Preface

We are thankful to Almighty ALLAH Who has given us an opportunity to write the book , named 'Applied Mathematics-I ' as Textbook, intending to cover the new syllabus for the first year students of Diploma of Associate Engineer (DAE)

Throughout the text, emphasis is on correct methods of computation, correct method for transposition of formulae, logical layout of solutions, neatness and clarity of arrangement of material and the systematic use of all the normal mathematical and other tables.

Topics covered include algebra, trigonometry, vectors & Phasors Algebra, Complex numbers, Number Systems, Boolean Algebra, Straight Line and The Circle.

Normally students face difficulty in solving complicated problems because they do not make a systematic attempt. We have attempted to help the students to overcome the difficulty by providing detailed instructions for an orderly approach. Difficult procedures and types of problems appearing in the exercise are illustrated by carefully explained examples. In the presentation of these illustrated examples, we have avoided unnecessary explanations. It is hoped that this book will help to give students a good foundation in old and new techniques.

Students are reminded that in order to acquire a proper understanding of the subject and its application, it is necessary to learn a number of sound basic rules and methods.No scientific or engineering subject can be fully comprehended and satisfactorily studied without a sound mathematical background.

We would like to express , sincere and thanks to Mr.Jawad Ahmed Qureshi Chief Operating Officer TEVTA,Engr. Mr.Azhar Iqbal Shad G.M Academic,Engr.Mazhar Abbas Naqvi Manager (Curriculum) and Engr. Syed Muhammad Waqar ud- Din Deputy Director(technical) Curriculum Section Acade mics Wing, who took keen interest and inspired us for the completion of this task.

We made every effort to make the book valuable both for students and teachers , however we shall gratefully welcome to receive any suggestion for the further improvement of the book.

Authors.

A Textbook of

APPLIED MATHEMATICS-I

Math–123

For

First year
Diploma of Associate Engineer
(DAE)

Including:
Objective Type
and
Short Questions

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8.8 Graphical Representation

Chapter 1
Quadratic Equations

1.1 Equation:

An equation is a statement of equality '=' between two expression for particular values of the variable. For example

$5x + 6 = 2$, x is the variable (unknown)

The equations can be divided into the following two kinds:

Conditional Equation:

It is an equation in which two algebraic expressions are equal for particular value/s of the variable e.g.,

a) $2x = 3$ is true only for $x = 3/2$

b) $x^2 + x - 6 = 0$ is true only for $x = 2, -3$

Note: for simplicity a conditional equation is called an equation.

Identity:

It is an equation which holds good for all value of the variable e.g;

a) $(a + b)\, x \equiv ax + bx$ is an identity and its two sides are equal for all values of x.

b) $(x + 3)(x + 4) \equiv x^2 + 7x + 12$ is also an identity which is true for all values of x.

For convenience, the symbol '=' shall be used both for equation and identity.

1.2 Degree of an Equation:

The degree of an equation is the highest sum of powers of the variables in one of the term of the equation. For example

$2x + 5 = 0$	1st degree equation in single variable
$3x + 7y = 8$	1st degree equation in two variables
$2x^2 - 7x + 8 = 0$	2nd degree equation in single variable
$2xy - 7x + 3y = 2$	2nd degree equation in two variables
$x^3 - 2x^2 + 7x + 4 = 0$	3rd degree equation in single variable
$x^2y + xy + x = 2$	3rd degree equation in two variables

1.3 Polynomial Equation of Degree n:

An equation of the form

$$a_nx^n + a_{n-1}x^{n-1} + \text{----------------} + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \text{--------------(1)}$$

Where n is a non-negative integer and a_n, a_{n-1}, -------------, a_3, a_2, a_1, a_0 are real constants, is called polynomial equation of degree n. Note that the degree of the equation in the single variable is the highest power of x which appear in the equation. Thus

$3x^4 + 2x^3 + 7 = 0$

$x^4 + x^3 + x^2 + x + 1 = 0$, $x^4 = 0$

are all fourth-degree polynomial equations.

By the techniques of higher mathematics, it may be shown that nth degree equation of the form (1) has exactly n solutions (roots). These roots may be real, complex or a mixture of both. Further it may be shown that if such an equation has complex roots, they occur in pairs of conjugates complex numbers. In other words it cannot have an odd number of complex roots.

A number of the roots may be equal. Thus all four roots of $x^4 = 0$ are equal which are zero, and the four roots of $x^4 - 2x^2 + 1 = 0$

Comprise two pairs of equal roots (1, 1, -1, -1).

1.4 Linear and Cubic Equation:

The equation of first degree is **called linear equation.**

For example,

i) $x + 5 = 1$ (in single variable)

ii) $x + y = 4$ (in two variables)

The equation of third degree is **called cubic equation.**

For example,

i) $a_3x^3 + a_2x^2 + a_1x + a0 = 0$ (in single variable)

ii) $9x^3 + 5x^2 + 3x = 0$ (in single variable)

iii) $x^2y + xy + y = 8$ (in two variables)

1.5 Quadratic Equation:

The equation of second degree is called quadratic equation. The word quadratic comes from the Latin for “square”, since the highest power of the unknown that appears in the equation is square. For example

$2x^2 – 3x + 7 = 0$ (in single variable)

$xy – 2x + y = 9$ (in two variable)

Standard form of quadratic equation

The standard form of the quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are constants with $a \neq 0$.

If $b \neq 0$ then this equation is called **complete quadratic equation** in x.

If $b = 0$ then it is called a **pure or incomplete quadratic equation** in x.

For example, $5 x^2 + 6 x + 2 = 0$ is a complete quadratic equation is x.

and $3x^2 – 4 = 0$ is a pure or incomplete quadratic equation.

1.6 Roots of the Equation:

The value of the variable which satisfies the equation is called the root of the equation. A quadratic equation has two roots and hence there will be two values of the variable which satisfy the quadratic equation. For example the roots of $x^2 + x – 6 = 0$ are 2 and -3.

1.7 Methods of Solving Quadratic Equation:

There are three methods for solving a quadratic equation:

i) By factorization

ii) By completing the square

iii) By using quadratic formula

i) Solution by Factorization:

Method:

Step I: Write the equation in standard form.

Step II: Factorize the quadratic equation on the left hand side if possible.

Step III: The left hand side will be the product of two linear factors. Then equate each of the linear factor to zero and solve for values of x. These values of x give the solution of the equation.

Example 1:

Solve the equation $3x^2 + 5x = 2$

Solution:

$3x^2 + 5x = 2$

Write in standard form $3x^2 + 5x – 2 = 0$

Factorize the left hand side $3x^2 + 6x - x - 2 = 0$

$3x(x + 2) -1(x + 2) = 0$

$(3x - 1)(x + 2) = 0$

Equate each of the linear factor to zero.

$3x - 1 = 0$ or $x + 2 = 0$

$3x = 1$ or $x = -2$

$x = \frac{1}{3}$

$x = \frac{1}{3}, -2$ are the roots of the Equation.

Solution Set = $\{\frac{1}{3}, -2\}$

Example 2:

Solve the equation $6x^2 - 5x = 4$

Solution:

$6x^2 - 5x = 4$

$6x^2 - 5x - 4 = 0$

$6x^2 - 8x + 3x - 4 = 0$

$2x(3x - 4) + 1(3x - 4) = 0$

$(2x + 1)(3x - 4) = 0$

$\therefore$ Either $2x + 1 = 0$ or $3x - 4 = 0$

Which gives $2x = -1$ which gives $3x = 4$

$\Rightarrow x = -\frac{1}{2}$ $\Rightarrow x = \frac{4}{3}$

$\therefore$ Required Solution Set = $\left\{-\frac{1}{2}, \frac{4}{3}\right\}$

ii) Solution of quadratic equation by Completing the Square

Method:

Step I: Write the quadratic equation is standard form.

Step II: Divide both sides of the equation by the co-efficient of x^2 if it is not already 1.

Step III: Shift the constant term to the R.H.S.

Step IV: Add the square of one-half of the co-efficient of x to both side.

Step V: Write the L.H.S as complete square and simplify the R.H.S.

Step VI: Take the square root on both sides and solve for x.

Example 3:

Solve the equation $3x^2 = 15 - 4x$ by completing the square.

Solution: $3x^2 = 15 - 4x$

Step I Write in standard form: $3x^2 + 4x - 15 = 0$

Step II Dividing by 3 to both sides: $x^2 + \frac{4}{3}x - 5 = 0$

Step III Shift constant term to R.H.S: $x^2 + \frac{4}{3}x = 5$

Step IV Adding the square of one half of the co-efficient of x. i.e., $\left(\frac{4}{6}\right)^2$ on both sides:

$$x^2 + \frac{4}{3}x + \left(\frac{4}{6}\right)^2 = 5 + \left(\frac{4}{6}\right)^2$$

Step V: Write the L.H.S. as complete square and simplify the R.H.S.
:

$$\left(x + \frac{4}{6}\right)^2 = 5 + \frac{16}{36}$$

$$= \frac{180 + 16}{36}$$

$$\left(x + \frac{4}{6}\right)^2 = \frac{196}{36}$$

Step VI: Taking square root of both sides and Solve for x

$$\sqrt{\left(x + \frac{4}{6}\right)^2} = \sqrt{\frac{196}{36}}$$

$$x + \frac{4}{6} = \pm\frac{14}{6}$$

$$x + \frac{4}{6} = \pm\frac{7}{3}$$

$$x + \frac{4}{6} = \frac{7}{3}, \qquad x + \frac{4}{6} = -\frac{7}{3}$$

$$\Rightarrow \quad x = \frac{7}{3} - \frac{4}{6} \qquad \Rightarrow \quad x = -\frac{7}{3} - \frac{4}{6}$$

$$x = \frac{10}{6}, \qquad x = \frac{-18}{6}$$

$$x = \frac{5}{3}, \qquad x = -3$$

Hence, the solution set = $\left\{-3, \frac{5}{3}\right\}$

Example 4:

Solve the equation $a^2 x^2 = ab\,x + 2b^2$ by completing the square.

Solution:

$$a^2 x^2 = ab\,x + 2b^2$$

$$a^2 x^2 - ab\,x - 2b^2 = 0$$

Dividing both sides by a^2, we have

$$x^2 - \frac{bx}{a} - \frac{2b^2}{a^2} = 0$$

$$x^2 - \frac{bx}{a} = \frac{2b^2}{a^2}$$

Adding the square of one half of the co-efficient of x i.e., $\left(-\frac{b}{2a}\right)^2$ on both sides.

$$x^2 - \frac{bx}{a} + \left(-\frac{b}{2a}\right)^2 = \frac{2b^2}{a^2} + \left(-\frac{b}{2a}\right)^2$$

$$\left(x - \frac{b}{2a}\right)^2 = \frac{2b^2}{a^2} + \frac{b^2}{4a^2}$$

$$\left(x - \frac{b}{2a}\right)^2 = \frac{8b^2 + b^2}{4a^2}$$

$$\left(x - \frac{b}{2a}\right)^2 = \frac{9b^2}{4a^2}$$

Taking square root on both sides

$$x - \frac{b}{2a} = \pm\frac{3b}{2a}$$

$$x - \frac{b}{2a} = \frac{3b}{2a} \qquad\qquad x - \frac{b}{2a} = \frac{3b}{2a}$$

$$\Rightarrow \quad x = \frac{b}{2a} + \frac{3b}{2a} \qquad\qquad \Rightarrow \quad x = \frac{b}{2a} - \frac{3b}{2a}$$

$$\Rightarrow \quad x = \frac{b + 3b}{2a} \qquad\qquad \Rightarrow \quad x = \frac{b - 3b}{2a}$$

$$\Rightarrow \quad x = \frac{4b}{2a} \qquad\qquad \Rightarrow \quad x = -\frac{2b}{2a}$$

$$\Rightarrow \quad x = \frac{2b}{a} \qquad\qquad \Rightarrow \quad x = -\frac{b}{a}$$

Solution Set = $\left\{\frac{2b}{a}, -\frac{b}{a}\right\}$

iii) Derivation of Quadratic formula

Consider the standard form of quadratic equation $ax^2 + b x + c = 0$.
Solve this equation by completing the square.

$$ax^2 + bx + c = 0$$

Dividing both sides by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = o$$

Take the constant term to the R.H.S

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

To complete the square on L.H.S. add $\left(\frac{b}{2a}\right)^2$ to both sides.

$$X^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square root of both sides

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}}$$ which is called the **Quadratic formula.**

Where, a = co-efficient of x^2 , b = coefficient of x , c = constant term
Actually, the Quadratic formula is the general solution of the quadratic equation $ax^2 + bx + c = 0$

Note: $\frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$, $\frac{-b -\sqrt{b^2 - 4ac}}{2a}$ are also called roots of the quadratic equation

Method:

To solve the quadratic equation by Using Quadratic formula:

Step I: Write the Quadratic Equation in Standard form.
Step II: By comparing this equation with standard form $ax^2 + bx + c = 0$ to identify the values of a , b , c.
Step III: Putting these values of a, b, c in Quadratic formula

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$$ and solve for x.

Example 5:

Solve the equation $3x^2 + 5x = 2$

Solution:

$$3x^2 + 5x = 2$$
$$3x^2 + 5x - 2 = 0$$

Composing with the standard form $ax^2 + bx + c = 0$, we have a = 3 , b = 5, c = -2.

Putting these values in Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{(5)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{-5 \pm \sqrt{25 + 24}}{6}$$

$$x = \frac{-5 \pm 7}{6}$$

$$x = \frac{-5 + 7}{6} \quad \text{or} \quad x = \frac{-5 - 7}{6}$$

$$x = \frac{2}{6} \quad \text{or} \quad x = \frac{-12}{6}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -2$$

$$\text{Sol. Set} = \left\{\frac{1}{3},\ 2\right\}$$

Example 6:

Solve the equation $15x^2 - 2ax - a^2 = 0$ by using Quadratic formula:

Solution:

$$15x^2 - 2ax - a^2 = 0$$

Comparing this equation with General Quadratic Equation

Here, $a = 15$, $b = -2a$, $c = -a^2$

Putting these values in Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2a) \pm \sqrt{(-2a)^2 - 4(15)(-a^2)}}{2(15)}$$

$$= \frac{-(-2a) \pm \sqrt{4a^2 + 60a^2}}{30}$$

$$= \frac{2a \pm 8a}{30}$$

$$x = \frac{2a + 8a}{30} \quad \text{or} \quad x = \frac{2a - 8a}{30}$$

$$x = \frac{10a}{30} \quad \text{or} \quad x = \frac{-6a}{30}$$

$$x = \frac{a}{3} \quad \text{or} \quad x = -\frac{a}{5}$$

$$\text{Sol. Set} = \left\{\frac{a}{3},\ -\frac{a}{5}\right\}$$

Example 7:

Solve the equation $\dfrac{1}{2x - 5} + \dfrac{5}{2x - 1} = 2$ by using Quadratic formula.

Solution:

$$\frac{1}{2x - 5} + \frac{5}{2x - 1} = 2$$

Multiplying throughout by $(2x - 5)(2x - 1)$, we get

$$(2x - 1) + 5(2x - 5) = 2(2x - 5)(2x - 1)$$
$$2x - 1 + 10x - 25 = 8x^2 - 24x + 10$$
$$8x^2 - 36x + 36 = 0$$
$$2x^2 - 9x + 9 = 0$$

Comparing this equation with General Quadratic Equation

Here, $a = 2$, $b = -9$, $c = 9$

Putting these values in the Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{(-9) \pm \sqrt{(-9)^2 - 4(2)(9)}}{2(2)}$$
$$= \frac{9 \pm \sqrt{81 - 72}}{4}$$
$$= \frac{9 \pm 3}{4}$$

$x = \dfrac{9 + 3}{4}$ or $x = \dfrac{9 - 3}{4}$

$x = \dfrac{12}{4}$ or $x = \dfrac{6}{4}$

$x = 3$ or $x = -\dfrac{3}{2}$

Sol. Set $\left\{3, -\dfrac{3}{2}\right\}$

Exercise 1.1

Q.1. Solve the following equations by factorization.

(i). $x^2 + 7x = 8$ (ii). $3x^2 + 7x + 4 = 0$

(iii). $x^2 - 3x = 2x - 6$ (iv). $3x^2 - 1 = \dfrac{1}{5}(1 - x)$

(v). $(2x + 3)(x + 1) = 1$ (vi). $\dfrac{1}{2x - 5} + \dfrac{5}{2x - 1} = 2$

(vii). $\dfrac{4}{x-1} - \dfrac{5}{x+2} = \dfrac{3}{x}$ (viii). $\dfrac{1}{a+b+x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}$

(ix). $abx^2 + (b^2 - ac)x - bc = 0$ (x). $(a + b)x^2 + (a + 2b + c)x + (b + c) = 0$

(xi). $\dfrac{a}{ax-1} + \dfrac{b}{bx-1} = a + b$ (xii). $\dfrac{x+2}{x-1} + 2\dfrac{2}{3} = \dfrac{x+3}{x-2}$

Q.2. Solve the following equations by the method of completing the square.

(i). $x^2 - 6x + 8 = 0$ (ii). $32 - 3x^2 = 10x$

(iii). $(x - 2)(x + 3) = 2(x + 11)$ (iv). $x^2 + (a + b)x + ab = 0$

(v). $x + \dfrac{1}{x} = \dfrac{10}{3}$ (vi). $\dfrac{10}{x-5} + \dfrac{10}{x+5} = \dfrac{5}{6}$

(vii). $2x^2 - 5bx = 3b^2$ (viii). $x^2 - 2ax + a^2 - b^2 = 0$

Q.3 Solve the following equations by using quadratic formula.

(i). $2x^2 + 3x - 9 = 0$ (ii). $(x + 1)^2 = 3x + 14$

(iii). $\dfrac{1}{x+1} + \dfrac{1}{x+2} + \dfrac{1}{x+3} = \dfrac{3}{x}$ (iv). $x^2 - 3\left(x + \dfrac{25}{4}\right) = 9x - \dfrac{25}{2}$

(v). $x^2 + (m-n)x - 2(m-n)^2 = 0$ (vi) $mx^2 + (1 + m)x + 1 = 0$

(vii) $abx^2 + (2b - 3a)x - 6 = 0$ (viii). $x^2 + (b-a)x - ab = 0$

(ix) $\dfrac{x}{x+1} + \dfrac{x+1}{x+2} + \dfrac{x+2}{x+3} = 3$

Q.4 The sum of a number and its square is 56 . Find the number.

Q.5 A projectile is fired vertically into the air. The distance (in meter) above the ground as a function of time (in seconds) is given by $s = 300 - 100t - 16t^2$. When will the projectile hit the ground?

Q.6 The hypotenuse of a right triangle is 18 meters. If one side is 4 meters longer than the other side, what is the length of the shorter side ?

Answers 1.1

Q.1. (i). $\{1, -8\}$ (ii). $\left\{-1, \dfrac{-4}{3}\right\}$ (iii). $\{2, 3\}$

(iv). $\{\frac{3}{5}, -\frac{2}{3}\}$ (v). $\left\{-2, -\dfrac{1}{2}\right\}$ (vi). $\left\{-\frac{3}{2}, 3\right\}$

(vii). $\left\{-\frac{1}{2}, 3\right\}$ (viii). {-a, -b} (ix). $\{-\frac{b}{a}, \frac{c}{b}\}$

(x). $\{-1, -\frac{b+c}{a+b}\}$ (xi). $\{\frac{a+b}{ab}, \frac{2}{a+b}\}$ (xii). $\{\frac{13}{4}, \frac{1}{2}\}$

Q.2. (i). {2, 4} (ii). $\{2, -\frac{16}{3}\}$ (iii). $\{\frac{1\pm\sqrt{113}}{2}\}$

(iv). $\{-a, -b\}$ (v). $\{3, \frac{1}{3}\}$ (vi). {-1, 25}

(vii). $\{3b, -\frac{b}{2}\}$ (viii) $\{(a+b), (a-b)\}$

Q.3. (i). $\{\frac{3}{2}, -3\}$ (ii). $\{\frac{1\pm\sqrt{53}}{2}\}$ (iii). $\{\frac{-11\pm\sqrt{13}}{6}\}$

(iv). $\{\frac{25}{2}, -\frac{1}{2}\}$ (v). $\{m-n, -2(m-n)\}$ (vi) $\left\{-1, \frac{-1}{m}\right\}$

(vii). $\{-\frac{2}{a}, \frac{3}{b}\}$ (viii). $\{-b, a\}$ (ix) $\left\{\frac{-6+\sqrt{3}}{3}, \frac{-6-\sqrt{3}}{3}\right\}$

Q.4. 7 , - 8 Q.5. 8.465 seconds Q.6. 10.6 m

1.8 Classification of Numbers

1. The Set N of Natural Numbers:

Whose elements are the counting, or natural numbers:

N = {1, 2, 3, - - - - - - - - }

2. The Set Z of Integers:

Whose elements are the positive and negative whole numbers and zero:

Z = {- - - - - - , -2, -1, 0, 1, 2, - - - - - - }

3. The Set Q of Rational Numbers: Whose elements are all those numbers that can be represented as the quotient of two integers $\frac{a}{b}$, where $b \neq 0$. Among the elements of Q are such numbers as $-\frac{3}{4}, \frac{18}{27}, \frac{5}{1}, -\frac{9}{1}$. In symbol

$$Q = \left\{\frac{a}{b} \mid a, b \in Z, b \neq 0\right\}$$

Equivalently, rational numbers are numbers with terminating or repeating decimal representation, such as

1.125, 1.52222, 1.56666, 0.3333

4. The Set Q□ of Irrational Numbers:

Whose elements are the numbers with decimal representations that are non-terminating and non-repeating. Among the elements of this set are such numbers as $\sqrt{2},\ -\sqrt{7},\ \pi$.

An irrational number cannot be represented in the form $\frac{a}{b}$, where $a, b \in Z$. In symbols,

$Q' = \{\text{irrational numbers}\}$

5. The Set R of Real Numbers:

Which is the set of all rational and irrational numbers:

$$R = \{x \mid x \in Q \cup Q'\}$$

6. The set I of Imaginary Numbers:

Whose numbers can be represented in the form x + yi, where x and y are real numbers, $y \neq 0$ and $i = \sqrt{-1}$

$$I = \{x + yi \mid x, y \in R, y \neq 0, i = \sqrt{-1}\}$$

If x = 0, then the imaginary number is called a pure imaginary number.

An imaginary number is defined as, a number whose square is a negative i.e, $\sqrt{-1},\ \sqrt{-3},\ \sqrt{-5}$

7. The set C of Complex Numbers:

Whose members can be represented in the form x + y i, where x and y real numbers and $i = \sqrt{-1}$:

$$C = \{x + yi \mid x, y \in R, i = \sqrt{-1}\}$$

With this familiar identification, the foregoing sets of numbers are related as indicated in Fig. 1.

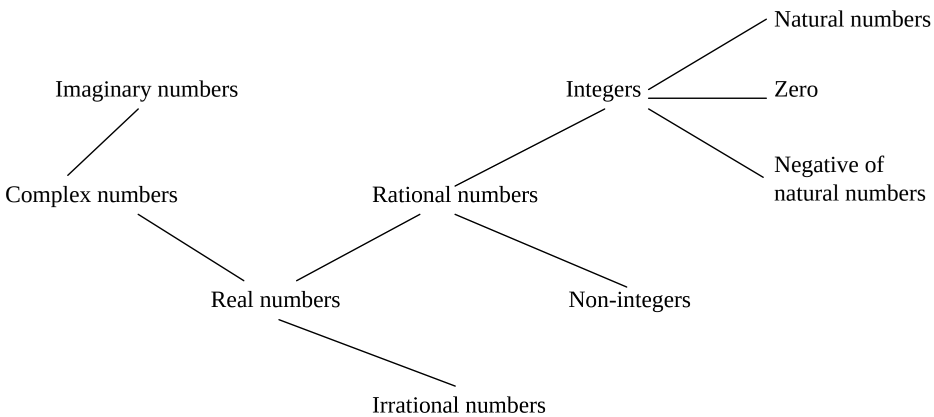


Fig. 1

Hence, it is clear that $N \subseteq Z \subseteq Q \subseteq R \subseteq C$

1.9 Nature of the roots of the Equation $ax^2 + bx + c = 0$

The two roots of the Quadratic equation $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ which appear under radical sign is called the Discriminant (Disc.) of the quadratic equation. i.e., Disc = $b^2 - 4ac$

The expression $b^2 - 4ac$ discriminates the nature of the roots, whether they are real, rational, irrational or imaginary. There are three possibilities.

(i)$b^2 - 4ac < 0$ (ii) $b^2 - 4ac = 0$ (iii) $b^2 - 4ac > 0$

(i) If $b^2 - 4ac < 0$, then roots will be imaginary and unequal.

(ii) If $b^2 - 4ac = 0$, then roots will be real, equal and rational.
(This means the left hand side of the equation is a perfect square).

(iii) If $b^2 - 4ac > 0$, then two cases arises:

(a) $b^2 - 4ac$ is a perfect square, the roots are real, rational and unequal.
(This mean the equation can be solved by the factorization).

(b) $b^2 - 4ac$ is not a perfect square, then roots are real, irrational and unequal.

Example 1:

Find the nature of the roots of the given equation

$$9x^2 + 6x + 1 = 0$$

Solution:

$$9x^2 + 6x + 1 = 0$$

Here $a = 9$, $b = 6$, $c = 1$

Therefore , Discriminant $= b^2 - 4ac$

$= (6)^2 - 4(9)(1)$

$= 36 - 36$

$= 0$

Because $b^2 - 4ac = 0$

$\therefore$ roots are equal, real and rational.

Example 2:

Find the nature of the roots of the Equation

$$3x^2 - 13x + 9 = 0$$

Solution:

$$3x^2 - 13x + 9 = 0$$

Here $a = 3$, $b = -13$, $c = 9$

Discriminant $= b^2 - 4ac$

$= (-13)^2 - 4(3)(9)$

$= 169 - 108 = 61$

Disc = $b^2 - 4ac = 61$ which is positive

Hence the roots are real, unequal and irrational.

Example 3:

For what value of "K" the roots of $Kx^2 + 4x + (K - 3) = 0$ are equal.

Solution:

$$Kx^2 + 4x + (K - 3) = 0$$

Here $a = K$, $b = 4$, $c = K - 3$

Disc $= b^2 - 4ac$

$= (4)^2 - 4(K)(K - 3)$
$= 16 - 4K^2 + 12K$

The roots are equal if $b^2 - 4ac = 0$

i.e. $16 - 4K^2 + 12K = 0$
$4K^2 - 3K - 4 = 0$
$K^2 - 4K + K - 4 = 0$
$K(K - 4) + 1(K - 4) = 0$

Or $K = 4, -1$

Hence roots will be equal if $K = 4, -1$

Example 4:

Show that the roots of the equation

$2(a + b)x^2 - 2(a + b + c)x + c = 0$ are real

Solution: $2(a + b)x^2 - 2(a + b + c)x + c = 0$

Here, $a = 2(a + b)$, $b = -2(a + b + c)$, $c = c$

Discriminant $= b^2 - 4ac$

$= [-2(a + b + c)]^2 - 4[2(a + b) c]$

$= 4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ac) - 8(ac + bc)$

$= 4(a^2 + b^2 + c^2 + 2ab + \cancel{2bc} + \cancel{2ac} - \cancel{2ac} - \cancel{2bc})$

$= 4(a^2 + b^2 + c^2 + 2ab)$

$= 4[(a^2 + b^2 + 2ab) + c^2]$

$= 4[(a + b)^2 + c^2]$

Since each term is positive, hence

Disc > 0 Hence , the roots are real.

Example 5:

For what value of K the roots of equation $2x^2 + 5x + k = 0$ will be rational.

Solution:

$2x^2 + 5x + k = 0$

Here, $a = 2$, $b = 5$, $c = k$

The roots of the equation are rational if

Disc $= b^2 - 4ac = 0$

So, $5^2 - 4(2)k = 0$

$25 - 8k = 0$

$k = \dfrac{25}{8}$ Ans

Exercise 1.2

Q1. Find the nature of the roots of the following equations

(i) $2x^2 + 3x + 1 = 0$ (ii) $6x^2 = 7x + 5$

(iii) $3x^2 + 7x - 2 = 0$ (iv) $\sqrt{2}\, x^2 + 3x - \sqrt{8} = 0$

Q2. For what value of K the roots of the given equations are equal.

(i) $x^2 + 3(K + 1)x + 4K + 5 = 0$ (ii) $x^2 + 2(K - 2)x - 8k = 0$

(iii) $(3K + 6)x^2 + 6x + K = 0$ (iv) $(K + 2)x^2 - 2Kx + K - 1 = 0$

Q3. Show that the roots of the equations

(i) $a^2(mx + c)^2 + b^2x^2 = a^2 b^2$ will be equal if $c^2 = b^2 + a^2m^2$

(ii) $(mx + c)^2 = 4ax$ will be equal if $c = \frac{a}{m}$

(iii) $x^2 + (mx + c)^2 = a^2$ has equal roots if $c^2 = a^2 (1 + m^2)$.

Q4. If the roots of $(c^2 – ab)x^2 – 2(a^2 – bc)x + (b^2 – ac) = 0$ are equal then prove that $a^3 + b^3 + c^3 = 3abc$

Q5. Show that the roots of the following equations are real

(i) $x^2 – 2 (m+\frac{1}{m})x + 3 = 0$

(ii) $x^2 – 2ax + a^2 = b^2 + c^2$

(iii) $(b^2 – 4ac)x^2 + 4(a + c)x – 4 = 0$

Q6. Show that the roots of the following equations are rational

(i) $a(b – c)x^2 + b(c – a)x + c(a – b) = 0$

(ii) $(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$

(iii) $(a + b)x^2 – ax – b) = 0$

(iv) $p x^2 - (p – q) x – q = 0$

Q7. For what value of 'K' the equation $(4–k) x^2 + 2(k+2) x + 8k + 1 = 0$ will be a perfect square.

(Hint : The equation will be perfect square if Disc. $b^2 – 4ac = 0$)

Answers 1.2

Q1. (i) Real, rational, unequal (ii) unequal, real and rational

(iii) ir-rational, unequal, real (iv) Real, unequal, ir-rational

Q2. (i) $1, \frac{-11}{9}$ (ii) - 2 (iii) 1, -3 (iv) 2

Q7. 0, 3

1.10 Sum and Product of the Roots

(Relation between the roots and Co-efficient of $ax^2 + b x + c = 0$)

The roots of the equation $ax^2 + bx + c = 0$ are

$$\propto = \frac{-b\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\beta = \frac{-b-\sqrt{b^2 - 4ac}}{2a}$$

Sum of roots:

Add the two roots

$$\propto +\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b\ \ -b}{2a}$$

$$= \frac{-2b}{2a} \qquad = -\frac{b}{a}$$

Hence, sum of roots = $\propto +\beta = \dfrac{-\text{ Co-efficient of x}}{\text{Co-efficient of x}^2}$

Product of roots:

$$\propto \beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) x \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) \quad =$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac}\ )^2}{2\ a^2}$$

$$= \frac{b^2-b^2+4ac}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$a\beta = \frac{c}{a}$$

i.e. product of roots = $\propto \beta = \dfrac{-\text{ Constant term}}{\text{Co-efficient of x}^2}$

Example 1:

Find the sum and the Product of the roots in the Equation $2x^2 + 4 = 7x$

Solution:

$2x^2 + 4 = 7x$

$2x^2 - 7x + 4 = 0$

Here a = 2, b = -7, c = 4

Sum of the roots = $-\dfrac{b}{a} = -\left(-\dfrac{7}{2}\right) = \dfrac{7}{2}$

Product of roots = $\dfrac{c}{a} \quad = \dfrac{4}{2} = 2$

Example 2:

Find the value of "K" if sum of roots of

$(2k - 1)x^2 + (4K - 1)x + (K + 3) = 0$ is $\frac{5}{2}$

Solution:

$(2k - 1)x^2 + (4K - 1)x + (K + 3) = 0$

Here $a = (2k - 1)$, $b = 4K - 1$, $c = K + 3$

Sum of roots $= -\frac{b}{a}$

$\frac{5}{2} = -\frac{(4K - 1)}{(2K - 1)}$ $\quad\because$ Sum of roots $= \frac{5}{2}$

$5(2K - 1) = -2(4K - 1)$

$10K - 5 = -8K + 2$

$10K + 8K = 5 + 5$

$18K = 7$

$K = \frac{7}{18}$

Example 3:

If one root of $4x^2 - 3x + K = 0$ is 3 times the other, find the value of "K".

Solution:

Given Equation is $4x^2 - 3x + K = 0$

Let one root be α, then other will be 3α.

Sum of roots $= -\frac{a}{b}$

$\alpha + 3\alpha = -\frac{(-3)}{4}$

$4\alpha = \frac{3}{4}$

$\alpha = \frac{3}{16}$

Product of roots $= \frac{c}{a}$

$\alpha(3\alpha) = \frac{K}{4}$

$3\alpha^2 = \frac{K}{4}$

$K = 12\alpha^2$

Putting the value of $\alpha = \frac{3}{16}$ we have

$K = 12\left(\frac{3}{16}\right)^2$

$= \frac{12x9}{256} = \frac{27}{64}$

Exercise 1.3

Q1. Without solving, find the sum and the product of the roots of the following equations.

(i) $x^2 - x + 1 = 0$ (ii) $2y^2 + 5y - 1 = 0$
(iii) $x^2 - 9 = 0$ (iv) $2x^2 + 4 = 7x$
(v) $5x^2 + x - 7 = 0$

Q2. Find the value of k, given that

(i) The product of the roots of the equation $(k + 1)x^2 + (4k + 3)x + (k - 1) = 0$ is $\frac{7}{2}$

(ii) The sum of the roots of the equation $3x^2 + k\,x + 5 = 0$ will be equal to the product of its roots.

(iii) The sum of the roots of the equation $4\,x^2 + k\,x - 7 = 0$ is 3.

Q3. (i)If the difference of the roots of $x^2 - 7x + k - 4 = 0$ is 5, find the value of k and the roots.

(ii) If the difference of the roots of $6x^2 - 23x + c = 0$ is $\frac{5}{6}$, find the value of k and the roots.

Q4. If α, β are the roots of $ax^2 + bx + c = 0$ find the value of

(i) $\alpha^3 + \beta^3$ (ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (iii) $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = 0$

(iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ (v) $\frac{\alpha}{\beta} - \frac{\beta}{\alpha}$
(v)

Q5. If p, q are the roots of $2x^2 - 6x + 3 = 0$ find the value of
$(p^3 + q^3) - 3pq\,(p^2 + q^2) - 3pq\,(p + q)$

Q6. The roots of the equation $px^2 + qx + q = 0$ are α and β,

Prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$

Q7. Find the condition that one root of the equation $px^2 + qx + r = 0$ is square of the other.

Q8. Find the value of k given that if one root of $9x^2 - 15x + k = 0$ exceeds the other by 3. Also find the roots.

Q9. If α, β are the roots of the equation $px^2 + qx + r = 0$ then find the values of

(i) $\alpha^2 + \beta^2$ (ii) $(\alpha - \beta)^2$ (iii) $\alpha^3\beta + \alpha\beta^3$

Answers 1.3

Q1.(i) 1, 1 (ii) $-\frac{5}{2}, -\frac{1}{2}$ (iii) 0, - 9 (iv) $\frac{7}{2}$, 2 (v) $-\frac{1}{5}, -\frac{7}{5}$

Q2.(i) $\frac{7}{18}$ (ii) $-\frac{9}{5}$ (iii) - 12

Q3.(i) K =10, roots = 6, 1 (ii) $\propto = \frac{7}{3},\ \beta = \frac{3}{2};$ c = 21

Q4. (i) $\frac{-b^3+3abc}{a^3}$ (ii) $\frac{b^2-2ac}{c^2}$ (iii) $-\frac{b}{\sqrt{ac}}$ (iv) $\frac{3abc-b^3}{a^2c}$ (v) $\frac{-b\sqrt{b^2-4ac}}{ac}$

Q5. - 27 **Q7.** Pr (p + r)+q^3 = 3pqr **Q8.** K = - 14, roots are $-\frac{2}{3},\ \frac{7}{3}$

Q9. (i) $\frac{q^2-2pr}{p^2}$ (ii) $\frac{q^2-4pr}{p^2}$ (iii) $\frac{r(q^2-2pr)}{p^3}$

1.11 Formation of Quadratic Equation from the given roots :

Let α, β be the roots of the Equation $ax^2 + bx + c = 0$

The sum of roots $= \alpha + \beta = -\frac{b}{a}$ ……… (I)

Product of roots $= \propto .\beta = \frac{c}{a}$ ……… (II)

The equation is $ax^2 + bx + c = 0$

Divide this equation by a $\Rightarrow$ $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Or $x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$

From I and II this equation becomes

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Or x^2 – (Sum of roots) x + Product of roots = 0

Or x^2 – (S) x + (P) = 0

is the required equation, where S = $\alpha + \beta$ and P = $\alpha\ \beta$

Alternate method:-

Let α, β be the roots of the equation $a x^2 + b x + c = 0$

i.e., $x = \alpha$ and $x = \beta$

$\Rightarrow$ $x - \alpha = 0$ and $x - \beta = 0$

$\Rightarrow$ $(x - \alpha)(x - \beta) = 0$

$$x^2 - \alpha x - \beta x + \alpha\beta = 0$$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Or x^2 – (Sum of roots) x + Product of roots = 0

Or $x^2 - Sx + P = 0$

is the required equation, where $S = \alpha + \beta$ and $P = \alpha\ \beta$

Example 4:

Form a quadratic Equation whose roots are $3\sqrt{5}, -3\sqrt{5}$

Solution:

Roots of the required Equation are $3\sqrt{5}$ and $-3\sqrt{5}$

Therefore S = Sum of roots = $3\sqrt{5} - 3\sqrt{5}$

S = 0

P = Product of roots = $(3\sqrt{5})(-3\sqrt{5})$ = – 9 (5)

P = – 45

Required equation is

x^2 – (Sum of roots) x + (Product of roots) = 0

Or $x^2 - Sx + P = 0$

$x^2 - 0(x) + (-45) = 0$

$x^2 - 0 - 45 = 0$

$x^2 - 45 = 0$

Example 5:

If α, β are the roots of the equation $ax^2 + bx + c = 0$, find the equation whose **roots are** $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$**.**

Solution:

Because α, β are the roots of the Equation $ax^2 + bx + c = 0$

The sum of roots = $\alpha + \beta = -\frac{b}{a}$

Product of roots = $\alpha\ \beta = \frac{b}{a}$

Roots of the required equation are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

Therefore ,

S = sum of roots of required equation = $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta} \because (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{\left(-\frac{b}{a}\right)^2 - 2\frac{c}{a}}{\alpha\beta}$$

$$= \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c}{a}} \quad = \frac{b^2 - 2ac}{a^2} \text{ x } \frac{a}{c}$$

$$S = \frac{b^2 - 2ac}{ac}$$

P = Product of roots of required equation $= \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = \frac{\alpha\beta}{\beta\alpha}$

$P = 1$

Required equation is: $x^2 - Sx + P = 0$

$$x^2 + \left(\frac{b^2 - 2ac}{ac}\right) x + 1 = 0$$

$$acx^2 - (b^2 - 2ac)x + ac = 0$$

Exercise 1.4

Q1. Form quadratic equations with the following given numbers as its roots.

(i) 2, - 3 (ii) $3 + i$, $3 - i$ (iii) $2+\sqrt{3}$, $2-\sqrt{3}$

(iv) $-3+\sqrt{5}$, $-3-\sqrt{5}$ (v) $4 + 5i$, $4 - 5i$

Q2. Find the quadratic equation with roots

(i)Equal numerically but opposite in sign to those of the roots of the equation $3x^2 + 5x - 7 = 0$

(ii)Twice the roots of the equation $5x^2 + 3x + 2 = 0$

(iii)Exceeding by '2' than those of the roots of $4x^2 + 5x + 6 = 0$

Q3. Form the quadratic equation whose roots are less by '1' than those of $3x^2 - 4x - 1 = 0$

Q4. Form the quadratic equation whose roots are the square of the roots of the equation $2x^2 - 3x - 5 = 0$

Q5. Find the equation whose roots are reciprocal of the roots of the equation $px^2 - qx + r = 0$

Q6. If $\propto, \beta$ are the roots of the equation $x^2 - 4x + 2 = 0$ find the equation whose roots are

(i) α^2 , β^2 (ii) α^3 , β^3 (iii) $\propto + \frac{1}{\propto}, \beta + \frac{1}{\beta}$

(iv) $\alpha + 2$, $\beta + 2$

Q7. If α , β are the roots of $ax^2 + bx + c = 0$ form an equation whose roots are

(i) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (ii) $\frac{\alpha^2}{\beta}, \frac{\beta^2}{\alpha}$ (iii) $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$

Answers 1.4

Q1. (i) $x^2 + x - 6 = 0$ (ii) $x^2 - 6x + 10 = 0$
(iii) $x^2 - 4x + 1 = 0$ (iv) $x^2 + 6x + 4 = 0$ (v) $x^2 - 5x + 41 = 0$

Q2. (i) $3x^2 - 5x - 7 = 0$ (ii) $5x^2 - 6x + 8 = 0$
(iii) $4x^2 - 11x + 12 = 0$

Q3. $3x^2 + 2x - 2 = 0$ **Q4 .** $4x^2 - 29x + 25 = 0$

Q5. $rx^2 - qx + p = 0$ **Q6.** (i) $x^2 - 12x + 4 = 0$ (ii) $x^2 - 40x + 8 = 0$
(iii) $2x^2 - 12x + 17 = 0$ (iv) $x^2 - 8x + 14 = 0$

Q7. (i) $acx^2 - (b^2 - 2ac)x + ac = 0$ (ii) $a^2cx^2 + (b^3 - 3abc)x + ac^2 = 0$
(iii) $cx^2 - (2c - b)x + (a - b + c) = 0$

Summary

Quadratic Equation:

An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, where a, b , c $\in$ R and x is a variable, is called a quadratic equation.

If α, β are its roots then

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}, \ \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Nature of Roots:

(i) If $b^2 - 4ac > 0$ the roots are real and distinct.
(ii) If $b^2 - 4ac = 0$ the roots are real and equal.
(iii) If $b^2 - 4ac < 0$ the roots are imaginary.
(iv) If $b^2 - 4ac$ is a perfect square, roots will be rational, otherwise irrational.

Relation between Roots and Co-efficients

If α and β be the roots of the equation $ax^2 + bx + c = 0$

Then sum of roots = $\alpha + \beta = \frac{-b}{a}$

Product of roots = $\alpha\beta = \frac{c}{a}$

Formation of Equation

If α and β be the roots of the equation $ax^2 + bx + c = 0$ then we have

x^2 – (sum of roots)x + (product of roots) = 0

Short Questions

Write Short answers of the following questions:

Solve the following quadratic equations by factorization

Q.1 $x^2 + 7x + 12 = 0$

Q2. $x^2 - x = 2$

Q3. $x(x + 7) = (2x - 1)(x + 4)$

Q4. $6x^2 - 5x = 4$

Q5. $3x^2 + 5x = 2$

Q6. $2x^2 + x = 1$

Q7. $m x^2 + (1 + m) x + 1 = 0$

Solve the following equations by completing the square:

Q8. $x^2 - 2x - 899 = 0$

Q9. $2x^2 + 12x - 110 = 0$

Q10. $x^2 + 5x - 6 = 0$

Q11. $x^2 - 6x + 8 = 0$

Solve the following equations by quadratic formula :

Q12. $4x^2 + 7x - 1 = 0$

Q13. $9x^2 - x - 8 = 0$

Q14. $X^2 - 3x - 18 = 0$

Q15. $X^2 - 3x = 2x - 6$

Q16. $3x^2 - 5x - 2 = 0$

Q17. $16x^2 + 8x + 1 = 0$

Q18 Define discriminant

Discuss the nature of the roots of the equation:

Q19 $2x^2 - 7x + 3 = 0$

Q20. $x^2 - 5x - 2 = 0$

Q21. $x^2 + x + 1 = 0$

Q22. $x^2 - 2\sqrt{2}\,x + 2 = 0$

Q23. $9x^2 + 6x + 1 = 0$

Q24. $3x^2 - 13x + 9 = 0$

For what value of K the roots of the following equations are equal:

Q25 $Kx^2 + 4x + 3 = 0$

Q26. $2x^2 + 5x + K = 0$

Q27 Prove that the roots of the equation

$(a + b)x^2 - ax - b = 0$ are rational

Q28 Write relation between the roots and the coefficients of the quadratic equation

$ax^2 + bx + c = 0$

Q.29 If the sum of the roots of $4x^2 + kx - 7 = 0$ is 3, Find the value of k.

Q.30 Find the value of K if the sum of the roots of equation

$(2k - 1)x^2 + (4k - 1)x + (K + 3) = 0$ is 5/2

Find the sum and product of the roots of following equations:

Q31 $7x^2 - 5x + 4 = 0$

Q32. $x^2 - 9 = 0$

Q33. $9x^2 + 6x + 1 = 0$

Q34. For what value of k the sum of roots of equation $3x^2 + kx + 5 = 0$

may be equal to the product of roots?

Q35. If α, β are the roots of $x^2 - px - p - c = 0$ then prove that $(1+\alpha)(1+\beta) = 1 - c$

Write the quadratic equation for the following equations whose roots are :

Q.36 -2, -3

Q37. $i\sqrt{3}$, $-i\sqrt{3}$

Q38. $-2+\sqrt{3}$, $-2-\sqrt{3}$

Q.39 Form the quadratic equation whose roots are equal numerically but opposite in sign to those of $3x^2 - 7x - 6 = 0$

If α, β are the roots of the equation $x^2 - 4x + 2 = 0$ find equation whose roots are:

Q40. $\frac{1}{\alpha}, \frac{1}{\beta}$

Q41 $-\alpha$, $-\beta$

Answers

Q1. {- 3 , - 4} **Q2** {- 1 , 2} **Q3** { 2 , -2} **Q4** {4/3 , - ½} **Q5** {1 , -6}

Q6 {-1 , ½} **Q7** { -1 , - 1/m} **Q8** {-29 , 31} **Q9** {-11 , 5} **Q10** {1 , – 6}

Q11 {2 , 4} **Q12.** {1 , -6} **Q13** $\{\frac{-7-\sqrt{65}}{8}, \frac{-7+\sqrt{65}}{8}\}$ **Q14** {-8/9 , 1}

Q15 {6 , -3} **Q16** {2 , 3} **Q17** {2 , -1/3} **Q18** {-1/4 }

Q19. Roots are rational , real and unequal

Q20 Roots are irrational , real and unequal

Q21 Roots are imaginary **Q22** Roots are equal and real

Q23 Roots are equal and real **Q24** Roots are unequal , real and irrational

Q25. K = 4/3 **Q26.** K = 5 **Q29** K = -12 **Q30.** K = 7/18

Q31 S = 5/7 , P = 4/7 **Q32** S = 0 , P = - 9 **Q33** S = -2/3 , 1/9

Q34 K = - 5 **Q36** $x^2 + 5x + 6 = 0$ **Q37** $x^2 + 3 = 0$ **Q38** $x^2 + 4x + 1 = 0$

Q39 $3x^2 + 7x - 2 = 0$ **Q40** $2x^2 - 4x + 1 = 0$ **Q41** $x^2 + 4x + 2 = 0$

Objective Type Questions

Q1. Each question has four possible answers .Choose the correct answer and encircle it .

__1. The standard form of a quadratic equation is:
(a) $ax^2 + bx = 0$ (b) $ax^2 = 0$
(c) $ax^2 + bx + c = 0$ (d) $ax^2 + c = 0$

__2. The roots of the equation $x^2 + 4x - 21 = 0$ are:
(a) (7, 3) (b) (–7, 3)
(c) (–7, –3) (d) (7, –3)

__3. To make $x^2 - 5x$ a complete square we should add:
(a) 25 (b) $\frac{25}{4}$ (c) $\frac{25}{9}$ (d) $\frac{25}{16}$

__4. The factors of $x^2 - 7x + 12 = 0$ are:
(a) $(x - 4)(x + 3)$ (b) $(x - 4)(x - 3)$
(c) $(x + 4)(x + 3)$ (d) $(x + 4)(x - 3)$

__5. The quadratic formula is:
(a) $\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$ (b) $\frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$
(c) $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (d) $\frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$

__6. A second degree equation is known as:
(a) Linear (b) Quadratic
(c) Cubic (e) None of these

__7. Factors of $x^3 - 1$ are:
(a) $(x - 1)(x^2 - x - 1)$ (b) $(x - 1)(x^2 + x + 1)$
(c) $(x - 1)(x^2 + x - 1)$ (d) $(x - 1)(x^2 - x + 1)$

__8. To make $49x^2 + 5x$ a complete square we must add:
(a) $\left(\frac{5}{14}\right)^2$ (b) $\left(\frac{14}{5}\right)^2$
(c) $\left(\frac{5}{7}\right)^2$ (d) $\left(\frac{7}{5}\right)^2$

__9. $lx^2 + mx + n = 0$ will be a pure quadratic equation if:
(a) $l = 0$ (b) $m = 0$
(c) $n = 0$ (d) Both l, $m = 0$

__10. If the discrimnant $b^2 - 4ac$ is negative, the roots are:
(a) Real (b) Rational
(c) Irrational (d) Imaginary

__11. If the discriminant $b^2 - 4ac$ is a perfect square, its roots will be:
(a) Imaginary (b) Rational
(c) Equal (d) Irrational

__12. The product of roots of $2x^2 - 3x - 5 = 0$ is:

(a) $-\frac{5}{2}$ (b) $\frac{5}{2}$
(c) $\frac{2}{5}$ (d) $-\frac{2}{5}$

__13. The sum of roots of $2x^2 - 3x - 5 = 0$ is:
(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$
(c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

__14. If 2 and – 5 are the roots of the equation, then the equations is:
(a) $x^2 + 3x + 10 = 0$ (b) $x^2 - 3x - 10 = 0$
(c) $x^2 + 3x - 10 = 0$ (d) $2x^2 - 5x + 1 = 0$

__15. If ± 3 are the roots of the equation, then the equation is:
(a) $x^2 - 3 = 0$ (b) $x^2 - 9 = 0$
(c) $x^2 + 3 = 0$ (d) $x^2 + 9 = 0$

__16. If ‘S’ is the sum and ‘P’ is the product of roots, then equation is:
(a) $x^2 + Sx + P = 0$ (b) $x^2 + Sx - P = 0$
(c) $x^2 - Sx + P = 0$ (d) $x^2 - Sx - P = 0$

__17. Roots of the equation $x^2 + x - 1 = 0$ are:
(a) Equal (b) Irrational
(c) Imaginary (d) Rational

__18. If the discriminant of an equation is zero, then the roots will be:
(a) Imaginary (b) Real
(c) Equal (d) Irrational

__19. Sum of the roots of $ax^2 - bx + c = 0$ is:
(a) $-\frac{c}{a}$ (b) $\frac{c}{a}$
(c) $-\frac{b}{a}$ (d) $\frac{b}{a}$

__20. Product of roots of ax2 + bx – c = 0 is:
(a) $\frac{c}{a}$ (b) $-\frac{c}{a}$ (c) $\frac{a}{b}$ (d) $-\frac{a}{b}$

Answers

1.	c	2.	b	3.	b	4.	b	5.	c
6.	b	7.	b	8.	a	9.	b	10.	d
11.	b	12.	a	13.	b	14.	c	15.	b
16.	c	17.	b	18.	c	19.	d	20.	b

Chapter 2
Binomial Theorem

2.1 Introduction:

An algebraic expression containing two terms is called a binomial expression, Bi means two and nom means term. Thus the general type of a binomial is $a + b$, $x - 2$, $3x + 4$ etc. The expression of a binomial raised to a small positive power can be solved by ordinary multiplication , but for large power the actual multiplication is laborious and for fractional power actual multiplication is not possible. By means of binomial theorem, this work reduced to a shorter form. This theorem was first established by Sir Isaac Newton.

2.2 Factorial of a Positive Integer:

If n is a positive integer, then the factorial of '*n*' denoted by $n!$ or $\underline{|n}$ and is defined as the product of n +ve integers from n to 1 (or 1 to n)

i.e., $n! = n(n-1)(n-2) \ldots.. 3.2.1$

For example,

$$4! = 4.3.2.1 = 24$$

and $$6! = 6.5.4.3.2.1 = 720$$

one important relationship concerning factorials is that

$$(n + 1)! = (n + 1)\, n! \qquad (1)$$

for instance,

$$5! = 5.4.3.2.1$$
$$= 5(4.3.2.1)$$
$$5! = 5.4!$$

Obviously, $1! = 1$ and this permits to define from equation (1)

$$n! = \frac{(n+1)!}{n+1}$$

Substitute 0 for n, we obtain

$$0! = \frac{(0+1)!}{0+1} = \frac{1!}{1} = \frac{1}{1}$$

$$0! = 1$$

2.3 Combination:

Each of the groups or selections which can be made out of a given number of things by taking some or all of them at a time is called combination.

In combination the order in which things occur is not considered e.g.; combination of a, b, c taken two at a time are ab, bc, ca.

The numbers $\binom{n}{r}$ **or** ${}^{n}C_{r}$

The numbers of the combination of n different objects taken 'r' at a time is denoted by $\binom{n}{r}$ **or** ${}^{n}C_{r}$ and is defined as,

$$\binom{n}{r} = \frac{n!}{r!\ (n-r)!}$$

e.g, $$\binom{6}{4} = \frac{6!}{4!(6-4)!}$$

$$= \frac{6 \times 5 \times 4!}{4! \times 2!} = \frac{6 \times 5}{2 \times 1} = 15$$

Example 1: Expand $\binom{7}{3}$

Solution: $$\binom{7}{3} = \frac{7!}{3!(7-3)!}$$

$$= \frac{7.6.5.4!}{3.2.1.4!}$$

$$= 35$$

This can also be expand as

$$\binom{7}{3} = \frac{7.6.5}{3.2.1} = 35$$

If we want to expand $\binom{7}{5}$, then

$$\binom{7}{5} = \frac{7.6.5.4.3}{5.4.3.2.1} = 21$$

Procedure: Expand the above number as the lower number and the lower number expand till 1.

Method 2

For expansion of $\binom{n}{r}$ we can apply the method:

a. If r is less than (n – r) then take r factors in the numerator from n to downward and r factors in the denominator ending to 1.

b. If n – r is less than r, then take (n – r) factors in the numerator from n to downward and take (n – r) factors in the denominator ending to 1. For example, to expand $\binom{7}{5}$ again, here 7 – 5 = 2 is less than 5, so take two factors in numerator and two in the denominator as,

$$\binom{7}{5} = \frac{7.6}{2.1} = 21$$

Some Important Results

(i). $\binom{n}{0} = \frac{n!}{0!(\,n-0\,)!} = \frac{n!}{1\times n!} = 1$

(ii) $\binom{n}{n} = \frac{n!}{n!(\,n-n\,)!} = \frac{n!}{n!\times 0!} = \frac{n!}{n!\times 1} = 1$

(iii) $\binom{n}{r} = \binom{n}{n-r}$

For example

$$\binom{4}{0} = \binom{4}{4} = 1 \text{ as } \quad \frac{4!}{0!(4-0)!} = \frac{4!}{4!\,0!}$$

$$1 = 1$$

$$\binom{4}{3} = \binom{4}{1} = 4 \quad \text{as} \quad \frac{4!}{3!\,.1!} = \frac{4!}{1!.\;3!}$$

$$\frac{4.\;3!}{3!\,.1!} = \frac{4.\;3!}{1!.\;3!}$$

$$4 = 4$$

Note: The numbers $\binom{n}{r}$ **or** ${}^{n}C_{r}$ are also called binomial co-efficients

2.4 The Binomial Theorem:

The rule or formula for expansion of $(a + b)^n$, where n is any positive integral power , is called binomial theorem .

For any positive integral n

$$(a+b\,)^n = \binom{n}{0}a^n + \binom{n}{1}\,a^{n-1}b + \binom{n}{2}\,a^{n-2} + \binom{n}{3}a^{n-3}b^3 \dots\dots$$

$$+\binom{n}{r}a^{n-r}b^r \dots\dots\dots.. + \binom{n}{n}\,b^n \text{-------------------}(1)$$

or briefly , $(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

Remarks:- The coefficients of the successive terms are

$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, ----, \binom{n}{r} ----, \binom{n}{n}$

and are called **Binomial coefficients.**

Note : Sum of binomial coefficients is 2^n

Another form of the Binomial theorem:

$$(a + b)^n = a^n + \frac{n}{1!} a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \ldots\ldots +$$

$$\frac{n(n-1)(n-2) ------ (n-r+1)}{r!} a^{n-r} b^r + \ldots\ldots\ldots + b^n \text{------------------} \quad (2)$$

Note: Since,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

So, $$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{1!(n-1)!} = \frac{n}{1!}$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-2)!}{2!(n-2)!} = \frac{n(n-1)}{2!}$$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)(n-3)!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

$$\binom{n}{r} = \frac{n(n-1)(n-2)!\ldots\ldots(n-r+1)(n-r)!}{r!(n-r)!}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}$$

$$\binom{n}{n} = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n! \times 0!} = \frac{n!}{n! \times 1} = 1$$

The following points can be observed in the expansion of $(a + b)^n$

1. There are (n + 1) terms in the expansion.
2. The 1^{st} term is a^n and (n + 1)th term or the last term is b^n
3. The exponent of 'a' decreases from n to zero.
4. The exponent of 'b' increases from zero to n.
5. The sum of the exponents of a and b in any term is equal to index n.
6. The co-efficients of the term equidistant from the beginning and end of the expansion are equal as $\binom{n}{r} = \binom{n}{n-r}$

2.5 General Term:

The term $\binom{n}{r} a^{n-r}b^r$ in the expansion of binomial theorem is called the General term or (r + 1)th term. It is denoted by T_{r+1}. Hence

$$T_{r+1} = \binom{n}{r} a^{n-r}b^r$$

Note: The General term is used to find out the specified term or the required co-efficient of the term in the binomial expansion

Example 2: Expand $(x + y)^4$ by binomial theorem:

Solution:

$$(x+y)^4 = x^4 + \binom{4}{1} x^{4-1} y + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + y^4$$

$$= x^4 + 4x^3y + \frac{4 \text{ x } 3}{2 \text{ x } 1} x^2y^2 + \frac{4 \text{ x } 3 \text{ x } 2}{3 \text{ x } 2 \text{ x } 1} xy^3 + y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Example 3: Expand by binomial theorem $\left(a - \frac{1}{a}\right)^6$

Solution:

$$\left(a - \frac{1}{a}\right)^6 = a^6 + \binom{6}{1} a^{6-1}\left(-\frac{1}{a}\right)^1 + \binom{6}{2} a^{6-2}\left(-\frac{1}{a}\right)^2 + \binom{6}{3} a^{6-3}\left(-\frac{1}{a}\right)^3 +$$

$$\binom{6}{4} a^{6-4}\left(-\frac{1}{a}\right)^4 + \binom{6}{5} a^{6-5}\left(-\frac{1}{a}\right)^5 + \binom{6}{6} a^{6-6}\left(-\frac{1}{a}\right)^6$$

$$= a^6 + 6a^5\left(-\frac{1}{a}\right) + \frac{6 \text{ x } 5}{2 \text{ x } 1} a^4\left(-\frac{1}{a^2}\right) + \frac{6 \text{ x } 5 \text{ x } 4}{3 \text{ x } 2 \text{ x } 1} a^3\left(-\frac{1}{a^3}\right) +$$

$$\frac{6 \text{ x } 5 \text{ x } 4 \text{ x } 3}{4 \text{ x } 3 \text{ x } 2 \text{ x } 1} a^2\left(-\frac{1}{a^4}\right)+\frac{6 \text{ x } 5 \text{ x } 4 \text{ x } 3 \text{ x } 2}{5 \text{ x } 4 \text{ x } 3 \text{ x } 2 \text{ x } 1} a\left(-\frac{1}{a^5}\right)^5+\left(-\frac{1}{a^6}\right)$$

$$= a^6 - 6a^4 + 15a^2 - 20 + \frac{15}{a^2} - \frac{6}{a^5} + \frac{1}{a^6}$$

Example 4: Expand $\left(\frac{x^2}{2} - \frac{2}{x}\right)^4$

Solution:

$$\left(\frac{x^2}{2} - \frac{2}{x}\right)^4 = \left(\frac{x^2}{2}\right)^4 + \binom{4}{1}\left(\frac{x^2}{2}\right)^{4-1}\left(\frac{-2}{x}\right)^1 + \binom{4}{2}\left(\frac{x^2}{2}\right)^{4-2}\left(\frac{-2}{x}\right)^2$$

$$+ \binom{4}{3}\left(\frac{x^2}{2}\right)^{4-3}\left(\frac{-2}{x}\right)^3 + \binom{4}{4}\left(\frac{x^2}{2}\right)^{4-4}\left(\frac{-2}{x}\right)^4$$

$$= \frac{x^4}{16}+4\left(\frac{x^2}{2}\right)^3\left(-\frac{2}{x}\right)+\frac{4\,.\,3}{2\,.\,1}\left(\frac{x^2}{2}\right)^2\left(\frac{4}{x^2}\right)+$$

$$\frac{4\,.\,3\,.\,2}{3\,.\,2\,.\,1}\left(\frac{x^2}{2}\right)\left(-\frac{8}{x^3}\right)+\frac{16}{x^4}$$

$$= \frac{x^8}{16}-4.\frac{x^8}{8}.\frac{2}{x}+6.\frac{x^4}{4}.\frac{4}{x^2}-4\frac{x^2}{2}.\frac{8}{x^3}+\frac{16}{x^4}$$

$$= \frac{x^8}{16}-x^5+6x^2-\frac{16}{x}+\frac{16}{x^4}$$

Example 5: Expand $(1.04)^5$ by the binomial formula and find its value to two decimal places.

Solution:

$$(1.04)^5 = (1 + 0.04)^5$$

$$(1 + 0.04)^5 = (1)^5+\binom{5}{1}(1)^{5-1}(0.04)+\binom{5}{2}(1)^{5-2}(0.04)^2+\binom{5}{3}$$

$$(1)^{5-3}(0.04)^3+\binom{5}{4}(1)^{5-4}(0.04)^4+(0.04)^5$$

$$= 1. + 0.2 + 0.016 + 0.00064 + 0.000128 + 0.000\,000\,1024$$

$$= 1.22$$

Example 6: Find the eighth term in the expansion of $\left(2x^2-\frac{1}{x^2}\right)^{12}$

Solution: $\left(2x^2 - \frac{1}{x^2}\right)^{12}$

The General term is, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Here $T_8 = ?$ $a = 2x^2$, $b = -\frac{1}{x^2}$, $n = 12$, $r = 7$,

Therefore , $T_{7+1} = \binom{12}{7}\left(2x^2\right)^{12-7}\left(-\frac{1}{x^2}\right)^7$

$$T_8 = \frac{12 . 11 . 10 . 9 . 8 . 7 . 6}{7 . 6 . 5 . 4 . 3 . 2 . 1}\left(2x^2\right)^5 \frac{(-1)^7}{x^{14}}$$

$$T_8 = 793 \text{ x } 32x^{10} \frac{(-1)}{x^{14}}$$

$$T_8 = -\frac{25\,344}{x^4}$$

Eighth term $= T_8 = -\frac{25\,344}{x^4}$

.

2.6 Middle Term in the Expansion $(a + b)^n$

In the expansion of $(a + b)^n$, there are $(n + 1)$ terms.

Case I :

If n is even then $(n + 1)$ will be odd, so $\left(\frac{n}{2} + 1\right)$th term will be the only one middle term in the expension .

For example, if $n = 8$ (even), number of terms will be 9 (odd), therefore, $\left(\frac{8}{2} + 1\right) = 5^{th}$ will be middle term.

Case II:

If n is odd then $(n + 1)$ will be even, in this case there will not be a single middle term, but $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2} + 1\right)$th term will be the two middle terms in the expension.

For example, for $n = 9$ (odd), number of terms is 10 i.e. $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+1}{2}+1\right)$th i.e. 5^{th} and 6^{th} terms are taken as middle terms and these middle terms are found by using the formula for the general term.

Example 7: Find the middle term of $\left(1-\frac{x^2}{2}\right)^{14}$ **.**

Solution:

We have n = 14, then number of terms is 15.

$\therefore \left(\frac{14}{2}+1\right)$ i.e. 8^{th} will be middle term.

a = 1, b = $-\frac{x^2}{2}$, n = 14, r = 7, T_8 = ?

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$T_{7+1} = \binom{14}{7}(1)^{14-7}\left(-\frac{x^2}{2}\right)^7 = \frac{14!}{7!7!}(-1)^7\frac{x^{14}}{2^7}$$

$$T_8 = \frac{14.13.12.11.10.9.8.7!}{7.6.5.4.3.2.1\ 7!}\ \frac{(-1)}{128}.x^{14}$$

$$T_8 = -(2)(13)(11)(2)(3)\ \frac{1}{128}.\ x^{14}$$

$$T_8 = -\frac{429}{16}\ x^{14}$$

Example 8 : Find the coefficient of x^{19} in $(2x^3 - 3x)^9$.

Solution:

Here, a = $2x^3$, b = $-3x$, n = 9

First we find r.

Since $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$= \binom{9}{r}\left(2x^3\right)^{9-r}(-3x)^r$$

$$= \binom{9}{r} 2^{9-r}(-3)^r x^{27-3r}.x^r$$

$$= \binom{9}{r} 2^{9-r}(-3)^r .x^{27-2r} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

But we require x^{19}, so put

$19 = 27 - 2r$

$2r = 8$

$r = 4$

Putting the value of r in equation (1)

$$T_{4+1} = \binom{9}{r} 2^{9-4}(-3)^4 x^{19}$$

$$= \frac{9.8.7.6.5}{4.3.2.1} . 2^5 . 3^4 x^{19}$$

$$= 630 \text{ x } 32 \text{ x } 81 x^{19}$$

$$T_5 = 1632960 x^{19}$$

Hence the coefficient of x^{19} is 1632960

Example 9: Find the term independent of x in the expansion of $\left(2x^2 + \frac{1}{x}\right)^9$.

Solution:

Let T_{r+1} be the term independent of x.

We have $a = 2x^2$, $b = \frac{1}{x}$, $n = 9$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r = \binom{9}{r} \left(2x^2\right)^{9-r} \left(\frac{1}{x}\right)^r$$

$$T_{r+1} = \binom{9}{r} 2^{9-r} . x^{18-2r} . x^r$$

$$T_{r+1} = \binom{9}{r} 2^{9-r} . x^{18-3r} \ldots\ldots\ldots\ldots\ldots (1)$$

Since T_{r+1} is the term independent of x i.e. x^0.

$\therefore$ power of x must be zero.

i.e. $18 - 3r = 0 \Rightarrow r = 6$

put in (1)

$$T_{r+1} = \binom{9}{6} 2^{9-6} . x^0 = \frac{!9}{!6!3} 2^3 .1$$

$$= \frac{{}^3\cancel{9} . \cancel{8}^4 . 7.6!}{6! . 3 . \cancel{2} . 1} . 8 . 1 = 672$$

Exercise 2.1

1. Expand the following by the binomial formula.

(i) $\left(x+\frac{1}{x}\right)^4$ (ii) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^5$ (iii) $\left(\frac{x}{2}-\frac{2}{y}\right)^4$

(iv) $(2x-y)^5$ (v) $\left(2a-\frac{x}{a}\right)^7$ (vi) $\left(\frac{x}{y}-\frac{y}{x}\right)^4$

(vii) $(-x+y^{-1})^4$

2. Compute to two decimal places of decimal by use of binomial formula.

(i) $(1.02)^4$ (ii) $(0.98)^6$ (iii) $(2.03)^5$

3. Find the value of

(i) $(x+y)^5+(x-y)^5$ (ii) $(x+\sqrt{2})^4+(x-\sqrt{2})^4$

4. Expanding the following in ascending powers of x

(i) $(1-x+x^2)^4$ (ii) $(2+x-x^2)^4$

5. Find

(i) the 5^{th} term in the expansion of $\left(2x^2-\frac{3}{x}\right)^{10}$

(ii) the 6^{th} term in the expansion of $\left(x^2+\frac{y}{2}\right)^{15}$

(iii) the 8^{th} term in the expansion of $\left(\sqrt{x}+\frac{2}{\sqrt{x}}\right)^{12}$

(iv) the 7^{th} term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$

6. Find the middle term of the following expansions

(i) $\left(3x^2+\frac{1}{2x}\right)^{10}$ (ii) $\left(\frac{a}{2}-\frac{b}{3}\right)^{11}$ (iii) $\left(2x+\frac{1}{x}\right)^7$

7. Find the specified term in the expansion of

(i) $\left(2x^2-\frac{3}{x}\right)^{10}$: term involving x^5

(ii) $\left(2x^2 - \frac{1}{2x}\right)^{10}$: term involving x^5

(iii) $\left(x^3 + \frac{1}{x}\right)^7$: term involving x^9

(iv) $\left(\frac{x}{2} - \frac{4}{x}\right)^8$: term involving x^2

(v) $\left(\frac{p^2}{2} + 6q^2\right)^{12}$: term involving q^8

8. Find the coefficient of

(i) x^5 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{10}$

(ii) x^{20} in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^{16}$

(iii) x^5 in the expansion of $\left(2x^2 - \frac{1}{3x}\right)^{10}$

(iv) b^6 in the expansion of $\left(\frac{a^2}{2} + 2b^2\right)^{10}$

9. Find the constant term in the expansion of

(i) $\left(x^2 - \frac{1}{x}\right)^9$ (ii) $\left(\sqrt{x} + \frac{1}{3x^2}\right)^{10}$

10. Find the term independent of x in the expansion of the following

(i) $\left(2x^2 - \frac{1}{x}\right)^{12}$ (ii) $\left(2x^2 + \frac{1}{x}\right)^9$

Answers 2.1

1. (i) $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

(ii) $\frac{32}{243}x^5 - \frac{40}{27}x^3 + \frac{20}{3}x - \frac{15}{x} + \frac{135}{8x^3} - \frac{243}{32x^5}$

(iii) $\frac{x^4}{16} - \frac{x^3}{y} + \frac{6x^2}{y^2} - \frac{6x}{y^3} + \frac{16}{y^4}$

(iv) $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$

(v) $128a^7 - 448a^5x + 672a^3x^2 - 560ax^3 + 280\dfrac{x^4}{a} -$

$84\dfrac{x^5}{a^3} + 14\dfrac{x^6}{a^5} - \dfrac{x^7}{a^7}$

(vi) $\dfrac{x^8}{y^8} - 8\dfrac{x^6}{y^6} + 28\dfrac{x^2}{y^2} - 56\dfrac{x^2}{y^2} + 70 - 56\dfrac{y^2}{x^2} + 28\dfrac{y^4}{x^4} - 8\dfrac{y^6}{x^6} + \dfrac{y^8}{x^8}$

(vii) $x^4 - 4x^3y^{-1} + 6x^2y^{-2} - 4xy^{-3} + y^{-4}$

2. (i) 1.14 (ii) 0.88 (iii) 34.47
3. (i) $2x^5 + 20x^3y^2 + 10xy^4$ (ii) $2x^4 + 24x^2 + 8$
4. (i) $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$
   (ii) $16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$
5. (i) $1088640x^8$ (ii) $\dfrac{3003}{32}x^{20}y^5$ (iii) $\dfrac{101376}{x}$ (iv) $\dfrac{\mathbf{10500}}{\boldsymbol{x^3}}$
6. (i) $1913.625x^5$ (ii) $-\dfrac{77a^6b^5}{2592} + \dfrac{77a^5b^6}{3888}$ (iii) $\dfrac{280}{x} + 560x$
7. (i) $-1959552x^5$ (ii) $-252x^5$ (iii) $35x^9$ (iv) $-112x^2$
   (v) $\dfrac{880}{9}p^{16}q^8$
8. (i) -1959552 (ii) 46590 (iii) 33.185 (iv) $\dfrac{15}{2}a^{14}$
9. (i) 84 (ii) 5
10. (i) 7920 (ii) 672

2.7 Binomial Series

Since by the Binomial formula for positive integer n, we have

$$(a + b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots\ldots + b^n \quad \ldots\ldots\ldots (2)$$

put a = 1 and b = x, then the above form becomes:

$$(1 + x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + x^n$$

if n is –ve integer or a fractional number (-ve or +ve), then

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\infty \qquad (3).$$

The series on the R.H.S of equation (3) is called binomial series.
This series is valid only when x is numerically less than unity
i.e.,$|x| < 1$otherwise the expression will not be valid.

Note: The first term in the expression must be unity. For example, when n is not a positive integer (negative or fraction) to expand $(a + x)^n$, we shall have to write it as, $(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$ and then apply the binomial series, where $\left|\frac{x}{a}\right|$ must be less than 1.

2.8 Application of the Binomial Series; Approximations:

The binomial series can be used to find expression approximately equal to the given expressions under given conditions.

Example 1: If x is very small, so that its square and higher powers can be neglected then prove that

$$\frac{1+x}{1-x} = 1 + 2x$$

Solution:

$\frac{1+x}{1-x}$ this can be written as $(1 + x)(1 - x)^{-1}$

$= (1 + x)(1 + x + x^2 + \ldots\ldots\ldots$ higher powers of x)

$= 1 + x + x +$ neglecting higher powers of x.

$= 1 + 2x$

Example 2: Find to four places of decimal the value of $(1.02)^8$

Solution:

$$(1.02)^8 = (1 + 0.02)^8$$

$$= (1 + 0.02)^8$$

$$= 1 + \frac{8}{1}(0.02) + \frac{8.7}{2.1}(0.02)^2 + \frac{8.7.6}{3.2.1}(0.02)^3 + \ldots$$

$$= 1 + 0.16 + 0.0112 + 0.000448 + \ldots$$

$$= 1.1716$$

Example 3: Write and simplify the first four terms in the expansion of $(1 - 2x)^{-1}$.

Solution:

$(1 - 2x)^{-1}$

$= [1 + (-2x)]^{-1}$

Using $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \text{-----}$

$$= 1 + (-1)(-2x) + \frac{(-1)(-1-1)}{2!}(-2x)^2 + -----$$

$$\frac{(-1)(-1-1)(-1-2)}{3!}(-2x)^3 + -----$$

$$= 1 + 2x + \frac{(-1)(-2)}{2.1}4x^2 + \frac{(-1)(-2)(-3)}{3.2.1}(-8x^3) + -----$$

$$= 1 + 2x + 4x^2 + 8x^3 + ----$$

Example 4: Write the first three terms in the expansion of $(2 + x)^{-3}$

Solution :

$$(2+x)^{-3} = (2)^{-3}\left(1 + \frac{x}{2}\right)^{-3}$$

$$= (2)^{-3}\left[1 + (-3)\left(\frac{x}{2}\right) + \frac{(-3)(-3-1)}{2!}\left(\frac{x}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{8}\left[1 - \frac{3}{2}x + 3x^2 + ---\right]$$

Root Extraction:

The second application of the binomial series is that of finding the root of any quantity.

Example 5: Find square root of 24 correct to 5 places of decimals.
Solution:

$$\sqrt{24} = (25-1)^{1/2}$$

$$= (25)^{1/2}\left(1-\frac{1}{25}\right)^{1/2}$$

$$= 5\left(1-\frac{1}{5^2}\right)^{1/2}$$

$$= 5\left[1+\frac{1}{2}\left(-\frac{1}{5^2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{5^2}\right)^2 + --\right]$$

$$= 5\left[1-\frac{1}{2.5^2}-\frac{1}{2^3.5^4}-\frac{1}{2^4.5^6}- ----\right]$$

$$= 5\,[1-(0.02 + 0.0002 + 0.000004 + ---)]$$

$= 4.89898$

Example 6: evaluate $\sqrt[3]{29}$ to the nearest hundredth.
Solution :

$$\sqrt[3]{29} = (27+2)^{1/3} = \left[27\left(1+\frac{2}{27}\right)\right]^{1/3} = 3\left[1+\frac{2}{27}\right]^{1/3} + \ldots\ldots$$

$$= 3\left[1+\frac{1}{3}\left(\frac{2}{27}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}\left(\frac{2}{27}\right)^2+\ldots\ldots\ldots\right]$$

$$= 3\left[1+\frac{2}{81}+\frac{1}{2}\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(\frac{2}{27}\right)^2+\ldots\ldots\ldots\right]$$

$$= 3\,[1+0.0247-0.0006\ldots\ldots\ldots\ldots\ldots]$$

$$= 3\,[1.0212] = 3.07$$

Exercise 2.2

Q1: Expand upto four terms.

(i) $(1-3x)^{1/3}$ (ii) $(1-2x)^{-3/4}$ (iii) $(1+x)^{-3}$

(iv) $\dfrac{1}{\sqrt{1+x}}$ (v) $(4+x)^{1/2}$ (vi) $(2+x)^{-3}$

Q2: Using the binomial expansion, calculate to the nearest hundredth.

(i) $\sqrt[4]{65}$ (ii) $\sqrt{17}$ (iii) $(1.01)^{-7}$

(iv) $\sqrt{28}$ (v) $\sqrt{40}$ (vi) $\sqrt{80}$

Q3: Find the coefficient of x^5 in the expansion of

(i) $\dfrac{(1+x)^2}{(1-x)^2}$ (ii) $\dfrac{(1+x)^2}{(1-x)^3}$

Q4: If x is nearly equal to unity, prove that

$$\frac{mx^n - n\,x^m}{x^n - x^m} = \frac{1}{1-x}$$

Answers 2.2

Q1: (i) $1-x\ -x\ -\dfrac{5}{3}x^3 + ----$ (ii) $1+\dfrac{3}{2}x+\dfrac{21}{8}x^2+\dfrac{77}{16}x^3+----$

(iii) $1-3x+6x^2-10^3----$ (iv) $1-\dfrac{1}{2}x+\dfrac{3}{8}x^2-\dfrac{5}{16}x^3+----$

(v) $2+\frac{x}{2}-\frac{x^2}{64}+\frac{x^3}{512}+----$ (vi) $\frac{1}{8}\left[1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{4}x^3\right]$

Q2:	(i) 2.84	(ii) 4.12	(iii) 0.93
	(iv) 5.29	(v) 6.32	(vi) 8.94
Q3:	(i) 20	(ii) 61	

Summary

Binomial Theorem

An expression consisting of two terms only is called a binomial expression. If n is a positive index, then

1. The general term in the binomial expansion is $T_{r-1} = {}^nC_r a^{n-r} b^r$
2. The number of terms in the expansion of $(a + b)^n$ is n + 1.
3. The sum of the binomial coefficients in the expansion of $(a + b)^n$ is 2^n. i.e. ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$
4. The sum of the even terms in the expansion of $(a + b)^n$ is equal to the sum of odd terms.
5. When n is even, then the only middle term is the $\left(\frac{n+2}{2}\right)$th term.
6. When n is odd, then there are two middle terms viz $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms.

Note: If n is not a positive index.

i.e. $(a + b)^n = a^n \left(1+\frac{n}{a}\right)^n$

$$= a^n\left[1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+--------\right]$$

1. Here n is a negative or a fraction, the quantities ${}^nC_1,\ {}^nC_2$ ------- here no meaning at all. Hence co-efficients can not be represented as ${}^nC_1,\ {}^nC_2$---------.
2. The number of terms in the expansion is infinite as n is a negative or fraction.

Short Questions

Write the short answers of the following

Expand by Bi-nomial theorem Q.No. 1 to 4

Q.1 $(2x - 3y)^4$ Q.2 $\left(\frac{x}{y} + \frac{y}{x}\right)^4$

Q.3 $\left(\frac{x}{2} - \frac{2}{y}\right)^4$ **Q.4** $\left(x + \frac{1}{x}\right)^4$

Q5 State Bi-nomial Theorem for positive integer n

Q.6 State Bi-nomial Theorem for n negative and rational.

Calculate the following by Binomial Theorem up to two decimal places.

Q.7 $(1.02)^{10}$ **Q.8** $(1.04)^5$

Q.9 Find the 7^{th} term in the expansion of $\left(x - \frac{1}{x}\right)^9$

Q.10 Find the 6^{th} term in the expansion of $(x + 3y)^{10}$

Q.11 Find 5^{th} term in the expansion of $(2x - \frac{x^2}{4})^7$

Expand to three term

Q.12 $(1 + 2x)^{-2}$ **Q.13** $\frac{1}{(1 + x)^2}$

Q.14 $\frac{1}{\sqrt{1+x}}$ **Q.15** $(4 - 3x)^{1/2}$

Q.16 Using the Binomial series calculate $\sqrt[3]{65}$ to the nearest hundredth.

Which will be the middle term/terms in the expansion of

Q.17 $(2x+3)^{12}$ **Q. 18** $\left(x+\frac{3}{x}\right)^{15}$?

Q19 Which term is the middle term or terms in the Binomial expansion of $(a+b)^n$

(i) When n is even (ii) When n is odd

Answers

Q1. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

Q.2 $\frac{x^4}{y^4} + 4\frac{x^2}{y^2} + 6 + 4\frac{y^2}{x^2} + 4\frac{y^4}{x^4}$ **Q.3** $\frac{x^4}{16} - \frac{x^3}{y} + \frac{6x^2}{y^2} - \frac{16x}{y^3} + \frac{16}{y^4}$

Q.4 $(x)^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$ **Q.7** 1.22 **Q.** 8 1.22

Q.9 $\frac{84}{x^3}$ Q.10 $61236\,x^5y^5$ **Q.11** $\frac{35}{32}x^{11}$

Q.12 $1 - 4x + 12x^2 + \ldots\ldots\ldots$ **Q.13** $1 - 2x + 3x^2 + \ldots\ldots\ldots$

Q.14 $1 - \frac{x}{2} + \frac{3}{8}x^2 + \ldots\ldots\ldots$ **Q.15** $2 - \frac{3x}{4} - \frac{9x^2}{64} + \ldots\ldots\ldots$

Q.16 4.02 **Q.17** $T_7 = \binom{12}{6}(2x)^6(3)^6$

Q.18 $T_8 = \binom{15}{7}(3)^7x$ and $T_9 = \binom{15}{8}\frac{(3)^8}{x}$

Q.19 (i) $\left(\frac{n}{2}+1\right)$ (ii) $\left(\frac{n+1}{2}\right)$ and $\left(\frac{n+1}{2}+1\right)$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Third term of $(x + y)^4$ is:

(a) $4x^2y^2$ (b) $4x^3y$ c) $6x^2y^2$ (d) $6x^3y$

__2. The number of terms in the expansion $(a + b)^{13}$ are:

(a) 12 (b) 13 (c) 14 (d) 15

__3. The value of $\binom{n}{r}$ is:

(a) $\dfrac{n!}{r!(n-r)!}$ (b) $\dfrac{n}{r(n-r)}$ (c) $\dfrac{n!}{r!(n-r)}$ (d) $\dfrac{n!}{(n-r)!}$

__4. The second last term in the expansion of $(a + b)^7$ is:

(a) $7a^6b$ (b) $7ab^6$
(c) $7b^7$ (d) 15

__5. $\binom{6}{4}$ will have the value:

(a) 10 (b) 15 (c) 20 (d) 25

__6. $\binom{3}{0}$ will have the value:

(a) 0 (b) 1 (c) 2 (d) 3

__7. In the expansion of $(a + b)^n$ the general term is:

(a) $\binom{n}{r}a^rb^r$ (b) $\binom{n}{r}a^{n-r}b^r$

(c) $\binom{n}{r-1}a^{n-r+1}b^{r-1}$ (d) $\binom{n}{r}a^{n-r-1}b^{r-1}$

__8. In the expansion of $(a + b)^n$ the term $\binom{n}{r}a^{n-r}b^r$ will be:

(a) nth term (b) rth term
(c) (r + 1)th term (d) None of these

__9. In the expansion of $(a + b)^n$ the rth term is:

(a) ${}^nC_ra^rb^r$ (b) ${}^nC_ra^{n-r}b^r$

(c) $^{n}C_{r}a^{n-r+1}b^{r-1}$ (d) $^{n}C_{r}a^{n-r-1}b^{r-1}$

__10. In the expansion of $(1 + x)^n$ the co-efficient of 3^{rd} term is:

(a) $\binom{n}{0}$ (b) $\binom{n}{1}$ (c) $\binom{n}{2}$ (d) $\binom{n}{3}$

__11. In the expansion of $(a + b)^n$ the sum of the exponents of a and b in any term is:

(a) n (b) n – 1 (c) n + 1 (d) None of these

__12. The middle term in the expansion of $(a + b)^6$ is:

(a) $15a^4b^2$ (b) $20a^3b^3$ (c) $15a^2b^4$ (d) $6ab^5$

__13. The value of $\binom{n}{n}$ is equal to:

(a) Zero (b) 1 (c) n (d) -n

__14. The expansion of $(1 + x)^{-1}$ is:

(a) $1 - x - x^2 - x^3 + \ldots$

(b) $1 - x + x^2 - x^3 + \ldots$

(c) $1 - \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

(d) $1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots$

__15. The expansion of $(1 - x)^{-1}$ is:

(a) $1 + x + x^2 + x^3 + \ldots$

(b) $1 - x + x^2 - x^3 + \ldots$

(c) $1 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

(d) $1 - \frac{1}{1!}x + \frac{1}{2!}x^3 - \frac{1}{3!}x^3 + \ldots$

__16. Binomial series for $(1 + x)^n$ is valid only when:

(a) $x < 1$ (b) $x < -1$

(c) $|x| < 1$ (d) None of these

__17. The value of $\binom{2n}{n}$ is:

(a) $\frac{2n}{n!\,n!}$ (b) $\frac{(2n)!}{n!\,n!}$

(c) $\dfrac{(2n)!}{n!}$ (d) $\dfrac{(2n)!}{n(n-1!)}$

__18. The middle term of $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)^4$ is:

(a) $\dfrac{4x^2}{y^2}$ (b) 6 (c) 8 (d) $\dfrac{4x}{y}$

Answers

1.	c	2.	c	3.	a	4.	b	5.	b
6.	b	7.	b	8.	c	9.	c	10.	c
11.	a	12.	b	13.	b	14.	b	15.	a
16.	b	17.	d	18.	c	19.	b	20.	b

Chapter 3
Fundamentals of Trigonometry

3.1 Introduction:

The word “trigonometry” is a Greek word. Its mean “measurement of a triangle”. Therefore trigonometry is that branch of mathematics concerned with the measurement of sides and angle of a plane triangle and the investigations of the various relations which exist among them. Today the subject of trigonometry also includes another distinct branch which concerns itself with properties relations between and behavior of trigonometric functions.

The importance of trigonometry will be immediately realized when its applications in solving problem of mensuration, mechanics physics, surveying and astronomy are encountered.

3.2 Types of Trigonometry:

There are two types of trigonometry

(1) Plane Trigonometry (2) Spherical Trigonometry

1. Plane Trigonometry

Plane trigonometry is concerned with angles, triangles and other figures which lie in a plane.

2. Spherical Trigonometry

Spherical Trigonometry is concerned with the spherical triangles, that is, triangles lies on a sphere and sides of which are circular arcs.

3.3 Angle:

An angle is defined as the union of two non-collinear rays which have a common end-points.

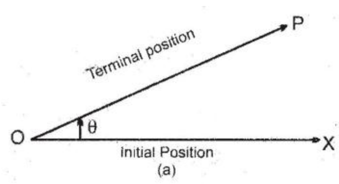


Fig. 4.1

An angle is also defined as it measures the rotation of a line from one position to another about a fixed point on it.

In figure 5.1(a)the first position OX is called initial line (position)and second position OP is called terminal line or generating line(position) of $\angle XOP$.

If the terminal side resolves in anticlockwise direction the angle described is positive as shown in figure (i)

If terminal side resolves in clockwise direction, the angle described is negative as shown in figure (ii)

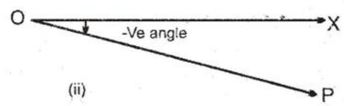


Fig. 4.3

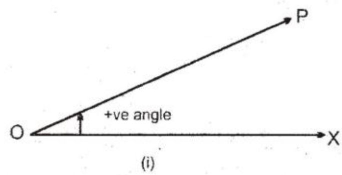


Fig. 4.2

3.4 Quadrants:

Two mutually perpendiculars straight lines xox and yoy divide the plane into four equal parts, each part is called quadrant.

Thus XOY, X′OY, X′OY′ and XOY′ are called the Ist, IInd, IIIrd and IVth quadrants respectively.

In first quadrant the angle vary from 0^o to 90^o in anti-clockwise direction and from – 270^o to – 360^o in clockwise direction.

In second quadrant the angle vary from 90^o to 180^o in anti-clockwise direction and – 180^o to – 270^o in clockwise direction.

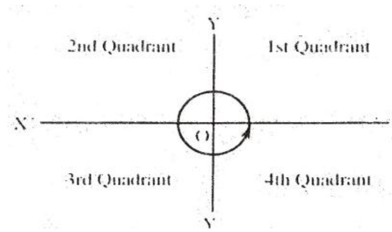


Fig. 4.4

In third quadrant the angle vary from 180^o to 270^o in anti-clockwise direction and from – 90^o to – 180^o in clockwise direction.

In fourth quadrant the angle vary from 270^o to 360^o in anti-clockwise direction and from – 0^o to – 90^o in clockwise direction.

3.5 Measurement of Angles:

The size of any angle is determined by the amount of rotations. In trigonometry two systems of measuring angles are used.

(i) Sexagesimal or English system (Degree)

(ii) Circular measure system (Radian)

(i) Sexagesimal or English System (Degree)

The sexagesimal system is older and is more commonly used. The name derive from the Latin for “sixty”. The fundamental unit of angle measure in the sexagesimal system is the degree of arc. By definition, when a circle is divided into 360 equal parts, then

One degree = $\frac{1}{360}$ th part of a circle.

Therefore, one full circle = 360 degrees.

The symbol of degrees is denoted by $(\ )^0$.

Thus an angle of 20 degrees may be written as 20^o.
Since there are four right angles in a complete circle .

One right angle = $\frac{1}{4}$ circle = $\frac{1}{4}(360^o) = 90^o$

The degree is further subdivided in two ways, depending upon whether we work in the common sexagesimal system or the decimal sexagesimal system. In the common sexagesimal system, the degree is subdivided into 60 equal parts, called minutes, denoted by the symbol$(\ )'$, and the minute is further subdivided into 60 equal parts, called second, indicated by the symbol $(\ )''$. Therefore

1 minute = 60 seconds
1 degree = 60 minutes = 3600 seconds
1 circle = 360 degrees = 21600 minutes = 12,96,000 sec.

In the decimal sexagesimal system, angles smaller than 1^o are expressed as decimal fractions of a degree. Thus one-tenth $\left(\frac{1}{10}\right)$ of a degree is expressed as 0.1^o in the decimal sexagesimal system and as $6'$ in the common sexagesimal system; one-hundredth $\left(\frac{1}{100}\right)$ of a degree is 0.01^o in the decimal system and 36" in the common system; and $47\frac{1}{9}$ degrees comes out $(47.111\ \ldots)^o$ in the decimal system and $47^o6'40''$ in the common system.

(ii) Circular measure system (Radian)

This system is comparatively recent. The unit used in this system is called a Radian. The Radian is define "The angle subtended at the centre of a circle by an arc equal in length to the radius of the circle." As shown in fig., Arc AB is equal in length to the radius $\overline{OB}$ of the circle. The subtended, $\angle AOB$ is then one radian.

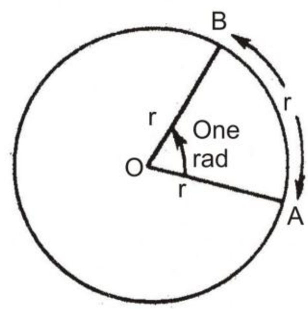


Fig. 4.5

i.e. m $\angle AOB$ = 1 radian.

3.6 Relation between Degree and Radian Measure:

Consider a circle of radius r, then the circumference of the circle is $2\pi r$. By definition of radian,

An arc of length ′r′ subtends an angle = 1 radian

$\therefore$ An arc of length $2\pi r$ subtends an angle = 2π radian

Also an arc of length $2\pi r$ subtends an angle = 360^o

Then 2π radians = 360^o

Or π radians = 180^o

$$1 \text{ radians} = \frac{180^o}{\pi}$$

$$1 \text{ radians} = \frac{180}{3.1416}$$

Or 1 radians = 57.3^o

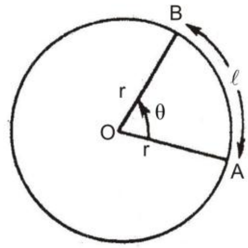


Fig. 4.6

Therefore to convert radians into degree,

we multiply the number of radians by $\frac{180^o}{\pi}$ or 57.3.

Now Again, $360^o = 2\pi$ radians

$$1^o = \frac{2\pi r}{360^o} \text{ radians}$$

$$1^o = \frac{\pi}{180^o}$$

Or $1^o = \frac{3.1416}{180^o}$

1^o = 0.01745 radians

Therefore, to convert degree into radians, we multiply the number of degrees by $\frac{\pi}{180}$ or 0.0175.

Note: One complete revolution = $360^o = 2\pi$ radius.

3.7 Relation between Length of a Circular Arc and the adian Measure of its Central Angle:

Let "l" be the length of a circular arc $\overline{AB}$ of a circle of radius r, and θ be its central angle measure in radians. Then the ratio of l to the circumference $2\pi r$ of the circle is the same as the ratio of θ to 2π.

Therefore

$$l : 2\pi r = \theta : 2\pi$$

Or $\frac{l}{2\pi r} = \frac{\theta}{2\pi}$

$$\frac{l}{r} = \theta$$

$l = \theta r$, where θ is in radian

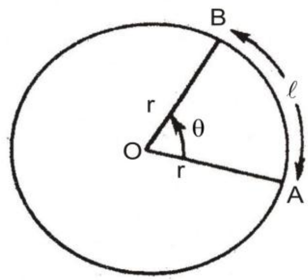


Fig. 4.7

Note:

If the angle will be given in degree measure we have to convert it into Radian measure before applying the formula.

Example 1:

Convert 120^o into Radian Measure.

Solution:

120^o

$$120^o = 120 \text{ x } \frac{\pi}{180}$$

$$= \frac{2\pi}{3} = \frac{2(3.1416)}{3} = 2.09 \text{ rad.}$$

Example 2:

Convert $37^o\ 25'\ 38''$ into Radian measure.

Solution:

$$37^o\ 25'\ 28'' = 37^o + \frac{25}{60} + \frac{28}{3600}$$

$$= 37^o + \frac{5^o}{12} + \frac{7^o}{900}$$

$$= 37^o + \frac{382^o}{900}$$

$$= 37 + \frac{181^o}{450} = \frac{16831^o}{450} = \frac{16831}{450} \text{ x } \frac{\pi}{180}$$

$$= \frac{16831(3.14160)}{81000} = \frac{52876.26}{81000}$$

$$37^o\ 25'\ 28'' = 0.65 \text{ radians}$$

Example 3:

Express in Degrees:

(i) $\frac{5\pi}{3}$ rad (ii) 2.5793 rad (iii) $\frac{\pi}{6}$ rad (iv) $\frac{\pi}{3}$ rad

Solution:

(i) Since 1 rad $= \frac{180}{\pi}$ deg

$\therefore \frac{5\pi}{3}$ rad $= \frac{5\pi}{3} \text{ x } \frac{180}{\pi}$ deg = 5(60) deg = 300^o

(ii) 2.5793 rad = 2.5793 (57^o .29578) $\therefore$ 1 rad = 57^o .295778

= 147^o .78301 = 147.78 (two decimal places)

(iii) $\frac{\pi}{6}$ rad $= \frac{\pi}{6} \text{ x } \frac{12}{\pi}$ deg = 30^o

(iv) $\frac{\pi}{3}$ rad $= \frac{\pi}{3} \text{ x } \frac{12}{\pi}$ deg = 60^o

Example 4:

What is the length of an arc of a circle of radius 5 cm. whose central angle is of 140^o?

Solution: l = length of an arc = ?

r = radius = 5 cm

θ = 140^o

Since 1 deg = 0.01745 rad

$\therefore$ θ = 140 x 0.01745 rad = 2.443 rad

$\therefore$ l = $r\theta$

l = (5)(2.443) = 12.215 cm

Example 5:

A curve on a highway is laid out an arc of a circle of radius 620m. How long is the arc that subtends a central angle of 32^o?

Solution: r = 620m l = ? $\theta = 32^o = 32 \text{ x } \frac{\pi}{180}$ rad

$l = 620 \text{ x } 32 \text{ x } \frac{\pi}{180}$ = 346.41 m

Example 6:

A railway Train is traveling on a curve of half a kilometer radius at the rate of 20 km per hour through what angle had it turned in 10 seconds?

Solution:

Radius = r = $\frac{1}{2}$km, θ = ?

We know s = vt

v = velocity of Train = 20km/hour = $\frac{20}{3600}$ km/sec.

$v = \frac{1}{180}$ km/sec

l = Distance traveled by train in 10 seconds = $\frac{1}{180}$ x10 km/sec

$l = \frac{1}{18}$

Since $l = r\theta$

$\Rightarrow$ $\frac{1}{18} = \frac{1}{2}\theta$

$$\theta = \frac{2}{18} = \frac{1}{9}\text{rad}$$

Example7

The moon subtends an angle of 0.5^o as observed from the Earth. Its distance from the earth is 384400 km. Find the length of the diameter of the Moon.

Solution:

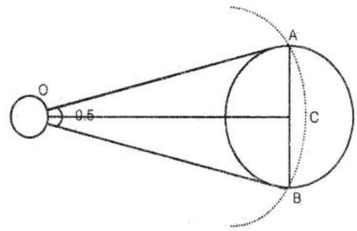


Fig. 4.8

l = AB = diameter of the Moon = ?

as angle 0.5 is very small.

i.e. AB (arc length) consider as a straight line AB

θ = $0.5^o = 0.5 \times 0.01745$ rad = 0.008725 rad

r = OC = d = distance between the earth and the moon

r = OC = 3844000 km

Since $l = r\theta$

l= 384400 x 0.008725 = 3353.89 km

Exercise 3.1

Q1. Convert the following to Radian measure

(i) 210^o (ii) 540^o (iii) $42^o\ 36'\ 12''$ (iv) $24^o32'30''$

Q2. Convert the following to degree measure:

(i) $\frac{5\pi}{4}$ rad (ii) $\frac{2\pi}{3}$ rad (iii) 5.52 rad (iv) 1.30 rad

Q3. Find the missing element l, r, θ when:

(i) $l = 8.4$ cm, $\theta = 2.8$ rad

(ii) $l = 12.2$ cm, r = 5cm

(iii) r = 620m, $\theta = 32^o$

Q4. How far a part are two cities on the equator whose longitudes are 10^oE and 50^oW? (Radius of the Earth is 6400km)

Q5. A space man land on the moon and observes that the Earth´s diameter subtends an angle of $1^o\ 54'$ at his place of landing. If the Earth´s radius is 6400km, find the distance between the Earth and the Moon.

Q6. The sun is about 1.496×10^8 km away from the Earth. If the angle subtended by the sun on the surface of the earth is 9.3×10^{-3} radians approximately. What is the diameter of the sun?

Q7. A horse moves in a circle, at one end of a rope 27cm long, the other end being fixed. How far does the horse move when the rope traces an angle of 70^o at the centre.

Q8. Lahore is 68km from Gujranwala. Find the angle subtended at the centre of the earth by the road. Joining these two cities, earth being regarded as a sphere of 6400km radius.

Q9. A circular wire of radius 6 cm is cut straightened and then bend so as to lie along the circumference of a hoop of radius 24 cm. find measure of the angle which it subtend at the centre of the hoop

Q10. A pendulum 5 meters long swings through an angle of 4.5^0. through what distance does the bob moves ?

Q11. A flywheel rotates at 300 rev/min. If the radius is 6 cm. through what total distance does a point on the rim travel in 30 seconds ?

Answers 3.1

Q1 (i) 3.66 rad (ii) 3π (iii) 0.74 rad (iv) 0.42 rad

Q2. (i) 225^o (ii) 120^o (iii) $316^o\,16'\,19''$ (iv) $74^o\,29'\,4''$

Q3. (i) r = 3cm (ii) θ = 2.443 rad (iii) l = 346.4 meters

Q4. 6704.76 km Q 5. 386240 km

Q6. 1.39×10^6 km Q7. 33 m Q8. $36'\,43''$ Q9. $\pi/2$ or 90^0

Q10. 0.39 m Q11. 5657 cm

3.8 Trigonometric Function and Ratios:

Let the initial line OX revolves and trace out an angle θ. Take a point P on the final line. Draw perpendicular PM from P on OX:

$\angle XOP = \theta$, where θ may be in degree or radians.

Now OMP is a right angled triangle,

We can form the six ratios as follows:

$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{b}, \frac{c}{a}$

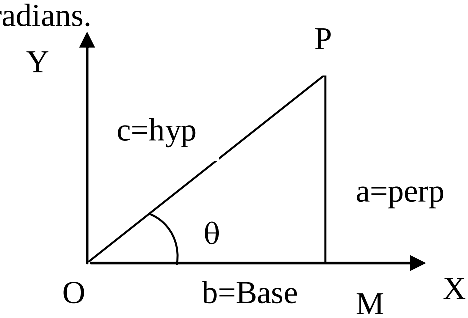


Fig. 3.10

In fact these ratios depend only on the size of the angle and not on the triangle formed. Therefore these ratios called **Trigonometric ratios or trigonometric functions** of angle θ

and defined as below: θ

$$\textbf{Sin } \theta = \frac{a}{c} = \frac{MP}{OP} = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$$

$$\text{Cos } \theta = \frac{b}{c} = \frac{OM}{OP} = \frac{\text{Base}}{\text{Hypotenuse}}$$

$$\tan \theta = \frac{a}{b} = \frac{MP}{OM} = \frac{\text{Perpendicular}}{\text{Base}}$$

$$\text{Cot } \theta = \frac{b}{a} = \frac{OM}{MP} = \frac{\text{Base}}{\text{Perpendicular}}$$

$$\text{Sec } \theta = \frac{c}{b} = \frac{OP}{OM} = \frac{\text{Hypotenuse}}{\text{Perpendicular}}$$

$$\text{Cosec } \theta = \frac{c}{a} = \frac{OP}{PM} = \frac{\text{Hyponenuse}}{\text{Perpendicular}}$$

3.9 Reciprocal Functions:

From the above definition of trigonometric functions, we observe that

(i) $\text{Sin } \theta = \dfrac{1}{\text{Cosec } \theta}$ or, $\text{Cosec } \theta = \dfrac{1}{\text{Sin } \theta}$ i.e. Sin θ and Cosec θ are reciprocal of each other.

(ii) $\text{Cos } \theta = \dfrac{1}{\text{Sec } \theta}$ or, $\text{Sec } \theta = \dfrac{1}{\text{Cos } \theta}$ i.e. Cos θ and Sec θ are reciprocals of each other.

(iii) $\tan \theta = \dfrac{1}{\text{Cot } \theta}$ or, $\text{Cot } \theta = \dfrac{1}{\tan \theta}$ i.e. tan θ and Cot θ are reciprocals of each other.

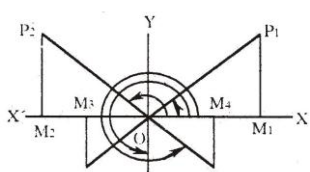
We can also see that;

$$\tan \theta = \frac{\text{Sin } \theta}{\text{Cos } \theta} \quad \text{and} \quad \text{Cot } \theta = \frac{\text{Cos } \theta}{\text{Sin } \theta}$$

3.10 Rectangular Co-ordinates and Sign Convention:

In plane geometry the position of a point can be fixed by measuring its perpendicular distance from each of two perpendicular called co-ordinate axes. The horizontal line (x-axis) is also called abscissa and the vertical line(y-axis) is called as ordinate.

Distance measured from the point O in the direction OX and OY are regarded as positive, while in the direction of OX′ and OY′ are



considered negative.

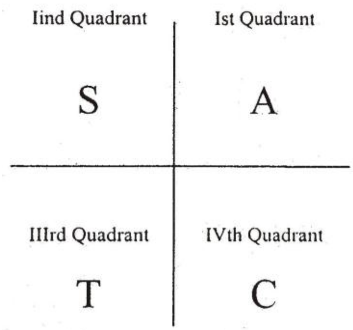
Thus in the given figure OM_1, OM_4, MP_1 and M_2P_2 are positive, while OM_2, OM_3, M_3P_3 and M_4P_4 are negative.

The terminal line i.e., OP_1, OP_2, OP_3, and OP_4 are positive in all the quadrants.

Fig. 3.11

3.11 Signs of Trigonometric Functions:

The trigonometric ratios discussed above have different signs in different quadrants. Also from the above discussion we see that OM and MP changes their sign in different quadrants. We can remember the sign of trigonometric function by “ACTS” Rule or CAST rule. In “CAST” C stands for cosine A stands for All and S stands for Sine and T stands for Tangent.



First Quadrant:

In first quadrant sign of all the trigonometric functions are positive i.e., sin, cos, tan, Cot, Sec, Cosec all are positive.

Second Quadrant:

In second quadrant sine and its inverse cosec are positive. The remaining four trigonometric function i.e., cos, tan, cot, sec are negative.

Third Quadrant:

In third quadrant tan and its reciprocal cot are positive the remaining four function i.e., Sin, cos, sec and cosec are negative.

Fourth Quadrant:

In fourth quadrant cos and its reciprocal sec are positive, the remaining four functions i.e., sin, tan, cot and cosec are negative.

3.12 Trigonometric Ratios of Particular Angles:

1. Trigonometric Ratios of 30^o or $\dfrac{\pi}{6}$:

Let the initial line OX revolve and trace out an angle of 30^o. Take a point P on the final line. Draw PQ perpendicular from P on OX. In 30^o right angled triangle, the side opposite to the 30^o angle is one-half the length of the hypotenuse, i.e., if PQ = 1 unit then OP will be 2 units.

From fig. OPQ is a right angled triangle

$\therefore$ By Pythagorean theorem, we have

$$(OP)^2 = (OQ)^2 + (PQ)^2$$
$$(2)^2 = (OQ)^2 + (1)^2$$
$$4 = (OQ)^2 + 1$$

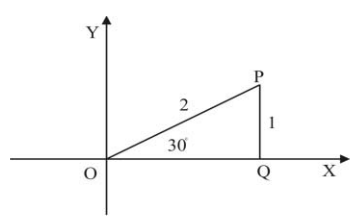


Fig. 4.12

$(OQ)^2 = 3$

$(OQ) = \sqrt{3}$

Fig. 3.12

Therefore $\quad \text{Sin } 30^o = \dfrac{\text{Prep.}}{\text{Hyp}} = \dfrac{PQ}{OP} = \dfrac{1}{2}$

$$\text{Cos } 30^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OQ}{OP} = \frac{\sqrt{3}}{2}$$

$$\tan 30^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PQ}{OQ} = \frac{1}{\sqrt{3}}$$

$$\text{Cot } 30^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OQ}{PQ} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\text{Sec } 30^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OQ} = \frac{2}{\sqrt{3}}$$

$$\text{Cosec } 30^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PQ} = \frac{2}{1} = 2$$

2. Trigonometric ratios of 45^o Or $\frac{\pi}{4}$

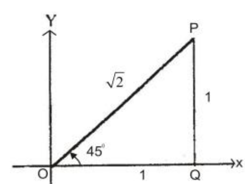


Fig. 4.13

Let the initial line OX revolve and trace out an angle of 45^o. Take a point P on the final line. Draw PQ perpendicular from P on OX . In 45^o right angled triangle the length of the perpendicular is equal to the length of the base

i.e., if PQ = 1 unit. then OQ = 1 unit

From figure by Pythagorean theorem.

$(OP)^2 = (OQ)^2 + (PQ)^2$

$(OP)^2 = (1)^2 + (1)^2 = 1 + 1 = 2$

$OP = \sqrt{2}$

Therefore $\quad \text{Sin } 45^o = \dfrac{\text{Prep.}}{\text{Hyp}} = \dfrac{PQ}{OP} = \dfrac{1}{\sqrt{2}}$

$$\text{Cos } 45^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OQ}{OP} = \frac{1}{\sqrt{2}}$$

$$\tan 45^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PQ}{OQ} = \frac{1}{1} = 1$$

$$\text{Cot } 45^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OQ}{PQ} = \frac{1}{1} = 1$$

$$\text{Sec } 45^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OQ} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\text{Cosec } 45^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PQ} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

3. Trigonometric Ratios of 60^o or $\frac{\pi}{3}$

Let the initial line OX revolve and trace out an angle of 60^o. Take a point P on the final line. Draw PQ perpendicular from P on OX. In 60^o right angle triangle the length of the base is one-half of the Hypotenuse.

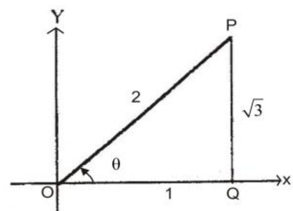


Fig. 4.14

i.e., OQ = Base = 1 unit

then, OP= Hyp = 2 units

from figure by Pythagorean Theorem:

$(OP)^2 = (OQ)^2 + (PQ)^2$

$(2)^2 = (1)^2 + (PQ)^2$

$4 = 1 (PQ)^2$

$(PQ)^2 = 3$

$PQ = \sqrt{3}$

Therefore $\text{Sin } 60^o = \frac{\text{Prep.}}{\text{Hyp}} = \frac{PQ}{OP} = \frac{\sqrt{3}}{2}$

$$\text{Cos } 60^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OQ}{OP} = \frac{1}{2}$$

$$\tan 60^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PQ}{OQ} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\text{Cot } 60^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OQ}{PQ} = \frac{1}{\sqrt{3}}$$

$$\text{Sec } 60^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OQ} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\text{Cosec } 60^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PQ} = \frac{2}{\sqrt{3}} =$$

Trigonometric ratios of 0°

Let the initial line revolve and trace out a small angle nearly equal to zero 0^o. Take a point P on the final line. Draw PM perpen-dicular on OX.

PM = 0

and OP = 1, OM = 1

(Because they just coincide x-axis)

Therefore from figure.

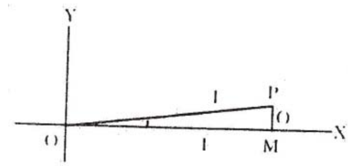


Fig. 4.15

$$\text{Sin } 0^o = \frac{\text{Prep.}}{\text{Hyp}} = \frac{PM}{OP} = \frac{0}{1} = 0$$

$$\text{Cos } 0^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OM}{OP} = \frac{1}{1} = 1$$

$$\tan 0^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PM}{OM} = \frac{0}{1} = 0$$

$$\text{Cot } 0^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OM}{PM} = \frac{1}{0} = \infty$$

$$\text{Sec } 0^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OM} = \frac{1}{1} = 1$$

$$\text{Cosec } 0^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PM} = \frac{1}{0} = \infty$$

Trigonometric Ratio of 90°

Let initial line revolve and trace out an angle nearly equal to 90^o. Take a point P on the final line. Draw PQ perpendicular from P on OX.

OQ = 0, OP = 1, PQ = 1 (Because they just coincide y-axis).

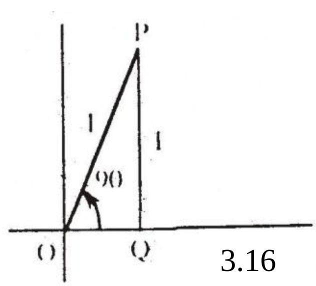
Therefore $\text{Sin } 90^o = \frac{\text{Prep.}}{\text{Hyp}} = \frac{PQ}{OP} = \frac{1}{1} = 1$

$$\text{Cos } 90^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OQ}{OP} = \frac{0}{1} = 0$$

$$\tan 90^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PQ}{OQ} = \frac{1}{0} = \infty$$

$$\text{Cot } 90^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OQ}{PQ} = \frac{0}{1} = 0$$

$$\text{Sec } 90^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OQ} = \frac{1}{0} = \infty$$



3.16

Fig. 4.16

$$\text{Cosec } 90^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PQ} = \frac{1}{1} = 1$$

Table for Trigonometrical Ratios of Special angle

$\frac{\text{Angles}}{\text{Ratios}}$	0^o	30^o Or $\frac{\pi}{6}$	45^o Or $\frac{\pi}{4}$	60^o Or $\frac{\pi}{3}$	90^o Or $\frac{\pi}{2}$
Sin θ	$\sqrt{\frac{0}{4}} = 0$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$	$\sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}} = 1$
Cos θ	$\sqrt{\frac{4}{4}} = 1$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$	$\sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$	$\sqrt{\frac{0}{4}} = 0$
Tan θ	$\sqrt{\frac{0}{4}} = 0$	$\sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{2}} = 1$	$\sqrt{\frac{3}{1}} = \sqrt{3}$	$\sqrt{\frac{4}{0}} = \propto$

Example 1:

If $\cos\theta = \frac{5}{13}$ and the terminal side of the angle lies in the first quadrant find the values of the other five trigonometric ratio of θ.

Solution:

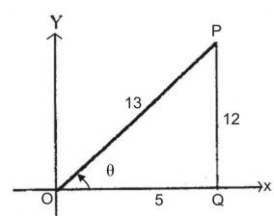


Fig. 4.17

In this cause $\cos\theta = \frac{5}{13}$

$$\cos\theta = \frac{\text{Base}}{\text{Hyp}} = \frac{5}{13}$$

From Fig. $(OP)^2 = (OQ)^2 + (PQ)^2$

$(13)^2 = (5)^2 + (PQ)^2$

$169 = 25 + (PQ)^2$

$(PQ)^2 = 169 - 25$

$= 144$

$PQ = \pm 12$

Because θ lies in the first quadrant

i.e., $\sin\theta = \frac{12}{13}$ $\because$ All the trigonometric ratios will be positive.

$\text{Cos }\theta = \frac{5}{13}$

$\tan\theta = \frac{12}{5}$, $\sec\theta = \frac{13}{5}$

$$\cot\theta = \frac{5}{12}, \qquad \operatorname{cosec}\theta = \frac{13}{12}$$

Example 2:

Prove that $\cos 90^o - \cos 30^o = -2\sin 60^o \sin 30^o$

Solution:

$$\text{L.H.S} = \cos 90^o - \cos 30^o$$

$$= 0 - \frac{\sqrt{3}}{2}$$

$$\text{L.H.S} = -\frac{\sqrt{3}}{2}$$

$$\text{R.H.S} = -2\sin 60^o \sin 30^o$$

$$= -2.\frac{\sqrt{3}}{2} \text{ x } \frac{1}{2}$$

$$\text{R.H.S} = -\frac{\sqrt{3}}{2}$$

Hence L.H.S = R.H.S

Example 3:

Verify that $\sin^2 30^o + \sin^2 60^o + \tan^2 45^o = 2$

Solution:

$$\text{L.H.S} = \sin^2 30^o + \sin^2 60^o + \tan^2 45^o$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (1)^2$$

$$= \frac{1}{4} + \frac{3}{4} + 1$$

$$= \frac{1+3+4}{4}$$

$$= \frac{8}{4}$$

$$\text{L.H.S} = 2 = \text{R.H.S}$$

Exercise 3.2

Q.1 If $\sin\theta = \frac{2}{3}$, and the terminal side of the angle lies in the second quadrant, find the remaining trigonometric ratios of θ.

Q.2 If $\sin\theta = \frac{3}{8}$, and the terminal side of the angle lies in the second quadrant, find the remaining trigonometric ratios.

Q.3 If $\cos\theta = -\frac{\sqrt{3}}{2}$, and the terminal side of the angle lies in the third quadrant, find the remaining trigonometric ratios of θ.

Q.4 If $\tan\theta = \frac{3}{4}$, and the terminal side of the angle lies in the third quadrant, find the remaining trigonometric ratios of θ.

Q.5 If $\tan\theta = -\frac{1}{3}$, and the terminal side of the angle lies in the second quadrant, find the remaining trigonometric ratios of θ.

Q.6 If $\cot\theta = \frac{4}{3}$, and the terminal side of the angle is not in the first quadrant, find the trigonometric ratios of 0.

Q.7 If $\cot\theta = \frac{2}{3}$, and the terminal side of the angle does not lies in the first quadrant, find the trigonometric ratios of θ.

Q.8 If $\sin\theta = \frac{4}{5}$,and $\frac{\pi}{2} < \theta < \pi$ find the trigonometric ratios of θ

Q.9 If $\sin\theta = \frac{7}{25}$, find $\cos\theta$, if angle θ is an acute angle.

Q.10 If $\sin\theta = \frac{5}{6}$, find $\cos\theta$, if angle θ is an obtuse angle.

Q.11 Prove that:

(i) $\sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6} = \sin\frac{\pi}{2}$

(ii) $4\tan 60^\circ \tan 30^\circ \tan 45^\circ \sin 30^\circ \cos 60^\circ = 1$

(iii) $2\sin 45^\circ + \frac{1}{2}\operatorname{cosec} 45^\circ = \frac{3}{\sqrt{2}}$

(iv) $\cos 90^\circ - \cos 30^\circ = -2\sin 60^\circ \sin 30^\circ$

(v) $\sin^2\frac{\pi}{6}+\sin^2\frac{\pi}{3}+\tan^2\frac{\pi}{4}=2$

Q.12 $\sin^2\frac{\pi}{6}:\sin^2\frac{\pi}{4}:\sin^2\frac{\pi}{3}:\sin^2\frac{\pi}{2}=1:2:3:4$

Q.13 Evaluate

(i) $\cos 30^o \cos 60^o - \sin 30^o \sin 60^o$

(ii) $\dfrac{\tan 60^o - \tan 30^o}{1+\tan 60^o \tan 30^o}$

Answers 3.2

Q.1 $\text{Sin }\theta = \frac{2}{3}$ $\quad$ $\text{Cot }\theta = -\frac{\sqrt{5}}{2}$

$COS\theta = -\frac{\sqrt{5}}{3}$ $\quad$ $\sec\theta = -\frac{3}{\sqrt{5}}$

$\tan\theta = -\frac{2}{\sqrt{5}}$ $\quad$ $\text{Cosec }\theta = \frac{3}{2}$

Q.2 $\text{Sin }\theta = \frac{3}{8}$ $\quad$ $\text{Cot }\theta = -\frac{\sqrt{55}}{3}$

$\text{Cos }\theta = -\frac{\sqrt{55}}{8}$ $\quad$ $\text{Sec }\theta = -\frac{8}{\sqrt{55}}$

$\tan\theta = -\frac{3}{\sqrt{55}}$ $\quad$ $\text{Cosec }\theta = \frac{8}{3}$

Q.3 $\text{Sin }\theta = \frac{-1}{2}$ $\quad$ $\text{Cot }\theta = \sqrt{3}$

$\text{Cos }\theta = -\frac{\sqrt{3}}{2}$ $\quad$ $\text{Sec }\theta = -\frac{2}{\sqrt{3}}$

$\tan\theta = \frac{1}{\sqrt{3}}$ $\quad$ $\text{Cosec }\theta = -2$

Q.4 $\text{Sin }\theta = -\frac{3}{5}$ $\quad$ $\text{Cot }\theta = \frac{4}{3}$

$\text{Cos }\theta = -\frac{4}{5}$ $\quad$ $\text{Sec }\theta = -\frac{5}{4}$

$\tan\theta = \frac{3}{4}$ $\quad$ $\text{Cosec }\theta = -\frac{5}{3}$

Q.5 $\text{Sin } \theta = \frac{1}{\sqrt{10}}$ $\text{Cot } \theta = -3$

$\text{Cos } \theta = -\frac{3}{\sqrt{10}}$ $\text{Sec } \theta = -\frac{\sqrt{10}}{3}$

$\tan \theta = -\frac{1}{3}$ $\text{Cosec } \theta = \sqrt{10}$

Q.6 $\text{Sin } \theta = -\frac{3}{5}$ $\text{Cot } \theta = \frac{4}{3}$

$\text{Cos } \theta = -\frac{4}{5}$ $\text{Sec } \theta = -\frac{5}{4}$

$\tan \theta = \frac{3}{4}$ $\text{Cosec } \theta = -\frac{5}{3}$

Q.7 $\text{Sin } \theta = \frac{-3}{\sqrt{10}}$ $\text{Cot } \theta = \frac{2}{3}$

$\text{Cos } \theta = -\frac{2}{\sqrt{13}}$ $\text{Sec } \theta = -\frac{\sqrt{13}}{2}$

$\tan \theta = \frac{3}{2}$ $\text{Cosec } \theta = -\frac{\sqrt{13}}{3}$

Q.8 $\text{Sin } \theta = \frac{4}{5}$ $\text{Cot } \theta = -\frac{3}{4}$

$\text{Cos } \theta = -\frac{3}{4}$ $\text{Sec } \theta = -\frac{5}{3}$

$\tan \theta = -\frac{4}{3}$ $\text{Cosec } \theta = \frac{5}{4}$

Q.9 $\cos \theta = \frac{24}{25}$

Q.10 $\cos \theta - \frac{\sqrt{11}}{6}$

Q.13 (i) 0 (ii) $\frac{1}{\sqrt{3}}$

3.13 Fundamental Identities:

For any real number θ, we shall derive the following three fundamental identities

(i) $\cos^2\theta + \sin^2\theta = 1$
(ii) $\sec^2\theta = 1 + \tan^2\theta$
(iii) $\text{Cosec}^2\theta = 1 + \cot^2\theta$

Proof :

Consider an angle $\angle XOP = \theta$ in the standard position. Take a point P on the terminal line of the angle θ. Draw PQ perpendicular from P on OX.

From fig., ΔOPQ is a right angled triangle. By pythagoruse theorem

$$(OP)^2 = (OQ)^2 + (PQ)^2$$

Or, $z^2 = x^2 + y^2$

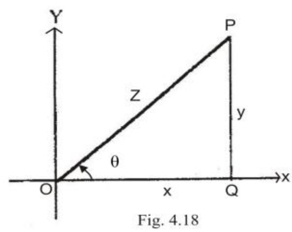


Fig. 4.18

(i) Dividing both sides by z^2

then $$\frac{z^2}{z^2} = \frac{x^2}{z^2} + \frac{y^2}{z^2}$$

$$1 = \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2$$

$$1 = (\cos\theta)^2 + (\sin\theta)^2$$

$$1 = \cos^2\theta + \sin^2\theta$$

or, $\cos^2\theta + \sin^2\theta = 1$

(ii) Dividing both sides of Eq. (i) by x^2, we have

$$\frac{z^2}{x^2} = \frac{x^2}{x^2} + \frac{y^2}{x^2}$$

$$\left(\frac{z}{x}\right)^2 = 1 + \left(\frac{y}{x}\right)^2$$

$$(\sec\theta)^2 = 1 + (\tan\theta)^2$$

$$\sec^2\theta = 1 + \tan^2\theta$$

(iii) Again, dividing both sides of Eq (i) by y^2, we have

$$\frac{z^2}{y^2} = \frac{x^2}{y^2} + \frac{y^2}{y^2}$$

$$\left(\frac{z}{y}\right)^2 = \left(\frac{x}{y}\right)^2 + 1$$

$(\text{Cosec } \theta)^2 = (\text{Cot } \theta)^2 + 1$

$\text{Cosec}^2 \theta = \text{Cot}^2 \theta + 1$

$\text{Cosec}^2 \theta = 1 + \text{Cot}^2 \theta$

Example 1:

Prove that $\dfrac{\text{Sin x}}{\text{Cosec x}} + \dfrac{\text{Cos x}}{\text{sec x}} = 1$

Solution:

$$\text{L.S.H.} = \frac{\text{Sin x}}{\text{Cosec x}} + \frac{\text{Cos x}}{\text{sec x}}$$

$$= \text{Sin x}.\frac{1}{\text{Cosec x}} + \text{Cos x}.\frac{1}{\text{sec x}} \because \quad \frac{1}{\text{Cosec x}} = \text{Sin x}$$

$$= \text{Sin x. Sin x + Cos x. Cos x} \because \quad \frac{1}{\text{Sec x}} = \text{Cos x}$$

$$= \text{Sin}^2 \text{ x} + \text{Cos}^2 \text{ x}$$

$= 1$

$= \text{R.H.S}$

Example 2:

Prove that $\dfrac{\text{Sec x} - \text{Cos x}}{1 + \text{Cos x}} = \text{Sec x} - 1$

Solution:

$$\text{L.H.S} = \frac{\text{Sec x} - \text{Cos x}}{1 + \text{Cos x}}$$

$$= \frac{\frac{1}{\text{Cos x}} - \text{Cos x}}{1 + \text{Cos x}}$$

$$= \frac{\frac{1 - \text{Cos}^2 \text{ x}}{\text{Cos x}}}{1 + \text{Cos x}} = \frac{1 - \text{Cos}^2 \text{ x}}{\text{Cos x}(1 + \text{Cos x})}$$

$$= \frac{(1 - \text{Cos x})(1 + \text{Cos x})}{\text{Cos x}(1 + \text{Cos x})}$$

$$= \frac{1 - \text{Cos x}}{\text{Cos x}} = \frac{1}{\text{Cos x}} - \frac{\text{Cos x}}{\text{Cos x}}$$

$= \text{Sex x} - 1 = \text{R.H.S.}$

Example 3:

prove that $$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$$

Solution: L.H.S. $= \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}} \qquad = \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{(1-\sin\theta)}{\cos\theta} \qquad = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta - \tan\theta \qquad = \text{R.H.S.}$$

Exercise 3.3

Prove the following Identities:

Q.1 $1 - 2\,\text{Sin}^2\,\theta = 2\,\text{Cos}^2\,\theta - 1$

Q.2 $\text{Cos}^4\,\theta - \text{Sin}^4\,\theta = 1 - 2\,\text{Sin}^2\theta$

Q.3 $$\frac{1}{\text{Cosec}^2\,\theta} + \frac{1}{\text{Sec}^2\,\theta} = 1$$

Q.4 $$\frac{1}{\tan\theta + \text{Cot}\,\theta} = \text{Sin}\,\theta.\text{Cos}\,\theta$$

Q.5 $$(\text{Sec}\,\theta - \tan\theta)^2 = \frac{1-\text{Sin}\,\theta}{1+\text{Sin}\,\theta}$$

Q.6 $$(\text{Cosec}\,\theta - \text{Cot}\,\theta)^2 = \frac{1-\text{Cos}\,\theta}{1+\text{Cos}\,\theta}$$

Q.7 $(1-\text{Sin}^2\theta)(1+\tan^2\theta)=1$

Q.8 $$\frac{1}{1+\text{Sin}\,\theta} + \frac{1}{1-\text{Sin}\,\theta} = 2\text{Sec}^2\theta$$

Q.9 $$\sqrt{\frac{1-\text{Sin}\,\theta}{1+\text{Sin}\,\theta}} = \text{Sec}\theta - \tan\theta$$

Q.10 $$\sqrt{\frac{1+\text{Cos}\,\theta}{1-\text{Cos}\,\theta}} = \text{Cosec}\,\theta + \cot\theta$$

Q.11 $\dfrac{1-\tan A}{1+\tan A}=\dfrac{\text{Cot } A-1}{\text{Cot } A+1}$

Q.12 $\dfrac{\text{Cot}^2\theta-1}{\text{Cot}^2\theta+1}=2\text{Cos}^2\theta-1$

Q.13 $\dfrac{\tan\theta}{1-\text{Cot }\theta}+\dfrac{\text{Cot }\theta}{1-\tan\theta}=\text{Sec }\theta\text{ Cosec }\theta+1$

Q.14 $\dfrac{\text{Sec }\theta-\tan\theta}{\text{Sec }\theta+\tan\theta}=1-2\text{Sec }\theta\tan\theta+2\tan^2\theta$

Q.15 $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\dfrac{(1-\tan\theta)^2}{(1-\cot\theta)^2}$

Q.16 $\text{Cosec } A+\text{Cot } A=\dfrac{1}{\text{Cosec } A-\text{Cot } A}$

Q.17 $\dfrac{1}{\text{Sec }\theta+\tan\theta}=\dfrac{1-\text{Sin }\theta}{\text{Cos }\theta}=\text{Sec } x-\tan x$

Q.18 $(1-\tan\theta)^2+(1-\text{Cot }\theta)^2=(\text{Sec }\theta-\text{Cosec }\theta)^2$

Q.19 $\dfrac{\text{Cos}^3t-\text{Sin}^3t}{\text{Cos } t-\text{Sin } t}=1+\text{Sin } t\text{ Cos } t$

Q.20 $\text{Sec}^2 A+\tan^2 A=(1-\text{Sin}^4 A)\text{ Sec}^4 A$

Q.21 $\dfrac{\text{Sec } x-\text{Cos } x}{1+\text{Cos } x}=\text{Sec } x-1$

Q.22 $\dfrac{1+\text{Sin }\theta+\text{Cos }\theta}{1+\text{Sin }\theta-\text{Cos }\theta}=\dfrac{\text{Sin }\theta}{1-\text{Cos }\theta}$

Q.23 $\dfrac{\text{Sin } x+\text{Cos } x}{\tan^2 x-1}=\dfrac{\text{Cos}^2 x}{\text{Sin } x-\text{Cos } x}$

Q.24 $(1+\text{Sin }\theta)(1-\text{Sin }\theta)=\dfrac{1}{\text{Sec}^2\theta}$

Q.25 $\dfrac{\tan\theta}{\text{Sec }\theta-1}+\dfrac{\tan\theta}{\text{Sec }\theta+1}=2\text{Cosec }\theta$

Q.26 $\dfrac{\cot\theta\cos\theta}{\cot\theta+\cos\theta}=\dfrac{\cot\theta-\cos\theta}{\cot\theta\cos\theta}$

Q.27 If $m=\tan\theta+\text{Sin }\theta$ and $n=\tan\theta-\text{Sin }\theta$ than prove that $m^2-n^2=4\sqrt{m\,n}$

3.14 Graph of Trigonometric Functions:

In order to graph a function = f(x), we give number of values of x and obtain the corresponding values of y. The several ordered pairs (x, y) are obtained we plotted these points by a curve we get the required graph.

3.14.1 Graph of Sine Let, y = Sinx where, $0^o \le x \le 360^o$

Or where, $0 \le x \le 2\pi$

1. Variations

Quadrants

	1st	2nd	3^{rd}	4^{th}
x Sinx	0 to 90^o + ve, Increase from 0 to 1	90^o to 180^o + ve, decrease from 1 to 0	180^o to 270^o $-$ve, decrease from 0 to -1	270^o to 360^o $-$ve, Increases from -1 to 0

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
Sinx	0	0.5	0.87	1	0.87	0.5	0

x	210^o	240^o	270^o	300^o	330^o	360^o
Sinx	-0.50	$-.87$	-1	$-.87$	$-.5$	0

3. Graph in Figure (3.19):

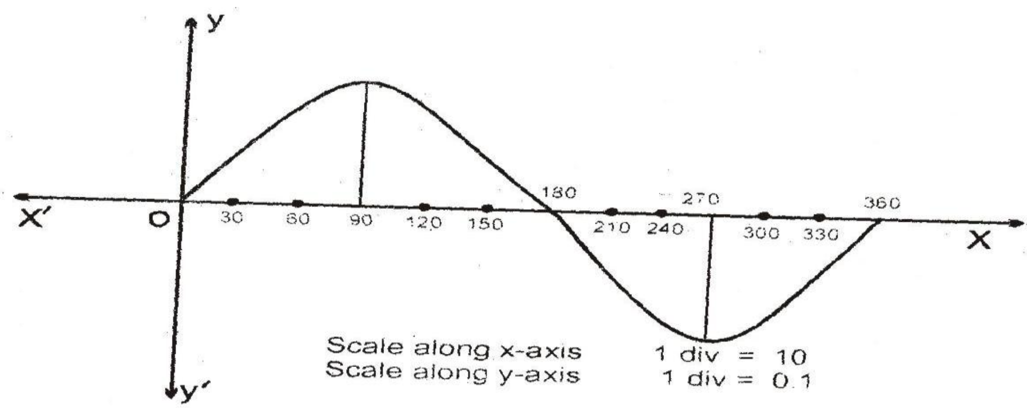


Fig. 4.19

3.14.2 Graph of Cosine

Let, y = Cosx where, $0^o \le x \le 360^o$

Or where, $0 \le x \le 2\pi$

1. Variations

Quadrants

	1st	2nd	3^{rd}	4^{th}
x y = Cosx	0 to 90^o + ve, decrease from 1 to 0	90^o to 180^o $-$ ve, decrease from 0 to -1	180^o to 270^o $-$ve, increase from -1 to 0	270^o to 360^o + ve, increases from 0 to 1

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = Cosx	1	0.87	0.5	0	-0.5	-0.87	-1

x	210^o	240^o	270^o	300^o	330^o	360^o
y = Cosx	-0.87	-0.5	0	0.5	0.87	1

3. Graph in Figure (3.20):

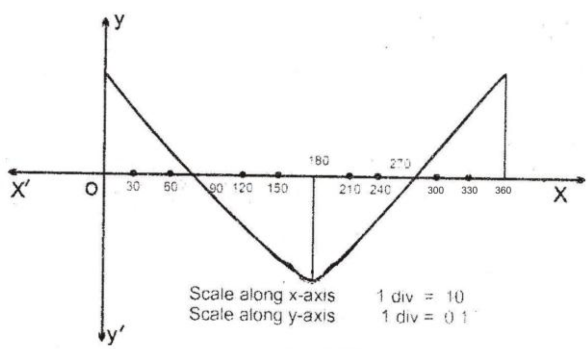


Fig. 4.20

Fig. 3.20

3.14.3 Graph of tanx

Let, y = tanx where, $0^o \leq x \leq 360^o$

Or where, $0 \leq x \leq 2\pi$

1. Variations

Quadrants

	1^{st}	2nd	3rd	4^{th}
x y = tanx	0 to 90^o + ve, Increase from 0 to $\propto$	90^o to 180^o $-$ve, increase from $-\propto$ to 0	180^o to 270^o + ve, increase from 0 to $\propto$	270^o to 360^o $-$ve, increases from $-\propto$ to 0

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = tanx	0	0.58	1.73	$\propto$	-1.73	-0.58	0

x	210^o	240^o	270^o	300^o	330^o	360^o
y = tanx	+ .58	1.73	$-\propto$, $+\propto$	-1.73	-2.58	0

3. Graph in Figure (3.21):

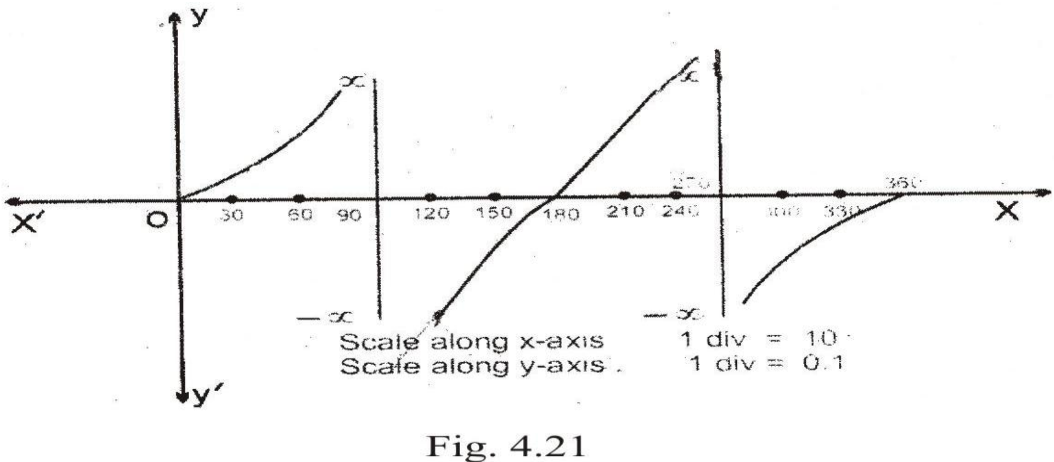


Fig. 3.21

3.14.4 Graph of Cotx:

Let, y = cotx where, $0^o \le x \le 360^o$

1. Variations

Quadrants

	1^{st}	2nd	3rd	4^{th}
x y = Cotx	0 to 90^o + ve, Increase from $\propto$ to 0	90^o to 180^o $-$ve, increase from 0 to $-\propto$	180^o to 270^o + ve, increase from $\propto$ to 0	270^o to 360^o $-$ve, increases from 0 to $-\propto$

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = Cotx	$\propto$	1.73	0.58	0	-0.58	-1.73	$-\propto$

x	210^o	240^o	270^o	300^o	330^o	360^o
y = Cotx	1.73	0.58	0	-0.58	-1.73	$\propto$

3. Graph in Figure (3.22):

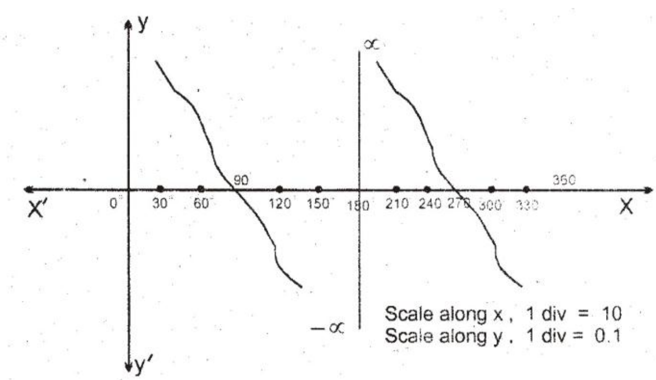


Fig. 4.22
Fig. 3.22

3.14.5 Graph of Secx:

Let, y = secx where, $0^o \le x \le 360^o$

1. Variations

Quadrants

	1^{st}	2nd	3rd	4^{th}
x y = Secx	0 to 90^o + ve, Increase from 1 to $\propto$	90^o to 180^o $-$ve, increase from $-\propto$ to -1	180^o to 270^o $-$ve, increase from -1 to $-\propto$	270^o to 360^o + ve, increases from $-\propto$ to 1

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = Secx	1	1.15	2	$+\propto$	-2	1.15	1

x	210^o	240^o	270^o	300^o	330^o	360^o
y = Secx	-1.15	-2	$\propto$	2	1.15	1

3. Graph in Figure (3.23):

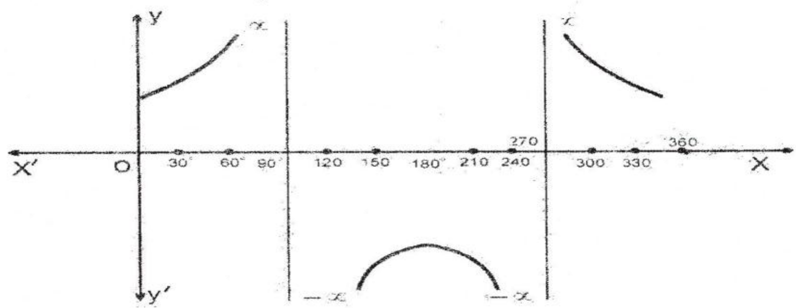


Fig. 4.23

3.14.6 Graph of Cosecx: Let, y = Cosecx ,where $0^o \le x \le 360^o$

1. Variations

Quadrants

	1^{st}	2nd	3rd	4^{th}
x y = Cosecx	0 to 90^o + ve, Increase from $\propto$ to 1	90^o to 180^o + ve, increase from 1 to $\propto$	180^o to 270^o $-$ve, increase from $-\propto$ to -1	270^o to 360^o $-$ ve, increases from -1 to $-\propto$

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = Cosecx	$\propto$	2	1.15	1	1.15	2	$\propto$

x	210^o	240^o	270^o	300^o	330^o	360^o
y = Cosecx	-2	-1.15	-1	-1.15	-2	$-\propto$

3. Graph in Figure (3.24):

Fig.3.2

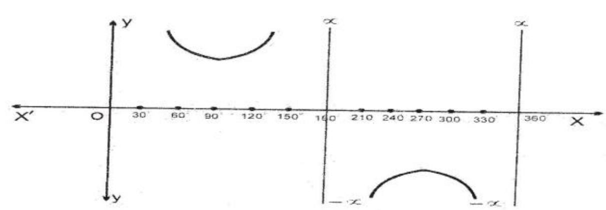


Fig. 4.24

Exercise 3.4

Q.1 Draw the graph of tan 2A as A varies from 0 to π .

Q.2 Plot the graph of 1 – Sin x as x varies from 0 to 2π .

Q.3 Draw the graphs for its complete period.

(i) $y = \frac{1}{2}\text{Sin } 2x$ (ii) $y = \text{Sin } 2x$

(iii) $y = \frac{1}{2}\cos x$

Summary

Trigonometry means measurement of triangles.

1. Radian is an angle subtended at the center of a circle by an arc of the circle equal in length to its radius.

   i.e. π Radian = 180 degree

   1 rad = $57^{o}\ 17'\ 45''$

   1 degree = 0.01745 radian

2. Length of arc of the circle, $l = s = r\theta$

3. Trigonometric functions are defined as:

$$\text{Sin}\theta = \frac{AP}{OP}, \text{Cosec}\theta = \frac{OP}{AP}$$

$$\text{Cos}\theta = \frac{OA}{OP}, \text{Sec}\theta = \frac{OP}{OA}$$

$$\tan\theta = \frac{AP}{OA}, \text{Cot}\theta = \frac{OA}{AP}$$

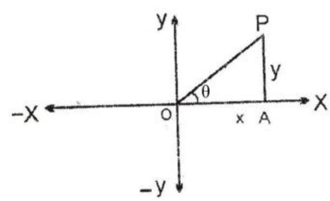


Fig. 4.25

4. Relation between trigonometric ratios:

(i) $\text{Sec}\theta = \frac{1}{\text{Cos}\theta}$ (ii) $\text{Cosec}\theta = \frac{1}{\text{Sin}\theta}$

(iii) $\text{Cot}\theta = \frac{1}{\tan\theta}$ (iv) $\text{Cos}\theta = \frac{1}{\text{Sec}\theta}$

(v) $\text{Sin}\theta = \frac{1}{\text{Cosec}\theta}$ (vi) $\tan\theta = \frac{1}{\text{Cot}\theta}$

(vii) $\text{Sin}^2\theta + \text{Cos}^2\theta = 1$ (viii) $\text{Sec}^2\theta = 1 + \tan^2\theta$

(ix) $\text{Cosec}^2\theta = 1 + \text{Cot}^2\theta$

5. Signs of the trigonometric functions in the Four Quadrants.

Quadrant	I	II	III	IV
Positive	All +ve	Sin θ , Cosec θ	tan θ , cot θ	Cos θ , Sec θ
Negative	Nil	Cos θ Sec θ tan θ Cot θ	Cos θ Sec θ Sin θ Cosec θ	Sin θ Cosec θ tan θ Cot θ

Short Questions

Write the short answers of the following:

Q.1: **Define degree and radians measure**

Q.2: Convert into radius measure.

(a) 120^o, (b) $22\frac{1}{2}^o$, (c) $12^o\ 40'$, (d) $42^o\ 36'\ 12''$

Q.3: Convert into degree measure

(a) $\frac{\overline{\wedge}}{2}$ rad, (b) 0.726 rad. (c) $\frac{2\overline{\wedge}}{3}$ rad.

Q.4: Prove that $\ell = r\theta$

Q.5: What is the length of an arc of a circle of radius 5 cm whose central angle is 140^o ?

Q.6: Find the length of the equatorial arc subtending an angle 1^o at the centre of the earth taking the radius of earth as 6400 KM.

Q.7: Find the length of the arc cut off on a circle of radius 3 cm by a central angle of 2 radius.

Q.8: Find the radius of the circle when $\ell = 8.4$ cm, $\theta = 2.8$ rad

Q.9: If a minute hand of a clock is 10 cm long, how far does the tip of the hand move in 30 minutes?

Q.10 Find x, if $\tan^2 45^o - \cos^2 60^o = x \sin 45^o \cos 45^o. \tan 60^o$.

Q.11: Find r when $l = 33$ cm. $\theta = 6$ radian

Q12: Prove that $2 \sin 45^o + \frac{1}{2} \operatorname{cosec} 45^o = \frac{3}{\sqrt{2}}$

Q.13: Prove that $\tan^2 30^o + \tan^2 45^o + \tan^2 60^o = \frac{13}{3}$

Q.12: Prove that $\dfrac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} = \sqrt{3}$

Q.13: prove that $\cos 30^o \operatorname{Cos} 60^o - \sin 30^o \sin 60^o = 0$

Q.14: Prove that $\operatorname{Cos} 90^o - \cos 30^o = -2\sin 60^o \sin 30^o$

Q.15: Prove that $\operatorname{Sin}^2\theta + \cos^2\theta = 1$

Q.16: Prove that: $1 + \tan^2\theta = \sec^2\theta$

Q.17: Prove that $1 + \cot^2\theta = \operatorname{Cosec}^2\theta$

Q.18: Prove that: $(1 + \operatorname{Sin}\theta)(1 - \operatorname{Sin}\theta) = \dfrac{1}{\operatorname{Sec}^2\theta}$

Q.19: Show that: $\operatorname{Cot}^4\theta + \operatorname{Cot}^2\theta = \operatorname{Cosec}^4\theta - \operatorname{cosec}^2\theta$

Q.20: Prove that: $\operatorname{Cos}\theta + \tan\theta \operatorname{Sin}\theta = \operatorname{Sec}\theta$

Q.21: Prove that $1 - 2\operatorname{Sin}^2\theta = 2\operatorname{Cos}^2\theta - 1$

Q.22: $\operatorname{Cos}^4\theta - \sin^4\theta = 1 - 2\sin^2\theta$

Q.23: $\dfrac{1}{1 + \operatorname{Sin}\theta} + \dfrac{1}{1 - \operatorname{SIn}\theta} = 2\operatorname{Sec}2\theta$

Answers

2. (a) 2.09 rad (b) 0.39 rad (c) 0.22 rad (d) 0.74 radius

3. (a) 90^o (b) $41^0\ 35'\ 48''$ (c) 120 degree

5. 12.21 cm. 6. 111.7 Km 7. 6 cm

8. 3cm. 9. 31.4 cm 10. $\dfrac{\sqrt{3}}{2}$ 11. 5.5 cm.

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. One degree is equal to:

(a) π rad (b) $\frac{\pi}{180}$ rad

(c) $\frac{180}{\pi}$ rad (d) $\frac{\pi}{360}$ rad

__2. 15^o is equal to:

(a) $\frac{\pi}{6}$ rad (b) $\frac{\pi}{3}$ rad

(c) $\frac{\pi}{12}$ rad (d) $\frac{\pi}{15}$ rad

__3. 75^o is equal to

(a) $\frac{\pi}{12}$ rad (b) $\frac{2\pi}{3}$ rad

(c) $\frac{4\pi}{3}$ rad (d) $\frac{5\pi}{12}$ rad

__4. One radian is equal to:

(a) 90^o (b) $\left(\frac{90}{\pi}\right)^o$

(c) 180^o (d) $\left(\frac{180}{\pi}\right)^o$

__5. The degree measure of one radian is approximately equal to:

(a) 57.3 (b) 57.2

(c) 57.1 (d) 57.0

__6. $\frac{2\pi}{3}$ radians are equal to:

(a) 60^o (b) 90^o

(c) 120^o (d) 150^o

__7. The terminal side of θ lies in 4^{th} quadrant, sign of the sin θ will be:

(a) Positive (b) Negative

(c) Both +ve and – ve (d) None of these

__8. The terminal side of θ lies in 4^{th} quadrant, both $\sin\theta$ and $\tan\theta$ are:

(a) $\sin\theta > 0, \tan\theta > 0$ (b) $\sin\theta > 0, \tan\theta < 0$

(c) $\sin\theta < 0, \tan\theta < 0$ (d) $\sin\theta < 0, \tan\theta > 0$

__9. A circle is equal to 2π rad and also to 360^o, then:

(a) $360^o = 2\pi$ rad (b) $360^o = \frac{3}{4}\pi$ rad

(c) $360^o = \frac{\pi}{6}$ rad (d) None of a, b & c

__10. π rad is equal to:

(a) 360^o (b) 270^o

(c) 180^o (d) 90^o

__11. The relation between are l, radius r and central angle θ rad is:

(a) $l = \frac{\theta}{r}$ (b) $l = \frac{r}{\theta}$

(c) $l = r\,\theta$ (d) $l = r^2\,\theta$

__12. If l = 12cm and r = 3 cm, then θ is equal to:

(a) 36 rad (b) 4 rad

(c) $\frac{1}{4}$ rad (d) 18 rad

__13. An angle subtended at the centre of a circle by an arc equal to the radius of the circle is called:

(a) Right angle (b) Degree

(c) Radian (d) Acute angle

__14. The radian measure of the angle described by a wheel in 5 revolution is:

(a) 5π (b) 10π

(c) 15π (d) 20π

__15. If an are of a circle has length l and subtends an angle θ, then radius ′r′ will be:

(a) $\frac{\theta}{l}$ (b) $\frac{l}{\theta}$

(c) $l\,\theta$ (d) $l + \theta$

__16. If sin x = $\frac{\sqrt{3}}{2}$ and the terminal ray of x lies in 1st quadrant, then cosx is equal to:

(a) $\frac{1}{\sqrt{2}}$ (b) $-\frac{1}{2}$

(c) $\frac{1}{2}$ (d) $-\frac{1}{\sqrt{2}}$

__17. If $\sin\theta = \frac{3}{5}$ and the terminal side of the angle lies in 2^{nd} quadrant, then $\tan\theta$ is equal to:

(a) $\frac{4}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{5}{4}$ (d) $-\frac{3}{4}$

__18. If $\sin\theta$ is +ve and $\cos\theta$ is –ve, then the terminal side of the angle lies in:

(a) 1^{st} quad (b) 2^{nd} quad (c) 3^{rd} quad (d) 4^{th} quad

__19. If $\sin\theta$ is +ve and $\tan\theta$ is-ve, then the terminal side of the angle lies in

(a) 1^{st} quad (b) 2^{nd} quad
(c) 3^{rd} quad (4) 4^{th} quad

__20. If $\sin\theta = \frac{2}{\sqrt{7}}$ and $\cos\theta = -\frac{1}{\sqrt{7}}$, then $\cot\theta$ is equal to:

(a) 2 (b) -1
(c) $-\frac{1}{2}$ (d) –2

21. $\sec^2\theta + \text{cosec}^2\theta$ is equal to:

(a) $\sec^2\theta\,\text{cosec}^2\theta$ (b) $\sin\theta\ \cos\theta$
(c) $2\sec^2\theta$ (d) $2\,\text{cosec}^2\theta$

Answers

1.	b	2.	c	3.	d	4.	d	5.	a
6.	c	7.	b	8.	c	9.	a	10.	c
11.	c	12.	b	13.	c	14.	b	15.	b
16.	c	17.	b	18.	b	19.	d	20.	c
21.	a								

Chapter 4
General Identities

4.1 Introduction:

In the previous chapters, we have dealt with functions of one angle. In this chapter we will discuss the trigonometrical ratios of the sum and difference of any two angles in terms of the ratios of these angles themselves. We will also derives several formulas for this purpose and point out some of their more elementary uses.

4.2 Distance formula:

Let P(x_1 , y_1) and Q(x_2 , y_2) be two points . If “d” denotes the distance between them, then

$$d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e., sum of the square of the difference of x-coordinates and y- coordinates and then the square roots.

Example1: Find distance between the points P(5 , 7) and Q(-3 , 4)

Solution:

$$d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(3 - 5)^2 + (4 - 7)^2}$$

$$= \sqrt{(-2)^2 + (-3)^2}$$

$$= \sqrt{4+9} \quad = \quad \sqrt{13}$$

4.3 Fundamental law of trigonometry

Let α and β be any two angles (real numbers), then

$$\text{Cos}(\alpha - \beta) = \cos\alpha \cos\beta \ - \ \sin\alpha \sin\beta$$

Which is called the **Fundamental law of trigonometry**

Proof: for convenience, let us assume that $\alpha > \beta > 0$

Consider a unit circle with centre at origin O.

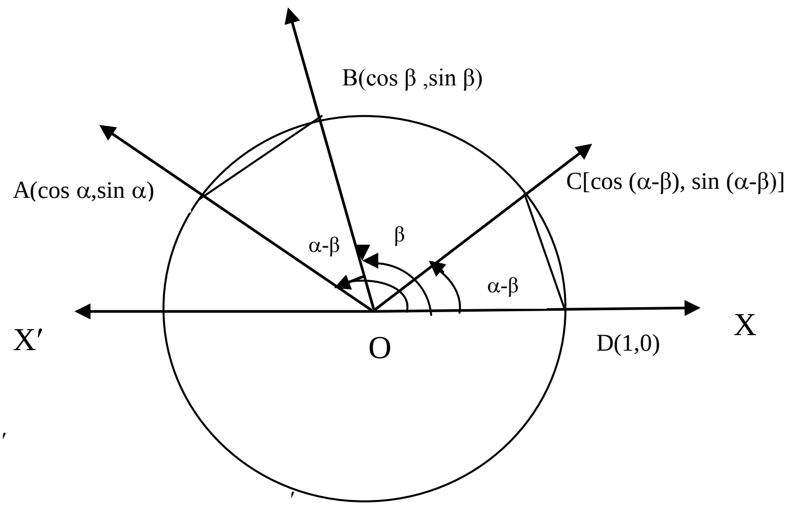
Let the terminal side of angles α and β

cut the unit circle at A and B respectively.

Evidently $\angle AOB = \alpha - \beta$. Take a point C

on the unit circle so that $\angle XOC = \angle AOB = \alpha - \beta$

Join A , B and C , D.



Now angle α , β and $\alpha - \beta$ are in standard position.

$\therefore$ The coordinates of A are $(\cos\alpha , \sin\alpha)$

The coordinates of B are $(\cos\beta , \sin\beta)$

The coordinates of C are $[\cos(\alpha-\beta) , \sin(\alpha - \beta)]$

and the coordinates D are $(1 , 0)$

Now Δ AOB and ΔCOD are congruent.

$\therefore \quad |AB| = |CD|$

$\Rightarrow \quad |AB|^2 = |CD|^2$

Using the distance formula , we have:

$$(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = [\cos (\alpha - \beta) - 1]^2 + [\sin (\alpha - \beta) - 0]^2$$

$$\Rightarrow \cos^2 \alpha + \cos^2\beta - 2\cos \alpha \cos \beta + \sin^2 \alpha + \sin^2 \beta - 2\sin \alpha \sin \beta$$

$$= \cos^2 (\alpha - \beta) + 1 - 2 \cos (\alpha - \beta) + \sin^2 (\alpha - \beta)$$

$$\Rightarrow 2 - 2 (\cos \alpha \cos \beta + \sin \alpha \sin \beta) = 2 - 2 \cos(\alpha - \beta)$$

Hence,

$$\boxed{\mathbf{\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \;\ldots\ldots\ldots\ldots\ldots(1)}}$$

Note: Although we have proved this law for $\alpha > \beta > 0$, it is true for all values of α and β

Suppose we know the values of sin and cos of two angles α and β, we can find $\cos(\alpha - \beta)$ using this law as explained in the following example:

Example 1:

Find the value of Cos 15^o.

Solution:

$$\text{Cos } 15^o = \text{Cos }(45^o - 30^o)$$
$$= \text{Cos } 45^o \text{ Cos } 30^o + \text{Sin } 45^o \text{ Sin } 30^o$$
$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

4.4 Deductions from fundamental law:

(i) Prove that : $\text{Cos}(-\beta) = \text{Cos}\,\beta$

Put $\alpha = 0$ in above equation (1) , then

$$\text{Cos }(0-\beta) = \text{Cos } 0 \text{ Cos } \beta + \text{Sin } 0 \text{ Sin}\,\beta$$
$$\text{Cos}(-\beta) = 1.\ \text{Cos } \beta + 0.\ \text{Sin}\,\beta \qquad \because \text{Cos } 0 = 1$$
$$\text{Cos}(-\beta) = \text{Cos}\,\beta \qquad \because \text{Sin } 0 = 0$$

(ii) Prove that : $\text{Cos}\left(\frac{\pi}{2} - \beta\right) = \text{Sin}\,\beta$

Putting $\alpha = \pi/2$ in equation

$$\text{Cos}(\alpha-\beta) = \text{Cos}\alpha\ \text{Cos}\beta + \text{Sin}\alpha\ \text{Sin}\beta$$
$$\text{Cos}\left(\frac{\pi}{2}-\beta\right) = \text{Cos}\frac{\pi}{2}\ \text{Cos}\beta + \text{Sin}\frac{\pi}{2}\ \text{Sin}\beta$$
$$= 0\ .\ \text{Cos}\beta + 1\ .\ \text{Sin}\beta$$
$$\boxed{\text{Cos}\left(\frac{\pi}{2}-\beta\right) = \text{Sin}\,\beta}$$

(iii) Prove that : $\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha$

Put $\beta = -\frac{\pi}{2}$ in equation

$$\mathrm{Cos}(\alpha - \beta) = \mathrm{Cos}\alpha\ \mathrm{Cos}\beta - \mathrm{Sin}\alpha\ \mathrm{Sin}\beta$$

$$\mathrm{Cos}[(\alpha - (-\frac{\pi}{2})] = \cos\alpha \,.\, \cos(-\frac{\pi}{2}) + \sin\alpha \,.\, \sin(-\frac{\pi}{2})$$

$$\cos(\alpha + \frac{\pi}{2}) = \cos\alpha \,.\, 0 + \sin\alpha \,.\, (-1)$$

$$\boxed{\cos(\alpha + \frac{\pi}{2}) = -\sin\alpha}$$

(iv) Prove that: $\sin(-\beta) = -\sin\beta$

By (iii) we have $\cos(\frac{\pi}{2} + \beta) = -\sin\beta$

replace β by $-\beta$

$$\cos(\frac{\pi}{2} - \beta) = -\sin(-\beta)$$

$$\sin\beta = -\sin(-\beta) \qquad [\text{ by (ii)}]$$

$$\boxed{\mathbf{\sin(-\beta) = -\sin\beta}}$$

(v) Prove that : $\sin(\frac{\pi}{2} + \alpha) = \cos\alpha$

we know that $\mathrm{Cos}\left(\frac{\pi}{2} - \beta\right) = \mathrm{Sin}\ \beta$

putting $\beta = \frac{\pi}{2} + \alpha$ in above equation, we get

$$\mathrm{Cos}[\frac{\pi}{2} - (\frac{\pi}{2} + \alpha)] = \sin(\frac{\pi}{2} + \alpha)$$

$\Rightarrow \quad \cos(-\alpha) = \sin(\frac{\pi}{2} + \alpha)$

$\cos\alpha = \sin(\frac{\pi}{2} + \alpha)$

$$\boxed{\sin(\frac{\pi}{2} + \alpha) = \cos\alpha}$$

(vi) Prove that : $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Since $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

replacing β by $-\beta$, we get

$\cos[\alpha-(-\beta)] = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$

{because $\cos(-\beta) = \cos\beta$,

$\sin(-\beta) = \sin\beta$}

$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

(vii) Prove that : $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

We know that

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

replace α by $\frac{\pi}{2} + \alpha$, we get

$\cos[(\frac{\pi}{2} + \alpha) + \beta] = \cos(\frac{\pi}{2} + \alpha)\cos\beta - \sin(\frac{\pi}{2} + \alpha)\sin\beta$

$\cos[\frac{\pi}{2} + (\alpha + \beta)] = -\sin\alpha\cos\beta - \cos\alpha\sin\beta$

$-\sin(\alpha + \beta) = -[\sin\alpha\cos\beta + \cos\alpha\sin\beta]$

$$\boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

(viii) Prove that: $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

We know that

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

replacing β by $-\beta$, we get

$$\sin(\alpha - \beta) = \sin\alpha \cos(-\beta) + \cos\alpha \sin(-\beta)$$

{because $\cos(-\beta) = \cos\beta$,
$\sin(-\beta) = \sin\beta$}

$$\boxed{\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta}$$

(ix) Prove that: $\text{Sin}\left(\frac{\pi}{2}+\beta\right) = -\text{Cos }\beta$

Put $\alpha = \frac{\pi}{2}$ in

$$\text{Sin}(\alpha+\beta) = \text{Sin}\alpha \text{ Cos}\beta + \text{Cos}\alpha \text{ Sin}\beta$$

$$\text{Sin}\left(\frac{\pi}{2}+\beta\right) = \text{Sin}\frac{\pi}{2} \text{ Cos}\beta + \text{Cos}\frac{\pi}{2} \text{ Sin}\beta$$

$$= 0 \,.\, \text{Cos}\beta + 0 \,.\, \text{Sin}\beta$$

$$\boxed{\text{Sin}\left(\frac{\pi}{2}+\beta\right) = \text{Cos }\beta}$$

(x) Prove that: $\text{Sin}\left(\frac{\pi}{2}-\beta\right) = \text{Cos}\beta$

Put $\alpha = \frac{\pi}{2}$ in

$$\text{Sin}(\alpha-\beta) = \text{Sin}\alpha \text{ Cos}\beta - \text{Cos}\alpha \text{ Sin}\beta$$

$$\text{Sin}\left(\frac{\pi}{2}-\beta\right) = \text{Sin}\frac{\pi}{2}\ \text{Cos}\beta - \text{Cos}\frac{\pi}{2}\text{Sin}\beta$$

$$\text{Sin}\left(\frac{\pi}{2}-\beta\right) = 1.\text{Cos}\beta - 0.\text{Sin}\beta$$

$$\boxed{\text{Sin}\left(\frac{\pi}{2}-\beta\right) = \text{Cos}\beta}$$

(xi) $\tan(\alpha+\beta) = \dfrac{\text{Sin}(\alpha+\beta)}{\text{Cos}(\alpha+\beta)}$

$$= \frac{\text{Sin}\ \alpha\ \text{Cos}\beta + \text{Cos}\ \alpha\ \text{Sin}\ \beta}{\text{Cos}\alpha\ \text{Cos}\beta - \text{Sin}\ \alpha\ \text{Sin}\ \beta}$$

Divide numerator and denominator by $\text{Cos}\alpha\ \text{Cos}\beta$, we get

$$\tan(\alpha+\beta) = \frac{\dfrac{\text{Sin}\ \alpha\ \text{Cos}\beta}{\text{Cos}\alpha\ \text{Cos}\beta} + \dfrac{\text{Cos}\ \alpha\ \text{Sin}\ \beta}{\text{Cos}\alpha\ \text{Cos}\beta}}{1 - \dfrac{\text{Sin}\ \alpha\ \text{Cos}\beta}{\text{Cos}\alpha\ \text{Cos}\beta}}$$

$$\boxed{\mathbf{tan}(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

(xii) $\text{Cot}(\alpha+\beta) = \dfrac{1}{\tan(\alpha+\beta)}$

$$= \frac{1-\tan\alpha\ \tan\beta}{\tan\alpha + \tan\beta}$$

$$= \frac{1 - \dfrac{1}{\text{Cot}\alpha\ \text{Cot}\beta}}{\dfrac{1}{\text{Cot}\alpha} + \dfrac{1}{\text{Cot}\beta}}$$

$$\textbf{Cot}(\alpha+\beta)=\frac{\text{Cot}\alpha\ \text{Cot}\beta-1}{\text{Cot}\alpha+\text{Cot}\beta}$$

Similarly, $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

and $\cot(\alpha-\beta)=\dfrac{\cot\alpha\ \cot\beta+1}{\cot\alpha-\cot\beta}$

($\tan\theta$ and Cot θ are odd functions).

Note :

(1) If θ is added or subtracted from odd multiple of right angle ($\pi/2$), the trigonometric ratios change into co-ratios and vice-versa.

e.g.,

Sin $(\frac{\pi}{2}-\theta)=\cos\theta$, cos $(\frac{\pi}{2}-\theta)=\sin\theta$, tan $(\frac{\pi}{2}-\theta)=\cot\theta$

Sin$(\frac{3\pi}{2}-\theta)=-\cos\theta$, cos$(\frac{3\pi}{2}-\theta)=-\sin\theta$, tan$(\frac{3\pi}{2}-\theta)=\cot\theta$

(2) If θ is added or subtracted from an even multiple of right angle ($\pi/2$), the trigonometric ratios shall remain the same.

e.g.,

Sin $(\pi-\theta)=\sin\theta$, cos $(\pi-\theta)=-\cos\theta$, tan $(\pi-\theta)=-\tan\theta$

Sin$(2\pi-\theta)=-\sin\theta$, cos$(2\pi-\theta)=\cos\theta$, tan$(2\pi-\theta)=-\tan\theta$

Example 2:

Show that Cos $(180^o+\theta)=-$Cos θ

Solution:

$$\begin{aligned}\text{L.H.S} &= \text{Cos }(180^o+\theta)\\ &= \text{Cos }180^o\ \text{Cos}\theta-\text{Sin }180^o\ \text{Sin }\theta\\ &= (-1)\ \text{Cos}\theta-(0)\ \text{Sin}\theta\\ &= \text{Cos}\theta-0\\ &= -\text{Cos}\theta=\text{R.H.S}\end{aligned}$$

Example 3:

If $\text{Sin}\alpha=\dfrac{4}{5}$ and $\text{Sin}\beta=\dfrac{12}{13}$, neither terminal ray of α nor β is in the first quadrant, find Sin$(\alpha+\beta)$.

Solution:

Because $$\text{Cos}\alpha = \sqrt{1-\text{Sin}^2\alpha}$$

$$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}}$$

$$\text{Cos}\alpha = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

Since α and β does not lie in 1st quadrant and Sin α and Sin β is positive, therefore α, β lies is 2nd quadrant and in 2nd quadrant Cos α is negative is

$$\text{Cos}\alpha = -\frac{3}{5}$$

Then $$\text{Cos}\beta = \sqrt{1-\text{Sin}^2\beta}$$

$$= \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}}$$

$$= \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \pm\frac{5}{13}$$

$$\therefore\ \text{Cos}\beta = -\frac{5}{13} \qquad \because \beta \text{ lies in } 2^{nd} \text{ quadrant}$$

Now $$\text{Sin}(\alpha+\beta) = \text{Sin}\alpha\ \text{Cos}\beta + \text{Cos}\alpha\ \text{Sin}\beta$$

$$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right)$$

$$= -\frac{20}{65} - \frac{36}{65} = \frac{-20-36}{65} = -\frac{56}{65}$$

$$\text{Sin}(\alpha+\beta) = -\frac{56}{65}$$

Example 4:

Express $4\,\text{Sin}\ \theta + 7\,\text{Cos}\ \theta$ in the form r Sin $(\theta + \phi)$, where the terminal side of θ and ϕ are in the first quadrant.

Solution: Multiplying and Dividing the expression by

$$r = \sqrt{(4)^2 + (7)^2} = \sqrt{16+49} = \sqrt{65}$$

$$4\ \text{Sin}\ \theta + 7\text{Cos}\ \theta = \sqrt{65}\left[\frac{4}{\sqrt{65}}\text{Sin}\ \theta + \frac{7}{\sqrt{65}}\text{Cos}\ \theta\right]$$

$$= \sqrt{65}\left[\text{Sin }\theta\left(\frac{4}{\sqrt{65}}\right) + \text{Cos }\theta\left(\frac{7}{\sqrt{65}}\right)\right] \text{.. (1)}$$

Since $\quad r\text{ Sin}(\theta + \phi) = r(\text{Sin }\theta \text{ Cos }\phi + \text{Cos }\theta \text{ Sin }\phi)$ .. (2)

Where $\quad 0 < \phi < \dfrac{\pi}{2}$

Let $\quad \text{Cos}\phi = \dfrac{4}{\sqrt{65}}$ and $\text{Sin}\phi = \dfrac{4}{\sqrt{65}}$, then

$$4\text{ Sin }\theta + 7\text{ Cos }\theta = \sqrt{65}\left[\text{Sin}\theta\text{ Cos}\phi + \text{Cos}\theta\text{ Sin}\phi\right]$$

$$= \sqrt{65}\left[\text{Sin }(\theta + \phi)\right]$$

Where $\text{Cos}\phi = \dfrac{4}{\sqrt{65}}$ and $\text{Sin}\phi = \dfrac{4}{\sqrt{65}}$

i.e., $\quad \tan\phi = \dfrac{7}{4}$

$$\phi = \tan^{-1}\frac{7}{4}$$

Example 5:

Find the value of Sin 75^o.

Solution: $\quad \text{Sin}(75^o) = \text{Sin }(45^o + 30^o)$

$$= \text{Sin }45^o\text{ Cos }30^o + \text{Cos }45^o\text{ Sin }30^o$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Example 6:

$$\text{Sin }(\alpha+\beta).\text{ Sin}(\alpha-\beta) = \text{Cos}^2\beta - \text{Cos}^2\alpha$$

Solution: $\quad$ L.H.S $= \text{Sin }(\alpha+\beta).\text{ Sin}(\alpha-\beta)$

$$= \left[\text{Sin}\alpha\text{ Cos}\beta + \text{Cos}\alpha\text{ Sin}\beta\right]$$

$$\left[\text{Sin}\alpha\text{ Cos}\beta - \text{Cos}\alpha\text{ Sin}\beta\right]$$

$$= \text{Sin}^2\alpha\text{ Cos}^2\beta - \text{Cos}^2\alpha\text{ Sin}^2\beta$$

$$= (1 - \cos^2\alpha)\cos^2\beta - \cos^2\alpha\,(1 - \cos^2\beta)$$

$$= \text{Cos}^2\beta - \text{Cos}^2\alpha\text{ Cos}^2\beta - \text{Cos}^2\alpha + \text{Cos}^2\alpha\text{ Cos}^2\beta$$

$$= \text{Cos}^2\beta - \text{Cos}^2\alpha$$

= R.H.S

Exercise 4.1

Q.1 Find the value of (i) $\cos 75^0$ (ii) $\sin 15^0$ (iii) $\sin 105^0$
(iv) $\cos 105^0$ (v) $\tan 105^0$.

Q.2 Prove that:

(i) $\sin(180^0 - \theta) = \sin\theta$ (ii) $\cos(270^0 + \theta) = \sin\theta$
(iii) $\tan(180^0 + \theta) = \tan\theta$ (iv) $\sin(360^0 - \theta) = -\sin\theta$
(v) $\cot(360^0 + \theta) = \cot\theta$ ((vi) $\tan(90^0 + \theta) = -\cot\theta$

Q.3 Show that:

(i) $\sin(x - y)\cos y + \cos(x - y)\sin y = \sin x$
(ii) $\cos(x + y)\cos y + \sin(x + y)\sin y = \cos x$
(iii) $\cos(A + B)\sin(A - B) = \sin A\cos A - \sin B\cos B$
(iv) $$\frac{\tan(x+y) - \tan x}{1 + \tan(x+y)\tan x} = \frac{\sin x}{\cos y}$$

Q.4 Suppose that A , B and C are the measure of the angles of a triangle such that $A + B + C = \pi$, prove that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Q.5 Prove that:

(i) $\sin\theta + \cos\theta = \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$

(ii) $\sqrt{3}\cos\theta - \sin\theta = 2\cos(\theta + 30^0)$

(iii) $\tan(45^0 - \theta) = \dfrac{1 - \tan\theta}{1 + \tan\theta}$

(iv) $\tan(45^0 + \theta) = \dfrac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta}$

(v) $$\frac{\tan(\alpha + \beta)}{\cot(\alpha - \beta)} = \frac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha\,\tan^2\beta}$$

(vi) $\cot(\alpha + \beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$

(vii) $$\frac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$$

(viii) $\dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

(ix) $\tan(x + \frac{\pi}{4}) - \tan\left(x - \frac{3\pi}{4}\right) = 0$

(x) $\text{Cos}(\frac{\pi}{3} + x) - \sin\left(\frac{\pi}{6} - x\right)$ **= 0**

Q.6 Prove that:

(i) $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$

(ii) $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$

Q.7 If $\text{Sin}\,\alpha = \frac{4}{5}$ and $\text{Sin}\,\beta = \frac{12}{13}$, both α and β are in the 1^{st} quadrant find:

(i) $\sin(\alpha-\beta)$ (ii) $\cos(\alpha+\beta)$

Q.8 If $\cos A = \frac{1}{5}$ and $\cos B = \frac{1}{2}$ A and B be acute angles , find the value of: (i) $\sin(A+B)$ (ii) $\cos(A-B)$

Q.9 If $\tan\alpha = \frac{3}{4}$ and $\sec\beta = \frac{13}{5}$ and neither α nor β is in the 1^{st} quaderant , find $\sin(\alpha+\beta)$

Q.10 Prove that : $\dfrac{\sin\alpha}{\sec 4\alpha} + \dfrac{\cos\alpha}{\text{cosec}\, 4\alpha} = \sin 5\alpha$

Q.11 If $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, prove that $\tan(\alpha-\beta) = (1-n)\tan\alpha$

Q.12 If α , β and γ are the angle of triangle ABC , then prove that

(i) $\sin(\alpha+\beta) = \sin\gamma$ (ii) $\cos(\alpha+\beta) = -\sin\gamma$

(iii) $\tan(\alpha+\beta) + \tan\gamma = 0$

Q.13 (i)If $\cos\alpha = \frac{1}{7}$, $\cos\beta = \frac{13}{14}$, then prove that $\alpha - \beta = 60^0$, where the terminal rays of α and β are in 1st quadrants.

(ii)If If $\tan\alpha = \frac{5}{6}$ and $\tan\beta = \frac{1}{11}$, then prove that $\alpha + \beta = 45^0$, where the terminal rays of α and β are in 1st quadrants.

Q.14 Express the following in the form of $r\sin(\theta + \phi)$, where the terminal rays of θ is in the 1st quadrants. Be sure to specify ϕ :

(i) $4\sin\theta + 3\cos\theta$ (ii) $\sqrt{3}\sin\theta + \sqrt{7}\cos\theta$

(iii) $5\sin\theta - 4\cos\theta$ (iv) $\sin\theta + \cos\theta$

Answers 4.1

Q.1 (i) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (iii) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(iv) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (v) $\frac{\sqrt{3}+1}{1-\sqrt{3}}$

Q.7 (i) $-\frac{16}{25}$ (ii) $-\frac{33}{65}$

Q.8 (i) $\frac{\sqrt{24}+\sqrt{3}}{10}$ (ii) $\frac{1+6\sqrt{2}}{10}$

Q.9 $\frac{33}{65}$

Q.14 (i) $5\sin(\theta + \phi)$, $\phi = \tan^{-1}\left(\frac{3}{4}\right)$

(ii)$10\sin(\theta + \phi)$, $\phi = \tan^{-1}\left(\sqrt{\frac{7}{3}}\right)$

(iii) $\sqrt{41}\sin(\theta+\phi)$, $\phi=\tan^{-1}\left(\frac{4}{5}\right)$

(iv) $\sqrt{2}\sin(\theta+\phi)$, $\phi=\tan^{-1}(1)$

4.5 Double Angle Identities:

We know that:

$$\text{Sin}(\alpha+\beta)=\text{Sin}\alpha\ \text{Cos}\ \beta+\ \text{Cos}\alpha\ \text{Sin}\beta$$

Putting $\beta=\alpha$, we have

$$\boxed{\text{Sin}\ 2\alpha=2\ \text{Sin}\alpha\ \text{Cos}\alpha}$$

$$\text{Sin}(\alpha+\alpha)=\text{Sin}\alpha\ \text{Cos}\alpha+\ \text{Cos}\alpha\ \text{Sin}\alpha$$

Also $\text{Cos}(\alpha+\alpha)=\ \text{Cos}\alpha\ \text{Cos}\beta-\text{Sin}\alpha\ \text{Sin}\beta$

Again putting $\beta=\alpha$ in this formula, we have

$$\text{Cos}(\alpha+\alpha)=\ \text{Cos}\alpha\ \text{Cos}\alpha-\text{Sin}\alpha\ \text{Sin}\alpha$$

$$\boxed{\text{Cos}2\alpha=\ \text{Cos}^2\alpha-\text{Sin}^2\alpha}$$

$$=\ \text{Cos}^2\alpha-(1-\text{Cos}^2\alpha)=\ \text{Cos}^2\alpha-1+\text{Cos}^2\alpha$$

$$\boxed{\text{Cos}\ 2\alpha=\ 2\text{Cos}^2\alpha-1}$$

Again $\text{Cos}\ 2\alpha=\ \text{Cos}^2\alpha-\text{Sin}^2\alpha$

$$=1-\text{Sin}^2\alpha-\text{Sin}^2\alpha$$

$$\boxed{\text{Cos}\ 2\alpha=1-2\ \text{Sin}^2\alpha}$$

Now, $\tan(\alpha+\beta)=\dfrac{\tan\ \alpha+\tan\ \beta}{1-\tan\ \alpha\ \tan\ \beta}$

Putting $\beta=\alpha$ in this formula, we have

$$\tan(\alpha+\alpha)=\frac{\tan\ \alpha+\tan\ \alpha}{1-\tan\alpha\ \tan\ \alpha}$$

$$\boxed{\tan\ 2\alpha=\frac{2\ \tan\ \alpha}{1-\tan^2\alpha}}$$

4.6 Half Angle identities:

We know that:

$$\text{Cos}\ 2\alpha=1-2\ \text{Sin}^2\alpha$$

Therefore $$2\,\text{Sin}^2\alpha = 1 - \text{Cos}\,2\alpha$$

$$\text{Sin}^2\alpha = \frac{1-\text{Cos}\,2\alpha}{2}$$

$$\text{Sin}\,\alpha = \pm\sqrt{\frac{1-\text{Cos}\,2\alpha}{2}}$$

Putting $2\alpha = \theta \Rightarrow \alpha = \frac{\theta}{2}$ in this formula

We have $$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos 2(\frac{\theta}{2})}{2}}$$

$$\boxed{\text{Sin}\,\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \ldots\ldots\ldots.(i)}$$

Also we know that

$$\text{Cos}\,2\alpha = 2\,\text{Cos}^2\alpha - 1$$

$\therefore$ $$\text{Cos}^2\alpha = \frac{1+\text{Cos}\,2\alpha}{2}$$

Or $$\text{Cos}\,\alpha = \pm\sqrt{\frac{1+\text{Cos}\,2\alpha}{2}}$$

Put $2\alpha = \theta \Rightarrow \alpha = \frac{\theta}{2}$ in this formula we have

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos 2(\frac{\theta}{2})}{2}}$$

$$\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta)}{2}} \quad \ldots\ldots\ldots.(ii)}$$

Now , $$\tan\frac{\theta}{2} = \frac{\text{Sin}\,\frac{\theta}{2}}{\text{Cos}\,\frac{\theta}{2}}$$

From (i) and (ii) $$= \frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}}$$

$$\boxed{\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\text{Cos}\ \theta}{1+\text{Cos}\ \theta}}}$$

Example 1:

If $\text{Sin}\ \theta = \frac{4}{5}$ and the terminal ray of θ is in the second quadrant. Find the value of (i) $\text{Sin}\ 2\theta$ (ii) $\text{Cos}\ \frac{\theta}{2}$

Solution:

Because $$\text{Cos}\ \theta = \sqrt{1-\text{Sin}^2\theta}$$
$$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}}$$
$$= \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

$\text{Cos}\ \theta = -\frac{3}{5}$ because the terminal ray of θ is in 2nd quadrant.

(i) $$\text{Sin}\ 2\ \theta = 2\ \text{Sin}\ \theta\ \text{Cos}\ \theta$$
$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

(ii) $$\text{Cos}\ \frac{\theta}{2} = \sqrt{\frac{1+\text{Cos}\ \theta}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{\frac{2}{5}}{2}}$$
$$= \sqrt{\frac{2}{10}} = \frac{1}{\sqrt{5}}$$

4.7 Triple angle identities:

(i) $\text{Cos}\ 3\theta = 4\ \text{Cos}^3\theta - 3\ \text{Cos}\ \theta$

(ii) $\text{Sin}\ 3\theta = 3\ \sin\theta - 4\ \sin^3\theta$

(iii) $$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

Prove that :

(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\begin{aligned}
\text{L.H.S.} &= \cos 3\theta \\
&= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta) \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \quad = \text{R.H.S.}
\end{aligned}$$

(ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\begin{aligned}
\text{L.H.S.} &= \sin 3\theta \\
&= \sin(2\theta + \theta) \\
&= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\
&= 2\sin\theta\cos\theta\cos\theta + (1 - 2\sin^2\theta)\sin\theta \\
&= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta \quad = \text{R.H.S.}
\end{aligned}$$

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$$\begin{aligned}
\text{L.H.S.} &= \tan 3\theta \\
&= \tan(2\theta + \theta)
\end{aligned}$$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta} \qquad = \frac{\dfrac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta}{1 - \dfrac{2\tan\theta}{1 - \tan^2\theta}\tan\theta}$$

$$\left(\because \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}\right)$$

$$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - \tan^2\theta - 2\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \qquad = \text{R.H.S.}$$

Example 2:

Show that $\text{Cosec}\ 2\theta - \text{Cot}\ 2\theta = \tan\theta$

Solution:

$$\text{L.H.S} = \text{Cosec}\ 2\theta - \text{Cot}\ 2\theta$$

$$= \frac{1}{\text{Sin}\ 2\theta} - \frac{\text{Cos}\ 2\theta}{\text{Sin}\ 2\theta}$$

$$= \frac{1-\text{Cos}\ 2\theta}{\text{Sin}\ 2\theta}$$

$$= \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} \qquad \because\ \text{Cos}\ 2\theta = 1-2\ \text{Sin}^2\theta$$

$$= \frac{2\ \text{Sin}^2\theta}{2\ \text{Sin}\ \theta\ \text{Cos}\ \theta} = \frac{\text{Sin}\ \theta}{\text{Cos}\ \theta}$$

$$\because\ \text{Sin}\ 2\theta = 2\ \text{Sin}\ \theta\ \text{Cos}\ \theta$$

$$= \tan\theta = \text{R.H.S}$$

Example 3:

Using Half angle formula find

(i)Sin 210^o (ii) Cos 210^o (iii) tan 210^0

Solution:

(i) $$\text{Sin}\frac{\theta}{2} = \pm\sqrt{\frac{1-\text{Cos}\ \theta}{2}}$$

$$\text{Sin}\ 210^o = \pm\sqrt{\frac{1-\text{Cos}\ 420^o}{2}} = \pm\sqrt{\frac{1-\text{Cos}\ 60^o}{2}}$$

$$= \pm\sqrt{\frac{1-\frac{1}{2}}{2}} = \pm\sqrt{\frac{\frac{1}{2}}{2}} = \pm\sqrt{\frac{1}{4}}$$

$$\text{Sin}\ 210^o = \pm\frac{1}{2}$$

(ii) $$\text{Cos}\frac{\theta}{2} = \pm\sqrt{\frac{1+\text{Cos}\ \theta}{2}}$$

$$\text{Cos}\ 210^o = \pm\sqrt{\frac{1+\text{Cos}\ 420}{2}} = \pm\sqrt{\frac{1+\text{Cos}\ 60^o}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{1}{2}}{2}} = \pm\sqrt{\frac{\frac{3}{2}}{2}} = \pm\frac{\sqrt{3}}{\sqrt{4}}$$

$$\text{Cos } 210^o = \pm\frac{\sqrt{3}}{2}$$

(iii) $$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\cos 420^o}{1+\cos 420^o}} \quad (\theta = 420^o)$$

$$\tan\frac{420^o}{2} = \sqrt{\frac{1-\cos 60^o}{1+\cos 60^o}} = \sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}}$$

$$\tan 210^o = \sqrt{\frac{\frac{2-1}{2}}{\frac{2+1}{2}}} = \sqrt{\frac{1}{2}\times\frac{2}{3}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

Example 4:

Prove that $\dfrac{\text{Sin } 2\text{ A}}{\text{Sin A}} - \dfrac{\text{Cos } 2\text{ A}}{\text{Cos A}} = \text{Sec A}$

Solution:

$$\text{L.H.S} = \frac{\text{Sin } 2\text{ A}}{\text{Sin A}} - \frac{\text{Cos } 2\text{ A}}{\text{Cos A}}$$

$$= \frac{\text{Sin } 2\text{ A Cos A} - \text{Cos } 2\text{ A Sin A}}{\text{Sin A Cos A}}$$

$$= \frac{\text{Sin }(2\text{A}-\text{A})}{\text{Sin A Cos A}} = \frac{\text{Sin A}}{\text{Sin A Cos A}}$$

$$= \frac{1}{\text{Cos A}}$$

$= \text{Sec A}$

Exercise 4.2

Q.1 If $\text{Cos }\theta = \dfrac{4}{5}$ and the terminal ray of θ is in the first quadrant find the value of

(i) $\text{Sin}\dfrac{\theta}{2}$ (ii) $\text{Cos}\dfrac{\theta}{2}$ (iii) $\tan\dfrac{\theta}{2}$

Q.2 If $\text{Sin}\,\theta = \dfrac{4}{5}$ and the terminal ray of θ is in the first quadrant, find the value of

(i) $\text{Sin}\,2\theta$ (ii) $\text{Cos}\,2\theta$

Q.3 If $\text{Cos}\,\theta = -\dfrac{5}{13}$ and the terminal side of θ is in the second quadrant, find the value of

(i) $\text{Sin}\dfrac{\theta}{2}$ (ii) $\text{Cos}\dfrac{\theta}{2}$

Q.4 If $\tan\,\theta = -\dfrac{1}{5}$, the terminal ray of θ lies in the second quadrant, then find:

(i) $\text{Sin}\,2\,\theta$ (ii) $\text{Cos}\,2\,\theta$

Prove the following identities:

Q.5 $\cos\theta = 2\cos^2\dfrac{\theta}{2} - 1$

Q.6 $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

Q.7 $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

Q.8 $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta}$

Q.9 $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

Q.10 $\dfrac{1+\sin 2A - \cos 2A}{1+\sin 2A + \cos 2A} = \tan A$

Q.11 $\sec 2A + \tan 2A = \dfrac{\cos A + \sin A}{\cos A - \sin A}$

Q.12 $\text{cosec}\,2\theta + \cot 2\theta = \cot\theta$

Q.13 $\dfrac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$

Q.14 $\dfrac{\sin 3\theta}{\cos\theta} + \dfrac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

Q.15 $\dfrac{\cot^2\theta - 1}{\text{cosec}^2\theta} = \cos 2\theta$

Q.16 $\cos^4\theta - \sin^4\theta = \dfrac{1}{\sec 2\theta}$

Q.17 $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta$

Q.18 $(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$

Q.19 $\sin^4\theta = \dfrac{3}{8} - \dfrac{1}{2}\cos 2\theta + \dfrac{1}{8}\cos 4\theta$

Q.20 Compute the value of sin $\dfrac{\pi}{12}$ from the function of $\dfrac{\pi}{6}$

Answers 4.2

Q.1 (i) $\dfrac{1}{\sqrt{10}}$ (ii) $\dfrac{3}{\sqrt{10}}$ (iii) $\dfrac{1}{3}$

Q.2 (i) $\dfrac{24}{25}$ (ii) $-\dfrac{7}{25}$

Q.3 (i) $\dfrac{3}{\sqrt{13}}$ (ii) $\dfrac{2}{\sqrt{13}}$

Q.4 (i) $-\dfrac{5}{13}$ (ii) $\dfrac{12}{13}$

Q.21 $\dfrac{\sqrt{2-\sqrt{3}}}{2}$

4.8 Conversion of sum or difference to products:

We know that

$$\text{Sin}(\alpha+\beta)=\text{Sin }\alpha\text{ Cos }\beta+\text{ Cos }\alpha\text{ Sin }\beta \;\ldots\ldots\; (1)$$

$$\text{Sin}(\alpha-\beta)=\text{Sin }\alpha\text{ Cos }\beta-\text{Cos }\alpha\text{ Sin }\beta \;\ldots\ldots\; (2)$$

$$\text{Cos}(\alpha+\beta)=\text{Cos }\alpha\text{ Cos }\beta-\text{Sin }\alpha\text{ Sin }\beta \;\ldots\ldots\; (3)$$

$$\text{Cos}(\alpha-\beta)=\text{Cos }\alpha\text{ Cos }\beta+\text{Sin }\alpha\text{ Sin }\beta \;\ldots\ldots\; (4)$$

Adding (1) and (2), we get

$$\text{Sin}(\alpha+\beta)+\text{Sin}(\alpha+\beta)=2\text{Sin }\alpha\text{ Cos }\beta \qquad \ldots\ldots \quad (5)$$

Subtracting (2) from (1)

$$\text{Sin}(\alpha+\beta)-\text{Sin}(\alpha-\beta)=2\text{Cos }\alpha\text{ Sin }\beta \qquad \ldots\ldots \quad (6)$$

Adding (3) and (4), we have

$$\text{Cos}(\alpha+\beta)+\text{Cos}(\alpha-\beta)=2\text{Cos }\alpha\text{ Cos}\beta \qquad \ldots\ldots \quad (7)$$

Subtracting (4) from (3), we have

$$\text{Cos}(\alpha+\beta)-\text{Cos}(\alpha-\beta)=-2\text{Sin }\alpha\text{ Sin }\beta \qquad \ldots\ldots \quad (8)$$

With the help of (5), (6), (7) and (8), we have get another set of important formulas

Let $\alpha+\beta=\text{A}$ and $\alpha-\beta=\text{B}$

Adding these, we have

$$2\,\alpha=\text{A}+\text{B} \qquad \Rightarrow \alpha=\frac{\text{A}+\text{B}}{2}$$

Subtracting these, we have

$$2\,\beta=\text{A}-\text{B} \qquad \Rightarrow \beta=\frac{\text{A}-\text{B}}{2}$$

Now putting these values of α and β in formulas from (5) to (8), we get

$$\text{Sin A}+\text{Sin B}=2\text{ Sin}\frac{\text{A}+\text{B}}{2}\text{Cos}\frac{\text{A}-\text{B}}{2} \qquad (9)$$

$$\text{Sin A}-\text{Sin B}=2\text{ Cos}\frac{\text{A}+\text{B}}{2}\text{Sin}\frac{\text{A}-\text{B}}{2} \qquad (10)$$

$$\text{Cos A}+\text{Cos B}=2\text{ Cos}\frac{\text{A}+\text{B}}{2}\text{Cos}\frac{\text{A}-\text{B}}{2} \qquad (11)$$

$$\text{Cos A}-\text{Cos B}=-2\text{ Sin}\frac{\text{A}+\text{B}}{2}\text{Sin}\frac{\text{A}-\text{B}}{2} \qquad (12)$$

4.9 Converting Products to Sum or Difference:

If we write the formulas given in (5) to (8) in reverse order, we have

$$2\text{ Sin }\alpha\text{ Cos }\beta=\text{ Sin}(\alpha+\beta)+\text{Sin}(\alpha-\beta) \qquad (13)$$

$$2\text{ Cos }\alpha\text{ Sin }\beta = \text{Sin}(\alpha+\beta)-\text{Sin}(\alpha-\beta) \qquad (14)$$
$$2\text{ Cos }\alpha\text{ Cos }\beta = \text{Cos}(\alpha+\beta)+\text{Cos}(\alpha-\beta) \qquad (15)$$
$$2\text{ Sin }\alpha\text{ Sin }\beta = \text{Cos}(\alpha-\beta)-\text{Cos}(\alpha+\beta) \qquad (16)$$

The formulas from (13) to (16) express products into sum or difference.

Example 1:

Express $\text{Sin}8\,\theta+\text{Sin}4\,\theta$ as products.

Solution:

We use the formula

$$\text{Sin A} + \text{Sin B} = 2\text{ Sin }\frac{A+B}{2}\text{ Cos}\frac{A-B}{2}$$

$$\text{Sin}8\,\theta\ \text{Sin}4\,\theta = 2\text{ Sin}\frac{8\theta+4\theta}{2}\text{ Cos}\frac{8\theta-4\theta}{2}$$

$$= 2\text{ Sin}\frac{12\,\theta}{2}\text{ Cos}\frac{4\,\theta}{2}$$

$$\text{Sin}8\,\theta\ \text{Sin}4\,\theta = 2\text{ Sin}6\,\theta\ .\ \text{Cos}2\,\theta$$

Example 2:

Express Sin θ + Sin3 θ + Sin5 θ + Sin7 θ as a product.

Solution:

$$\text{Sin }\theta+\text{Sin}3\,\theta+\text{Sin}5\,\theta+\text{Sin}7\,\theta$$
$$= (\text{Sin}7\,\theta+\text{Sin }\theta)+(\text{Sin}5\,\theta+\text{Sin}3\,\theta)$$
$$= 2\text{ Sin}\left(\frac{7\theta+\theta}{2}\right)\text{Cos}\left(\frac{7\theta-\theta}{2}\right)+2\text{Sin}\left(\frac{5\theta+3\theta}{2}\right)\text{Cos}\left(\frac{5\theta-3\theta}{2}\right)$$
$$= 2\text{ Sin}\frac{8\,\theta}{2}\text{ Cos}\frac{6\,\theta}{2}+2\text{ Sin}\frac{8\,\theta}{2}\text{Cos}\frac{2\,\theta}{2}$$
$$= 2\text{Sin}4\,\theta\text{ Cos }3\,\theta+2\text{ Sin}4\,\theta\text{ Cos }\theta$$
$$= 2\text{Sin}4\,\theta\,[\text{Cos }3\,\theta+\text{Cos }\theta]$$
$$= 2\text{ Sin}4\,\theta\left[2\text{ Cos}\frac{3\,\theta+\theta}{2}\text{Cos}\frac{3\,\theta-\theta}{2}\right]$$
$$= 2\text{ Sin}4\,\theta\left[2\text{ Cos}\frac{4\,\theta}{2}\text{Cos}\frac{2\,\theta}{2}\right]$$
$$= 2\text{ Sin}4\,\theta\,[2\text{ Cos}2\,\theta\text{ Cos }\theta]$$
$$= 4\text{ Sin}4\,\theta\text{ Cos}2\,\theta\text{ Cos }\theta$$

Example 3:

Prove that Sin 10^o Sin 30^o Sin 50^o Sin 70^o = $\frac{1}{16}$

Solution:

L.H.S $= \text{Sin } 10^o \text{ Sin } 30^o \text{ Sin } 50^o \text{ Sin } 70^o$

$= \text{Sin } 30^o \text{ Sin } 10^o \text{ Sin } 50^o \text{ Sin } 70^o$

$= \frac{1}{2}[\text{Sin } 10^o \text{ Sin } 50^o] \text{ Sin } 70\text{o}$ because $\text{Sin } 30^o = \frac{1}{2}$

$= \frac{1}{4}[2 \text{ Sin } 10^o \text{ Sin } 50^o] \text{ Sin } 70^o$

Since, $2 \text{ Sin}\alpha \text{ Sin}\beta = \text{Cos}(\alpha-\beta)-\text{Cos}(\alpha+\beta)$

$$= \frac{1}{4}\left[\text{Cos}(10^o-50^o)-\text{Cos}(10^o+50^o)\right]\text{Sin}70^o$$

$$= \frac{1}{4}\left[\text{Cos}(-40^o)-\text{Cos } 60^o\right]\text{Sin}70^o$$

$$= \frac{1}{4}\left[\text{Cos}40^o-\frac{1}{2}\right]\text{Sin}70^o = \frac{1}{4}\left[\frac{2\text{Cos}40^o-1}{2}\right]\text{Sin}70^o$$

$$= \frac{1}{8}\left[2\text{Sin}70^o \text{ Cos}40^o-\text{Sin}70^o\right]$$

We know $2 \text{ Sin}\alpha \text{ Cos}\beta = \text{Sin}(\alpha+\beta)+\text{Sin}(\alpha-\beta)$

$$= \frac{1}{8}\left[\text{Sin}(70^o+40^o)+\text{Sin}(70^o-40^o)-\text{Sin } 70^o\right]$$

$$= \frac{1}{8}\left[\text{Sin } 110^o+\text{Sin}30^o-\text{Sin}70^o)\right]$$

$$= \frac{1}{8}\left[\text{Sin}(180^o-70^o)+\frac{1}{2}-\text{Sin}70^o\right]$$

$$= \frac{1}{8}\left[\text{Sin } 70^o+\frac{1}{2}-\text{Sin}70^o\right] = \frac{1}{8}\left(\frac{1}{2}\right) = \frac{1}{16} = \text{R.H.S}$$

Example 4:

Prove that $\frac{\text{Sin A + 2Sin3A + Sin5A}}{\text{Sin3A + 2Sin5A + Sin7A}} = \frac{\text{Sin3A}}{\text{Sin5A}}$

Solution:

$$\text{L.H.S} = \frac{\text{Sin A + 2Sin3A + Sin5A}}{\text{Sin3A + 2Sin5A + Sin7A}}$$

$$= \frac{\text{(Sin5A + Sin A) + 2Sin3A}}{\text{(Sin7A + Sin3A) + 2Sin5A}}$$

$$= \frac{2\text{Sin}\dfrac{5A + A}{2}\text{Cos}\dfrac{5A - A}{2} + 2\text{Sin}3A}{2\text{Sin}\dfrac{7A + 3A}{2}\text{Cos}\dfrac{7A - 3A}{2} + 2\text{Sin}5A}$$

$$= \frac{2\text{Sin}3A\ \text{Cos}\ 2A + 2\text{Sin}3A}{2\text{Sin}5A\ \text{Cos}2A + 2\text{Sin}5A}$$

$$= \frac{2\text{Sin}3A\ (\text{Cos}\ 2A + 1)}{2\text{Sin}5A\ (\text{Cos}2A + 1)}$$

$$= \frac{\text{Sin}3A}{\text{Sin}5A} = \text{R.H.S}$$

Exercise 4.3

Q.1 Express each of the following sum or difference as products.

(i) $\text{Sin}5\,\theta - \text{Sin}\,\theta$ (ii) $\text{Cos}\,\theta - \text{Cos}5\,\theta$

(iii) $\text{Cos}12\,\theta - \text{Cos}4\,\theta$ (iv) $\text{Sin}\dfrac{50}{3} - \text{Sin}\dfrac{50}{6}$

(v) $\text{Cos}\left(\dfrac{\alpha+\beta}{2}\right)+\text{Cos}\left(\dfrac{\alpha-\beta}{2}\right)$ (vi) $\text{Sin}4\,\theta + \text{Sin}2\,\theta$

Q.2 Express each of the following products as sum or difference.

(i) $2\ \text{Sin}3\,\theta\ \text{Cos}\,\theta$ (ii) $\text{Sin}3\,\theta\ .\ \text{Cos}5\,\theta$

(iii) $\text{Cos}3\,\theta\ .\ \text{Cos}5\,\theta$ (iv) $\text{Sin}\dfrac{\alpha+\beta}{2}\text{Cos}\dfrac{\alpha-\beta}{2}$

Q.3 Express $\text{Sin}3\,\theta + \text{Sin}5\,\theta + \text{Sin}7\,\theta + \text{Sin}9\,\theta$ as a product.

Prove the following identities:

Q.4 $\dfrac{\sin 5\theta - \sin 3\theta}{\cos 5\theta + \cos 3\theta} = \tan\theta$ Q5. $\dfrac{\sin 5\theta + \sin 3\theta}{\cos 5\theta - \cos 3\theta} = -\cot\theta$

Q.6 $\dfrac{\text{Cos}\,\beta + \text{Cos}\,9\beta}{\text{Sin}\,\beta + \text{Sin}\,9\beta} = \cot 5\beta$ Q.7 $\dfrac{\text{Sin}\,3\theta - \text{Sin}\,\theta}{\text{Cos}^2\theta - \text{Sin}^2\theta} = 2\text{Sin}\,\theta$

Q.8 $\dfrac{\text{Sin}\,A + \text{Sin}\,B}{\text{Cos}\,A + \text{Cos}\,B} = \tan\dfrac{A+B}{2}$

Q.9 $\dfrac{\text{Cos}\,2\theta - \text{Cos}\,6\theta}{\text{Cos}\,2\theta + \text{Cos}\,6\theta} = \tan 4\theta\,\tan 2\theta$

Q.10 $\dfrac{\text{Cos }\alpha - \text{Cos }\beta}{\text{Cos }\alpha + \text{Cos }\beta} = -\dfrac{\tan\left(\dfrac{\alpha+\beta}{2}\right)}{\cot\left(\dfrac{\alpha-\beta}{2}\right)}$

Q.11 $\dfrac{\text{Sin }\theta + \text{Sin }2\theta + \text{Sin }3\theta}{\text{Cos }\theta + \text{Cos }2\theta + \text{Cos }3\theta} = \tan 2\theta$

Q.12 $\text{Sin }5\theta + 2\sin 3\theta + \sin\theta = 4\sin 3\theta\cos^2\theta$

Q.13 Show that:

(i) $\dfrac{\text{Sin }75^o - \text{Sin }15^o}{\text{Cos }75^o + \text{Cos }15^o} = \dfrac{1}{\sqrt{3}}$

(ii) $\text{Sin }20^o + \text{Sin }40^o = \text{Cos }10^o$

(iii) $\text{Cos }80^o + \text{Cos }40^o = \text{Cos}20^o$

(iv) $\text{Cos }20^o + \text{Cos }100^o + \text{Cos }140^o = 0$

Prove that.

Q.14 $\text{Sin }20^o\ \text{Sin }40^o\ \text{Sin }60^o\ \text{Sin }80^o = \dfrac{3}{16}$

Q.15 $\text{Sin }10^o\ \text{Sin }30^o\ \text{Sin }50^o\ \text{Sin }70^o = \dfrac{1}{16}$

Q.16 $\text{Sin }20^o\ \text{Sin }40^o\ \text{Sin }80^o\ \text{Sin }90^o = \dfrac{\sqrt{3}}{8}$

Q.17 $\text{Cos }20^o\ \text{Cos }40^o\ \text{Cos }60^o\ \text{Cos }80^o = \dfrac{1}{16}$

Answer 4.3

Q.1 (i) $2\text{ Cos }3\theta\text{ Sin }2\theta$ (ii) $2\text{ Sin }3\theta\text{ Sin }2\theta$

(iii) $2\text{ Cos }8\theta\text{ Cos }4\theta$ (iv) $2\text{ Cos }\dfrac{150}{12}\text{ Sin}\dfrac{50}{12}$

(v) $2\text{ Cos }\dfrac{\alpha}{2}\text{ Cos}\dfrac{\beta}{2}$ (vi) $2\text{ Sin }3\theta\text{ Cos }\theta$

Q.2 (i) $\text{Sin }4\theta + \text{Sin }2\theta$ (ii) $\dfrac{1}{2}[\text{Sin }8\theta - \text{Sin }2\theta]$

(iii) $\dfrac{1}{2}[\text{Cos }8\theta + \text{Cos }2\theta]$ (iv) $\dfrac{1}{2}[\text{Sin}\alpha + \text{Sin}\beta]$

Q.3 $4\text{Cos }\theta\text{ Sin }6\theta\text{ Cos }2\theta$

Short Questions

Write the short answers of the following:

Q.1 Prove that: $\cos(-\beta) = \cos\beta$

Q.2 Prove that: $\sin(-\theta) = -\sin\theta$

Q.3 Prove that: $\tan(-\theta) = -\tan\theta$

Q.4 $\cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta$

Q.5 Prove that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

Q.6 $\sin(\pi + \theta) = -\sin\theta$

Q.7 Show that: $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$

Q.8 $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta$

Q.9 Prove that: $\sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

Q.10 Prove that: $\tan(45^0 + \theta)\tan(45^0 - \theta) = 1$

Q.11 Express: $\sin x \cos 2x - \sin 2x \cos x$ as single term

Q.12 Express: $\cos(a+b)\cos(a-b) - \sin(a+b)\sin(a-b)$ as single term.

Q.13 Prove that: $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

Q.14 Prove that: $\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$

Q.15 Prove that: $\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$

Q.16 Prove that: $\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}$

Q.17 If $\sin\theta = \frac{4}{5}$ and the terminal side of θ lies in 1st quadrant,find $\cos\frac{\theta}{2}$

Q.18 Prove that: $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

Q.19 Prove that: $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$

Q.20 Express the sum as product: $\cos 12\theta + \cos 4\theta$

Q.21 Express $\cos\theta - \cos 4\theta$ as product:

Q.22 Express as sum or difference $2\cos 5\theta \sin 3\theta$

Q.23 Express as sum or difference $\cos 3\theta \cos\theta$

Q.24 Express $\sin(x + 30^0) + \sin(x - 30^0)$ as product

Q.25 Find $\cos\theta$ if $\sin\theta = \frac{7}{25}$ and angle θ is an acute angle.

Answers

Q.11 $-\sin x$ Q.12 $\cos 2a$ Q.17 $\frac{2}{\sqrt{5}}$

Q.20 $2\cos 8\theta \cos 4\theta$ Q.21 $2\sin\frac{5\theta}{2}\sin\frac{3\theta}{2}$

Q.22 $\sin 8\theta - \sin 2\theta$ Q.23 $\frac{1}{2}[\cos 4\theta + \cos 2\theta]$

Q.24 $2\sin x \cos 30^0$ Q.25 $\frac{24}{25}$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Sin $(\alpha+\beta)$ is equal to:

(a) Sin α Cos $\beta +$ Cos α Sin β

(b) Cos α Cos $\beta -$ Sin α Sin β

(c) Sin α Cos $\beta -$ Cos α Sin β

(d) Cos α Cos $\beta +$ Sin α Sin β

__2. Cos $(\alpha-\beta)$ is equal to:

(a) Cos α Cos $\beta -$ Sin α Sin β

(b) Cos α Cos $\beta +$ Sin α Sin β

(c) Cos α Sin $\beta -$ Sin α Cos β

(d) Sin α Cos $\beta +$ Cos α Sin β

__3. tan $(45^{o} - x)$ is equal to:

(a) $\dfrac{\text{Cos x + Sin x}}{\text{Cos x} - \text{Sin x}}$ (b) $\dfrac{1+\tan x}{1-\tan x}$

(c) $\dfrac{1+\cot x}{1-\cot x}$ (d) $\dfrac{\text{Cos x} - \text{Sin x}}{\text{Cos x + Sin x}}$

__4. $\cos\left(\dfrac{\pi}{2}+\theta\right)$ is equal to:

(a) Cos θ (b) –Cos θ

(c) Sin θ (d) –Sin θ

__5. Sin (90o – 0) is equal to:

(a) –Sin θ (b) Sin θ

(c) –Cos θ (d) Cos θ

__6. $\text{Sin}(\pi - \text{x})$ is equal to:

(a) –Sin x (b) Sin x

(c) Cos x (d) –Cos x

__7. $\tan\left(\dfrac{\pi}{2}+\theta\right)$ is equal to:

(a) Tan θ (b) Cot θ

(c) – Cot θ (d) – Tan θ

__8. $\text{Cos}(\pi+\theta)$ is equal to:

(a) Cos θ (b) –Sin θ

(c) –Cos θ (d) Sin θ

__9. $\text{Cos}\left(\frac{\pi}{2}+\theta\right)$ is equal to:

(a) Cos θ (b) Sin θ
(c) –Cos θ (d) –Sin θ

__10. $\tan\left(\frac{3\pi}{2}+\theta\right)$ is equal to:

(a) Tan θ (b) − Tan θ
(c) Cot θ (d) − Cot θ

__11. $\frac{\text{Sin}(\alpha+\beta)}{\text{Cos }\alpha\text{ Sin }\beta}$ is equal to:

(a) $\tan\alpha-\tan\beta$ (b) $\tan\alpha+\tan\beta$
(c) $\text{Sin }\alpha+\text{Sin }\beta$ (d) $\text{Sin }\alpha-\text{Sin }\beta$

__12. $\text{Sin }2\alpha$ is equal to:

(a) $\text{Cos}^2\alpha-\text{Sin}^2\alpha$ (b) $\text{Cos }2\alpha$
(c) $1-\text{Cos}^2\alpha$ (d) $2\text{Sin }\alpha\text{ Cos}\alpha$

__13. $2\text{Cos}^2\frac{\theta}{2}$ is equal to:

(a) $1+\text{Cos}\theta$ (b) $1-\text{Cos}\theta$
(c) $1+\text{Sin }\theta$ (d) $1-\text{Sin }\theta$

__14. $\text{Cos}(\alpha+\beta)-\text{Cos}(\alpha-\beta)$ is equal to:

(a) $2\text{ Sin }\alpha\text{ Cos }\beta$ (b) $2\text{ Cos }\alpha\text{ Sin }\beta$
(c) $2\text{ Cos }\alpha\text{ Cos }\beta$ (d) $-2\text{ Sin }\alpha\text{ Sin }\beta$

__15. Cos A – Cos B is equal to:

(a) $2\text{Cos}\frac{A+B}{2}\text{Cos}\frac{A-B}{2}$ (b) $-2\text{Sin}\frac{A+B}{2}\text{Sin}\frac{A-B}{2}$
(c) $2\text{Sin}\frac{A+B}{2}\text{Cos}\frac{A-B}{2}$ (d) $2\text{Cos}\frac{A+B}{2}\text{Sin}\frac{A-B}{2}$

__16. Sin (A + B) – Sin (A –B) is equal to:

(a) 2 Sin A Cos B (b) 2 Cos A Cos B
(c) –2 Sin A Sin B (d) 2 Cos A Sin B

__17. Cos (A – B) – Cos (A+B) is equal to:

(a) 2 Sin A Sin B (b) –2 Sin A Sin B
(c) 2 Cos A Cos B (d) 2 Cos A Sin B

__18. $\text{Sin }5\theta-\text{Sin }2\theta$ is equal to:

(a) 2Sin 3θ Cos 2θ (b) 2Cos 3θ Sin 2θ
(c) 2Cos 3θ Cos 2θ (d) –2Cos 3θ Sin 2θ

__19. $\sin 5\theta + \sin\theta$ is equal to:

(a) $2\sin 3\theta \cos 2\theta$ (b) $-2\cos 3\theta \sin 2\theta$

(c) $2\cos 3\theta \sin 2\theta$ (d) $2\sin 3\theta \sin 2\theta$

__20. $2\sin 6\theta \cos 2\theta$ is equal to:

(a) $\sin 8\theta + \sin 4\theta$ (b) $\sin 8\theta - \sin 4\theta$

(c) $\cos 8\theta + \cos 4\theta$ (d) $\cos 8\theta - \cos 4\theta$

Answers

1.	a	2.	b	3.	d	4.	d	5.	d
6.	b	7.	c	8.	c	9.	d	10.	d
11.	b	12.	d	13.	a	14.	d	15.	a
16.	d	17.	a	18.	c	19.	a	20.	a

Chapter 5
Solution of Triangles

5.1 Solution of Triangles:

A triangle has six parts in which three angles usually denoted by α, β, γ and the three sides opposite to α, β, γ denoted by a, b, c respectively. These are called the elements of Triangle. If any three out of six elements at least one side are given them the remaining three elements can be determined by the use of trigonometric functions and their tables.

This process of finding the elements of triangle is called the solution of the triangle.

First we discuss the solution of right angled triangles i.e. triangles which have one angle given equal to a right angle.

In solving right angled triangle γ denotes the right angle. We shall use the following cases

Case-I:

When the hypotenuse and one Side is given.

Let a & c be the given side and hypotenuse respectively. Then angle α can be found by the relation.

$$\text{Sin } \alpha = \frac{a}{c}$$

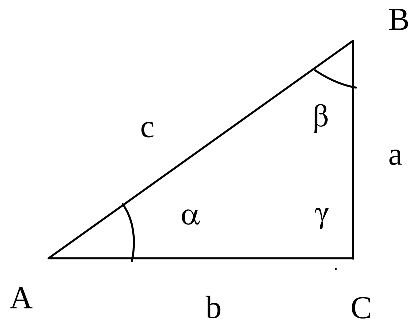


Fig.5.1

Also angle β and side "b" can be obtained by the relations

$\beta = 90^o - \alpha$ and $\text{Cos } \alpha = \dfrac{b}{c}$

Case-II:

When the two sides a and b are given. Here we use the following relations to find α , β & c.

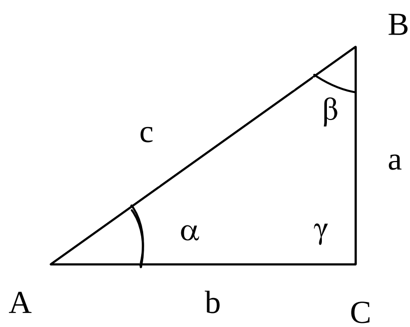


Fig. 5.2

$\text{Tan } \alpha = \dfrac{a}{b}$, $\beta = 90^o - \alpha$,

$c = \sqrt{a^2 + b^2}$

Case-III:

When an angle α and one of the sides b , is given. The sides a , c and β are found from the following relations.

Tan $\alpha = \frac{a}{b}$ and Cos $\alpha = \frac{b}{c}$, $\beta = 90^o - \alpha$

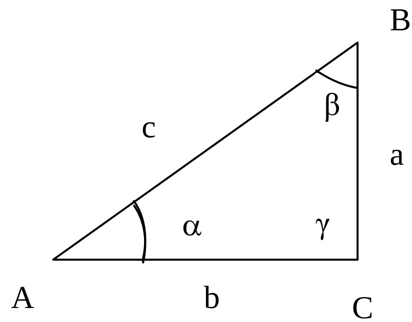


Fig.5.3

Case-IV:

When an angle α and the hypotenuse ‘c’ is given.The sides a , b and β can be found from the following relations.

Sin $\alpha = \frac{a}{c}$, Cos $\alpha = \frac{b}{c}$ and $\beta = 90^o - \alpha$

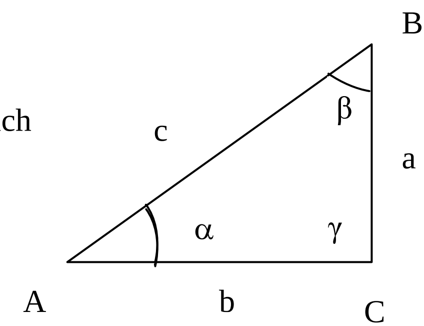


Fig.5.4

Example-1:

Solve the right triangle ABC in which

$\alpha = 34^o\ 17'$, b=31.75, $\gamma = 90^o$

Solution:

Given that

$\alpha = 34^o\ 17'$, b = 31.75, $\gamma = 90^o$

We have to find

a = ? c = ? β = ?

$$\text{Tan } \alpha = \frac{a}{b}$$

$$\text{Tan } 34^o\ 17' = \frac{a}{31.75}$$

$\Rightarrow$ a = 31.75 tan 34^o 17’

a = 31.75 (0.6817) = 21.64

Also $\cos\alpha = \frac{b}{c}$

$$\Rightarrow \quad \text{Cos } 34^o\ 17' = \frac{31.75}{c}$$

$$C = \frac{31.75}{\text{Cos } 34^o\ 17'} \quad \Rightarrow \quad \beta = 90^o - 34^o\ 17' = 55^o\ 43'$$

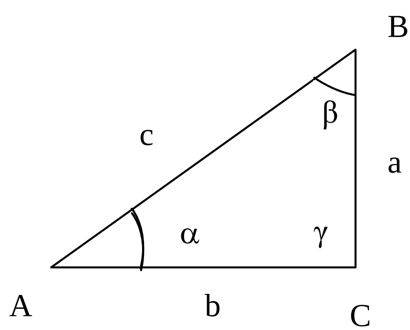


Fig.5.5

Example 2:

Solve the right ΔABC in which $\gamma = 90^o$, a = 450, b = 340

Solution:

$a = 450, \quad b = 340, \quad \gamma = 90^o,$

$c = ? = \alpha = ? \quad \beta = ?$

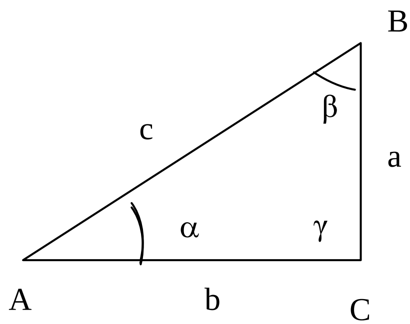


Fig. 5.6

$$\text{Tan}\,\alpha = \frac{a}{b}$$

$$\text{Tan}\,\alpha = \frac{450}{340} = 1.3231$$

$$\Rightarrow \quad \alpha = 52^o\ 56'$$

$$\beta = 90^o - \alpha = 90^o - 52^o\ 56' = 37^o\ 4'$$

By Pythagoras theorem:

$$C^2 = a^2 + b^2 = (450)^2 + (340)^2 = 318100$$

$$C = 564$$

Exercise 5.1

Solve the right triangle ABC in which $\gamma = 90^o$

(1)	$a = 250,$	$\alpha = 42^o\ 25'$	(2)	$a = 482,$	$\alpha = 35^o\ 36'$
(3)	$a = 5\ ,$	$c = 13$	(4)	$b = 312,$	$\alpha = 23^o\ 42'$
(5)	$a = 212,$	$\beta = 40^o\ 55'$	(6)	$c = 232,$	$\beta = 52^o\ 46'$
(7)	$c = 540,$	$a = 380$			

Answers 5.1

1.	$\beta = 47^o\ 35'$	$b = 273.63$	$c = 370.64$
2.	$\beta = 54^o\ 24'$	$b = 673.25$	$c = 828.01$
3.	$b = 12$	$\alpha = 22^o\ 37'$	$\beta = 67^o\ 23'$
4.	$a = 136.96$	$c = 340.72$	$\beta = 66^o\ 18'$
5.	$\alpha = 49^o\ 05'$	$b = 183.74$	$c = 280.5$
6.	$a = 184.72$	$b = 140.37$	$\alpha = 37^o\ 14'$
7.	$b = 383.61$	$\alpha = 44^o\ 44'$	$\beta = 45^o\ 16'$

5.2 Application of Right Angled Triangles

(Measurement of Heights and Distances)

Sometimes we deal with problems in which we have to find heights and distances of inaccessible objects.

The solution of these problems are generally the same as that of solving the right triangles.

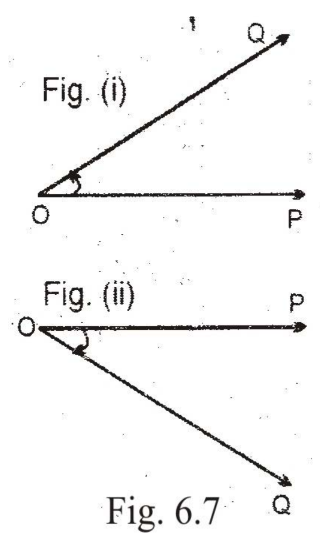


Fig. 5.7

5.3 Angle of Elevation and Depression:

If O be the eye of the observer, Q the position of the object and OP a horizontal line through O then:

i. If Q be above OP, then <POQ is called angle of elevation is shown in Figure (1)

ii. If Q be below OP, then <POQ is called angle of depression is shown in Figure (2)

Example 1:

Find the distance of man from the foot of tower 100m high if the angle of elevation of its top as observed by the man is $52^o\ 30'$.

Solution:

Let, A be the position of man and B be the foot of tower BC. Height of tower = BC = 100m in right Δ AB

$$\text{Tan } 52^o\ 32' = \frac{BC}{AB}$$

$$1.3032 = \frac{100}{AB} \Rightarrow AB = \frac{100}{1.3032} = 78.73m$$

AB = distance of man from the foot of tower = 76.73m

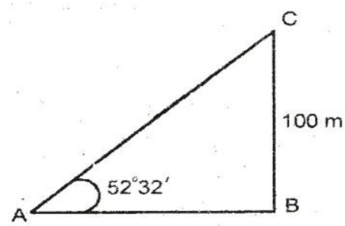


Fig. 6.8

Example 2:

From the two successive positions on the straight road 1000 meters apart man observes that the angle of elevation of the top a directly ahead of him are of $12^o\ 10'$ and $42^o\ 35'$. How high is the tower above the road.

Solution:

Let, A and D be the two successive positions of a man on the road.

AD = 1000m (Given)

Let BC = height of tower = h = ?

And DB = xm

In ΔABC

$$\tan 12^o\ 10' = \frac{BC}{AB}$$

$$0.2156 = \frac{h}{(x + 1000)}$$

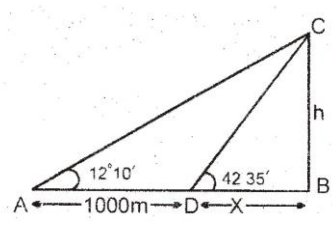


Fig. 6.9

Fig. 5.9

$h = 0.2156\ (x + 1000)$ …………(1)

In ΔDBC

$$\tan 42^o\ 35' = \frac{BC}{DB} = \frac{h}{x}$$

$0.9190 = \frac{h}{x}$

$x = \frac{h}{0.9190}$

Put in (1)

$h = 0.2156\left[\frac{h}{0.9190} + 100\right]$

$h = \frac{0.2156}{0.9190}h + \frac{(0.2156)(100)}{0.9190}$

h = 0.2346h + 215.6

h = 0.2346h = 2156

0.7654h = 215.6

$h = \frac{2156}{.7654} = 28.168$

Example 3:

Measure of the angle of elevation of the top of a flag staff observed from a point 200 meters from its foot is :

Solution:

Let height of flog staff = BC = h = ?

A = point of observation

In right ΔABC

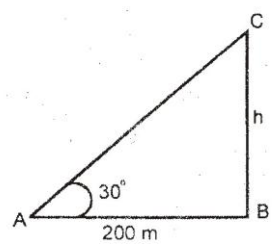


Fig. 6.10

$\tan 30^o = \frac{h}{200} \Rightarrow$ h = 200 (0.577)

h = 115.4m

Example 4:

Find the measures of the angle of elevation of the top of a tree 400 meters high, when observed from a point 250 meters away from the foot of the base.

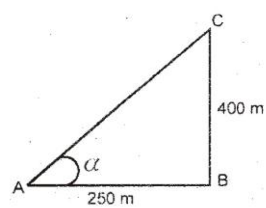
Solution: Given that:

Height of tree = BC = 400m

AB = 250m

Let

$\angle BAC = \alpha$ = ?



$\angle$BAC= angle of elevation of top of the tree

Fig. 6.11

$\tan\alpha = \frac{BC}{AB} = \frac{400}{250} = 1.6$

$\alpha = \tan^{-1}(1.6) = 58^o$

Example 5:

The measure of the angle of depression of an airport as observed by a pilot while flying at a height of 5000 meters is $40^o\ 32'$. How far is the pilot from a point directly over the airport?

Solution:

The pilot is at the height of C

BC = 5000m

From right $\angle ABC$

$$\tan 40^o\ 32' = \frac{5000}{x}$$

$$x = \frac{5000}{\tan 40^o 32'} = \frac{5000}{0.8551} = 584736\text{m}$$

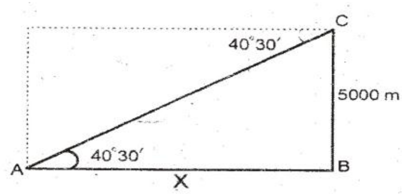


Fig. 6.12
Fig. 5.12

Example 6: From a point on the ground the measure of angle of elevation of the top of tower is 30^o. On walking 100 meters towards the tower the measure of the angle is found to be of 45^o. Find the height of the tower.

Solution:

Let BC = height of tower

= h = ?

And DB = x m

AD = 100 m

AB = 100 + x

In right ΔABC

$$\text{Tan}30^o = \frac{BC}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{100+x}$$

$100 + x = \sqrt{3}\,h$ …………………. (1)

In right ΔBDC

$$\tan 45^o = \frac{h}{x}$$

$$1 = \frac{h}{x}$$

$x = h$ ……………….. (2)

Put x = h in (1)

$100 + h = \sqrt{3}\,h$

$1.7321h - h = 100$

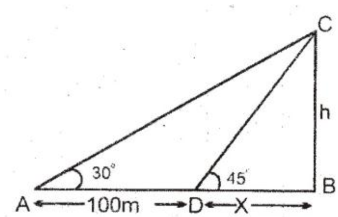


Fig. 6.13

$$h = \frac{100}{0.7321} = 136.60\text{m}$$

Example 7:

A pole being broken by the wind, its top struck ground at an angle of 30^o and at a distance of 10m from the foot of the pole. Find the whole height of the pole.

Solution:

Let BC = h = height of pole = ?

AD = CD

In right Δ ABD

$$\text{Tan}30^o = \frac{BD}{10}$$

BD = 10tan30^o = 10(0.5774) = 5.77m

Also

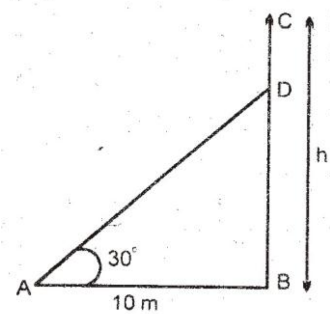


Fig. 6.14

$$\text{Cos}30^o = \frac{AB}{AD} \Rightarrow AD = \frac{10}{\text{Cos}30^o} = \frac{10}{0.8660} = 11.55\text{m}$$

Height of pole = h = BD + AD

$\therefore$ AD = CD

h = 11.55 + 5.77 = 17.32m

Exercise 5.2

Q1. How far is a man from the foot of tower 150 meters high, if the measure of the angle of elevation of its top as observed by him is $40^o\,30'$.

Q2. The shadow of a building is 220 meters when the measure of the angle of elevation of the sun is 35^o. Find the height of the building.

Q3. The measure of the angle of elevation of a kite is 35. The string of the kite is 340 meters long. If the sag in the string is 10 meters. Find the height of the kite.

Q4. A man 18dm. tall observes that the angle of elevation of the top of a tree at a distance of 12m from the man is 32^0.What is the height of the tree?

Q5. On walking 300 meters towards a tower in a horizontal line through its base, the measure of the angle of elevation of the top changes from 30^o to 60^o. Find the height of the tower.

Q6. The measure of the angle of elevation of the top of a cliff is 25. On walking 100 meters straight towards the cliff, the measure of the angle of elevation of the top is 48^o. Find the height of the cliff.

Q7. From two points A and B. 50 meters apart and in the line with a tree, the measures of the angles of elevation of the top of the tree are 30^o and 40^o respectively. Find the height of the tree.

Q8. Two men on the opposite sides of a tower observe that the measures of the angles of elevation of the tower as observed by them separately are 15^o and 25^o respectively. If the height of the tower is 150 meters. Find the distance between the observers.

Q9. From a light-house, angles of depression of two ships on opposite of the light-house are observed to be 30^o and 45^o. If the height of the light house be 300m. Find the distance between the ships of the line joining them passes through foot of light-house.

Q10. The measure of angle elevation of the top of a tower is 30^o from a point on the ground. On retreating 100 meters, the measure of the angle of elevation is found to be 15^o. Find the height of the tower.

Q11. From the top of a hill 200 meters high, the angles of depression of the top and bottom of a tower are observed to be 30^o and 60^o respectively. Find the height of the tower.

Q12. A television antenna is on the roof of a building. From a point on the ground 36m from the building, the angle of elevation of the top and the bottom of the antenna are 51^o and 42^o respectively. How tall is the antenna?

Q13. A ladder 20 meter long reaches the distance of 20 meters, from the top of a building. At the foot of the ladder the measure of the angle of elevation of the top of the building is 60^o. Find the height of the building.

Q14. A man standing on the bank of a canal observes that the measure of the angle of elevation of a tree is 60^0.On retreating 40m from the bank , he finds the measure the angle of elevation of the tree as 30^0.Find the height of the tree and the width of the canal.

Q15. Two buildings A and B are 100m apart. The angle of elevation from the top of the building A to the top of the building B is 20^0. The angle of elevation from the base of the building B to the top of the building A is 50^0. Find the height of the building B.

Answers 5.2

(1)	175.63m	(2)	154.05m	(3)	h = 189.29m
(4)	9.6m	(5)	h = 259.81m	(6)	h = 80.37m
(7)	h = 17.10m	(8)	881.58m	(9)	819.6m
(10)	49.98m	(11)	133.3m	(12)	12.1m
(13)	h = 30m	(14)	34.64m ; 20m	(15)	155.5 m

5.4 Law of Sines:

In any triangle, the length of the sides are proportional to the sines of measures of the angle opposite to those sides. It means

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma}$$

Proof: Let one angle of the triangle say β be acute, then γ will be either acute, obtuse or right as in figure 1, 2, 3.

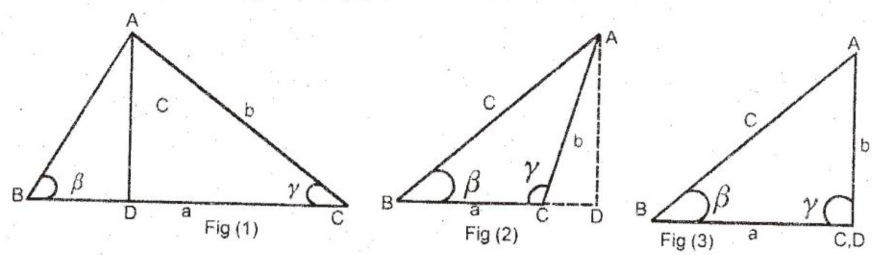


Fig. 6.15

Draw $AD \perp BC$ or BC produced.

Then from ΔABC (for all figures)

$$\frac{AD}{AB} = \text{Sin}\beta \qquad \therefore\ AD = c\text{Sin}\beta \ldots\ldots\ldots\ldots\ldots.. \quad (1)$$

If γ is acute in figure (1) $\frac{AD}{AC} = \text{Sin}\ \gamma \qquad \Rightarrow AD = b\ \text{Sin}\ \gamma$

If γ is obtuse in figure (2) $\frac{AD}{AC} = \text{Sin}\ (180 - \gamma) = \text{Sin}\ \gamma$

$$\Rightarrow AD = b\ \text{Sin}\ \gamma$$

If γ is right in figure (3) $\frac{AD}{AC} = 1 = \text{Sin}\ 90^{o} = \text{Sin}\ \gamma$

$$AD = b\ \text{Sin}\ \gamma$$

In each case we have

$$AD = b\ \text{Sin}\ \gamma \ldots\ldots\ldots\ldots \quad (2)$$

From (1) & (2), we have

It can similarly be proved that:

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta}, \text{ Similarly, } \frac{a}{\text{Sin}\alpha} = \frac{c}{\text{Sin}\ \gamma}$$

Hence ,

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma}$$

This is known as law of sines.

Note: we use sine formula when

i. one side and two angles are given

ii. two sides and the angle opposite one of them are given

Example 1:

In any ΔABC

$a = 12$, $b = 7$, $\alpha = 40^o$ Find β

Solution:

By law of sines $\dfrac{a}{\text{Sin}\alpha} = \dfrac{b}{\text{Sin}\beta} = \dfrac{c}{\text{Sin}\gamma}$

$$\Rightarrow \frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta} = \frac{12}{\text{Sin}40^o} = \frac{7}{\text{Sin}\beta}$$

$$\Rightarrow \text{Sin}\beta = \frac{7\text{Sin}40^o}{12} = \frac{7(0.6429)}{12}$$

$\text{Sin}\ \beta = 0.3750$

$$\Rightarrow \beta = \text{Sin}^{-1}(.3750) \qquad \Rightarrow \beta = 22^o1'$$

Example 2:

In any ΔABC, $b = 24$, $c = 16$

Find the ratio of $\text{Sin}\beta$ to $\text{Sin}\gamma$

Solution:

By law of sines

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma}$$

$$\Rightarrow \quad \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma} \Rightarrow \frac{\text{Sin}\beta}{\text{Sin}\gamma} = \frac{b}{c} = \frac{24}{16} = \frac{3}{2}$$

Example 3:

A town B is 15 km due North of a town A. The road from A to B runs North 27^o, East to G, then North 34^o, West to B. Find the distance by road from town A to B.

Solution:

Given that: $c = 15$ km $\quad \alpha = 24^o$, $\quad \beta = 34^o$

We have to find

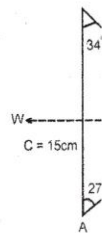
Distance from A to B by road.

Since $\alpha + \beta + \gamma = 180^o \quad \Rightarrow \quad 27^o + 34^o + \gamma = 180^o$

$$\gamma = 119^o$$

By law of sines:

$$\frac{a}{\text{Sin}\alpha}=\frac{b}{\text{Sin}\beta}=\frac{c}{\text{Sin}\gamma}$$



$$\Rightarrow \quad \frac{a}{\text{Sin}\alpha}=\frac{c}{\text{Sin}\gamma} \quad \Rightarrow \quad a=\frac{\text{Sin}\alpha}{\text{Sin}\gamma}$$

$$a=\frac{15\text{Sin}27^o}{\text{Sin}119^o}=\frac{15(0.4539)}{0.8746}=7.78$$

Fig. 5.16

$$\text{Also } \frac{b}{\text{Sin}\beta}=\frac{c}{\text{Sin}\gamma} \quad \Rightarrow \quad \frac{b}{\text{Sin}34^o}=\frac{15}{\text{Sin}119^o}$$

$$b=\frac{15\text{Sin}34^o}{\text{Sin}119^o}=\frac{15(0.5592)}{0.8746}=9.59$$

Thus distance from A to B by road:

$= b + a = 9.59 + 7.78 \qquad = 17.37\text{km}$

Exercise 5.3

In any triangle ABC if:

Q1. $a = 10$ $\quad b = 15$ $\quad \beta = 50^o$ Find α

Q2. $a = 20$ $\quad c = 32$ $\quad \gamma = 70^o$ Find α

Q3. $a = 3$ $\quad b = 7$ $\quad \beta = 85^o$ Find α

Q4. $a = 5$ $\quad c = 6$ $\quad \alpha = 45^o$ Find γ

Q5. $a = 20\sqrt{3}$ $\quad \alpha = 75^o$ $\quad \gamma = 60^o$ Find c

Q6. $a = 211.3$ $\quad \beta = 48^o16'$ $\quad \gamma = 71^o\,38'$ Find b

Q7. $a = 18$ $\quad \alpha = 47^o$ $\quad \beta = 102^o$ Find c

Q8. $a = 475$ $\quad \beta = 72^o15'$ $\quad \gamma = 43^o\,30'$ Find b

Q9. $a = 82$ $\quad \beta = 57^o$ $\quad \gamma = 78^o$ Find a

Q10. $\alpha = 60^o$ $\quad \beta = 45^o$ Find the ratio of b to c

Q11. Two shore batteries at A and B, 840 meters apart are firing at a target C. The measure of angle ABC is 80^o and the measure of angle BAC is 70^o. Find the measures of distance AC and BC.

Answers 5.3

1. $\alpha = 30^o\,42'\,37''$ 2. $\alpha = 35^o\,37'\,58''$

3.	$\alpha = 25^o\ 16^/\ 24^{//}$	4.	$\gamma = 58^o\ 3^/$
5.	c = 31.06	6.	b= 181.89
7.	c = 12.68	8.	b = 449.22
9.	a = 69.13	10.	0.7319
11.	1578.68, 1654.46m		

5.5 The Law of Cosines:

This law states that “the square of any sides of a triangle is equal to the sum of the squares of the other two sides minus twice their product times the cosine of their included angle. That is

$$a^2 = b^2 + c^2 - 2bc \cos\alpha$$
$$b^2 = c^2 + a^2 - 2ac \cos\beta$$
and $$c^2 = a^2 + b^2 - 2ab \cos\gamma$$

Proof:

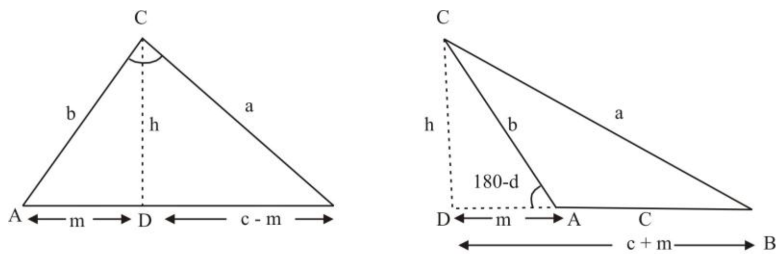


Fig. (I) Fig. (II)

Fig. 6.17

Fig.5.17

Let β be an acute angle of ΔABC, draw CD$\perp$AB

Let AD = m and CD = h

In right triangle BCD, we have

$$(BC)^2 = (BD)^2 + (CD)^2$$
$$a^2 = (BD)^2 + h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

(i) If α is an acute angle, then from (i)

In right triangle ACD,

$$\text{Sin}\,\alpha = \frac{h}{b} \quad \Rightarrow h = b\ \text{Sin}\ \alpha$$

and $$\text{Cos}\,\alpha = \frac{m}{b} \quad \Rightarrow m = b\ \text{Cos}\ \alpha$$

So, $\boxed{BD = c - m = c - b\cos\alpha}$

Putting the values of h and BD in equation (1)

$$\begin{aligned} a^2 &= (c - b\cos\alpha)^2 + (b\sin\alpha)^2 \\ &= c^2 - 2bc\cos\alpha + b^2\cos^2\alpha + b^2\sin^2\alpha \\ &= c^2 - 2bc\cos\alpha + b^2(\cos^2\alpha + \sin^2\alpha) \end{aligned}$$

$= c^2 - 2bc\cos\alpha + b^2$

$\boxed{a^2 = b^2 + c2 - 2bc\cos\alpha}$

(ii) If α in an obtuse angle, then from fig (ii)

In right triangle ACD,

$$\text{Sin}\,(180 - \alpha) = \frac{h}{b}$$

$$\text{Sin}\,\alpha = \frac{h}{b} \quad \Rightarrow h = b\sin\alpha$$

and $$\cos(180 - \alpha) = \frac{m}{b}$$

$$-\cos\alpha = \frac{m}{b} \quad \Rightarrow m = -b\cos\alpha$$

So, $\boxed{BD = c + m = c - b\cos\alpha}$

Putting the values of h and BD in equation (1)

$$a^2 = (c - b\cos\alpha)^2 + (b\sin\alpha)^2$$

we get, $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Similarly we obtain

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

and $c^2 = b^2 + c^2 - 2bc\cos\alpha$

Also when three sides are given, we find

$$\text{Cos}\,\alpha = \frac{b^2+c^2-a^2}{2bc},\ \text{Cos}\,\beta = \frac{a^2+c^2-b^2}{2ac} \text{ and}$$

$$\text{Cos}\,\gamma = \frac{a^2+b^2-c^2}{2ab}$$

Note: we use the cosine formula, when

(i) Two sides and their included angle are given.

(ii) When the three sides are given.

Example 1: In any by using the law of cosines

$a = 7$, $c = 9$, $\beta = 112^0$ Find b

Solution: By law of cosines

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$b^2 = (7)^2 + (9)^2 - 2(7)(9)\,\text{Cos}\,112^o$

$= 49 + 81 - 126(.3746)$

$b^2 = 130 + 47.20 = 177.2$

$b = 13.31$

Example 2: Two man start walking at the same time from a cross road, both walking at 4 km/hour. The roads make an angle of measure 80^o with each other. How far apart will they be at the end of the two hours?

Solution: Let, A be the point of starting of two man V = 4 km/hour

Distance traveled by two men after 2 hours = vt

= 4 x 2 = 8km

Thus, we have to find BC = a = ?

By law of cosine:

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$a^2 = (8)^2 + (8)^2 - 2(8)(8)\cos 80^o = 128 - 128\,(0.1736)$

$a^2 = 105.77 \quad \Rightarrow \quad a = 10.28km$

Thus, two men will apart 10.28 km after two hours.

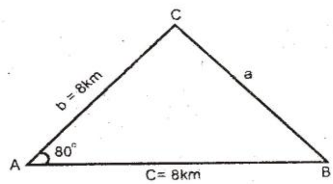


Fig. 6.18

Fig. 5.18

Exercise 5.4

In any triangle ABC by using the law of cosines:

1. a = 56 c = 30 $\beta = 35^o$ Find b
2. b = 25 c = 37 $\alpha = 65^o$ Find a
3. b = 5 c = 8 $\alpha = 60^o$ Find a
4. a = 212 c = 135 $\beta = 37^o\,15^/$ Find b
5. a = 16 b = 17 $\gamma = 25^o$ Find c
6. a = 44 b = 55 $\gamma = 114^o$ Find c
7. a = 13 b = 10 c = 17 Find α and β
8. Three villages P, Q and R are connected by straight roads. Measure PQ is 6 km and the measure QR is 9km. The measure of the angle between PQ and QR is 120^o. Find the distance between P and R.
9. Two points A and B are at distance 55 and 32 meters respectively from a point P. The measure of angle between AP and BP is 37^o. Find the distance between B and A.
10. Find the cosine of the smallest measure of an angle of a triangle with 12, 13 and 14 meters as the measures of its sides.

Answers 5.4

1.	b = 35.83	2.	a = 34.83	3.	a =7
4.	b = 132.652	5.	c = 7.21	6.	c = 83.24
7.	$\alpha = 49^o\,40^/\,47^{//}$		$\beta = 35^o\,54^/\,30^{//}$		
8.	13.08km	9.	35.18m	10.	52^o 37’

5.6 Solution of Oblique Triangles:

Definition:

The triangle in which have no right angle is called oblique triangle.

A triangle has six elements (i.e. three sides and three angles) if any three of a triangle are given, provided at least one of them is a side, the remaining three can be found by using the formula discussed in previous articles i.e. law of sines and law of cosines.

There are four important cases to solve oblique triangle.

Case I: Measure of one side and the measures of two angles.

Case II: Measure of two sides and the measures of the angle included by them.

Case III: When two sides and the angle opposite to one of them is given.

Case IV: Measure of the three sides.

Example 1:

Solve the ABC with given data.

$a = 850, \quad \alpha = 65^o, \quad \beta = 40^o$

Solution:

Given that:

$a = 850, \quad \alpha = 65^o, \quad \beta = 40^o$

$b = ? \quad c = ? \quad \gamma = ?$

Since, $\alpha + \beta + \gamma = 180^o$

$65^o + 40^o + \gamma = 180 \ \gamma \quad \Rightarrow \ = 75^o$

By law of sines to find b:

$$\frac{a}{\text{Sin } \alpha} = \frac{b}{\text{Sin } \beta}$$

$$\frac{850}{\text{Sin } 65^o} = \frac{b}{\text{Sin } 40^o} \Rightarrow \quad b = \frac{850 \text{ Sin } 40^o}{\text{Sin } 65^o}$$

$$b = \frac{850(0.64.28)}{0.9063} = 602.85$$

To find c, by law of Sines

$$\frac{b}{\text{Sin } \beta} = \frac{a}{\text{Sin } \alpha} \quad \Rightarrow c = \frac{b \text{ Sin } \gamma}{\text{Sin } \beta}$$

$$c = \frac{602.85 \text{ Sin } 75^o}{\text{Sin } 40^o} = \frac{602.85(0.9659)}{0.6428}$$

$= \quad 905.90$

Example2:

Solve the triangle with given data:

$a = 45 \quad b = 34 \quad \gamma = 52^o$

Solution:

Given $a = 45$ $\quad b = 34$ $\quad \gamma = 52^o$

$c = ?$ $\quad \beta = ?$ $\quad \gamma = ?$

To find c, we use law of cosines

$c^2 = a^2 + b^2 - 2ab \text{ Cos } \gamma$

$c^2 = (45)^2 + (34)^2 - 2(45)(34) \text{ Cos } 52^o$

$c^2 = 2025 + 1156 - (3060)(0.6157)$

$c^2 = 1297 \quad \Rightarrow \quad \boxed{c = 36.01}$

To find α, we use law of sines

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\gamma} \quad \Rightarrow \quad \frac{45}{\text{Sin}\alpha} = \frac{36.01}{\text{Sin}52^o}$$

$$\text{Sin}\alpha = \frac{45 \text{ Sin}52^o}{36.01} = \frac{45(0.7880)}{36.01} = 0.9847$$

$\alpha = \text{Sin}^{-1}(0.9847) = 97^o\ 58'\ 39''$

To find β, we use $\alpha + \beta + \gamma = 180^o$

$79^o\ 58'\ 39'' + \beta + 52^o = 180^o$

$\beta = 48^o\ 01'\ 20''$

Exercise 5.5

Solve the triangle ABC with given data.

Q1. $c = 4$ $\quad \alpha = 70^o$ $\quad \gamma = 42^o$

Q2. $a = 464$ $\quad \beta = 102^o$ $\quad \gamma = 23^o$

Q3. $b = 85$ $\quad \beta = 57^o 15'$ $\quad \gamma = 78^o 18'$

Q4. $b = 56.8$ $\quad \alpha = 79^o 31'$ $\quad \beta = 44^o 24'$

Q5. $b = 34.57$ $\quad \alpha = 62^o 11'$ $\quad \beta = 63^o 22'$

Q 6. Find the angle of largest measure in the triangle ABC where:

(i) $a = 224$ $\quad b = 380$ $\quad c = 340$

(ii) $a = 374$ $\quad b = 514$ $\quad c = 425$

Q7. solve the triangle ABC where:

(i) $a = 74$ $\quad b = 52$ $\quad c = 47$

(ii) $a = 7$ $\quad b = 9$ $\quad c = 7$

(iii) $a = 2.3$ $\quad b = 1.5$ $\quad c = 2.7$

Answers 5.4

Q1.	$a = 5.62$	$b = 5.54$	$\beta = 68^o$	
Q2.	$\alpha = 55^o$	$b = 454$	$c = 221.31$	
Q3.	$\alpha = 44^o27'$	$a = 70.78$	$c = 98.97$	
Q4.	$a = 79.82$	$c = 67.37$	$\gamma = 56^o\,0'$	
Q5.	$a = 34.20$	$c = 31.47$	$\gamma = 54^o\,2'$	
Q6.	(i) $81^o\,55'\,57''$	(ii) $79^o\,47'\,53''$		
Q7.	(i)	$\alpha = 96^o37'$	$\beta = 44^o16'$	$\gamma = 39^o\,07'$
	(ii)	$\alpha = 50^o$	$\beta = 80^o$	$\gamma = 50^o$
	(iii)	$\alpha = 58^o21'$	$\beta = 33^o45'$	$\gamma = 87^o\,55'$

Summary

1. **Right Triangle:**
   A triangle which has one angle given equal to a right angle.
2. **Oblique Triangle:**
   The triangle in which have no right angle is called oblique triangle.
3. **Law of Sines**
   In any ΔABC, the measures of the sides are proportional to the sines of the opposite angles.
   i.e. $\dfrac{a}{\text{Sin}\alpha} = \dfrac{b}{\text{Sin}\beta} = \dfrac{c}{\text{Sin}\gamma}$
4. **Law of Cosines**
   (i) $a^2 = b^2 + c^2 - 2bc\,\text{Cos}\,\alpha$
   (ii) $b^2 = a^2 + c^2 - 2ac\,\text{Cos}\,\beta$
   (iii) $c^2 = a^2 + b^2 - 2ab\,\text{Cos}\,\gamma$
   (iv) $\text{Cos}\,\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$
   (v) $\text{Cos}\,\beta = \dfrac{a^2 + c^2 - b^2}{2ac}$
   (vi) $\text{Cos}\,\gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$

Angle of Elevation:

The angle AOP which the ray from an observer's eye at O to an object P at O to an object P at a higher level, makes with horizontal ray OA through O is called the angle of elevation.

Angle of Depression:

The angle AOP which the ray from an observer's eye at O to an object at P at a lower level makes with the horizontal ray OA through O is called the angle of depression.

Short Questions

Write the short answers of the following:

Q.1: Define the law of sine.

Q.2: Define the Laws of cosines

Q.3: In right triangle ABC, $\gamma = 90^o$, $a = 5$, $c = 13$ then find the value of angle α .

Q.4: Given that $\gamma = 90^o$, $\alpha = 35^{o,}$ $a = 5$, find angle β

Q.5: In right triangle ABC $b = 6$, $\alpha=35^o$, $\gamma =90^o$, Find side 'a'

Q.6: Given that $\alpha = 30^o$, $\gamma = 135^o$, and $c = 10$, find a

Q.7: In any triangle ABC, if $a =20$, $c = 32$ and $c =70^o$, Find A.

Q.8: In any triangle ABC if $a = 9$, $b = 5$, and $\gamma = 32^{o,}$ Find c.

Q.9: The sides of a triangle are 16, 20 and 33 meters respectively. Find its greatest angle.

Q.10: Define angle of elevation and depression.

Q.11: A string of a flying kite is 200 meters long, and its angle of elevation is 60^o. Find the height of the kite above the ground taking the string to be fully stretched.

Q.12: A minaret stands on the horizontal ground. A man on the ground, 100 m from the minaret, Find the angle of elevation of the top of the minaret to be 60^o. Find its height.

Q.13: The shadow of Qutab Minar is 81m long when the measure of the angel of elevation of the sun is $41^o\ 31'$. Find the height of the Qutab Minar.

Q.14: In any triangle ABC in which

$b = 45$, $c =34$, $\alpha = 52^o$, find a

Q.15: In any triangle ABC is which

$a = 16$, $b = 17$, $\gamma = 25^o$ find c

Q.16: In any triangle ABC in which

$a = 5$, $c =6$, $\alpha = 45^o$ Find $\sin\gamma$

Q.17: $b =25$, $c =37$ $a =65^o$ find a

Q.18: $a = 16$, $b = 17$, $\gamma = 25^o$ find c

Q.19: $a = 3$, $b = 7$, $\beta = 85^o$, Find α.

Answers

3. $22^o\,37'$. 4. $\beta = 55^o$ 5. $a = 4.2$ 6. $a = 7.07$

7. $A = 35^o\,77'\,58''$ 8. $c = 5.48$ 9. $\gamma = 132^o\,34'$ 11. $h = 173.2$ m

12. $h = 173.20$ m 13. $h = 71.66$ m 14. $a = 36.04$ 15. $c = 7.21$

16. $\gamma = 58^o\,3'$ 17. $a = 34.82$ 18 $c = 7.21$ 19 $\alpha = 25^o 14'\,14''$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Law of sines is:

(a) $\dfrac{a}{\sin B} = \dfrac{b}{\sin A} = \dfrac{c}{\sin c}$ (b) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

(c) $\dfrac{a}{\sin B} = \dfrac{b}{\sin A} = \dfrac{c}{\sin C}$ (d) $\dfrac{a}{\sin B} = \dfrac{b}{\sin C} = \dfrac{a}{\sin A}$

__2. In a triangle ABC $\angle A = 70^o$, $\angle B = 60^o$, then $\angle C$ is:

(a) 30^o (b) 40^o (c) 50^o (d) 60^o

__3. When angle of elevation is viewed by an observer, the object is:

(a) Above (b) Below
(c) At the same level (d) None of these

__4. If b = 2, A = 30^o, B = 45^o, then a is equal to:

(a) 2 (b) $\sqrt{2}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $\dfrac{2}{\sqrt{3}}$

__5. If a = 2, b = 2, A = 30^o, then B^o is:

(a) 45^o (b) 30^o (c) 60^o (d) 90^o

__6. If in a triangle ABC, the sides b, c and angle A are given, then the side a is:

(a) $a^2 = b^2 + c^2 + 2bc \cos A$ (b) $a^2 = b^2 - c^2 - 2ab \cos A$
(c) $a^2 = b^2 + c^2 - 2bc \cos A$ (d) $a^2 = b^2 - c^2 + 2ab \cos A$

__7. In a triangle ABC, the law of cosine is:

(a) Cos A= $\dfrac{b^2 + c^2 + a^2}{2bc}$ (b) Cos A= $\dfrac{b^2 + c^2 - a^2}{2bc}$

(c) Cos A= $\dfrac{b^2 + c^2 + a^2}{2ab}$ (d) Cos A= $\dfrac{b^2 + c^2 - a^2}{2ac}$

__8. If in a triangle ABC, b = 2, c = 2, A = 60^o, then side a is:

(a) 2 (b) 3 (c) 4 (d) 5

__9. If in a triangle ABC, a = 1, b = $\sqrt{2}$, C = 60^o, then side c is:

(a) $\sqrt{2}$ (b) 2 (c) 1 (d) 3

__10. If in a triangle ABC, b = 2, c = 3, a = 1, then cos A is:

(a) 1 (b) 2 (c) 3 (d) 4

__11. If in a triangle ABC, a = 3, b = 4, c = 2, then cos C is:

(a) $\dfrac{1}{2}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{7}{8}$ (d) 3

__12. If b sin C = c sin B, then, a sin B is equal to:

(a) c sin A (b) b sin c (c) b sin A (d) b sin B

__13. In a right triangle if one angle is 30^o, then the other will be:
(a) 45^o (b) 50^o (c) 60^o (d) 75^o

__14. In a right triangle if one angle is 45^o, then the other will be:
(a) 45^o (b) 50^o (c) 60^o (d) 75^o

__15. If $B = 90^o$, $b = 2$, $A = 30^o$, then side a is:
(a) 4 (b) 3 (c) 2 (d) 1

__16. If $c = 90^o$, $a = 1$, $c = 2$, then angle A is:
(a) 90^o (b) 60^o (c) 45^o (d) 30^o

__17. If $c = 90^o$, $b = 1$, $c = \sqrt{2}$, then side a is:
(a) 1 (b) 2 (c) $\sqrt{2}$ (d) 3

__18. If $c = 90^o$, $b = 1$, $c = \sqrt{2}$, then angle A is:
(a) 15^o (b) 30^o (c) 45^o (d) 60^o

__19. The distance of a man from the foot of a tower, 100m high if the angle of elevation of its top as observed by the man is 30^o is:
(a) 50m (b) 100m (c) 150m (d) 200m

__20. A pilot at a distance of 50m, measure the angle of depression of a tower 30^o, how far is the plane from the tower:
(a) 50m (b) 25m (c) 20m (d) 10m

Answers

Q1:

1.	b	2.	c	3.	a	4.	d	5.	b
6.	c	7.	b	8.	a	9.	c	10.	a
11.	c	12.	c	13.	c	14.	a	15.	d
16.	d	17.	a	18.	c	19.	d	20.	b

Chapter 6
Vectors and Scalars

6.1 Introduction:

In this chapter we shall use the ideas of the plane to develop a new mathematical concept, vector. If you have studied physics, you have encountered this concept in that part of physics concerned with forces and equilibrium.

Physicists were responsible for first conceiving the idea of a vector, but the mathematical concept of vectors has become important in its own right and has extremely wide application, not only in the sciences but in mathematics as well.

6.2 Scalars and Vectors:

A quantity which is completely specified by a certain number associated with a suitable unit without any mention of direction in space is known as scalar. Examples of scalar are time, mass, length, volume, density, temperature, energy, distance, speed etc. The number describing the quantity of a particular scalar is known as its magnitude. The scalars are added subtracted, multiplied and divided by the usual arithmetical laws.

A quantity which is completely described only when both their magnitude and direction are specified is known as vector. Examples of vector are force, velocity, acceleration, displacement, torque, momentum, gravitational force, electric and magnetic intensities etc. A vector is represented by a Roman letter in bold face and its magnitude, by the same letter in italics. Thus **V** means vector and V is magnitude.

6.3 Vector Representations:

A vector quantity is represented by a straight line segment, say $\overrightarrow{PQ}$. The arrow head indicate the direction from P to Q. The length of the Vector represents its magnitude. Sometimes the vectors are represented by single letter such as V or $\vec{V}$. The magnitude of a vector is denoted by |V| or by just **V**, where $|\vec{V}|$ means modulus of $\vec{V}$ which is a positive value

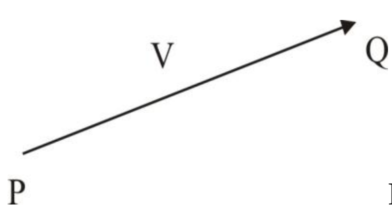


Fig. 1

6.4 Types of Vectors:

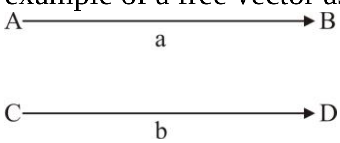
1. Unit Vector:

A vector whose magnitude is unity i.e., 1 and direction along the given vector is called a unit Vector. If $\vec{a}$ is a vector then a unit vector in the direction of $\vec{a}$, denoted by $\hat{a}$ (read as a cap), is given as,

$$\hat{a} = \frac{\vec{a}}{|a|} \quad \text{or} \quad \vec{a} = |a|\,\hat{a}$$

2. Free Vector:

A vector whose position is not fixed in space. Thus, the line of action of a free vector can be shifted parallel to itself. Displacement is an example of a free vector as shown in figure 1:



3. Localized or Bounded Vectors:

A vector which cannot be shifted parallel to itself, i.e., whose line of action is fixed is called a localized or bounded vector. Force and momentum are examples of localized vectors.

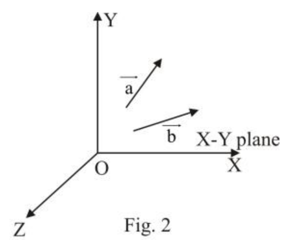


Fig. 2

4. Coplanar Vectors:

The vectors which lies in the same plane are called coplanar vectors, as shown in Fig. 2.

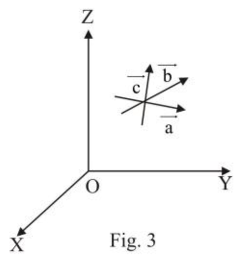


Fig. 3

5. Concurrent Vectors:

The vectors which pass through the common point are called concurrent vectors. In the figure no.3 vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are called concurrent as they pass through the same point.

6. Negative of a Vector:

The vector which has the same magnitude as the vector $\vec{a}$ but opposite in direction to $\vec{a}$ is called the negative to $\vec{a}$. It is represented by $-\vec{a}$. Thus of $\overrightarrow{AB} = -\vec{a}$ then $\overrightarrow{BA} = -\vec{a}$

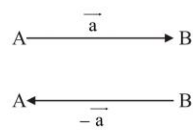


Fig. 4

7. Null or Zero Vector:

It is a vector whose magnitude is zero. We denote the null vector by O. The direction of a zero vector is arbitrary.

The vectors other than zero vectors are proper vectors or non-zero vectors.

8. Equal Vectors:

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude and direction. If $\vec{a}$ and $\vec{b}$ are equal vectors then $\vec{a} = \vec{b}$

9. Parallel and Collinear Vectors:

The vectors $\vec{a}$ and $\vec{b}$ are parallel if for any real number n,

$\vec{a} = n\,\vec{b}$. If

(i) n > 0 then the vectors $\vec{a}$ and $\vec{b}$ have the same direction.

(ii) n < 0 then $\vec{a}$ and $\vec{b}$ have opposite directions.

Now, we can also define collinear vectors which lie along the same straight line or having their directions parallel to one another.

10. Like and Unlike Vectors:

The vectors having same direction are called like vectors and those having opposite directions are called unlike vectors.

11. Position Vectors (PV):

If vector $\overrightarrow{OA}$ is used to specify the position of a point A relative to another point O. This $\overrightarrow{OA}$ is called the position vector of A referred to O as origin. In the figure 4 $\vec{a} = \overrightarrow{OA}$ and $\overrightarrow{OB} = \vec{b}$ are the position vector (P.V) of A and B respectively. The vector $\overrightarrow{AB}$ is determined as follows:

By the head and tail rules,

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

Or $\quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

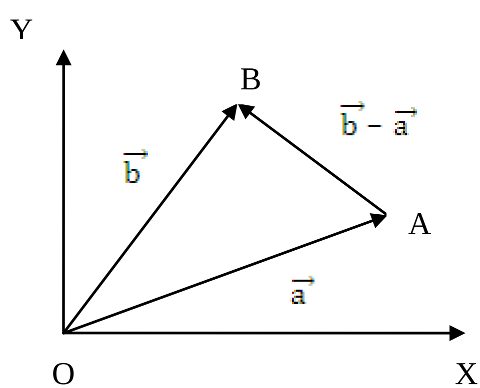


Fig. 5

6.5 Addition and Subtraction of Vectors:

1. Addition of Vectors:

Suppose $\overrightarrow{a}$ and $\overrightarrow{b}$ are any two vectors. Choose point A so that $\overrightarrow{a} = \overrightarrow{OA}$ and choose point C so that $\overrightarrow{b} = \overrightarrow{AC}$. The sum, $\overrightarrow{a} + \overrightarrow{b}$ of $\overrightarrow{a}$ and $\overrightarrow{b}$ is the vector is the vector $\overrightarrow{OC}$. Thus the sum of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is performed by the Triangle Law of addition.

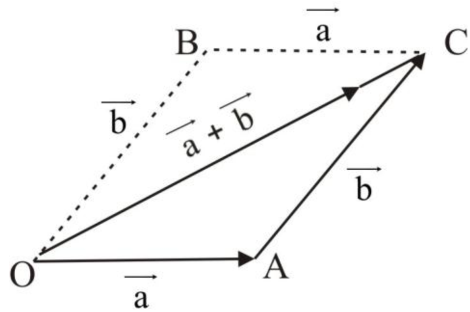


Fig. 6

2. Subtraction of Vectors:

If a vector $\vec{b}$ is to be subtracted from a vector $\vec{a}$, the difference vector $\vec{a} - \vec{b}$ can be obtained by adding vectors $\vec{a}$ and $-\vec{b}$.

The vector $-\vec{b}$ is a vector which is equal and parallel to that of vector but its arrow-head points in opposite direction. Now the vectors $\vec{a}$ and $-\vec{b}$ can be added by the head-to-tail rule. Thus the line $\overrightarrow{AC}$ represents, in magnitude and direction, the vector $\vec{a} - \vec{b}$.

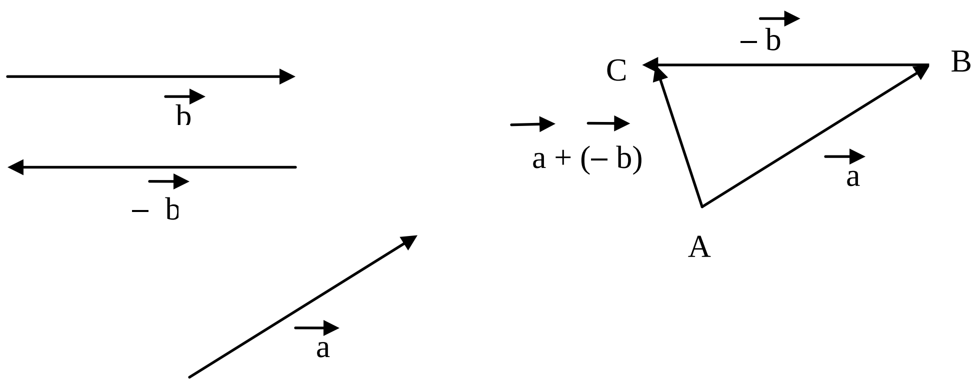


Fig . 7

Properties of Vector Addition:

i. Vector addition is commutative

i.e., $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{a} + \overrightarrow{b}$ where $\vec{a}$ and $\overrightarrow{b}$ are any two vectors.

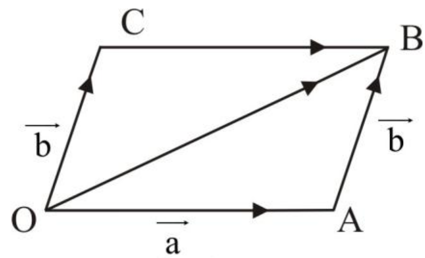


Fig. 8

(ii) Vectors Addition is Associative:

i.e. $\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$

where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors.

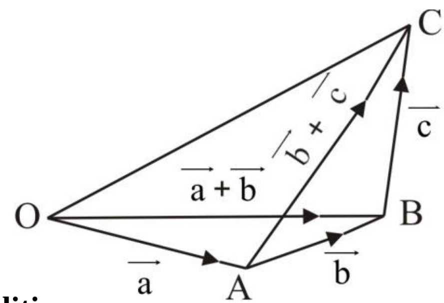


Fig.9

(iii) $\vec{O}$ is the identity in vectors addition:

For every vector $\vec{a}$

$\vec{a} + \vec{O} = \vec{a}$

Where $\vec{O}$ is the zero vector.

Remarks: Non-parallel vectors are not added or subtracted by the ordinary algebraic Laws because their resultant depends upon their directions as well.

6.6 Multiplication of a Vector by a Scalar:

If $\vec{a}$ is any vectors and K is a scalar, then $K\vec{a} = \vec{a}K$ is a vector with magnitude $|K|.\ |a|$ i.e., $|K|$ times the magnitude of $\vec{a}$ and whose direction is that of vector $\vec{a}$ or opposite to vector $\vec{a}$ according as K is positive or negative resp. In particular $\vec{a}$ and $-\vec{a}$ are opposite vectors.

Properties of Multiplication of Vectors by Scalars:

1. The scalar multiplication of a vectors satisfies

$$m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$$

2. The scalar multiplication of a vector satisfies the distributive laws

i.e., $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

and $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Where m and n are scalars and $\vec{a}$ and $\vec{b}$ are vectors.

6.7 The Unit Vectors i, j, k (orthogonal system of unit Vectors):

Let us consider three mutually perpendicular straight lines OX, OY and OZ. These three mutually perpendicular lines determine uniquely the position of a point. Hence these lines may be taken as the co-ordinates axes with O as the origin.

We shall use i, j and k to denote the Unit Vectors along OX, OY and OX respectively.

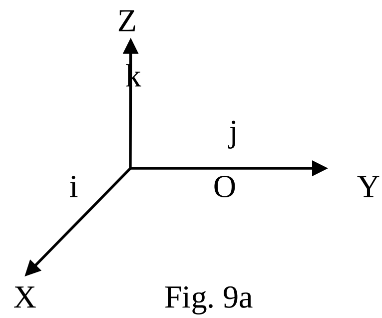


Fig. 9a

6.8 Representation of a Vector in the Form of Unit Vectors i, j and k.

Let us consider a vector $\vec{r} = \overrightarrow{OP}$ as shown in fig. 11. Then x i, y j and z k are vectors directed along the axes,

$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \overrightarrow{OA} + \overrightarrow{OB} \qquad \text{because}$$

and $\overrightarrow{OQ} = xi + yi$

Because $\overrightarrow{QP} = zk$

$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$

and $\vec{r} = \overrightarrow{OP} = xi + yj + zk$

Here the real numbers x, y and z are the components of Vector $\vec{r}$ or the co-ordinates of point P in the direction of OX, OY and OZ respectively. The vectors xi, yj and zk are called the resolved parts of the vector $\vec{r}$ in the direction of the Unit vectors i, j and k respectively.

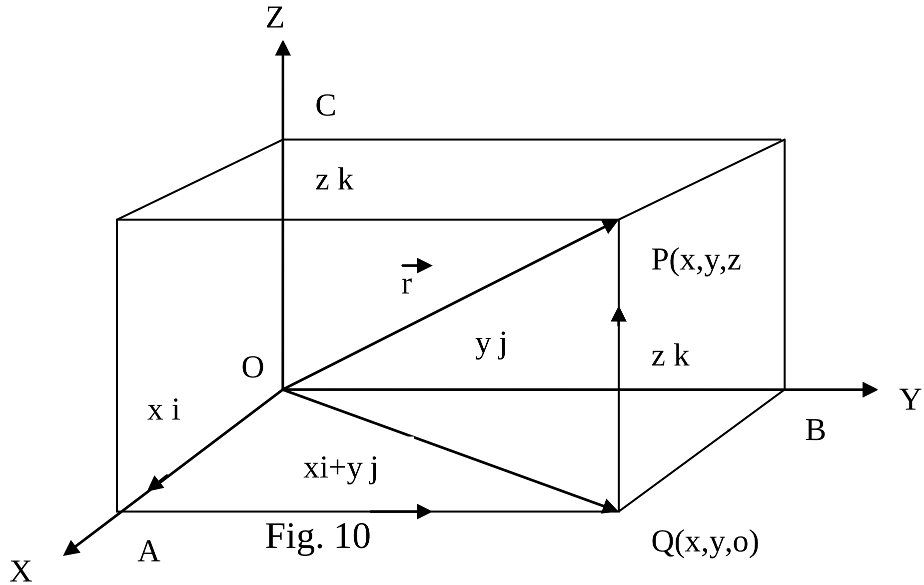


Fig. 10

6.9 Components of a Vector when the Tail is not at the Origin:

Consider a vector $\vec{r} = \overrightarrow{PQ}$ whose tail is at the point $P(x_1, y_1, z_1)$ and the head at the point $Q(x_2, y_2, z_2)$. Draw perpendiculars PP′ and QQ′ on x-aixs.

$P'Q' = x_2 - x_1$ = x-component of $\vec{r}$

Now draw perpendiculars PP^o and QQ^o on y-axis.

Then $P^oQ^o = y_2 - y_1$ = y-component of $\vec{r}$

Similarly $z_2 - z_1$ = z-component of $\vec{r}$

Hence the vector $\vec{r}$ can be written as,

$\vec{r} = \overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Or, $\vec{r} = \overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

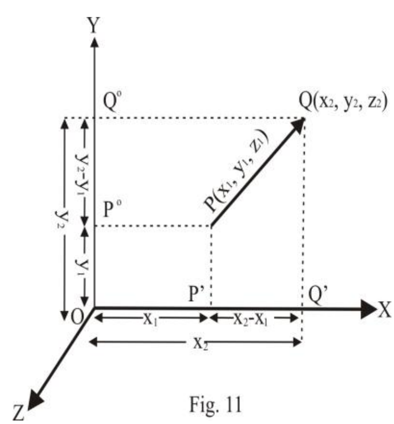


Fig. 11

6.10 Magnitude or Modulus of a Vector:

Suppose x, y and z are the magnitude of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ as shown in fig. 10.

In the right triangle OAQ, by Pythagorean Theorem

$$OQ^2 = x^2 + y^2$$

Also in the right triangle OQP, we have

$$OP^2 = OQ^2 + QP^2$$
$$OP^2 = x^2 + y^2 + z^2$$

Or $$|\vec{r}| = |OP| = \sqrt{x^2 + y^2 + z^2}$$

Thus if $\vec{r} = \overrightarrow{PQ} = xi + yj + zk$

Then , its magnitude is

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

If $\vec{r} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Then $|\vec{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 1:

If $P_1 = P(7, 4, -1)$ and $P_2 = P(3, -5, 4)$, what are the components of P_1P_2? Express $\overrightarrow{P_1\,P_2}$ in terms of i, j and k.

Solution:

x-component of $P_1\,P_2 = x_2 - x_1 = 3 - 7 = -4$

y-component of $P_1\,P_2 = y_2 - y_1 = -5 - 4 = -9$

and z-component of $P_1\,P_2 = z_2 - z_1 = 4 - (-1) = 5$

also $\overrightarrow{P_1\,P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

$$\overrightarrow{P_1\,P_2} = -4i - 9j + 5k$$

Example 2: Find the magnitude of the vector

$$|\bar{u}| = \frac{3}{5}i - \frac{2}{5}j + \frac{2\sqrt{3}}{5}k$$

Solution:

$$|\bar{u}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 + \left(\frac{2\sqrt{3}}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + \frac{4}{25} + \frac{12}{25}} = \sqrt{\frac{25}{25}}$$

$$|\bar{u}| = 1$$

Note: Two vectors are equal if and only if the corresponding components of these vectors are equal relative to the same co-ordinate system.

Example 3:

Find real numbers x, y and z such that

$xi + 2yj - zk + 3i - j = 4i + 3k$

Solution:

Since $(x + 3)i + (2y - 1)j + (-z)k = 4i + 3k$

Comparing both sides,we get

$x + 3 = 4,\ 2y - 1 = 0, \qquad -z = 3$

$x = 1, \quad y = \frac{1}{2}, \quad z = -3$

Note 2: If $\vec{r_1} = x_1 i + y_1 j + z_1 k$

$\vec{r_2} = x_2 i + y_2 j + z_2 k$

Then the sum vector =

$$\vec{r_1} + \vec{r_2} = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$$

Or $$\vec{r_1} + \vec{r_2} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

Example 4:

$\vec{a} = 3i - 2j + 5k$ and $\vec{b} = -2i - j + k$.

Find $2\vec{a} - 3\vec{b}$ and also its unit vector.

Solution:

$2\vec{a} - 3\vec{b} = 2(3i - 2j + 5k) - 3(-2i - j + k)$

$= 6i - 4j + 10k + 6i + 3j - 3k$

$= 12i - j + 7k$

If we denote $2\vec{a} - 3\vec{b} = \vec{c}$, then $\vec{c} = 12i - j + 7k$

and $|\vec{c}| = \sqrt{12^2 + (-1)^2 + 7^2} = \sqrt{144 + 1 + 49} = \sqrt{194}$

Therefore, $$\hat{c} = \frac{\vec{c}}{|c|} = \frac{12i - j + 7k}{\sqrt{194}}$$

$$\hat{c} = \frac{12}{\sqrt{194}} i - \frac{1}{\sqrt{194}} j + \frac{1}{\sqrt{194}} k$$

Note 3: Two vectors $\vec{r_1} = x_1 i + y_1 j + z_1 k$ and $\vec{r_2} = x_2 i + y_2 j + z_2 k$ are parallel if and only if $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.

6.11 Direction Cosines:

Let us consider that the vector $\vec{r} = \overrightarrow{OP}$ which makes angles α, β and γ with the coordinate axes OX, OY and OZ respectively. Then Cos α, Cos β and Cos γ are called the direction cosines of the vector $\overrightarrow{OP}$. They are usually denoted by l, m and n respectively.

If $\overrightarrow{OP} = \vec{r} = xi + yj + zk$, then x, y and z are defined as the direction ratios of the vector $\vec{r}$ and $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. Since the angles A, B and C are right angles (by the fig. 11), so in the right triangles.

OAP, OBP and OCP the direction cosines of $\vec{r}$ can be written as,

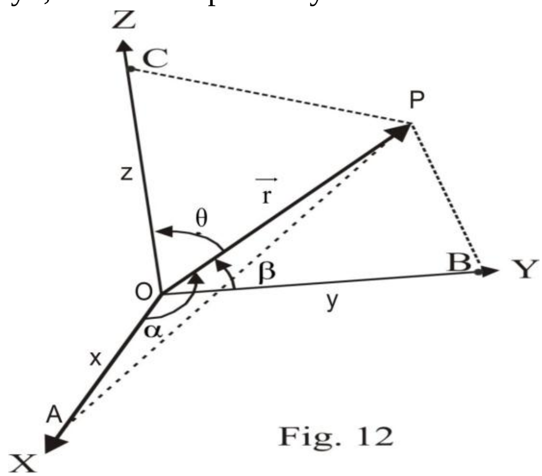


Fig. 12

$$l = \cos\alpha = \frac{x}{|r|} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$m = \cos\beta = \frac{y}{|r|} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

and $$n = \cos\gamma = \frac{z}{|r|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Note 1: Since the unit vector $\hat{r} = \frac{\vec{r}}{|r|} = \frac{xi + yj + zk}{|r|}$

$$\hat{r} = \frac{x}{|r|}i + \frac{y}{|r|}j + \frac{z}{|r|}k$$

$$\hat{r} = \text{Cos}\,\alpha\ i + \text{Cos}\,\beta\ j + \text{Cos}\,\gamma\ k$$

Or $$\hat{r} = li + mj + nk$$

Therefore the co-efficient of i, j and k in the unit vector are the direction cosines of a vector.

Note 2: $$l^2 + m^2 + n^2 = \frac{x^2}{|r|^2} + \frac{y^2}{|r|^2} + \frac{z^2}{|r|^2}$$

$$= \frac{x^2 + y^2 + z^2}{|r|^2} = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1$$

Example 5:

Find the magnitude and direction cosines of the vectors
3i + 7j – 4k, i – 5j – 8k and 6i – 2j + 12k.

Solution:

Let $\overrightarrow{a} = 3i + 7j - 4k$

$\overrightarrow{b} = i - 5j - 8k$

$\overrightarrow{c} = 6i - 2j + 12k$

Now $\hat{a} = \dfrac{\overrightarrow{a}}{|a|} = \dfrac{3i + 7j - 4k}{\sqrt{74}}$

$= \dfrac{3}{\sqrt{74}}i + \dfrac{7}{\sqrt{74}}j - \dfrac{4}{\sqrt{74}}k$

So the direction cosines of $\overrightarrow{a}$ are: $\dfrac{3}{\sqrt{74}}, \dfrac{7}{\sqrt{74}}, -\dfrac{4}{\sqrt{74}}$

Similarly the direction cosines of $\overrightarrow{b}$ are: $\dfrac{1}{\sqrt{90}}, -\dfrac{5}{\sqrt{90}}, -\dfrac{8}{\sqrt{90}}$

and the direction cosines of $\overrightarrow{c}$ are: $\dfrac{6}{\sqrt{184}}, -\dfrac{2}{\sqrt{184}}, \dfrac{12}{\sqrt{184}}$

Exercise 6.1

Q.1 If $\overrightarrow{a} = 3i - j - 4k$, $b = -2i + 4j - 3k$ and $\overrightarrow{c} = i + 2j - k$.
Find unit vector parallel to $3\overrightarrow{a} - 2\overrightarrow{b} + 4\overrightarrow{c}$.

Q.2 Find the vector whose magnitude is 5 and which is in the direction of the vector 4i – 3j + k.

Q.3 For what value of m, the vector 4i + 2j – 3k and mi – j + $\sqrt{3}$k have same magnitude?

Q.4 Given the points A = (1, 2, −1), B = (−3, 1, 2) and C = (0, −4, 3)
(i) find $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{AC}$ (ii) Show that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Q.5 Find the lengths of the sides of a triangle, whose vertices are A = (2, 4, –1), B = (4, 5, 1), C = (3, 6, –3) and show that the triangle is right angled.

Q.6 If vectors 3i + j – k and λ i – 4j + 4k are parallel, find the value of λ .

Q.7 Show that the vectors 4i – 6j + 9k and –6i + 9j – $\frac{27}{2}$ k are parallel.

Q.8 Find real numbers x, y and z such that

(a) 7xi + (y – 3)j + 6k = 10i + 8j – 3zk

(b) (x + 4)i + (y – 5)j + (z – 1)k = 0

Q.9 Given the vectors $\vec{a} = 3i - 2j + 4k$ and $\vec{b} = 2i + j + 3k$ find the magnitude and direction cosines of

(i) $\vec{a} - \vec{b}$ (ii) $3\vec{a} - 2\vec{b}$

Q.10 If the position vector of $\vec{A}$ and $\vec{B}$ 5i – 2j + 4k and i + 3j + 7k respectively, find the magnitude and direction cosines of $\overrightarrow{AB}$.

Answers 6.1

Q.1 $\frac{1}{\sqrt{398}}(17i - 3j - 10k)$ Q.2 $\frac{5}{\sqrt{26}}(4i - 3j + k)$

Q.3 ± 5 Q.4 $(-4, -1, 3), (3, -5, 1), (-1, -6, 4)$

Q.5 AB = AC = 3, BC = $3\sqrt{2}$ Q.6 $\lambda = -12$

Q.8 (a) $x = \frac{10}{7}$ y = 11 z = - 2 (b) x = -4 , y = 5 , z = 1

Q.9 (a) $\sqrt{11}; \frac{1}{\sqrt{11}} \frac{-3}{\sqrt{11}} \frac{1}{\sqrt{11}}$ (b) $5\sqrt{5}; \frac{1}{\sqrt{5}} \frac{-8}{5\sqrt{5}} \frac{6}{5\sqrt{5}}$

Q.10 $\sqrt{50}; \frac{-4}{\sqrt{50}}, \frac{5}{\sqrt{50}}, \frac{3}{\sqrt{50}}$

6.12 Product of Vectors:

1. Scalar Product of two Vectors:

If $\vec{a}$ and $\vec{b}$ are non-zero vectors, and θ is the angle between them, then the scalar product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and read as $\vec{a}$ dot $\vec{b}$. It is defined by the relation

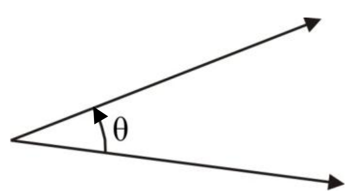


Fig.14

$$\vec{a} \cdot \vec{b} = |a|\ |b| \cos\theta \qquad \ldots\ldots\ldots\ldots\ldots \qquad (1)$$

If either $\vec{a}$ or $\vec{b}$ is the zero vector, then $\vec{a} \cdot \vec{b} = 0$

Remarks:

i. The scalar product of two vectors is also called the dot product because the "." used to indicate this kind of multiplication. Sometimes it is also called the inner product.

ii. The scalar product of two non-zero vectors is zero if and only if they are at right angles to each other. For $\vec{a}\cdot\vec{b} = 0$ implies that $\cos\theta = 0$, which is the condition of perpendicularity of two vectors.

Deductions:

From the definition (1) we deduct the following:

i. If $\vec{a}$ and $\vec{b}$ have the same direction, then

$\theta = 0^o \Rightarrow \cos 0^o = 1$

$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|$

ii. If $\vec{a}$ and $\vec{b}$ have opposite directions, then

$\theta = \pi \Rightarrow \cos\pi = -1$

$\therefore \quad \vec{a}\cdot\vec{b} = -|\vec{a}|\,|\vec{b}|$

Iii $\vec{a}\cdot\vec{b}$ will be positive if $0 \le \theta < \frac{\pi}{2}$

and negative if, $\frac{\pi}{2} < \theta \le \pi$

iv The dot product of $\vec{a}$ and $\vec{b}$ is equal to the product of magnitude of $\vec{a}$ and the projection of $\vec{b}$ on $\vec{a}$. This illustrate the geometrical meaning of $\vec{a}\cdot\vec{b}$. In the fig. 15 $|\vec{b}|\cos\theta$ is the projection of $\vec{b}$ on $\vec{a}$.

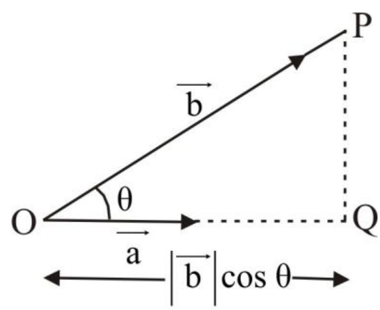


Fig. 15

v From the equation (1)

$$\vec{b}\cdot\vec{a} = |\vec{b}|\,|\vec{a}|\cos\theta$$

$$= |\vec{a}|\,|\vec{b}|\cos\theta$$

$$\vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b}$$

Hence the dot product is commutative.

Corollary 1:

If $\vec{a}$ be a vector, then the scalar product $\vec{a} \cdot \vec{a}$ can be expressed with the help of equation (1) as follows:

$$\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}| \text{Cos } 0^o = |\vec{a}|^2$$

Or $$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

This relation gives us the magnitude of a vector in terms of dot product.

Corollary 2:

If i, j and k are the unit vectors in the directions of X –, Y – and Z – axes, then from eq. (2)

$$i^2 = i.i = |i| \ |i| \ \text{Cos}0^o$$

$$i^2 = 1$$

so $i^2 = j^2 = k^2 = 1$

and $i \cdot j = j \cdot i = 0$ Because $\text{Cos}90^o = 0$

$$i \cdot k = k \cdot i = 0$$

$$k \cdot i = i \cdot k = 0$$

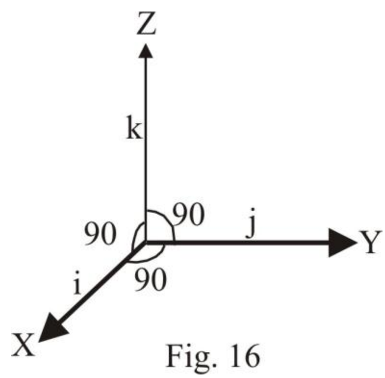


Fig. 16

Corollary 3:

(Analytical expression of $\vec{a} \cdot \vec{b}$)

Scalar product of two vectors in terms of their rectangular components.

For the two vectors

$$\vec{a} = a_1i + a_2j + a_3k$$

and $$\vec{b} = b_1i + b_2j + b_3k$$

the dot product is given as,

$$\vec{a} \cdot \vec{b} = (a_1i + a_2j + a_3k) \cdot (b_1i + b_2j + b_3k)$$

$$= a_1b_1 + a_2b_2 + a_3b_3 \text{ as } i^2 = j^2 = k^2 = 1$$

$$\text{and } i \cdot j = j \cdot k = k \cdot i = 0$$

Also $\vec{a}$ and $\vec{b}$ are perpendicular if and only if $a_1b_1 + a_2b_2 + a_3b_3 = 0$

Example 6:

If $\vec{a} = 3i + 4j - k$, $\vec{b} = -2i + 3j + k$ find $\vec{a} \cdot \vec{b}$

Solution:

$$\vec{a} \cdot \vec{b} = (3i + 4j - k) \cdot (-2i + 3j + k)$$

$$= -6 + 12 - 1$$

$$= 5$$

Example 7:

For what values of λ, the vectors 2i – j + 2k and 3i + 2λj are perpendicular?

Solution:

Let $\vec{a}$ = 2i – j + 2k and $\vec{b}$ = 3i + 2λj

Since $\vec{a}$ and $\vec{b}$ are perpendicular,

So $\vec{a} \cdot \vec{b} = 0$

$(2i - j + 2k) \cdot (3i + 2\lambda j) = 0$

$6 - 2\lambda = 0$

Or $\lambda = 3$

Example 8:

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where $\vec{a}$ = i + 2j – k and $\vec{b}$ = –i + j –2k.

Solution:

As $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \text{ Cos } \theta$

Therefore $\text{Cos } \theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| . |\vec{b}|}, \; \vec{a} \cdot \vec{b} = -1 + 2 + 2 = 3$

$|\vec{a}| = \sqrt{1+4+1} = \sqrt{6}, \quad |\vec{b}| = \sqrt{1+1+4} = \sqrt{6}$

$$\text{Cos } \theta = \frac{3}{\sqrt{6}\sqrt{6}}$$

$$\text{Cos } \theta = \frac{3}{6} = \frac{1}{2}$$

$$\theta = \text{Cos}^{-1}\frac{1}{2} = 60^{o}$$

Example 9:

Consider the points A, B, C, D where coordinates are respectively (1, 1, 0), (−1, 1, 0), (1, −1, 0), (0, −1, 1). Find the direction cosines of AC and BD and calculate the angle between them.

Solution:

Now we have A(1, 1, 0), B(−1, 1, 0), C(1, −1, 0), D(0, −1, 1)

$\vec{a} = \overline{AC} = (1-1)i + (-1-1)j + (0-0)k = -2j$

$\therefore$ Unit vector along AC = $\dfrac{\overrightarrow{AC}}{|AC|} = \dfrac{-2j}{2} = -j$

$\therefore$ The direction cosines of AC are 0, −1, 0

Now

$$\vec{b} = \overrightarrow{BD} = (0+1)i + (-1-1)j + (1-0)k = i - 2j + k$$

Unit vector along BD = $\dfrac{\overrightarrow{BD}}{|BD|} = \dfrac{i-2j+k}{\sqrt{1+4+1}} = \dfrac{i-2j+k}{\sqrt{6}}$

$\therefore$ The direction cosines of BD are:

$$\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$$

Let, θ be the angle between AC and BD then:

$$\text{Cos }\theta = \frac{\vec{a}\,.\,\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{\overrightarrow{AC}\,.\,\overrightarrow{BD}}{|\overrightarrow{AC}|\,|\overrightarrow{BD}|}$$

$$= \frac{(-2j).(i-2j+k)}{(2)\sqrt{6}}$$

$$= \frac{(-2)(-2)}{(2)\sqrt{6}} = \frac{2}{\sqrt{6}}$$

$$\theta = \text{Cos}^{-1}\left(\frac{2}{\sqrt{6}}\right)$$

Example 10:

Show that if $|a + b| = |a - b|$ then a and b are perpendicular.

Solution:

We have $|a + b| = |a - b|$

$\therefore$ $|a + b|^2 = |a - b|^2$ taking square.

$a^2 + b^2 + 2a.b = a^2 + b^2 - 2a.b$

$4a.b = 0$ or $a.b = 0$

Hence $\bar{a}$ and $\bar{b}$ are perpendicular.

2. Vector Product:

If $\vec{a}$ and $\vec{b}$ are non-zero vectors and θ is the angle between $\vec{a}$ and $\vec{b}$, then the vector product of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is the vector $\vec{c}$ which is perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$. It is defined by the relation,

$$\vec{c} = \vec{a} \times \vec{b} = \left(|\vec{a}|\,|\vec{b}|\,\text{Sin }\theta\right)n$$

Where $|\vec{a}||\vec{b}|\text{Sin}\,\theta$ is the magnitude of $\vec{c}$ and n is the Unit Vector in the direction of $\vec{c}$. The direction of $\vec{c}$ is determined by the right hand rule.

The vector product is also called the 'cross product' or 'Outer product' of the vectors.

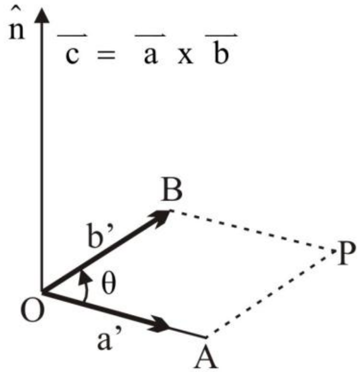


Fig. 17

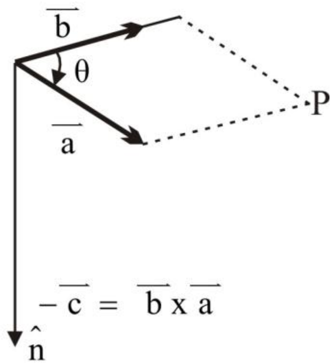


Fig. 18

Remarks:

If we consider $\vec{b} \times \vec{a}$, then $\vec{b} \times \vec{a}$ would be a vector which is opposite in the direction to $\vec{a} \times \vec{b}$.

Hence $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Which gives that $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ in general

Hence the vector product is not commutative.

Deductions:

The following results may be derived from the definition.

i. The vector product of two non-zero vectors is zero if $\vec{a}$ and $\vec{b}$ are parallel, the angle between $\vec{a}$ and $\vec{b}$ is zero. Sin $0^o = 0$, Hence $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 0$.

For a x b = 0 implies that $\text{Sin}\,\theta = 0$ which is the condition of parallelism of two vectors. In particular $\vec{a} \times \vec{a} = 0$. Hence for the unit vectors i, j and k,

$$i \times i = j \times j = k \times k = 0$$

ii. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then $\vec{a} \times \vec{b}$ is a vector whose magnitude is $|\vec{a}||\vec{b}|$ and whose direction is such that the vectors a, b, a x b form a right-handed system of three mutually perpendicular

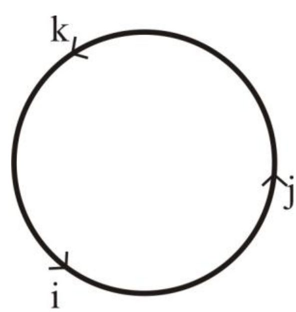


Fig. 19

vectors. In particular i x j = (1) (1) Sin 90^o k (k being perpendicular to i and j) = k
Similarly j x i = $-$k, i x k = $-$j, k x j = $-$i
Hence the cross product of two consecutive unit vectors is the third unit vector with the plus or minus sign according as the order of the product is anti-clockwise or clockwise respectively.

iii. Since $\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \text{Sin}\,\theta$ ...... (2)

Which is the area of the parallelogram whose two adjacent sides are $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$.

Hence, area of parallelogram OABC = $\left|\vec{a} \times \vec{b}\right|$

and area of triangle OAB = $\frac{1}{2}\left|\vec{a} \times \vec{b}\right|$

If th vertices of a parallelogram are given, then

area of parallelogram OABC = $\left|\overline{OA} \times \overline{OB}\right|$

and, area of triangle OAB = $\frac{1}{2}\left|\overline{OA} \times \overline{OB}\right|$

iv. If n is the unit vector in the directions of $\vec{c} = \vec{a} \times \vec{b}$ then

$$n = \frac{\vec{c}}{|c|} = \frac{\vec{a} \times \vec{b}}{|a \times b|}$$

or $$n = \frac{\vec{a} \times \vec{b}}{|a||b|\text{Sin}\,\theta}$$

from equation (2) we also find.

$$\text{Sin}\,\theta = \frac{\left|\vec{a} \times \vec{b}\right|}{|a||b|}$$

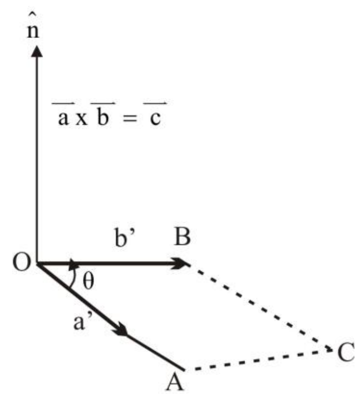


Fig. 20

6.13 Rectangular form of $\vec{a} \times \vec{b}$
(Analytical expression of $\vec{a} \times \vec{b}$)

If $\vec{a} = a_1i + a_2j + a_3k$

and $\vec{b} = b_1i + b_2j + b_3k$

then $\vec{a} \times \vec{b} = (a_1i + a_2j + a_3k) \times (b_1i + b_2j + b_3k)$
$= (a_1b_2k - a_1b_3j - a_2b_1k + a_2b_3j + a_3b_1j - a_3b_2j)$
$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$

This result can be expressed in determinant form as

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 11:

If $\vec{a} = 2i + 3j + 4k$ $\vec{b} = I - j + k$, Find

(i) $\vec{a} \times \vec{b}$

(ii) Sine of the angle between these vectors.

(iii) Unit vector perpendicular to each vector.

Solution:

(i) $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = i(3 + 4) - j(2 - 4) + k(-2 - 3)$$
$$= 7i + 2j - 5k$$

(ii) $$\text{Sin}\,\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{7^2 + 2^2 + (-5)^2}}{\sqrt{2^2 + 3^2 + 4^2}.\sqrt{1^2 + (-1)^2 + 1^2}}$$

$$= \frac{\sqrt{78}}{\sqrt{29}\sqrt{3}}$$

$$\text{Sin}\,\theta = \sqrt{\frac{26}{29}}$$

(iii) If $\hat{n}$ is the unit vector perpendicular to $\vec{a}$ and $\vec{b}$ then

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{7i + 2j - 5k}{\sqrt{78}}$$

Example 12:

$\vec{a} = 3i + 2k$, $\vec{b} = 4i + 4j - 2k$

$\vec{c} = i - 2j + 3k$, $\vec{d} = 2i - j + 5k$

Compute $(\vec{d} \times \vec{c}).(\vec{a} - \vec{b})$

Solution:

$$\vec{d} \times \vec{c} = \begin{vmatrix} i & j & k \\ 2 & -1 & 5 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= i(-3 + 10) - j(6 - 5) + k(-4 + 1)$$
$$= 7i - j - 3k$$

Also $\vec{a} - \vec{b} = -i - 4j + 4k$

Hence $(\vec{d} \times \vec{c}).(\vec{a} - \vec{b}) = (7i - j - 3k).(-i - 4j + 4k)$

$= -7 + 4 - 12$

$= -15$

Example 13:

Find the area of the parallelogram with adjacent sides,

$\vec{a} = i - j + k$, and $\vec{b} = 2j - 3k$

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 2 & -3 \end{vmatrix}$$

$= i(3 - 2) - j(-3 - 0) + k(2 + 0)$

$= i + 3j + 2k$

Area of parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{1 + 9 + 4}$

$= \sqrt{14}$ square unit.

Example 14:

Find the area of the triangle whose vertices are

A(0, 0, 0), B(1, 1, 1) and C(0, 2, 3)

Solution:

Since $\overrightarrow{AB} = (1 - 0, 1 - 0, 1 - 0)$

$= (1, 1, 1)$

and $\overrightarrow{AC} = (0 - 0, 2 - 0, 3 - 0)$

$\overrightarrow{AC} = (0, 2, 3)$

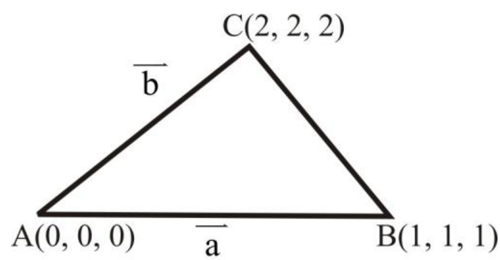


Fig. 21

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix}$$

$= i(3 - 2) - j(3 - 0) + k(2 - 0)$

$= i - 3j + 2k$

Area of the triangle ABC $= \dfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \dfrac{1}{2}\sqrt{1^2 + (-3)^2 + 2^2}$

$= \dfrac{\sqrt{14}}{2}$ square unit

Example 15:

Prove by the use of cross-product that the points

A(5, 2, -3), B(6, 1, 4), C(-2, -3, 6) and D(-3, -2, -1) are the vertices of a parallelogram.

Solution:

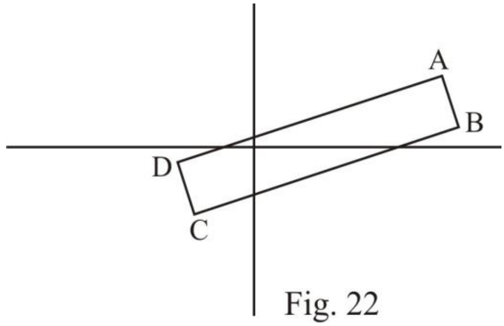


Fig. 22

Since $\overrightarrow{AB} = (1, -1, 7)$

$\overrightarrow{DC} = (+1, -1, +7)$

$\overrightarrow{BC} = (-8, -4, 2)$

and $\overrightarrow{AD} = (-8, -4, 2)$

$$\overrightarrow{AB} \times \overrightarrow{DC} = \begin{vmatrix} i & j & k \\ 1 & -1 & 7 \\ 1 & -1 & 7 \end{vmatrix}$$

$$= i(-7-7) - j(+7-7) + k(1-1)$$

$\overrightarrow{AB} \times \overrightarrow{DC} = 0$, so, $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are parallel.

Also $$\overrightarrow{BC} \times \overrightarrow{AD} = \begin{vmatrix} i & j & k \\ -8 & -4 & 2 \\ -8 & -4 & 2 \end{vmatrix}$$

$$= i(0) - j(0) + k(0)$$

$\overrightarrow{BC} \times \overrightarrow{AD} = 0$, so, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel.

Hence the given points are the vertices of a parallelogram.

Exercise 6.2

Q.1 Find $\overrightarrow{a} . \overrightarrow{b}$ and $\overrightarrow{a} \times \overrightarrow{b}$

(i) $\overrightarrow{a} = 2i + 3j + 4k$ $\quad$ $\overrightarrow{b} = i - j + k$

(ii) $\overrightarrow{a} = i + j + k$ $\quad$ $\overrightarrow{b} = -5i + 2j - 3k$

(iii) $\overrightarrow{a} = -i - j - k$ $\quad$ $\overrightarrow{b} = 2i + j$

Q.2 Show that the vectors $3i - j + 7k$ and $-6i + 3j + 3k$ are at right angle to each other.

Q.3 Find the cosine of the angle between the vectors:

(i) $\overrightarrow{a} = 2i - 8j + 3k$ $\quad$ $\overrightarrow{b} = 4j + 3k$

(ii) $\overrightarrow{a} = i + 2j - k$ $\quad$ $\overrightarrow{b} = -j - 2k$

(iii) $\overrightarrow{a} = 4i + 2j - k$ $\quad$ $\overrightarrow{b} = 2i + 4j - k$

Q.4 If $\overrightarrow{a} = 3i + j - k$, $\overrightarrow{b} = 2i - j + k$ and $\overrightarrow{c} = 5i + 3k$, find $(2\overrightarrow{a} + \overrightarrow{b}).\overrightarrow{c}$.

Q.5 What is the cosine of the angle between $\overline{P_1P_2}$ and $\overline{P_3P_4}$

If $P_1(2,1,3)$, $P_2(-4, 4, 5)$, $P_3(0, 7, 0)$ and $P_4(-3, 4, -2)$?

Q.6 If $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$, prove that:

$$\vec{a}\cdot\vec{b} = \frac{1}{2}\left[\left|\vec{a}+\vec{b}\right|^2 - \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2\right]$$

Q.7 Find $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})$ if $\vec{a} = i + 2j + 3k$ and $\vec{b} = 2i - j + k$.

Q.8 Prove that for every pair of vectors $\vec{a}$ and $\vec{b}$

$$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2$$

Q.9 Find x so that $\vec{a}$ and $\vec{b}$ are perpendicular,

(i) $\vec{a} = 2i + 4j - 7k$ and $\vec{b} = 2i + 6j + xk$

(ii) $\vec{a} = xi - 2j + 5k$ and $\vec{b} = 2i - j + 3k$

Q.10 If $\vec{a} = 2i - 3j + 4k$ and $\vec{b} = 2j + 4k$

Find the component or projection of $\vec{a}$ along $\vec{b}$.

Q.11 Under what condition does the relation $(\vec{a}.\vec{b})^2 = \vec{a}^{\,2}\,\vec{b}^{\,2}$ hold for two vectors $\vec{a}$ and $\vec{b}$.

Q.12 If the vectors 3i + j – k and λ I – 4 j + 4 k are parallel , find value of λ.

Q.13 If $\vec{a} = i - 2j + k$, $\vec{b} = i + 2j - 4k$, $\vec{c} = 2i - 3j + k$ Evaluate:

(i) $(\vec{a} \times \vec{b}).(\vec{a} \times \vec{c})$ (ii) $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$

Q.14 If $\vec{a} = i + 3j - 7k$ and $\vec{b} = 5i - 2j + 4k$. Find:

(i) $\vec{a}.\vec{b}$ (ii) $\vec{a} \times \vec{b}$

(iii) Direction cosines of $\vec{a} \times \vec{b}$

Q.15 Prove that for the vectors $\vec{a}$ and $\vec{b}$

(i) $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a}.\vec{b}\right|^2 = \left|\vec{a}\right|^2\left|\vec{b}\right|^2$

(ii) $(\vec{a} - \vec{b}) \times (\vec{a} \times \vec{b}) = 2(\vec{a} \times \vec{b})$

Q.16 Prove that for vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$[\vec{a} \times (\vec{b} + \vec{c})] + [\vec{b} \times (\vec{c} + \vec{a})] + [\vec{c} \times (\vec{a} + \vec{b})] = 0$$

Q.17 Find a vector perpendicular to both the lines AB and CD, where A is (0, 2, 4), B is (3, –1, 2), C is (2, 0, 1) and D is (4, 2, 0).

Q.18 Find $\left|(\overrightarrow{a} \times \overrightarrow{b}) \times c\right|$ if $\overrightarrow{a} = i - 2j - 3k$, $\overrightarrow{b} = 2i + j - k$, $\overrightarrow{c} = i + 3j - 2k$.

Q.19 Find the sine of the angle and the unit vector perpendicular to each:

(i) $\overrightarrow{a} = i + j + k$ and $\overrightarrow{b} = 2i + 3j - k$

(ii) $\overrightarrow{a} = 2i - j + k$ and $\overrightarrow{b} = 3i + 4j - k$

Q.20 Given $\overrightarrow{a} = 2i - j$ and $\overrightarrow{b} = j + k$, if $\left|\overrightarrow{c}\right| = 12$ and $\overrightarrow{c}$ is perpendicular to both $\overrightarrow{a}$ and $\overrightarrow{b}$, write the component form of $\overrightarrow{c}$.

Q.21 Using cross product, find the area of each triangle whose vertices have the following co-ordinates:

(i) (0, 0, 0),(1, 1, 1),(0, 0, 3)

(ii) (2, 0, 0), (0, 2, 0), (0, 0, 2)

(iii) (1, -1, 1),(2, 2, 2), (4, -2, 1)

Q.22 Find the area of parallelogram determined by the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ $\overrightarrow{a} = i + 2j + 3k$ and $\overrightarrow{b} = -3i - 2j + k$.

Answers 6.2

Q.1 **(i)** 3; 7i – 2j – 5k **(ii)** –6, –5i – 2i + 7k **(iii)** –3; –i –2j + k

Q.3 **(i)** $-\dfrac{33}{5\sqrt{77}}$ **(ii)** 0 **(iii)** $\dfrac{17}{21}$ **Q.4** 37

Q.5 $\dfrac{5}{7\sqrt{22}}$ **Q.7** 8 **Q.9(i)**4 **(ii)** $-\dfrac{17}{2}$ **Q.10** $\sqrt{5}$

Q.11 [0, 11] **Q.12** $\lambda = -12$ **Q.13** (i) 15 (ii)i – 2j + k

Q.14 **(i)** –29 **(ii)** –2i – 39j – 17k **(iii)** $-\dfrac{2}{\sqrt{1814}}, \dfrac{-39}{\sqrt{1814}}, \dfrac{-17}{\sqrt{1814}}$

Q.17 7i – j + 12k **Q.18** $5\sqrt{26}$

Q.19 **(i)** $\sqrt{\dfrac{13}{21}}, \dfrac{-4i + 3j + k}{\sqrt{26}}$ **(ii)** $\sqrt{\dfrac{155}{156}}, \dfrac{-3i + 5j + 11k}{\sqrt{155}}$

Q.20 –4i – 8j + 8k

Q.21 **(i)** $\dfrac{3\sqrt{2}}{2}$ sq. unit. **(ii)** $2\sqrt{3}$ sq. unit.

(iii) $\dfrac{\sqrt{110}}{2}$ sq. unit. Q.22 $\sqrt{180}$ sq. unit

Summary

A vector is a quantity which has magnitude as well as direction while scalar is a quantity which has only magnitude. Vector is denoted as $\overrightarrow{AB}$ or $\overrightarrow{OP}$.

1. If P (x, y, z) be a point in space, then the position vector of P relative to 0 = $\overrightarrow{OP}$.
2. Unit coordinator vectors x, j, k are taken as unit vector s along axis $\overrightarrow{OP}$ = xi + yj + zk.
3. Magnitude of a vector. i.e. $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$
4. Unit vector of a (non-zero vector), then $\hat{a} = \dfrac{\overrightarrow{a}}{|a|}$
5. Direction cosines of $\overrightarrow{OP}$ = xi + yj + zk then,

$$\cos\alpha = \frac{x}{|\overrightarrow{OP}|}.\ \mathrm{Cos}\beta = \frac{y}{|\overrightarrow{OP}|}.\mathrm{Cos}\gamma = \frac{z}{|\overrightarrow{OP}|}$$

Scalar product:

The scalar product of two vector $\overrightarrow{a}$ and $\overrightarrow{b}$ is defined as $\overrightarrow{a}\ .\ \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\,\mathrm{Cos}\,\theta$

1. If a.b = 0, vectors are perpendicular.
2. i.j = j.k = i.k = zero while i.i = j.j = k.k = 1
3. $\overrightarrow{a}\ .\ \overrightarrow{b} = (a_1i + a_2j + a_3k).(b_1i + b_2j + b_3k) = a_1b_1 + a_2b_2 + a_3b_3$

Vector product:

The vector or cross product of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ denoted $\overrightarrow{a}\ x\ \overrightarrow{b}$ and is defined as: $\overrightarrow{a}\ x\ \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\,\mathrm{Sin}\,\theta\ n$, $\mathrm{Sin}\,\theta = \dfrac{|a\ x\ b|}{|a||b|}$

1. $n = \dfrac{\overrightarrow{a}\ x\ \overrightarrow{b}}{|a\ x\ b|}$ unit vector.
2. $\overrightarrow{a}\ x\ \overrightarrow{b} = 0$. $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel or collinear.
3. i x j = j x j = k x k = 0 and i x j = k.j x k = i, k x i = j
4. $\overrightarrow{a}\ x\ \overrightarrow{b} = -\overrightarrow{b}\ x\ \overrightarrow{a}$
5. $\overrightarrow{a}\ x\ \overrightarrow{b} = (a_1i + a_2j + a_3k)\ x\ (b_1i + b_2j + b_3k)$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Short Questions

Write the short answers of the following:

Q.1: What is scalar? Give examples.
Q.2: What is a vector? Give example.
Q.3: What is unit vector?
Q.4: Find the formula for magnitude of the vector $\vec{r} = xi + yj + zk$

Q.5: Find the magnitude of vector $-2i - 4j + 3k$

Q.6: What are parallel vectors?
Q.7: Find α, so that $|\alpha i + (\alpha + 1)j + 2k| = 3$
Q.8: If cosα , cosβ , cosγ are direction cosines of a vector

$\vec{r} = xi + yj + zk$, then show that $\cos^2\alpha + \cos^2\beta + co^2\gamma = 1$

Q.9: Find the unit vector along vector $4i - 3j - 5k$.

Q.10: Find the unit vector parallel to the sum of the vectors

$\bar{a} = [2, 4, -5]$, $\vec{b} = [1, 2, 3]$

Q.11: Given the vectors, $\bar{a} = 3i - 2j + k$, $\bar{b} = 2i - 4j - 3k$

$\bar{c} = -i + 2j + 2k$, Find a +b + c

Q.12: Given the vectors $\bar{a} = 3i + j - k$ and $\bar{b} = 2i + j - k$,find magnitude

of $3\bar{a} - \bar{b}$

Q.13: Find a vector whose magnitude is 2 and is parallel to $5i + 3j + 2k$.

Q.14: Define scalar product of two vectors.

Q.15 Find $\bar{a} . \bar{b}$ if $\bar{a} = i + 2j + 2k$, $\bar{b} = 3i - 2j - 4k$

Q.16: Find (a +b) . (a – b) if

$\bar{a} = 2i + 2j + 3k$, $\bar{b} = 2i - j + k$

Q.17: Define Vector product.

Q.18: If $\bar{a} = 2i + 3j + 4k$, $\bar{b} = i - j + k$

Find $|\vec{a} \times \vec{b}|$

Q.19: Find the area of parallelogram with adjacent sides,

$\bar{a} = 7i - j + k$ and $\bar{b} = 2j - 3k$

Q.20: For what value of λ, the vectors $2i - j + 2k$ and $3i + 2\lambda j$ are perpendicular.

Q.21: Under what conditions does the relation $\bar{a} \cdot \bar{b} = |\bar{a}|\,|\bar{b}|$ hold?

Q.22: Find scalars x and y such that $x(i + 2j) + y(3i + 4j) = 7$

Q.23: Prove that if $\bar{a} = i + 3j - 2k$ and $\bar{b} = i - j - k$, then $\bar{a}$ and $\bar{b}$ are perpendicular to each other.

Answers

5. $\sqrt{29}$

7. 1 , -2

9. $\dfrac{4i - 3j - 5k}{5\sqrt{2}}$

10. $\dfrac{3i + 6j - 2k}{7}$

11. $4i - 4j + 0k$

12. $\sqrt{54}$

13. $\dfrac{10i + 6j + 4k}{\sqrt{38}}$

15. -9

16. 8

18. $\sqrt{78}$

19. $\sqrt{14}$ sq. unit

20. $\lambda = 3$

21 $\theta = 0^{o}$

22. $x = \dfrac{-1}{2}$, $y = 5/2$

Objective Type Question

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Magnitude of the vector 2i – 2j – k is:
(a) 4 (b) 3 (c) 2 (d) 1

__2. Unit vector of i + j + k is:
(a) i + j + k (b) $\frac{1}{3}$(i + j + k)
(c) $\frac{1}{\sqrt{3}}$(i + j + k) (d) $\frac{1}{2}$ (i + j + k)

__3. Unit vector of i – 2j – 2k is:
(a) i – 2j – 2k (b) $\frac{1}{3}$(i – 2j – 2k)
(c) $\frac{1}{\sqrt{3}}$(i – 2j – 2k) (d) $\frac{1}{2}$ (i – 2j – 2k)

__4. If $\hat{i}$, $\hat{j}$ and k are orthogonal unit vectors, then j x i is:
(a) k (b) $-$k (c) 1 (d) -1

__5. The magnitude of a vector $\hat{i}-3\hat{j}+5k$ is:
(a) 3 (b) 25 (c) 35 (d) $\sqrt{35}$

__6. In *l*, m and n are direction cosine of a vector, then:
(a) $l^2 - m^2 - n^2 = 1$ (b) $l^2 - m^2 + n^2 = 1$
(c) $l^2 + m^2 - n^2 = 1$ (d) $l^2 + m^2 + n^2 = 1$

__7. If θ is the angle between the vector $\vec{a}$ and $\vec{b}$, then cos θ is:
(a) $\vec{a} \cdot \vec{b}$ (b) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$
(c) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ (d) $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

__8. If $\vec{a}$ = $a_1j + a_2j + a_3k$, $\vec{b}$ = $b_1i + b_2j + b_3k$, then $\vec{a} \cdot \vec{b}$ is:
(a) $a_1b_1j + a_2b_2j + a_3b_3k$ (b) $a_1b_1 + a_2b_2 + a_3b_3$
(c) $a_1b_2j + a_2b_3j + a_3b_1k$ (d) None of these

__9. $\vec{a} \cdot \vec{b}$ = 0 implies that $\vec{a}$ and $\vec{b}$ are:
(a) Perpendicular (b) Parallel

(c) Non-parallel (d) Oblique

__10. If $\overrightarrow{a} = i + j + k$ and $\overrightarrow{b} = -i - j - mk$ are perndicular then m will be equal to:
(a) 1 (b) -2 (c) ± 1 (d) ± 3

__11. $\overrightarrow{a} \cdot \overrightarrow{b}$ is a:
(a) Vector quantity (b) Scalar quantity
(c) Unity (d) None of these

__12. $\overrightarrow{a} \cdot \overrightarrow{a}$ is equal to:
(a) 1 (b) a^2 (c) $\left|\overrightarrow{a}\right|$ (d) None of these

__13. If $\overrightarrow{a} = 2i - 3j + k$ and $\overrightarrow{b} = -i + 2j + 7k$ then $\overrightarrow{a} \cdot \overrightarrow{b}$ is equal to:
(a) -1 (b) -2 (c) -3 (d) -4

__14. If $\overrightarrow{a} \times \overrightarrow{b} = 0$ then $\overrightarrow{a}$ and $\overrightarrow{b}$ is:
(a) Non-parallel (b) Parallel
(c) Perpendicular (d) None of these

__15. The cross product of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is:
(a) $\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|\cos\theta$ (b) $\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|\sin\theta$
(c) $\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|\sin\theta\ \hat{n}$ (d) $\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|\cos\theta\ \hat{n}$

__16. If $\hat{n}$ is the unit vector in the direction of $\overrightarrow{a} \times \overrightarrow{b}$, then $\hat{n}$ is:
(a) $\dfrac{\overrightarrow{a} \times \overrightarrow{b}}{\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|}$ (b) $\dfrac{\overrightarrow{a} \cdot \overrightarrow{b}}{\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|}$ (c) $\dfrac{\left|\overrightarrow{a} \times \overrightarrow{b}\right|}{\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|\sin\theta}$ (d) $\dfrac{\overrightarrow{a} \times \overrightarrow{b}}{\left|\overrightarrow{a} \times \overrightarrow{b}\right|}$

__17. $\left|\overrightarrow{a} \times \overrightarrow{b}\right|$ is area of the figure called:
(a) Triangle (b) Rectangle (c) Parallelogram (d) Sector

__18. $\overrightarrow{a} \times \overrightarrow{b}$ is equal to:
(a) $-\overrightarrow{b} \times \overrightarrow{a}$ (b) $\overrightarrow{b} \times \overrightarrow{a}$ (c) $\left|\overrightarrow{a} \times \overrightarrow{b}\right|$ (d) $\left|\overrightarrow{b} \times \overrightarrow{a}\right|$

__19. If $\overrightarrow{a}$ and $\overrightarrow{b}$ are collinear vectors, then:
(a) $\overrightarrow{a} \times \overrightarrow{b} = 0$ (b) $\overrightarrow{a} \cdot \overrightarrow{b} = 0$
(c) $\overrightarrow{a} - \overrightarrow{b} = 0$ (d) $\overrightarrow{a} + \overrightarrow{b} = 0$

__20. $\left|\overrightarrow{a} \times \overrightarrow{b}\right|$ is a:
(a) Vector quantity (b) Scalar quantity
(c) Unity (d) None of these

Answers

1.	b	2.	c	3.	b	4.	b	5.	d
6.	d	7.	b	8.	b	9.	a	10.	c
11.	b	12.	b	13.	a	14.	b	15.	c
16.	d	17.	c	18.	a	19.	a	20.	b

Chapter – 7
PHASORS ALGEBRA

Vectors, in general, may be located anywhere in space. We have restricted ourselves thus for to vectors which are all located in one plane (co planar vectors), but they may still be anywhere in that plane. Such general vectors are referred to as free vectors. Our primary interest in vectors, however, relates to their application in the solution of AC circuits. For this purpose, we do not require the generality of free vectors, and we may restrict ourselves still further to vectors which all originate from a fixed point in the plane (i.e., the origin of our coordinate axes) and whose direction rotates about this point. Such vectors are called phasors.

The value of dealing with phasors, rather than vector in general, lies in the fact that phasors can be represented by complex numbers, with j (imaginary number with $j = \sqrt{-1}$) interpreted as an operator. Thus all the special mathematics of vectors, in the case of phasors, becomes simply a matter of the arithmetic of complex numbers

7.2 J as an Operator:

An operator is a symbol for a mathematical .operation. We have defined the imaginary number j ($j = i = \sqrt{-1}$), and from it built up the system of imaginary and complex numbers. In this chapter, we shift our view point slightly and consider j as an operator which is going to operate on real numbers.

Let us look at graphical behaviour of a real number which operated upon by j repeatedly. In fig. 1, we show the axis of real numbers (R axis) horizontal and the axis of real numbers affected by j (j axis) vertical to it or at right angles to it. Starting with the real number 2, we multiply by (i.e., operate with) j once and arrive at 2j. Multiplying by j again, we arrive at $2j^2 = -2$, A third multiplication by j yields $2j^3 = -2j$. The fourth multiplication by j yields $2j^4 = 2$, which brings us back to our staring point.

From- this example, we note that-the graphical effect of j as an operator is to rotate a point counter clockwise (CCW) about the origin, along a circle of constant radius, through an of 90°.

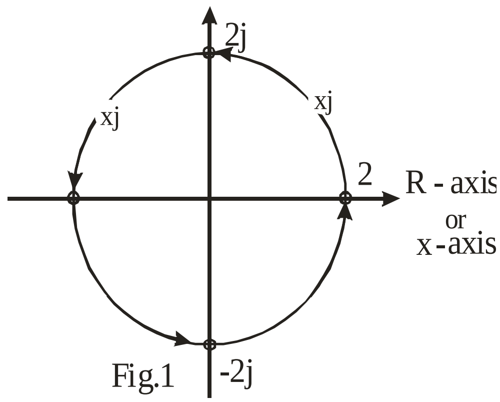


Fig.1

If we consider a quantity of the form a + jb (a and b real numbers) as representing a vector E whose tail is fixed at the origin and whose point is located by the coordinates (a, ib), the effect of operating on this vector E with j is to rotate it counter clockwise 90° from its initial position. Operating with j twice i.e., $j^2E = -E$, rotates the original vector E, through 180° counter clockwise. Operating with j three times., i.e, $j^3E = -jE$,, rotates the original vector through 270^0 counter clockwise, which is equivalent to rotating through 90° clockwise. Four successive operations with j i.e, $j^4E=E$, rotates the original vector through 360° counter clockwise, which is the same vector E. These rotations are shown in Fig. 2.

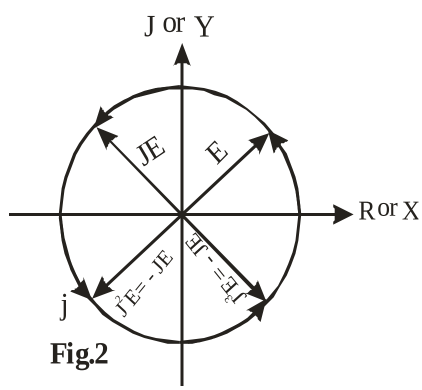


Fig.2

For vectors which rotates about some fixed point in a plane (called phasors), this concept of ‘j’ leads to a neat and simple algebraic way of performing vector operations.

7.3 Mathematical Representation of Phasors (or complex numbers):

A Phasor can be represented graphically in the various forms such as:

(i) Rectangular or Cartesian form

(ii) Trigonometric and polar form.

(iii) Exponential form.

(i) Rectangular or Cartesian Form:

A Phasor Z can be expressed in terms of its x-component ‘x’ and y-component ‘y’ as shown in Fig.3.

Mathematically it is written as,

$$Z = x + j\,y$$

where $j = \sqrt{-1}$, known as operator, it indicates that the component y is perpendicular to the component x. In the Phasor or complex number

Z = x + jy, where x and y are called **real and imaginary** part of Z respectively. But in Electrical Engineering these are known as **in-phase**

(or active)and **quardrature**(or reactive) component respectively. The Phasor or complex numbers (or vectors) are shown in Fig. 3 and represented as,

$$z_2 = -x_2 + jy_2$$

$$z_3 = -x_3 - jy_2$$

and $$z_4 = x_4 - jy_2$$

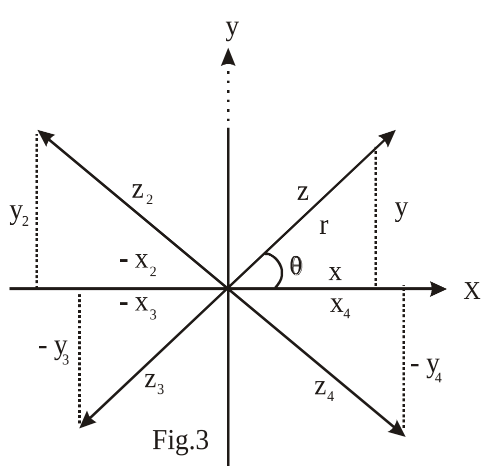


Fig.3

The numerical value (or magnitude) of Z is denoted by r or |Z| or |x + jy|, and is given by

$$r = |Z| = |x + jy| = \sqrt{x^2 + y^2}$$

The argument or amplitude of Z, denoted by arg (Z), is an angle θ with the positive x-axis, and is given by

$$\theta = \tan^{-1}\frac{y}{x}$$

Note: In mathematics $\sqrt{-1}$ is denoted by i, but in electrical engineering j is adopted because the letter 'j' is used for representing current.

(ii) Trigonometric and Polar Form:

From Fig.4, we see that

$$\text{Cos}\ \theta = \frac{x}{r} \qquad \text{and} \qquad \text{Sin}\ \theta = \frac{y}{r}$$

Hence, $x = r\ \text{Cos}\ \theta$ and $y = r \sin\theta$

Therefore, the complex number

$$Z = x + jy$$

$$= r\ \text{Cos}\ \theta + j\ r \sin\theta$$

$$Z = r\ (\cos\theta + j \sin\theta)$$

The general form of this equation is

$$Z = r\ (\cos\theta \pm j\ \text{Sin}\ \theta)$$

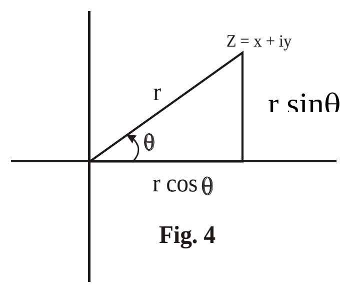


Fig. 4

This is called the Trigonometric form of the complex number Z.

If we simply write $r(\cos\theta + j\sin\theta) = r\angle\theta$

Then $Z = r\angle\theta$

In general, $Z = r\angle\pm\theta$

This is called the polar form (or Modulus argument form) of the complex number Z. Trigonometric and polar forms are the same, but the polar form is simply a short hand or symbolic style of writing the Trigonometric form.

(iii) Exponential Form:

A very interesting and useful relation was discovered by the great Swiss mathematician Euler. Stated as an equation

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This equation is known as Euler's equation

(The derivation of this relationship is given at the end of this chapter).

If we apply this relationship to the trigonometric form of a complex number Z, then

$$Z = r(\cos\theta + j\sin\theta)$$

$$Z = r\,e^{j\theta}$$

In general, $Z = \pm\ r\ e^{j\theta}$

This relation is very useful for multiplication and division of complex numbers.

Hence, we get $Z = x + jy = r(\cos\theta + j\sin\theta) = r\angle\theta = r\,e^{j\theta}$

7.4 Conjugate Complex Numbers:

Two complex numbers are called the conjugate of each other if their real parts are equal and their imaginary parts differ only in sign. The conjugate of a complex number Z = x + iy, is denoted by $\bar{z}$ and is given as

$\bar{Z} = x - iy$.

Example 1: Express the following in polar form:

(a) **(b)** $4 - j5$ **(c)** $-1 - j$

Solution:

(a) $1 + j\sqrt{3}$

Here $x = 1,$ $y = \sqrt{3}$

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^o$$

So, polar form is $= r\angle\theta = 2\angle 60^o$

Its trigonometric form is $= r(\cos\theta + j\sin\theta)$

$= 2(\cos 60^o + j\sin 60^o)$

(b) $4 - j5$

Here $x = 4,$ $y = -5$

$$r = \sqrt{(4)^2 + (-5)^2} = \sqrt{16 + 25} = \sqrt{41}$$

$$\theta = \tan^{-1}\left(-\frac{5}{4}\right) = 308^o\, 40'$$

Hence, polar form is $r \angle \theta = \sqrt{41} \angle 308^o\, 40'$

(c) $-1 - j$

Here $x = -1, \quad y = -1$

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-5}{-4}\right) = 225^o$$

Hence, $r \angle \theta = \sqrt{2} \angle 225^o$

Example 2: Express the following in rectangular form.

(a) $10 \angle 3.5^o$ **(b)** $450 \angle 94^o$ **(c)** $12.3 \angle -45^o$

Solution:

(a) $10 \angle 3.5^o$ $=10(\cos 3.5^o + j \sin 3.5^o)$

$= 10\,(0.998 + j\, 0.06)$

$= 9.98 + j\, 0.61$

(b) $450 \angle 94^o$ $= 450\,(\cos 94^o + j\, 94^o)$

$= 450(-0.698 + j\, 0.998)$

$=31.4 + j\, 449.1$

(c) $12.3 \angle 45^o$ $= 12.3\,(\cos(-45^o) + j\,(-45^o))$

$= 12.3(0.707 + j\, 0.707)$

$=8.696 - j\, 8.696$

Example 3: Given that $z = 2e^{-j\pi}$, write the other forms.

Solution:

Here $r = 2, \quad \theta = -\frac{\pi}{6} = -30^o$

So, polar form is $r\angle\theta = 2 \angle -30^o$

Trigonometric form is

$r = (\cos\theta + j \sin\theta)$ $= 2\,(\cos(-30^o) + j \sin(-30^o))$

$= 2\,(\cos 30^o - j \sin 30^o)$

Rectangular form is

$$x + jy = 2\,(0.866 - j\,0.5)$$
$$= 1.632 - j$$

7.5 Addition and Subtraction of Complex Numbers (Or vectors):

Addition and subtraction of complex numbers can be performed conveniently only when both numbers are in the rectangular form. Suppose we are given two complex numbers:

$$Z_1 = x_1 + jy_1 \quad , \quad Z_2 = x_2 + jy_2$$

(i) Addition: $Z = Z_1 + Z_2 = (x_1 + jy_1) + (x_2 + jy_2)$

$$= (x_1 + x_2) + j\,(y_1 + y_2)$$

The magnitude of $Z = |Z| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)}^2$

The argument or amplitude of Z is $\theta = \tan^{-1}\left(\dfrac{y_1 + y_2}{x_1 + x_2}\right)$

(ii)Subtraction : $Z = Z_1 - Z_2 = (x_1 + jy_1) - (x_2 + jy_2)$

$$= (x_1 - x_2) + j\,(y_1 - y_2)$$

The magnitude of $Z = |Z| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

The argument or amplitude of Z is $\theta = \tan^{-1}\left(\dfrac{y_1 - y_2}{x_1 - x_2}\right)$

7.6 Multiplication and Division :

1- Multiplication:

(i)In Rectangular Form:

Since $Z = Z_1\ Z_2 = (x_1 + jy_1)\ (x_2 + jy_2)$

$$= x_1 x_2 + y_1 y_2\, j^2 + jx_1\, y_2 + jx_2\, y_1$$
$$= (x_1 x_2 - y_1 y_2) + j\,(x_1\, y_2 + x_2\, y_1)$$

The magnitude of $Z = |Z| = \sqrt{(x_1 x_2 - y_1 y_2)^2 + j(x_1 y_2 + x_2 y_1)^2}$

The argument or amplitude of Z $=\theta = \tan^{-1}\left(\frac{x_1y_2 + x_2y_1}{x_1x_2 - y_1y_2}\right)$

(ii)In Polar Form:

Since $Z_1 = x_1 + jy_1 = r_1 \angle \theta_1 = r_1 e^{j\theta 1}$

$Z_2 = x_2 + iy_2 = r_2 \angle \theta_2 = r_2 e^{j\theta 2}$

$Z_1 Z_2 = r_1 \angle \theta_1 \times r_2 \angle \theta = r_1 e^{j\theta 1} \times r_2 e^{j\theta 2}$

$= r_1 r_2 e^{j(\theta 1+\theta 2)}$

$Z_1 Z_2 = r_1 r_2 \angle \theta_1 + \theta_2$

(iii)In Trigonometric Form:

Since, $Z_1 Z_2 = r_1 r_2 \angle \theta_1 + \theta_2$

$Z_1 Z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + j \sin((\theta_1+\theta_2)]$

2- Division

(i) In Rectangular Form:

Since $\frac{Z_1}{Z_2} = \frac{x_1 + jy_1}{x_2 + jy_2}$

Multiplying and divide by the conjugate $z_2 - jy_2$ in order to made the denominator real.

$$\frac{Z_1}{Z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} \times \frac{x_2 - jy_2}{x_2 - jy_2}$$

$$= \frac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

Which is x + jy form

Note :

Generally, the result be expressed in the form x + jy

(ii)In Polar Form:

Since $Z_1 = x_1 + jy_1 = r_1 \angle \theta_1 = r_1 e^{j\theta 1}$

$Z_2 = x_2 + iy_2 = r_2 \angle \theta_2 = r_2 e^{j\theta 2}$

So $\frac{Z_1}{Z_2} = \frac{r_1 \angle \theta_1}{r_2 \angle \theta_2} = \frac{r_1 e^{j\theta 1}}{r_2 e^{j\theta 2}}$

$= \frac{r_1}{r_2} e^{j(1\theta_1 - \theta_2)}$

$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$

(iii)In Trigonometric Form:

Since $\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2} [\text{Cos } (\theta_1 - \theta_2) + j \text{ Sin } (\theta_1 - \theta_2)]$$

Example 4: Find the product of $1 + j$, $1 + j\sqrt{3}$, $\sqrt{3} - j$ and express the result in polar form, trigonometry form and rectangular form.

Solution:

First we express each number is polar form.

$1 + j = \sqrt{2} \angle 45^o$

$1 + j\sqrt{3} = 2 \angle 60^o$

and $\sqrt{3} - j = 2 \angle -30^o$

Hence, the product of the three numbers is

$\sqrt{2} \angle 45^o \times 2\angle 60^o \times 2 \angle -30^o$

$= \sqrt{2} \times 2 \times 2 \angle 45^o + 60^o - 30^o$

$= 4\sqrt{2} \angle 75^o$-------------------------------- **polar form.**

Or $= 4\sqrt{2}$ (cos 76^o + j sin 75^o)-----**trigonometry form.**

Or $= 4\sqrt{2}$ (0.259 + j 0.966)

$= 1.465 + j\ 5.465$----------------- **rectangular form.**

Example 5: Given $Z_1 = 8\angle -30^o$ and $Z_2 = 2\angle -60^o$ find $\frac{Z_1}{Z_2}$ and express the result in polar form, exponential form, trigonometry form and rectangular form.

Solution:

Since $\frac{Z_1}{Z_2} = \frac{8\angle -30^o}{2\angle -60^o}$

$= \frac{8}{2} \angle -30^o + 60^o$

$= 4\angle 30^o$--- **polar form.**

$Z = r\angle\theta = 4e^{j\pi/6}$-----------------------------**exponential form.**

$Z = 4(\cos 30^0 + j\sin 30^0)$------------------ **trigonometry form.**

$Z = 4(0.866 + j0.5)$

$= 3.464 + j0.5$ ------------------------------**rectangular form.**

Example 6: Simplify $\frac{3+5j}{4+3j}$ to the form a + ib.

Solution: $\frac{3+5i}{4+3j}$

Multiply and divide by conjugate 4 – 3j of denominator

so, $\frac{3+5j}{4+3j} \times \frac{4-3j}{4-3j}$

$= \frac{(12+15)+j(20-9)}{16+9}$

$= \frac{27+j11}{25}$

$= \frac{27}{25} + j\frac{11}{25}$

$= 1.08 + j0.44$

Example 7: Perform the indicated operation and given the result in rectangular form ,polar form ,exponential form and trigonometry form.

$$\frac{(1+\sqrt{3}j)(\sqrt{3}+j)}{1+j}$$

Solution:

$$Z = \frac{(1 + j\sqrt{3})(\sqrt{3} + j)}{1 + j} = \frac{(\sqrt{3} - \sqrt{3}) + 4i}{1 + j}$$

$$= \frac{4j}{1 + j} = \frac{4j}{1 + j} \times \frac{1 - j}{1 - j}$$

$$= \frac{4 + 4j}{1 + 1} = 2 + 2j \text{------------------ } \textbf{rectangular form.}$$

$$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{2}{2}\right) = 45^o$$

Hence, $Z = r\angle\theta = 2\sqrt{2}\ \angle 45^o$ ---------------------**polar form.**

$Z = r\,e^{j\theta} = 2\sqrt{2}\,e^{j\,\pi/4}$ ---------------**exponential form.**

$Z = 2\sqrt{2}\,(\cos 45^0 + j \sin 45^0)$---------- **trigonometry form.**

Example 8: **Using $Z = \cos\theta + j \operatorname{Sin}\theta$ and $Z^3 = \operatorname{Cos}^3\theta + j \sin 3\theta$, by expanding $(\cos\theta + j \operatorname{Sin}\theta)^3$ and equating real and imaginary parts show that:**

(a) $\operatorname{Cos} 3\theta = 4\cos^3\theta - 3\cos\theta$

(b) $\operatorname{Sin} 3\theta = 3\sin\theta - 4\sin^3\theta$

Solution:

Since $\cos 3\theta + j \sin 3\theta = Z^3$

So, $\cos 30 + j \operatorname{Sin} 30 = (\cos\theta + j\sin\theta)^3$

Expand by binomial theorem

$=\cos^3\theta + 3\cos^2\theta\,(j\sin\theta) + 3\cos\theta\,(j\sin\theta)^2 + (j\sin\theta)^3$

Put $j^2 = -1$ and $j^3 = -j$

$= \cos^3\theta + j\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - j\sin^3\theta$

$=(\cos^3\theta - 3\cos\theta\sin^2\theta) + j(3\cos^2\theta\sin\theta - \sin^3\theta)$

Comparing real and imaginary parts on both sides, we get

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$

$= \cos^2\theta - 3\cos\theta\,(1 - \cos^2\theta)$

$\operatorname{Cos} 3\theta = 4\cos^3\theta - 3\cos\theta$

(b) $\text{Sin } 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$

$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta)$

$= 3\sin\theta - 4\sin^3\theta$

7.8 Powers and Roots of the Complex Numbers (Vectors): (De Moivre's Theorem)

If we square the complex number

$$Z = r(\cos\theta + j\sin\theta)$$

We get, $$Z^2 = [r(\cos\theta + j\sin\theta)][r(\cos\theta + j\sin\theta)]$$

$$= r^2(\cos 2\theta + j\sin 2\theta)$$

Further more

$$Z^3 = Z(Z^2) = [r(\cos\theta + j\sin\theta)][r^2(\cos 2\theta + j\sin 2\theta)]$$

$$= r^3(\cos 3\theta + j\sin 3\theta)$$

A repeated application of this process leads to the following theorem if,

$$Z = r(\cos\theta + j\sin\theta) = r\angle\theta$$

Then $$Z^n = r^n(\cos n\theta + j\sin n\theta) = r^n\angle n\theta$$

And $$Z^{1/n} = r^{1/n}\left(\cos\frac{\theta}{n} + j\sin\frac{\theta}{n}\right)$$

Or $$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + j\sin\frac{\theta}{n}\right) = \sqrt[n]{r}\angle\frac{\theta}{n}$$

Which is the nth roots of Z.

Note:

The fact that

$$Z^{1/n} = r^{1/n}\angle\frac{\theta}{n} = r^{1/n}\angle\frac{\theta + K360^o}{n}$$

Where, $K = 0, \pm 1, \pm 2, \ldots\ldots\ldots$ enable use to find n distinct nth roots of the complex number by assigning to K the values 0, 1, 2, 3, ………n – 1.

Example 9: **Simplify** **(a)** $(\sqrt{3} + j)^7$

(b) $\sqrt{3 - j4}$ **and express the result in a + jb form**

Solution:

(a) $\sqrt{3} + j = 2 \angle 30^o$, which is a polar form

Then $(\sqrt{3} + j)^7 = (2\angle 30^o)^7 = 2^7 \angle 7\,(30^o)$

$= 128 \angle 210^o$

$= 128\left[(\cos 210^o + j \sin 210^o\right]$

$= 128\left(-\dfrac{\sqrt{3}}{2} - j\dfrac{1}{2}\right)$

$= 64\,(-\sqrt{3} - j)$

$= -64\sqrt{3} - j64$

(c) Express first 3 – 4j in polar form:

$3 - j4 = 5 \angle -53.13^o$

So, $\sqrt{3 - j4} = (5 \angle -53.13^o)^{1/2}$

$= \sqrt{5}\angle \dfrac{-53.13^0 + k\,360^0}{2}$, where k = 0 , 1

Hence the roots are

$= \sqrt{5}\angle \dfrac{-53.13^0}{2}$ and $\sqrt{5}\angle \dfrac{-53.13^0 + 360^0}{2}$

$\sqrt{5} \angle -26.57^o$	$= \sqrt{5} \angle 153.03^0$
$= \sqrt{5}\,(\cos(-26.57^o) + j \sin(-26.57^o))$	$= \sqrt{5}\,(\cos 153.03^0 + j\sin 153.03^0)$
$= \sqrt{5}\,(0.89 - j\,0.45)$	$= \sqrt{5}\,(-0.89 + j\,0.45)$
$= 2 - j$	$= -2 + j$

Example 10: Find the four fourth roots of Z = 2 + $\sqrt{3}$ j in the form r $\angle\theta$.

Solution:

Write Z $= 2 + j2\sqrt{3}$ in polar form.

Z $= 2 + 2\sqrt{3}\,j = 4\angle 60^o$

Now, the fourth roots of Z is

$\omega = Z^{1/4} = (4 \angle 60^o)^{1/4} = 4^{1/4} \angle \dfrac{60^o}{4}$

$$\omega_k = 4^{1/4} \angle \frac{60^o + K\,360^o}{4}$$

$$\omega_k = \sqrt{2} \angle \frac{60^o + K\,360^o}{4}$$

Taking K = 0, 1, 2, 3, in turn, gives the roots

$$\omega_o = \sqrt{2} \angle \frac{60^o}{4} = \sqrt{2} \angle 15^o$$

$$\omega_1 = \sqrt{2} \angle \frac{60^o + 360^o}{4} = \sqrt{2} \angle 105^o$$

$$\omega_2 = \sqrt{2} \angle \frac{60^o + 720^o}{4} = \sqrt{2} \angle 195^o$$

$$\omega_3 = \sqrt{2} \angle \frac{60^o + 1090^o}{4} = \sqrt{2} \angle 285^o$$

7.9 Principle Roots:

The n distinct nth roots of x + jy are equal spaced about the circumference of a circle of radius $r^{1/n} = (\sqrt{x^2+y^2})^{1/n}$, the smallest argument(for K = 0) being $\frac{\theta}{n}$. The root that is obtained by using this smallest value of θ is called the Principle root.

i.e., The principle root is: $Z^{1/n} = r^{1/n} \angle \frac{\theta}{n}$

Example 11: Solve the equation, $x^3 + j4 = 4\sqrt{3}$.

Solution:

$x^3 = 4\sqrt{3} - 4j$,convert into polar form.

$$x^3 = 8\angle -30^o$$

$$x^3 = 8\angle 330^o$$

$$x = (8\angle 330^o)^{1/3}$$

$$= 2 \angle \frac{330^o + K\,360^o}{3}$$

$$x_o = 2\angle 110^o$$

$$x_1 = 2\angle 230^o$$

$$x_2 = 2\angle 350^o$$

Hence, the solution set is: $2\angle 110^o$, $2\angle 230^o$, $2\angle 350^o$

7.10 Derivation of Euler's Relation:

By calculus Techniques, it can be shown that the sine and cosine functions of any angle x (in radian) are given by an expression containing an infinite series of terms.

$$\text{Sin } x = x - \frac{x^3}{3!} + \frac{x^3}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\text{Cos } x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots\ldots\ldots\ldots\ldots \quad (2)$$

By the same kind of Technique, e^x can also be expressed as an infinite series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots\ldots\ldots\ldots\ldots \quad (3)$$

Substitute $x = j\theta$, we get,

$$e^{j\theta} = 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \frac{j^5\theta^5}{5!} + \frac{j^6\theta^6}{6!} + \ldots\ldots\ldots\ldots\ldots$$

And substituting values for power of j

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\ldots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\ldots\ldots\ldots\right)$$

From equations (1) and (2)

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Exercise 7

Q.1: If A = 200 + j 425 and B = 150 – j 275 , find

(a) A + B (b) A – 2B

Q.2: Express the following in the forms a + jb and $r \angle \theta$

(a) $(5 + j4)^2$ (b) (7–j2) (4 + j5)+ (3 – j)(5 – j2)

(c) $\dfrac{2 + j}{2 - j} + \dfrac{2 - j}{2 + j}$ (d) $\dfrac{(3+j3)(5-j3)}{3-j4}$

Q.3: Express each of the following in Rectangular form i.e., a + jb form.

(a) $3 \angle 45^o$ (b) $86 \angle -115^o$

(c) $2 \angle \dfrac{\pi}{6}$ rad (d) $3 \angle 0^o$

Q.4: Evaluate the following expressions:

(a) $(5 \angle 45°)(3 \angle 36^o)$ (b) $(1.1 \angle 30^o)(2.3 \angle 17^o)(2.8 \angle 74^o)$

(c) $\dfrac{3.7 \angle 17^o}{6.5 \angle 48^o}$ (d) $\dfrac{(8.7 \angle 76^o)(6.8 \angle 62^o)(1.2 \angle -67^o)}{(8.9 \angle 74^o)(1.9 \angle 24^o)}$

Q.5: Evaluate the following expressions:

(a) $(1 + j)^3$ (b) $(-2 + 3j)^4$

(c) $\left[\left(2 \angle \dfrac{\pi}{4}\right)\left(3 \angle \dfrac{\pi}{4}\right)\right]^2$ (d) $\left(\dfrac{2\sqrt{2} \angle 30^o}{\sqrt{3} \angle 15^o}\right)^2$

Q.6: Express the following in a + jb form.

(a) $5e^{j\theta}$ (b) $10e^{j60^o}$ (c) $5\ e^{j\pi/3}$

Q.7: Find the indicated roots of the following:

(a) $\sqrt{5 + j8}$ (b) fifth roots of $-\sqrt{3} - j$

Q.8: Find all the n of the nth roots of the following:

(a) $Z = 32 \angle 45^o$; n = 5 (b) $Z = -16\sqrt{3} + j16$; n =5

Q.9: Given $\omega = u + jv$ and $Z = x + jy$ and $\omega = 3Z^2$, express u and v in terms of x and y.

Q.10: The resultant impedance Z of two parallel circuits is given by $\dfrac{1}{Z} = \dfrac{1}{Z_1} + \dfrac{1}{Z_2}$.

If $Z_1 = 5 - j3$ and $Z_2 = 3 + j5$ express Z in the form a + jb.

Q.12: In alternating current theory, the voltage V, current I and impedance Z may all be complex numbers and the basic relation between V, I, and Z is $I = \frac{V}{Z}$. Find a+ jb form.

(i)	I	When	V= 10,	Z = 4 + j3
(ii)	V	When	I = 3 + j8,	Z = 10 + j5
(iii)	Z	When	I = 8 + j5 ,	V =20 – j8

Q.13: Solve the following equations:

(a) $x^2 = -j16$ (b) $x^5 = 16 - j16\sqrt{3}$

(Hints: find all n of the nth roots).

Answers 7

Q.1: (a) 350 + j15 (b) -100 + j975

Q.2: (a) 9 + j40 ; 41 $\angle$ 77°18′

(b) 51 + j16; 53.46$\angle$ 17°25′

(c) 1.2 + j0; 1.2 $\angle$ 0 (d) $\frac{1}{25}$(59 + j87); 4.288$\angle$ 55°51′

Q.3: (a) $\frac{3}{\sqrt{2}} + j\frac{3}{\sqrt{2}}$ (b) - 36.3 – j78

(c) $\sqrt{3} + j$ (d) 3 + oj

Q.4: (a) 15 $\angle$ 81° (b) 7.08 $\angle$ 121°

(c) 5.69 $\angle$ - 31° (d) 4.34 $\angle$ -27°

Q.5: (a) $2\sqrt{2}$ $\angle$ 135° (b) 169 $\angle$ 134° 40′

(c) 36 $\angle$ 180° (d) 2.667 $\angle$ 30°

Q.6: (a) 5 + j0 (b) 5 - j8.66 (c) 2.5 + j4.33

Q.7: (a) 3.907 $\angle$ 29° , 3.907 $\angle$ 209°

(b) $2^{1/5}\angle(42^o + K72^o)$, k = 0, 1, …….4

Q.8: (a) 2 $\angle$ 9° ; 2 $\angle$ 81°; 2 $\angle$ 153°; 2 $\angle$ 225° ; 2 $\angle$ 297°

	(b)	$2 \angle 30°$; $2 \angle 102°$; $2 \angle 174^o$; $2 \angle 246°$; $2 \angle 318°$				
Q.9:		$U = 3(x^2 - y^2)$, $V = 6xy$				
Q.10:		$Z = 4 + j$				
Q.12:	(i)	$1.6 – j1.2$	(ii)	$-10 + j95$	(iii)	$1.348 – j1.843$
Q.13:	(a)	$4 \angle 135^o$; $4 \angle 315^o$	or	$-2.83 + j2.83$; $2.83 – j2.83$		
	(b)	$2 \angle 60°$; $2 \angle 132°$; $2 \angle 204°$; $2 \angle 276°$; $2 \angle 348°$				

Short Questions

Write the short answers of the following:

Q1. Write the phasor (vector) $Z = a + jb$ in Trigonometric and Exponential form.

Q2. Express $\sqrt{2} \angle 45^0$ in Rectangular form. (i.e., $a + j\, b$)

Q3. Express $\sqrt{3} + j$ in Polar form.

Q4. Express $Z = e^{j\, \pi/3}$ in Rectangular form. (i.e., $a + j\, b$)

Q5. Find the product of $Z_1 = 2\angle 15^0$, $Z_2 = -1\angle 30^0$

Q6. Given that $Z_1 = 4\angle 60^0$ and $Z_2 = 2\angle 30^0$ find $\frac{Z_1}{Z_2}$

Q7. If $A = 2 + j\,3$ and $B = 8 + j\,5$, then find $A + B$

Q8. Simplify $(2 + j\,3)\,(4 - j\,2)$.

Q9. If $A = 20\angle 60^0$ and $B = 5\angle 30^0$, then find AB.

Q10. Simplify $\dfrac{(5\angle 45^0)(6\angle 60^0)}{3\angle 30^0}$

Q11. Write the conjugate and modulus of- $2 + j$.

Q12. Write the conjugate and modules of $\frac{-2}{3} - j\,\frac{4}{9}$

Simplify the Phasor(vector) and write the result in Rectangular form.

Q13. $(7-j2) - (4 + j5)$

Q14. $(-5+j3)\,(2 - j3)$

Q15. $\dfrac{-9+j4}{8-j3}$

Q16. $\dfrac{1}{4-j5} - \dfrac{1}{5-j4}$

Answers

Q1. $Z = r\,(\cos\theta + j\sin\theta)$, $Z = r\,e^{j\theta}$

Q2. $1 + j$

Q3. $2(\cos 30^0 + j\sin 30^0)$

Q4. $1 + j\sqrt{3}$

Q5. $-\sqrt{2} + j\sqrt{2}$

Q6. $\frac{1}{2}\angle 30^0$

Q7. $10 + j\,8$

Q8. $14 + j\,8$

Q9. $100\angle 90^0$

Q10. $10\angle 75^0$

Q11. $-2 - j$, $5\sqrt{5}$

Q12. $-\frac{2}{3} + j\,\frac{4}{9}$, $\frac{\sqrt{52}}{9}$

Q13. $3 - j7$

Q14. $-1 + j21$

Q15. $-\frac{84}{73} + j\,\frac{5}{73}$

Q16. $-\frac{1}{41} - j\,\frac{1}{41}$

OBJECTIVE TYPE QUESTIONS

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. The value of j^3 E is:

(a) jE (b) -JE (c) E (d) -E

__2. The trigonometric form of Z = a + jb is:

(a) |Z|, (cos θ - j sin θ) (b) |Z| (cos θ + j sin θ)

(c) (cos θ + sin θ) (d) (cos θ - j sin θ)

__3. The trigonometric form of $e^{-j\theta}$ is:

(a) (cos θ + j sin θ) (b) (cos θ - j sin θ)

(c) (cos θ + sin θ) (d) (cos θ - sin θ)

__4. If 2(cos 60° - j sin 60°) then he exponential form is:

(a) $2e^{j60^o}$ (b) $2\,e^{-j60^o}$ (c) $2e^{\,60^o}$ (d) $2\,e^{-60^o}$

__5. If E (cos θ + j sin θ) then the exponential form is:

(a) $Ee^{j\theta}$ (b) $Ee^{-j\theta}$ (c) $e^{j\theta}$ (d) $e^{-j\theta}$

__6. The trigonometric form of $1 + j\sqrt{3}$ is:

(a) 2(cos 60° -- j sin 60°) (b) 2(cos 30° - j sin 30°)

(c) 2(cos 60° + j sin 60°) (d) 2(cos 30° + j sin 30°)

__7. The polar form of -1- j is:

(a) $2 \angle 225°$ (b) $-2 \angle 225^o$

(c) $-\sqrt{2}\ \angle 225°$ (d) $\sqrt{2} \angle 225°$

__8. a + jb form of $2 \angle \frac{\pi}{6}$ is:

(a) $1 + j\sqrt{3}$ (b) $1 - j\sqrt{3}$

(c) $\sqrt{3} + j$ (d) $\sqrt{3} - j$

__9. If Z = 1 – j, then arg Z is:

(a) 45^o (b) 135^o (c) 225^o (d) 315^o

__10. If $Z_1 = 8 \angle -30^o$ and $Z_2 = 2 \angle -60^o$, then $\frac{Z_1}{Z_2}$ is:

(a) $4 \angle 30^o$ (b) $4 \angle -30^o$

(c) $4 \angle 90^o$ (d) $4 \angle -90^o$

Answers

1.	b	2.	b	3.	b	4.	a	5.	a
6.	c	7.	d	8.	c	9.	d	10.	a

Chapter – 8
COMPLEX NUMBERS

7.1 Introduction:

The real number system had limitations that were at first accepted and later overcome by a series of improvements in both concepts and mechanics. In connection with, quadratic, equations we encountered the concept of imaginary number and the device invented for handling it, the notation $i^2 = -1$or $i = \sqrt{-1}$. In this chapter we continue the extension of the real number system to include imaginary' numbers. The extended system is called the complex number system.

7.2 Complex Number:

A complex number is a number of the form a + bi, where a and b are real and $i^2 = -1$ or $i = \sqrt{-1}$. The letter 'a' is called the real part and 'b' is called the imaginary part of a + bi. If a = 0, the number ib is said to be a purely imaginary number and if b = 0, the number a is real. Hence, real numbers and pure imaginary numbers are special cases of complex numbers. The complex numbers are denoted by Z , i.e.,

$$Z = a + bi.$$

In coordinate form, $Z = (a, b)$.

Note : Every real number is a complex number with 0 as its imaginary part.

7.3 Properties of Complex Number:

(i) The two complex numbers a + bi and c + di are equal if and only if a = c and b =d for example if.

$$x - 2 + 4yi = 3 + 12\,i$$

Then $x - 2 = 3$ and $y = 3$

(ii) If any complex number vanishes then its real and imaginary parts will separately vanish.

For, if $a + ib = 0$, then

$$a = -ib$$

Squaring both sides

$$a^2 = -b^2$$

$$a^2 + b^2 = 0$$

Which is possible only when $a = 0$, $b = 0$

7.4 Basic Algebraic Operation on Complex Numbers:

There are four algebraic operations on complex numbers.

(i) **Addition:**

If $Z_1 = a_1 + b_1 i$ and $Z_2 = a_2 + b_2 i$, then

$$Z_1 + Z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$$
$$= (a_1 + a_2) + i(b_1 + b_2)$$

(ii) **Subtraction:**

$$Z_1 - Z_2 = (a_1 + b_1 i) - (a_2 + b_2 i)$$
$$= (a_1 - a_2) + i(b_1 - b_2)$$

(iii) **Multiplication:**

$$Z_1\ Z_2 = (a_1 + b_1 i) . (a_2 + b_2 i)$$
$$= a_1 a_2 + b_1 b i^2 + a_1 b_2 i + b_1 a_2 i$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

(iv) **Division:**

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

Multiply Numerator and denominator by the number $a_2 - b_2 i$ in order to make the denominator real.

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} \times \frac{a_2 - b_2 i}{a_2 - b_2 i}$$
$$= \frac{(a_1 a_2 + b_1 b_2) + i(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2}$$
$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\ \frac{b_1 a_2 - b_1 b_2}{a_2^2 + b_2^2}$$

Generally result will be expressed in the form $a + ib$.

Example 1: Add and subtract the numbers $3 + 4i$ and $2 - 7i$.

Solution:

Addition: $(3 + 4i) + (2 - 7i) = (3 + 2) + i(4 - 7) = 5 - 3i$

Subtraction: $(3 + 4i) - (2 - 7i) = (3 - 2) + i(4 + 7) = 1 + 11i$

Example 2: Find the product of the complex numbers: $3 + 4i$ and $2 - 7i$.

Solution: $(3 + 4i)\,(2 - 7i) = 6 - 21i + 8i - 28i^2$

$$= 6 + 28 - 13i$$

$$= 34 - 13i$$

Example 3: Divide $3 + 4i$ by $2 - 7i$.

Solution: $\dfrac{3 + 4i}{2 - 7i} = \dfrac{3 + 4i}{2 - 7i} \times \dfrac{2 + 7i}{2 + 7i}$

$$= \frac{6 + 28i^2 + i(21 + 8)}{4 + 49}$$

$$= \frac{-22 + 29i}{53}$$

$$= \frac{-22}{53} + i\,\frac{29}{53}$$

Example 4: Express $\dfrac{(2 + i)\,(1 - i)}{4 - 3i}$ in the form of $a + ib$.

Solution: $\dfrac{(2 + i)(1 - i)}{4 - 3i} = \dfrac{(2 + 1) + i(1 - 2)}{4 - 3i} = \dfrac{3 - i}{4 - 3i}$

$$= \frac{3 - i}{4 - 3i} \times \frac{4 + 3i}{4 + 3i} = \frac{(12 + 3) + i(9 - 4)}{16 + 9}$$

$$= \frac{15 + i(5)}{25} = \frac{15}{25} + \frac{5}{25}\,i$$

$$= \frac{3}{5} + \frac{1}{5}\,i$$

Example 5: Separate into real and imaginary parts: $\dfrac{1 + 4i}{3 + i}$.

Solution :

$$\frac{1 + 4i}{3 + i} = \frac{1 + 4i}{3 + i} \times \frac{3 - i}{3 - i}$$

$$= \frac{(3 + 4) + i(12 - 1)}{9 + 1} = \frac{7 + 11i}{10}$$

$$= \frac{7}{10} + \frac{11}{10}i = \frac{7}{10} + \frac{11}{10}i$$

Here, real part = a = $\frac{7}{10}$

And imaginary part = b = $\frac{11}{10}$

Extraction of square roots of a complex number:

Example 6: Extract the square root of the complex numbers 21 – 20i.

Solution:

Let $a + ib = \sqrt{21 - 20i}$

Squaring both sides

$$(a + ib)^2 = 21 - 20\,i$$

$$a^2 - b^2 + 2abi = 21 - 20\,i$$

Comparing both sides

$$a^2 - b^2 = 21 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$2ab = -20 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (2) $b = -\frac{10}{a}$ Put b in equation (1),

$$a^2 - \frac{100}{a^2} = 21$$

$$a^4 - 21a^2 - 100 = 0$$

$$(a^2 - 25)(a^2 + 4) = 0$$

$$a^2 = 25 \quad \text{or} \quad a^2 = -4$$

$$a = \pm 5 \quad \text{or} \quad a = \pm\sqrt{-4} = \pm 2i$$

But a is not imaginary, so the real value of a is

$$a = 5 \quad \text{or} \quad a = -5$$

The corresponding value of b is

$b = -2 \quad$ or $\quad b = 2$

Hence the square roots of $21 - 20i$ are:

$5 - 2i \quad$ and $\quad -5 + 2i$

Factorization of a complex numbers:

Example 7: Factorise: $a^2 + b^2$

Solution:

We have
$$a^2 + b^2 = a^2 - (-b^2)$$
$$= a^2 - (i^2 b^2) \quad , \quad i^2 = -1$$
$$= (a)^2 - (ib)^2$$
$$= (a + ib)(a - ib)$$

7.5 Additive Inverse of a Complex Number:

Let $Ƶ = a + ib$ be a complex number, then the number $-a - ib$ is called the additive inverse of Ƶ. It is denoted by $-Ƶ$ i.e.,

$-Ƶ = -a - ib.$ Also $Ƶ - Ƶ = 0$

Example 8: Find the additive inverse of $2 - 5i$

Solution: Let $Ƶ = 2 - 5i$

Then additive inverse of Ƶ is:

$-Ƶ = -(2 - 5i) = -2 + 5i.$

7.6 Multiplicative inverse of a complex number:

Let $a + ib$ be a complex number, then $x + iy$ is said to be multiplicative inverse of $a + ib$ if

$$(x + iy)(a + ib) = 1$$

Or
$$x + iy = \frac{1}{a + ib}$$
$$= \frac{1}{a + ib} \times \frac{a - ib}{a - ib}$$
$$x + iy = \frac{a - ib}{a^2 + b^2}$$
$$x + iy = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$$

So $\quad x = \dfrac{a}{a^2 + b^2} \qquad y = -\dfrac{b}{a^2 + b^2}$

Hence multiplicative inverse of (a, b) is $\left(\dfrac{a}{a^2 + b^2}, -\dfrac{b}{a^2 + b^2}\right)$

Example 9: Find the multiplicative inverse of 4 + 3i or (4, 3).

Solution:

The multiplicative inverse of 4 + 3i is: $\dfrac{1}{4 + 3i}$

Since, $\dfrac{1}{4 + 3i} = \dfrac{1}{4 + 3i} \times \dfrac{3 - 3i}{4 - 3i}$

$$= \frac{4 - 3i}{16 + 9} = \frac{4}{25} - \frac{3}{25} i = \left(\frac{4}{25}, -\frac{3}{25}\right)$$

7.7 Conjugate of a complex number:

Two complex numbers are called the conjugates of each other if their real parts are equal and their imaginary parts differ only in sign.

If Z = a + bi, the complex number a – bi is called the conjugate of Z. it is denoted by $\bar{Z}$.

i.e., $\bar{Z} = \overline{a + bi} = a - bi$

Moreover $Z.\bar{Z} = (a + bi)(a - bi) = a^2 + b^2$

$Z + \bar{Z} = 2a$ and $Z - \bar{Z} = 2bi$

Theorem: If Z_1 and Z_2 are complex numbers, then

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(ii) $\overline{z_1 + z_2} = \bar{z}_1 \,.\, \bar{z}_2$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Proof:

Let $Z_1 = a_1 + bi$ and $Z_2 = a_2 + b_2i$

(i) $Z_1 + Z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$

$$\overline{z_1 + z_2} = (a_1 + a_2) - i\,(b_1 + b_2)$$
$$= (a_1 - ib_1) + (a_2 - ib_2)$$
$$= \overline{z_1} + \overline{z_2}$$

(ii) $Z_1 Z_2 = (a_1 + ib_1)\,(a_2 + ib_2)$

$$= (a_1\,a_2 - b_1\,b_2) + i\,((a_1b_2 + b_1b_2)$$

$$\overline{z_1 \,.\, z_2} = (a_1a_2 - b_1b_2) - i(a_1b_2 + b_1a_2)$$
$$= a_1a_2 - ia_1b_2 - ib_1\,a_2 - b_1b_2$$
$$= a_1\,(a_2 - ib_2) - ib_1a_2 + i^2\,b_1\,b_2$$
$$= a_1\,(a_1 - ib_2) - ib\,(a_2 - ib_2)$$
$$= a_1(a_2 - ib_2) - ib_1(a_2 - ib_2)$$
$$= (a_1 - ib_1)\,(a_2 - ib_2)$$
$$= \overline{z_1} \,.\, \overline{z_2}$$

(iii) $\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2}$

$$= \frac{(a_1 + ib_1)\,(a_2 - ib_2)}{a_2^2 + b_2^2}$$

$$\frac{z_1}{z_2} = = \frac{(a_1a_2 + b_1b_2)}{a_2^2 + b_2^2} + i\,\frac{(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{(a_1 + ib_1)\,(a_2 - ib_2)}}{a_2^2 + b_2^2} = \frac{\overline{(a_1 + ib_1)}\,\overline{(a_2 - ib_2)}}{a_2^2 + b_2^2} \text{ by (ii)}$$

$$= \frac{(a_1 - ib_1)\,(a_2 + ib_2)}{(a_2 - ib_2)\,(a_2 + ib_2)} = \frac{a_1 - ib_1}{a_2 - ib_2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

Example 10: Evaluate $(3 + 4i)\,(\overline{3 + 4i})$

Solution:

$$(3 + 4i)\,(\overline{3 + 4i}) = (3 + 4i)\,(3 - 4i) = 9 + 16 = 25$$

Example 11: Find the conjugate of the complex number 2i(– 3 + 8i)

Solution:

$$2i\,(-3 + 8i) = -16 - 6i$$

$$\overline{2i(-3 + 8i)} = -16 + 6i$$

Alternately $\overline{2i(-3 + 8i)} = \overline{2i}\;\overline{(-3 + 8i)}$

By above theorem $= 2i\,(-3 - 8i)$

$= -16 + 6i$

Exercise 7.1

Q.1: Write each ordered pair (complex number) in the form: a + bi.

(i) (2, 6) (ii) (5, – 2)

(iii) (– 7, – 3) (iv) (4, 0)

Q.2: Write each complex number as an ordered pair.

(i) (2 + 3i) (ii) (- 3 + i) (iii) (4i) (iv)(0)

Q.3: Find the value of x and y in each of the following:

(i) $x + 3i + 3 = 5 + yi$ (ii) $x + 2yi = ix + y + 1$

(iii) $(x, y)\,(1, 2) = (-1, 8)$ (iv) $(x, -y)\,(3, -4) = (3, -29)$

(v) $(2x - 3y) + i(x - y)\,6 = 2 - i\,(2x - y + 3)$

Q.4: Simplify the following:

(i) $(2 - 3i) + (1 + 2i)$ (ii) $(3 + 5i) - (5 - 3i)$

(iii) $(9 + 7i) - (-9 + 7i) + (-18 + i)$ (iv) $(2 - 3i)(3 + 5i)$

(v) $(4 - 3i)^2$ (vi) $(3 + 4i)(4 + 3i)(2 - 5i)$

(vii) $(-1 + i\sqrt{3})^3$ (viii) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{-3}$

(ix) $\frac{1}{8 + 3i} + \frac{1}{8 - 3i}$ (x) $\frac{2 + i}{1 - 3i}$

(xi) $\frac{(3 + 4i)(1 - 2i)}{1 + i}$ (xii) $\frac{2 + \sqrt{-1}}{3 - \sqrt{-4}}$

(xiii) $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

Q.5: Find the conjugate of each of the following:

(i) $-2 + 3i$ (ii) $(1 + i)(-2 - i)$

(iii) $-3i(2 + 5i)$ (iv) $(-5 + 3i)(2 - 3i)$

Q.6: Reduce of the following to the form a + bi.

(i) $\frac{3 - i}{3 + 2i}$ (ii) $\frac{(2 + 3i)(3 + 2i)}{4 - 3i}$

(iii) $(2 - 3i)^2\,(\overline{3 + 4i})$ (iv) $(\overline{4 + i})(\overline{-1 + 3i})$

(v) $(1 - i)^2(1 + i)$ (vi) $\frac{\sqrt{-3}}{1 - \sqrt{-7}}$

(vii) $(2 + \sqrt{-3})(2 - \sqrt{-3})$

Q.7: Factorize the following:

(i) $4m^2 + 9n^2$ (ii) $49a^2 + 625b^2$

(iii) $\frac{a^2}{9} + \frac{b^2}{81}$

Q.8: Find the multiplicative inverse of the following:

(i) $(-3, 4)$ (ii) $(\sqrt{2}, -\sqrt{5})$

(iii) $\dfrac{2}{1+\sqrt{-1}}$ (iv) $-6-3i$

(v) $4-\sqrt{-7}$

Q.9: Extract the square root of the following complex numbers:

(i) $-3+4i$ (ii) $8-6i$

(iii) $24+10\,i$

Q.10: Prove that : $\dfrac{1}{\text{Cos}\theta - i\,\text{Sin}\theta} = \text{Cos}\,\theta + i\,\text{Sin}\,\theta$

Q.11: Express $x^2+y^2=a^2$ in terms of conjugate co-ordinates.

Q.12: The resultant impedance Z of two parallel circuits of impedances z_1 and z_2 is given by the formula $\dfrac{1}{z}=\dfrac{1}{z_1}+\dfrac{1}{z_2}$, Find the resultant impedance z when

$z_1 = 3+4i$, $z_2 = 4-2i$

Answers 7.1

(1)	(i)	$2+6i$	(ii)	$5-2i$	(iii)	$-7-3i$	(iv)	4
(2)	(i)	$(2,3)$	(ii)	$(-3,1)$	(iii)	$(0,4)$	(iv)	$(0,0)$
(3)	(i)	$x=2,\ y=3$	(ii)	$x=2,\ y=1$	(iii)	$x=3,\ y=2$		
	(iv)	$x=5,\ y=3$	(v)	$x=-\dfrac{23}{10},\ y=-\dfrac{11}{5}$				
(4)	(i)	$3+i$	(ii)	$-2+8i$	(iii)	i		
	(iv)	$21+i$	(v)	$7-24i$	(vi)	$125+50\,i$		
	(vii)	8	(viii)	1	(ix)	$\dfrac{16}{73}$		
	(x)	$-\dfrac{1}{10}+\dfrac{7}{10}i$	(xi)	$\dfrac{(9-13i)}{2}$	(xii)	$\dfrac{4}{13}+\dfrac{7}{13}i$		
	(xiii)	1						
(5)	(i)	$-2-3i$	(ii)	$-1+3i$	(iii)	$15+6i$		
(6)	(i)	$\dfrac{(7-9i)}{13}$	(ii)	$\dfrac{(-39+52i)}{25}$	(iii)	$-6-17i$		
	(iv)	$-7-11i$	(v)	$2-2i$	(vi)	$\dfrac{-\sqrt{21}}{8}+\dfrac{\sqrt{3}}{8}i$		
	(vii)	7						

(7) (i) $(2m + 3ni)(2m - 3ni)$ (ii) $(7a+25bi)(7a - 25bi)$

(iii) $\left(\frac{a}{3} + \frac{ib}{9}\right)\left(\frac{a}{3} - \frac{ib}{9}\right)$ or $\frac{1}{9}\left(a + \frac{ib}{3}\right)$

(8) (i) $\left(-\frac{3}{25}, -\frac{4}{25}\right)$ (ii) $\left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$ (iii) $\frac{1}{2} + \frac{1}{2}i$

(iv) $-\frac{6}{25} + i\frac{3}{45}$ (v) $\frac{4}{23} + \frac{\sqrt{7}}{23}i$

(9) (i) $\pm(1 + 2i)$ (ii) $\pm(3 - i)$ (iii) $\pm(5 + i)$

(11) $z\bar{z} = a^2$ (12) $z = 3.018 + 0.566i$

7.8 Graphical Representation:

Since a complex number Z = a + ib can also be represented by an ordered pair (a, b), each point in the plane can be viewed as the graph of a complex number. Thus, the graph of the complex number (a, b) or a + ib, is as shown in fig. 1. Since the real part a of a + ib taken as the x-coordinate of P, in this context the x-axis is called the real axis. Similarly, since the imaginary part b of a + ib is taken as y-coordinate of P, the y-axis is called the imaginary axis.

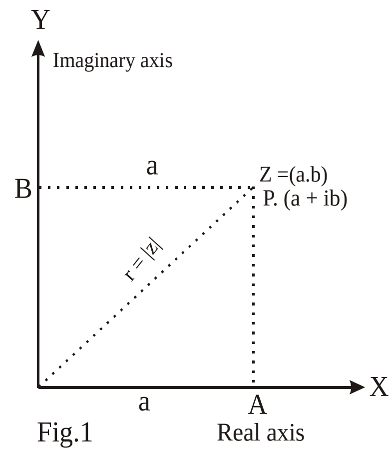


Fig.1

A plane on which complex numbers are thus represented is often called a complex plane. It is also sometimes called an Argand or Gauss Plane, after the French Mathematician Jean Robert Argand and the great German Mathemation Carl Friedrich Gauss.

7.9 Modulus of a Complex Number:

The Modulus or the absolute value of the complex number Z = a + ib is denoted by r, |Z| or |a + ib| and is given by,

$$r = |Z| = |a + ib| = \sqrt{a^2 + b^2}$$

Thus the modulus |a + ib| is just the distance from the origin to the point (a , b)

Note:

Since $z\bar{z} = a^2 + b^2 = |z|^2$, so $|z| = \sqrt{z\bar{z}}$

Example 11: Find The modulus of (3, – 5) and – 7 – i

Solution:

$$|(3, -5)| = |3 - 5i| = \sqrt{3^2 + (-5)^2} = \sqrt{9 + 25}$$
$$= \sqrt{34}$$

And $$|-7 - i| = \sqrt{(-7)^2 + (-1)^2} = \sqrt{49 + 1}$$
$$= \sqrt{50}$$

Theorem:

For complex numbers z_1 and z_2

(i) $|z_1 . z_2| = |z_1|\, |z_2|$

(ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(iii) $|z_1 + z_2| \le |z_1| + |z_2|$

(iv) $|z_1 - z_2| \ge |z_1| - |z_2|$

Proof:

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$

(i) $$|z_1\, z_2| = |(a_1 + ib_1)(a_2 + ib_2)|$$
$$= |(a_1 a_2 - b_1b_2) + i\,(a_1b_2 + b_1a_2)|$$
$$= \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2}$$
$$= \sqrt{a_1^2a_2^2 + b_1^2b_2^2 + a_1^2b_2^2 + b_1^2\, a_2^2}$$
$$= \sqrt{a_1^2(a_2^2 + b_2^2) + b_1^2\,(a_2^2 + b_2^2)}$$
$$= \sqrt{(a_1^2 + b_1^2)\,(a_2^2 + b_2^2)}$$
$$= \sqrt{a_1^2 + b_1^2}\ .\ \sqrt{a_2^2 + b_2^2}$$
$$|z_1\, z_2| = |z_1|\, |z_2|$$

(ii) $$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{a_2 - ib_2}{a_2 - ib_2}$$

$$= \frac{(a_1a_2 + b_1b_2) + i(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2}$$

$$\left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2}\right)^2 + \left(\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}\right)^2}$$

$$= \sqrt{\frac{a_1^2\, a_2^2 + b_1^2\, b_2^2 + b_1^2\, a_2^2 + a_1^2\, b_2^2}{(a_2^2 + b_2^2)^2}}$$

$$= \sqrt{\frac{(a_1^2 + b_1^2)\,(a_2^2 + b_2^2)}{(a_2^2 + b_2^2)^2}}$$

$$= \sqrt{\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}} = \sqrt{\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

(iii) $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$$= (a_1 + a_2) + i(b_1 + b_2)$$

$$|z_1 + z_2| = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$$

$$= \sqrt{a_1^2 + a_2^2 + 2a_1a_2 + b_1^2 + b_2^2 + 2b_1b_2}$$

$$= \sqrt{(a_1^2 + b_1^2) + (a_2^2 + b_2^2) + 2(a_1a_2 + b_1b_2)}$$

Squaring both sides:

$$|z_1 + z_2|^2 = (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + 2(a_1a_2 + b_1b_2)$$

$$= |z_1|^2 + |z_2|^2 + 2\sqrt{(a_1a_2 + b_1b_2)^2}$$

$$= |z_1|^2 + |z_2|^2 + 2\sqrt{a_1^2\, a_2^2 + b_1^2 b_2^2 + 2a_1a_2\, b_1b_2}$$

as $2a_1\, a_2\, b_1\, b_2 \le a_1^2 b_2^2 + b_1^2 a_2^2$

So $|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2$

$$+ 2\sqrt{a_1^2\, a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + b_1^2 a_2^2}$$

$$\le |z_1|^2 + |z_2|^2 + 2\sqrt{(a_1^2 + b_1^2)\,(a_2^2 + b_2^2)^2}$$

$$\le |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2|$$

$$|z_1 + z_2|^2 \le \left[|z_1| + |z_2|\right]^2$$

So $|z_1 + z_2| \le |z_1| + |z_2|$

(iv) $|z_1| = |(z_1 - z_2) + z_2|$

$\le |z_1 - z_2| + |z_2|$ by (iii)

$|z_1| - |z_2| \le |z_1 - z_2|$

Or $|z_1 - z_2| \ge |z_1| - |z_2|$

7.10 Polar form of a complex number

In the fig. 2, we join the point P with the origin, we obtain the line r and the angle θ. Then the numbers or order pair (r, θ) are called the polar coordinates of the point P to distinguish them from the rectangular co-ordinates (x, y).

We call r the absolute value or modulus of z and θ, the angle from the positive real axis to this line, as the argument or amplitude of z and is denoted by arg z i.e., θ = arg z.

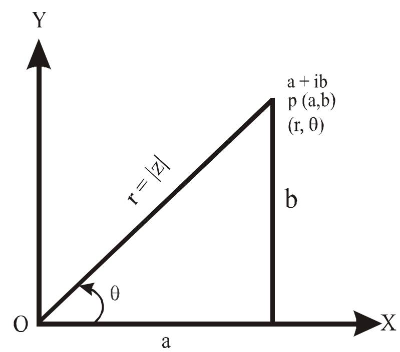


Fig.2

By use of Pythagorean theorem we have

$$\cos\theta = \frac{a}{r} \quad , \quad \sin\theta = \frac{b}{r}$$

$$a = r\cos\theta \quad , \quad b = r\sin\theta$$

$$r = |z| = \sqrt{a^2 + b^2} \quad ; \quad r \ge 0$$

and, $\tan\theta = \dfrac{b}{a}$

$$\theta = \arg Z = \tan^{-1}\left(\frac{b}{a}\right)$$

Therefore, the complex number

$$Z = a + ib$$

$$= r\cos\theta + ir\sin\theta$$

$$Z = r(\cos\theta + i\sin\theta) \ldots\ldots\ldots\ldots\ldots..(1)$$

This is sometimes written as $Z = r \operatorname{Cis}\theta$

The right hand side of equation (1) is called the Trigonometric or polar form of Z.

The argZ has any one of an infinite number of real values differing by integral multiple $2k\pi$, where $k = 0, \pm 1, \pm 2, \ldots\ldots\ldots$ The values satisfying the relation. $\pi \le \theta \le \pi$ is called the principle value of the arg Z, denoted by Arg. Z.

Thus $\arg Z = \text{Arg } Z + 2k\pi$

Example 12: Express the complex number $1 + i\sqrt{3}$ in polar form.

Solution:

Here, we have $a = 1,\ b = \sqrt{3}$

$$r = \sqrt{a^2 + b^2} = \sqrt{1+3} = 2$$

and $\tan\theta = \dfrac{b}{a} = \sqrt{3}$

$$\theta = \tan^{-1}\sqrt{3} = 60^o$$

Hence $1 + i\sqrt{3} = 2\,[\text{Cos } 60^o + i \sin 60^o] = 2 \text{ Cis } 60^o$

Example13: Write $4(\text{Cos}225^o + i\ \text{Sin}225^o)$ in rectangular form.

Solution:

Since $\text{Cos } 225^o = \dfrac{-1}{\sqrt{2}}$ and $\text{Sin } 225^o = \dfrac{-1}{\sqrt{3}}$

So, $4\,(\text{Cos } 225^o + i \text{ Sin } 225^o) = 4\left(-\dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}\right)$

$$= -\frac{4}{\sqrt{2}} - i\frac{4}{\sqrt{2}}$$

$$= -2\sqrt{2} - 2\sqrt{2}\,i$$

Alternately, $a = r \text{ Cos } \theta = 4\left(-\dfrac{1}{\sqrt{2}}\right) = -2\sqrt{2}$

And $b = r \text{ Sin } \theta = 4\left(-\dfrac{1}{\sqrt{2}}\right) = -2\sqrt{2}$

$$a + ib = -2\sqrt{2} - 2\sqrt{2}\ i$$

Example 14: Find the magnitude (modulus) and argument of $(4 + 7i)(3 - 2i)$

Solution:

$$(4 + 7i)(3 - 2i) = (12 + 14) + i(21 - 8)$$
$$= 26 + 13i$$
$$r = \sqrt{26^2 + 13^2} = \sqrt{676 + 169} = 29.1$$
$$\tan\theta = \frac{13}{26} = \frac{1}{2}$$
$$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26^o\, 34'$$

Example 15: Find z such that $|Z| = \sqrt{2}$ and arg $Z = \frac{\pi}{4}$.

Solution:

Since, $|Z| = \sqrt{2}$, $\theta = \arg z = \frac{\pi}{4}$

$$a = r \cos\theta = \sqrt{2}\ \cos\frac{\pi}{4} = \sqrt{2}\ .\ \frac{1}{\sqrt{2}} = 1$$

and $b = r \sin\theta = = \sqrt{2}\ \sin\frac{\pi}{4} = \sqrt{2}\ .\ \frac{1}{\sqrt{2}} = 1$

$$Z = a + bi = 1 + i$$

Example 16: Show that the equation $|z + i| = 4$ represents a circle and find its centre and radius.

Solution:

$|Z + i| = 4$

Or $|x + iy + i| + I = 4$

Or $|x + i(y + 1)| = 4$

Or $\sqrt{x^2 + (y + 1)^2} = 4$

Or $x^2 + (y + 1)^2 = 16$

Which represents the circle with centre at (0, -1) and radius 4.

Exercise 7.2

Q.1: Find the magnitude (Modulus) of the following:

(i) -2 (ii) 3 + 2i (iii) 5i (iv) (2, 0)

(v) (-2, 1) (vi) (-2,-1) (vii) $\frac{1 + 2i}{2 - i}$

(viii) $\frac{(3 - 5i)(1 + i)}{4 + 2i}$

Q.2: Express of the following complex number in the polar (Trigonometric) form:

(i) $2 + 2\sqrt{3}\, i$ (ii) $1 - i$ (iii) $-1 - i$

(iv) $-1 + i\sqrt{3}$ (v) $-\sqrt{3} + i$ (vi) $-3 - 4i$

(vi) $8 - 15i$ (viii) -2 (ix) $2\sqrt{3} - 2i$

(x) $\left(\frac{2 + i}{3 - i}\right)^2$ (xi) $\frac{1 + 2i}{1 - 3i}$

Q.3: Write each complex number in the form a + bi

(i) $4 \text{ Cis } 240^o$ (ii) $3 \text{ Cis } 300^o$

(iii) $6 \text{ Cis } (-30^o)$ (iv) $12\text{Cis } 420^o$

Q.4: Find the magnitude and the principle argument of:

(i) $5 - 7i$ (ii) $8 + 5i$ (iii) $(5 - 7i)(8 + 5i)$

(iv) $\frac{5 - 7i}{8 + 5i}$ (v) $\frac{1 + i}{1 + i}$

Q.5: Find z such that:

(i) $|Z| = 8\sqrt{2}$, $\arg Z = \frac{\pi}{4}$ (ii) $|Z| = 5$, $\arg Z = -\frac{\pi}{2}$

(iii) $|Z| = 2$, $\arg Z = \frac{\pi}{6}$ (iv) $|Z| = \sqrt{6}$, $\arg Z = -\frac{\pi}{3}$

(v) $|Z| = \frac{1}{3}$, $\arg Z = \frac{\pi}{3}$ (vi) $|Z| = \sqrt{3}$, $\arg Z = -\pi$

Answers 7.2

Q.1:	(i)	2	(ii)	$\sqrt{13}$	(iii)	5	(iv)	2
	(v)	$\sqrt{5}$	(vi)	$\sqrt{5}$	(vii)	1	(viii)	$\sqrt{\frac{17}{5}}$
Q.2:	(i)	$4 \text{ Cis } 60^o$	(ii)	$\sqrt{2}\text{ Cis } 315^o$	(iii)	$\sqrt{2}\text{ Cis } 225^o$		
	(iv)	$2 \text{ Cis } 120^o$	(v)	$2 \text{ Cis } 150^o$	(vi)	$5 \text{ Cis } 233^o\ 10'$		
	(vii)	$17 \text{ Cis } 298^o$	(viii)	$2 \text{ Cis } \pi$	(ix)	$4 \text{ Cis } 330^o$		
	(x)	$\frac{1}{2}\text{Cis}\frac{\pi}{2}$	(xi)	$\frac{1}{\sqrt{2}}\text{ Cis } 135^o$				
Q.3:	(i)	$-2 -2\sqrt{3}\, i$	(ii)	$\frac{3}{2} - \frac{3\sqrt{3}}{2}\, i$	(iii)	$3\sqrt{3} - 3i$		
	(iv)	$6 + 6\sqrt{3}\, i$						
Q.4:	(i)	$8.602,\ -54^o\ 28'$	(ii)	$9.434,\ 32^o$				
	(iii)	$81.16,\ -22^o\ 28'$	(iv)	$0.912,\ -86^o\ 28'$				
	(vi)	$1,\ 90^o$						
Q.5:	(i)	$8 + 8i$	(ii)	$-5i$	(iii)	$\sqrt{3} + i$		
	(iv)	$\frac{\sqrt{3}}{\sqrt{2}} - \frac{3}{\sqrt{2}}\, i$	(v)	$\frac{1}{6} + \frac{\sqrt{3}}{6}\, i$	(vi)	$-\sqrt{3}$		

7.11 Multiplication and Division of Complex Numbers in Polar Form:

Generally addition and subtraction are better dealt with using Cartesian forms, but the multiplication and division of complex numbers can be found quite easily when the complex numbers are written in Trigonometric or polar form.

Multiplication:

Let $Z_1 = r_1 (\text{Cos } \theta_1 + i \text{ Sin } \theta_1)$

$Z_2 = r_2 (\text{Cos } \theta_1 + i \text{ Sin } \theta_1)$

$Z_1 . Z_2 = [r_1(\text{Cos } \theta_1 + i \text{ Sin } \theta_1)]\ [r_2(\text{Cos } \theta_2 + i \text{ Sin } \theta_2)]$

$=r_1r_2\ [(\text{Cos}\ \theta_1\ \text{Cos}\ \theta_2 - \text{Sin}\ \theta_1\ \text{Sin}\ \theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$Z_1\ Z_2\ = r_1r_2[\text{Cos}\ (\theta_1 + \theta_2) + i\ \text{Sin}\ (\theta_1 + \theta_2)]$

Or $Z_1\ Z_2\ = r_1r_2\ \text{Cis}\ (\theta_1 + \theta_2)$

Hence the absolute value of the product of two complex numbers is the product of their absolute values and the argument of the product of two complex numbers is the sum of their arguments.

Note: Since the product of two complex numbers is itself a complex number, we may find the product of any number of complex numbers by repeated application of this theorem.

i.e. $Z_1\ Z_1\ = r_1r_1[\text{Cos}\ (\theta_1 + \theta_1) + i\ \text{Sin}\ (\theta_1 + \theta_1)]$

$Z_1^{\ 2}\ =[r_1^{\ 2}(\text{Cos}\ (2\theta_1) + i\ \text{Sin}\ (2\theta_1)$

Or $Z_1^{\ 2}\ =r_1^{\ 2}\ \text{Cis}\ 2\theta_1$

Similarly $Z^3\ =r^3\ \text{Cis}\ 3\theta,$ for $Z = r\ (\text{Cos}\ \theta + i\ \text{Sin}\ \theta)$

A repeated application of this process, we get,

$$Z^n = r^n\ (\text{Cos}\ n\ \theta + i\ \text{Sin}\ n\ \theta) = r^n\ \text{Cis}\ n\ \theta$$

Similarly it $Z^{1/n}$ is the n-th root of Z, then put n = 1/n in above equation. then,

$$Z^{1/n} = r^{\ 1/n}\ (\text{Cos}\frac{\theta}{n} + i\ \text{Sin}\frac{\theta}{n}) = r^{1/n}\ \text{Cis}\frac{\theta}{n}$$

Example 17: Multiply $4(\text{Cis}\ 30^o + i\ \text{Sin}\ 30^o)$ by $2\ (\text{Cis}\ 60^o + i\ \text{Sin}\ 60^o)$

Solution:

Let $Z_1 = 4\ (\ \text{Cos}\ 30^o + i\ \text{Sin}\ 30^o)$

And $Z_2 = 2\ (\ \text{Cos}\ 60^o + i\ \text{Sin}\ 60^o)$

Then $Z_{1.}\ Z_2 = 4 \times 2[\text{Cos}\ (30^o + 60^o) + i\ \text{Sin}\ (30^o + 60^o)]$

$Z_{1.}\ Z_2 = 8\ (\text{Cis}\ 90^o + i\ \text{Sin}\ 90^o) = 8\ \text{Cis}\ 90^o$

Example 18: Write $\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^3$ in the form a + bi

Solution: Let , $Z = \frac{-1}{2} - \frac{i\sqrt{3}}{2}$, first we write it in polar form

$$r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1, \qquad \theta = \tan^{-1}\left(\frac{\frac{-\sqrt{3}}{2}}{-\frac{1}{2}}\right)$$

$$\theta = \tan^{-1}\sqrt{3}$$

$$\theta = 240^o$$

$$Z = r\,[\text{Cos}\,\theta + i\,\text{Sin}\,\theta) = 1\ (\text{Cos}\,240^o + i\,\text{Sin}\,240^o)$$

$$Z = \left(-\frac{1}{3} - \frac{\sqrt{3}}{2}\right)^3 = 1^3\,[(\text{Cos}\,3\,(240^o) + i\,\text{Sin}\,3\,(240^o)]$$

$$= \text{Cos}\,720^o + i\,\text{Sin}\,720^o = 1$$

$$\frac{Z_1}{Z_2} = \frac{r_1(\text{Cos}\,\theta_1 + i\,\text{Sin}\,\theta_1)}{r_2\,(\text{Cos}\,\theta_2 + i\,\text{Sin}\,\theta_2)}$$

$$= \frac{r_1(\text{Cos}\,\theta_1 + i\,\text{Sin}\,\theta_1)}{r_2\,(\text{Cos}\,\theta_2 + i\,\text{Sin}\,\theta_2)} \times \frac{(\text{Cos}\,\theta_2 - i\,\text{Sin}\,\theta_2)}{(\text{Cos}\,\theta_2 - i\,\text{Sin}\,\theta_2)}$$

$$= \frac{r_1\left[(\text{Cos}\,\theta_1 \cos\,\theta_2 + \text{Sin}\,\theta_1\,\text{Sin}\,\theta_2) + i(\text{Sin}\,\theta_1\,\text{Cos}\,\theta_2 - \cos\,\theta_1 \text{Sin}\,\theta_2)\right]}{r_2(\text{Cos}^2\,\theta_2 + \text{Sin}^2\,\theta_2)}$$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\left[\,\text{Cos}\,(\theta_1 - \theta_2) + i\,\text{Sin}(\theta_1 - \theta_2)\right]$$

Or $$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\,\text{Cis}\,(\theta_1 - \theta_2)$$

Hence the absolute value of the quotient of two complex numbers is the quotient of their absolute values and the argument of the quotient is the angle of the dividend minus the angle of the divisor.

Example 19:

$$\frac{8\,\text{Cis}\,540^o}{2\,\text{Cis}\,225^o} = 4\,\text{Cis}\,(540^o - 225^o)$$

$$= 4\,\text{Cis}\,(315^o)$$

$$= 4\,(\text{Cos}\,315^o + i\,\text{Sin}\,315^o)$$

Since $\text{Cos}\,315^o = \frac{1}{\sqrt{2}}$ and $\text{Sin}\,315^o = -\frac{1}{\sqrt{2}}$

So $$\frac{8\,\text{Cis}\,540^o}{2\,\text{Cis}\,225^o} = 4\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i\right)$$

$= 2\sqrt{2} - 2\sqrt{2}\ i$

Example 20: Find the quotient of $1 + \sqrt{3}$ i and 1 + i.

Solution:

We first write each of them in polar form.

Let $Z_1 = 1 + \sqrt{3}\ i$ and $Z_2 = 1 + i$

$r_1 = \sqrt{1+3} = 2$ and $r_2 = \sqrt{1+1} = \sqrt{2}$

$\theta_1 = \tan^{-1}\sqrt{3} = 60^o$ and $\theta_2 = \tan^{-1}(1) = 45^o$

So, $Z_1 = 2(\text{Cos } 60^o + i \text{ Sin } 60^o)$ and $Z_2 = \sqrt{2}(\text{Cos } 45^o + i \text{ Sin } 45^o)$

Now,
$$\frac{z_1}{z_2} = \frac{2(\text{Cos } 60^o + i \text{ Sin } 60^o)}{\sqrt{2}(\text{Cos } 45^o + i \text{ Sin } 45^o)}$$
$$= \sqrt{2}[\text{Cos }(60^o - 45^o) + i \text{ Sin }(60^o - 45^o)]$$
$$= \sqrt{2}[\text{Cos } 15^o + i \text{ Sin } 15^o] = \sqrt{2} \text{ Cis } 15^o$$

Exercise 7.3

Q.1: Perform the indicated operations in each and express the results in the form a + ib.

(i) $[3(\text{Cos } 22^o + i \text{ Sin } 22^o] [2(\text{Cos } 8^o + i \text{ Sin } 8^o]$

(ii) $[4(\text{ Cos } 29^o + I \text{ Sin } 29^o)]\left[\frac{1}{2}(\text{Cos } 16^o + i \text{ Sin } 16^o)\right]$

(iii) $(\sqrt{5} \text{ Cis } 28^o)(\sqrt{5} \text{ Cis } 8^o)(2 \text{ Cis } 9^o)$

(iv) $\dfrac{6(\text{Cos } 51^o + i \text{ Sin } 51^o)}{2(\text{Cos } 21^o + i \text{ Sin } 21^o)}$

(v) $\dfrac{10(\text{Cos } 143^o + i \text{ Sin } 143^o)}{5(\text{Cos } 8^o + i \text{ Sin } 8^o)}$

(vi) $\dfrac{(\text{Cis } 180^o)(6 \text{ Cis } 99^o)}{3(\text{Cos } 39^o + i \text{ Sin } 39^o)}$

(vii) $\dfrac{15(\text{Cos } 48^o + i \text{ Sin } 48^o)}{(3 \text{ Cis } 46^o)(2 \text{ Cis } 32^o)}$

Q.2: Perform the indicated operations and give the results in polar form.

(i) $(1 + i)(1 - \sqrt{3}\ i)$ (ii) $(-\sqrt{3} - i)(-1 + i)$

(iii) $(1 + i)^4$ (iv) $(1 - i\sqrt{3})^2$

(v) $\sqrt{\dfrac{1 + i}{1 - i}}$ (vi) $\dfrac{-1 - i}{-1 + i}$

(vii) $\dfrac{(1 + i\sqrt{3})(\sqrt{3} + i)}{1 + i}$

Q.3: Show that $\left|\dfrac{1 + 2i}{2 - i}\right| = 1$

Answers 7.3

Q.1: (i) $3(\sqrt{3} + i)$ (ii) $\sqrt{2}(1 + i)$ (iii) $5\sqrt{2}(1 + i)$

(iv) $1.5(\sqrt{3} + i)$ (v) $\sqrt{2}(-1 + i)$ (vi) $-5(1 + i\sqrt{3})$

(vii) $1.25(\sqrt{3} - i)$

Q.2: (i) $2\sqrt{2}$ Cis 345^o (ii) $2\sqrt{2}$ Cis 345^o

(iii) Cos 180^o (iv) 4 Cis 240^o

(v) Cis 45^o (vi) Cis 90^o

(vii) $2\sqrt{2}$ Cis 45^o

Short Questions

Write the short answers of the following:

Q.1: Write the conjugate and modulus of- 2 + i

Q.2: Write the conjugate and modulus of $\frac{-2}{3} - \frac{4}{9} i$

Q.3: Simplify the complex numbers:

(i)$(2 + 5i) + (-3 + i)$ (ii)$(7-2i) - (4 + 5i)$ (iii) $(-5+3i)(2 - 3i)$

(iv) $\frac{-9 + 4i}{8 - 3i}$

Q.4: Find the addition inverse of (3, - 8).

Q.5: Find the conjugate and modulus of $\frac{1 + i}{1 - i}$

Q.6: Prove that if $Z = \bar{Z}$, then $\bar{Z}$ is real.

Q.7: Find the values of x and y from the equation.

$(2x - y - 1) - i(x - 3y) = (y - x) - i(2 - 2y)$

Q.8: Show that $\left|\frac{1 + 2i}{2 - i}\right| = 1$

Q.9: If $Z = 2 + 3i$ Prove that $Z\bar{Z} = 13$

Q.10: Find the multiplication inverse of (-3, 4)

Q.11: Find the conjugate and modulus of complex number $\frac{1 + 2i}{2 - i}$

Q.12: Factorize $36a^2 + 100b^2$

Q.13: Factorize $2x^2 + 5y^2$

Q.14: Factorize $9a^2 + 64b^2$

Q.15: Write complex number $-\sqrt{2} + \sqrt{6}\, i$ in polar (trigonometric)form.

Q.16: Express complex number $3 - \sqrt{3}\, i$ in polar(trigonometric)form.

Express the following complex number in the form x + *i* y

Q.17: When |Z| = 6 and arg Ƶ $= \frac{3\overline{\wedge}}{4}$

Q.18: When |Z| = 3 and arg Ƶ $= -\frac{\pi}{2}$

Q.19: Express |Z| = 2, arg Ƶ $= \frac{\pi}{3}$

Q.20: Show that Ƶ ^{2} + $\overline{Ƶ}^{\,2}$ is a real number

Answers

Q1. $-2 - i$, $\sqrt{5}$ **Q2.** $-\frac{2}{3} + \frac{4}{9}i$, $\frac{\sqrt{52}}{9}$

Q3. (i) $1 + 6i$ (ii) $3 - 7i$ (iii) $-1 + 21i$ (iv) $-\frac{84}{73} + \frac{5}{73}i$

Q4. $-3 + 8i$ **Q5.** i , 1

Q7. x = –5 , y –7 **Q10.** $-\frac{3}{25} - \frac{4}{25}i$ **Q11.** $-i$, 1

Q12. (6a-10bi) (6a + 10bi) **Q13.** $(\sqrt{2}\,x - \sqrt{5}\,yi)(\sqrt{2}\,x + \sqrt{5}\,yi)$

Q14. (3a – 8bi)(3a +8bi) **Q15.** $2\sqrt{2}\,[\cos 60^o - i \sin 60^o]$

Q16. $2\sqrt{3}\,[\cos 30^o - i \sin 30^o]$ **Q17.** $-3\sqrt{2} + 3\sqrt{2}\,i$

Q18. $0 - 3i$ **Q19.** $1 + i\sqrt{3}$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. The additive inverse of a + ib is:
(a) -a + ib (b) a – ib (c) –a – ib (d) a + ib

__2. Sum of -3 + 5i and 4 - 7i is:
(a) 1 – 2i (b) -1 – 2i (c) 1-12i (d) -7 +12i

__3. Conjugate of (2 + 3i) + [1 – i] is:
(a) 3 - 2i (b) 3 + 4i (c) 3 – 4i (d) 3 + 2i

__4. Modulus of 3 + 4i is:
(a) 47 (b) 16 (c) 5 (d) 3

__5. Product of 2 + 3i and 2 – 3i is:
(a) $\sqrt{13}$ (b) 13 (c) $\sqrt{2}$ (d) $\sqrt{-5}$

__6. Ordered pair form of -3 - 2i is:
(a) (3, 2) (b) (-3, -2) (c) (-3, 2) (d) (3, -2)

__7. If z = a + ib then Z+ $\overline{Z}$ is equal to:
(a) 2a (b) 2b (c) 0 (d) 2a + 2ib

__8. (1 + 2i) (3 – 5i) is equal to:
(a) 13 + i (b) -2 - i (c) -4 + 3i (d) 3i

__9. i[2 – i] is equal to:
(a) 1 + 2i (b) $2i + i^2$ (c) 3 (d) 3i

__10. If z= a + ib then, $\overline{Z}$ is equal to:
(a) a + ib (b) a – ib (c) a + b (d) a – b

ANSWERS

1.	c	2.	a	3.	a	4,	c	5.	b
6.	b	7.	a	8.	a	9.	a	10.	b

Chapter 9
Partial Fractions

9.1 Introduction: A fraction is a symbol indicating the division of integers. For example, $\frac{13}{9}, \frac{2}{3}$ are fractions and are called Common Fraction. The dividend (upper number) is called the numerator N(x) and the divisor (lower number) is called the denominator, D(x).

From the previous study of elementary algebra we have learnt how the sum of different fractions can be found by taking L.C.M. and then add all the fractions. For example

i) $\frac{1}{x-1} + \frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$

ii) $\frac{2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x-2} = \frac{9x^2+5x-3}{(x+1)^2(x-2)}$

Here we study the reverse process, i.e., we split up a single fraction into a number of fractions whose denominators are the factors of denominator of that fraction. These fractions are called **Partial fractions.**

9.2 Partial fractions :

To express a single rational fraction into the sum of two or more single rational fractions is called **Partial fraction resolution**.

For example,

$$\frac{2x+x^2-1}{x(x^2-1)} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$$

$\frac{2x+x^2-1}{x(x^2-1)}$ is the resultant fraction and $\frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$ are its partial fractions.

9.3 Polynomial:

Any expression of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real constants, if $a_n \neq 0$ then P(x) is called polynomial of degree n.

9.4 Rational fraction:

We know that $\frac{p}{q}$, $q \neq 0$ is called a rational number. Similarly the quotient of two polynomials $\frac{N(x)}{D(x)}$ where $D(x) \neq 0$, with no common factors, is called a rational fraction. A rational fraction is of two types:

9.5 Proper Fraction:

A rational fraction $\dfrac{N(x)}{D(x)}$ is called a proper fraction if the degree of numerator N(x) is less than the degree of Denominator D(x).

For example

(i) $\dfrac{9x^2 - 9x + 6}{(x-1)(2x-1)(x+2)}$

(ii) $\dfrac{6x + 27}{3x^3 - 9x}$

9.6 Improper Fraction:

A rational fraction $\dfrac{N(x)}{D(x)}$ is called an improper fraction if the degree of the Numerator N(x) is greater than or equal to the degree of the Denominator D(x)

For example

(i) $\dfrac{2x^3-5x^2-3x-10}{x^2 - 1}$

(ii) $\dfrac{6x^3-5x^2-7}{3x^2-2x-1}$

<u>Note:</u> An improper fraction can be expressed, by division, as the sum of a polynomial and a proper fraction.

For example:

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Which is obtained as, divide $6x^2 + 5x^2 - 7$ by $3x^2 - 2x - 1$ then we get a polynomial (2x+3) and a proper fraction $\dfrac{8x - 4}{x^2 - 2x - 1}$

9.7 Process of Finding Partial Fraction:

A proper fraction $\dfrac{N(x)}{D(x)}$ can be resolved into partial fractions as:

(I) If in the denominator D(x) a linear factor (ax + b) occurs and is non-repeating, its partial fraction will be of the form $\dfrac{A}{ax + b}$,where A is a constant whose value is to be determined.

(II) If in the denominator D(x) a linear factor (ax + b) occurs n times, i.e., $(ax + b)^n$, then there will be n partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots\ldots+\frac{A_n}{(ax+b)^n}$$

,where A_1, A_2, A_3 - - - - - - - - - A_n are constants whose values are to be determined

(III) If in the denominator D(x) a quadratic factor $ax^2 + bx + c$ occurs and is non-repeating, its partial fraction will be of the form

$\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants whose values are to be determined.

(IV) If in the denominator a quadratic factor $ax^2 + bx + c$ occurs n times, i.e., $(ax^2 + bx + c)^n$,then there will be n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+$$

$$\text{- - - - - - - - - - -}+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Where A_1, A_2, A_3 - - - - - - - - A_n and B_1, B_2, B_3 - - - - - - - B_n are constants whose values are to be determined.

Note: The evaluation of the coefficients of the partial fractions is based on the following theorem:

If two polynomials are equal for all values of the variables, then the coefficients having same degree on both sides are equal, for example , if

$px^2 + qx + a = 2x^2 - 3x + 5 \quad \forall\ x$, then

$p = 2, \quad q = -3$ and $a = 5$.

9.8 Type I

When the factors of the denominator are all linear and distinct i.e., non repeating.

Example 1:

Resolve $\frac{7x-25}{(x-3)(x-4)}$ into partial fractions.

Solution:

$$\frac{7x-25}{(x-3)(x-4)}=\frac{A}{x-3}+\frac{B}{x-4} \text{ -------------------(1)}$$

Multiplying both sides by L.C.M. i.e., $(x - 3)(x - 4)$, we get

$7x - 25 = A(x - 4) + B(x - 3)$ -------------- (2)

$7x - 25 = Ax - 4A + Bx - 3B$

$7x - 25 = Ax + Bx - 4A - 3B$

$7x - 25 = (A + B)x - 4A - 3B$

Comparing the co-efficients of like powers of x on both sides, we have

$7 = A + B$ and

$-25 = -4A - 3B$

Solving these equation we get

$A = 4$ and $B = 3$

Hence the required partial fractions are:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Alternative Method:

Since $7x - 25 = A(x - 4) + B(x - 3)$

Put $x - 4 = 0, \Rightarrow x = 4$ in equation (2)

$7(4) - 25 = A(4 - 4) + B(4 - 3)$

$28 - 25 = 0 + B(1)$

$B = 3$

Put $x - 3 = 0 \Rightarrow x = 3$ in equation (2)

$7(3) - 25 = A(3 - 4) + B(3 - 3)$

$21 - 25 = A(-1) + 0$

$-4 = -A$

$A = 4$

Hence the required partial fractions are

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Note : The R.H.S of equation (1) is the identity equation of L.H.S

Example 2:

write the identity equation of $\dfrac{7x-25}{(x-3)(x-4)}$

Solution : The identity equation of $\dfrac{7x-25}{(x-3)(x-4)}$ is

$$\frac{7x-25}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

Example 3:

Resolve into partial fraction: $\dfrac{1}{x^2 - 1}$

Solutios: $\dfrac{1}{x^2 - 1} = \dfrac{A}{x - 1} + \dfrac{B}{x + 1}$

$1 = A(x + 1) + B(x - 1)$ (1)

Put $x - 1 = 0, \Rightarrow x = 1$ in equation (1)

$1 = A(1 + 1) + B(1 - 1) \Rightarrow A = \dfrac{1}{2}$

Put $x + 1 = 0$, ⇨ $x = -1$ in equation (1)

$1 = A(-1+1) + B(-1-1)$

$1 = -2B$, ⇨ $B = \frac{1}{2}$

$$\frac{1}{x^2 - 1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Example 4:

Resolve into partial fractions $\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Solution:

This is an improper fraction first we convert it into a polynomial and a proper fraction by division.

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x+3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Let $\frac{8x - 4}{x^2 - 2x - 1} = \frac{8x - 4}{(3x+1)} = \frac{A}{x-1} + \frac{B}{3x+1}$

Multiplying both sides by $(x - 1)(3x + 1)$ we get

$$8x - 4 = A(3x + 1) + B(x - 1) \qquad \text{(I)}$$

Put $x - 1 = 0$, $\Rightarrow$ $x = 1$ in (I), we get

The value of A

$$8(1) - 4 = A(3(1) + 1) + B(1 - 1)$$
$$8 - 4 = A(3 + 1) + 0$$
$$4 = 4A$$
$$\Rightarrow \qquad A = 1$$

Put $3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$ in (I)

$$8\left(-\frac{1}{3}\right) - 4 = B\left(-\frac{1}{3} - 1\right)$$

$$-\frac{8}{3} - 4 = \left(-\frac{4}{3}\right)$$

$$-\frac{20}{3} = -\frac{4}{3}B$$

$$\Rightarrow \qquad B = \frac{20}{3} \text{ x } \frac{3}{4} = 5$$

Hence the required partial fractions are

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{1}{x - 1} + \frac{5}{3x + 1}$$

Example 5:

Resolve into partial fraction $\dfrac{8x - 8}{x^3 - 2x^2 - 8x}$

Solution: $\dfrac{8x - 8}{x^3 - 2x^2 - 8x} = \dfrac{8x - 8}{x(x^2 - 2x - 8)} = \dfrac{8x - 8}{x(x - 4)(x + 2)}$

Let $\dfrac{8x - 8}{x^3 - 2x^2 - 8x} = \dfrac{A}{x} + \dfrac{B}{x - 4} + \dfrac{C}{x + 2}$

Multiplying both sides by L.C.M. i.e., $x(x - 4)(x + 2)$

$$8x - 8 = A(x - 4)(x + 2) + Bx(x + 2) + Cx(x - 4) \qquad \text{(I)}$$

Put $x = 0$ in equation (I), we have

$8(0) - 8 = A(0 - 4)(0 + 2) + B(0)(0 + 2) + C(0)(0 - 4)$

$-8 = -8A + 0 + 0$

$\Rightarrow$ $A = 1$

Put $x - 4 = 0 \Rightarrow x = 4$ in Equation (I), we have

$8(4) - 8 = B(4)(4 + 2)$

$32 - 8 = 24B$

$24 = 24B$

$\Rightarrow$ $B = 1$

Put $x + 2 = 0 \Rightarrow x = -2$ in Eq. (I), we have

$8(-2) - 8 = C(-2)(-2 - 4)$

$-16 - 8 = C(-2)(-6)$

$-24 = 12C$

$\Rightarrow$ $C = -2$

Hence the required partial fractions

$$\frac{8x - 8}{x^3 - 2x^2 - 8x} = \frac{1}{x} - \frac{1}{x - 4} - \frac{2}{x + 2}$$

Exercise 9.1

Resolve into partial fraction:

Q.1 $\dfrac{2x + 3}{(x - 2)(x + 5)}$ Q.2 $\dfrac{2x + 5}{x^2 + 5x + 6}$

Q.3 $\dfrac{3x^2 - 2x - 5}{(x - 2)(x + 2)(x + 3)}$ Q.4 $\dfrac{(x - 1)(x - 2)(x - 3)}{(x - 4)(x - 5)(x - 6)}$

Q.5 $\dfrac{x}{(x-a)(x-b)(x-c)}$ Q.6 $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

Q.7 $\dfrac{2x^3-x^2+1}{(x+3)(x-1)(x+5)}$ Q.8 $\dfrac{1}{(1-x)(1-2x)(1-3x)}$

Q.9 $\dfrac{6x+27}{4x^3-9x}$ Q.10 $\dfrac{9x^2-9x+6}{(x-1)(2x-1)(x+2)}$

Q.11 $\dfrac{x^4}{(x-1)(x-2)(x-3)}$ Q.12 $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$

Answers 9.1

Q.1 $\dfrac{1}{x-2}+\dfrac{1}{x+5}$ Q.2 $\dfrac{1}{x+2}+\dfrac{1}{x+3}$

Q.3 $\dfrac{3}{20(x-2)}-\dfrac{11}{4(x-2)}+\dfrac{28}{5(x+3)}$

Q.4 $1+\dfrac{3}{x-4}-\dfrac{24}{x-5}+\dfrac{30}{x-6}$

Q.5 $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-a)(b-c)(x-b)}+\dfrac{c}{(c-b)(c-a)(x-c)}$

Q.6 $\dfrac{a^2}{(a-b)(a-c)(1-ax)}+\dfrac{b^2}{(b-a)(b-c)(1-bx)}+\dfrac{c^2}{(c-b)(c-a)(1-cx)}$

Q.7 $2+\dfrac{31}{4(x+3)}+\dfrac{1}{12(x-1)}-\dfrac{137}{6(x+5)}$

Q.8 $\dfrac{1}{2(1-x)}-\dfrac{4}{(1-2x)}+\dfrac{9}{2(1-3x)}$

Q.9 $\dfrac{3}{x}+\dfrac{4}{2x-3}+\dfrac{2}{2x+3}$

Q.10 $\dfrac{2}{x-1}-\dfrac{3}{2x-1}+\dfrac{4}{x+12}$

Q.11 $x+6+\dfrac{1}{2(x-1)}-\dfrac{16}{x-2}+\dfrac{81}{2(x-3)}$

Q.12 $1+\frac{1}{x}-\frac{1}{5(x-1)}-\frac{8}{5(2x+3)}$

9.9 Type II:

When the factors of the denominator are all linear but some are repeated.

Example 1:

Resolve into partial fractions: $\frac{x^2-3x+1}{(x-1)^2(x-2)}$

Solution:

$$\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$$

Multiplying both sides by L.C.M. i.e., $(x-1)^2(x-2)$, we get

$$x^2-3x+1=A(x-1)(x-2)+B(x-2)+C(x-1)^2 \quad \text{(I)}$$

Putting $x-1=0 \Rightarrow x=1$ in (I), then

$(1)^2-3(1)+1=B(1-2)$

$1-3+1=-B$

$-1=-B$

$\Rightarrow B=1$

Putting $x-2=0 \Rightarrow x=2$ in (I), then

$(2)^2-3(2)+1=C(2-1)^2$

$4-6+1=C(1)^2$

$\Rightarrow -1=C$

Now $x^2-3x+1=A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)$

Comparing the co-efficient of like powers of x on both sides, we get

$A+C=1$

$A=1-C$

$=1-(-1)$

$=1+1=2$

$\Rightarrow A=2$

Hence the required partial fractions are

$$\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{1}{x-2}$$

Example 2:

Resolve into partial fraction $\frac{1}{x^4(x+1)}$

Solution

$$\frac{1}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1}$$

Where A, B, C, D and E are constants. To find these constants multiplying both sides by L.C.M. i.e., x^4 (x + 1), we get

$1 = A(x^3)(x + 1) + Bx^2 (x + 1) + Cx (x + 1) + D(x + 1) + Ex^4$ (I)

Putting $x = -1$ in Eq. (I)

$1 = E(-1)^4$

$\Rightarrow$ $E = 1$

Putting x = 0 in Eq. (I), we have

$1 = D(0 + 1)$

$1 = D$

$\Rightarrow$ $D = 1$

$1 = A(x^4 + x^3)+B(x^3 + x^2)+C(x^2 + x)+D(x + 1) + Ex$

Comparing the co-efficient of like powers of x on both sides.

Co-efficient of x^3 : $A + B = 0$ ……… (i)

Co-efficient of x^2 : $B + C = 0$ ……… (ii)

Co-efficient of x : $C + D = 0$ ……… (iii)

Putting the value of D = 1 in (iii)

$C + 1 = 0$

$\Rightarrow$ $C = -1$

Putting this value in (ii), we get

$B - 1 = 0$

$\Rightarrow$ $B = 1$

Putting B = 1 in (i), we have

$A + 1 = 0$

$\Rightarrow$ $A = -1$

Hence the required partial fraction are

$$\frac{1}{x^4(x+1)} = \frac{-1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x+1}$$

Example 3:

Resolve into partial fractions $\dfrac{4 + 7x}{(2 + 3x)(1 + x)^2}$

Solution:

$$\frac{4+7x}{(2+3x)(1+x)^2} = \frac{A}{2+3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

Multiplying both sides by L.C.M. i.e., $(2 + 3x)(1 + x)^2$

We get $4 + 7x = A(1 + x)^2 + B(2 + 3x)(1 + x) + C(2 + 3x)$ …. (I)

Put $2 + 3x = 0$ $\Rightarrow$ $x = -\frac{2}{3}$ in (I)

Then $4+7\left(-\frac{2}{3}\right) = A\left(1-\frac{2}{3}\right)^2$

$$4-\frac{14}{3} = A\left(-\frac{1}{3}\right)^2$$

$$-\frac{2}{3} = \frac{1}{9}A$$

$\Rightarrow$ $A=\frac{-2}{3} \text{ x } \frac{9}{1} = -6$

$A = -6$

Put $1 + x = 0$ $\Rightarrow$ $x = -1$ in eq. (I), we get

$4 + 7(-1) = C(2 - 3)$

$4 - 7 = C(-1)$

$-3 = -C$

$\Rightarrow$ $C = 3$

$4 + 7x = A(x^2 + 2x + 1) + B(2 + 5x + 3x^2) + C(2 + 3x)$

Comparing the co-efficient of x^2 on both sides

$A + 3B = 0$

$-6 + 3B = 0$

$3B = 6$

$\Rightarrow$ $B = 2$

Hence the required partial fraction will be

$$\frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2}$$

Exercise 4.2

Resolve into partial fraction:

Q.1 $\frac{x+4}{(x-2)^2(x+1)}$ Q2. $\frac{1}{(x+1)(x^2-1)}$

Q.3 $\frac{4x^3}{(x+1)^2(x^2-1)}$ Q.4 $\frac{2x+1}{(x+3)(x-1)(x+2)^2}$

Q.5 $\frac{6x^2-11x-32}{(x+6)(x+1)^2}$ Q.6 $\frac{x^2-x-3}{(x-1)^3}$

Q.7 $\dfrac{5x^2+36x-27}{x^4-6x^3+9x^2}$ Q.8 $\dfrac{4x^2-13x}{(x+3)(x-2)^2}$

Q.9 $\dfrac{x^4+1}{x^2(x-1)}$ Q.10 $\dfrac{x^3-8x^2+17x+1}{(x-3)^3}$

Q.11 $\dfrac{x^2}{(x-1)^3(x+2)}$ Q.12 $\dfrac{2x+1}{(x+2)(x-3)^2}$

Answers4.2

Q.1 $-\dfrac{1}{3(x-2)}+\dfrac{2}{(x-2)^2}+\dfrac{1}{3(x+1)}$

Q.2 $\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}-\dfrac{1}{2(x+1)^2}$

Q.3 $\dfrac{1}{2(x-1)}+\dfrac{7}{2(x+1)}-\dfrac{5}{(x+1)^2}+\dfrac{2}{(x+1)^3}$

Q.4 $\dfrac{5}{4(x+3)}+\dfrac{1}{12(x-1)}-\dfrac{4}{3(x+2)}+\dfrac{1}{(x+2)^2}$

Q.5 $\dfrac{10}{x+6}-\dfrac{4}{x+1}-\dfrac{3}{(x-1)^2}$

Q.6 $\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}-\dfrac{3}{(x-1)^3}$

Q.7 $\dfrac{2}{x}-\dfrac{3}{x^2}-\dfrac{2}{(x-3)}+\dfrac{14}{(x-3)^2}$

Q.8 $\dfrac{3}{x+3}+\dfrac{1}{x-2}-\dfrac{2}{(x-2)^2}$

Q.9 $x+1-\dfrac{1}{x}-\dfrac{1}{x^2}+\dfrac{2}{x-1}$

Q.10 $1+\dfrac{1}{x-3}-\dfrac{4}{(x-3)^2}+\dfrac{7}{(x-3)^3}$

Q.11 $\dfrac{4}{27(x-1)}+\dfrac{5}{9(x-1)^2}+\dfrac{1}{3(x-1)^3}-\dfrac{4}{27(x+2)}$

Q.12 $-\dfrac{3}{25(x+2)}+\dfrac{3}{25(x-3)}+\dfrac{7}{5(x-3)^2}$

9.10 Type III:

When the denominator contains ir-reducible quadratic factors which are non-repeated.

Example 1:

Resolve into partial fractions $\dfrac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by L.C.M. i.e., $(x + 3)(x^2 + 1)$, we get

$$9x - 7 = A(x^2 + 1) + (Bx + C)(x + 3) \qquad \text{(I)}$$

Put $x + 3 = 0 \Rightarrow x = -3$ in Eq. (I), we have

$9(-3) -7 = A((-3)^2 + 1) + (B(-3) + C)(-3 + 3)$

$-27 -7 = 10A + 0$

$A = -\dfrac{34}{10} \Rightarrow \boxed{A=-\dfrac{17}{5}}$

$$9x - 7 = A(x^2 + 1) + B(x^2 + 3x) + C(x + 3)$$

Comparing the co-efficient of like powers of x on both sides

$A + B = 0$

$3B + C = 9$

Putting value of A in Eq. (i)

$-\dfrac{17}{5} + B = 0 \Rightarrow \boxed{B=\dfrac{17}{5}}$

From Eq. (iii)

$C = 9 - 3B = 9 - 3\left(\dfrac{17}{4}\right)$

$= 9-\dfrac{51}{5} \Rightarrow \boxed{C = -\dfrac{6}{5}}$

Hence the required partial fraction are

$$\frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

Example 2:

Resolve into partial fraction $\dfrac{x^2+1}{x^4+x^2+1}$

Solution:

Let $\dfrac{x^2+1}{x^4+x^2+1} = \dfrac{x^2+1}{(x^2-x+1)(x^2+x+1)}$

$$\frac{x^2+1}{(x^2-x+1)(x^2+x+1)} = \frac{Ax+B}{(x^2-x+1)} + \frac{Cx+D}{(x^2+x+1)}$$

Multiplying both sides by L.C.M. i.e., $(x^2-x+1)(x^2+x+1)$

$x^2 + 1 = (Ax + B)(x^2 + x + 1) + (Cx + D)(x^2 - x + 1)$

Comparing the co-efficient of like powers of x, we have

Co-efficient of x^3 : $A + C = 0$ ………… (i)

Co-efficient of x^2 : $A + B - C + D = 1$ ……….. (ii)

Co-efficient of x : $A + B + C - D = 0$ ……….. (iii)

Constant $B + D = 1$ ……….. (iv)

Subtract (iv) from (ii) we have

$A - C = 0$ …………… (v)

$A = C$ …………… (vi)

Adding (i) and (v), we have

$A = 0$

Putting A = 0 in (vi), we have

$C = 0$

Putting the value of A and C in (iii), we have

$B - D = 0$ …………… (vii)

Adding (iv) and (vii)

$2B = 1 \quad \Rightarrow \quad B = \dfrac{1}{2}$

from (vii) B = D, therefore

$D = \dfrac{1}{2}$

Hence the required partial fraction are

$$\frac{0x+\frac{1}{2}}{(x^2-x+1)} + \frac{0x+\frac{1}{2}}{(x^2+x+1)}$$

i.e., $\dfrac{1}{2(x^2-x+1)} + \dfrac{1}{2(x^2+x+1)}$

Exercise 4.3

Resolve into partial fraction:

Q.1 $\dfrac{x^2+3x-1}{(x-2)(x^2+5)}$ Q.2 $\dfrac{x^2-x+2}{(x+1)(x^2+3)}$

Q.3 $\dfrac{3x+7}{(x+3)(x^2+1)}$

Q.4 $\dfrac{1}{(x^3+1)}$

Q.5 $\dfrac{1}{(x+1)(x^2+1)}$

Q.6 $\dfrac{3x+7}{(x^2+x+1)(x^2-4)}$

Q.7 $\dfrac{3x^2-x+1}{(x+1)(x^2-x+3)}$

Q.8 $\dfrac{x+a}{x^2(x-a)(x^2+a^2)}$

Q.9 $\dfrac{x^5}{x^4-1}$

Q.10 $\dfrac{x^2+x+1}{(x^2-x-2)(x^2-2)}$

Q.11 $\dfrac{1}{x^3-1}$

Q.12 $\dfrac{x^2+3x+3}{(x^2-1)(x^2+4)}$

Answers 4.3

Q.1 $\dfrac{1}{x-2}+\dfrac{3}{x^2+5}$

Q.2 $\dfrac{1}{x+1}-\dfrac{1}{x^2+3}$

Q.3 $-\dfrac{1}{5(x+3)}+\dfrac{x+12}{5(x^2+1)}$

Q.4 $\dfrac{1}{3(x+1)}-\dfrac{(x-2)}{3(x^2-x+1)}$

Q.5 $\dfrac{1}{2(x+1)}-\dfrac{x-1}{2(x^2+1)}$

Q.6 $\dfrac{13}{28(X-2)}-\dfrac{1}{12(X+2)}-\dfrac{8X+31}{21(X^2+X+1)}$

Q.7 $\dfrac{1}{x+1}+\dfrac{2x-2}{x^2-x+3}$

Q.8 $\dfrac{1}{a^3}\left[\dfrac{1}{X-a}+\dfrac{x}{X^2+a^2}-\dfrac{2}{X}-\dfrac{a}{X^2}\right]$

Q.9 $x+\dfrac{1}{4(x-1)}+\dfrac{1}{4(x+1)}-\dfrac{x}{2(x^2+1)}$

Q.10 $\dfrac{1}{3(x+1)}+\dfrac{7}{6(x-2)}-\dfrac{3x+2}{2(x^2-2)}$

Q.11 $\dfrac{1}{3(x-1)}-\dfrac{x+2}{3(x^2+x+1)}$

Q.12 $\dfrac{7}{10(x-1)}-\dfrac{1}{10(x+1)}-\dfrac{3x-1}{5(x^2+4)}$

9.11 Type IV: Quadratic repeated factors

When the denominator has repeated Quadratic factors.

Example 1:

Resolve into partial fraction $\dfrac{x^2}{(1-x)(1+x^2)^2}$

Solution:

$$\frac{x^2}{(1-x)(1+x^2)^2} = \frac{A}{1-x} + \frac{Bx+C}{(1+x^2)} + \frac{Dx+E}{(1+x^2)^2}$$

Multiplying both sides by L.C.M. i.e., $(1-x)(1+x^2)^2$ on both sides, we have

$$x^2 = A(1+x^2)^2 + (Bx+C)(1-x)(1+x^2) + (Dx+E)(1-x) \quad \ldots\ldots(i)$$

$$x^2 = A(1+2x^2+x^4) + (Bx+C)(1-x+x^2-x^3) + (Dx+E)(1-x)$$

Put $1 - x = 0 \Rightarrow x = 1$ in eq. (i), we have

$$(1)^2 = A(1+(1)^2)^2$$

$$1 = 4A \quad \Rightarrow \quad \boxed{A = \frac{1}{4}}$$

$$x^2 = A(1+2x^2+x^4) + B(x-x^2+x^3-x^4) + C(1-x+x^2-x^3) + D(x-x^2) + E(1-x) \quad \ldots\ldots\ldots\ldots \quad (ii)$$

Comparing the co-efficient of like powers of x on both sides in Equation (II), we have

Co-efficient of x^4 : $A - B = 0$ ……………. (i)

Co-efficient of x^3 : $B - C = 0$ ……………. (ii)

Co-efficient of x^2 : $2A - B + C - D = 1$ ……. (iii)

Co-efficient of x : $B - C + D - E = 0$ ……. (iv)

Co-efficient term : $A + C + E = 0$ ……………. (v)

from (i), $B = A$

$$\Rightarrow \quad B = \frac{1}{4} \qquad \because \quad A = \frac{1}{4}$$

from (i) $B = C$

$$\Rightarrow \quad C = \frac{1}{4} \qquad \because \quad C = \frac{1}{4}$$

from (iii) $D = 2A - B + C - 1$

$$= 2\left(\frac{1}{4}\right) - \frac{1}{4} + \frac{1}{4} - 1$$

$$\Rightarrow \quad \boxed{D = -\frac{1}{2}}$$

from (v) $E = -A - C$

$$E = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Hence the required partial fractions are by putting the values of A, B, C, D, E,

$$\frac{\frac{1}{4}}{1-x} + \frac{\frac{1}{4}x + \frac{1}{4}}{1+x^2} + \frac{-\frac{1}{2}x - \frac{1}{2}}{(1+x^2)^2}$$

$$\frac{1}{4(1-x)} + \frac{(x+1)}{4(1+x^2)} - \frac{x+1}{2(1+x^2)^2}$$

Example 2:

Resolve into partial fractions $\frac{x^2 + x + 2}{x^2(x^2+3)^2}$

Solution:

Let $$\frac{x^2 + x + 2}{x^2(x^2+3)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2+3} + \frac{Ex + F}{(x^2+3)^2}$$

Multiplying both sides by L.C.M. i.e., $x^2(x^2+3)^2$, we have

$$x^2 + x + 2 = Ax(x^2+3)^2 + B(x^2+3)^2$$
$$+(cx + D)x^2(x^2+3) + (Ex + F)(x^2)$$

Putting x = 0 on both sides, we have

$2 = B\,(0 + 3)^2$

$2 = 9B \quad \Rightarrow$ $\boxed{B = \frac{2}{9}}$

Now $$x^2 + x + 2 = Ax(x^4 + 6x^2 + 9) + B(x^4 + 6x^2 + 9)$$
$$+C(x^5 + 3x^2) + D(x^4 + 3x^2) + E(x^3) + Fx^2$$

$$x^2 + x + 2 = (A + C)x^5 + (B + D)x^4 + (6A + 3C + E)x^3$$
$$+(6B+3D+F)x^2 + (x+9B)$$

Comparing the co-efficient of like powers of x on both sides of Eq. (I), we have

Co-efficient of x^5 : $A + C = 0$ ……… (i)

Co-efficient of x^4 : $B - D = 0$ ……… (ii)

Co-efficient of x^3 : $6A + 3C + E = 0$ ……… (iii)

Co-efficient of x^2 : $6B + 3D + F = 1$ ……… (iv)

Co-efficient of x : 9A = 1 ……… (v)

Co-efficient term : 9B = 1 ……… (vi)

from (v) 9A = 1

$$\Rightarrow \boxed{A = \frac{1}{9}}$$

from (i) A + C = 0

C = –A

$$\Rightarrow \boxed{C = -\frac{1}{9}}$$

from (i) B + D = 0

D = –B

$$\Rightarrow \boxed{D = -\frac{2}{9}}$$

from (iii) 6A + 3C + E =

$$6\left(\frac{1}{9}\right)+3\left(-\frac{1}{9}\right)+E=0$$

$$E = \frac{3}{9}-\frac{6}{9}$$

$$\Rightarrow \boxed{E = -\frac{1}{3}}$$

from (iv) 6B + 3D + F = 1

F = 1 – 6B – 3D

$$=1-6\left(\frac{2}{9}\right)-3\left(\frac{2}{9}\right)$$

$$=1-\frac{12}{9}+\frac{6}{9}$$

$$\Rightarrow \boxed{F = \frac{1}{3}}$$

Hence the required partial fractions are

$$\frac{\frac{1}{9}}{x}+\frac{\frac{2}{9}}{x^2}+\frac{-\frac{1}{9}x-\frac{2}{9}}{x^2+3}+\frac{-\frac{1}{3}x+\frac{1}{3}}{(x^2+3)^2}$$

$$=\frac{1}{9x}+\frac{2}{9x^2}-\frac{x+2}{9(x^2+3)}-\frac{x-1}{3(x^2+3)^2}$$

Exercise 4.4

Resolve into Partial Fraction:

Q.1 $\dfrac{7}{(x+1)(x^2+2)^2}$

Q.2 $\dfrac{x^2}{(1+x)(1+x^2)^2}$

Q.3 $\dfrac{5x^2+3x+9}{x(x^2+3)^2}$

Q.4 $\dfrac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}$

Q.5 $\dfrac{2x^4-3x^2-4x}{(x+1)(x^2+2)^2}$

Q.6 $\dfrac{x^3-15x^2-8x-7}{(2x-5)(1+x^2)^2}$

Q.7 $\dfrac{49}{(x-2)(x^2+3)^2}$

Q.8 $\dfrac{8x^2}{(1-x^2)(1+x^2)^2}$

Q.9 $\dfrac{x^4+x^3+2x^2-7}{(x+2)(x^2+x+1)^2}$

Q.10 $\dfrac{x^2+2}{(x^2+1)(x^2+4)^2}$

Q.11 $\dfrac{1}{x^4+x^2+1}$

Answers 4.4

Q.1 $\dfrac{7}{9(x+1)}-\dfrac{7x-7}{9(x^2+2)}-\dfrac{7x-7}{3(x^2+2)^2}$

Q.2 $\dfrac{1}{4(1+x)}-\dfrac{x-1}{4(1+x^2)}+\dfrac{x-1}{2(1+x^2)^2}$

Q.3 $\dfrac{1}{x}-\dfrac{x}{x^2+3}+\dfrac{2x+3}{(x^2+3)^2}$

Q.4 $\dfrac{2}{x-1}+\dfrac{2x-1}{x^2+x+1}+\dfrac{3}{(x^2+x+1)^2}$

Q.5 $\dfrac{1}{3(x+1)}+\dfrac{5(x-1)}{3(x^2+2)}-\dfrac{2(3x-1)}{(x^2+1)^2}$

Q.6 $-\dfrac{2}{2x-5}+\dfrac{x+3}{1+x^2}+\dfrac{x-2}{(1+x^2)^2}$

Q.7 $\dfrac{1}{x-2}-\dfrac{x+2}{x^2+3}-\dfrac{7x+14}{(x^2+3)^2}$

Q.8 $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}-\dfrac{4}{(1+x^2)^2}$

Q.9 $\frac{1}{x+2}+\frac{2x-3}{(x^2+x+1)^2}-\frac{1}{x^2+x+1}$

Q.10 $\frac{1}{9(x^2+1)}-\frac{1}{9(x^2+4)}+\frac{2}{3(x^2+4)^2}$

Q.11 $-\frac{(x-1)}{2(x^2-x+1)}+\frac{(x+1)}{2(x^2+x+1)}$

Summary

Let N(x) ≠ and D(x) ≠ 0 be two polynomials. The $\frac{N(x)}{D(x)}$ is called a proper fraction if the degree of N(x) is smaller than the degree of D(x).

For example: $\frac{x-1}{x^2+5x+6}$ is a proper fraction.

Also $\frac{N(x^1)}{D(x)}$ is called an improper fraction of the degree of N(x) is greater than or equal to the degree of D(x).

For example: $\frac{x^5}{x^4-1}$ is an improper fraction.

In such problems we divide N(x) by D(x) obtaining a quotient Q(x) and a remainder R(x) whose degree is smaller than that of D(x).

Thus $\frac{N(x)'}{D(x)}=Q(x)+\frac{R(x)}{D(x)}$ where $\frac{R(x)'}{D(x)}$ is proper fraction.

Types of proper fraction into partial fractions.

Type 1: Linear and distinct factors in the D(x)

$$\frac{x-a}{(x+a)(x+b)}=\frac{A}{x+a}+\frac{B}{x+b}$$

Type 2: Linear repeated factors in D(x)

$$\frac{x-a}{(x+a)(x^2+b^2)}=\frac{A}{x+a}+\frac{Bx+C}{x^2+b^2}$$

Type 3: Quadratic Factors in the D(x)

$$\frac{x-a}{(x+a)(x^2+b)^2}=\frac{A}{x+a}+\frac{Bx+C}{x^2+b^2}$$

Type 4: Quadratic repeated factors in D(x):

$$\frac{x-a}{(x^2+a^2)(x^2+b^2)}=\frac{Ax+B}{x^2+a^2}+\frac{Cx+D}{x^2+b^2}+\frac{Ex+F}{(x^2+b^2)^2}$$

Short Questions:

Write the short answers of the following:

Q.1: What is partial fractions?

Q.2: Define proper fraction and give example.

Q.3: Define improper fraction and given one example:

Q.4: Resolve into partial fractions $\frac{2x}{(x - 2)(x + 5)}$

Q.5: Resolve into partial fractions: $\frac{1}{x^2 - x}$

Q.6: Resolve $\frac{7x + 25}{(x + 3)(x + 4)}$ into partial fraction.

Q.7: Resolve $\frac{1}{x^2 - 1}$ into partial fraction:

Q.8: Resolve $\frac{x^2 + 1}{(x + 1)(x - 1)}$ into partial fractions.

Q.9: Write an identity equation of $\frac{8x^2}{(1 - x^2)(1 + x^2)^2}$

Q.10: Write an identity equation of $\frac{2x + 5}{x^2 + 5x + 6}$

Q.11: Write identity equation of $\frac{x - 5}{(x + 1)(x^2 + 3)}$

Q.12: Write an identity equation of $\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Q.13: Write an identity equation of $\frac{(x - 1)(x - 2)(x - 3)}{(x - 4)(x - 5)(x - 6)}$

Q.14: Write an identity equation of $\frac{x^5}{x^4 - 1}$

Q.15: Write an identity equation of $\frac{2x^4 - 3x^2 - 4x}{(x + 1)(x^2 + 2)^2}$

Q16. Form of partial fraction of $\frac{1}{(x + 1)(x - 2)}$ is ___________.

Q.17. Form of partial fraction of $\frac{1}{(x + 1)^2(x - 2)}$ is ___________.

Q.18. Form of partial fraction of $\frac{1}{(x^2 + 1)(x - 2)}$ is ___________.

Q.19. Form of partial fraction of $\frac{1}{(x^2 + 1)(x - 4)^2}$ is ___________.

Q.20. Form of partial fraction of $\dfrac{1}{(x^3 - 1)(x^2 + 1)}$ is __________.

Answers

Q4. $\dfrac{4}{7(x-2)} - \dfrac{10}{7(x+5)}$ Q5. $\dfrac{-1}{x} + \dfrac{1}{x-1}$

Q6. $\dfrac{4}{x+3} + \dfrac{3}{x+4}$ Q7. $\dfrac{1}{x^2-1} = \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

Q8. $1 + \dfrac{1}{x+1} + \dfrac{1}{x-1}$ Q9. $\dfrac{A}{1-x} + \dfrac{B}{1+x} + \dfrac{Cx+D}{1+x^2} + \dfrac{Ex+F}{(1+x^2)^2}$

Q10. $\dfrac{A}{x+2} + \dfrac{B}{x+3}$ Q11. $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+3}$

Q12. $(2x+3) + \dfrac{A}{x-1} + \dfrac{B}{3x+1}$ Q13. $1 + \dfrac{A}{4-4} + \dfrac{B}{x-5} + \dfrac{C}{x-6}$

Q14. $x + \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$ Q15. $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2} + \dfrac{Dx+E}{(x^2+2)^2}$

Q16. $\dfrac{A}{x+1} + \dfrac{B}{x-2}$ Q17.

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

Q18. $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-2}$ Q19.

$$\frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Q20. $\dfrac{A}{(x-1)} + \dfrac{Bx+C}{(x^2+x+1)} + \dfrac{Dx+E}{x^2+1}$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. If the degree of numerator N(x) is equal or greater than the degree of denominator D(x), then the fraction is:
(a) proper (b) improper
(c) Neither proper non-improper (d) Both proper and improper

_2. If the degree of numerator is less than the degree of denominator, then the fraction is:
(a) Proper (b) Improper
(c) Neither proper non-improper (d) Both proper and improper

__3. The fraction $\frac{2x+5}{x^2+5x+6}$ is known as:
(a) Proper (b) Improper
(c) Both proper and improper (d) None of these

__4. The number of partial fractions of $\frac{6x+27}{4x^3-9x}$ are:
(a) 2 (b) 3
(c) 4 (d) None of these

__5. The number of partial fractions of $\frac{x^3-3x^2+1}{(x-1)(x+1)(x^2-1)}$ are:
(a) 2 (b) 3
(c) 4 (d) 5

__6. The equivalent partial fraction of $\frac{x+11}{(x+1)(x-3)^2}$ is:
(a) $\frac{A}{x+1}+\frac{B}{(x-3)^2}$ (b) $\frac{A}{x+1}+\frac{B}{x-3}$
(c) $\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$ (d) $\frac{A}{x+1}+\frac{Bx+C}{(x-3)^2}$

__7. The equivalent partial fraction of $\frac{x^4}{(x^2+1)(x^2+3)}$ is:
(a) $\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3}$ (b) $\frac{Ax+B}{x^2+1}+\frac{Cx}{x^2+3}$
(c) $1+\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3}$ (d) $\frac{Ax}{x^2+1}+\frac{Bx}{x^2+3}$

__8. Partial fraction of $\frac{2}{x(x+1)}$ is:

(a) $\frac{2}{x} - \frac{1}{x+1}$ (b) $\frac{1}{x} - \frac{2}{x+1}$

(c) $\frac{2}{x} - \frac{2}{x+1}$ (d) $\frac{2}{x} + \frac{2}{x+1}$

__9. Partial fraction of $\frac{2x+3}{(x-2)(x+5)}$ is called:

(a) $\frac{2}{x-2} + \frac{1}{x+5}$ (b) $\frac{3}{x-2} + \frac{1}{x+5}$

(c) $\frac{2}{x-2} + \frac{3}{x+5}$ (d) $\frac{1}{x-2} + \frac{1}{x+5}$

__10. The fraction $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$ is called:

(a) Proper (ii) Improper

(c) Both proper and Improper (iv) None of these

Answers:

1.	b	2.	a	3.	a	4.	b	5.	c
6.	c	7.	c	8.	c	9.	d	10.	B

Chapter 10

Number Systems and Arithmetic Operations

10.1 The Decimal Number System:

The Decimal number system is a number system of base or radix equal to 10, which means that there are 10, called Arabic numerals, symbols used to represent number : 0, 1, 2, 3,…….,9 , which are used for counting.

To represent more than nine units, we must either develop additional symbols or use those we have in combination. When used in combination, the value of the symbol depends on its position in the position in the combination of symbols. We refer to this as positional notation and refer to the position as having a weight designated as units, tens, hundreds, thousands, and so on.

The units symbol occupies the first position to the left of the decimal point is represented as 10^0. The second position is represented as 10^1, and so forth. To determine what the actual number is in each position, take the number that appears in the position, and multiply it by 10^X, where x is the power representation.

This is expressed mathematically of the first five positions as

10^4	10^3	10^2	10^1	10^0
Ten thousands	thousands	hundreds	tens	units

For example the value of the combination of symbols 435 is determined by adding the weight of each position as

$$4 \times 10^2 + 3 \times 10^1 + 5 \times 10^o$$

Which can be written as

$$4 \times 100 + 3 \times 10 + 5 \times 1$$

Or $\quad 400 + 30 + 5 = 435$

The position to the right of the decimal point carry a positional notation and corresponding weight as well. The exponents to the right of the decimal point are negative and increase in integer steps starting with-1. This is expressed mathematically for each of the first four positions as;

weight	10^{-1}	$1\ 0^{-2}$	10^{-3}	10^{-4}
	tenths	hundredths	thousandths	ten thousandths

For example the value of the combination of symbols, 249.34 determined by adding the weight of each position as

$$2 \times 10^2 + 4 \times 10^1 + 9 \times 10^\circ + 3 \times 10^{-1} + 4 \times 10^{-2}$$

Or $$200 + 40 + 9 + \frac{3}{10} + \frac{4}{100}$$

Or $$200 + 40 + 9 + 0.3 + 0.04$$

$$= 249.34$$

10.2 The Binary Number System:

The binary number system is a number system of base or radix equal to 2, which means that there are two symbols used to represent number : 0 and 1.

A seventeenth-century German Mathematician, Gottfriend Wilhelm Von Leibniz, was a strong advocate of the binary number system. The binary number system has become extremely important in the computer age.

The symbols of the binary number system are used to represent number in the same way as in the decimal system symbol is used individually; then the symbols are use combination. Since there are only two symbols, we can represent two numbers , 0 and 1, with individual symbols. The position of the 1 or 0 in a binary number system indicates its weight or value within the number. We then combine the 1 with 0 and with itself to obtain additional numbers.

10.3 Binary and Decimal Number Correspondence:

Here are first 15 equivalence decimal and binary numbers:

Decimal Number	**Binary Numbers**
0	0000
1	0001
2	0010
3	0011

4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111

An easy way to remember that how to write a binary sequence such as in the above table for a four-bits example is as follows:

1. The right most column in the binary number begins with a 0 and alternate each bit.

2. The next column begin with two O's and alternate every two bits.

3. The next column begin with four 0's and alternate every four bits.

4. The next column begin with eight-O's and alternate every eight bits.

It is seen that it takes at least four bits from 0 to 15. The formula to Count the decimal number with n bits, beginning with zero is:

Highest decimal number = $2^n - 1$

for example, with two bits we can count the decimal number from 0 to 3 as,

$2^2 - 1 = 3$

For three bits, the decimal number is from 0 to 7, as,

$2^3 - 1 = 7$

The same type of positional weighted system is used with binary numbers as in the decimal system, The base 2 is raised to power equal to the number of positions away from the binary point The weight and designation of the several positions are as follows:

Power equal to position Base

weight	4	3	2	1	0	-1	-2	-3
positional notation	2	2	2	2	2	2	2	2
(decimal value)	16	8	4	2	1	0.5	0.25	0.125

when the symbols 0 and 1 are used to represent binary number, each symbol is called a binary digit or a bit. Thus the binary number 1010 is a four-digit binary number or a 4-bit binary number,

10.4 Binary-to-Decimal Conversion:

Since we are programmed to count in the decimal number system, it is only natural that we think in terms of the decimal equivalent value when we see a binary number. The conversion process is straight forward and is done as follows: Multiply binary digit (1 or 0) in each position by the weight of the position and add the results. The following examples explain the process.

Example 1: Convert the following binary number to their decimal equivalent. (a) 1101 (b) 1001

Solution:

(a) 1101 $= (1 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^o)$

$= 8 + 4 + 0 + 1 = 13$

(b) 1001 $= (1 \times 2^3) + (0 \times 2^2) + (0 \times 2^1)(1 \times 2^0)$

$= 8 + 0 + 0 + 1 = 9$

Example 2: Convert the following binary numbers to their decimal equivalent. (a) 0.011 (b) 0.111

Solution:

(a) 0.011 $= (0 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3})$

$= 0 + \frac{1}{4} + \frac{1}{8}$

$= 0.25 + 0.125 = 0.375$

(b) 0.111 $= (1 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3})$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$= 0.5 + 0.25 + 0.125 = 0.875$

Example 3: Convert the binary number 110.011 to its decimal equivalent.

Solution:

$$110.011 = (1 \times 2^2) + (1 \times 2^1) + (0 \times 2^\circ) + (0 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3})$$

$$= 4 + 2 + 0 + 0 + \frac{1}{4} + \frac{1}{8}$$

$$= 4 + 2 + 0.25 + 0.125 = 6.375$$

10.5 Decimal-to-Binary Conversion:

It is frequently necessary to convert decimal numbers to equivalent binary numbers. The two most frequently used methods for making the conversion are the

- Repeated division-by-2 or multiplication-by-2 method.

Which is discussed below:

10.6 Repeated Division-by-2 Or Multiplication-by-2 Method:

To convert a decimal whole number to an equivalent number in a new base, the decimal number is repeatedly divided by the new base. For the case of interest here, the new base is 2, hence the repeated division by 2. Repeated division by 2 means that the original number is divided by 2, the resulting quotient is divides by 2, and each resulting quotient thereafter is divided by 2 until the quotient is 0. The remainder resulting from each division forms binary number. The first remainder to be produced is called the least significant bit (LSB) and the last remainder is called most significant bit (MSB).

When converting decimal fraction to binary, multiply repeatedly by 2 any fractional part. The equivalent binary number is formed from the

1 or 0 in the units position. The following examples illustrate the procedure.

Example 4: Convert the decimal number 17 to binary.

Solution:

2	17	
2	8 – 1 →	L.S.B.
2	4 – 0	
2	2 – 0	
2	1 – 0	
	0 – 1 →	M.S.B.

Therefore, 17 =10001

Example 5: Convert the decimal number 0.625 to binary.

Solution:

$0.625 \times 2 = 1.250$	1 (M S B)
$0.250 \times 2 = 0.500$	0
$0.25 \times 2 = 1.00$	1 (L S B)

Therefore, $0.625 = 0.101$

Note: Any further multiplication by 2 in example 5 will equal to 0; therefore the multiplication can be terminated. However, this, is not so. Often it will be necessary to terminate the multiplication when an acceptable degree of accuracy is obtained. The binary number obtained will then be an approximation.

Example 6: Convert the number 0.6 to binary:

Solution: Carry

$0.6 \times 2 = 1.2$	1 (M S B)
$0.2 \times 2 = 0.4$	0
$0.4 \times 2 = 0.8$	0
$0.8 \times 2 = 1.6$	1
$0.6 \times 2 = 1.2$	1 (LSB)

Therefore, $0.6 = 0.10011$

10.7 Double-Dibble Technique:

To convert a binary integer to a decimal integer we make use of double-dibble technique. The verb dibble is a neologism (i.e., a made-up-word) which has found wide spread acceptance among programmer's and other computer-oriented persons. To dibble a number is to double it and then add 1. The double-dibble technique for converting a binary integr (whole-number) goes a follows:

Begin by setting the first 'results equal to 1. If the second digit of the binary number is a zero then double this 1 (= 2) and if the second binary digit is a 1, then dibble this 1 (= 3) to obtain the second result; continue to double or dibble the successive results according to whether the successive binary digits are 0 or 1; the result corresponding to the last binary digit is the decimal equivalent of the binary integer.

Example 7: Convert 110101101 to a decimal number.

Solution:

<table>
<tr><td colspan="2">Binary digits</td><td>1</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td rowspan="2">Results</td><td>double</td><td rowspan="2">1</td><td></td><td>6</td><td></td><td>26</td><td></td><td></td><td>214</td><td></td></tr>
<tr><td>double</td><td>3</td><td></td><td>13</td><td></td><td>53</td><td>107</td><td></td><td>429</td></tr>
</table>

Thus $110101101 = 429$

10.8 The Octal Number System:

The octal number system is used extensively in digital work because it is easy to convert from octal to binary, vice versa. The octal system has a base; or radix, of 8, which means that there are eight symbols which are used to form octal numbers. Therefore, the single-digit numbers of the octal, number system are

0 1, 2, 3, 4, 5, 6, 7

To count beyond 7, a 1 is carried to the next higher-order column and combined with each of the other symbols, as in the decimal system. The weight of the different positions far the octal system is the base raised to the appropriate power, as shown below

weight	3	2	1	0	-1	-2
positional notation	8	8	8	8	8	8
(decimal value)	512	64	8	1	$\frac{1}{8}$	$\frac{1}{64}$

Octal numbers look just like decimal numbers except that the symbols 8 and 9 are not used. To distinguish between octal and decimal numbers, we must subscript the numbers with their bas. for example, $20_8 = 16_{10}$.

The following table shows octal numbers 0 through 37 and their decimal equivalent.

Octal numbers and their Decimal equivalent

Octal	Decimal	Octal	Decimal	Octal	Decimal	Octal	Decimal
0	0	10	8	20	16	30	24
1	1	11	9	21	17	31	25
2	2	12	10	22	18	32	26
3	3	13	11	23	19	33	27
4	4	14	12	24	20	34	28
5	5	15	13	25	21	35	29
6	6	16	14	26	22	36	30
7	7	17	15	27	23	37	31

10.9 Octal-to-Decimal Conversion:

Octal numbers are converted to their decimal equivalent by multiplying the weight of each position by the digit in that position and adding the products. This is illustrated in the following examples.

Example 8: Convert the following octal numbers to their decimal, equivalent.

Solution: (a) 35_8 (b) 100_8 (c) 0.24_8

(a) $35_8 = (3 \times 8^1) + (5 \times 8^o)$
$= 24 + 5 = 29$

(a) $100_8 = (1 \times 8^2) + (0 \times 8^1) + (0 \times 8^o)$
$= 64 + 0 + 0 = 64_{10}$

(b) $0.24_8 = (2 \times 8^{-1}) + (4 \times 8^{-2})$
$= \frac{2}{8} + \frac{4}{64}$
$= 0.3125_{10}$

10.10 Decimal-to-Octal Conversion:

To convert decimal numbers to their octal equivalent, the following procedures are employed:

- Whole-number conversion: Repeated division-by-8.
- Fractional number conversion: Repeated multipli-cation-by-8.

10.11 Repeated Division-by-8 Method:

The repeated-division by 8 method of converting decimal to octal applies only to whole numbers. The procedure is illustrated in the following example.

Example 8: Convert the following decimal numbers to their octal equivalent:

(a) 245 (b) 175

Solution:

8	245	
8	30 – 5	(LSD)
8	3 – 6	
	0 – 6 (MSD)	

8	175
8	21 – 7
8	2 – 5
	0 – 2

Therefore, $245_{10} = 365_8$, therefore , $175_{10} = 257_8$

10.12 Repeated Multiplication-by-8 Method:

To convert decimal fractions to their Octal equivalent requires repeated Multiplication by 8, as shown in the following example.

Example 9: Convert the decimal fraction 0.432 to octal equivalent .

Solution:

		Carry
0.432×8	$= 3.456$	3(MSD)
0.456×8	$= 3.648$	3
0.648×8	5.184	5
0.184×8	$= 1.472$	1 (LSD)

The first carry is nearest the octal point, therefore,

$$0.432_{10} = 0.3351_8$$

The conversion to octal is not precise, since there is a remainder. If greater accuracy is required, we simply continue multiplying by 8 to obtained more octal digits.

10.13 Octal-to-Binary-Conversion:

The primary reason for our interest in octal numbers lies in entering and outputting computer data and because of the ease of octal-to-binary conversion. Computers recognize only binary information and may be programmed using only, 1's and O's.

Converting numbers from octal to binary can be done essentially by inspection. Since there are eight symbols used for counting in the octal system and eight combinations of three binary digits that corresponds to these single-digit octal numbers, we can assign a binary three-digit combination to each single-digit octal number as shown in the Table given below:

10.14 Octal and Binary Number Correspondence.

Octal	Binary
0	0 0 0
1	1 0 1
2	0 1 0
3	0 1 1
4	1 0 0
5	1 0 1
6	1 1 0
7	1 1 1

To convert from octal to binary, simply replace each octal digit with the corresponding thee-digit binary number, as illustrated in the following example.

Example 10: Convert the following octal numbers to their binary equivalent.
(a) 247_8 (b) 501_8

Solution:

(a)	2	4	7	Octal
	010	100	111	binary

Thus, $247_8 = 010100111_2$

(b)	5	0	1	octal
	101	000	001	binary

Thus, $501_8 = 101000001_2$

10.15 Binary-to-Octal Conversion:

In printing out octal numbers, the modern electronic digital computer performs a binary-to-octal conversion. This is a simple procedure. The binary number is divided into groups to three bits, counting to the right and to the left from the binary point and then each group Of three is interpreted as an octal digit; as shown in above table.

Example 11: Convert 11010101 . 01101 to an octal-number.

Solution:	011	010	101	.	011	10
	011	010	101		011	010
	3	2	5		3	2

Therefore $11010101.0110_1 = 325.32_8$

10.16 Binary Arithmetic:

Binary arithmetic includes the basic arithmetic operations of addition, subtraction, multiplication and division. The following sections present the rules that apply to these operations when they are performed on binary numbers.

Binary Addition:

Binary addition is performed in the same way as addition in the decimal-system and is, in fact, much easier to master. Binary addition obeys the following four basic rules:

$$\begin{array}{r} 0 \\ +\,0 \\ \hline 0 \end{array} \qquad \begin{array}{r} 0 \\ +\,1 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +\,0 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +\,1 \\ \hline 10 \end{array}$$

The results of the last rule may seem some what strange, remember that these are binary numbers. Put into words, the last rule states that

binary one + binary one = binary two = binary "one zero"

When adding more than single-digit binary number, carry into, higher order columns as is done when adding decimal numbers. For example 11 and 10 are added as follows:

$$\begin{array}{r} 11 \\ +\,10 \\ \hline 101 \end{array}$$

In the first column (L S C or 2°) '1 plus 0 equal 1. In the second column (2^1) 1 plus 1 equals 0 with a carry of 1 into the third column (2^2).

When we add 1 + 1.+ 1 (carry) produces 11, recorded as 1 with a carry to the next column.

Example 12: Add (a) 111 and 101 (b) 1010, 1001 and 1101.

Solution:

(a)
$$\begin{array}{r} {\scriptstyle (1)(1)} \\ 1\,1\,1 \\ 1\,0\,1 \\ \hline 1\,1\,0\,0 \end{array}$$

(b)
$$\begin{array}{r} {\scriptstyle (2)(1)(1)(1)} \\ 1\;0\;1\;0 \\ 1\;0\;0\;1 \\ 1\;1\;0\;1 \\ \hline 1\,0\,0\,0\,0 \end{array}$$

Binary Subtraction:

Binary subtraction is just as simple as addition subtraction of one bit from another obey the following four basic rules

0 – 0 = 0

1 – 1 =0

1 – 0 = 1

10 – 1 = 1 with a transfer (borrow) of 1.

When doing subtracting, it is sometimes necessary to borrow from the next higher-order column. The only it will be necessary to borrow is when we try to subtract a 1 from a 0. In this case a 1 is borrowed from the next higher-order column, which leaves a 0 in that column and creates a

10 i.e., 2 in the column being subtracted. The following examples illustrate binary subtraction.

Example 13: Perform the following subtractions.

(a) 11 - 01 , (b) 11-10 (c) 100 - 011

Solution:

(a) $\begin{array}{r} 11 \\ -\,01 \\ \hline 10 \end{array}$ (b) $\begin{array}{r} 11 \\ -\,10 \\ \hline 01 \end{array}$ (c) $\begin{array}{r} 100 \\ -\,011 \\ \hline 001 \end{array}$

Part (c) involves to borrows, which handled as follows. Since a 1 is to be subtracted from a 0 in the first column, a borrow is required from the next higher-order column. However, it also contains a 0; therefore, the second column must borrow the 1 in the third column. This leaves a 0 in the third column and place a 10 in the second column. Borrowing a 1 from 10 leaves a 1 in the second column and places a 10 i.e, 2 in the first column:

When subtracting a larger number from a smaller number, the results will be negative. To perform this subtraction, one must subtract the smaller number from the larger and prefix the results with the sign of the larger number.

Example 14: Perform the following subtraction 101 – 111.

Solution:

Subtract the smaller number from the larger.

$$\begin{array}{r} 111 \\ -\,101 \\ \hline 010 \end{array}$$

Thus 1 0 1 – 1 1 1 = - 010 = - 10

Binary multiplication:

Binary multiplication is performed in the same manner as decimal multiplication. It is much easier, since there are only two possible results of multiplying two bits. The Binary multiplication obeys the four basic rules.

$0 \times 0 = 0$

$0 \times 1 = 0$

$1 \times 0 = 0$

$1 \times 1 = 1$

Example 15: Multiply the following binary numbers.

(a) 101 × 1 1 (b) 1101 × 10

(c) 1010 × l 0 1 (d) 1 0 1 l × 1 0 1 0

Solution:

(a)
$$\begin{array}{r} 101 \\ \underline{1\,1} \times \\ 101 \\ \underline{101} \\ 1111 \end{array}$$

(b)
$$\begin{array}{r} 11101 \\ \underline{1\,0} \times \\ 0000 \\ \underline{1101} \\ 11010 \end{array}$$

(c)
$$\begin{array}{r} 1010 \\ \underline{101} \times \\ 1010 \\ 0000 \\ \underline{1010} \\ \underline{110010} \end{array}$$

(d)
$$\begin{array}{r} 1011 \\ \underline{1010} \times \\ 0000 \\ 1011\times \\ 0000\times\times \\ \underline{1011\times\times\times} \\ 1101110 \end{array}$$

Multiplication of fractional number is performed in the same way as with fractional numbers in the decimal numbers.

Example 16: Perform the binary multiplication 0.01 × 11.
Solution:

$$\begin{array}{r} 0.01 \\ \underline{1\,1} \times \\ 01 \\ \underline{01\times} \\ 0.11 \end{array}$$

Binary Division:

Division in the binary number system employees the same procedure as division in the decimal system, as will be seen in the following examples.

Example 17: Perform the following binary division.

(a) 110 ÷ 11 (b) 1100 ÷ 11

Solution:

(a)	10 11 | 110 11 0 0 0 0 0 0	(b)	100 11 | 11000 11 0 0 0 0 0 0 0 0 0 0

Binary division problems with remainders are also treated the same as in the decimal system, as illustrates the following example.

Example 18: Perform the following binary division:

(a) 1111 ÷ 110 (b) 1100 ÷ 101

Solution:

(a)	10. 1 110 | 1 11 1.00 1 10 1 10 1 10 0 0 0	(b)	10. 011 110 | 1 100.00 1 01 100 000 1000 101 110 101 1 (reminder)

EXERCISE 10

Q.1: Convert the following binary numbers to decimal equivalent.

(a) 100 (b) 11010 (c) 10110010 (d) 1.001
(e) 110100.010011 (f) 11010.10110 (g) 1000001.111

Q.2: Convert the following decimal numbers to binary equivalent.

(a) 16 (b) 247 (c) 962.84 (d) 0.0132
(e) 6.74

Q.3: Convert the following octal numbers to their decimal equivalent:

(a) 14_8 (b) 236_8 (c) 1432_8 (d) 0.43_8

(e) 0.254_8 (f) $(16742.3)_8$ (g) $(206.104)_8$

Q.4: Convert the following decimals numbers to their octal equivalent:

(a) 29 (b) 68 (c) 125 (d) 243.67
(e) 419.95 (f) 645.7 (g) 39.4475 (h) 49.21875

Q.5: Convert the following octal numbers to their binary equivalent:

(a) 13_8 (b) 27_8 (c) 65_8 (d) 124.375_8
(e) 217.436_8

Q.6: Convert the following binary numbers to their octal equivalent:

(a) 10110010 (b) 10101101

(c) 1110101 (d) 10111.101

(e) 111010.001

Q.7: Convert 18 × 24 to binary form and then perform binary multiplication.

Q.8: Convert 58.75 ÷ 23.5 to binary form and then perform operation.

Q.9: Add the following binary numbers:

(a)
```
   1 1
 + 1 1
 …. …..
```

(b)
```
   1011
 + 1101
 ……….
```

(c)
```
   101011
 + 110101
 ………...
```

(d)
```
   1110101
 + 1011111
 …………
```

Q.10: Add the following binary numbers:

(a)
```
  1011
+ 1101
  1011
  …….
```

(b)
```
  1010110
  1110101
+ 1001010
…..……….
```

(c)
```
  1010110
  1111011
+ 1011111
 ……….
```

(d)
```
  10110110
  11010101
+ 11010110
 ………...
```

Q.11: Subtract the following binary numbers:

(a)
```
  1 1
– 1 0
…. …..
```

(b)
```
  111
– 101
……….
```

(c)
```
  101011
– 100101
 …...........
```

(d)
```
  11100001
– 10011110
 …………
```

Q.12: Multiply the following binary numbers:

(a) 11×11 (b) 101×10 (c) 110×101

(d) 1011×1101 (e) 1010×101

Q.13: Divide the following binary numbers:

(a) $110 \div 10$ (b) $10110 \div 10$

(d) $101101 \div 1.1$ (e) $11001.11 \div 1101$

Answers 10

Q.1: (a) 4 (b) 26 (c) 178 (d) 1.125

(e) 52.2969 (f) 26.6875 (g) 65.875

Q.2: (a) 10000_2 (b) 11110111_2 (c) 1111000010.11010111_2

(d) 0.000000110110_2 (e) 110.1011110_2

Q.3: (a) 12 (b) 158 (c) 794 (d) 0.5469

	(e)	0.3359	(f)	7650.375	(g)	134.1328		
Q.4:	(a)	35_8	(b)	104_8	(c)	175_8	(d)	363.527_8
	(e)	643.2631_8	(f)	$(1205.043)_8$	(g)	47.345_8		
	(h)	61.61_8						
Q.5:	(a)	001011	(b)	010111	(c)	110101		
	(d)	001010100.011111101	(e)	010001111.100011110				
Q.6:	(a)	262_8	(b)	255_8	(c)	165_8	(d)	27.5_8	(e)	72.1_8
Q.7:	110110000_2							
Q.8:	10.1_2							
Q.9:	(a)	110	(b)	11000	(c)	1100000	(d)	11010100
Q.10:	(a)	100011	(b)	100010101	(c)	100110000		
	(d)	1001100001						
Q.11:	(a)	01	(b)	10	(c)	110	(d)	1000011
Q.12:	(a)	1001	(b)	1100	(c)	11110	(d)	110010
	(e)	10001111						
Q.13:	(a)	11	(b)	1011	(c)	01.111111	(d)	11110

Short Questions

Write the short answers of the following:

Q.1: Define Binary Number.

Q.2: Define Octal numbers.

Q.3: Define Decimal number.

Q.4: Convert binary number 10101_2 to decimal numbers.

Q.5: Convert binary numbers 11111_2 to decimal numbers.

Q.6: Convert 110011.11_2 to decimal numbers.

Q.7: Add the binary number $110_2 + 1011_2$

Q.8: Add binary numbers $10011.1_2 + 11011.01_2$

Q.9: Multiply the binary numbers $111_2 \times 101_2$

Q.10: Subtract the binary numbers $1100_2 - 1001_2$

Q.11: Divide the binary numbers $1000_2 \div 100$

Q.12: Convert binary 101101_2 to octal nos.

Q.13: Convert binary numbers 10110111_2 to octal numbers.

Q.14: Convert binary numbers 11 0 11 0 . 0 11 to octal numbers.

Q.15: Convert octal numbers $103\,.\,45_8$ to binary number.

Q.16: Convert octal number 107_8 binary nos.

Q.17: Find the octal equivalent to $(359.325)_{10.}$

Q.18: Convert the decimal numbers 932_{10} to octal numbers.

Note: LB means least bit and GB means greatest bit

Answers

Q4	21	**Q5.**	31	**Q6.**	51.75
Q7.	11000_2	**Q8.**	101110.11_2	**Q9.**	100011_2
Q10	0011_2	**Q11**.	11_2	**Q12.**	55_8
Q13.	267_8	**Q14.**	66.3_8	**Q15.**	$001000\ 0\ 11.100\ \ 101_2$
Q16.	$001\ \ 000\ \ 111_2$	**Q17.**	$(547\ .\ 246)_8$	**Q18.**	1644_8

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. $(11)_2$ to decimal is:

(a) 3 (b) 5 (c) 4 (d) None of these

__2. $(11)_2$ to decimal is:

(a) $\frac{3}{2}$ (b) 2.2 (c) 3 (d) None of these

__3. 25 when converted to octal is:

(a) $(31)_8$ (b) $(2.5)_8$ (c) $(13)_8$ (d) None of these

__4. $(11)_2 + (101)_2$ is equal to:

(a) $(1100)_2$ (b) $(121)_2$ (c) $(112)_2$ (d) None of these

__5. Addition of $(45)_8$ and $(73)_8$ is:

(a) $(140)_8$ (b) $(118)_8$ (c) $(110)_8$

__6. $(11)_8 \times (7)_8$ is equal to:

(a) $(105)_8$ (b) $(77)_8$ (c) $(43)_8$ (d) None of these

__7. Number of digits in a binary system are:

(a) 2 (b) 7 (c) 10 (d) None of these

__8. $(11)_2 \times (11)_2$ is equal to:

(a) $(1101)_2$ (b) $(1001)_2$ (c) $(121)_2$ (d) None of these

__9. $(11)_8$ to decimal is equal to:

(a) 9 (b) 88 (c) 110 (d) None of these

__10. Conversion of 9 to binary system is:

(a) $(1001)_2$ (b) $(101)_2$ (c) $(11)_2$ (d) None of these

ANSWERS:

1.	a	2.	a	3.	d	4.	a	5.	a
6.	b	7.	a	8.	a	9.	a	10.	a

Chapter - 11
Boolean Algebra

11.1 Introduction:

George Boole, a nineteenth-century English Mathematician, developed a system of logical algebra by which reasoning can be expressed mathematically. In 1854, Boole published a classic book, “An Investigation of the Laws of thought” on which he founded the Mathematical theories of Logic and Probabilities,

Boole’s system of logical algebra, now called Boolean algebra, was investigated as a tool for analyzing and designing relay switching circuits by Claude E. Shannon at the Massachusetts institute of Technology in 1938. Shannon, a research assistant in the Electrical Engineering Department, wrote a thesis entitled “A” symbolic Analysis of Relay and Switching Circuits. As a result of his work, Boolean algebra is now, used extensively in the analysis and design of logical circuits. Today Boolean algebra is the backbone of computer circuit analysis.

11.2 Two Valued Logical Symbol:

Aristotle made use of a two valued logical system in devising a method for getting to the truth, given a set of true assumptions. The symbols that are used to represent the two levels of a two valued logical system are 1 and 0. The symbol 1 may represent a closed switch, a true statement, an “on” lamp, a correct action, a high voltage, or many other things. The symbol “O” may represent on open switch, a false statement, an “off” lamp, an incorrect action, a low voltage, or many other things.

For the electronics circuits and signals a logic 1 will represent closed switch, a high voltage, or an “on” lamp, and a logic 0 will represent an open switch, low voltage, or an “off” lamp. These describe the only two states that exist in digital logic systems and will be used to represent the in and out conditions of logic gates.

11.3 Fundamental Concepts of Boolean Algebra:

Boolean algebra is a logical algebra in which symbols are used to represent logic levels. Any symbol can be used, however, letters of the alphabet are generally used. Since the logic levels are generally associated with the symbols 1 and 0, whatever letters are used as variables that can take the values of 1 or 0.

Boolean algebra has only two mathematical operations, addition and multiplication. These operations are associated with the OR gate and the AND gate, respectively.

11.4 Logical Addition:

When the + *(*the logical addition) symbol is placed between two variables, say X and Y, since both X and Y can take only the role 0 and 1, we can define the + Symbol by listing, all possible combinations for X and Y and the resulting value of X + Y.

The possible input and out put combinations may arranged as follows:

$$0 + 0 = 0$$
$$0 + 1 = 1$$
$$1 + 0 = 1$$
$$1 + 1 = 1$$

This table represents a standard binary addition, except for the last entry. When both' X and Y represents 1's, the value of X + Y is 1. The symbol + therefore does not has the "Normal" meaning, but is a Logical addition symbol. The plus symbol (+) read as "OR", therefore X +Y is read as X or Y.

This concept may be extended to any number of variables for example A + B + C +D = E Even if A, B, C and D all had the values 1, the sum of the values i.e. is 1.

11.5 Logical Multiplication:

We can define the "." (logical multiplication) symbol or AND operator by listing all possible combinations for (input) variables X and Y and the resulting (output) value of X. Y as,

$$0 . 0 = 0$$
$$0 . 1 = 0$$
$$1 . 0 = 0$$
$$1 . 1 = 1$$

Note : Three of the basic laws of Boolean algebra are the same as in ordinary algebra; the commutative law, the associative law and the distributive law.

The commutative law for addition and multiplication of two variables is written as,

$$A + B = B + A$$

And $$A . B = B . A$$

The associative law for addition and multiplication of three variables is written as,

$$(A + B) + C = A + (B + C)$$

And $$(A .B) . C = A. (B. C)$$

The distributive law for three variables involves' both addition and multiplication and is written as,

$$A (B+ C) = A B + AC$$

Note that while either '+' and '.' s can be used freely. The two cannot be mixed without ambiguity in the absence of further rules.

For example does A . B + C means (A . B) + C or A . (B+ C)? These two form different values for A = O, B = 1 and C = 1, because we have

$$(A . B) + C = (0.1) + 1 = 1$$

and $$A . (B + C) = 0 . (1 + 1) = 0$$

which are different. The rule which is used is that '.' is always performed before '+'. Thus X . Y + Z is (X.Y) + Z.

11.6 Logic Gates:

A logic gate is defined as a electronics circuit with two or more input signals and one output signal. The most basic logic Circuits are OR gates, AND gates, and invertors or NOT gates. Strictly speaking, invertors are not logic gates since they have only one input signal; however They are best introduced at the same time as basic gates and will therefore be dealt in this section.

OR Gate:

An OR gate is a logic circuit with two or more input signals and one output signal. The output signal will be high (logic 1) if any one input signal is high (logic 1). OR gate performs logical addition

The symbol for the logic OR gate is

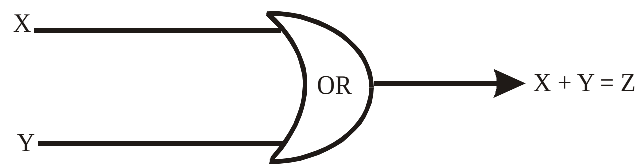
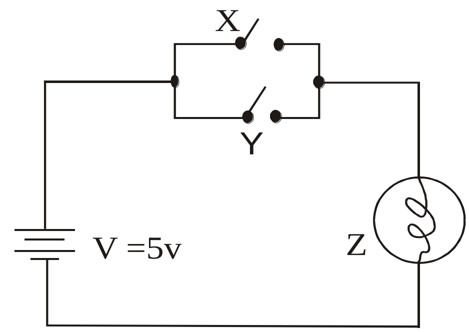


Fig. 1

A circuit that will functions as an OR gate can be implemented in several ways. A mechanical OR gate can be fabricated by connecting two switches in parallel as shown in figure 2.

Fig. 2



Truth Table for a switch circuit operation as an OR gate.

Table – 1

Switch X	Switch Y	Output Z
Open	Open	0
Open	Closed	5V
Closed	Open	5V
Closed	Closed	5V

Note that for the switch circuit were use diodes and resistors, Transistors and resistors and other techniques to control the voltage and resistance.

Note: If the switch is "on", it is represented by 1, and if, it is "off", it is represented by 0.

Truth Table for a Two-input **OR** gate.

Table - 2

In Put		**Out Put**
X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

Truth table for a three in put **OR** gate.

Table – 3

A	**B**	**C**	**X**
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

No. of combinations = 2^n, where n is number of variables.

AND Gate:

An **AND** gate is a logic circuit with two or more input signals and one output signal. The output signal of an **AND** gate is high (logic 1) only if all inputs signals are high (Logic 1).

An **AND** gate performs logical multiplication on inputs. The symbol for **AND** gate is

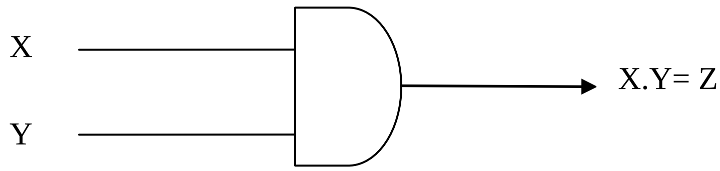


Fig.3

A circuit that will functions as an **AND** gate can be implemented in several ways. A mechanical **AND** gate can be fabricated by connecting two switches in series as show in fig. 4

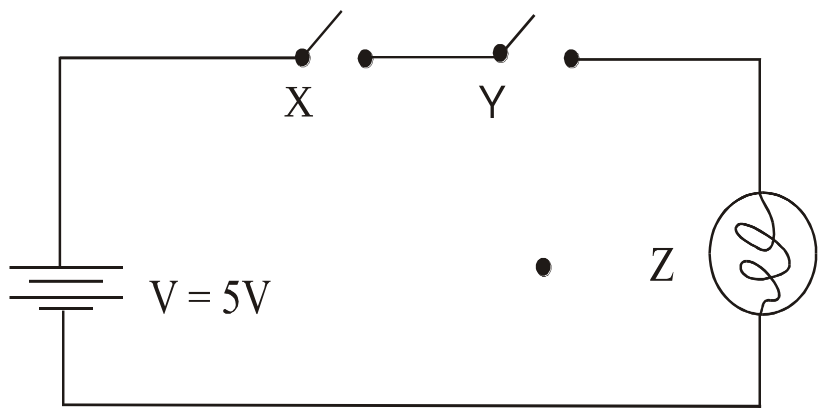


Fig.4

Truth Table for a switch circuit operation as an AND gate.

Table – 4

Switch X	**Switch Y**	**Output Z**
Open	Open	0
Open	Closed	0
Closed	Open	0
Closed	Closed	5V

Truth Table for a Two-input **AND** gate

Table - 5

In Put		Out Put
X	**Y**	**Z**
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table for a three input **AND** gate.

Table 6

Inputs			Output
A	**B**	**C**	**X**
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Complementation:

The logical operation of complementary or inverting a variable is performed in the Boolean Algebra. The purpose of complementation is to invert the, input signal, since there are only two values that variables can assume in two-value logic system, therefore if the input is 1, the output is 0 and if the input is 0 the output is 1. The symbol used to represent complementation of a variable is a bar (-) above the variable, for example the complementation of A is written as $\overline{A}$ and is read as "complement of A" or "A not".

Since variables can only be equal to 0 or 1, we can say that

$\overline{O} = 1$ Or $\overline{1} = O$

Also $\overline{\overline{O}} = O$ Or $\overline{\overline{1}} = 1$

Invertors Or NOT gate:

An inventor is a gate with only one

input signal and one output signal; the output signal is always the opposite or complement of the input signal.

An invertor is also called a **NOT** gate because the output not the same as the input.

Symbol of inverter or **NOT** gate is

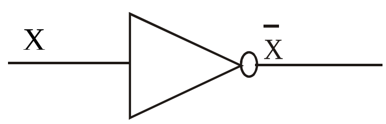


Fig.5 (i)

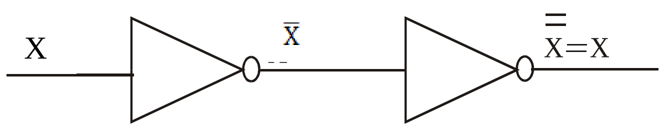


Fig.5 (ii)
Fig.5(ii) (Two invertors in series)

Fig. 5

The circle at the output or input indicates inversion. It also distinguish between the symbol for the **NOT** gate or the symbol for an operational amplifier or certain types of buffers, because the symbol -►- can also be used for diode.

Truth Table for a **NOT** circuit

Table – 7

In put	Out put
0	1
1	0

NOTE :A word is a group (or string) of binary bits that represents a closed instruction or data,

Example 1: How many input words in the Truth Table of an 6 - input OR gate? Which' input word produce a high output?

Solution:

The total number of input word's = $2^n = 2^6 = 32$, where n is number of inputs. In an OR gate 1 or more-high inputs produce a high output. Therefore the word of 000000 results in low outputs all other input words produce a high output.

11.7 Basic Duality in Boolean Algebra:

We state the duality theorem without proof. Starting with a Boolean relation, *we* can derive another Boolean relation by

1. Changing each **OR (+)** sign to an **AND (.)** sign
2. Changing each **AND (.)** sign to an **OR (+)** sign.
3. Complementary each 0 and 1

For instance

$$A + 0 = A$$

The dual relation is $A \,.\, 1 = A$

Also since $A (B + C) = AB + AC$ by distributive law. Its dual relation is $A + B C = (A + B) (A + C)$

11.8 Fundamental Laws and Theorems of Boolean Algebra:

$$\left.\begin{array}{ll} 1. & X + 0 = X \\ 2. & X + 1 = 1 \\ 3. & X + X = X \\ 4. & X + \overline{X} = 1 \end{array}\right\} \text{OR operations}$$

5.	$X . 0 = 0$	AND operations
6.	$X . 1 = X$	
7.	$X . X = X$	
8.	$X . \overline{X} = 0$	
9.	$\overline{\overline{X}} = X$	Double complement
10.	$X + Y = Y + X$	Commutative laws
11.	$XY = YX$	
12.	$(X + Y) + Z = X + (Y + Z)$	Associative laws
13.	$(X . Y). Z = X. (Y. Z)$	
14.	$X (Y + Z) = XY + XZ$	Distribution Law
15.	$X + Y .Z = (X + Y) . (X + Z)$	Dual of Distributive Law
16.	$X + XZ = X$	Laws of absorption
17.	$X (X + Z) = X$	
18.	$X + \overline{X} Y = X + Y$	Identity Theorems
19.	$X (\overline{X} + Y) = X.Y$	
20.	$\overline{X+Y} = \overline{X} . \overline{Y}$	De Morgan's Theorems
21.	$\overline{X.Y} = \overline{X} + \overline{Y}$	

Proof of Boolean Algebra Rules:

Every rule can be proved by the application of rules and by perfect Induction.

Rule 15:

(i) This rule does not apply to normal algebra We follow:

$(X + Y) (X + Z) = XX + XZ + YX + YZ$

$= X + XZ + YX + YZ, \qquad X.X = X$

$= X(1 + Z) + YX + YZ$

$= X + YX + YZ,$ $\quad 1 + Z = 1$

$= X(1 + Y) + Y$ Ƶ

$= X + YZ$ $\quad 1 + Y = 1$

(ii) Proof by Perfect induction Method:

Truth Table-8 for the R.H.S. $(X + Y)(X + Z)$ and for L.H.S. $X + YZ$

X	Y	Z	X+Y	X+Z	YZ	(X+Y)(X+Z)	X+YZ
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	0	1	1
1	0	0	1	1	0	1	1
1	0	1	1	1	0	1	1
1	1	0	1	1	1	1	1
1	1	1	1	1	1	1	1

R.H.S. = L.H.S.

Rule .16 $X + XZ = X$

L.H.S. $= X + XZ = X(1 + Z) = X.\,1 = X,$ $\quad I + Z = 1$

$= $ R.H.S.

Rule 17: $X(X + Z) = X$

L.H.S. $= X(X + Z)$

$= X\,X + XZ$ By distributive law

$= X + XZ,$ as $X.X = X$

$= X(1 + Z),$ As $1 + Z = 1$

$= X\,.\,1$

$= X$

L.H.S. = R.H.S.

Rule 18: **(i)** $X + \overline{X}\,Y = X + Y$

L.H.S. $= X + \overline{X}\,Y = (X + \overline{X}) . (X + Y)$ By rule 15 dual Of distributive law.

$= 1 . (X + Y)$ as $X + \overline{X} = 1$
$= X + Y$
L.H.S. = R.H.S.

(ii) Proof by perfect Induction Method:

Truth Table 9 for L.H.S. $X + \overline{X}\,Y$ and for R.H.S. $X + Y$

X	Y	$\overline{X}$	$\overline{X}\,Y$	$X + \overline{X}\,Y$	$X + Y$
0	0	1	0	0	0
0	1	1	1	1	1
1	0	0	0	1	1
1	1	0	0	1	1

L.H.S. =R.H.S.

Rule 19:

(i) $X.(\overline{X} + Y) = X . Y$

L.H.S. $= X(\overline{X} + Y) = X\overline{X} + X\,Y$ By distributive law

$= 0 + XY$ as $X . \overline{X} = 0$
$= X\,Y$

L.H.S. = R.H.S.

(ii) Proof by Perfect Induction Method:

Truth Table 10 for L.H.S. $X.(\overline{X} + Y)$ and for R.H.S. $X.Y$.

X	Y	$\overline{X}$	$\overline{X} + Y$	$X(\overline{X} + Y)$	$X . Y$
0	0	1	1	0	0
0	1	1	1	0	0
1	0	0	0	0	0
1	1	0	1	1	1

L.H.S. = R.H.S.

11.9 De Morgan's Theorems:

(i) $\overline{X + Y} = \overline{X} \cdot \overline{Y}$

(ii) $\overline{X \cdot Y} = \overline{X} + \overline{Y}$

Proof: (i) By Perfect induction

(i)Truth Table 11 for L.H.S. $\overline{X + Y}$ and for R.H.S. $\overline{X} \cdot \overline{Y}$

X	Y	$\overline{X}$	$\overline{Y}$	X +Y	$\overline{X + Y}$	$\overline{X} \cdot \overline{Y}$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

L.H.S. =R.H.S.

(ii)Truth Table 12 **for L.H.S.** $\overline{X \cdot Y}$ and for R.H.S. $\overline{X} + \overline{Y}$

X	Y	$\overline{X}$	$\overline{Y}$	X .Y	$\overline{X \cdot Y}$	$\overline{X} + \overline{Y}$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

L.H.S. = R.H.S.

Rules:

3rd and 7th called idempotent. These shows that Boolean algebra is idempotent.

i.e. A + A = A and A. A = A

Proof:

The variable A can have only the value 0 or 1.

(3) If A =0, then 0 + 0 = 0

If A = 1, then 1 + 1 = 1

(7) If A = 0, then 0.0. = 0

If A =1, then 1 . 1 =1

Rule 2: X +1 =1

If X = 0 then 0 + 1 = 1

If X = 1, then 1 + 1 =1

Rule 5: $X \,.\, 0 = 0$

If $X = 0,$ Then $0.0 = 1$

If $X= 1,$ Then $1\,.0 = 1$

Rule 9: $\overline{\overline{X}} = X,$ i.e., the Boolean algebra is involuted.

If $X = 0,$ Then $\overline{O} = 1$ and $\overline{1} = 0$

So $\overline{\overline{O}} = \overline{1} = 0$

If $X=1,$ Then $\overline{1} = 0$ and $\overline{O} = 1$

So $\overline{\overline{1}} = \overline{O} = 1$

Similarly we can prove the remaining rules by setting the values of variables as 0 and 1 or by perfect induction

Example:2: Express the Boolean function

$$XY + YZ + \overline{Y}\,Z = XY + Z$$

Solution:

$$
\begin{aligned}
\text{L.H.S.} &= XY + YZ + \overline{Y}\,Z \\
&= XY+Z(Y + \overline{Y}\,) \\
&= XY + Z.1 \\
&= XY + Z \\
\text{L.H.S} &= \text{R.H.S.}
\end{aligned}
$$

Example 3: Find the complement of the expression: X + YZ and verified the result by perfect induction.

Solution:

$$\overline{X+ YZ} = \overline{X}\,.\,\overline{YZ}$$

$$= \overline{X}\,.(\overline{Y} + \overline{Z}\,) \text{ by DeMorgan's Law}$$

This relation can be verified by perfect induction.

Truth Table 13 for L.H.S. $\overline{X+YZ}$ and for R.H.S. $\overline{X} . (\overline{Y} + \overline{Z})$

X	Y	Z	$\overline{X}$	$\overline{Y}$	$\overline{Z}$	YZ	X+YZ	$\overline{Y}+\overline{Z}$	$\overline{X+YZ}$	$\overline{X}(\overline{Y}+\overline{Z})$
0	0	0	1	1	1	0	0	1	1	1
0	0	1	1	1	0	0	0	1	1	1
0	1	0	1	0	1	0	0	1	1	1
0	1	1	1	0	0	1	1	0	0	0
1	0	0	0	1	1	0	1	1	0	0
1	0	1	0	1	0	0	1	1	0	0
1	1	0	0	0	1	0	1	1	0	0
1	1	1	0	0	0	1	1	0	0	0

L.H.S. = R.H.S.

Example 4: Find the complement of $\overline{A}B + C\overline{D}$, (b) AB +CD = 0

Solution:

(a) $\overline{\overline{A}B + C\overline{D}} = (\overline{\overline{A}B}) . (\overline{C\overline{D}})$

$= (\overline{\overline{A}} + \overline{B}) . (\overline{C} + \overline{\overline{D}})$

$(A + \overline{B}) . (\overline{C} + D)$

(b) $AB + CD = 0$

Taking complement on both sides.

$= \overline{AB + CD} = \overline{O}$

$= \overline{AB} . \overline{CD} = 1$

$(\overline{A} + \overline{B}) . (\overline{C} + \overline{D}) = 1$

Example 5: Simplify the Boolean expressions:

(i) $(X + Y)(X + \overline{Y})(\overline{X} + Z)$

(ii) $XYZ + X\overline{Y}Z + XY\overline{Z}$

Solution:

(i) First simplify $(X + Y)(X + \overline{Y})$

$(X + Y)(X + \overline{Y}) = XX + X\overline{Y} + YX + Y\overline{Y}$

$= X + X\overline{Y} + YX + O$, as $XX = X$

as $Y\overline{Y} = 0$

$= X + X(\overline{Y} + Y)$, as $\overline{Y} + Y = 1$
$= X + X . 1$, as $X .1 = X$
$= X + X$
$= X$

Now $(X + Y)(X + \overline{Y})(\overline{X} + Z)$

$= X(\overline{X} + Z)$

$= X\overline{X} + XZ$, by distributive law

$= 0 + XZ$
$= XZ$

(ii) $XYZ + X\overline{Y}Z + XY\overline{Z}$

$= XZ(Y + \overline{Y}) + XY\overline{Z}$

$= XZ + XY\overline{Z}$, as $Y + \overline{Y} = 1$

$= X(Z + Y\overline{Z})$

$= X[(Z + Y).(Z + \overline{Z})]$, (By Rule 15 dual of distributive

$= X[(Z + Y). 1] = X(Z + Y)$

$= X(Y + Z)$, by commutative law.

Example 6: **Minimize the following expression by use of Boolean rules.**

(a) $X = A\,B\,C + \overline{A}\,B + A\,B\,\overline{C}$

(b) $X = \overline{A}\,B\,\overline{C} + A\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C}$

(c) $AB + \overline{A}\,C + B\,C = AB + \overline{A}\,C$

(d) $(A + B)\,(\overline{A} + C)\,(B + C) = (A + B)\,(\overline{A} + C)$

Solution:

(a) $X = ABC + \overline{A}\,B + AB\overline{C}$

$= ABC + AB\,\overline{C} + \overline{A}\,B$

$= AB\,(C + \overline{C}) + \overline{A}\,B$

$= AB + \overline{A}\,B$ as $C + \overline{C} = 1$

$= (A + \overline{A})\,B.$

$= 1.\,B$

$= B$

(b) $X = \overline{A}\,B\overline{C} + A\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C}$

$= \overline{A}\,B\overline{C} + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C}$ as $\overline{A} + \overline{A} = \overline{A}$

$= \overline{A}\,B\overline{C} + (A + \overline{A})\,\overline{B}\,\overline{C}$

$= \overline{A}\,B\overline{C} + 1\,.\,B\overline{C}$

$= (\overline{A}\,B + \overline{B})\,\overline{C}$

$= [(\overline{A} + \overline{B})\,.\,(\overline{B} + \overline{B})]\,\overline{C}$ by the dual of distribution, rules 15

$= (\overline{A} + \overline{B})\,.\,1]\,\overline{C}$

$= (\overline{A} + \overline{B})\,\overline{C}$

(c) L.H.S. $= AB + \overline{A}C + BC$

$= AB + \overline{A}C + BC$

$= AB + \overline{A}C + 1.BC$ as $1 = A + \overline{A}$

$= AB + \overline{A}C + (A + \overline{A})BC$

$= AB + \overline{A}C + ABC + \overline{A}BC$, by distributive law

$= AB + ABC + \overline{A}C + \overline{A}BC$, by commutative law

$= AB(1 + C) + \overline{A}C(1 + B)$, AS $1 + X = 1$

$= AB + \overline{A}C$

L.H.S. = R.H.S.

(d) L.H.S. $= (A + B)(\overline{A} + C)(B + C)$

$= (A\overline{A} + AC + B\overline{A} + BC)(B + C)$

$= (0 + AC + B\overline{A} + BC)(B + C)$

$= (AC + B\overline{A} + BC)(B + C)$

$= [AC + B(\overline{A} + C)](B + C)$

$= ABC + ACC + BB(\overline{A} + C) + BC(\overline{A} + C)$

$= ABC + AC + B(\overline{A} + C) + BC(\overline{A} + C)$

$= AC(B + 1) + B(\overline{A} + C)(1 + C)$

$= AC + B(\overline{A} + C)$

$= A\overline{A} + AC + B(\overline{A} + C)$ as $A\overline{A} = 0$

$= A(\overline{A} + C) + B(\overline{A} + C)$ or by rule 19.

$= (A + B)(\overline{A} + C)$
L.H.S. $=$ R.H.S.

11.10 Sum of Product (Minterm):

The S**um of Product** means that the products of the variables that are seperated by a plus sign. The variables can be complemented or uncomplemented, **for example,**

$$AB + A\overline{B} + \overline{A}B + \overline{A}\,\overline{B} + AB\overline{C} + A\overline{B}C + A\overline{B}C$$

11.11 Product of sum (Maxterm):

The P**roduct of Sum** means that the sum of variables that are seperated by a multiplication sign. For example,

$$(A + B)(\overline{A} + B)(A + \overline{B})(\overline{A} + \overline{B}),$$

$$(A + \overline{B} + C)(A + \overline{B} + \overline{C})(\overline{A} + \overline{B} + C)$$

11.12 Fundamental Products:

The products that produce a high (1) output are called Fundamental products. For example, for the two input variables A and B.

We have four possible combination's, which are shown in the table below and the fundamental product's corresponding to each:

Truth Table 14 Two Variables

Table 14

A	B	Fundamental Product	Output Z
0	0	$\overline{A}\,\overline{B}$	1
0	1	$\overline{A}B$	1
1	0	$A\overline{B}$	1
1	1	AB	1

For three input variables or signals a similar idea is applied. Whenever the input variable is 0, the same variable is complemented in the fundamental product.

Truth Table 15. Three variables

A	B	C	Output Z	Fundamental Product	Output for product	Sum terms	Output for Sum
0	0	0	0	$\overline{A}\,\overline{B}\,\overline{C}$	1	$A+B+C$	0
0	0	1	0	$\overline{A}\,\overline{B}\,C$	1	$A+B+\overline{C}$	0
0	1	0	1	$\overline{A}\,B\overline{C}$	1	$A+\overline{B}+C$	0
0	1	1	1	$\overline{A}\,BC$	1	$A+\overline{B}+\overline{C}$	0
1	0	0	0	$A\overline{B}\,\overline{C}$	1	$\overline{A}+B+C$	0
1	0	1	0	$A\overline{B}\,C$	1	$\overline{A}+B+\overline{C}$	0
1	1	0	1	$AB\overline{C}$	1	$\overline{A}+\overline{B}+C$	0
1	1	1	0	ABC	1	$\overline{A}+\overline{B}+\overline{C}$	0

Sum of product(SOP) $= \overline{A}\,B\,\overline{C} + \overline{A}\,B\,C + A\,B\overline{C}$

Product of sum(POS) $= (A + B + C)\,(A + B + \overline{C})\,(\overline{A} + B + C)$

$$(\overline{A} + B + \overline{C})\,(\overline{A} + \overline{B} + \overline{C})$$

Note: See remarks for sum of product and product of sums.

Remarks:

(1) A sum of product (minterm) is obtained as follows: For each row of the truth table for which the out put is 1, the Boolean term is the product of variables that are equal to 1 and the complement of variable that are equal to 0. The sum of these products is the desired Boolean equation.

(2) A product of sum expression is obtained as follows: each row of the truth table for which the output is 0, the Boolean term is the sum of the variables that are equal 0 plus the complement of the variables that are equal to 1. The

product of these sum is the desired Boolean equation.

Example 7: Find the sum-of-products and product of sums equations from the given truth Table - 16.

Table 16

A	B	C	Output Functional Values
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

Sum-of-Product Equation

$$X = \overline{A}\,\overline{B}\,C + \overline{A}\,B\,\overline{C} + A\,\overline{B}\,\overline{C} + AB\overline{C}$$

Product-of-Sums Equation:

$$Y = (A + B + C)(A + \overline{B} + \overline{C})(\overline{A} + B + \overline{C})(\overline{A} + \overline{B} + \overline{C})$$

NOTE: *The Boolean expression from the truth table is the sum of product (minterms)terms for which the output is i.e.,*

$$X = \overline{A}\,\overline{B}\,C + \overline{A}\,B\,\overline{C} + A\,\overline{B}\,\overline{C} + AB\overline{C}$$

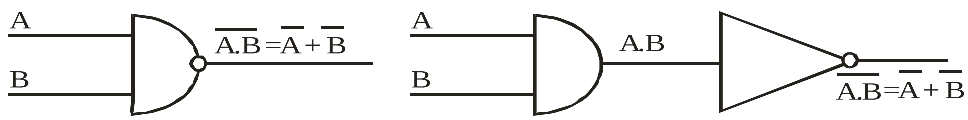
11.13 NAND and NOR gates:

DeMorgan’s theorems form two new gates NAND and NOR gates. These gates are the most popular and most widely used logic gates. Since any logic circuit can be constructed using only NAND and NOR gates, they are often referred to as the Universal building blocks.

NAND gates:

This NAND (or not AND) gate is an AND gate followed by a NOT circuit: The operation of the NAND gate is described by one of

DeMorgan’s Theorem, which states that $\overline{A.B} = \overline{A} + \overline{B}$. The NAND gate has two or more input signals but only one output signal. Any input must be high to get a high output



NANDGae (a) Standard Symbol (b) Logical meaning of NAND Gate

Fig. 6(a) (b)

NAND- Gate truth Table 17

Table 17

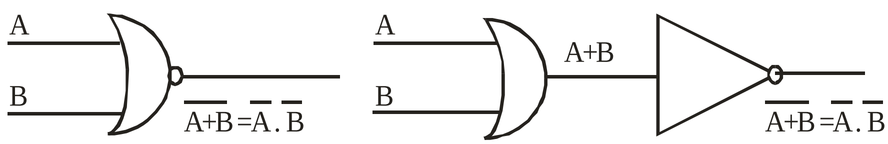
A	B	A.B.	$\overline{A.B}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

NOR-Gate:

The NOR (or not OR) gate is an OR-gate followed by a NOT circuit. The operation of the NOR-gate is described by DeMorgan's theorem, which states that.

$$\overline{A + B} = \overline{A} \,.\, \overline{B}$$

The NOR gate has two or more input signals but only one output signal. All inputs must be low to get a high output.



NAND Gate (c) Standard Symbol (d) Logical meaning of NAND Gate

Fig. 6(C)(d)

Table 18

A	B	A + B	$\overline{A + B}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

NAND and NOR Gates in Two level Network

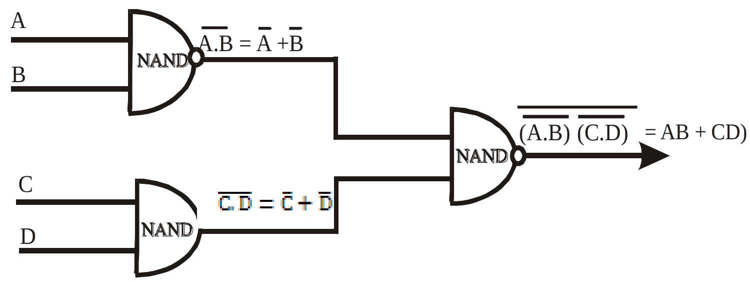


Fig. 7 (i) NAND Gates in Two - Level Networks

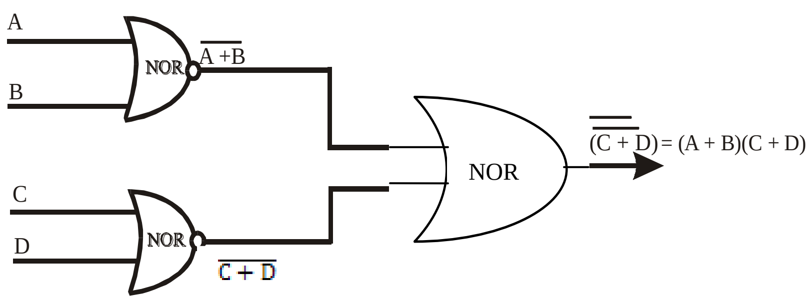


Fig. 7: NAND and NOR Gates in Two - Level Networks

Fig. 7

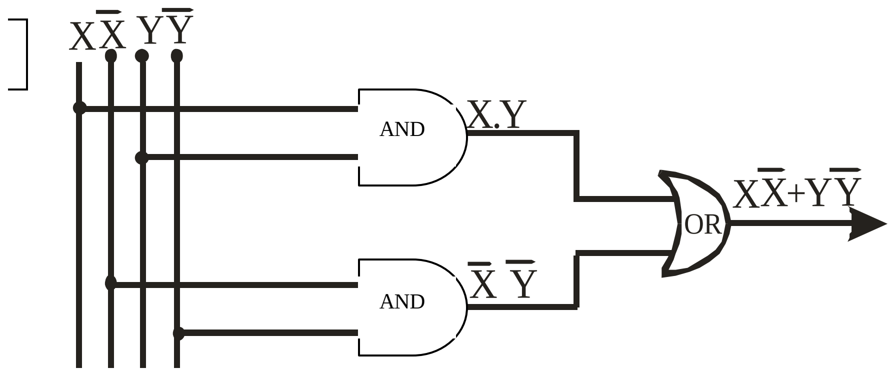
11.14 Combination of Gates:

The OR gate and AND gates and invertors can be interconnected to form gatting or logic networks, in the switching theory, these are also called combinational networks. The Boolean algebra expression corresponding to a given Network can be driven by systematically progressing from input to output on the gates.

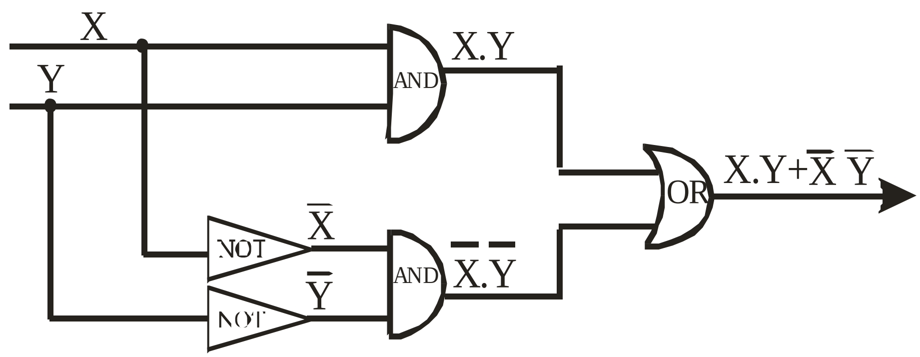
A net work that forms (i) $(X.Y) + (\overline{X}\,.\overline{Y})$

and another net work that forms (ii) $(X + Y).\,(\overline{X} + \overline{Y})$ are shown as

(i)



Or,



(ii)

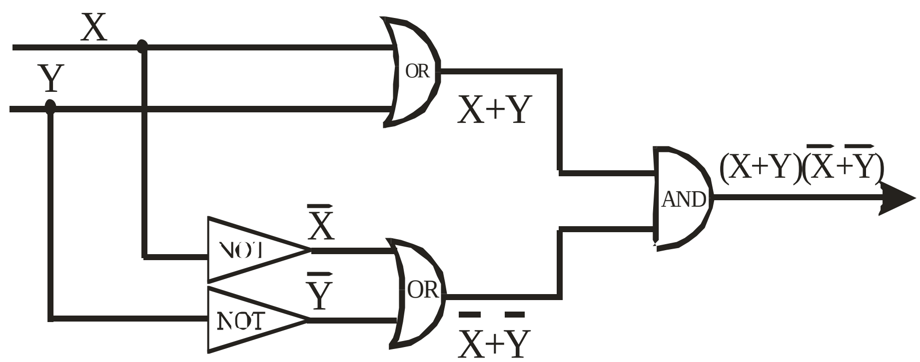


Fig. 8

11.15 Boolean Expression and Logic Diagrams:

Boolean expressions are frequently written to describe mathematically the behavior of a logic circuit. Using a truth table and the Boolean expression, one can determine which combinations of input signals cause the output signal.

Example 8: **Write the Boolean expression that describes mathematically the behavior of logic circuit shown in fig.10. Use a truth table to determine what input conditions produce a logic 1 output.**

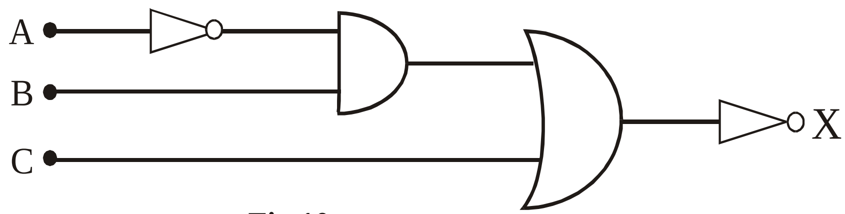


Fig.10

Solution:

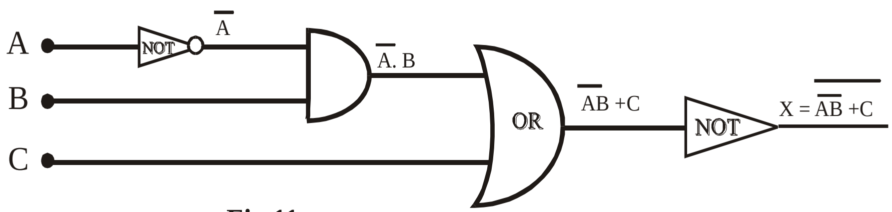


Fig.11

Fig.11 Circuit showing solution for example 8

Solution:

Truth Table 18 for the Circuit in Fig.11

A	B	C	$\overline{A}$	$\overline{A}\,B$	$\overline{A}\,B + C$	$\overline{\overline{A}\,B + C}$
0	0	0	1	0	0	1
0	0	1	1	1	1	0
0	1	0	1	1	1	0
0	1	1	1	1	1	0
1	0	0	0	0	0	1
1	0	1	0	1	1	0
1	1	0	0	0	0	1
1	1	1	0	1	1	0

Thus the input conditions those produce a logic 1 output are : 0 0 0 , 100, 110

Example 9: **Given the Boolean expression**

$X = AB + ABC + A\overline{B}\,\overline{C} + A\overline{C}$

(a) Draw the logic diagram for the expression.

(b) Minimize the expression.

(c) Draw the logic diagram for the reduced expression.

Solution: (a)The logic diagram is shown in the Fig. 12.

(b) $X = AB + ABC + A\overline{B}\,\overline{C} + A\overline{C}$

$= AB\,(1 + C) + A\overline{C}\,(\overline{B} + 1)$

$= AB.\,1 + A\overline{C}\,.1 = AB + A\overline{C}$

$= A(B + \overline{C})$

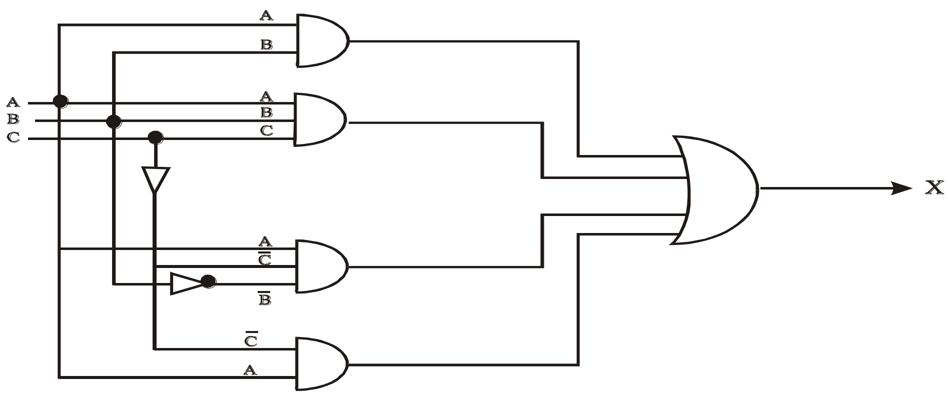


FIG. 12

(c)

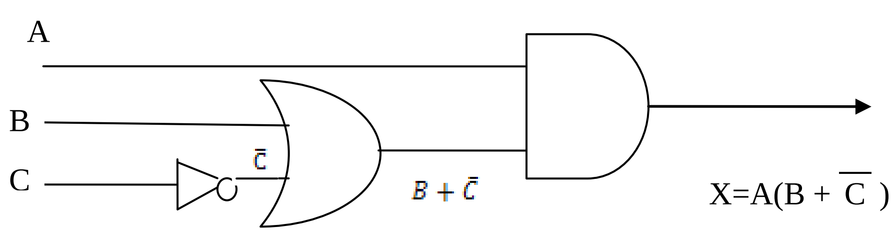


Fig.13 Minimize diagram for example 9.

11.16 Karnaugh Maps:

Many engineers and technicians prefer to use Karnaugh Maps to minimize the Boolean expressions instead of Boolean Algebra. This section tells you how to construct. Here we use the Karnaugh maps to minimize expressions containing up to four Variables.

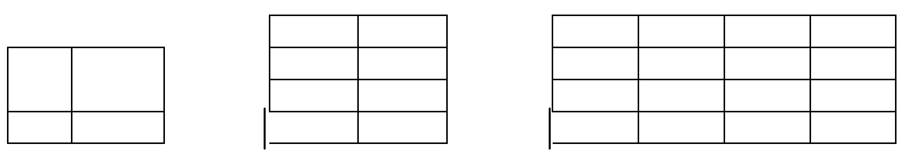
A Karnaugh map is a graphical form of a *truth* table and consists of a square or rectangular array of adjacent cells or blocks. The number of cells in a particular map depends on the number of variables in the Boolean expression to be minimized. The number of cells for a particular map is determined from expression.

$$N = 2^n$$

Where N = number of cells required for the Karnaugh map.

n = number of variables in the Boolean expression.

The configuration of the Karnaugh map for two, three and four variable expression is shown in Fig. In the Karnaugh map, each variable and its complement are assigned half of the cells in the map. The assigned cells consists of adjacent rows or columns.



(a)Two-variables map (b) Three variables map (c) Four Variables map

The sides of the map are labeled to show cell assignment as shown for the two variables map in Fig. 16. The two left hand cell beneath A are assigned to A and the two right hand cells or assigned to $\overline{A}$. Moving horizontally, the top two cells are assigned to variable B and the bottom two cells are assigned to $\overline{B}$. Each cell assigned a unique address, which is specified by the row column in which the cell resides. Fig. 17 shows which variable share each cell in fig. 16.

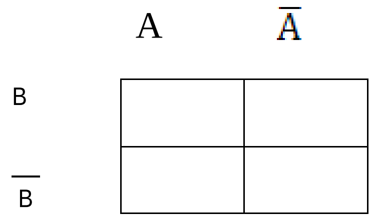


Fig. 16

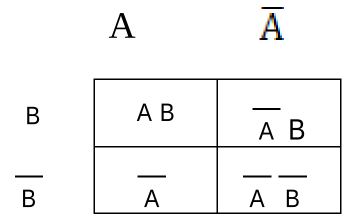
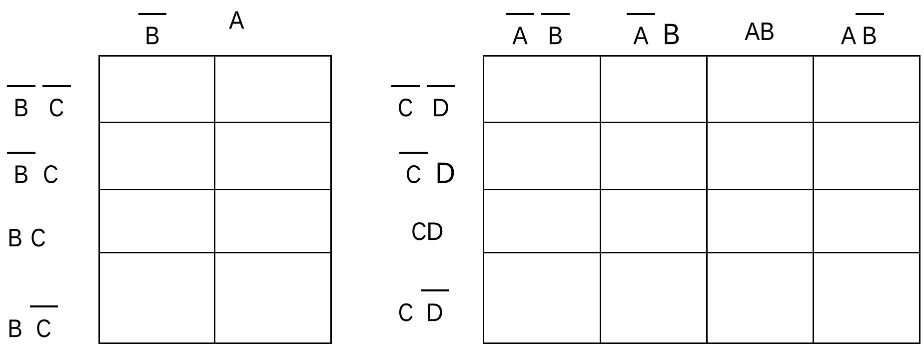


Fig. 17

The maps showing cell assignments for three and four variables expression are shown in Fig.18. The sides of the maps may be labeled in only convenient way. Two cells of a map are considered to be adjacent as long as their respective addressed differed by no more than one variable.

For example, ABCD and ABC$\overline{D}$ are addresses of adjacent cells. Diagonal cells are not adjacent, even though they shore a common corner, because their addresses differe by more than one variable.



(a) (b)

Fig. 18

After learning how to draw a Karnaugh map, the next step is to plot the given expression. The given expression must be in the sum-of-products form. The expression A$\overline{B}$ C +ABC is the sum of two products, or, two minterms, and is therefore in the correct form; however, the expression (A+C) (B+C) is not in the correct form and cannot be plotted in this form.

To plot an expression, we identify with a 1 each cell addressed by a minterm (product term). After placing a l in cells addressed by each minterm, adjacent cells containing all are enclosed either singularly, in pairs, or in groups of 4, 8 or 16 (integral power of 2). We enclose the largest number of adjacent cells containing a ‘l’ as possible – as long as the enclosed ‘l’s equal 2 raised to an integral power. The enclosed, ‘l’s are collectively called a group or an implicant.

The desired minimized Boolean expression is obtained from the Karnaugh map by applying the following two steps:

1. All 1's must be included in at least one group. It is permissible, and desirable, to enclosed a l more than once if it facilitates enlarging another enclosure.
2. Each group represents a minterm. The sum of the minterms that represent these groups is the minimized Boolean expression in sum-of-products form corresponding to the given logic function.

The use of these rules is illustrated in the following example:-

Example 10: Plot the Boolean express $X = AB + A\bar{B} + BC$ and minimize expression from the Map.

Solution:

Since the expression contains three variables, we need a Karnaugh map containing cells equal to

$N = 2^3 = 8$

	$\bar{A}$	A
$\bar{B}\,\bar{C}$		1
$\bar{B}\,C$		1
$B\,C$	1	1
$B\,\bar{C}$		1

Fig.19

To plot the expression, we start by identifying all cells that are common to AB by placing 1 in these cells. If we repeat this procedure for the terms $A\bar{B}$ and B C and placing a l in cells common to the minterms, we have plotted the function as shown in Fig. 19.

To minimized Boolean expression, all 1,s must be enclosed at least once. Greatest minimization is achieved by enclosing the largest number of adjacent l's possible, even if same have already enclosed.

In Fig. 19, the 1's making up the larger group are common only to A, and the 1's making up the smaller group are common to B and C; therefore, the minimized expression is

$$X = A + BC$$

Algebraic Proof:

The sum-of-products equation corresponding to larger group is

$$Y = A\overline{B}\,\overline{C} + A\overline{B}C + AB\overline{C} + ABC$$

$$= \mathbf{A}\,(\overline{B}\,\overline{C} + \overline{B}C + B\overline{C} + BC)$$

$$= \mathbf{A}\{\overline{B}(\overline{C} + C) + B(\overline{C} + C)$$

$$= A(\overline{B} + B)$$

$$= A$$

The sum-of-products equation corresponding to the smaller group is

$$Z = \overline{A}BC + ABC$$

$$=(\overline{A} + A)BC = BC$$

Example 11: Minimize the following Boolean expression by use of the Karnaugh map.

$$X = B\overline{C} + B\overline{D} + AB + AD + AC + C\overline{D}$$

Solution:

The number of cells are $N = 2^4 = 16$

To plot the expression, place 1 in the cells common to each term in the expression: Fig: 20

	$\overline{A}\,\overline{B}$	$\overline{A}B$	AB	$A\overline{B}$
$\overline{C}\,\overline{D}$		1	1	
$\overline{C}D$		1	1	1
CD			1	1
$C\overline{D}$	1	1	1	1

Fig.20

The minimized expressions is:

$$X = B\overline{C} + AD + C\overline{D}$$

NOTE: On four-variable maps, l's is cells on opposite sides of the map may be enclosed since the map is continuous, like a cylinder, in both the horizontal and vertical planes

Example 12: Minimize the following Boolean expression by use of the Karnaugh map:-

$$X = A\overline{C}D + \overline{A}B\overline{C}D + \overline{A}\,\overline{B}D + A\overline{B}CD$$

Solution:

Fig.21 shows the Karnaugh map for the expression.

The minimized expression is

$$X = \overline{B}D + \overline{C}D$$

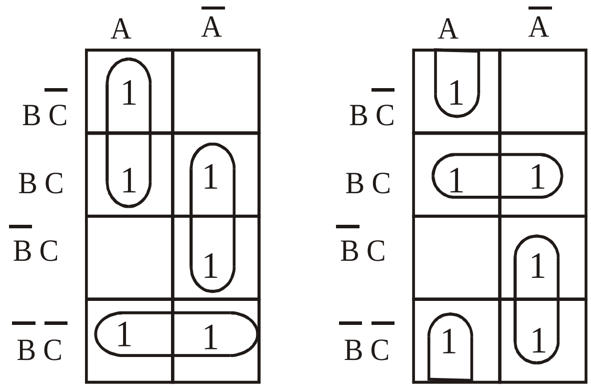
	$A\overline{B}$	AB	$\overline{A}B$	$\overline{A}\,\overline{B}$
$C\overline{D}$				
CD	1			1
$\overline{C}D$	1	1	1	1
$\overline{C}D$	1	1		1

Fig.21

11.17 Non-Unique Group:

Occasionally, we find a Karnaugh map for which more than one set of groups exists. This situation is illustrated in Fig.19. One can readily see that both maps yield Boolean expression with three two-variable minterms. These equivalent expressions, described the same function; however, they differ in the way in which variables are combined. Boolean functions that can be described by two or more equivalent minimized Boolean expressions are referred to as non-unique. On the other hand, functions that described by a single minimized Boolean expressions are referred as unique.

When a non-unique Boolean function is to be implemented with logic gates, it generally does not matter which of the possible minimized expressions is implemented. However, occasionally one expression may be preferred because certain variables or minterms are more accessible or may be available from an existing circuit.



(a) (b)

Fig. 19: Boolean expressions from non-unique groups.

(a) $X_1 = AB + \bar{A}C + \bar{B}\,\bar{C}$, (b) $X_2 = A\bar{C} + BC + \bar{A}\,\bar{B}$

Example 12: Read out some of the possible Boolean expressions for the Karnaugh map shown in Fig. 20.

	$\bar{A}\bar{B}$	$\bar{A}B$	AB	$A\bar{B}$
$\bar{C}\bar{D}$		1	1	
$\bar{C}D$		1	1	
CD	1	1		1
$C\bar{D}$	1	1	1	

Fig.20

Solution: Fig.20 shows several possible combinations of enclosure and the resulting Boolean expression.

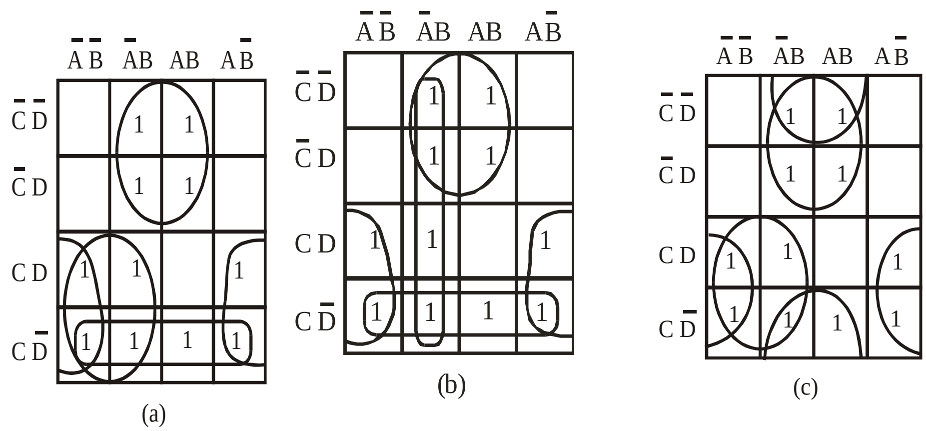


Fig.21 Various combinations of enclosures.

(a) $X = B\overline{C} + \overline{A}C + \overline{B}C + C\overline{D}$

(b) $X = B\overline{C} + \overline{B}C + \overline{A}B + C\overline{D}$

(c) $X = B\overline{C} + \overline{A}C + \overline{B}C + B\overline{D}$

11.18 Redundant Groups:

After you finish encircling groups, there is one more thing to do before writing the simplified Boolean equation: eliminate any group whose l’s are completely over lapped by other groups. (A group whose 1’s are all overlapped by other groups is called redundant group).

11.19 Dont’ Care States:

Using the truth table, we, can list each combination of input variables that should cause a high output to exist. For some Boolean functions, the output corresponding to certain combinations of input variables does not matter. This usually occurs because certain combinations of input variables cannot exist. Also, there are times that we do not really care what value a function may take on. In both instances we call such a term a don't care state. Don’t-care states, means, that we do not care whether the entry in a **karnaugh** map corresponding to a certain combination of variables is a 1 or a 0. We shall use the symbol X for don’t care states.

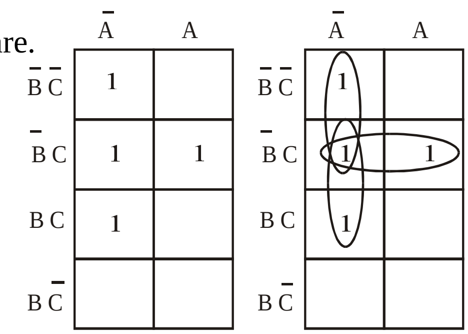
Don’t-care states can be very important in the minimization process. Since we can assign either a 1 or 0 to a don’t-care condition, we can choose which ever value will provide a larger enclosure and therefore a simpler, more economical circuit.

Example 13: A logic circuit is to be constructed that will implement the Boolean expression:

$$X = A\bar{B}C + \bar{A}BC + \bar{A}\bar{B}\bar{C}$$

Plot this expression on a Karnaugh map and reduce the expression if the term $\bar{A}\bar{B}C$ is a don't care.

Solution:



(a) (b)

Fig.22 (a) Karnaugh map for given expression (b) Karnaugh map, illustrating the use of don't care states.

The expression corresponding to the map in Fig. 22 (b) is :

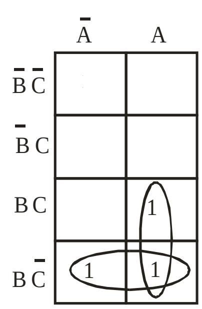
$$X = \bar{A}\bar{B} + \bar{A}C + \bar{B}C$$

Therefore, assigning a value of 1 to the don't care State allowed us to reduce the original expression significantly.

11.20 For the given truth table minimize the Boolean expression using Karnaugh map.

Consider the truth table. The Fundamental products for these 1 out put are $\bar{A}B\bar{C}$, $AB\bar{C}$ and ABC. Enter these 1's on the Karnaugh map.

A	B	C	Output Y
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1



Karanaugh map for the TruthTable

Fig.23

The sum-of-product form for the Boolean expression from the truth table is

$$X = \overline{A}\,B\,\overline{C} + AB\overline{C} + ABC$$

To minimize this Boolean expression from the Karnaugh map, we find that

$$X = B\,\overline{C} + A\,B$$

Exercise 11

Q.1: Prepare a truth table for the following Boolean expression:

(i) $XYZ + \overline{X}\,\overline{Y}\,\overline{Z}$ (ii) $ABC + A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C}$

(iii) $(A + D)(B + C)$ (iv) $(A + B)(A + C)(\overline{A} + \overline{B})$

(v) $AB + \overline{A}\,\overline{B}$

Q.2: Simplifying the following with the help of Boolean algebra Rules:

(i) $AB + AC + ABC$ (ii) $AB + A(\overline{B} + C) + AB\overline{C}$

(iii) $\overline{A}\,BC + A\overline{B}\,C + ABC + AB\overline{C} + \overline{A}\,\overline{B}\,\overline{C}$

(iv) $A\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,B\overline{C} + \overline{A}\,\overline{B}\,C$

Q.3: Minimize the following expressions:

(a) $X = W\,\overline{Z}\,(W + Y) + WY(\overline{Z} + \overline{W})$

(b) $X = \overline{(A+B)\,(\overline{C})}$

(c) $X = \overline{(A\overline{B}\,C + B\overline{C})}$

(d) $X = (A\overline{B}\,C + \overline{ABC})\,C$

Q.4: Convert the following expression to sum-of-product form.

(i) $(A + B)(\overline{B} + C)(\overline{A} + C)$

(ii) $(A + C)(A\overline{B} + AC)(\overline{A}\,\overline{B} + \overline{C})$

Q.5: Convert the following expression to product-of-sum form:

(i) $A + \overline{A}B + \overline{A}C$ (ii) $(\overline{AB + \overline{\overline{C}}}) + A\overline{C}B + B$

(iii) $AB + ABC + A\overline{B}\,\overline{C} + A\overline{C}$ (iv) $AB + \overline{A}\,\overline{B}$

Q.6: Express the given function in the product of sums, also draw its circuit and truth table.

$X = A\overline{B}C + \overline{A}\,\overline{B}\,\overline{C} + AB$

Q.7: Draw a logic circuit using only NAND gates for which output expression is X = A C + B C.

Q.8: Draw a logic circuit using only NOR gates for which the output expression is $\mathbf{X = A\overline{C} + \overline{B}C}$.

Q.9: Prove the following by use of a truth table:

$\overline{A}B\overline{A} + \overline{A}BC + \overline{A}\,\overline{B}C = \overline{A}B + \overline{A}C$

Q.10: Draw the circuit diagrams of the following:

(i) $F = X\overline{Y}Z + XYZ + \overline{X}YZ$

(ii) $F = AB + (A\overline{B} + \overline{A}B)$

Q.11: Use the Karnaugh map to minimize the following expressions.

(i) $X = AB + A\overline{B} + B\overline{C} + \overline{A}C$

(ii) $X = \overline{A}C + B\overline{C} + \overline{B}\,\overline{C} + AC$

(iii) $X = ABC + \overline{A}BC + A\overline{B}C + \overline{A}\,\overline{B}\,\overline{C}$

(iv) $X = \overline{A}\,\overline{B}\,\overline{C} + ABC + A\overline{B}\,\overline{C} + \overline{A}B\overline{C} + AB\overline{C} + A\overline{B}C + \overline{A}\,\overline{B}C$

(v) $X = \overline{A.B} + (A + B)(\overline{A} + C) + A(\overline{A} + C)$

Q.12: For the given truth tables, find out logical expressions using Boolean algebra and and minimize these expressions by Boolean Rules or Karnaugh map techniques:

(i) Truth Table 20

A	B	C	Output Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

(ii) Truth Table 21

A	B	C	D	Output Y
0	0	0	0	0
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	0

Answers:11

Q.1(i)

X	Y	Z	$\bar{X}$	$\bar{Y}$	$\bar{Z}$	XYZ	$\bar{X}\,\bar{Y}\,\bar{Z}$	$XYZ + \bar{X}\,\bar{Y}\,\bar{Z}$
0	0	0	1	1	1	0	1	1
0	0	1	1	1	0	0	0	0
0	1	0	1	0	1	0	0	0
0	1	1	1	0	0	0	0	0
1	0	0	0	1	1	0	0	0
1	0	1	0	1	0	0	0	0
1	1	0	0	0	1	0	0	0
1	1	1	0	0	0	1	0	1

(ii)

A	B	C	$\bar{A}$	$\bar{B}$	$\bar{C}$	ABC	$A\,\bar{B}\,\bar{C}$	$\bar{A}\,\bar{B}\,\bar{C}$	$ABC + A\,\bar{B}\,\bar{C} + \bar{A}\,\bar{B}\,\bar{C}$
0	0	0	1	1	1	0	0	1	1
0	0	1	1	1	0	0	0	0	0
0	1	0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0	0	0
1	0	0	0	1	1	0	1	0	1
1	0	1	0	1	0	0	0	0	0
1	1	0	0	0	1	0	0	0	0
1	1	1	0	0	0	1	0	0	1

(iii)

A	B	C	D	A +D	B + C	(A + B)(B + C)
0	0	0	0	0	0	0
0	0	0	1	1	0	0
0	0	1	0	0	1	0
0	0	1	1	1	1	1
0	1	0	0	0	1	0
0	1	0	1	1	1	1
0	1	1	0	0	1	0
0	1	1	1	1	1	0
1	0	0	0	1	0	1
1	0	0	1	1	0	1
1	0	1	0	1	1	1
1	0	1	1	1	1	1
1	1	0	0	1	1	1
1	1	0	1	1	1	1
1	1	1	0	1	1	1
1	1	1	1	1	1	1

(iv)

A	B	C	$\overline{A}$	$\overline{B}$	A+B	A+C	$\overline{A}+\overline{B}$	$(A+B)(A+C)(\overline{A}+\overline{B})$
0	0	0	1	1	0	0	1	0
0	0	1	1	1	0	1	1	0
0	1	0	1	0	1	0	1	0
0	1	1	1	0	1	1	1	1
1	0	0	0	1	1	1	1	1
1	0	1	0	1	1	1	1	1
1	1	0	0	0	1	1	0	0
1	1	1	0	0	1	1	0	0

(v)

A	B	$\overline{A}$	$\overline{B}$	AB	$\overline{A}\,\overline{B}$	$AB+\overline{A}\,\overline{B}$
0	0	1	1	0	1	1
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	1	0	0	1	0	1

Q.2: (i) $A(B+C)$ (ii) A (iii) $\overline{A} + A(\overline{B}C + B)$

(iv) $\overline{B}\,\overline{C} + \overline{A}\,\overline{C} + \overline{A}\,\overline{B}$

Q.3: (a) $W\overline{Z}$ (b) $\overline{A}\,\overline{B} + C$ (C) BC (d) $(\overline{A} + \overline{B})C$

Q.4: (i) $BC + AC$ (ii) $A\overline{B}$

Q.5: (i)

A	B	C	$\overline{A}$	$\overline{A}B$	$\overline{A}.C$	$A+\overline{A}B+\overline{A}C$
0	0	0	1	0	0	0
0	0	1	1	0	1	1
0	1	0	1	1	0	1
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	0	0	1
1	1	0	0	0	0	1
1	1	1	0	0	0	1

Hence product-of-sum is $(A + B + C)$

(ii) $A + B + C$ (iii) $A(B + \overline{C})$

A	B	$\overline{A}$	$\overline{B}$	AB	$\overline{A}\,\overline{B}$	$AB+\overline{A}\,\overline{B}$
0	0	1	1	0	1	1
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	1	0	0	1	0	1

Hence Product of sums $=(A+\overline{B})(\overline{A}+B)$

Q.6:

A	B	C	$\overline{A}$	$\overline{B}$	$\overline{C}$	$A\overline{B}C$	$\overline{A}\,\overline{B}\,\overline{C}$	AB	$A\overline{B}C+\overline{A}\,\overline{B}\,\overline{C}+AB$
0	0	0	1	1	1	0	1	1	1
0	0	1	1	1	0	0	0	0	0
0	1	0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0	0	0
1	0	0	0	1	1	0	0	0	0
1	0	1	0	1	0	1	0	0	1
1	1	0	0	0	1	0	0	1	1
1	1	1	0	0	0	0	0	1	1

Hence Product-of-sums $=(A+B+\overline{C})\,(A+\overline{B}+C)\,(A+\overline{B}+\overline{C})\,(\overline{A}+B+C)$

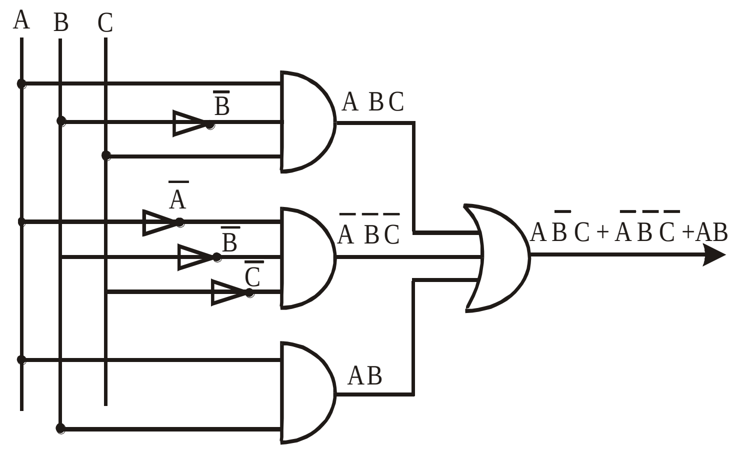
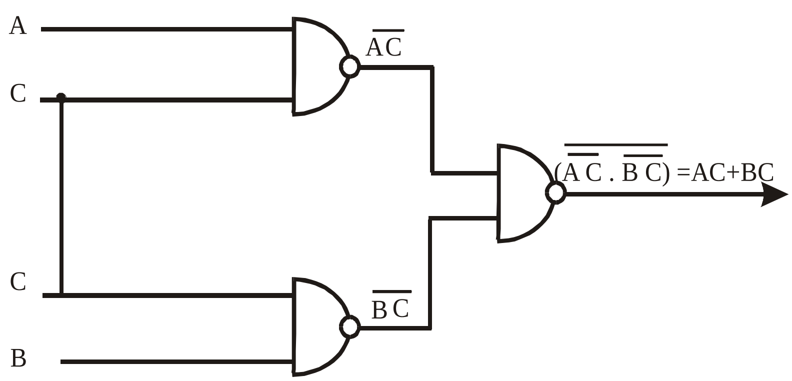


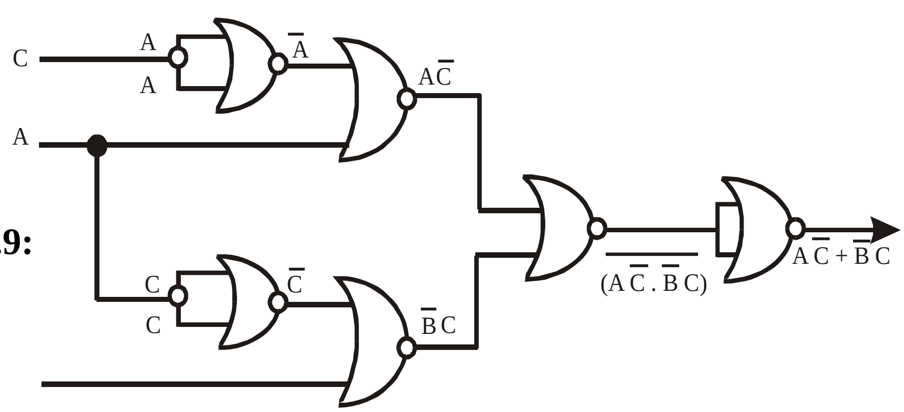
Fig.Circuit

Q.7:



Q.7: Fig..

Q.8:



Q.9:

Q.8: Fig..

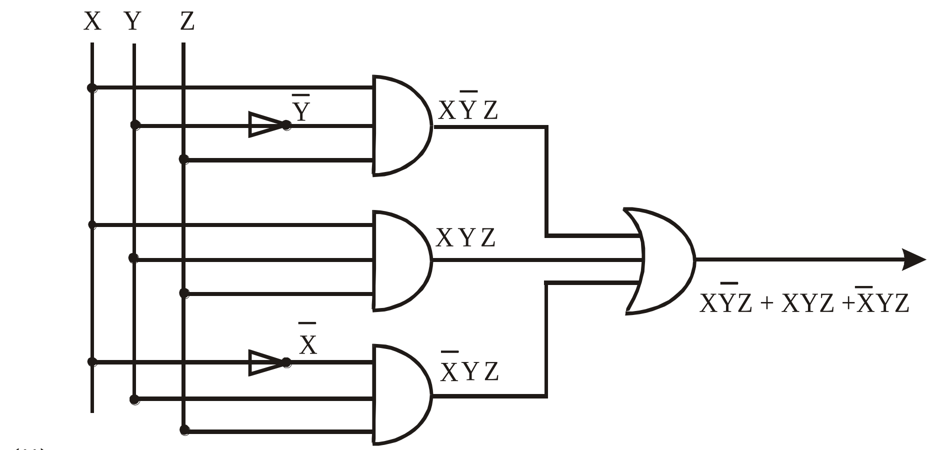
Truth Table

A	B	C	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{A}B$	$\bar{A}C$	$\bar{A}B\bar{C}$	$\bar{A}BC$	$\bar{A}\bar{B}C$	$\bar{A}\bar{B}C + \bar{A}BC + \bar{A}\bar{B}C$	$\bar{A}B + \bar{A}C$
0	0	0	1	1	1	0	0	0	0	0	0	0
0	0	1	1	1	0	0	1	0	0	1	1	1
0	1	0	1	0	1	1	0	1	0	0	1	1
0	1	1	1	0	0	1	1	0	1	0	1	1
1	0	0	0	1	1	0	0	0	0	0	0	0
1	0	1	0	1	0	0	0	0	0	0	0	0
1	1	0	0	0	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0

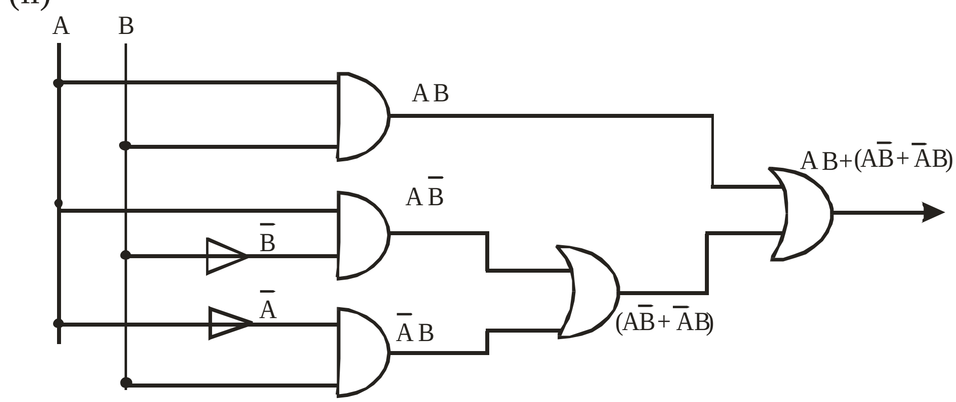
L.H.S. = R.H.S.

Q.10

(i)



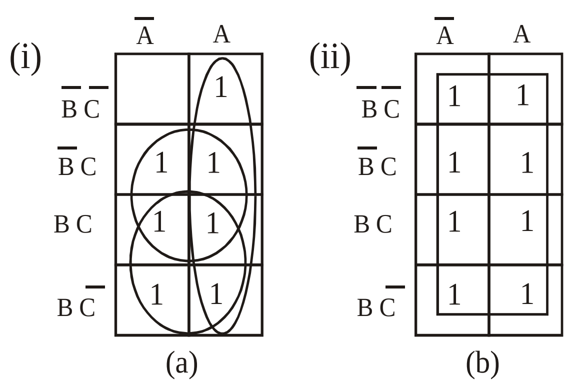
(ii)



Q.10. Fig. (i) and (ii)

Q.11.

(a) X



=C + B +A

(b) $X = 1$

(iii)

	$\overline{A}$	A
$\overline{B}\,\overline{C}$		
$\overline{B}\,C$	1	1
$B\,C$	1	1
$B\,\overline{C}$		

(c)

(iv)

	$\overline{A}$	A
$\overline{B}\,\overline{C}$	1	1
$\overline{B}\,C$	1	1
$B\,C$		1
$B\,\overline{C}$	1	1

(d)

(c) $X = C$

(d) $X = A + \overline{B} + \overline{C}$

(v)

	$\overline{A}$	A
$\overline{B}\,\overline{C}$	1	1
$\overline{B}\,C$	1	1
$B\,C$	1	1
$B\,\overline{C}$	1	

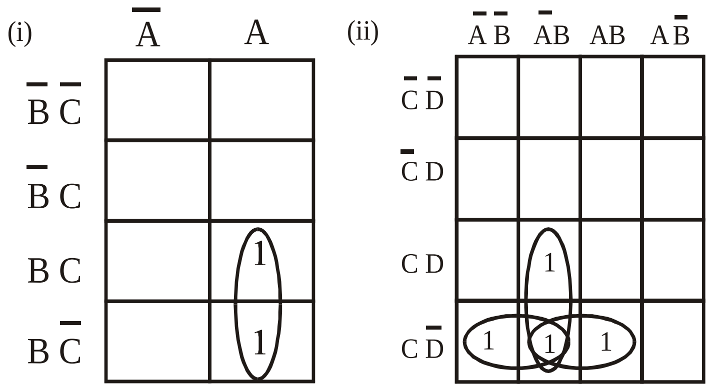
(e)

(a) $X = \overline{A} + \overline{B} + \overline{C}$

Q.12: Boolean expressions by Truth table are:

(i) $X = A\,B\,\overline{C} + ABC$

(ii) $X = \overline{A}\,\overline{B}\,C\,\overline{D} + \overline{A}BC\overline{D} + \overline{A}BCD + ABC\overline{D}$



(a) (b)

(a) $X = AB$ (b) $X = \overline{A}\,C\,\overline{D} + \overline{A}\,BC + BC\,\overline{D}$

Are the logical minimized expressions after applying Karnaugh map.

Short Questions

Q.1: Define Logic Gates.

Q.2: Define following gates and draw logic circuit diagram

(a) OR Gate (b) AND Gate

(c) NOT Gate (d) NAND Gate

(e) NOR Gate

Q.3: Prove by Boolean Algebra X + X Ƶ = X

Q.4: Prove by Boolean Algebra rules X (X + Y) = X

Q.5: Prove by Boolean Algebra Rules X + $\overline{X}$ Y = X + Y

Prove that by Boolean Algebra Rules.

Q.6: AB +AC + ABC= AB+AC

Q.7: XY + YƵ + $\overline{Y}$ Ƶ = XY + Ƶ

Q.8: X ($\overline{X}$ + Y) = XY

Q.9: X + Y Ƶ = (X +Y) (X + Ƶ)

Q.10: Construct a logic diagram for expression A. B + C

Q.11: Construct a logic diagram for expression $\overline{AB} = \overline{A} . \overline{B}$

Q.12: Construct a logic diagram for expression A. B + B.C

Q.13: Construct a logic diagram for expression B. (A +C)

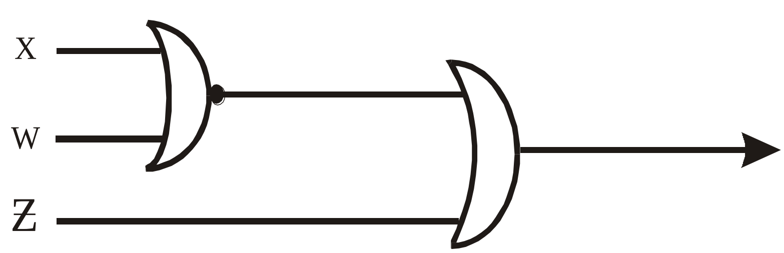
Q.14: Find truth table of X + Y = Y + X

Q.15: Prepare a truth table of XY= YX

Q.16: Prepare a truth table X(X +Y) = X

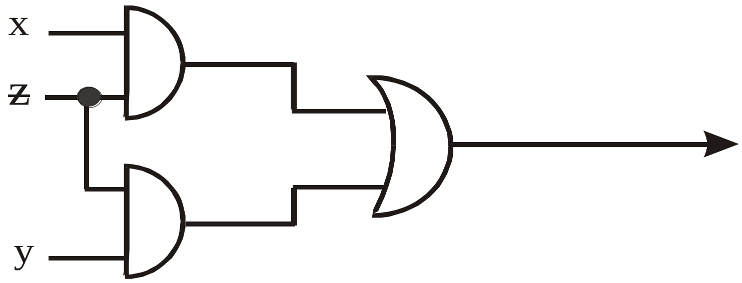
Q.17: Prepare a truth table of X + X Ƶ = X

Q.18: Obtain logic expression for logic diagram

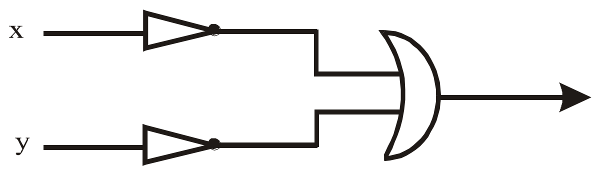


Q.19: Obtain the logic expression for logic diagram.

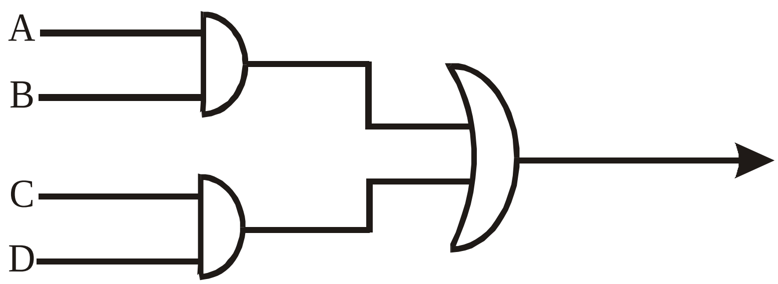
Q.20: Obtain the logic expression for logic diagram.



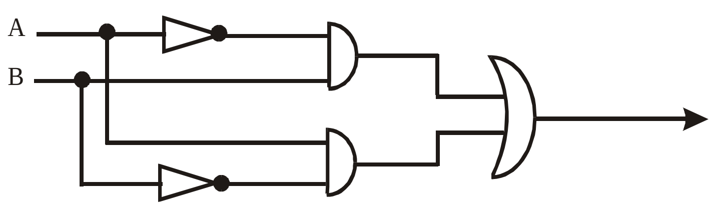
Q.21: Obtain the logic expression for logic diagram.



Q.22: Obtain the logic expression for logic diagram.

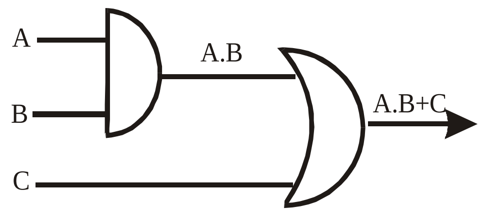


Q.23: Obtain the logic expression for logic diagram.

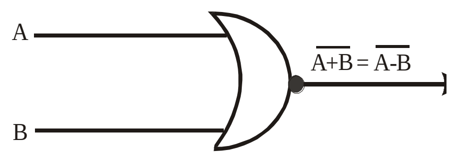


Answers

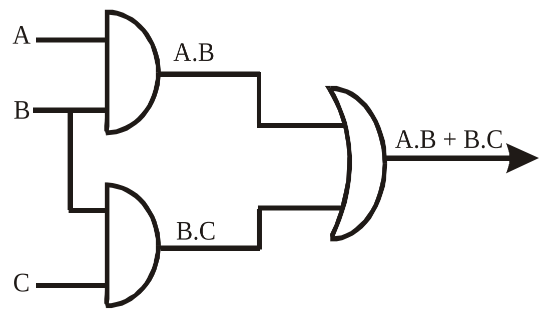
Q10.



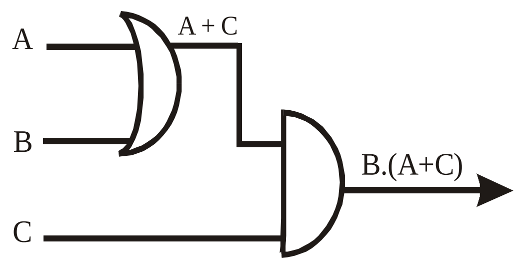
Q11.



Q12.



Q13.



Q14.

X	Y	X +Y	Y +X
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

Q15.

X	Y	XY	YX
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

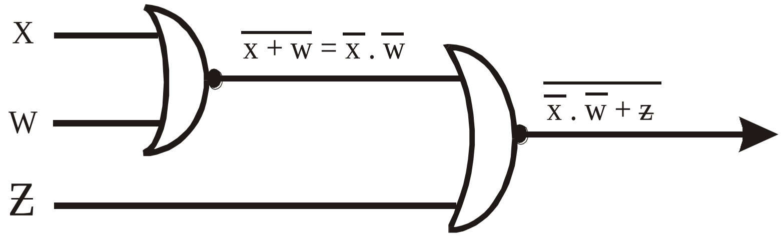
Q16.

X	Y	X +Y	X(X+Y)	X
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	1	1

Q17.

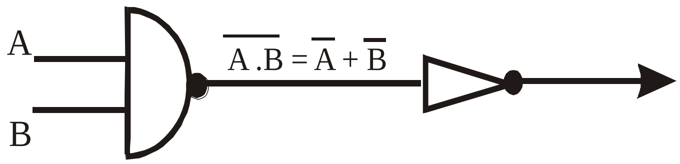
X	Ƶ	XƵ	X+XƵ	X
0	0	0	0	0
0	1	0	0	0
1	0	0	1	1
1	1	1	1	1

Q18.



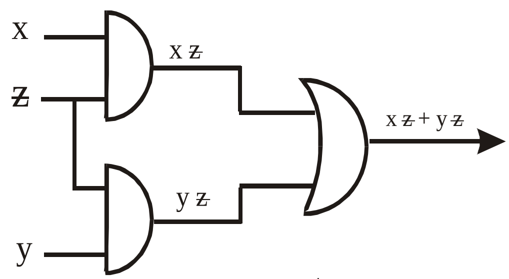
Expression = $\overline{\overline{x} . \overline{w} + z}$

Q19.



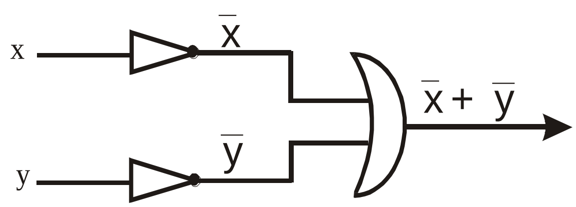
Expression = $\overline{\overline{A} + \overline{B}}$ = A .B

Q20.



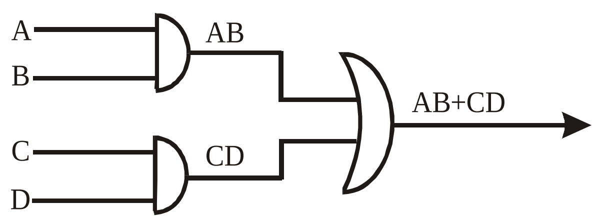
Expression = x z + y z

Q21.



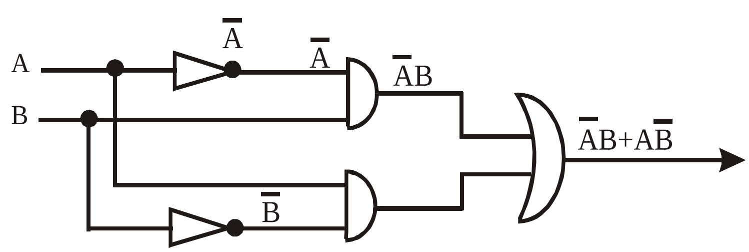
Expression = $\overline{x} + \overline{y}$

Q22.



Expression = AB+CD

Q23.



Expression = $\overline{A}B+A\overline{B}$

OBJECTIVE TYPE QUESTIONS

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

1. Boolean Algebra $X + \overline{X}Y$ is equal to:

   (a) X (b) $\overline{X}$ (c) $X + Y$ (d) $\overline{X} + Y$

2. According to Boolean Algebra $X + \overline{X}$ is equal to:

   (a) X (b) $\overline{X}$ (c) 0 (d) 1

3. In Boolean Algebra $\overline{X+ Y}$ is equal to:

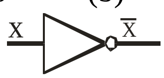
   (a) $\overline{X}+ \overline{Y}$ (b) $\overline{X} . \overline{Y}$ (C) $X .Y$ (d) $X + Y$

4. If the switch is on it is represent by:

   (a) 0 (b) 1 (c) OR (d) NOT

5. If the switch is off it is represented by:

   (a) 0 (b) 1 (c) OR (d) NOT

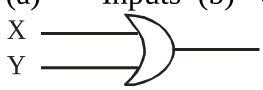
6. Symbol 

is used for:

   (a) NOT gate (13) NOR (c) OR(d) NAND

7. An AND gate performs logical multiplication on:

   (a) Inputs (b) Outputs (c) OR gates (d) NOR gates

8. 

$X + Y$ is the symbol for the logic:

   (a) OR gate (b) NOR gate (c) NAND gate (d) AND gate

9. $X.(\overline{X} + Y)$ equal to:

   (a) $X \; Y$ (b) $X . \overline{X}$ (c) $X. \overline{X} +X.Y$ (d) $X + Y$

10. $X +XZ$ is equal to:

    (a) Z (b) X (c) $X + Z$ (d)

ANSWERS

1.	c	2.	d	3.	b	4.	b	5.	b
6.	a	7.	a	8.	a	9.	a	10.	b

<><><>

Chapter 12

The Straight Line

(Plane Analytic Geometry)

12.1 Introduction:

Analytic- geometry was introduced by Rene Descartes (1596 – 1650) in his La Geometric published in 1637. Accordingly, after the name of its founder, analytic or co-ordinate geometry is often referred to as Cartesian geometry. It is essentially a method of studying geometry by mean of algebra. Its main purpose was to show how a systematic use of coordinates (real numbers) could vastly simplify geometric arguments. In it he gave a simple technique of great flexibility for the solution of a variety of problems.

12.2 Rectangular Coordinates:

Consider two perpendicular lines X′X and Y′Y intersecting point in the point O (Fig. 1). X'X is called the x-axis and Y'Y the y-axis and together they form a rectangular coordinate system. The axes divide the plane into four **quadrants** which are usually labeled as in trigonometry. The point O is called the origin, When numerical scales are established on the axes, positive distances x (abscissa) are drawn to the right of the origin, negative distance to the left; positive distance y (ordinates) are drawn upwards and negative distances downwards to the origin. Thus OX and OY have positive direction while OX′, OY′ have negatives direction.

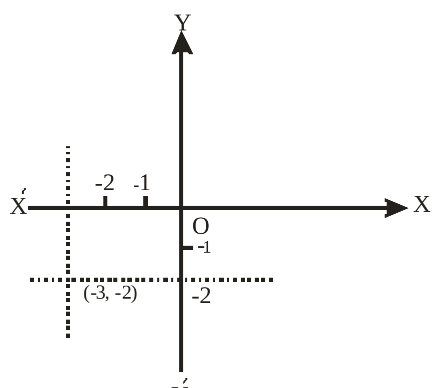


Fig 12.1

We now consider an arbitrary point P in the plane and the lines through P parallel to the axes. (These parallel lines might coincide with an axis if P is on the axis). The line through P parallel to the y-axis will intersect the x-axis at a point corresponding to some real number a. This number is called the x-coordinate (or abscissa) of P. the line through P parallel to the x-axis will intersect the y-axis at a point corresponding to some real number b. this number is called the y-coordinate (or ordinate) of P. The real numbers a and b are the coordinates of P and we indicate the point and coordinates by P (a, b) or by (a, b). In the Fig.1 the point P (-3,-2) is platted frequently. We shall refer to the order pair of real numbers (a, b) as a point. The coordinates a and b of a point (a, b) are called the Rectangular coordinates or Cartesian coordinates.

12.3 Polar Coordinates:

The Cartesian coordinates that we have been using specify the location of a point in the plane by giving the directed distances of the point from a pair of fixed perpendicular lines, the axes. There is an alternative coordinate system that is frequently used in the plane, in which the location of a point is specified in a different way.

In a plane, consider a fixed ray PB and any point A(Fig. 2) we can describe the location of A by giving the distance r from p to A and specifying the angle θ (measured in degree or radian). By stating the order pair (r, θ) we clearly identify the location of A. The components of such an ordered pair are celled polar coordinates of A. The fixed ray PB is called the polar axis and the initial point of the polar axis is called the pole of the system.

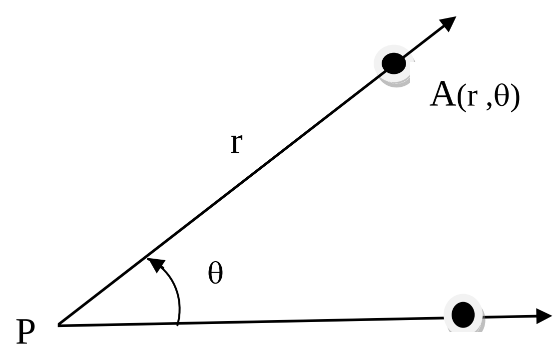


Fig.12.2

In polar coordinate system each point in the plane has infinitely many pair of polar coordinates. In the first place, if (r, θ) are polar coordinate of A, then so are (r, $\theta + K360^o$ for each k∈j (Fig 3a). In the second place, if we let – r < 0 denote the directed distance from P to A along the negative extension of the ray PA' in the direction opposite that of PA (Fig. 3b). Then we see that also $(-r, \theta + 180^o)$ and more generally $(-r, \theta + 180° + K360°)$, $K \in J$, are polar coordinates of A'. The pole P itself represented by $(0, \theta)$ for any θ whatsoever.

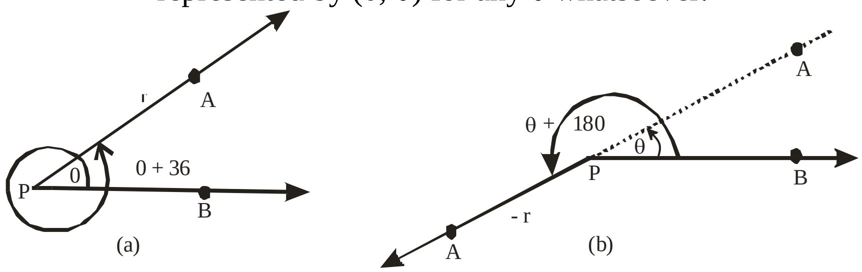


Fig 12,3

12.4 Relation between Rectangular and Polar Coordinates:

If the pole P in a rectangular coordinates system is also the origin O in a Cartesian coordinates system, and if the polar axis coincides with the positive x-axis of the cartesian system, then the coordinates (x, y) can be expressed in terms of the polar coordinates (r, θ) by the following equations (Fig. 4).

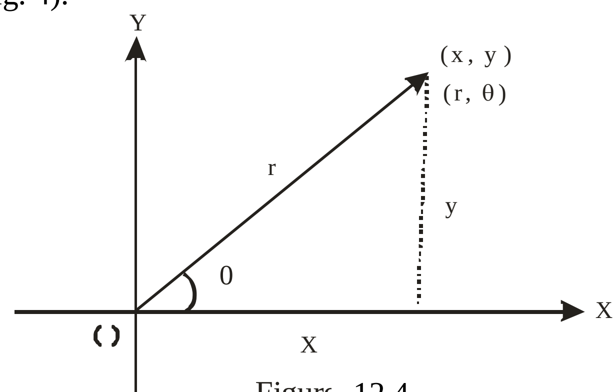


Figure 12.4

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

conversely, we have

$$r = \underline{+}\sqrt{x^2 + y^2}$$

and $$\tan\theta = \frac{y}{x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Or, $\theta = \tan^{-1} \dfrac{y}{x}$

The sets of equations (1) and (2) enable us to find rectangular coordinates for a point when given a pair of polar coordinates and vice versa.

Example 1:

Find the rectangular coordinates of the point with polar coordinates (4, 30^o)

$$x = r \cos\theta \qquad = 4 \cos 30^o \qquad = 4\,\frac{\sqrt{3}}{2} \qquad = 2\sqrt{3}$$

$$y = r \sin\theta \qquad = 4 \sin 30^o \qquad = 4\,\frac{1}{2} \quad = 2$$

The rectangular coordinates are $(2\sqrt{3}, 2)$

Example 2:

Find a pair of polar coordinates for the point with cartesian coordinates (7, -2).

Solution:

$$r = \sqrt{x^2 + y^2} = \sqrt{49 + 4} = \sqrt{53}$$

Noting that (7, -1) is in the fourth quadrant, so

$$\text{Tan}\,\theta = \frac{y}{x} = \frac{-2}{7}$$

$$\theta = \tan\text{-}1 \left(-\frac{2}{7}\right) \qquad = -16^o$$

12.5 The Distance Formula (distance between two points):

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points lying in the first quadrant. Let d be the distance between the points P and Q. Draw PR and QR parallel to the coordinates axes (Fig.12.5). By simple subtraction of abscissa, $PR = x_2 - x_1$; similarly subtracting ordinates, $QR = y_2 - y_1$.

Since PQR is a right triangle, so by Pythagorean theorem, we have.

$$(PQ)^2 = (PR)^2 + (QR)^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{Or} \quad = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This is known as the **distance formula.** The same formula holds true regardless of the quadrants in which the points lie.

Note: The distance d is given positive, because we are interested to find the numerical value of d and not its direction.

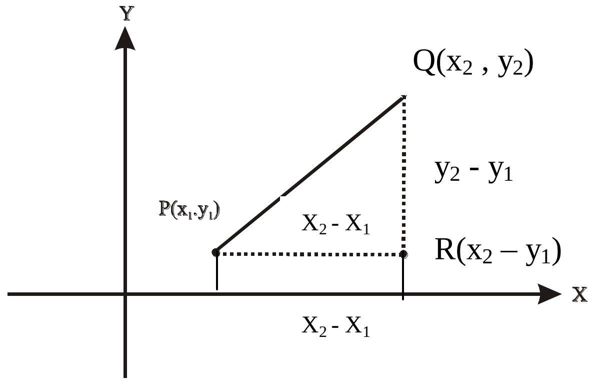


Figure 12.5

Example 3:

Find the distance between (–3, –2) and (–1,5).

Solution:

$$d = \sqrt{(-1 + 3)^2 + (5 + 2)^2}$$
$$= \sqrt{4 + 49} = \sqrt{53} = 7.42$$

Example 4:

Show that the points (–3, 1) (2,4) and (0, -4) are vertices of a right triangle.

Solution:

Let A (–3, 1), B(2,4) and C(0, –4), then

$$|AB| = \sqrt{(2 + 3)^2 + (4 - 1)^2} = \sqrt{25 + 9} = \sqrt{34} \text{ Units}$$
$$|BC| = \sqrt{(0 - 2)^2 + (-4 - 4)^2} = \sqrt{4 + 64} = \sqrt{68} \text{ Units}$$

And $|AC| = \sqrt{(0 + 3)^2 + (-4 - 1)^2} = \sqrt{9 + 25} = \sqrt{34}$ Units

since $|AB|^2 + |AC|^2 = |BC|^2$

So the given points are the vertices of a right triangle, with right angle at point A.

Example 5:

Show that the point (3, $\sqrt{7}$) is on a circle with centre at the origin and radius 4.

Solution:

Let A (O,O) and B (3, $\sqrt{7}$)

The distance between A and B is:

$$|AB| = \sqrt{(3-0)^2 + (\sqrt{7}-0))^2} = \sqrt{9+7} = 4$$

Which is the radius of the circle, so the given point (3, $\sqrt{7}$) lies on the circle.

Exercise 12.1

Q.1: Find the distance between:

(a) (–4,2) and (0,5) (b) (2, –2) and (2,7)

(c) (1–$\sqrt{2}$,1 –$\sqrt{3}$) and (1 + $\sqrt{2}$, 1 + $\sqrt{3}$

(d) (a, b) and (a + c, b + d)

Q.2: Show that the points

(a) A (2, 2), B(6, 6) and C(11,1) are the vertices of a right triangle.

(b) A(1, 0), B(–2, –3), C(2, –1) and D(5,2) are the vertices of a parallelogram.

(c) A (2, 3), B(0, –1) and C(–2, 1) are the vertices of an isoscles triangle.

Q.3: Is the point (0,4) inside or outside the circle of radius 4 with centre at (–3, 1) ?

Q.4: Determine y so that (0, y) shall be on the circle of radius 4 with centre at (–3, 1).

Q.5: The point (x, y) is on the x-axis and is 6 units away from the point (1, 4), find x and y.

Q.6: If one end of a line whose length is 13 Units is the point (4, 8) and the ordinate of the other end is 3. What is its abscissa?

Q.7: Find a point having ordinate 5 which is at a distance of 5 units from the point (2, 0).

Q.8: Find the value of y so that the distance between (1, y) and (–1, 4) is 2.

Q.9: Find the coordinates of the point that is equidistant from the points (2, 3), (0, –1) and (4, 5).

Q.10: Show that the points A(–3, 4), B(2,6) and C(0,2) are collinear. Find the values of AC: CB and AB: BC.

Answers

Q.1: **(a)** 5 **(b)** 9 **(c)** $2\sqrt{5}$ **(d)** $\sqrt{c^2 + d^2}$

Q.3: Outside **Q.4:** $1 \pm \sqrt{7}$

Q.5: $x = 1 \pm 2\sqrt{5}$, $y = 0$ **Q.6:** 16, –8 **Q.7:** (2,5)

Q.8: 4 **Q.9:** (11, –4)

Q.10: $3\sqrt{5} : 2\sqrt{5}$; $5\sqrt{5} : 2\sqrt{5}$

12.6 Segment of Line:

It is a part of a straight line between two points on it. The segment contains one end point or both end points. The sign of a line segment has a plus or minus sign according to some convention. Thus, if $P_1 P_2$ is the positive directed segment, then P_2P_1 has the negative sense and we write $P_1P_2 = – P_2P_1.$

12.7 The Ratio Formula (point of division):

Given a directed line segment such as P_1P_2 in Fig.6; to find the coordinates of the point P which divides internally P_1P_2 in a given ratio r_1: r_2. Let the coordinates of points P, P_1, and P_2

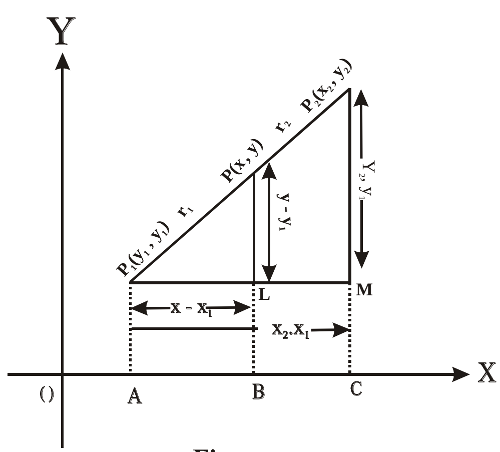


Figure 12.6

are (x, y), (x_1, y_1) and (x_2, y_2) respectively.

From points P_1, P and P_2 drawn P_1A, PB and P_2C perpendicular on x – axis. Also draw a line P_1M parallel to x-axis meeting PB at point L.

Now from the similar triangles P_1PL and P_1P_2M, we have

$$\frac{P_1L}{P_1M} = \frac{P_1P}{P_1P_2}$$

$$\frac{x - x_1}{x_2 - x_1} = \frac{r_1}{r_1 + r_2}$$

$$x - x_1 = \frac{r_1(x_2 - x_1)}{r_1 + r_2}$$

$$x = x_1 + \frac{r_1(x_2 - x_1)}{r_1 + r_2}$$

Or $$x = \frac{r_1x_2 + r_2 x_1}{r_1 + r_2} \; ; \; r_1 + r_2 \neq 0$$

Similarly, $$\frac{PL}{P_2M} = \frac{P_1P}{P_1P_2}$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{r_1}{r_1 + r_2}$$

$$y - y_1 = \frac{r_1(y_2 - y_1)}{r_1 + r_2}$$

$$y = y_1 + \frac{r_1(y_2 - y_1)}{r_1 + r_2}$$

Or $$y = \frac{r_1 y_2 + r_2 y_1}{r_1 + r_2} \; ; \quad r_1 + r_2 \neq 0$$

Hence coordinates of point P are:

$$\left(\frac{r_1 x_2 + r_2 x_1}{r_1 + r_2}, \frac{r_1 y_2 + r_2 y_1}{r_1 + r_2}\right)$$

Corollary 1:

If P divides P_1P_2 externally, then P_1 P and PP_2 are measured in opposite directions. So in the ratio $r_1 : r_2$ either r_1 or r_2 is negative. The corresponding coordinates of P for external point, are obtained just by giving negative sign to either r_1 or to r_2. So far external division the coordinates of P are.

$$\left(\frac{r_1\, x_2 - r_2\, x_1}{r_1 - r_2}, \frac{r_1\, y_2 - r_2\, y_1}{r_1 - r_2}\right) \qquad , \qquad \text{if } r_1 > r_2$$

Or
$$\left(\frac{r_2\, x_1 - r_2\, x_1}{r_1 - r_2}, \frac{r_1\, y_2 - r_2\, y_1}{r_2 - r_1}\right) \qquad , \qquad \text{if } r_2 > r_1$$

Corollary 2:

(Coordinate of Mid-point)

For the mid point P of the segment $P_1\, P_2$.

$r_1 = r_2 \Rightarrow$ 1 : 1. Therefore, the mid point P has the coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 7:

Find the coordinates of the mid point of the segment.

$P_1(3, 7)$, $P_2(-2, 3)$

Solution:

If (x, y) is the mid point of the segment, then

$$x = \frac{3 + (-2)}{2} = \frac{1}{2}$$

$$y = \frac{7 + 3}{2} = 5$$

$$P(x, y) = P\left(\frac{1}{2}, 5\right)$$

Example 8:

Find the coordinates of the point P which divides the segments P_1 (–2, 5), P_2(4, –1) in the ratio of

(a) $\frac{r_1}{r_2} = \frac{6}{5}$, (b) $\frac{r_1}{r_2} = -2$ (c) $\frac{r_1}{r_2} = -\frac{1}{3}$

Solution:

(a) $r_1 : r_2 = 6 : 5$

$$x = \frac{6(4) + 5(-2)}{6 + 5} = \frac{14}{11}$$

$$y = \frac{6(-1) + 5(5)}{6 + 5} = \frac{19}{11}$$

(b) $r_1 : r_2 = -2 : 1$

$$x = \frac{-2(4) + 1(-2)}{-2 + 1} = 10$$

$$y = \frac{-2(-1) + 1(5)}{-2 + 1} = -7$$

(c) $r_1 : r_2 = -1 : 3$

$$x = \frac{-1(4) + 3(-2)}{-1 + 3} = -5$$

$$y = \frac{-1(-1) + 3(5)}{-1 + 3} = 8$$

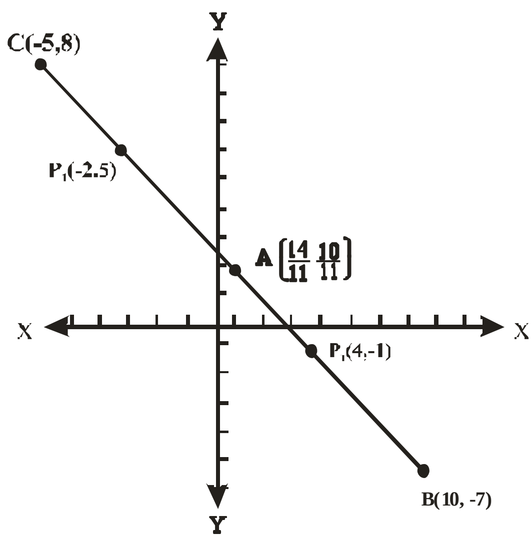


Figure 12.7

Example 9:

Find the ratio in which the line joining (–2, 2) and (4, 5)

is cut by the axis of y.

Solution:

Let the ratio be $r_1 : r_2$

$$x = \frac{4r_1 - 2r_2}{r_1 + r_2} \quad =$$

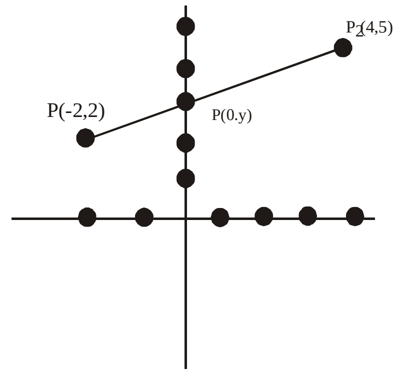


Figure 12.8

Since on y – axis, x = 0

So, $0 = \dfrac{4r_1 - 2r_2}{r_1 + r_2}$

Or $4r_1 - 2r_2 = 0$

Or $2r_1 = r_2$

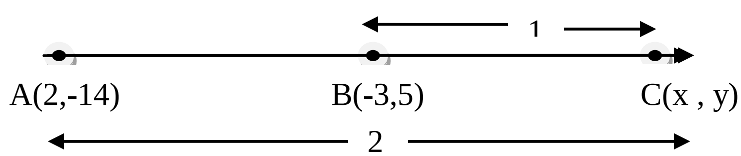
Or $\dfrac{r_1}{r_2} = \dfrac{1}{2}$

Or $r_1 : r_2 = 1 : 2$

Example 10:

Find the point reached by going from the point (2, –14) to the point (–3, 5) and then proceeding an equal distance beyond the latter point.

Solution:



$r_1 : r_2 = 2 : 1$

Let P (x, y) be a point which is to find.

$$x = \frac{r_1\, x_2 - r_2\, x_1}{r_1 - r_2} \quad = \quad \frac{2(-3) - 1(2)}{2 - 1} = -8$$

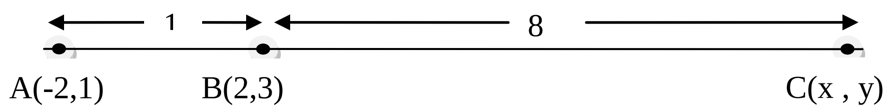
$$y = \frac{r_1 y_2 - r_2 y_1}{r_1 - r_2} = \frac{2(5) - 1(-14)}{2 - 1} = 24$$

Hence $P(x, y) = P(-8, 24)$

Example 11:

Let A (–2, 1), B(2, 3) and C (x, y) are collinear with B between A and C and if |BC| = 8|AB|,find the point C(x, y)

Solution:



Since $|BC| = 8\,|AB|$

$$\frac{|BC|}{|AB|} = \frac{8}{1}$$

So $|AC| : |CB| = 9 : 8 = r_1 : r_2$

Since. C(x, y) is external point, so by formula

$$x = \frac{r_1 x_2 - r_2 x_1}{r_1 - r_2}, \qquad y = \frac{r_1 y_2 - r_2 y_1}{r_1 - r_2}$$

$$x = \frac{9(2) - 8(-2)}{9 - 8}, \qquad y = \frac{9(3) - 8(-1)}{9 - 8}$$

$$x = 34, \qquad y = 35$$

Hence C (x, y) = C (34, 35)

Exercise 12.2

Q.1: Assuming that the points $P_1(2, 3)$, $P_2(4, 2)$ and $P_3(6, 1)$ are collinear, find the ratio $P_1P_2 : P_2P_3$.

Q.2: Obtain the ratio in which the point (3, – 2) divides the line joining the points (1, 4) and (– 3, 16).

Q.3: Find the point which is three fifth i.e. $\left(\frac{3}{5}\right)$ from the point (4,1) to the point (5,7).

Q.4: Find the point which is $\frac{7}{10}$ of the way from the point (4, 5) to the point (–6, 10).

Q.5: Find the point which is two third of the way from the point (5, 1) to the point (–2, 9)

Q.6: Let P(0, 4), Q (5, 0) and R(x, y) are collinear with P between R and Q and if |RP| = 10 |PQ|, find the coordinates of R(x, y).

Q.7: If A(–4, 2), B(6, –4) and C (x, y) are collinear with B between A and C and if |AC| = 5 |AB|, find the coordinates of C.

Q.8: Find the point of trisection of the median of the triangle with vertices at (– 1, –2), (4, 2) and (6, 3).

Q.9: Find the points trisecting the join of A(-1, 4) and B(6 , 2)

Q.10: Find the coordinates of the points that trisect the segment whose end points are (a, b) and (c, d).

Q.11: The mid points of the sides of a triangle are at (–1, 4), (5, 2) and (2, –1). Find its vertices.

Answers

Q.1: 1 : 1 internally **Q.2:** 1:3 externally

Q.3: $\left(\frac{23}{5}, \frac{23}{5}\right)$ **Q.4:** (–3, 8.5) **Q.5:**(1/3, 19/3)

Q.6: (–50, 44) **Q.7:** (46, –28) **Q.8:** (3, 1)

Q.9: (4/3 , 10/3) and (11/3 , 8/3)

Q.10: $\left(\frac{2a + c}{3}, \frac{2b + d}{3}\right); \left(\frac{a + 2c}{3}, \frac{b + 2d}{3}\right)$ **Q.11:** (–4, 1), (2,7)(8, –3)

12.8 Inclination and Slope of a Line:

The angle θ ($0 \leq \theta < 180^o$), measured counter clockwise from the positive x – axis to the line is called the inclination of the line or the angle of inclination of the line.

The tangent of this angle i.e. tan θ, is called the slope or gradient of the line. It is generally denoted by m. Thus m = tan θ, $0 \leq \theta < 180^o$. The slope of the line is positive or negative according as the angle of inclination is acute or obtuse.

Now $P_1 P_2 R$ is a right triangle,

Then $m = \tan\theta = \dfrac{P_1R}{P_2R}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Vertical cange}}{\text{Horizontal change}}$$

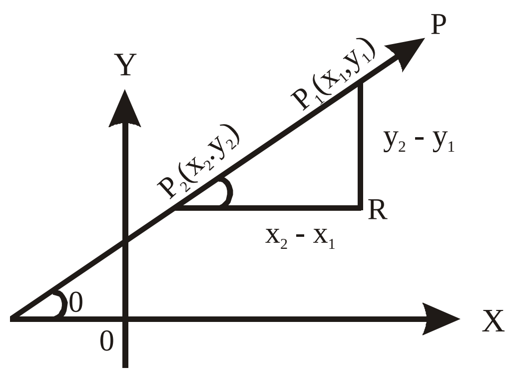


Figure 12.9

Note:

(i) $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$

(ii) The slope of a line parallel to x – axis is zero, because then $y_2 - y_1 = 0$.(fig.9)

(iii) The slope of x – axis is also zero.

(iv) The slope of a line parallel to y – axis is not defined, because then $x_2 - x_1 = 0$

(v) The slope of y – axis is also not defined.

12.9 Parallel and Perpendicular Lines:

The concept of slope is a convenient tool for studying parallel and perpendicular lines. The slopes of the vertical do not exist.

Theorem – I:

Two lines are parallel or coincide if and only if they have the same slope.

Proof:

Let ℓ_1 and ℓ_2 be two parallel lines. The inclinations of the lines are θ_1 and θ_2 respectively. Therefore

$$\theta_1 = \theta_2$$

Or $\tan\theta_1 = \tan\theta_2$

$m_1 = m_2$ i.e, the slopes of ℓ_1 and ℓ_2 are equal:

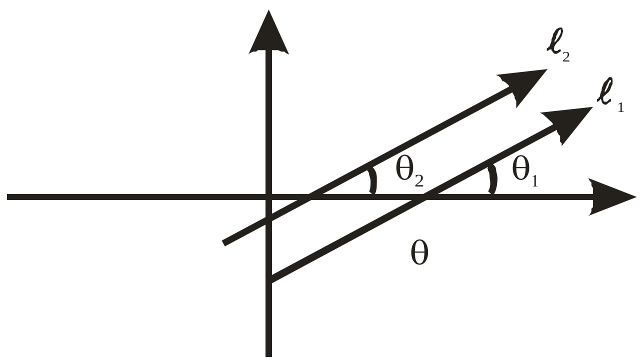


Figure 12.10

Conversely, if $m_1 = m_2$, then the two lines ℓ_1 and ℓ_2 are parallel.

Theorem 2:

Two lines are perpendicular if and only if the product of their slopes is –1.

Proof: Let ℓ_1 and ℓ_2 be two perpendicular lines with inclinations θ_1 and θ_2 respectively from fig.12.11

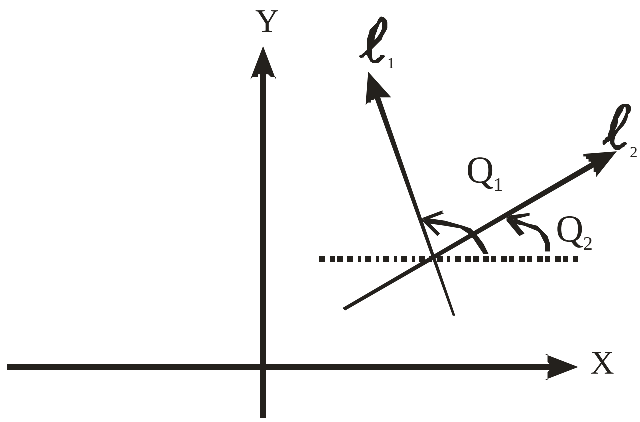


Figure 12.11

$$\theta_1 = 90^o + \theta_2$$

Or $\tan\theta_1 = \tan(90^o + \theta_2)$

Or $\tan\theta_1 = -\text{Cot}\,\theta_2$

Or $\tan\theta_1 = -\dfrac{1}{\tan\ \theta_2}$

Or $m_1 = -\dfrac{1}{m_2}$

Or $m_2\, m_1 = -1$

Conversely if $m_1 m_2 = -1$, then the two lines ℓ_1 and ℓ_2 are perpendicular.

Example 12: Find the slope of a line which is perpendicular to the line joining P_1 (2, 4), P_2(–2, 1).

Solution:

The slope of the line $P_1\, P_2$ is

$$m_1 = \frac{1-4}{-2-2} = \frac{3}{4}$$

Therefore the slope of a perpendicular line is $m_2 = -\dfrac{4}{3}$

Example 13: Show that the points A(–5, 3), B(6, 0) and C(5, 5) are the vertices of a right triangle.

Solution:

Slope of AB $= m_1 = \dfrac{0-3}{6+5} = -\dfrac{3}{11}$

$$\text{Slope of BC} = m_2 = \frac{5-0}{5-6} = -5$$

$$\text{Slope of AC} = m_3 = \frac{5-3}{5+5} = \frac{1}{5}$$

Since $m_2\, m_3 = -1$, so the sides AC and BC are perpendicular, with vertices C at right angle. Hence the given points are the vertices of a right triangle.

12.10 Angle Between Two Lines:

If θ is the angle between two lines ℓ_1 and ℓ_2 then from Fig.12

$$\theta \Rightarrow \theta_2 - \theta_1$$

$$\tan\theta = \tan(\theta_2 - \theta_1)$$

$$= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2\, m_1}$$

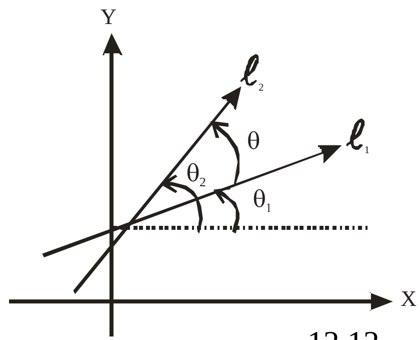


Figure 12.12

Example 14:

Find the angle from the line with slope $-\frac{7}{3}$ to the line with slope $\frac{5}{2}$

Solution:

Here $m_1 = -\frac{7}{3}$, $m_2 = \frac{5}{2}$

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2\, m_1} = = \frac{\frac{5}{2} - \left(-\frac{7}{3}\right)}{1 + \frac{5}{2}\left(-\frac{7}{3}\right)} = \frac{29}{-29} = -1$$

$$\theta = \tan^{-1}(-1) = 135^0$$

Example 15:

Show that the points (2 , 6) , (-8 , 1) and (-2 , 4) are collinear.

Solution:

Let the given points be A(2 , 6) , B (-8 , 1) and C (-2 , 4), then

$$\text{Slope of line AB} = \frac{1-6}{-8-2} = \frac{-5}{-10} = \frac{1}{2}$$

$$\text{Slope of line AC} = \frac{4-6}{-2-2} = \frac{-2}{-4} = \frac{1}{2}$$

$\therefore$ A , B and C are collinear

Exercise 12.3

Q.1: Show that the two lines passing through the given points are perpendicular.

(a) (0, −7), (8, – 5) and (5, 7), (8, – 5)

(b) (8, 0), (6, 6) and (–3, 3) , (6, 6)

Q.2: If a line ℓ_1 contains P (2, 6) and (0, y). Find y if ℓ_1 is parallel to ℓ_2 and that the slope of $\ell_2 = \frac{3}{4}$

Q.3: For the triangle A (1, 3), B(–2, 1), C(0, – 4), find

(a) Slope of a line perpendicular to AB.

(b) Slope of a line parallel to AC.

(c) Angle ABC

Q.4: Show that the given points are the vertices of a right triangle.

(a) (0, 6), (9, – 6) and (–3, 0)

(b) (1, –1), $\left(-\frac{39}{25}, 7\right)$ and $\left(\frac{29}{4}, 1\right)$

Q.5: Show that the given points are the vertices of a parallelogram.

(a) (–3, 1), (–1, 7), (2, 8) and (0, 2)

(b) (1, 0), (–2, –3), (2, –1) and (5, 2)

Q.6: Find the slopes of the sides and altitudes of the triangles whose vertices are the points (2, 3), (0, –1) and (–2, 1).

Q.7: Show that the points (2, 6), (–8, 1) and (–2, 4) are collinear by using slope.

Answers

Q.2: $\frac{9}{2}$ **Q.3**: (a) $-\frac{3}{2}$ (b) 7, (c) $\tan\theta = -\frac{19}{4}$

Q.6: $2, -1, \frac{1}{2}; -\frac{1}{2}, 1, -2$

12.11 Equation of a Straight Line:

Straight Line:

The line, in Euclidean geometry, which passes through two points in such a way that the length of the segment between the points is a minimum. Or the straight line is a curve with constant slope.

The equation of a line is an equation in x, y which is satisfied by every point of the line.

12.11.1 When Parallel to X – Axis:

Let ℓ be a line parallel to x – axis at a distance of 'b' units. Then the equation of the line ℓ is the locus of the point P (x, y) which moves such that it remains at a constant distance b units from the x – axis i.e., y coordinate of P is always equal to b. therefore.

y = b is the required equation, for example the equation of the line passing through (1,4)(2, 4) (3, 4) etc. is y = 4 or y – 4=0

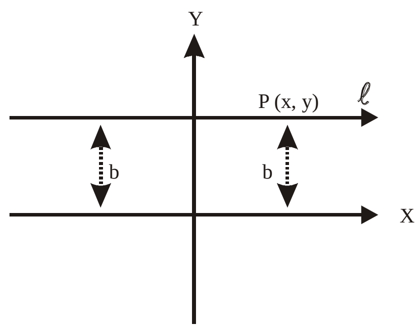


Figure 12.13

12.11.2. When parallel to Y – axis:

Let ℓ be a line parallel to Y – axis at a distance of 'a' units. Then the equation of the line ℓ is the locus of the point P (x, y) which moves such that it remains at a constant distance a units from the Y – axis i.e., X-Coordinates of P is always equal to a. Therefore.

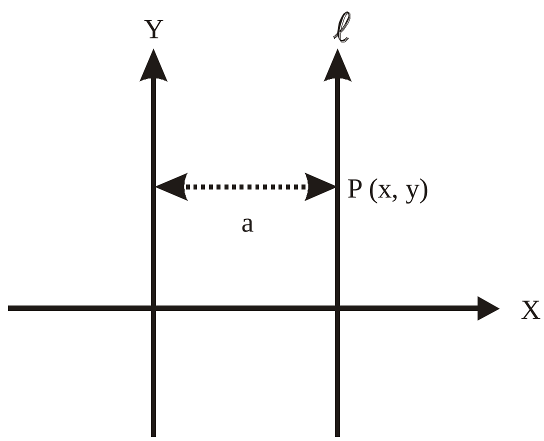


Figure 12.14

x = a is the required equation, for example the equation of the line passing through (3, 1), (3, 3) (3, – 5) etc. is x = 3 or x – 3 = 0

12.12 Three Important Forms of the Equation of a Line:

12.12.1 Point – slope form:

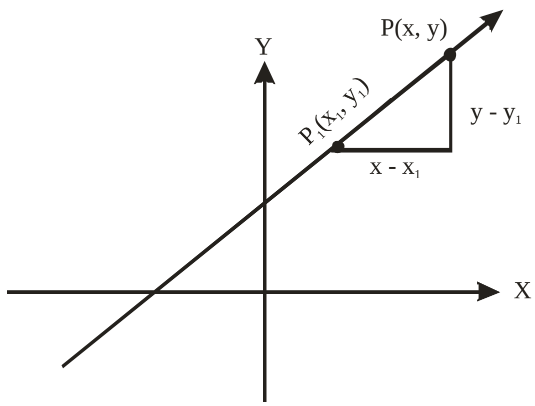
Suppose a line having slope m and passing through a given point (x_1, y_1) as shown in Fig. 15. If P(x, y) is any other point on the line, then the slope of the line is

$$\frac{y - y_1}{x - x_1} = m$$

From which $y - y_1 = m(x - x_1)$

This is called point slope form for a linear equation.

(1)



Figu 12.15

Corollary I:

Slope – Intercept Form:

Suppose a line having slope m, passing through a given point on the y – axis having coordinates (0, c) as show in Fig.16 substituting (0, C) in the point slope form of a linear equation.

$y - y_1 = m(x - x_1)$

(2)

We obtain $y - c = m(x - 0)$

From which $y = mx + c$

This equation is called slope – intercept form.

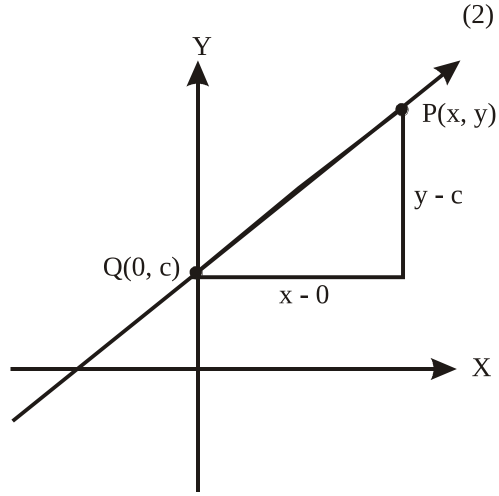


Figure 8.16

Where, m is the slope and C is the y- Intercept
Equation of the line passing through the origin is y = mx.

Corollary – II:

Two Point Form:

Suppose a line is passing through the points (x_1, y_1) and (x_2, y_2), then slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad 12.16$$

Using point slope form of a linear equation

$$y - y_1 = m(x - x_1)$$

We obtain $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad (3)$$

This is the two point form of the linear equation.

Example 14: Find the equation of the line passing through the point (–2, 3) and having slope $-\frac{1}{2}$

Solution:

By point – slope form, its equation is

$$y - 3 = -\frac{1}{2}(x + 2)$$

Or $2y - 6 = -x - 2$

Or $x + 2y - 4 = 0$

Example 15: Find an equation of the line with slope $-\frac{2}{3}$ and having y- intercept 3.

Solution:

By sloping-intercept form, we have

$$y = -\frac{2}{3}x + 3$$

Or $3y = -2x + 9$

Or $2x + 3y - 9 = 0$

Example 16: Write the equation, in standard form, of the line with the same slope as 2y – 3x = 5 and passing through (0, 5).

Solution:

Write the equation $2y - 3x = 5$ in slope- intercept form

$$y = \frac{3}{2}x + \frac{5}{2}\ , \quad \text{the slope } m = \frac{3}{2}$$

Now by point-slope form (Or slope- intercept form)

We have, $y - 5 = \frac{3}{2}\ (x - 0)$

$2y - 10 = 3x$

Or $3x - 2y + 10 = 0$

Example 17: Find the equation of the line through (–1, 2) and (3, –4)

Solution:

Equation of the line through two point is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

So $\frac{y - 2}{-4 - 2} = \frac{x + 1}{3 + 1}$

Or $\frac{y - 2}{-6} = \frac{x + 1}{4}$

Or $6x + 4y - 2 = 0$

Or $3x + 2y - 1 = 0$

OR Alternatively, slope of (–1, 2) and (3, –4) is

$$m = \frac{-4 - 2}{3 + 1} = \frac{-6}{4} = \frac{-3}{2}$$

Now by point-slope form, point (–1, 2), (Or (3, –4))

We have $y - 2 = -\frac{3}{2}\ (x + 1)$

$2y - 4 = -3x - 3$

Or $3x + 2y - 1 = 0$

12.12.2 Intercept Form:

Suppose a and b are the x and y-intercepts of a straight line of point A and B respectively. Then the coordinates of point A and B are (a,0) and (0, b) respectively.

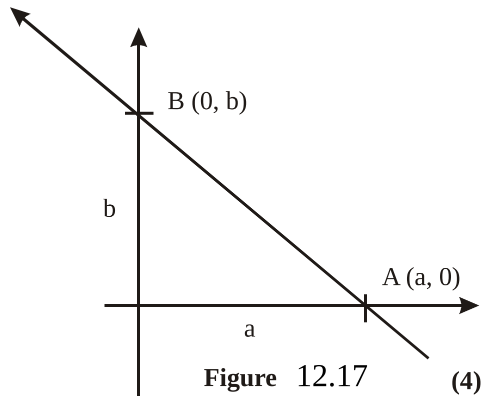


Figure 12.17 **(4)**

Therefore the slope of AB.

$$m = \frac{b-0}{0-a} = -\frac{b}{a}$$

Using point-slope from

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{b}{a}(x - a)$$

Divide both sides by b,

$$\text{Or} \quad \frac{y}{b} = -\frac{x}{a} + 1$$

$$\text{Or} \quad \frac{x}{a} + \frac{y}{b} = 1$$

This is called intercept form of a linear equation.

Example 18: Find the equation of the line passing through (–8,–1) and making equal intercepts on the coordinate axes.

Solution:

Here a = b, so equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1$$

Putting (x, y) = (–8, –1), we have

$$\frac{-8}{a} + \frac{-1}{a} = 1$$

$$\frac{-9}{a} = 1$$

$$a = -9$$

$$\frac{x}{-9} + \frac{y}{-9} = 1$$

$$\text{Or} \quad x + y + 9 = 0$$

Is the required equation.

Example 19: A line makes the positive intercepts on coordinate axes whose sum is 7. It passes through (– 3, 8) find the equation.

Solution:

If a and b are the positive intercepts, then

$$a + b = 7$$

Or $$b = 7 - a$$

By the Equation $$\frac{x}{a} + \frac{y}{b} = 1$$

We have $$\frac{x}{a} + \frac{y}{7-a} = 1$$

As this line is passing through the point (–3, 8), so

$$\frac{-3}{a} + \frac{8}{7-a} = 1$$

$$-21 + 3a + 8a = 7a - a^2$$

Or $$a^2 + 4a - 21 = 0$$

By solving this equation we get,

$$a = 3 \quad \text{or} \quad a = -7$$

Since a is positive, so $a = 3$

and $b = 4$

Hence $$\frac{x}{3} + \frac{y}{4} = 1$$

$$4x + 3y = 12$$

Or $4x + 3y - 12 = 0$ is the required equation.

12.12.3 Perpendicular or Normal Form:

Suppose P is the perpendicular length from the origin O to the point A on the line ℓ and θ is the angle of inclination of perpendicular P. The equation of the line ℓ which is passing through the point A can be found in terms of P and θ.

The coordinates of A are $(p \cos \theta, p \sin \theta)$.

The slope of AO is tan θ, since the line ℓ is perpendicular to OA, so slope of line ℓ is $-\dfrac{1}{\tan\theta} = -\cot\theta$

Using point slope form.

$$y - y_1 = m(x - x_1)$$

We have, $y - p\sin\theta = -\cot\theta\,(x - p\cos\theta)$

Or $y - p\sin\theta = -\dfrac{\cos\theta}{\sin\theta}(x - p\cos\theta)$

Or $y\sin\theta - p\sin^2\theta = -x\cos\theta + p\cos^2\theta$

Or $x\cos\theta + y\sin\theta = p$

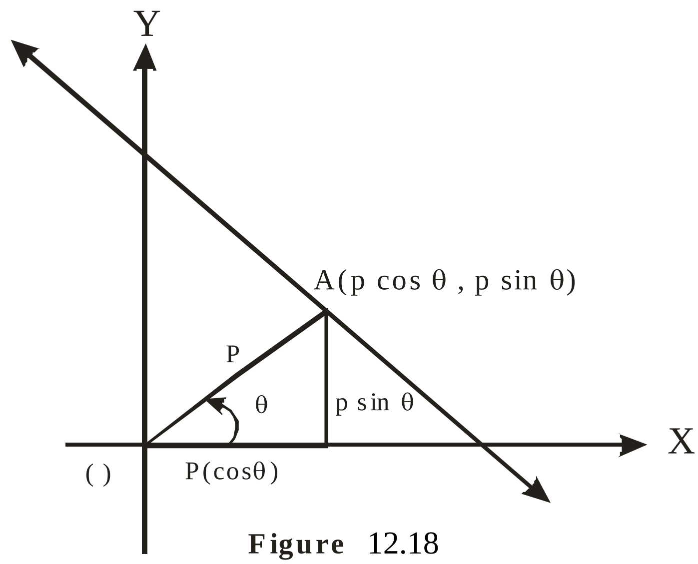


Figure 12.18

This is the perpendicular or Normal form of a linear equation.

Example 20: Find the equation of the line when $\theta = 45^o$ and $p = \dfrac{1}{\sqrt{2}}$.

Solution:

Since the equation of the Normal form is

$$x \cos\theta + y \sin\theta = p$$

Putting $\theta = 45^o$, and $p = \frac{1}{\sqrt{2}}$

$$x \cos 45^o + y \sin 45^o = \frac{1}{\sqrt{2}}$$

$$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Or $x + y = 1$ or $x + y - 1 = 0$ is the required equation.

Exercise 12.4

Q.1: Find equation for the lines:

(a) through (4 , 2) and (–5, –1)

(b) through (–1 , –1) with slope $-\frac{1}{2}$

(c) through $\left(-\frac{7}{3} \quad 0\right)$ and $\left(-\frac{5}{2} \quad 0\right)$

(d) through (–1, –2) and parallel to y – axis.

Q.2: Find the slope and y – intercept:

(a) $ax + by = b,\ b \neq 0$ (b) $\sqrt{2}\, x + (1 - \sqrt{2})\, y = 2$

Q.3: Determine the real number k so that the two lines $5x - 3y = 12$ and $kx - y = 2$ will be

(a) parallel (b) Perpendicular

Q.4: Show that the given points are collinear:

(a) (1, 0), (–4, – 12) and (2, –4)

(b) (– 4,4), (– 2,1) and (6, – 11)

Q.5: Find the equations of the medians of the triangle with vertices (–4, –6), (0, 10), (4, 2).

Q.6: Find the equations of the three altitudes of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3)

Q.7: A triangle has vertices at (0, 0), (a, b) and (c, d). Show that a line containing the mid points of the two of the sides of the triangle is parallel to the third side.

Q.8: Find the equation of the line which is perpendicular to the line $x + 2y = 7$ and having y – intercept 3.

Q.9: A line is parallel to the line $2x + 3y = 5$ and passes through (–1, 3). Find an equation for the line.

Q.10: Write an equation of the line parallel to $2x - 7y = 8$ and containing the origin.

Q.11: Find the line which is perpendicular to the line $4x + 7y = 5$ and which passes through (–1, 2).

Q.12: What is an equation of a line perpendicular to $5x - y = 4$ and containing the point (2, 3)?

Q.13: Find the equation of the line passing through (–1, 7) and perpendicular to the line through the points (2, 3) and (o, – 4).

Q.14: What are the x and y-intercepts of $3x + 4y = 12$?

Q.15: Find the equation of the line whose intercept on x- axis is three times its intercept on y-axis and which passes through the point (–1, 3).

Q.16: A line makes the negative intercepts on the coordinates axes whose sum is –10. It passes through (4, –9), find its equation.

Q.17: Find the equation of the perpendicular bisector of the line segment joining the points.

(a) (2, 4) and (6, 8) (b) (–4, 6) and (6, 10)

(c) 3, –2) and (5, 4)

Answers

Q.1: (a) $x - 3y + 2 = 0$ (b) $x + 2y + 3 = 0$

(c) $y = 0$ (d) $x = -1$

Q.2: (a) $m = \frac{-a}{b}$; y – intercept = 1

(b) $m = -\frac{\sqrt{2}}{1-\sqrt{2}}$; y – intercept $= \frac{2}{1-\sqrt{2}}$

Q.3: (a) $k=\frac{5}{3}$ (b) $-\frac{3}{5}$

Q.5: Equations of the medians : $x = 0$, $y = 2$ and $2x - y = -2$

Q.6: $(y_2 - y_1)\, y + (x_2 - x_1)\, x = y_3\,(y_2 - y_1) + x_3(x_2 - x_1)$,

$(y_3 - y_2)\, y + (x_3 - x_2)\, x = y_1\,(y_3 - y_2) + x_1\,(x_3 - x_2)$,

$(y_3 - y_1)\, y + (x_3 - x_1)\, x = y_2\,(y_3 - y_1) + x_2\,(x_3 - x_1)$

Q.8: $2x - y + 3 = 0$ **Q.9**: $2x + 3y = 7$

Q.10: $2x - 7y = 0$ **Q.11:** $7x - 4y + 15 = 0$

Q.12: $x + 5y = 17$ **Q.13:** $2x + 7y - 47 = 0$

Q.14: $a = 4$, $b = 3$ **Q.15:** $x + 3y - 8 = 0$

Q.16: $x + y + 5 = 0$

Q.17: **(a)** $x + y=10$ **(b)** $5x - 8y = 21$

(c) $3x - y + 11 = 0$

12.13 The General Linear Equation:

A linear equation in x and y is an equation of the form

$$Ax + By + C = 0 \qquad (1)$$

Where A and B are given real numbers and A and B are not both zero. The equation (1) is called the General linear equation because the graph of such on equation is always a straight line.

Theorem:

Every linear equation has a graph which is a straight line.

Proof:

Suppose we have a linear equation (first degree) in variables x and y.

$$Ax + By + C = 0$$

(i) If $B = 0$, then $A \neq 0$ and the equation is

$$Ax + C = 0$$

Or $$x = -\frac{C}{A}$$

The graph of this equation is a line parallel to the y – axis

(ii) If $B \neq 0$, then $A = 0$ and the equation is

$$By + C = 0$$

Or $$y = -\frac{C}{B}$$

The graph of this equation is a line parallel to the x – axis.

(iii) If $A \neq 0$, $B \neq 0$, then equation can be written in the form.

$$y = -\frac{A}{B}x - \frac{C}{B}$$

Again the graph of this equation is a straight line with slope $m = -\frac{A}{B}$ and y – intercept $c = -\frac{C}{B}$

Therefore, in all cases, the linear equation $Ax + By + C = 0$ represents a straight line.

12.14 <u>Reduction of General form Ax + By + C = 0 to other forms.</u>

(i) Reduction to slope – Intercept Form:

$$Ax + by + C = 0$$

Reduce to, $y = -\frac{A}{B}x - \frac{C}{B}$

Which is of the form $y = mx + c$

(ii) Reduction to Intercept Form:

$$Ax + By + C = 0$$

Or $Ax + By - C$

Or $\frac{A}{-C} + \frac{B}{-C} = 1$

Or $\frac{\frac{x}{-C}}{A} + \frac{\frac{y}{-C}}{A} = 1$

Which is of the form $\frac{x}{a} + \frac{y}{b} = 1$

(iii) Reduction to Perpendicular Form:

Comparing the equation

$$Ax + By + C = 0 \qquad (1)$$

With the perpendicular form

$$\text{Cos } \alpha.\ x + \sin\alpha\ y - P = 0 \qquad (2)$$

We have, $\frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = -\frac{C}{P} = k$

Because the co-efficients of (1) and (2) are proportional.

So $A = k\cos\alpha$, $B = k\sin\alpha$ and $C= -pk$ or $p = -\frac{C}{k}$

$$A^2 + B^2 = k^2(\cos^2\alpha + \sin^2\alpha)$$

Or $A^2 + B^2 = k^2$

Or $k = \pm\sqrt{A^2 + B^2}$

So $p = -\left(\frac{C}{\pm\sqrt{A^2 + B^2}}\right)$

As p must be positive, so sign of C and $\sqrt{A^2 + B^2}$ are opposite.

If C is positive then $k = -\sqrt{A^2 + B^2}$

Therefore $\cos\alpha = \frac{A}{K} = -\frac{A}{\sqrt{A^2 + B^2}}$

$\cos\alpha = \frac{B}{K} = -\frac{B}{\sqrt{A^2 + B^2}}$

and $p = -\left(\frac{C}{-\sqrt{A^2 + B^2}}\right) = \frac{C}{\sqrt{A^2 + B^2}}$

Putting these values in eq. (2), we get

$$-\frac{A}{\sqrt{A^2 + B^2}}\,x - \frac{B}{\sqrt{A^2 + B^2}}\,y - \frac{C}{\sqrt{A^2 + B^2}} = 0$$

Or $$\frac{A}{\sqrt{A^2 + B^2}}\,x + \frac{B}{\sqrt{A^2 + B^2}}\,y + \frac{C}{\sqrt{A^2 + B^2}} = 0$$

If C is negative, then $k = \frac{A}{\sqrt{A^2 + B^2}}$

So, $\cos\alpha = \frac{A}{\sqrt{A^2 + B^2}}$, $\sin\alpha = \frac{B}{\sqrt{A^2 + B^2}}$, $p = -\frac{C}{\sqrt{A^2 + B^2}}$

Putting these values in equation (2), we get.

$$\frac{A}{\sqrt{A^2 + B^2}}\,x + \frac{B}{\sqrt{A^2 + B^2}}\,y + \frac{C}{\sqrt{A^2 + B^2}} = 0$$

Hence the equation Ax + by + C = 0 is reduced in perpendicular form by dividing $+\sqrt{A^2 + B^2}$ or $-\sqrt{A^2 + B^2}$ accordingly C is negative or positive respectively.

Example 22: Reduce the equation 3x + 4y = 10 to the
(i) Slope-intercept from (ii) Intercept form
(ii) Normal form

Solution: (i)

$$3x + 4y = 10$$

$$4y = -3x + 10$$

$$y = -\frac{3}{4}x + \frac{5}{2}$$

Which is the slope-intercept form with slope $m = -\frac{3}{4}$, y – intercept = $C = \frac{5}{2}$.

(iii) $3x + 4y = 10$

$$\frac{3x}{10} + \frac{4y}{10} = 1$$

Or $$\frac{x}{\frac{10}{3}} + \frac{x}{\frac{5}{2}} = 1$$

Which is the intercepts form $\frac{x}{a} + \frac{y}{b} = 1$

With x – intercept $a = \frac{10}{3}$ and y – intercept $b = \frac{5}{2}$

(iii) $3x + 4y = 10$

$$k = \sqrt{A^2 + B^2} = \sqrt{9 + 16} = 5$$

Divide the equation by 5 $\Rightarrow \frac{3}{5}x + \frac{4}{5}y = 2$

Which is the perpendicular form

$$X \cos \alpha + y \sin \alpha = p$$

With $\cos \alpha = \frac{3}{5}$, $\sin \alpha = \frac{4}{5}$ and $p = 2$

$$\tan \alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1} \frac{4}{3}$$

Hence the given line is at a distance of 2 unit from the origin and is perpendicular from the origin on the line with angle of inclination $\alpha = \tan^{-1} \frac{4}{3}$.

12.15 Intersection of Two Lines:

Let $a_1x + b_1y + c_1 = 0$ (1)

$a_2x + b_2y + c_2 = 0$ (2)

be the two lines. Their point of intersection can be obtained by solving them simultaneously.

$$\frac{x}{b_1c_2 - b_2c_1} = -\frac{y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

Hence point of intersection is

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}\right)$$

If $a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ then the above coordinates have no meaning and the lines do not intersect but are parallel.

Hence the lines (1) and (2) intersect if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

12.16 Concurrent Lines and Point of Concurrency:

Three or more than three lines are said to be concurrent if these are intersecting at the same point. The point of intersection of these lines is called point of concurrency.

12.16.1 Condition of Concurrency of Three Lines:

Suppose the three line are

$$a_1x + b_1y + c_1 = 0 \qquad (1)$$

$$a_2x + b_2y + c_2 = 0 \qquad (2)$$

$$a_3x + b_3y + c_3 = 0 \qquad (3)$$

These lines will be concurrent if the point of intersection of any two lines satisfies the third line.

The point of intersection of (2) and (3) is

$$\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2} \quad , \frac{a_3c_2 - a_2c_3}{a_2b_3 - a_3b_2}\right)$$

Putting this point in equation (1)

$$a_1\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}\right) + b_1\left(\frac{a_3c_2 - a_2c_3}{a_2b_3 - a_3b_2}\right) + c_1 = 0$$

$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1\,(a_2b_3 - a_3b_2) = 0$

This equation can be written in the determinant form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Example 23: Show that the three lines

$x - y + 6 = 0$ ……………………….(1)

$2x + y - 5 = 0$ ……………………….(2)

$-x - 2y + 11 = 0$ ……………………….(2)

are concurrent. Also find the point of concurrency

Solution:

Since, $\begin{vmatrix} 1 & -1 & 6 \\ 2 & 1 & -5 \\ -1 & -2 & 11 \end{vmatrix}$

$= 1\,(11 - 10) + 1\,(22 - 5) + 6\,(-4 + 1)$

$= 1 + 17 - 18 = 0$

Hence the lines are concurrent.

For the point of concurrency, solving equations (1) and (2)
Adding equations (1) and (2), we get

$3x + 1 = 0$

Or $x = -\dfrac{1}{3}$, Put in (1)

$$y = \frac{1}{3} + 6 = \frac{19}{3}$$

So point of concurrency is $\left(-\frac{1}{3} \quad \frac{19}{3}\right)$

Example 24: Find K so that the lines

$x - 2y + 1 = 0$, $2x - 5y + 3 = 0$ and $5x + 9y + k = 0$ are concurrent.

Solution:

Since the lines are concurrent, so

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & -5 & 3 \\ 5 & 9 & k \end{vmatrix} = 0$$

$$1(-5k - 27) + 2(2k - 15) + 1(18 + 25) = 0$$

$$-5k - 27 + 4k - 30 + 43 = 0$$

$$-k - 14 = 0$$

Or $k = -14$

12.16.2 Condition that Three Points be Collinear:

The three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) will be collinear

If $$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

Or $(y_2 - y_1)(x_3 - x_2) = (y_3 - y_2)(x_2 - x_1)$

Or $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example 25: Show that the three points (1, 2), (7, 6), (4, 4) are collinear.

Solution:

Since $$\begin{vmatrix} 1 & 2 & 1 \\ 7 & 6 & 1 \\ 4 & 4 & 1 \end{vmatrix} = 0$$

$$= 1\,(6 - 4) - 2(7 - 4) + 1\,(28 - 24)$$

$$= 2 - 6 + 4 = 0$$

Therefore the points are collinear i.e., these lie on a line.

Exercise 12.5

Q.1: Reduce the given equations to

(i) Slope-intercept form (ii) Intercepts form
(iii) Normal form

(a) $3x + y = y^2$ (b) $6x - 5y = 15$

(c) $\sqrt{3} + \sqrt{6}\,y = \sqrt{2}$

Q.2: Determine p and α for the lines:
(a) $x - 5y + 3 = 0$ (b) $x + y + 3\sqrt{2} = 0$

Q.3: Show that the following lines are concurrent. Also find the point of concurrency.

(a) $3x - 5y + 8 = 0$, $x + 2y - 4 = 0$ and $4x - 3y + 4 = 0$

(b) $2x - 3y - 7 = 0$, $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$

(c) $5x + y + 11 = 0$, $x + 7y + 9 = 0$ and $2x + y + 5 = 0$

Q.4: Find K so that $x + y + 1 = 0$

$kx - y + 3 = 0$

and $4x - 5y + k = 0$ will be concurrent.

Q.5: Show that the altitudes of the triangle whose vertices are (–1, 2), (4, 3) and (1, –2) intersect at a point. Find the coordinates of the point of intersection.

(Hint: First find the equations of the altitudes, then show them concurrent).

Q.6: Find the equations of the medians of the triangle with vertices (–4, –6), (0, 10), (4, 2). Show that the medians meet in a point.

Q.7: Show that the given points are collinear

(a) (–4, 4), (–2,1) and (6, –11)

(b) (1, 9), (–2, 3) and (–5, –3)

Q.8: Find the value of k so that (1, –3), (–2, 5), (4, k) lie on a line.

Answers

Q.1: (a) (i) $y = -3x - 2$ (ii) $\frac{x}{-2/3} + \frac{y}{-2} = 1$

(iii) $\frac{-3}{2}x - \frac{y}{2} = 1$

(b) (i) $y = \frac{6}{5}x - 3$ (ii) $\frac{x}{15/6} + \frac{y}{-3} = 1$

(iii) $\frac{6}{\sqrt{61}}x - \frac{5}{\sqrt{61}}y, = \frac{15}{\sqrt{61}}$

(c) (i) $y = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{3}}$ (ii) $\frac{x}{\sqrt{\frac{2}{3}}} + \frac{y}{\sqrt{\frac{1}{3}}} = 1$

(iii) $\frac{1}{\sqrt{3}}x + \sqrt{\frac{2}{3}}\,y = \frac{\sqrt{2}}{3}$

Q.2: (a) $p = \frac{3}{\sqrt{26}}$, $\alpha = 92^o\,34'$ (b) $p = 3$, $\alpha = 225^o$

Q.3: (a) $\left(\frac{4}{11}, \frac{20}{11}\right)$ (b) (11,5) (c) (–2, 1)

Q.4: $k = -3 \pm 2\sqrt{10}$

Q.5: $\left(\frac{4}{11}, \frac{13}{11}\right)$ is the point of intersection.

Q.6: Equation of the medians : $x = 0$, $y = 2$ and $2x - y = -2$. The medians intersect at (0, 2).

Q.8: $k = -11$

12.17 The Distance from a Point to a Line:

To find the distance from a point $P_1(x_1, y_1)$ to the line $Ax + By + C = 0$, draw $P_1P_2 = d$ perpendicular on the line. The coordinates of P_2 are (x_2, y_2).

By distance formula

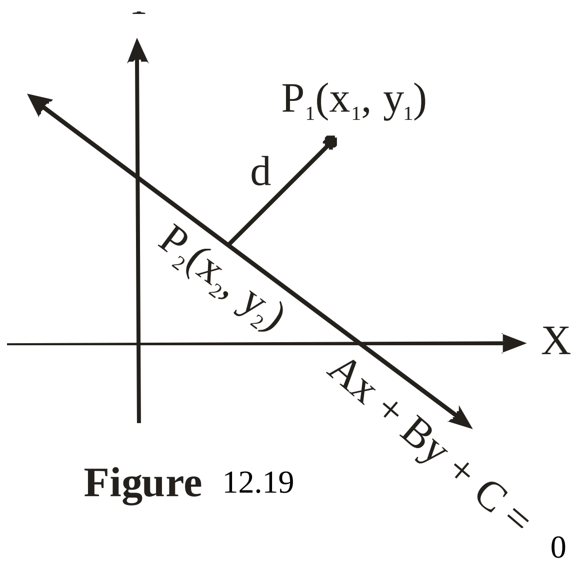


Figure 12.19

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (1)$$

We find the points $P_2(x_2, y_2)$

Since point $P_2(x_2, y_2)$ lies on the line $Ax + By + C = 0$

So $\quad Ax_2 + By_2 + C = 0 \qquad (2)$

From equation $Ax + By + C = 0$

$$y = -\frac{A}{B}x - \frac{C}{B}$$

The slope of lines is $-\dfrac{A}{B}$

The slope of perpendicular $\overline{P_1P_2}$ is m = $\frac{B}{A}$

The equation of perpendicular $\overline{P_1P_2}$ passing through (x_2, y_2) and (x_1, y_1).

Slope of P_1 P_2 = $\frac{B}{A}$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{B}{A}$$

Or $$(y_2 - y_1) = \frac{B}{A}(x_2 - x_1)$$

$$Ay_2 - Ay_1 = Bx_2 - Bx_1$$

Or $$Bx_2 - Ay_2 + Ay_1 - Bx_1 = 0 \qquad (3)$$

Now solving equation (2) and (3) for x_2 and y_2

$$Ax_2 + By_2 + C = 0 \qquad (2)$$

$$Bx_2 - Ay_2 + Ay_1 - Bx_1 = 0 \qquad (3)$$

$$\frac{x_2}{ABy_1 - B^2x_1 + AC} = -\frac{y^2}{A^2y_1 - ABx_1 - BC} = \frac{1}{-A^2 - B^2}$$

$$x_2 = \frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}$$

$$y_2 = \frac{A^2y_1 - ABx_1 - BC}{A^2 + B^2}$$

Putting x_2 and y_2 in equation (1)

$$|P_1 P_2| = \sqrt{\left(\frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2} - x_1\right)^2 + \left(\frac{A^2y_1 - ABx_1 - BC}{A^2 + B^2} - y_1\right)^2}$$

$$= \sqrt{\left(\frac{-ABy_1 - AC - A^2x_1}{A^2 + B^2}\right)^2 + \left(\frac{-ABx_1 - BC - B^2y_1}{B^2 - A^2y_1 - B^2y_1}\right)^2}$$

$$= \sqrt{\frac{\{-A\,(Ax_1 + By_1 + C)\}^2\ \} + \{\,-B\,(Ax_1 + By_1 + C)\}^2}{(A^2 + B^2)^2}}$$

$$= \sqrt{\frac{A^2\,(Ax_1 + By_1 + C)\}^2 + B^2\,(Ax_1 + By_1 + C)^2}{(A^2 + B^2)^2}}$$

$$= \sqrt{\frac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}}$$

Or $\quad d = |P_2\,P_2| \quad = \quad \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \qquad (4)$

Which is the required distance from a point P_1 (x_1, y_1) on the line $Ax + By + C = 0$

Remarks

If we take the expression

$$\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}$$

The Numerator of this expression i.e., $Ax_1 + By_1 + C$ will be positive, negative or zero depending upon the relative positions of the point $P_{1,}$ the line, and the origin. If P_1 (x_1, y_1) is any point, and

$$\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}$$

has the same sign as $\qquad \dfrac{C}{\sqrt{A^2 + B^2}}$

Then P_1 (x_1, y_1) and the origin are on the same side of the line. If the signs are different, then P_1 (x_1, y_1) is on the opposite side of the line from the origin.

For P_1 $(0, 0)$, we get the directed distance from the origin to the line, which may be positive, negative or zero.

Example 26: Find the distance from the point (–3, 2) to the line $2x - y + 4 = 0$, Is (–3, 2) on the same side of the line as the origin, or is it on the opposite side?

Solution:

$$d = \frac{|2(-3) - 2 + 4|}{\sqrt{2^2 + (-1)^2}} = \frac{|-4|}{\sqrt{5}}$$

$$d = \frac{4}{\sqrt{5}}$$

Because $\dfrac{Ax_1 - By_1 + C}{\sqrt{A^2 + B^2}} = \dfrac{-4}{\sqrt{5}}$

and $\dfrac{C}{\sqrt{A^2 + B^2}} = \dfrac{4}{\sqrt{5}}$

has different signs, so the point (–3, 2) and the origin are on opposite side of the line $2x - y + 4 = 0$ as shown in Fig. 20.

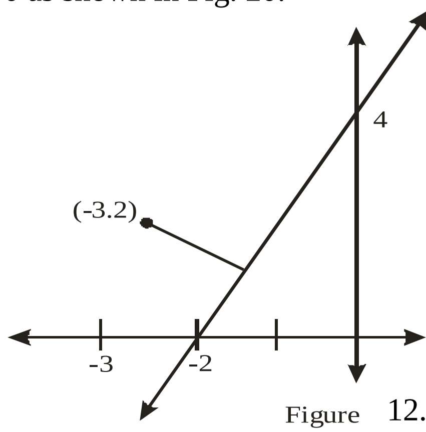


Figure 12.20

Example 27: An equation of a line is $4x - 3y + 12 = 0$. Find the coordinates of the point P_o (x_o, y_o) which is the foot of the perpendicular from the origin to the line.

Solution:

Since point P_o (x_o, y_o)

Lies on the line $4x - 3y + 12 = 0$

So,

$4x_o - 3y_o + 12 = 0$ …………..(1)

The slope of line (1) is

$y_o = \frac{4}{3} x_o + 4$

$m = \frac{4}{3}$

The slope of the perpendicular $\overline{OP}$ is

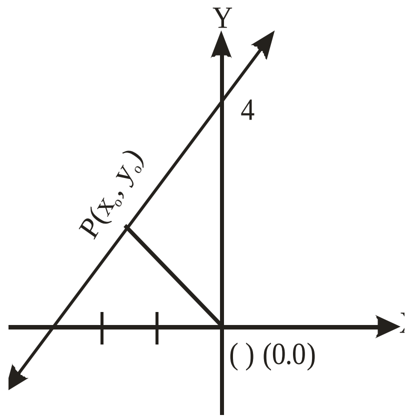


Figure 12.21

$$\frac{y_o - 0}{x_o - 0} = -\frac{3}{4}$$

Or $3x_o + 4y_o = 0$ ……………………..(2)

Solving equation (1) and (2) together, we find

$$\frac{x_o}{0-48} = -\frac{y_o}{0-36} = \frac{1}{16+9}$$

$$x_o = -\frac{48}{25}, \quad y_o = \frac{36}{25}$$

Hence the coordinates of P_o are : $\left(-\frac{48}{25} \quad \frac{36}{25}\right)$

Example 28: If P is the perpendicular distance of the origin from a line whose intercepts on the axes are a and b, show that.

$$\frac{1}{P^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

Solution:

The equation of the line in intercepts form is

$$\frac{x}{a} = \frac{y}{b} = 1$$

Or $\quad xb + ay - ab = 0$

If P is the perpendicular distance from the origin 0 (o,o) on the line bx + ay – ab = 0.

Then $\quad P = \frac{|b(o) + a(o) - ab|}{\sqrt{b^2 + a^2}} = \frac{|-ab|}{\sqrt{b^2 + a^2}}$

Or $\quad P = \frac{ab}{\sqrt{b^2 + a^2}}$

Or $\quad P^2 = \frac{a^2\, b^2}{b^2 + a^2}$

Or $\quad \frac{1}{P^2} = \frac{a^2 + b^2}{b^2\, a^2}$

Or $\quad \frac{1}{P^2} = \frac{1}{a^2} + \frac{1}{b^2}$

Example 29: Find the equations of the two lines (or find the locus of a point) which are parallel to and 3 units from the perpendicular bisector of the line segment (1, – 2), (– 3, 8).

Solution:

Mid-point of the segment (1, **–3)**

Slope of the segment is:

$$m = \frac{8 + 2}{-3\ -1} \quad = \quad \frac{10}{-4}$$

$$m = -\frac{5}{2}$$

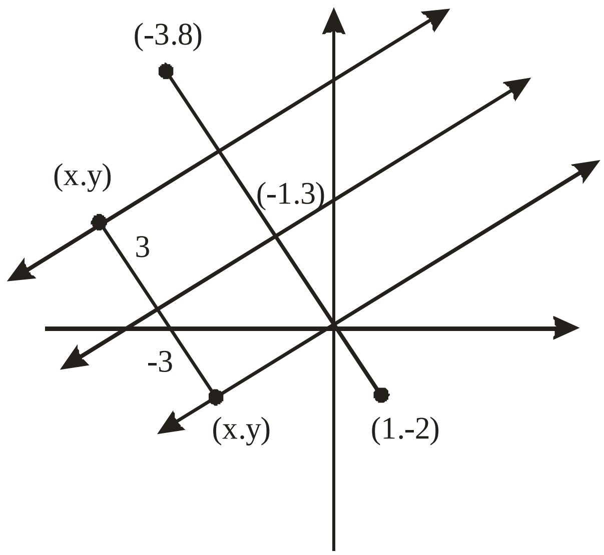


Figure 12.22

Slope of the perpendicular bisector is $m_1 = \frac{2}{5}$

Equation of the perpendicular bisector passing through (–1, 3)

$$y - 3 = \frac{2}{5}\ (x + 1)$$

$$5y - 15 = 2x + 2$$

$$2x - 5y + 17$$

Since the required equations are of a distance of 3 units from the perpendicular bisector $2x - 5y + 17 = 0$. So by the formula.

$$d = \frac{|Ax_1 + By + C|}{\sqrt{A^2 + B^2}}$$

$$3 = \frac{|2x - 5y + 17|}{\sqrt{2^2 + (-5)^2}}$$

$$3 = \frac{|2x - 5y + 17|}{\sqrt{29}}$$

Or $\quad |2x - 5y + 17| = 3\sqrt{29}$

Or $\quad 2x - 5y + 17 = \pm 3\sqrt{29}$

So the equation (for locus) are

$$2x - 5y + 17 - 3\sqrt{29} = 0$$

and $\quad 2x - 5y + (17 + 3\sqrt{29}) = 0$

Exercise 12.6

Q.1: Find the distance to the line $3x - 2y + 12 = 0$ from each of the following points:
(a) $(1, 3)$ (b) $(-1, 7)$ (c) $(-3, -2)$

Q.2: Which of the following points are on the same side of the line $x - 6y + 8 = 0$ as the origin?
(a) $(2, 3)$ (b) $(3, -2)$ (c) $(-2, -3)$ (d) $(-3,2)$

Q.3: If the vertices of a triangle are A (2, –1), B(–2,3) and C(0,3), Find the length of altitude from B to AC.

Q.4: The distance from the point $(-2, -3)$ to the line $3x - 4y + k = 0$ is $\frac{13}{5}$. Find the value of K.

Q.5: Write equation of the two lines (or locus of a point) parallel to the line through (1, 2) and (4, 6) which are 3 units distant from the given line.

Q.6: Find the locus of a point which moves so that it is always "one" unit from the line $3x - 4y + 7 = 0$.

Q.7: Find the equation of the two lines (or locus of point) parallel to the line $x - 6y + 8 = 0$ and a distance of $\frac{18}{\sqrt{37}}$ units from it.

Q.8: Find the distance between the parallel lines.

$3x - 4y + 11 = 0$ and $3x - 4y - 9 = 0$

(Hints: Take a point at one line and find the distance of this point on the other line)

Q.9: Find the perpendicular distance from the origin to the line passing through (1, 2) and perpendicular to the line $\sqrt{3}\ y = x + 4$.

Q.10: Find the locus of all points which are equidistant from the point (– 3, 8) and the line $4x + 9 = 0$

Answers

Q.1: (a) (a) $\frac{9}{\sqrt{13}}$ (b) $\frac{5}{\sqrt{13}}$ (c) $\frac{12}{\sqrt{13}}$ (d) $\frac{7}{\sqrt{13}}$

Q.2: (b), (c) **Q.3**: $4\sqrt{2}$ **Q.4:** k = 7, –19

Q.5: $4x - 3y - 13 = 0$, $4x - 3y + 17 = 0$

Q.6: $3x - 4y + 2 = 0$, $3x - 4y + 12 = 0$

Q.7: $x - 6y - 10 = 0$, $x - 6y + 26 = 0$

Q.8: 4 **Q.9:** $\frac{2 + \sqrt{3}}{2}$

Q.10: $16y^2 - 256y + 24x + 1087 = 0$

Short Questions

Write the short answers of the following:

Q.1: Write distance formula between two points and give one example.

Q.2: Find distance between the points (-3, 1) and (3, -2)

Q.3: Show that the points A(-1, -1), B (4, 1) and C(12, 4) lies on a straight line.

Q.4: Find the co-ordinate of the mid point of the segment P_1 (3,7), P_2 (-2, 3).

Q.5: Find the co-ordinates of the point P(x,y) which divide internally the segment through P_1 (-2,5) and P_2 (4, -1) of the ratio of $\frac{r_1}{r_2} = \frac{6}{5}$.

Q.6: If a line is extended from A(2, 3) through B(-2, 0) to a point C so that AC = 4 AB, find the co-ordinate of C.

Q.7: For the triangle whose vertices are A(0,1), B (7,2) and C(3,8). Find the length of the median from C to AB.

Q.8: If the mid point of a segment is (6,3) and one end point is(8, -4), what are the co-ordinates of the other end point.

Q.9: Find the angle between the lines having slopes -3 and 2

Q.10: Find the slope of a line which is perpendicular to the line joining P_1 (2, 4) and P_2 (-2, 1).

Q.11: Find the equation of a line through the point (3, -2) with slope $m = \frac{3}{4}$.

Q.12: Find the equation of a line through the points (− 1, 2) and (3, 4).

Q.13: Find an equation of the line with the following intercepts
$a = 2$, $b = -5$

Q.14: Find the equation of line having x – intercept -2 and y – intercept 3.

Q.15: Find the equation of a line whose perpendicular distance from the origin is 2 and inclination of the perpendicular is 225^o.

Q.16: Reduce the equation $3x + 4y – 2 = 0$ into intercept form.

Q.17: Find the equation of the line passing the point (1, –2) making an angle of 135^o with the x-axis.

Q.18: Find the points of intersection of the lines
$x + 2y - 3 = 0, 2x - 3y + 8 = 0$

Q.19: Show that the points (1,9), (-2, 3) and (-5, -3) are collinear.

Q.20: Show that the lines passing through the points (0, -7), (8, -5) and (5, 7), (8, -5) are perpendicular.

Q.21: Find the distance from the point (-2,1) to the line $3x + 4y - 12 = 0$

Answers

Q2.	$3\sqrt{5}$	**Q4.**	$\left(\frac{1}{2}, 5\right)$	**Q5.**	$P\left(\frac{14}{11}, \frac{19}{11}\right)$
Q6.	C (– 14, – 9)	**Q7.**	$\sqrt{\left(\frac{85}{2}\right)}$	**Q8.**	(4, 10)
Q9.	135^o	**Q10.**	$-\frac{4}{3}$	**Q11.**	$3x - 4y - 17 = 0$
Q12.	$x - 2y + 5 = 0$	**Q13.**	$5x - 2y = 10$	**Q14.**	$3x - 2y + 6 = 0$

Q15. $x + y + 2\sqrt{2} = 0$

Q16. $\frac{x}{2/3} + \frac{y}{1/2} = 1$, where $a = \frac{2}{3}$ and $b = \frac{1}{2}$

Q17. $x + y + 1 = 0$ Q18. (– 1, 2) Q21. $\frac{14}{5}$

<><><>

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Slope of the line $\frac{x}{a} + \frac{y}{b} = 1$ is:

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$ (c) $-\frac{b}{a}$ (d) $-\frac{a}{b}$

__2. y = 2 is a line parallel to:

(a) x – axis (b) y – axis (c) y = x (d) x = 3

__3. Eq. of the line in slope intercept form is;

(a) $\frac{x}{y} + \frac{y}{b} = 1$ (b) y = mx + c (c) $y - y_1 = m(x-1)$

(d) None of these

__4. Distance between (4, 3) and (7, 5) is:

(a) 25 (b) $\sqrt{13}$ (c) 5 (d) None of these

__5. Point (– 4, –5) lines in the quadrant:

(a) 1st (b) 2nd (c) 3rd (d) 4th

__6. When two lines are perpendicular:

(a) $m_1 = m_2$ (b) $m_1m_2 = -1$ (c) $m_1 = -m_2$

(d) None of these

__7. Ratio formula for y – coordinate is:

(a) $\frac{x_1r_2 + x_2 r_1}{r_1 + r_2}$ (b) $\frac{y_1r_2 + y_2 r_1}{r_1 + r_2}$ (c) $\frac{x - y}{2}$

(d) None of these

__8. Slope of the line through (x_1 , y_2) and (x_2, y_2)

(a) $\frac{x_1 + x_2}{y_1 + y_2}$ (b) (c) $\frac{y_2 - y_1}{x_2 - x_1}$

(d) None of these

__9. Given three points are collinear if their slopes are:

(a) Equal (b) Unequal (c) $m_1 m_2 = -1$

(d) None of these

__10. $y - y_1 = m(x - x_1)$ is the:

(a) Slope intercept form (b) Intercept form

(c) Point slope form (d) None of these

Answers

1.	c	**2.**	a	**3.**	b	**4.**	b	**5.**	c
6.	b	**7.**	b	**8.**	c	**9.**	a	**10.**	c

Chapter 13

The Circle

13.1 Circle:

A circle is the set of all points in a plane that are equally distants from a fixed point. The fixed point is called the centre of circle and the distance from the centre to any point on the circle is called the radius of the circle.

An equation of a circle is an equation in x and y which is satisfied by the coordinates of a point if and only if the point is on the circle.

13.2 Standard Form of the Equation of a Circle:

Let P(x,y) be a point in a plane which moves so that it is always a constant distance, called the radius r, from the fixed point (h, k), called the centre of the circle. Then by distance formula

$(x - h)^2 + (y - k)^2 = r^2$……..…(1)

Equation (1) is **called standard form** of the circle, with centre (h, k) and radius r.

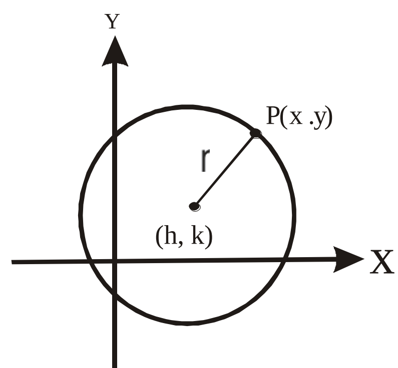


Figure 9.1

If the centre is at the origin (0, 0), equation (1) reduces to

$x^2 + y^2 = r^2$……………………..(2)

Also if the centre is at the origin (0,0) and radius is 1 (one),then the equation (1) reduces to the unit circle i.e;

$x^2 + y^2 = 1$

Note that any equation equivalent to equation (1) is also an equation of the circle. We may reduce the equation (1) to the form.

$x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0$ …………(3)

we observe that

(i) The equation (3) is second degree in x and y.

(ii) The coefficients of x^2 and y^2 are equal.

(iii) There is no product term xy.

Example 1: Find the equation of the circle with centre at (–2, 3) and radius 6.

Solution: From the standard form.

$(x - h)^2 + (y - k)^2 = r^2$, here (h, k) = (–2, 3) , r = 6

$(x + 2)^2 + (x - 3)^2 = 36$

Example 2: Find the centre and radius of the circle $x^2 + y^2 + x - 4y - \frac{7}{4} = 0$.

Solution: $x^2 + y^2 + x - 4y - \frac{7}{4} = 0$

Add. $\left(\frac{1}{2}\right)^2 + (2)^2$ on both sides

$$x^2 + x + \left(\frac{1}{2}\right)^2 + y^2 - 4y + (2)^2 = \frac{7}{4} + \frac{1}{4} + 4$$

$$(x + \frac{1}{2})^2 + (y - 2)^2 = 6$$

Comparing with $(x - h)^2 + (y - k)^2 = r^2$

Hence the centre is at $(-\frac{1}{2}, 2)$ and the radius is $\sqrt{6}$

Theorem: (General form of an equation of a circle)

The equation $x^2 + y^2 + 2gx + 2fy + c + 0$………………..(4)

represents a circle with centre (–g, –f) and radius $\sqrt{g^2 + f^2 - c}$

Proof :

Since , $x^2 + y^2 + 2gx + 2fy + c = 0$

$x^2 + 2gx + y^2 + 2fy = - c$

Add $g^2 + f^2$ on both sides

$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$

$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$……………….(5)

Comparing this equation with the standard form

i.e., $(x - h)^2 + (y - k)^2 = r^2$

We have $h = -g$, $k = -f$, $r = \sqrt{g^2 + f^2 - c}$

Thus equation (4) represents a circle with

centre $(h, k) = (-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$

Equation (4) is **called the General form of the circle**

Note: From equation (5) we find

$r^2 = g^2 + f^2 - c$

(i) If $r^2 > 0$, the circle is real.

(ii) If $r^2 = 0 \Rightarrow r = 0$, the circle is a point circle

(iii) If $r^2 < 0$, the circle is an imaginary circle.

These are **called special features** of equation of circle.

Example 3: What type of the circle is represented by $x^2 + y^2 + 2x - 4y + 8 = 0$.

Solution: Here, $g = 1, f = -2, c = 8$

Since $r^2 = g^2 + f^2 - c$

$= 1 + (-2)^2 - 8$

$r^2 = -3 < 0$

Hence the equation represents an imaginary circle.

Example 4: By comparing with the general form, find the centre and radius of the circle $2x^2 + 2y^2 - 5x + 4y - 7 = 0$.

Solution: The given circle has the equation

$$x^2 + y^2 - \frac{5}{2}x + 2y - \frac{7}{2} = 0$$

Comparing it with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

We have, $g = -\frac{5}{4}$, $f = 1$, $c = -\frac{7}{2}$

Hence the centre is $(-g, -f) = \left(\frac{5}{2}, -1\right)$

and radius $r = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\left(\frac{5}{4}\right)^2 + (-1)^2 - \left(-\frac{7}{2}\right)}$$

$$= \sqrt{\frac{25}{16} + 1 + \frac{7}{2}} = \sqrt{\frac{97}{4}}$$

13.3 Circle Determined by Three Conditions:

From the general form of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

We see that there are three effective constants g, f and c. In the general form three conditions can be imposed upon them which will determined a circle, unique or otherwise.

13.3.I Circle Through Three Points:

If we substitute the coordinates of any point, we get three linear equations, in the three unknowns, which may be solved simultaneously for g, f and c. By substituting these values in general form we get the equation of the circle.

13.3.2 Circles Tangent to Line:

Instead of specifying that the circle pass through certain points we may require that it is tangent to certain line or that its centre lie on a given line. Combinations of point and line conditions may be used to determine a circle (or circles).

Example 5: Find an equation of the circle passing through (9,– 7), (–3, –1) and (6, 2).

Solution:

There are at least two ways of proceeding.

Method –I:

The General form of an equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (1)$$

We must determine the constants g, f and c.

Since the circle through the three given points, so

Put point $(9, -7)$ $\Rightarrow 9^2 + (-7)^2 + 2(9)\,g + 2(-7)f + c = 0$ (2)

Put point $(-3,-1)$ $\Rightarrow (-3)^2 + (-1)^2 + 2(-3)g + 2\,(-1)f + c = 0$ (3)

Put point $(6, 2)$ $\Rightarrow (6)^2 + (2)^2 + 2\,(6)g + 2(2)\,f + c = 0$ (4)

Equivalent equations are

$18\,g - 14f + c = -130$ (2′)

$-6g - 2f + c = -10$ (3′)

$12g + 4f + c = -40$ (4′)

Subtracting equation (4′) from (2′) and (3′) we get,

$6g - 18f = -90$ ------------------------------------ (5)

$-18g - 6\,f = 30$ ------------------------------------ (6)

Dividing equation (6) by 3, we get

$-6g - 2f = 10$ ------------------------------------ (7)

Adding equation (5) and (7), we get ,

$-20\,f = -80$

$\boxed{f = 4}$

put $f = 4$ in equation (7) $\Rightarrow -6g - 8 = 10$

$-6g = 18$

$\boxed{\mathbf{g = -3}}$

put values of f and g in equation (3′)

$18 - 8 + c = -10$

$\boxed{\mathbf{C = -20}}$

Substituting g, f and c in equation (1), we get

$x^2 + y^2 - 6x + 8y - 20 + 0$

This method has the disadvantage that we must do more work to find the centre and radius of the circle.

Method –II:

This method is more geometric. It depends on the geometric fact that the centre of the circle is at the intersection of the perpendicular bisectors of segments joining the three given points. It sufficent to use only two of these perpendicular bisectors.

As we see from Fig. 2, the slope of

$$\ell \text{ is } \frac{-7+1}{9+3} = -\frac{1}{2}$$

and the slope of

$$n \text{ is } \frac{2+7}{6-9} = -3$$

The slope of the perpendicular bisector $\ell_\perp$ and $n_\perp$ are 2 and $\frac{1}{3}$ respectively.

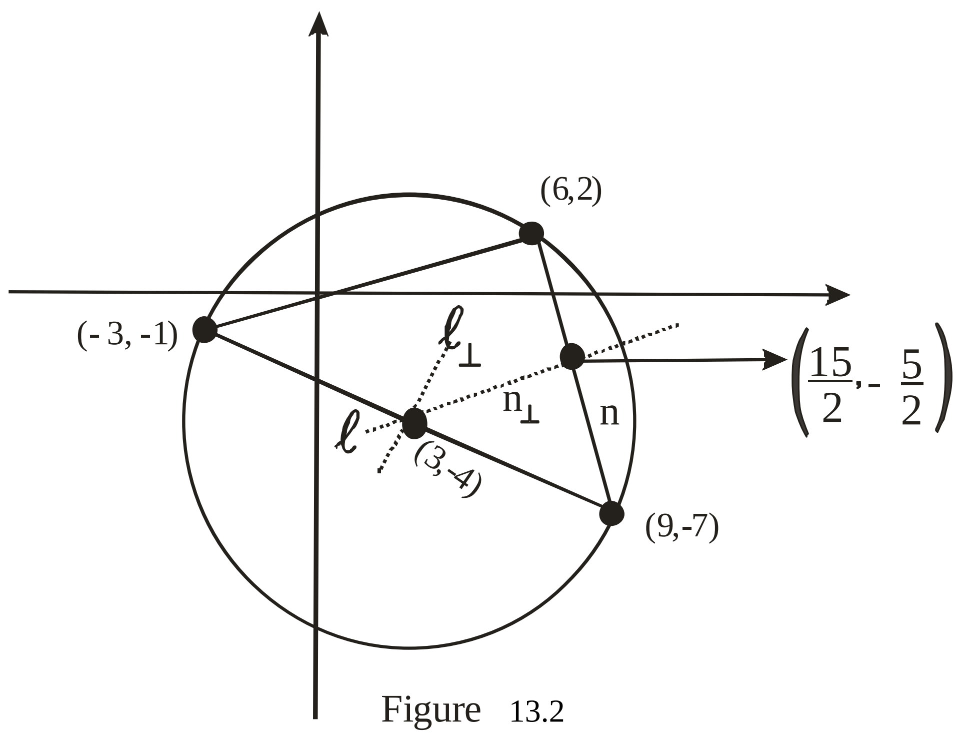


Figure 13.2

[Equation of line by point- slope foam is $y - y_1 = m(x - x_1)$]

Therefore the equation of $\ell_\perp$ and $n_\perp$ are

$\ell_{\perp}:$ $y + 4 = 2(x - 3)$ or $2x - y = 10$

$n_{\perp}:$ $y + \frac{5}{2} = \frac{1}{3}\left(x - \frac{15}{2}\right)$ or $x - 3y = 15$

Solving these two simultaneous equations we obtain $x = 3$, $y = -4$.

Hence the centre of the circle is $(3, -4)$

The radius is $r = \sqrt{(3+3)^2 + (-4+1)^2}$

$r = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

Put the $(h, k) = (3, -4)$ and $r = 3\sqrt{5}$ in equation $(x - h)^2 + (y - k)^2 = r^2$

We get , $(x - 3)^2 + (y + 4)^2 = (3\sqrt{5})^2$

Or $x^2 + y^2 - 6x + 8y - 20 = 0$

Example 6: Find the equation of the circle which is tangent to the y – axis, which passes through the point (–1, –1), and the centre of which is on the line 2x + y + 4 = 0.

Solution: If (h, k) is the centre of the circle.

Since the circle is tangent to y – axis so that radius is

$r = h$

As circle passes through point $(-1, -1)$, so

$(h + 1)^2 + (k + 1)^2 = h^2$

Or $k^2 + 2k + 2h + 2 = 0$ (1)

Since the centre (h, k) lines on the line $2x + y + 4 = 0$

So $2h + k + 4 = 0$ (2)

from (2) $h = \frac{-(k+4)}{2}$ (3)

Put in (1)

$k^2 + 2k + 2\left[\frac{-(k+4)}{2}\right] + 2 = 0$

Or $k^2 + 2k - k - 4 + 2 = 0$

Or $k^2 + k - 2 = 0$

$k = 1$ or $k = -2$

From (3) when $k = 1$, $h = -\frac{5}{2}$

When $k = -2$, $h = -1$

Hence there are two circles with centres $\left(-\frac{5}{2}, 1\right)$, the radius $r = \sqrt{\left(-1+\frac{5}{2}\right)^2 + (-1-1)^2} = \frac{5}{2}$

and (–1, –2), the radius $r = \sqrt{(-1+1)^2 + (-1+2)^2} = 1$

The equation of circles are

$$\left(x + \frac{5}{2}\right)^2 + (y-1)^2 = \left(\frac{5}{2}\right)^2$$

Or $x^2 + y^2 + 5x - 2y + 1 = 0$ (4)

And, $(x+1)^2 + (y+2)^2 = 1$

Or $x^2 + y^2 + 2x + 4y + 4 = 0$ (5)

Example7: Find the equation of the circle which contains the point (0, 1) and touches the line x + 2y + 2 = 0 at the point (4, –3).

Solution: If the (h, k) is the centre of the circle. Since the points A (0, 1) and B (4, – 3) lie on the circle, so

$$|OA| = |OB|$$

$$\sqrt{(h-0)^2 + (k-1)^2} = \sqrt{(h-4)^2 + (k+3)^2}$$

Or $h^2 + h^2 - 2k + 1 = h^2 + k^2 - 8h + 6k + 25$

Or $8h - 8k - 24 = 0$

Or $h - k - 3 = 0$……………………….(1)

Slope of the line $x + 2y + 2 = 0$

Is $m = -\frac{1}{2}$

Slope of perpendicular OB is 2.

Equation of the perpendicular OB on the line is

$$\frac{k+3}{h-4} = 2$$

Or $2h - k - 11 = 0$ (2)

Solving equation (1) and (2) we have

$$h = 8, \quad k = 5$$

So the centre is (8, 5) and radius $r = \sqrt{(8-0)^2 + (5-1)^2} = \sqrt{80}$

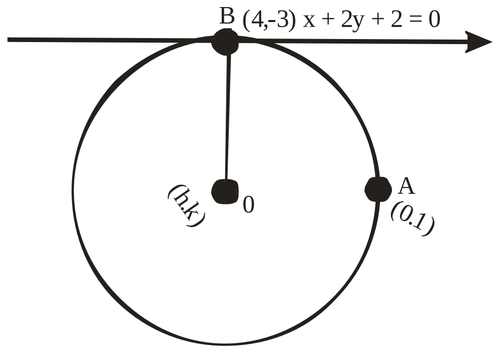


Figure 13.3

Hence the equation of the circle is

$$(x-8)^2 + (y-5)^2 = 80$$

Or $\quad x^2 + y^2 - 16x - 10y - 9 = 0$

Example 8: A circle is tangent to the x – axis at (5, 0) and is also tangent to the line y = x. Find the centre, radius, and an equation of the circle.

Solution: Since the circle is tangent to (5, 0), so the centre of the circle is (h, k) = (5, k).

The radius of the circle is r = k

The perpendicular distance of centre (5, k) from the line x – y = 0 is

$$k = \frac{|5-k|}{\sqrt{2}}$$

Or, $\quad |5 - k| = \sqrt{2}\, k$

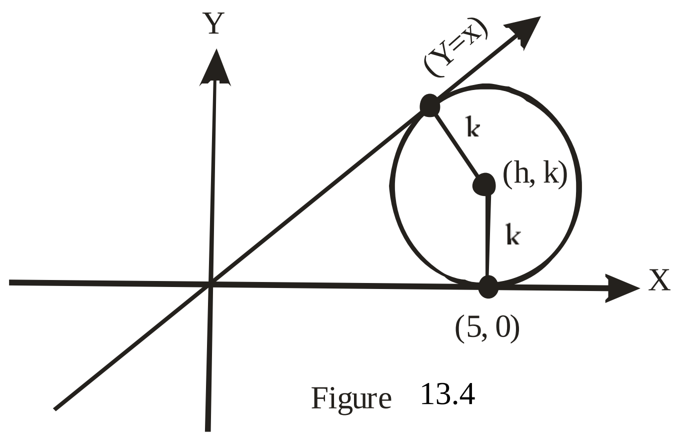


Figure 13.4

Squaring both sides,

$$k^2 - 10k + 25 = 2k^2$$

Or $$k^2 + 10k - 25 = 0$$

$$K = -5 + 2\sqrt{2} \qquad \text{or} \qquad 5 - 5\sqrt{2}$$

Hence the centre is at $\left(5, 5\left(\sqrt{2} - 1\right)\right)$

Radius is $r = 5\left(\sqrt{2} - 1\right)$

An equation is $(x - 5)^2 + \left[y - 5\left(\sqrt{2} - 1\right)\right]^2 = \left(5\left(\sqrt{2} - 1\right)\right)^2$

Or $x + y^2 - 10x - 10y\left(\sqrt{2} - 1\right) + 25 = 0$ ………………….(1)

When the center is at $\left(5, -5\left(\sqrt{2} + 1\right)\right)$

Radius is $r = 5\left(\sqrt{2} + 1\right)$

An equation is $(x - 5)^2 + \left(y + 5\left(\sqrt{2} + 1\right)\right)^2 = \left(5\left(\sqrt{2} + 1\right)\right)^2$

Or $x^2 + y^2 - 10x + 10y\left(\sqrt{2} + 1\right) + 25 = 0$ ………………….(2)

Exercise 13

Q.1: Find the equation of the circles with the given centres and radii.

(a) $(-1, 2), r = \sqrt{2}$ **(b)** $(-\sqrt{2}, -2), r = \sqrt{6}$
(c) $(0, 0), r = a$ **(d)** $(1, -3), r = 3$

Q.2: Find centres and radii of the circles with the following equations:

(a) $x^2 + y^2 - 6x + 6y = 0$ **(b)** $x^2 + y^2 - 4x + y - 1 = 0$
(c) $3x^2 + 3y^2 - 2x - 6y - 2 = 0$ **(d)** $(x + 2)^2 + (y - 1)^2 = 16$
(e) $x^2 + y^2 + -4x + 6y - 12 = 0$

Q.3: Find the equations of the circles:

(a) Passing through the points (1, 2), (0, –1) and (–1, 1).

(b) Passing through the points (0,1), (3, –3) and (3, –1).

(c) Through (–2,1), (–4, –3) and (3, 0).

(d) through (2, –1) and (–2,0) with center on $2x - y - 1 = 0$

(e) Through (–1, 2) and tangent to the axes.

(f) Through (3, 1) and touching the x – axis at (0,0).

Q.4: Find the equations of the following circles:

(a) through the point of intersection of the lines.

$2x - y + 7 = 0$ and $3x + y + 8 = 0$ with center at the origin

(b) Centre at the point of intersection of the lines.

$x - 2y + 4 = 0$ and $2x + y - 2 = 0$ with radius 4 units.

Q.5: Find the equations of the following circles:

(a) With center on the line $y = -x$ has radius 4 and passes through the origin.

(b) The circle touching the line $x = 2$ and $x = 12$ and passes through (4, 5).

(c) Through origin and whose intercepts on the axes are 3 and 4.

(d) through (–1, –2), (6, –1) and touching the x – axis.

Q.6: Find the equation of the following circles.

(a) Concentric with the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ with radius 6 units.

(b) Concentric with the circle $x^2 + y^2 - 7x + 8y = \frac{9}{2}$ and touches the y – axis.

Q.7: Find the equations of the following circles:

(a) Which is tangent to the positive x and y – axis and radius 5 units.

(b) Which touches both the axes of 4^{th} quadrant and has a radius of 5 units.

(c) Whose center is (–3, 2) and passes through the center of the circle $x^2 + y^2 - 4x + 8y - 16 = 0$

Q.8: Find which of the two circles $x^2 + y^2 - 3x + 4y = 0$ and $x^2 + y^2 - 6x - 8y = 0$ is greater.

Q.9: Find the equation of the circle having:

(*a*) (–2, 5) and (3, 4) as the end points of its diameter. Find also its centre and radius.

(b) (–3, 7) and (2,- 1) as the end points of its diameter. Find also its centre and radius.

Q.10: Find the equation of the circle whose center is at (–2, 5) and which touches the line $4x - 3y - 18 = 0$

Q.11: Show that the circles $x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y + 9 = 0$ touch externally.

Q.12: Show that the circles $x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ touch internally.

Answers

Q.1: (a) $x^2 + y^2 + 2x - 4y + 3 = 0$

(b) $x^2 + y^2 + 2\sqrt{2}\,x + 4y = 0$ (c) $x^2 + y^2 = a^2$

(d) $(x - 1)^2 + (y + 3)^2 = 16$

Q.2: (a) $(3, -3)$, $r = 3\sqrt{2}$ (b) $\left(2, -\frac{1}{2}\right)$, $r = \frac{1}{2}\sqrt{21}$

(c) $\left(\frac{1}{3}, 1\right)$, $r = 4/3$ (d) $(-2, 1)$, $r = 4$ (e) $(2, -3)$, $r = 5$

Q.3: (a) $x^2 + y^2 - x - y - 2 = 0$

(b) $3x^2 + 3y^2 - x + 12y - 15 = 0$

(c) $48x^2 + 48y^2 - 784x + 632y - 2440 = 0$

(d) $\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{85}{16}$

(e) $(x + 1)^2 + (y - 1)^2 = 1$ and $(x + 5)^2 + (y - 5)^2 = 25$

(f) $x^2 + y^2 - 10y = 0$

Q.4: (a) $x^2 + y^2 = 10$ (b) $x^2 + y^2 - 4y - 12 = 0$

Q.5: (a) $(x + 2\sqrt{2})^2 + (y - 2\sqrt{2})^2 = 16$

(b) $(x - 7)^2 + (y - 4)^2 = 25$ and $(x - 7)^2 + (y + 4)^2 = 25$

(c) $\left(x - \frac{3}{2}\right)^2 + (y - 2)^2 = \frac{25}{4}$

(d) $(x - 3)^2 + (y + 5)^2 = (5)^2$ and $(x - 23)^2 + (y + 145)^2 = (145)^2$

Q.6: (a) $x^2 + y^2 - 6x + 4y - 23 = 0$

(b) $x^2 + y^2 - 7x + 8y + 16 = 0$

Q.7: (a) $(x - 5)^2 + (y - 5)^2 = (5)^2$

(b) $(x - 5)^2 + (y + 5)^2 = (5)^2$

(c) $x^2 + y^2 + 6x - 4y - 48 = 0$

Q.8: **Second circle to greater than first.**

Q.9: (a) $x^2 + y^2 - x - 9y + 14 = 0$ (b) $4x^2 + 4y^2 + 4x - 24y - 141 = 0$

Q.10: $x^2 + y^2 + 6x - 10y - 47 = 0$

Short Questions

Write the short answers of the following

Q.1: Write the equation of circle with centre at (h, k) and radius r.

Q.2: Write the general form of the circle, also represent the centre and radius is this form.

Q.3: Find the equation of circle with centre (0, 0) and radius r.

Q.4: Find the equation of circle with centre (-3, 4) and radius 4.

Q.5: Find the equation of circle with centre on origin and radius is $\frac{1}{2}$.

Q.6: Find centre and radius of the circle $x^2 + y^2 + 9x - 7y - 33 = 0$

Q.7: Find the centre and radius of the circle $6x^2 + 6y^2 - 18y = 0$

Q.8: What type of circle is represented by $x^2 + y^2 - 2x + 4y + 8 = 0$

Q.9: Find the equation of circle with centre at (-1, 3) and tangent to x –axis.

Q.10: Find the equation of circle with centre (3, 0) and tangent to y – axis.

Q.11: Find the equation of the circle touches the lines at $x = 0$ and $x = 10$ and the centre is an x – axis.

Q.12: Reduce the equation into standard form $x^2 + y^2 - 4x + 6y - 12 = 0$

Q.13: Reduce the equation into standard form $2x^2 + 2y^2 - 5x + 4y - 7 = 0$

Q.14: Reduce the equation into standard form $x^2 + y^2 - 10y = 0$.

Q.15: Find the equation of circle centered at (-3, 2) and passes through the point (2, -4).

Q.16: Define the circle.

Answers

Q2. Centre $(-g, -f)$, $r = \sqrt{g^2 + f^2 - c}$

Q3. Equation of circle is: $x^2 + y^2 = r^2$

Q4. $x^2 + y^2 + 6x - 8y + 9 = 0$

Q5. $x^2 + y^2 - \frac{1}{4} = 0$

Q6. centre $= \left(-\frac{7}{2}, \frac{7}{2}\right)$, $r = \sqrt{\frac{131}{2}}$

Q7. Centre $= (0, \frac{3}{2})$, $r = \frac{3}{2}$

Q8. $r^2 < 0$ imaginary circle.

Q9. $x^2 + y^2 + 2x - 6y + 1 = 0$

Q10. $x^2 + y^2 - 6x = 0$

Q11. $x^2 + y^2 - 10x = 0$

Q12. $(x - 2)^2 + (y + 3)^2 = 5^2$

Q13. $\left(x - \frac{5}{4}\right)^2 + (y + 1)^2 = \left(\frac{\sqrt{41}}{8}\right)^2$

Q14. $(x - 0)^2 + (y - 5)^2 = 5^2$

Q15. $x^2 + y^2 + 6x - 4y - 48 = 0$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. General Equation of the circle is:

(a) $x^2 + y^2 + 2gx + 2fy + c = 0$ (b) $(x - h)^2 + (y - k)^2 = r^2$

(c) $x^2 + y^2 + x + y + 1 = 0$ (d) None of these

__2. Standard equation of the circle is:

(a) $x^2 + y^2 + 2gh + 2fy + c = 0$ (b) $(x - h)^2 + (y - k)^2 = r^2$

(c) $x^2 + y^2 + x + y + 1 = 0$ (d) None of these

__3. Straight line from center to the circumference is:

(a) Circle (b) Radius

(c) Diameter (d) None of these

__4. Radius of the circle $x^2 + y^2 = 1$ is:

(a) 1 (b) (0, 0)

(c) 2 (d) None of these

__5. Radius of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is:

(a) c (b) c^2

(c) $\sqrt{g^2 + f^2 - c}$ (d) None of these

__6. Centre of the circle $x^2 + y^2 - 2x - 4y = 8$ is:

(a) (1, 2) (b) (2, 4)

(c) (1, 3) (d) None of these

__7. Radius of the circle $x^2 + y^2 - 2x - 4y = 8$ is:

(a) 8 (b) $\sqrt{8}$

(c) $\sqrt{12}$ (d) None of these

__8. Equation of the unit circle is:

(a) $x^2 + y^2 + 2x + 2y + 1 = 0$ (b) $x^2 + y^2 = 1$

(c) $x^2 + y^2 = r^2$ (d) None of these

__9. Radius of the circle $(x - 1)^2 + (y - 2)^2 = 16$ is:

(a) 2 (b) 1

(c) 4 (d) None of these

__10. Centre of the circle $(x - 1)^2 + (y - 2)^2 = 16$ is :

(a) (1, 2) (b) (2, 1)

(c) (4, 0) (d) None of these

Answers

1.	a	**2.**	b	**3.**	b	**4.**	a	**5.**	c
6.	a	**7.**	d	**8.**	b	**9.**	d	**10.**	a

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