Discrete Mathematics

Scribed Notes-10

Recap:

Functions

Functions are relations where every element of the domain appears in exactly one ordered pair.

Eg : **f : A ⟶B**

Where,

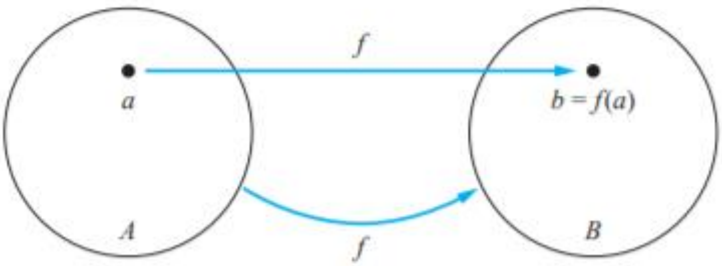
A is the domain of the function

B is the co-domain of the function

A ⟶B is called range of function

f is a function from A to B, we say that A is the **domain** of f and B is the **codomain** of f.

If f (a) = b, we say that b is the **image** of a and a is a **preimage** of b. The **range**, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.



The Function f Maps A to B.

Some functions never assign the same value to two different domain elements. These functions are said to be one-to-one.

TYPES OF FUNCTIONS:

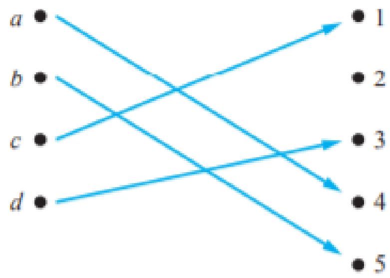
1. INJECTIVE FUNCTIONS

A function f is said to be one-to-one, or an injection, if and only if f (a) = f (b) implies that a = b for all a and b in the domain of f. A function is said to be injective if it is one-to-one.

In a nutshell, we can say that in an injective function two elements shouldn't match the same thing, every element has to match distinct elements.

$\forall a \, \forall b \, (f(a) = f(b) \rightarrow a = b)$

For injective functions, **|D|<=|C|**, where |D| is the cardinality of domain and |C| is the cardinality of codomain.



Injective function

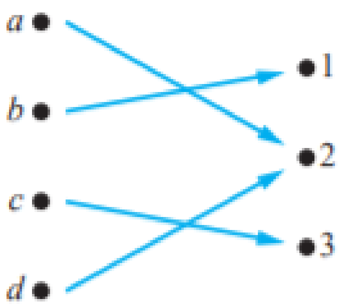
For some functions the range and the codomain are equal. That is, every member of the codomain is the image of some element of the domain. Functions with this property are called onto functions.

2. SURJECTIVE FUNCTIONS

A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.

A function f is onto if $\forall y \exists x (f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.

For surjective functions, **|D|=>|C|**, where |D| is the cardinality of domain and |C| is the cardinality of codomain.



Surjective function

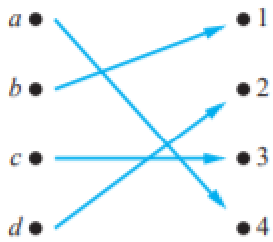
3. BIJECTIVE FUNCTIONS

The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto. We also say that such a function is bijective.

We can say that, bijective function is:

Injective AND Surjective.

It implies that, For bijective functions, **|D|=|C|**, where |D| is the cardinality of domain and |C| is the cardinality of codomain.



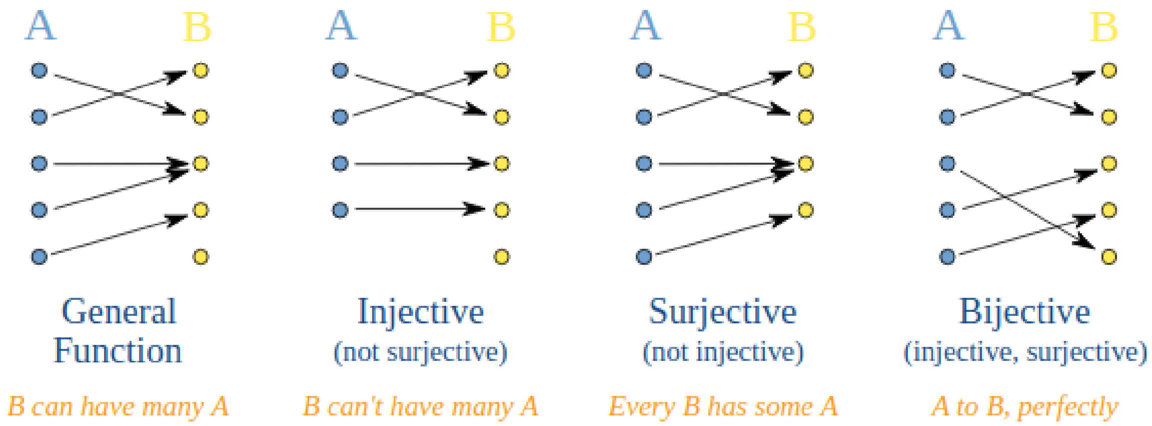
Bijective function

We can also conclude that,
If two sets have the same cardinality (i.e the number of elements in it) and there is an injective function between them, then it will mandatorily be a surjective function as well, thereby making it a Bijective function.

If Domain=Codomain then $\mathbf{f : D \longrightarrow C}$ is either a BIJECTIVE function or neither injective nor surjective. It cannot be solely an injective or surjective function.

If we have a function $f : \{1,2,3,4\} \longrightarrow \{1,2,3,4\}$
f(1) = 1, f(2) = 1, f(3) = 3, f(4) = 4
Then this function is neither Injective(1 and 2 in the domain have the same image) nor Surjective(2 in the codomain has no pre-image).

To summarise till now, we can see that:



Suppose that f : A → B.

To show that f is injective Show that if f (x) = f (y) for arbitrary x, y ∈ A with x = y, then x = y.

To show that f is not injective Find particular elements x, y ∈ A such that x = y and f (x) = f (y).

To show that f is surjective Consider an arbitrary element y ∈ B and find an element x ∈ A such that f (x) = y.

To show that f is not surjective Find a particular y ∈ B such that f (x) = y for all x ∈ A.

<u>COUNTING METHODS</u>

1. <u>INJECTIVE FUNCTIONS</u>

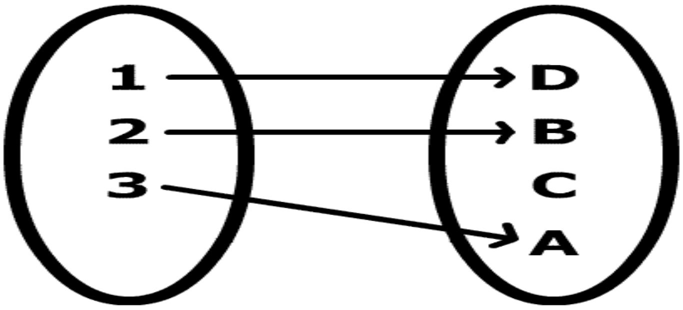
Suppose that there are 5 elements in domain A and 7 elements in codomain B.

The number of injective functions are:

m!/m-n!

Where m = Number of elements in codomain B,
n= Number of elements in domain A.

Therefore, 7!/7-5! = 7!/2!.

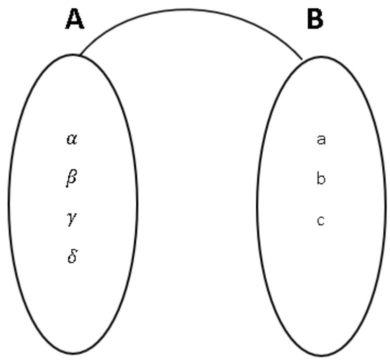


In the above example, the number of injective functions are = 4! / 1! = 24.

2. SURJECTIVE FUNCTIONS

Suppose that there are 4 elements in domain A and 3 elements in codomain B.

The number of surjective functions are:
Pairs will be(1,1,2) =(4!/1!*1!*2!*2!) *3!



In the above example, the number of surjective functions are = 36

COMPOSITION OF FUNCTIONS AND PERMUTATIONS

Function composition is an operation that takes two functions f and g and produces a function h such that $h(x) = g(f(x))$. In this operation, the function g is applied to the result of applying the function f to x. That is, the functions $f : X \to Y$ and $g : Y \to Z$ are **composed** to yield a function that maps x in X to $g(f(x))$ in Z.

Intuitively, if z is a function of y, and y is a function of x, then z is a function of x. The resulting *composite* function is denoted $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all x in X.

The composition of two permutations is also a permutation.

For example,
f(x)=2x+3 and g(x)=2x on domain=codomain= All positive integers,
then, for f(g(2)) firstly we will evaluate g(2) = 4 and then replace value of g(2) in f(g(2))

Which is equal to $f(4) = 11$

Meanwhile, for $g(f(2))$ firstly we will evaluate $f(2) = 7$ and then replace value of $f(2)$ in $g(f(2))$
Which is equal to $g(f(2)) = 14$

We see that, $f(g(2) \neq g(f(2))$
Which implies that composition of functions is **NOT COMMUTATIVE** in nature .

In an injective function **f**, if $x \in$ domain, $y \in$ codomain,
$x \neq y \Rightarrow f(x) \neq f(y)$

Now if we have another function **g,** then,
$g(f(x)) \neq g(f(y))$ iff $f(x) \neq f(y)$, which is only possible when $x \neq y$

Then we can conclude that,
$x \neq y \Rightarrow g(f(x)) \neq g(f(y))$ which is also an injective function.

It can be concluded that the **composition of two injective functions is also an injective function.**