for every such collection. Here the bar on top represents set complement. How many elements would be present in the union of all five sets?

   (a) 5
   (b) 10
   (c) 31 (correct answer)
   (d) 32

   The number of combinatiosn barring absent from all is $2^5 - 1 = 32$.

16. Consider the power set of an n element set where we can move from any subset to another subset in one step by one of three operations:

   - add any two elements to the subset
   - remove any two elements from the subset
   - take the complement of the subset

   We will be able to migrate from any subset to any other subset using these operations for which of the following values of n?

   (a) 10
   (b) 12
   (c) 13 (correct answer)
   (d) 14

   The indicated answer is the correct one because it is the only set with odd cardinality. Notice that the operation of adding two elements or deleting two elements will preserve the parity (odd even) of the resulting set. If the set has even cardinality then even the complement operation will preserve the parity. Thus we can never reach an even cardinality set starting from odd cardinality and vice versa. There is a certain level of similarity to the property of bipartite graphs covered in the lecture on proof techniques.

17. The number of injective functions (one-to-one) from a set D of size 4 to a set C of size 7 is

    (a) 24
    (b) 140
    (c) 840 (correct answer)
    (d) 5040

    This was explained in the lecture before. It is straight-forward. $7 \times 6 \times 5 \times 4 = 840$.

18. The number of surjective functions (onto) from a set D of size 5 to a set C of size 4 is

    (a) 240 (correct answer)
    (b) 120
    (c) 24
    (d) 625

    This is as explained in the tutorial before the exam. However, the numbers here are much smaller than the one covered in the tutorial and hence easier, by far. The answer is $\binom{5}{2} \times 4 \times 3! = 240$

19. Consider a finite set S with cardinality $|S| \geq 4$. Let us denote some parameters by the following code:

    - n_{surj}: number of surjective functions from S to S
    - n_{inj}: number of injective functions from S to S
    - n_{bij}: number of bijective functions from S to S
    - n_{func}: number of functions from S to S

    The correct sorted order of these four numbers is:

    (a) $n_{surj} < n_{inj} < n_{bij} < n_{func}$
    (b) $n_{inj} < n_{surj} < n_{bij} < n_{func}$
    (c) $n_{bij} < n_{surj} < n_{inj} < n_{func}$
    (d) $n_{bij} = n_{inj} = n_{surj} < n_{func}$ (correct answer)

    The indicated answer is obvious. We already saw that between two finite sets of equal cardinality every injective function is also surjective and vice versa. The functions are also, consequently, also bijective. Also clearly every function is not bijective. Thus the answer follows.

20. Consider two finite nonempty sets A and B with $|A| < |B|$. Which of the following is correct?

    (a) The number of injective functions from A to B is less than the number of surjective functions from B to A
    (b) The number of injective functions from A to B is greater than the number of surjective functions from B to A
    (c) The number of injective functions from A to B is equal to the number of surjective functions from B to A
    (d) The number greater value between the number of injective functions from A to B and the number of surjective functions from B to A cannot be determined, just by knowing that $|A| < |B|$. The specific values are needed. (correct answer)

    (a) We saw in the tutorrial. (b) If $|A| = 1$ and $|B| > 1$. The number of injective functions from A to B is $|B|$. The number of surjective functions from B to A is 1. (c) If$|A| = 2$ and $|B| = 3$, the number of injective functions from A to B is 6. The number of surjective functions from B to A is also 6. Thus equal. This leaves option (d) a combinaton of all the rest.