Discrete Mathematics (SC612)
Tutorial 3
8^{th} Oct0ber, 2021

1. Count the number of positive integers relatively prime to 300 in the set $\{1, \ldots, 300\}$.

   Hint: This is an application of the principle of inclusion & exclusion.

   solution:

   The total numbers is 300. $300 = 2^2 \times 3 \times 5^2$. Thus its prime factors are 2, 3 and 5. Thus its divisors which are multiples of 2 number 150. Its divisors that are multiples of 3 number 100. The number that are multiples of 5 is 60. The total is 310. The number that are multiples of 2 and 3 is 50. The numbers that are multiples of both 2 and 5 are 30. The number that are multiples of both 3 and 5 are 20. Thus the total of pairwise intersection is 100. The number that are multiples of 2, 3 and 5 is 10. Thus the total number of numbers that have common non-trivial divsors with 300 is 310-100+10=220. Thus number of numbers less than 300 and relatively prime to 300 is 300-220=80.

2. Consider strings of length exactly 20 over the alphabet $\Sigma = \{9, 1\}$. A monotic block is a maximal substring of identical characters. Example $s = 00111001110011111010$. The monotic block lengths reading from left to right are: 2,3,2,3,2,5,1,1,1. The total is 20. Count the number of strings such that the block lengths are strictly increasing from left to right. The example we gave does not qualify, because for example the third block is shorter than the second block. Also the the 8^{th} block and 9^{th} block have identical length. It is allowed that there is only one block, and the strings may start with either 0 or 1.

solution:

We can classify according to the length of the left most block, starting from the biggest possibility. Each such configuration will lead to an even number by exchanging the positioning of 0's and 1's. The configurations are:

(a) First block is length 20. This is just one.

(b) The first block is of length 9. This also leads to just one formation. (9+11)

(c) The first block is of length 8. This also leads to just one formation. (8+12)

(d) The first block is of length 7. This also leads to just one formation. (7+13)

(e) The first block is of length 6. This also leads to just one formation. (6+14)

(f) The first block is of length 5. This leads to three formations. (5+15, 5+6+9 and 5+7+8)

(g) The first block is of length 4. This leads to four formations (4+16, 4+5+11, 4+6+10, 4+7+9)

(h) The first blockis of length 3. This leads to eight formations. (3+17, 3+4+13, 3+5+12, 3+6+11, 3+7+10, 3+8+9, 3+4+5+8, 3+4+6+7)

(i) The first block of length 2. This leads to fifteen configurations: (2+18, 2+3+15, 2+4+14, 2+5+13, 2+6+12, 2+7+11, 2+8+10, 2+3+4+11, 2+3+5+10, 2+3+6+9, 2+3+7+8, 2+4+5+9, 2+4+6+8, 2+5+6+7, 2+3+4+5+6)

(j) The first block is of length 1. This leads to twenty-nine configurations:
(1+19,
$1+2+17, \ldots, 1+9+10,$
$1+2+3+14, \ldots, 1+2+8+9,$
$1+3+4+12, \ldots, 1+3+7+9,$
$1+4+5+10, 1+4+6+9, 1+4+7+8,$
$1+5+6+8,$
$1+2+3+4+10, \ldots, 1+2+3+6+8,$

$1+2+4+5+8, 1+2+4+6+7,$
$1+3+4+5+7)$

The total is 64 and the multiplication by 2 for the 0 1 exchange leads to 128 strings.

3. We saw in the lecture that for a grid consisting of $m+1$ vertical lines 0 to m, and $n+1$ horizontal lines 0 to n, the number of shortest paths from $(0,0)$ to (m,n) is $\binom{m+n}{m}$.

   (a) Find the smallest number of grid points (other than $(0,0)$ and (m,n)) that must be deleted, such that exactly one of the original shortest paths remain.

   solution:

   It is $\min\{m-1, n-1\}$. Without loss of generality assume the vertical direction is shorter than or equal to the horizontal direction. We need to remove all the elements on the second vertical line, except the top most one.

   (b) Is it possible to delete nodes in such a way that all the original shortest paths (length $m+n$) between $(0,0)$ and (m,n) are destroyed, but paths (of longer length) still remain. What is the smallest number of points to be deleted to achieve this?

   solution:

   Yes, it is possible as discussed during the tutorial. Remove the points $(1,0), (0,2), (1,2), (2,2), (3,1)$ and the shortest path get destroyed but longer paths remain.

4. Suppose you have a 6×6 grid (number of squares, not, number of parallel lines. Suppose you need to colour all the 36 squares with colours from $Col = \{Red, Blue, Green, Orange\}$ such that the squares immediately to the left, right, top and bottom of any square cannot be coloured the same colour as that square. How many ways are there to colour the entire grid? Can you first conclude that the number is a multiple of 24, even before making the actual calculation?

   solution:

   Once we get a valid colouring, even without making any adhoc alterations, one can systemmatically permute the four coloured squares's colours among themselves. Thus we have for each structurally different colouring, $4! = 24$ duplications by merely permuting the colours. Thus the final answer is a multiple of 24.

Let us consider the first third and fifth rows. They can be filled up in 4×3^5 ways each. Once we have filled these three rows, we will fill rows 2, 4 and 6 each from left to right. When we attempt to fill these rows, at each point we could have either 1, 2 or 3 choices. So we can lower and upper bound the number of colourings. It is difficult to count exactly.

Thus the number of colourings is between $(4 \times 3^5)^3$ and $(4 \times 3^5)^3 \times 3^{18}$

5. Suppose you are given a list of **ordered pairs** of integers, such that no two ordered pairs have the same integer in their first coordinates and also no two integers have the same integer in their second coordinates. You are required to make a hierarchical arrangement such that at the head must be the ordered pair with the smallest first coordinate. The remaining ordered pairs are to be partitioned into two disjoint sets, such that in one set all the elements have second coordinate smaller than the head's second coordinate, and the other set is such that all the elements have second coordinate larger than the head's second coordinate. You need to repeat the same proicedure on each of the two sets, until all the elements have been placed in the hierarchy. Perform this for the input: $(3, 8), (5, 6), (11, 2), (33, 29), (51, 22), (4, 43), (18, 18)$. In how many ways can this hierarchy be made in general?

   solution:

   Head of the hierarchy is the element $(3, 8)$.

   Left-set=$\{(5, 6), (11, 2)\}$

   Right-set=$\{(33, 29), (51, 22), (4, 43), (18, 18)\}$

   The left-set has at the top of its hierarchy, the element $(5, 6)$ The other element in the left-set goes to the left-set of the left-set.

   The right-set has at the top of its hierarchy the element $(4, 43)$

   All the remaining elements go into **its** left-set.

   The top of the hierarchy of the left-set of the right-set is the element $(18, 18)$.

The remaining two elements go to its right-set.

The top of the hierarchy among these two elements is $(33, 29)$.

The remaining element $(51, 22)$ is in its right-set.

Such sets have a unique formation. They are called **treaps** (a hybrid of binary seatch tree and binary heap).

6. In how many ways can n distinct integers be arranged in a sequence such that the longest number of consecutively increasing or decreasing elements is 2?

solution:

The answer is trivially 1 for 1 and 2 length sequences. For longer sequences, we merely write down a recurrence relation here, without solving it. Notice that the maximum element can be present at any position. However, after we place the maximum element at position i, the sequence $1, \ldots, i$ must end with a decrease, and the sequence $i + 1, \ldots, n$ must begin with an increase. Thus each of these sequences are inductive terms multiplied by a factor of $\frac{1}{2}$. The recurrence relation is therefore:

$$S_n = \frac{1}{4} \Sigma_{k=1}^{n} S_{k-1} \times S_{n-k}$$