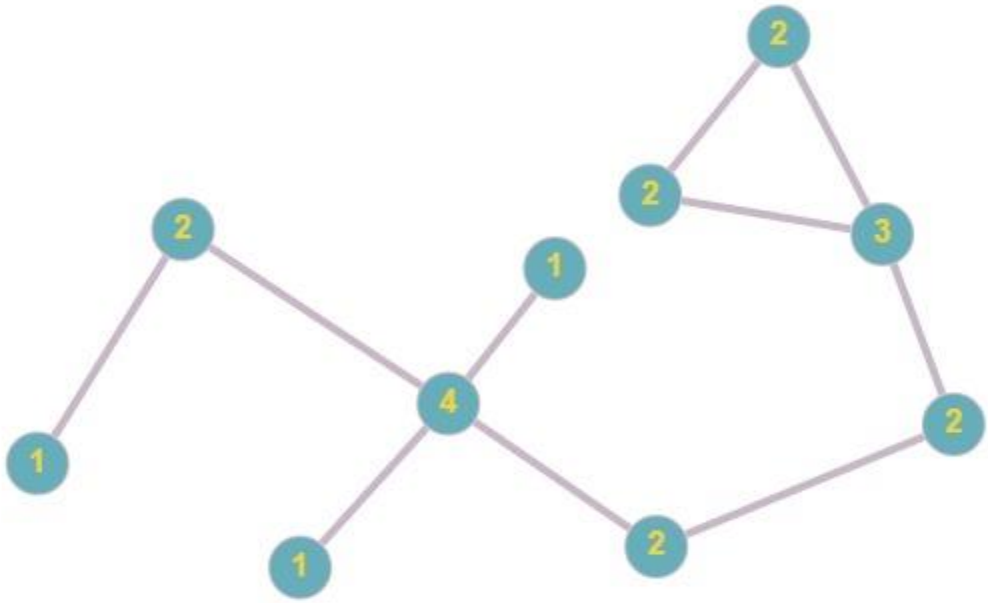
GRAPH THEORY

VERTEX DEGREE

Degree of a vertex is the number of edges falling on it. It tells us how many other vertices are adjacent to that vertex.

The vertices having zero degree are called isolated vertices. They don't have any other vertex connected to them.



DEGREE SEQUENCE

Degree sequence of undirected graph is list of degrees written in any order depending on application, there should be some standard order. (increasing/non-decreasing, decreasing/non-increasing neighbourhood, random based).

For Ex:

Decreasing Degree Sequence for above graph is as follows:

4,3,2,2,2,2,2,1,1,1

Neighbourhood/Connection Degree Sequence for above graph is as follows:

4,2,2,1,1,1,2,3,2,2

FINDING MAX ALGORITHM

Find_Max(A)

1. Max <- A[1]
2. Pos <- 1
3. For i <- 2 to n
4. If A[i] > Max
5. Then Max <- A[i]
6. Pos <- i
7. Return Max, Pos

MAXIMUM DEGREE OF GRAPH

Maximum Degree Denotion = Δ (Capital Delta)

Maximum Degree for above Graph G = Δ(G) = 4

MINIMUM DEGREE OF GRAPH

Minimum Degree Denotion = δ(Small Delta)

Minimum Degree for above Graph G = δ(G) = 1

AVERAGE DEGREE OF GRAPH

Formula : $\frac{\text{Sum of Vertex Degrees}}{\text{Number of Vertex}}$

Denotion : $d_{(\,avg\,)}$

Average Degree of above Graph is as follows :

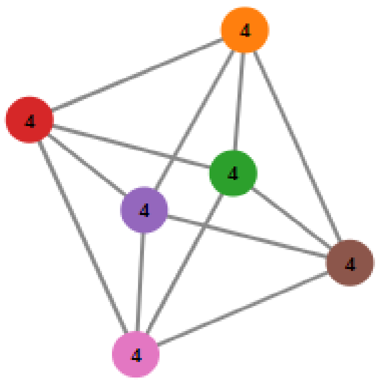
$$d_{(\,avg\,)} = \frac{4+3+2+2+2+2+2+1+1+1}{10} = \frac{20}{10} = 2$$

δ >= 0	Δ <= n-1
(no neighbours)	(every vertex is a neighbour)
There are total n numbers between above two notations	
0(no neighbours) ← n-1(every vertex is a neighbour)	
Therefore, proving by contradiction : ¬((0) ∧ (n-1))	
They are mutually exclusive.	
Thus, this two values cannot occur simultaneously.	

REGULAR GRAPHS

It is a special case where all vertices have same degree.

Therefore, min degree δ = max degree Δ.



$$\text{Degree Sequence} = (4, 4, 4, 4, 4, 4)$$

Theorem: The sum of degree of all vertices of a regular graph is twice the size of regular graph.

Mathematically,

$$\sum_{v\,\in V} \deg(v) = 2\,|\,E\,|$$

where, |E| stands for the number of edges in the graph (size of graph).

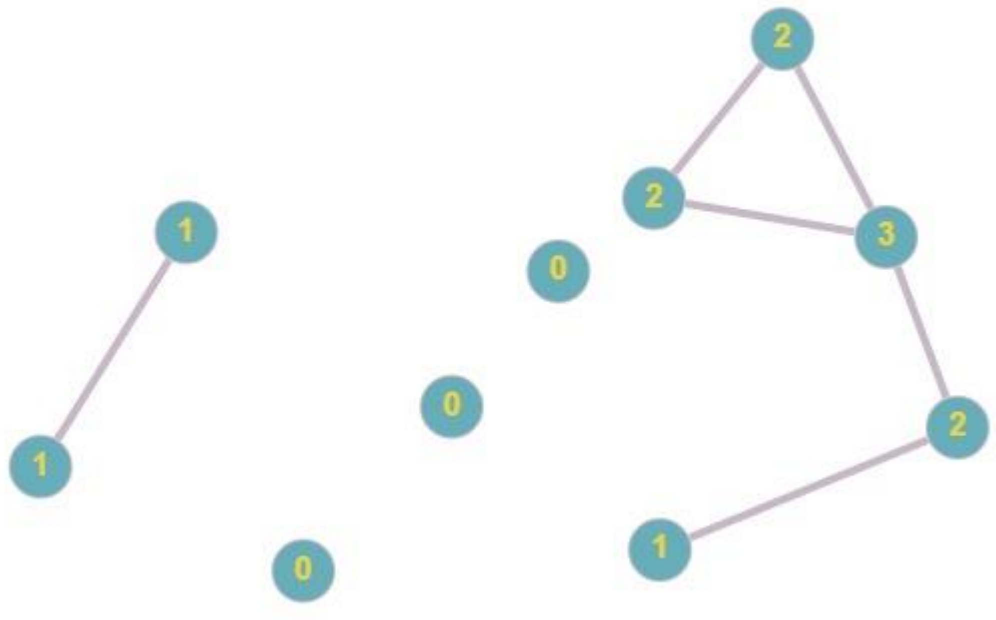
PROPERTIES OF DEGREE SEQUENCE

1. Total is always even number.
   (There must be two vertices of equal degree – extended application of pigeon hole principle)
2. Simple Graphs vertices are always between 0 and n-1.
3. They all cannot be distinct.
4. There are graphs which are identical and are known regular graphs.
5. When you delete a vertex there are several vertex degrees that are affected.
   (i.e., neighbours are affected and non-neighbours are not affected)

DELETING A VERTEX FROM GRAPH

Deleting a vertex having 4 as vertex degree in the above graph.

Updated graph (after deleting) is as follows:



Neighbourhood/Connection Degree Sequence will be as follows:

(Before) 4,2,2,1,1,1,2,3,2,2

(After) 1,1,0,0,1,2,3,2,2

If we delete a vertex, then it will affect several other vertices (dependence)

This makes degree sequence different from normal integer sequences.

REVERSE PROBLEM

You are given a degree sequence for a particular graph.

Degree Sequence → 4,3,2,2,2,2,2,1,1,1

Assume that we have to delete a vertex with vertex degree 4.

And create actual graph from that

Use RECURSION / INDUCTION to solve.

HAVEL-HAKIMI ALGORITHM

Step 1 : Right sequence in non-increasing order

4,3,2,2,2,2,2,1,1,1

Step 2 : Reduce sequence after deleting a vertex with vertex degree 4.

Assuming deleted vertex is adjacent to the immediately appearing neighbours in the list.

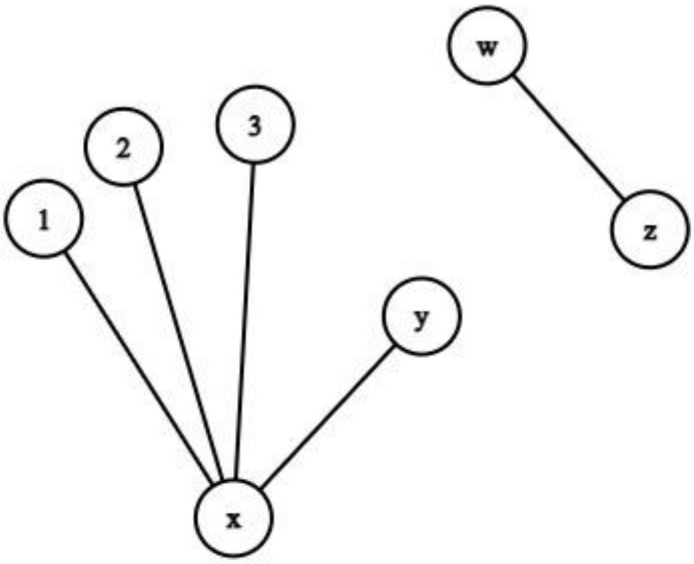
4,3,2,2,2,2,2,1,1,1

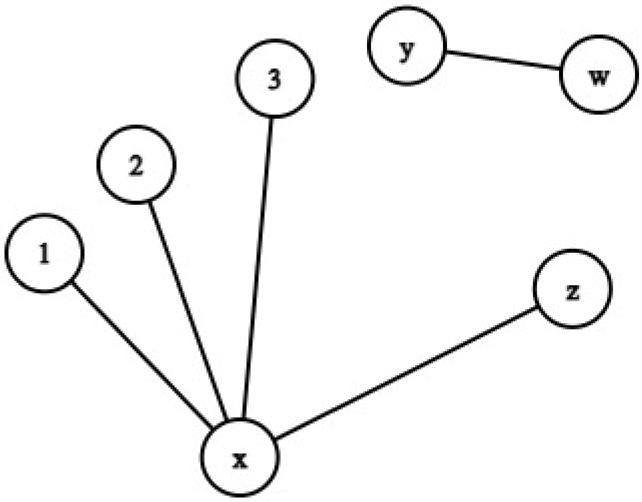
2,1,1,1,2,2,1,1,1

Now this creates ambiguity, there are chances that vertex with vertex degree 4 can be adjacent to some other vertex which is not present among immediate preceding 4 degree vertex (i.e. there might be a graph with different adjacency)

But this idea works.

Reason is as follows :





d(y) is bigger set than d(z)

If d(y)>d(z)

$(x,y) \notin E$

$(x,z) \in E$

If d(y) > d(z)

$\exists$ w(w, y) $\in$ E & (w,z) $\notin$ E

$\exists$ w(w, y) $\notin$ E & (w,z) $\in$ E (Incorrect Option)

If there is a set of 5 elements and a set of 3 elements. It is guaranteed that what is there in a 5 element set is not going to be present in a set of 3 elements.

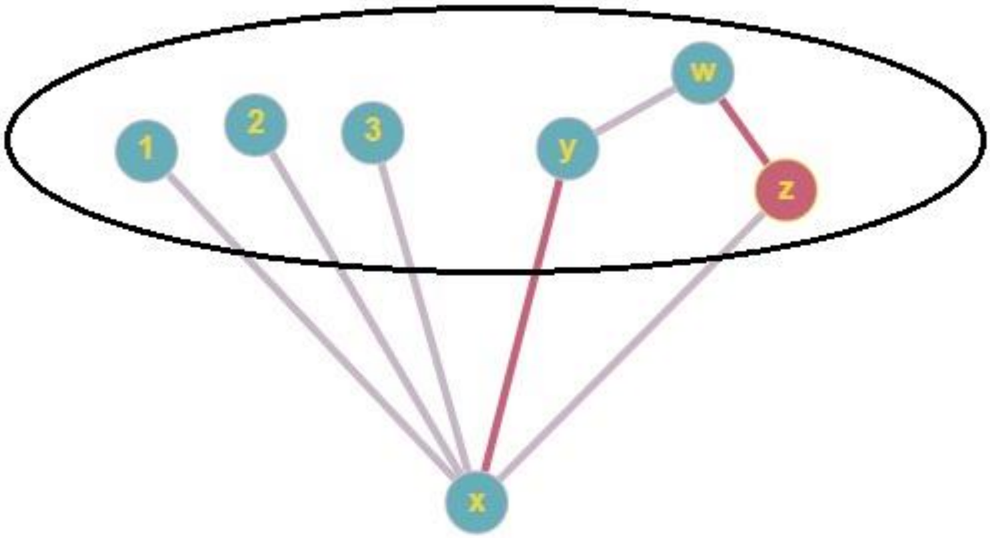
Using same example we can say that,

$N(Y) \subseteq N(Z)$

Suppose N(Y)'s every element also belongs to N(Z). Therefore, $N(Y) \subseteq N(Z)$. Cardinality of subset cannot be greater than set. Thus proving by contradiction as below:

$\neg (\forall x, (x \in N(y)) \Rightarrow (x \in N(z)))$

$\exists x, (x \in N(y) \wedge x \notin N(z))$



You can observe the pattern of in out (edges) in the above diagram.

Step 3 : 1. get degree sequences in nonnegative integers
2. delete the first vertex
For the degree of the vertex
Subtract connected vertex degree by 1
Repeat for every vertex until all vertex does not arrive to 0.
If any one vertex ends up with -1, it's an illegal graph. (i.e., graph doesn't exist)

4	3	2	2	2	2	2	1	1	1
v1	v2	v3	v4	v5	v6	v7	v8	v9	v10

Deleting V1

0	3 -1 = 2	2-1 = 1	2-1 = 1	2-1 = 1	2	2	1	1	1
v1	v2	v3	v4	v5	v6	v7	v8	v9	v10

0	2	1	1	1	2	2	1	1	1
v1	v2	v3	v4	v5	v6	v7	v8	v9	v10

Rearranging in non-increasing sequence.

2	2	2	1	1	1	1	1	1	0
v2	v6	v7	v3	v4	v5	v8	v9	v10	v1

Deleting V2

0	2-1 = 1	2-1 = 1	1	1	1	1	1	1	0
v2	v6	v7	v3	v4	v5	v8	v9	v10	v1

Rearranging in non-increasing sequence.

1	1	1	1	1	1	1	1	0	0
v6	v7	v3	v4	v5	v8	v9	v10	v1	v2

Deleting V6

0	1 - 1 = 0	1	1	1	1	1	1	0	0
v6	v7	v3	v4	v5	v8	v9	v10	v1	v2

Rearranging in non-increasing sequence.

1	1	1	1	1	1	0	0	0	0
v3	v4	v5	v8	v9	v10	v1	v2	v6	v7

Deleting V3

0	1 -1 =0	1	1	1	1	0	0	0	0

v3	v4	v5	v8	v9	v10	v1	v2	v6	v6

Rearranging in non-increasing sequence.

1	1	1	1	0	0	0	0	0	0
v5	v8	v9	v10	v1	v2	v6	v7	v3	v4

Deleting V5

0	0	1	1	0	0	0	0	0	0
v5	v8	v9	v10	v1	v2	v6	v7	v3	v4

Rearranging in non-increasing sequence.

1	1	0	0	0	0	0	0	0	0
v9	v10	v1	v2	v6	v7	v3	v4	v5	v8

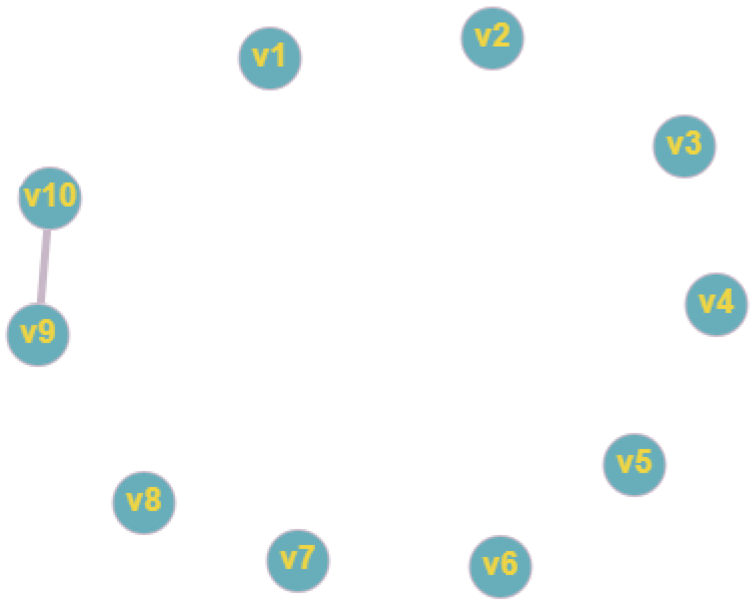
Deleting V9

0	1-1 =0	0	0	0	0	0	0	0	0
v9	v10	v1	v2	v6	v6	v3	v4	v5	v8

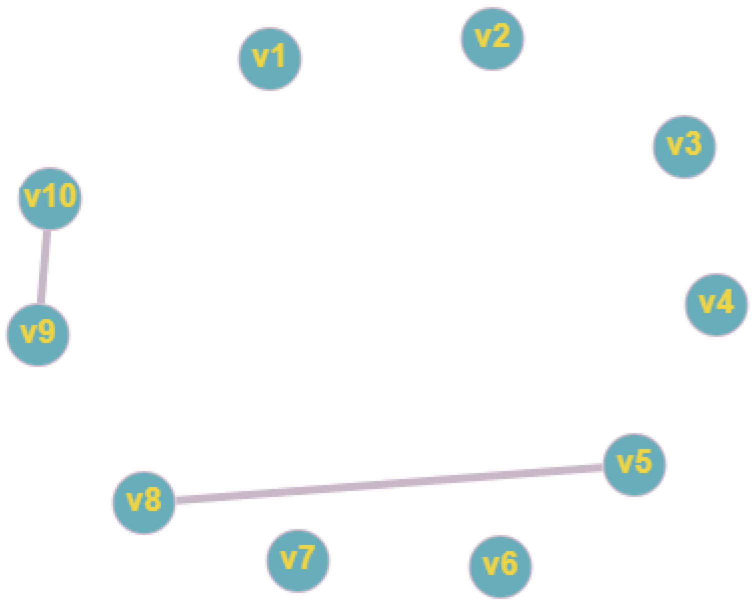
Rearranging in non-increasing sequence.

0	1-1 =0	0	0	0	0	0	0	0	0
v9	v10	v1	v2	v6	v6	v3	v4	v5	v8

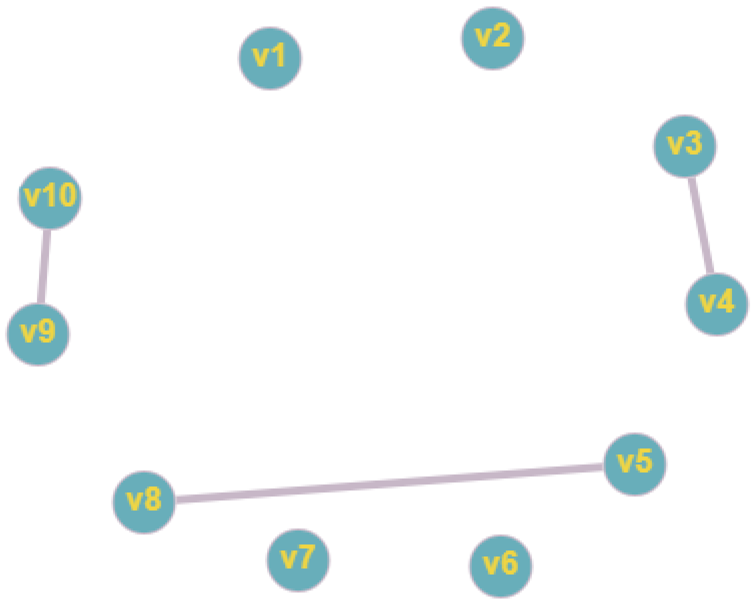
Step 4 : In step 3 v9 vertex has degree vertex 1 (when we are deleting v9) and is affecting v10. Therefore, we can say that v9 is connected with v10



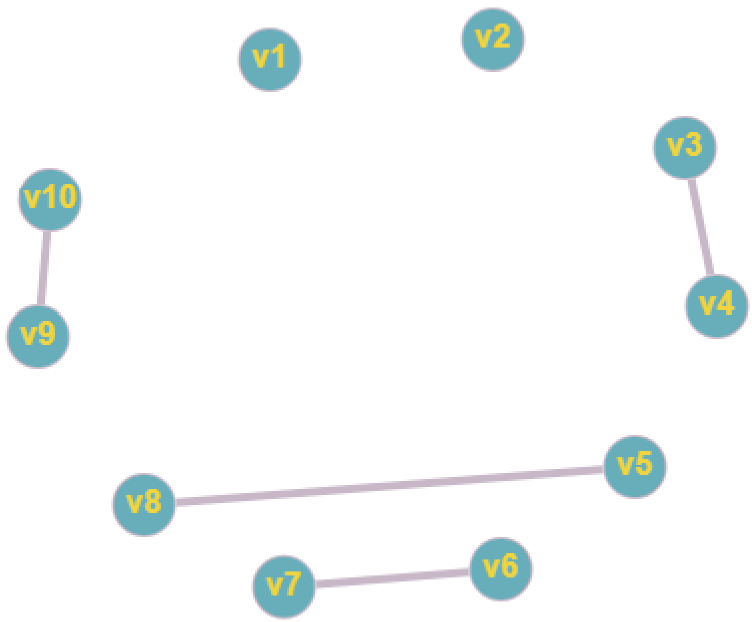
In step 3 v5 vertex has degree vertex 1 (when we are deleting v5) and is affecting v8. Therefore, we can say that v5 is connected with v8.



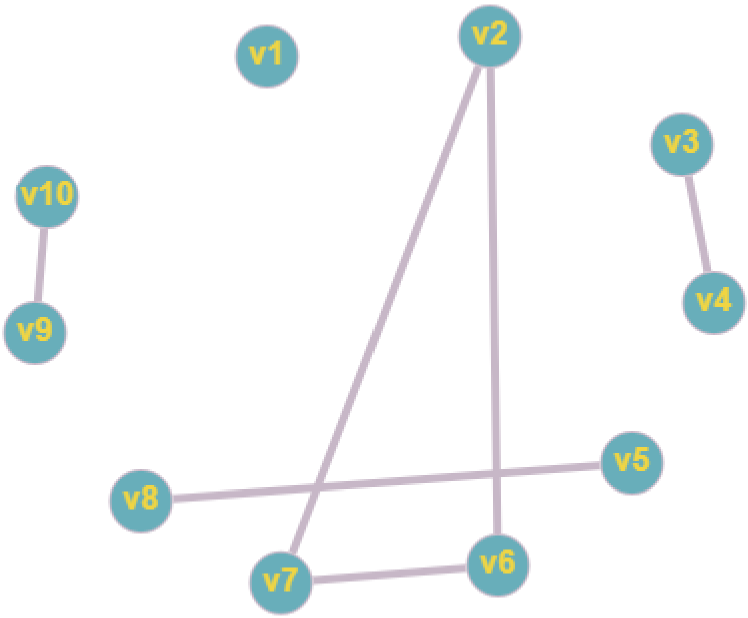
In step 3 v3 vertex has degree vertex 1 (when we are deleting v3) and is affecting v4. Therefore, we can say that v3 is connected with v4.



In step 3 v6 vertex has degree vertex 1 (when we are deleting v6) and is affecting v7. Therefore, we can say that v6 is connected with v7.

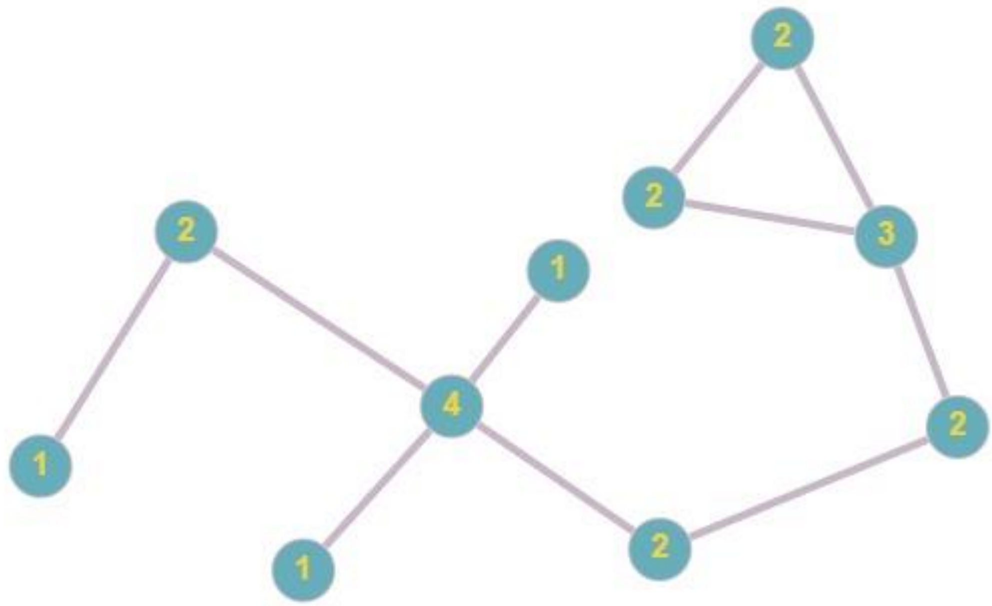


In step 3 v2 vertex has degree vertex 2 (when we are deleting v2) and is affecting v6 and v7. Therefore, we can say that v2 is connected with v6 and v7.

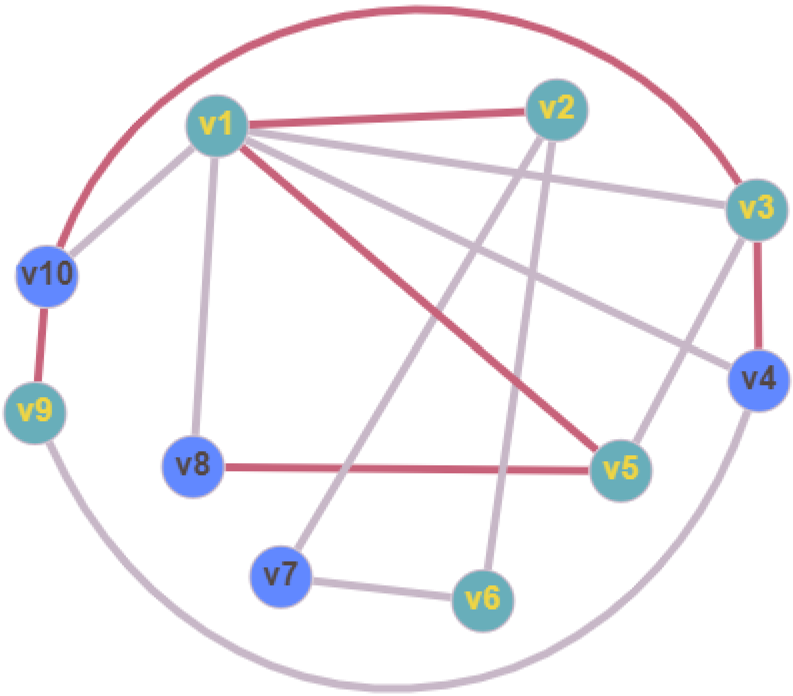


In step 3 v1 vertex has degree vertex 4 (when we are deleting v1) and is affecting v2, v3, v4 and v5. Therefore, we can say that v1 is connected with v2, v3, v4 and v5.

Step 5 :



Comparing above 2 graphs, we came to know they are not same and to make it similar we know that we have to switch 2 edges. So we are removing 2 edges that were present (and making them absent) adding other 2 edges which were absent (and making them present). Following this criteria we build the below graph that makes the above 2 graph similar from vertex degrees point of view.



Red Line indicates deleted lines.

Blue coloured vertex represents last 4 vertex reducing to 0 degree sequence.

Remove 2 connections that are present and add 2 connections that are absent among the loop of those vertex.

Havel-Hakimi's Theorem might result into varying graph from person to person.