Discrete Mathematics (SC612)
Tutorial 2
17^{th} September, 2021

1. Consider a generic relation defined over any list of positive integers, which says that $(x, y) \in R$ if and only if,

$$(((x < y) \wedge ((y - x) \leq 200)) \vee ((x - y) \geq 150))$$

This can be instantiated by considering any finite subset of the integers of your choice. When considered over all positive integers determine whether this relation is:

(a) reflexive/irreflexive or neither
solution:
Here it isan **irreflexive** relation. Ths is because there the boolean conditions are false when we set $x = y$, and thus there cannot be any (x, x) as an ordered pair in the relation.

(b) symmetric/anti-symmetric or neither
solution:
This relation is **neither symmetric nor anti-symmetric**. The explanation for not symmetric is:
$(200, 400) \in R$ but $(400, 200) \notin R$.
The explanation for not anti-symmetrc is:
$(100, 250) \in R$ and $(250, 100) \in R$.

(c) transitive
solution:
This relation is **NOT transitive**. $(200, 400) \in R$ and $(400, 600) \in R$ but $(200, 600) \notin R$

2. Suppose the size of a finite set $|S|$ divides the size of its power set $\mathcal{P}(S)$. What are the possible values of $|S|$?

   solution:

   We know that the cardinality of the power set of S, denoted by $\mathcal{P}(S)$ is $2^{|S|}$.

   Thus the only prime factor of $|\mathcal{P}(S)|$ is 2. Thus if $|S|$ divides $|\mathcal{P}(S)|$ then $|S|$ must be a power of 2. These are the possible vlaues of $|S|$: $1, 2, 4, 8, 16, \ldots$.

3. Construct a relation over a set of 8 elements that takes exactly 5 iterations to reach transitive closure.

   soluton:

   $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (7, 7), (8, 8)\}$

   On the fifth iterated version we get $(1, 6)$ in the relation and there wont be further changes, thereafter.

4. Consider the relation $R = \{(a, b), (b, c), (c, d), (d, e), (e, f)\}$ over the set $S = \{a, b, c, d, e, f\}$. What is the minimum number of ordered pairs to be added to R to transform it into a transitive relation? What it the minimum number of pairs to be deleted from R to transform it into a transitive relation?

   solution:

   We need to add $(a, c), (a, d), (a, e), (a, f), (b, d), (b, e), (b, f), (c, e), (c, f), (d, f)$. This is a total of 10 ordered pairs.

   We need to remove (b, c) and (d, e) in order to transform the relation into a transitive one. Thus we need to remove 2 pairs.

5. Suppose an equivalence relation over a set S contains exactly 79 ordered pairs. Give the minimum and maximum possible value of $|S|$ and also state which values in this range are possible.

   solution:

   Since a relation that is equivalence, must be reflexive, we cannot have more than 79 elements in the set. This is the maximum possible and can easily be achieved by only including the pairs of identical first and second coordinates. The maximum number of ordered pairs in a relation over a set of n elements is n^2. Thus to have 79 pairs we need at least $\sqrt{79}$ elements. This number is at least 9. We have to calculate more precisely to confirm whether 9 works or not.

Clearly we cannot lose just two ordered pairs from the 81 as removing would require removal of say (a, b) and (b, a) to preserve symmetry but (a, c) and (c, b) would still remain in the absence of (a, b) violating transitivity. Thus, 9 wont work. Effectively to respect reflexivity, symmetry and transitivity, what we require is that the points of the set are partitioned in such a way that each part with k points has k^2 ordered pairs. Thus we have to partition a candidate n into positive integers such that the sum of squares is exactly 79. We see that $79 = 64+9+4+1+1 = 8^2+3^2+2^2+1^2+1^2$. Here, $8+3+2+1+1 = 15$ which works.

We cannot have 78 elements as that leads to 78 ordered pairs with identical first and seond coordinates after which we cannot create just one ordered pair. With 77 we can simply augment the 77 ordered pairs with identical first and second coordinates with a single pair of distinct elements in both orders. Thus 77 also works. We cannot obtain any odd number consisting of distinct elements as the symmetry implies we will get groups of 2. Thus the number of elements to obtain 79 ordered pairs has to be an odd number. We can obtain the numbers $79, 77, 75, 73, 71, \ldots, 41$.

The 41 elements case consists of 3 isolated elements and 19 pairs. That leads to a total of 41+38=79. For 39, we can obtain 39+20+20 using complete graphs on 5 elements each (bidirectional).

For 37 it is 37+ 42 where the 42 is obtained by a bidirectional complete graph on 7 elements.

For 35 it is 35+ 44 where the 44 is obtained as 42+2 with the 42 coming from 7 elements and the two coming from 2 elements. Thus apart from the identical pairs we use a total of 7+2=9 of the 35 elements.

For 33 it is 33+46. 46=42+2+2 using 7+2+2=11 elements.

For 31 it is 31+48. Here the 48 comes as 42+6 with the 42 coming from 7 elements and the 6 coming from 3 elements.

For 29, we have 79=29+50. 50=42+2+2+2+2 the 42 comes from 7 elements and the four 2's come from two elements each using a total of

15 of the 29 elements.

For 27 we have 79=27+52. 52=42+10. 42 comes 7 and the 10 comes from 5 using a total of 15 of the 27 elements.

For 25 we get 79=25+54. 54=42+10+2 from 7+5+2=14 elements.

For 23 we get 79=23+56. The 56 is obtained from 8 elements.

For 21 we get 79=21+58. 58=56+2 obtained from 8+2= 10 elements.

For 19 we get 79=60+19. 60=56+2+2 obtained from 8+2+2=12 elements.

For 17 we get 79=62+17. 62=56+2+2+2 obtained from 8+2+2+2=14 elements.

Thus we have obtained every odd number from 15 to 79.

9 was shown to be impossible. We need to resolve the problem for 11 and 13.

$11^2 = 121$ so we cant get 79 with just one equivalence class. If we try with two, we cannot use 10 as one of them since $10^2 = 100 > 79$. Also 9 wont work because $9^2 = 81 > 79$. If we use 8, it accounts for64. With the remaining 3 we can obtain at most 6 and that is 70 which is not 79. With 7 we get 49 and the remaining 4 can generate at most 16 which is only 65. A 6,5 split generates at most $61(6^2 + 5^2)$. More than 2 parts will reduce the number of ordered pairs even further, thus 11 wont work.

For 13, we see $13^2 = 169 > 79$. The biggest part cannot exceed 8. If we take 8 it generates 64 leaving a further 15 to be generated by the remaining 5 elements. Retaining the remaining 5 elements in a single class gives 25. Splitting as 4, 1 yields 17 which is too much. Splitting as 3,2 yields 13 which is too few. Thus with biggest part 8 it wont work. If the biggest part is 7 that part yields 49. We need to generate 30 from the remaining 6 elements. 6 in one block generates 36 which is too many. Splitting as 5 1 generates 26 which is too few. Splitting as 4, 2 yields 20 which is again too small. Splitting as 3, 3 yields 18 which is too few. A three way split also wont work. The rest of the splits also wont work. Thus this is also a no instance.

6. We know that a set is a collection of **well defined, distinct** objects, and there is no further restriction. Thus we could have a set of sets (that

is the elements of the set are each sets, which are distinct). Consider such a set $\mathcal{A}$ of sets $\{S_1, \ldots, S_n\}$. Let us define a relation R over $\mathcal{A}$ whetre $(S_i, S_j) \in R$ if and only if $S_i \subseteq S_j$. Is the relation R:

(a) Reflexive, irreflexive or neither?

solution:

It is **reflexive**. This is because every element is a subset of itself.

(b) Symmetric, anti-symmetric or neither?

solution:

It is **anti-symmetric**. This is because if $A \subseteq B$ and $A \neq B$ implies $B \not\subseteq A$.

(c) Transitive?

solution:

Yes it is **transitive**. This is because $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$.

7. Consider a set of sets. We define a relation over this set where two elements are related if and only if their intersection is of size atleast 5.

(a) This relation is reflexive if and only if ________________________________

: solution

Each set has cardinality at least 5. This is because the intersection of a set with itself is the set itself and so for it to have size at least 5 the set must have size at least 5.

(b) Is this relation symmetric, anti-symmetric or neither, in general?

solution:

The relation is symmetric, since the relation is based on intersection and set intersection is a commutative operation.

(c) Is this relation transitive, in general?

solution:

No the relation is not transitive. Examole $S_1 = \{1, 2, 3, 4, 5\}$ $S_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $S_3 = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.
$|S_1 \cap S_2 = 5|$, $|S_2 \cap S_3| = 5$ but $|S_1 \cap S_3| = 0$.

8. Let $S = \{1, \ldots, 100\}$, the set of the first 100 positive integers. Define a relation R where $(x, y) \in R$ if and only if

$$((x = y) \vee ((|x - y| \leq 15) \wedge (|x - y| \geq 5)))$$

Find the cardinality of the largest subset X of S, such that the relation R restricted to the subset X is an equivalence relation. How many such subsets are there in S?

We can generate an equivalence relation using the following list:

$$1, 2, 3, 4, 5, 6, 22, 23, 24, 25, 26, 27, 43, 44, 45, 46, 47, 48, 64, 65, 66, 67, 68, 69, 85, 86, 87, 88, 89,$$

As explained during the tutorial, this is the maximum possible and yields 30 elements.

The number of such sequences is difficult to count precisely, but there is a lower bound of $\binom{16}{10}$. This is because the final gap of ten unused numbers in the sequence can be pushed to other elements occurring earlier inn any of the six zones where there is a gap.