Discrete Mathematics (SC612)
Tutorial 6
26^{th} November, 2021

1. Evaluate:

   (a) $5^{984861} \pmod{17}$

   solution:

   Let us calculate the powers of 5, mod 17 to see when it reaches the value 1. The sequence is $5, 8(= 5 \times 5 = 25 \bmod 17), 6(= 8 \times 5 = 40 \bmod 17), 13(= 6 \times 5 = 30 \bmod 17), 14(= 13 \times 5 = 65 \bmod 17), 2(= 14 \times 5 = 70 \bmod 17), 10, 16(= 10 \times 5 = 50 \bmod 17), 12(= 16 \times 5 = 80 \bmod 17), 9(= 12 \times 5 = 60 \bmod 17), 11(-9 \times 5 = 45 \bmod 17), 4(= 11 \times 5 = 55 \bmod 17), 3(= 4 \times 5 = 20 \bmod 17), 15, 7(= 15 \times 5 = 75 \bmod 17), 1(= 7 \times 5 = 35 \bmod 17)$.

   Rewriting without the detailed calculations, the sequence is: $5, 8, 6, 13, 14, 2, 10, 16, 12, 9, 11, 4, 3, 15, 7, 1,$ The 16th term in this sequence is the first occurrence of 1. Thus $5^{16} \cong 1 \bmod 17$. Thus to get the value of the given expression, it is simplest to get the remainder of the exponent when divided by 16. The exponent is $984861 = 61553 \times 16 + 13$. Therefre the given epression is equal to $5^{13} \bmod 17 = 3$ as seen in the list calculated earlier.

   (b) $7^{231987} \pmod{12}$

   solution:

   Here again, let us see see the cyclic pattern of powers of 7 mod 12. The sequence is $7, 1(= 7 \times 7 = 49 \bmod 12)$. Thus, $7^2 \cong 1 \bmod 12$. Thus the answer is either 1 or 7, if the exponent is even the answer is 1 if the exponent is odd the answer is 7. The given exponent, 231987 is an odd number, thus the answer is 7.

2. Solve the system of simultaneous congruences given below, for x.

$$\begin{aligned} x &\cong 4 \pmod 7 \\ x &\cong 2 \pmod 3 \\ x &\cong 3 \pmod 4 \\ x &\cong 4 \pmod 5 \\ x &\cong 9 \pmod{11} \end{aligned}$$

solution:

Step 1: Final Modulus The solution will be unique modulo the product of the individually modulii, since they are relatively prime. The product of the modulii is 4620.

Step 2: quotients We now take the five quotients of dividing the product of the modulii by the five individual modulii. These are: 660, 1540, 1155, 924, 420.

Step 3: inverses

$660^{-1} \bmod 7 = 4$,

$1540^{-1} \bmod 3 = 1$

$1155^{-1} \bmod 4 = 3$

$924^{-1} \bmod 5 = 4$

$420^{-1} \bmod 11 = 6$

Step 4: Original residues: 4,2,3,4,9

Step 5: Final calculation

$$x = 4 \times 4 \times 660 + 1 \times 2 \times 1540 + 3 \times 3 \times 1155 + 4 \times 4 \times 924 + 6 \times 9 \times 420 \bmod 4620$$

$$x = 10560 + 3080 + 10395 + 14784 + 22680 = 61499 = 1439 \bmod 4620$$

The solution is therefore $x = 1439 \bmod 4620$.

3. Is it possible to have a simple undirected graph G on n vertices, such that the degree sequence of G is the same as the degree sequence of $\overline{G}$ for:

   (a) $n = 4$

   (b) $n = 5$

   (c) $n = 6$

   (d) $n = 7$

   solution

   The complement sequence is obtained from the sequence by subtracting each term from $n - 1$.

   $1, 2, 2, 1$. The complement is $3 - 1, 3 - 2, 3 - 2, 3 - 1 = 2, 1, 1, 2$. And we know this is a legitimate degree sequence, of the path graph. So $n = 4$ is a yes instance.

   For $n = 5$, we have 2,2,2,2,2. The complement sequence is also the same, as can be easily checked. This is the degree sequence of a cycle.

   For $n = 6$, the number of edges of the complete graph is $\binom{6}{2} = 15$. If a graph and its consequence have the same degree sequence, then they have the same total degree and hence the same number of edges. That means 15 edges are partitioned into two eaual parts. This is not possible as 15 is an odd number.

   For $n = 7$, the total number of edges in the complete graph is 21. Thus, this is also a no instance.

4. (a) Is a dominating set of a graph always a vertex cover?

   solution

   No. Take the example of the triangle graph C_3. Any one of its vertices is a dominating set, but not a vertex cover.

   (b) Is a vertex cover of a graph always a dominating set?

   solution:

   Yes, provided there are no isolated vertices. In this case, take any vertex not in the vertex cover. All its neighbours must be in a vertex cover, in order to cover its incident edges. Thus this vertex is dominated by its neighbours, and thus a vertex cover is a dominating set in any graph withoug isolated vertices.

(c) What is the condition such that if D is a dominating set, then $V \setminus D$ is also a dominating set?

solution:

When there are isolated vertices, they must be present in every dominating set, since they cannot be dominated by other vertices. If a dominating set is **minimal** it means every vertex in that set dominates at least one vertex outside. Thus the complement set of vertices is also a dominating set. However, this is not true for non-minimal dominating sets.

5. Construct two 2×2 matrices A and B such that $A \neq B$, but $AC = BC$ where $C \neq 0$, the all 0's matrix.

   solution:

$$A = \begin{bmatrix} 15 & 1 \\ 3 & 13 \end{bmatrix}$$

$$B = \begin{bmatrix} 15 & 71 \\ 3 & 34 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

6. Construct a 2×2 matrix M such that its multiplication with the 2×1 vector with entries 1,4 results in the vector 1,4 itself.

   solution:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if you multiply it by

$$X = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

you get

$$X = \begin{bmatrix} a + 4b \\ c + 4d \end{bmatrix}$$

Thus $a + 4b = 1$ and $c + 4d = 4$. A solution is $a = 5, b = -1, c = -8, d = 3$.

Thus ,

$$M = \begin{bmatrix} 5 & -1 \\ -8 & 3 \end{bmatrix}$$