Discrete Mathematics (SC612)
Tutorial 1
27^{th} August, 2021

1. Find the number of satisfying assignments for each of the following formulae.

   (a) $p_1 \wedge p_2 \wedge p_3$
   solution: Only 1. This is evident from the definition/truth table. This is when each of the variables is set to true.

   (b) $\neg(p_1 \vee p_2 \vee p_3)$
   solution: Only 1. This is when all the variables are set to true.

   (c) $(p_1 \vee p_2) \Rightarrow (\neg p_1 \wedge \neg p_2 \wedge p_3)$
   solution: This formula has implication ($\Rightarrow$) as its top level operator. Thus it is true if either the premise is false, or the conclusion is true. The premise is false has 2 assignments. When the premise is true, the conclusion is false for this formula. Thus the total number of satisfying assignments for this formula is 2.

   (d) $(p_1 \Rightarrow (p_2 \Rightarrow p_3))$
   solution: Here, again the top level operator is an implication. Either the premise is false or the conclusion is true. The premise is false on four assignments. The conclusion is true on three assignments (not considering those which make the premise false. Thus, the number of satisfying assignments is $4 + 3 = 7$.

2. On how many assignments of truth values do the formulae $\phi_1 = ((p_1 \Rightarrow p_2) \Rightarrow p_3)$ and $\phi_2 = (p_1 \Rightarrow (p_2 \Rightarrow p_3))$ evaluate to the same truth value?
   solution:

p_1	p_2	p_3	$\phi_1 = ((p_1 \Rightarrow p_2) \Rightarrow p_3)$	$\phi_2 = (p_1 \Rightarrow (p_2 \Rightarrow p_3))$
F	F	F	F	T
F	F	T	T	T
F	T	F	F	T
F	T	T	T	T
T	F	F	T	T
T	F	T	T	T
T	T	F	F	F
T	T	T	T	T

From the truth table above, it is easy to see that the two formulae agree on their truth values under six assignments.

3. Let a, b and c be three atomic propositions in propositional logic. Suppose you are told that

(i) $a \vee (b \wedge c)$ is true, and

(ii) $(a \vee b) \wedge c$ is true.

Which of the individual truth values can be inferred from the given information?

solution:

$a = F, b = T, c = T$ makes both (i) and (ii) true.

$a = T, b = F, c = T$ also make both (i) and (ii) true.

Both truth values are possible for a and b to make both formulae true. However, $c = T$ is forced, otherwise the second formula wont hold. **Thus, we can determine the truth value of proposition c.**

4. Assume the truth of the statement "Every country has at least one citizen that knows at least one citizen of all other countries." Assume that every country has more than one citizen. Iceland and Norway are countries.

   (a) There is a person in Iceland who knows everyone in Norway.

   (b) There is a person in Iceland who knows no one in Norway.

   (c) There is a person in Iceland who knows someone in Norway.

   (d) Every person in Iceland knows at least one person from Norway.

   Which of rhe statements can be inferred from the initial information?

   solution: The correct answer is (c), $\forall$ countres can be instantiated with Iceland. The second $\forall$ other countries can be instantiated as Norway. $\exists$ can be instantiated by the specific choice of the citizen of Iceland, and similarly for Norway.

5. A boolean function is said to be symmetrical with respect to a propositional variable if substituting true or false for that propositional variable leads to an identical reduced formula in the remaining variables. On which propositional variables is the formula $\phi = (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge p_3)$ symmetrical?

   Solution:

   Symmetrical on p_1 because the reduced formula for both assignments to p_1 is the same:
   $$((\neg p_2 \wedge \neg p_3) \vee (p_2 \wedge p_3))$$

   On $p_2 = F$ we get:
   $$((\neg p_1 \wedge \neg p_3) \vee (p_1 \wedge \neg p_3))$$

   On $p_2 = T$ we get:
   $$((\neg p_1 \wedge p_3) \vee (p_1 \wedge p_3))$$

   Thus not symmetrical on p_2.

   On $p_3 = F$ we get:
   $$((\neg p_1 \wedge \neg p_2) \vee (p_1 \wedge \neg p_2))$$

On $p_3 = T$ we get:

$$((\neg p_1 \wedge p_2) \vee (p_1 \wedge p_2))$$

Thus not symmetrical on p_3.

6. Translate the boolean function $\phi = (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge p_3)$, which is in Disjunctive Normal Form (DNF) into its equivalent Conjunctive Normal Form (CNF):

   solution:

   This can be done in a routine fashion in two ways:

   (a) By constructing the truth table for the formula and then retrieving its CNF representation using the algorithm covered during the lecture.

   (b) By using some of the inference rules on boolean operators taught in class (like $\wedge$ distributes over $\vee$, and vice versa).

   There are four clauses, each containing three literals, in the given DNF formula. The corresponding CNF formula, when constructed directly by this rule will have 81 clauses each consisting of four literals. This is using the rule of product ver the four clauses. $3 \times 3 \times 3 \times 3 = 81$. You just take all possible clauses of four terms which takes one literal from each clause of the given DNF and make it a clause of CNF and then conjunct all the 81 clauses.

   The first clause would be:

$$(\neg p_1 \vee \neg p_1 \vee p_1 \vee p_1)$$

   . It is evident that this is always true, or a tautology. It is a clause of a CNF formula, and thus, it can be ignored and the rest of the clauses considered to determine the truth value. This is because, for any formula ϕ, $T \wedge \phi \equiv \phi$. It can thus be seen that this kind of clause will be eliminated and thus although technically there are 81 clauses, many of them will be irrelevant. Students are advised to understand why this first clause is irrelevant and then it will become easier to understand which of the 81 clauses will survive. If more than one clause is identical, then the duplicates can, likewise be eliminated. This follows from the fact that for any formula ϕ, $\phi \wedge \phi \equiv \phi$.

For ease of reading (also called readability) I will be presenting here, the individual clauses of the translated CNF formula, eliminating duplicate occurrences as well as tautologies. Thus, I will be presenting here the entire list of clauses but not writing them together with the $\wedge$ operator. The following is the list of clauses of the CNF:

(a) $\neg p_1 \vee \neg p_2 \vee p_3$

(b) $\neg p_1 \vee p_2 \vee \neg p_3$

(c) $p_1 \vee \neg p_2 \vee p_3$

(d) $p_1 \vee p_2 \vee \neg p_3$

I assure the reader/student I did not work out anywhere near 81 clauses. One can, by visual inspecton dismiss several clauses and immediately converge on these four. I will explain my thinking a bit here: In the first clause if I pick $\neg p_1$, then I know that in the last two clauses picking p_1 is useless as it will make the clause a tautology. Thus I must pick one of the literals associated with p_2 and p_3 from the last two clauses. The two clauses negate both these literals with respect to each other. Thus only two combinations are possible involving p_2 and p_3. A similar reasoning works when considering p_1 instead of $\neg p_1$. Thus although, theoretically the number of clauses (81) makes it appear daunting this method is probably easier than the truth tables method, provided you are alert mentally to dismiss repetitive cases and tautologies.

7. A boolean function is said to be symmetrical with respect to a propositional variable if substituting true or false for that propositional variable leads to an identical reduced formula in the remaining variables. For the following formulae determine in which variables they are symmetrical.

   (a) $(\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge p_3)$

   (b) $(\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge \neg p_2 \wedge p_3) \vee (p_1 \wedge p_2 \wedge p_3)$

   (c) $(\neg p_1 \wedge p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge \neg p_2 \wedge p_3)$

   (d) $(\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge p_3)$

   I will explain the solution method, rather than entire solutions. Each of the four formulae are over the three variables p_1, p_2, p_3. Thus one needs to determine a total of 12 statements, symmetrical or not, for each of the three variables in each of the four formulae. All the formulae are in the same form - DNF. In a particular formula for a particular variable, separate the clauses into two groups according to whether the variable is in the positive form or negative form in that variable. Now ignore that variable in all the clauses and see f the clauses in the other two variables that appear in the two parts are identical.

I will demonstrate how to do it for variable p_1 in formula (a).

The clauses with p_1 are:

- $(p_1 \wedge \neg p_2 \wedge \neg p_3)$
- $(p_1 \wedge p_2 \wedge p_3)$

Ignoring the variable p_1, the clauses reduce to:

- $(\neg p_2 \wedge \neg p_3)$
- $(p_2 \wedge p_3)$

The clauses with $\neg p_1$ are:

- $(\neg p_1 \wedge \neg p_2 \wedge \neg p_3)$
- $(\neg p_1 \wedge p_2 \wedge p_3)$

Ignoring the variable p_1, the clauses reduce to:

- $(\neg p_2 \wedge \neg p_3)$
- $(p_2 \wedge p_3)$

By inspection it can be seen that the reduced clauses are identical for the cases of p_1 and $\neg p_1$, and thus the formula (a) is symmetrical with respect to the variable p_1.

8. Construct a boolean formula over 3 propositional variables such that it has exactly 5 assignments that make it true. Write it down in the form of a truth table. How many such possible formulae are there? Convert the truth table into a formula with only $\wedge$ and $\neg$. Do the same with only $\vee$ and $\neg$.

   Let us construct an implication, which consists of a premise and a conclusion. We can make the premise and conclusion independent of each other by using disjoint variables for the two subformulae. We know tha an implication is true when either the conclusion is true or the premise is false. We can arrange the construction of a formula with exactly 5 assignments that make it true by making the conclusion true on four assignments, and false for the other four. The four where the conclusion is true, the overall formula is true. Thus, we have four satisfying assignments among those. From the remaining four, we know that the formula is true only if the premise i also false. We want only one assignment among the four to make the premise false. Thus we can use the disjunction of two variables, which is false only for one of the four assignments. With this reasoning the formula is:

$$((p_1 \vee p_2) \Rightarrow p_3)$$

p_1	p_2	p_3	$((p_1 \vee p_2) \Rightarrow p_3))$
F	F	F	T
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	F
T	T	T	T

Of course, as taught in the lecture, we know that we dont need any syntactic reasoning to construct a formula over three variables that has exactly five satisfying assignments. We can simply write down the truth table with eight rows corresponding to the eight assignments of truth values to the three variables and assign any five rows of the formula as true and the remaining three as false. In order to covert this semantic (truth table) formula into a syntactic one, we covered how we can transform it into either CNF (using the rows/assignments which make the formula false) or DNF (using the rows/assignments which make the formula true).

Thus the number of such formulae is simply $\binom{8}{5} = 56$.

Since, I constructed the formula with syntactic reasoning and not the crude semantica of the truth table, I can convert into the two required forms by using inference rules of syntactic formulae. Thus,

$$\neg(p_1 \vee p_2) \vee p_3$$

This is obtained using the standard rule for eliminating implication ($\Rightarrow$) using disjunction ($\vee$). To convert to a formula using only $\wedge$ and $\neg$, we should apply De Morgan's law.

Thus,

$$(\neg p_1 \wedge \neg p_2) \vee p_3$$

which is equivalent to

$$\neg(\neg(\neg p_1 \wedge \neg p_2) \wedge (\neg p_3))$$

9. Write a long formula with $\wedge$, $\neg$ and $\vee$ and convert it into a formula involving only $\neg$ and $\Rightarrow$.

   solution:

   $\phi = (((p \vee \neg q) \wedge r) \wedge (p \vee (\neg q \wedge \neg r)))$

   Let us translate this formula into one involving only $\neg$ and $\Rightarrow$.

   We will do this in two steps:

   (a) Eliminate $\wedge$ (De morgan s rule)

   $$\neg(\neg(\neg(\neg(p \vee \neg q) \vee \neg(r)) \vee \neg(p \vee \neg(q \vee r))))$$

   (b) Eliminate $\vee$ (implication rule)

   $$\neg((\neg((q \Rightarrow p) \Rightarrow \neg(r)) \Rightarrow \neg(((\neg q) \Rightarrow r) \Rightarrow p)))$$

10. Statement 1: $\exists x$, such that $\forall y$, $x - y = 5$, where $x \in \mathcal{R}$ and $y \in \mathcal{R}$.
    Statement 2: $\forall x$, $\exists y$, such that $x - y = 5$, where $x \in \mathcal{R}$ and $y \in \mathcal{R}$.

    Which of the two statements is correct?

    solution:

    Statement 1 is wrong. Statement 2 is correct. For any specific choice of x, take $y = x - 5$.

11. $\phi = ((p_1 \Rightarrow (p_2 \vee (\neg p_1 \wedge (p_3 \vee \neg(p_1 \Rightarrow p_3))))) \vee (\neg p_1 \wedge (p_2 \vee \neg p_3)))$. Write the cascading set of formulae involving fewer variables by setting first $p_1 = T$ and next, $p_1 = F$. Reduce these formulae further by setting p_2 to the two values in succession. If you get ϕ to be true at some stage make a note of it. This means the formula is true regardless of the truth values of unassigned variables.

    solution:

    Setting $p_1 = T$ gives:

    $$\phi = p_2$$

    Explanation: Highest level operator is $\vee$. The second disjunct has highest level operator as $\wedge$. One of the conjuncts of that is $\neg p_1$ which is false when $p_1 = T$. Thus thw whole clause is false. The formula thus reduces to the left operand of the top level operator. In the left, the top level operator is implication, where the premise p_1 is true. The formula thus reduces to the conclusion. The conclusion is a disjunction. The second disjunct is a conjunction, which has $\neg p_1$ as one of its two conjuncts. That dijunct therefore evaluates to false. The formula thus reduces to the first disjunct, which is p_2.

    Here, further setting p_2 determines the formula as that value. p_3 is already eliminated.

    Setting $p_1 = F$ gives:

    $$\phi = (p_2 \vee p_3) \vee (p_2 \vee \neg p_3) \equiv T$$

12. Consider the 2-player game where the players A and B play alternately. The initial configuration is a list $1, 3, 4, 2$. The play begins with player A. A each step the player to move is allowed to interchange the position of any two elements. However a move is illegal if it leads to the same configuration as has already occurred earlier in the game. If at any point the configuration 1,2,3,4 is reached then player A is the winner and if 4,3,2,1 is reached then player B is the winner (it doesn't matter which player made the move leading to the winning configurations, only the configurations matter). If a player to move doesnt have a legal move then the game ends in a draw. For the given configuration determine the outcome with best play (each player trying to win) by both players.

Write down the description of a win for a player in a 2-player strategy game in the form of a predicate logic formula.