Discrete Mathematics
Scribed Notes 6

Rule of Sum:

If there are A choices for one action, and B choices for another action and the two actions cannot be done at the same time, then there are A+B ways to choose one of these actions.

The rule of sum only applies to choices that are mutually exclusive, meaning that only one of the choices can be picked.

Example:

1. You go to a Stationery shop where there are 3 sizes of colored papers and 4 sizes of white papers. You need to buy one paper to make a poster. How many options do you have?
    - You will have 3+4 = 7 options to buy a paper.
2. In the same stationery shop, there are 3 types of pencils and 7 types of pens. In how many ways can you buy a writing tool?
    - You will have 3+7 = 10 options to buy a writing tool.

Rule Product:

It is the idea that if there are A ways of doing something and B ways of doing another thing, then there are A×B ways of performing both actions.

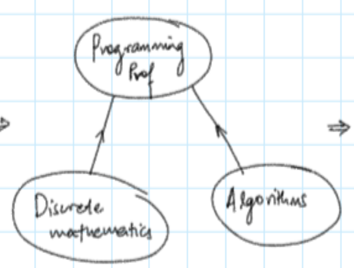
Example:

1. Referring to the same stationery shop example, now you need to make a poster for which you need a display medium and a writing tool. How many ways can you choose one display medium and a writing tool?
    - You will have 7×10 = 70 options for choosing a display medium and a writing tool.

Partial Order and Total Order:

	Professor Rating		
	Discrete Math	Algorithms	Programming
Student 1	3.6	2.7	4.5
Student 2	1.9	4.7	4.9

⇒



⇒

Discrete mathematics, Algorithms, Programming
or
Algorithms, Discrete mathematics, Programming

Let the above table represent the ratings of professors provided by each student. But unless there is a clear criteria for ordering these ratings we cannot choose the professor.

Let's say the Professor who gets the highest rating from a student is the best. From the table it is evident that the Programming professor is the best in this context but it will be different for different students. Also there is an ambiguity in the order of the remaining two professors.

The two of the possible orders in this context are :
Discrete mathematics, Algorithms, Programming
(3.6, 1.9), (2.7, 4.7), (4.5, 4.9)
OR
Algorithms, Discrete mathematics, Programming
(2.7, 4.7), (3.6, 1.9), (4.5, 4.9)

The above mentioned case is a case of **Partial Ordering**. It is called Partial ordering as it has some kind of ambiguity in the ordering.

On the other hand if we take the order of total average marks of students in each subject as follows:
Discrete mathematics - 7
Algorithms - 6
Programming - 8

If we need to order the above data it is simple and defined i.e. 6,7,8 (Algorithms, Discrete mathematics, Programming)
This kind of definite ordering is called **Total Ordering**.

Relation:

A relation R on a set S is a subset of S×S (cartesian product of the set with itself).

Let R be a relation on set S, which has ‘n’ elements in it.
Therefore $|S| = n$

$$\Rightarrow |S \times S| = n^2$$

⇒Number of Relations of $|S \times S|$= 2^{n^2}

- Each element of $|S \times S|$can either be present in R or not present in R
- Therefore each element from n^2 has 2 options
- So by the “Rule of Product” number of ways in which relation R can be formed are:

$$2 \times 2 \times 2 \times \ldots (n^2\ times) \times 2 = 2^{n^2}$$

Example:
As given in the figure there is a set S defined with six numbers and a relation R is defined on S such that the absolute difference between the numbers should be less than 50.

$$S = \{\overset{A}{1890}, \overset{B}{1920}, \overset{C}{1957}, \overset{D}{1978}, \overset{E}{2015}, \overset{F}{2036}\}$$

$$R = \{(a,b) : a \in S, b \in S, |a-b| < 5\}$$

or

$$R = \{(A,A), (A,B), (B,A), (B,C), (C,B), (B,B), (C,C), (C,D), (D,C), (D,D), (D,E), (E,D), (E,E), (E,F), (F,E), (F,F)\}$$

Note: here A to F symbols have been used for easier representation of the values.

Figure 1

Types of Relations:

1. **Reflexive Relation:**

A relation R on a set S is called reflexive if (a,a) ∈ R holds for every element a ∈ S .i.e. if set S = {a,b,c} then R = {(a,a), (b,b),(c,c)} is a reflexive relation.

$\forall a \in S, (a,a) \in R$ where R is a Relation on set S.

Example:

From figure 1, we can see that the relation R contains (A,A), (B,B), (C,C), (D,D), (E,E), (F,F) i.e. the relation R has all the elements (a,a) ∀ a∈S.

Number of Reflexive Relations on a Set:

Let R be a relation on set S, which has ‘n’ elements in it.

Therefore $|S| = n$

$$\Rightarrow |S \times S| = n^2$$

⇒Number of Relations of $|S \times S|$= 2^{n^2}

⇒Number of elements (a,a) in $|S \times S|$, $\forall a \in S = n$

If the relation R is reflexive then these n elements should compulsorily be present in the relation R. Thus these n elements have no option but to be present in the Relation. Therefore by the Rule of Product we can say that the number of ways relation R can be formed such that it is a Reflexive Relation are:

$$(1 \times 1 \times \ldots (n\ times) \times 1) \times (2 \times 2 \times \ldots ((n^2 - n)\ times) \times 2) = 1^n \times 2^{n^2-n}$$

⇒Number of Reflexive relations on set S = 2^{n^2-n}

2. **Irreflexive Relation**

A relation R on set S is said to be irreflexive if $(a, a) \notin R$ holds for all $a \in S$..i.e. if set S = {a,b,c} then R = {(a,b), (a,b),(a,c)} is an irreflexive relation.

$\forall a \in S, (a, a) \notin R$ where R is a relation on set S.

Example:
Let R be a relation on set A={1,2,3} such that R = {(1,2),(2,2),(3,1),(1,3)}. Is relation R reflexive or irreflexive?

- Relation R is not reflexive as $\exists a \in S$ such that $(a, a) \notin R$.
- Relation R is not irreflexive as $\exists a \in S$ such that $(a, a) \in R$.

Number of Irreflexive Relations on a Set:
Let R be a relation on set S, which has 'n' elements in it.
Therefore $|S| = n$

$$\Rightarrow |S \times S| = n^2$$

$\Rightarrow$Number of Relations of $|S \times S|$= 2^{n^2}
$\Rightarrow$Number of elements (a,a) in $|S \times S|$, $\forall a \in S = n$
If the relation R is irreflexive then these n elements should compulsorily be absent in the relation R. Thus these n elements have no option but to be absent in the Relation. Therefore by the Rule of Product we can say that the number of ways relation R can be formed such that it is a Irreflexive Relation are:

$$(1 \times 1 \times \ldots (n\ times) \times 1) \times (2 \times 2 \times \ldots ((n^2 - n)\ times) \times 2) = 1^n \times 2^{n^2 - n}$$ [redacted]

$\Rightarrow$Number of Irreflexive relations on set S = $2^{n^2 - n}$

Number of relations that are neither reflexive nor irreflexive = $2^{n^2} - 2^{n^2 - n + 1}$

3. **Symmetric Relation:**

A relation R on set S is called Symmetric if $(b, a) \in R$whenever $(a, b) \in R$ for all $a, b \in A$.

$$\forall a \in S, \forall b \in S, ((a, b) \in R) \Leftrightarrow ((a, b) \in R)$$

Or

$$[((a, b) \in R) \wedge ((b, a) \in R)] \vee [((a, b) \notin R) \wedge ((b, a) \notin R)]$$

Requirement to be a Symmetric Relation does not impose any restriction on the presence or absence of (a, a) $\forall a \in S$.

Example:
From figure 1, since R has {(A,B),(B,A),(B,C),(C,B),(C,D),(D,C),(D,E),(E,D),(E,F),(F,E)}

Number of Symmetric Relations on a Set:
Let R be a relation on set S, which has 'n' elements in it.
Therefore $|S| = n$

$$\Rightarrow |S \times S| = n^2$$

$\Rightarrow$Number of Relations of $|S \times S| = 2^{n^2}$
$\Rightarrow$Number of elements (a,a) in $|S \times S|$, $\forall a \in S = n$
$\Rightarrow$Number of elements (a, b)in $|S \times S|$, $\forall a \in S, \forall b \in S, a \neq b = (n^2 - n)$
In a symmetric relation if $(a, b)$is present then (b, a)should definitely be present and if (a, b) is absent then (b, a) shouldn't be present.
Thus, if we fix the presence or absence of half of $(n^2 - n)$elements then the presence or absence of the remaining half will be fixed.
Also the presence or absence of (a, a) $\forall a \in S$doesn't affect the Symmetricity of the Relation.
$\Rightarrow$Number of Symmetric relations = $2^{\frac{n^2-n}{2}+n} = 2^{\frac{n^2+n}{2}}$

4. Antisymmetric Relation:

A relation R on set S such that for all $a, b \in S$, if $(a, b) \in R$and $(b, a) \in R$, then $a = b$ is called Antisymmetric.

$$[((a, b) \in R) \wedge ((b, a) \notin R)] \vee [((a, b) \notin R) \wedge ((b, a) \in R)] \vee [((a, b) \notin R) \wedge ((b, a) \notin R)]$$

If proposition p means "presence of (a,b) in relation R" and proposition q means "presence of (b,a) in relation R" then $\neg(p \wedge q)$defines the behavior of an antisymmetric relation i.e. it will always be an antisymmetric relation except for the case where (a,b) and (b,a) both are present in relation R.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Number of Antisymmetric Relations on a set:
Let R be a relation on set S, which has 'n' elements in it.
Therefore $|S| = n$

$$\Rightarrow |S \times S| = n^2$$

$\Rightarrow$Number of Relations of $|S \times S| = 2^{n^2}$
$\Rightarrow$Number of elements (a,a) in $|S \times S|$, $\forall a \in S = n$
$\Rightarrow$Number of elements (a, b)in $|S \times S|$, $\forall a \in S, \forall b \in S, a \neq b = (n^2 - n)$

⇒Number of distinct pairs (a,b) irrespective of their order in $|S \times S|, \forall a \in S, \forall b \in S = \frac{(n^2-n)}{2}$ = C(n,2)

From the above truth table we can say that there are 3 options for each pair of distinct (a,b) , irrespective of their order, to make the relation an antisymmetric relation.
Again the presence or absence of (a,a) in R, $\forall a \in S$ doesn't affect the anti-symmetricity of the relation.

⇒Number of Antisymmetric relations on a set S = $3^{\frac{n^2-n}{2}} \times 2^n$

(where $3^{\frac{n^2-n}{2}}$ is the number of ways of choosing (a,b) pairs, from $|S \times S|$, where a≠b and 2^n is the number of ways of choosing (a,a) pairs from $|S \times S|$)