UNIT : 5

MULTIPLE INTEGRALS

DOUBLE INTEGRATION

Let $f(x,y)$ be a continuous function defined

On region R then double integration of $f(x,y)$

On R is denoted by $\iint_R f(x,y)dA$ and it is defined by,

$$\boxed{\iint_R f(x,y)dA = \lim_{\substack{n\to\infty \\ \delta A_r}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \quad , R \text{ is closed bounded region.}}$$

Fubini's Theorem

1:- If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b, c \le y \le d$,

Then $\boxed{\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dx\, dy = \int_a^b \int_c^d f(x,y)dy\, dx}$

which means when limits of integration are constants and function is continuous then order of integration has no importance.

2:- If $f(x,y)$ is continuous on a region $R: a \le x \le b, g(x) \le y \le h(x)$,then

$$\boxed{\iint_R f(x,y)dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x,y)dy\, dx \ , where\ g\ and\ h\ are\ continuous\ on\ [a,b]}$$

3:- If $f(x,y)$ is continuous on a region $R: g(y) \le x \le h(y), c \le y \le d$,then

$$\boxed{\iint_R f(x,y)dA = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x,y)dx\, dy \ , where\ g\ and\ h\ are\ continuous\ on\ [c,d]}$$

Example(1): Evaluate: $\int_0^2 \int_0^1 (x+y)\ dx\, dy$

Solution: Integrating first w.r.t. x keeping y constant we get

$$\int_0^2 \int_0^1 (x+y)\ dx\, dy = \int_0^2 \left(\frac{x^2}{2} + yx\right)_0^1 dy$$

$$= \int_0^2 \left(\frac{1}{2} + y\right) dy$$

$$= \left(\frac{y}{2} + \frac{y^2}{2}\right)_0^2$$

$$= (1 + 2)$$

$$= 3$$

<u>Example(2):</u> Evaluate: $\int_0^2 \int_0^x \left(\frac{1}{x}\right) dy\, dx$

<u>Solution:</u> Integrating first w.r.t. y keeping x constant we get

$$\int_0^2 \int_0^1 \left(\frac{1}{x}\right) dy\, dx \; == \int_0^2 \left(\frac{1}{x}\right) [y]^x{}_0 \, dx$$

$$= \int_0^2 \left(\frac{1}{x}\right) x \, dx$$

$$= \int_0^2 1 \; dx$$

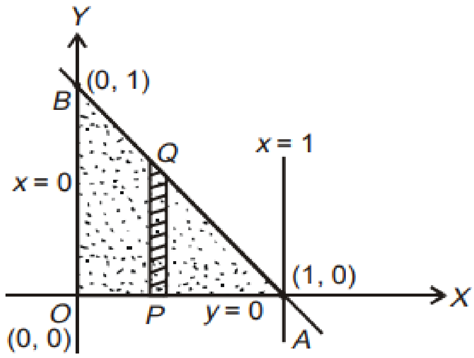
$$= [x]_0^2$$

$$= 2$$

Example (3): Evaluate $\iint_R e^{2x+3y} dxdy$ over the triangle bounded by the lines x = 0, y = 0 and x + y = 1.

Solution: Here, the region of integration is the triangle OABO as the line x + y = 1 intersects the axes at points (1, 0) and (0, 1). Thus, precisely the region R (say) can be expressed as:

$$0 \le x \le 1, \; 0 \le y \le 1 - x$$



$$I = \iint_R e^{2x+3y} dxdy$$

$$= \int_0^1 \left(\int_0^{1-x} e^{2x+3y} \, dy \right) dx$$

$$= \int_0^1 \left[\frac{1}{3} e^{2x+3y} \right]_0^{1-x} dx$$

$= \frac{1}{3}\int_0^1 (e^{3-x} - e^{2x})\, dx$

$= \frac{1}{3}\left[\frac{e^{3-x}}{-1} - \frac{e^{2x}}{2}\right]_0^1$

$= -\frac{1}{3}\left(e^2 + \frac{e^2}{2}\right) - \left(e^3 + \frac{1}{2}\right)$

$= \frac{1}{6}(2e^3 - 3e^2 + 1)$

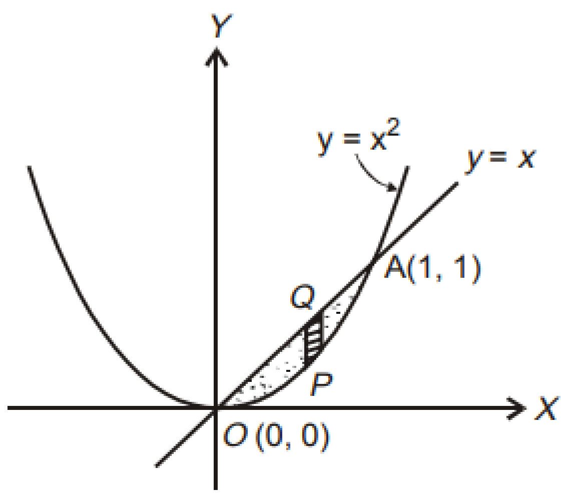
$= \frac{1}{6}(2e+1)(e-1)^2$

Example (4): Evaluate the integral $\iint_R xy(x+y)dxdy$ over the area between the curve $y = x^2$ and $y = x$.

Solution: We have $y = x^2$ and $y = x$ which implies $x^2 - x = 0$ i.e. either $x = 0$ or $x = 1$.

Further, if x = 0 then y = 0; if x = 1 then y = 1. Means the two curves intersect at points (0, 0), (1, 1). ∴ The region R of integration is doted and can be expressed as:

$0 \le x \le 1, x^2 \le y \le x.$



$\therefore \iint_R xy(x+y)dxdy = \int_0^1 (\int_{x^2}^{x} x\, y(x+y)dy)\ dx$

$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$

$= \int_0^1 \{ (\frac{x^4}{2} + \frac{x^4}{3}) - (\frac{x^6}{2} + \frac{x^7}{3})\}\, dx$

$$= \left\{\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24}\right\}_0^1$$

$$= \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{3}{56}$$

EXERCISE

Q-1. Evaluate the following integrals:

1. $\int_0^3 \int_0^4 (4 - y^2)dydx$ Ans. 16
2. $\int_0^1 \int_x^{x^2} xydydx$ Ans. $-\frac{1}{24}$
3. $\int_1^3 \int_1^x \frac{1}{xy} dxdy$ Ans. 0.603
4. $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dxdy}{\sqrt{1-x^2-y^2}}$ Ans. $\frac{\pi}{4}$
5. $\int_0^\pi \int_0^x x\sin y dydx$ Ans. $\frac{\pi^2}{2} + 2$
6. $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y} dydx$ Ans.2
7. $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2} dydx$ Ans. $1 - \ln 2$
8. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dxdy$ Ans. $e - 2$
9. $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} dydx$ Ans. $7(e-1)$
10. $\int_1^4 \int_{2x^2}^{3x^2} xe^{x^2+y} dydx$ Ans. $\frac{e^{64}}{8} - \frac{e^{48}}{6} - \frac{e^4}{8} + \frac{e^3}{6}$

Q-2 Evaluate the following integrals:

1. $\iint_R \frac{x}{y} dxdy$, where R is the region in the first quadrant bounded by the lines $y = x, y = 2x, x = 1, x = 2.$ Ans: $\frac{3}{2}\ log\ 2$ [1]
2. $\iint_R xy\ dxdy$, where R is the region in the positive quadrant for which $x + y \leq 1.$ Ans: $\frac{1}{24}$ [4]
3. $\iint_R x^2 + y^2\ dxdy$, where R is the triangularregion with vertices (0, 0), (1, 0) and (0, 1). Ans: $\frac{1}{6}$[1]
4. $\iint_R xy\ dxdy$, where R is the region bounded by the x-axis, the line $x = 2a$ and the curve $x^2 = 4ay.$ Ans. $\frac{a^4}{3}$ [4]

5. $\iint_R (x-1)\,dA$,where R is the region in the first quadrant enclosed between $y = x$ and $y = x^3$. Ans. $-\frac{7}{60}$ [2]

6. $\iint_R x(1+y^2)^{-1/2}\,dA$, where R is the region in the first quadrant enclosed $y = x^2, y = 4$ and $x = 0$. Ans. $\frac{\sqrt{17}-1}{2}$ [2]

7. $\iint_R xy\,dxdy$, where R is in the quadrant of the circle $x^2 + y^2 = a^2$, where $x \geq 0$ and $y \geq 0$. Ans. $\frac{a^4}{8}$ [4]

8. $\iint_R (x+y)^2 dxdy$, where R is the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ =1.
$Ans. \frac{1}{4}\pi ab(a^2 + b^2)$ [4]

9. $\iint_R x^2 dxdy$, where R is the region bounded by the curves $y = x$ and $y = x^2$. Ans. $\frac{1}{20}$[4]

10. $\iint_R x^2 dxdy$, where R is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x, y = 0$ and $x = 8$. Ans. 448 [4]

11. $\iint_R e^{x^2+y^2} dy\,dx$ where R is the region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$ Ans : $\frac{\pi}{2}(e-1)$ [1]

CHANG OF ORDER OF INTEGRATION

To evaluate double integrals by changing the order of integration becomes easier.

1:- first draw the region using given limits of integration.

2:- If it is given first to integrate w.r.t. x, then to change the limit draw a vertical strip line and determine the limits.

3:- If it is given first to integrate w.r.t.y, then to change the limit draw a horizontal strip line and determine the limit.

For example:-

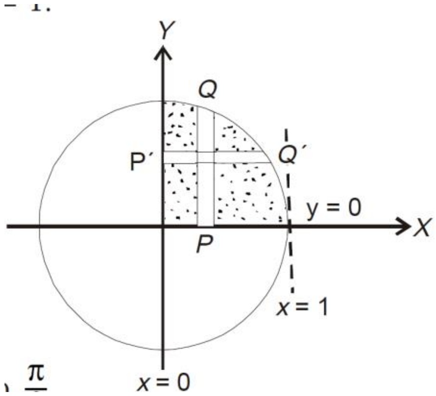
$$\boxed{\int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x,y)\,dx\,dy}$$

Remark:-While changing order of integration integrating function $f(x,y)$ remains unchanged.

Example:1 Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dydx$ by changing the order of integration.

Solution: In the above integral, y on vertical strip (say PQ) varies as a function of x and then the strip slides between $x = 0$ to $x = 1$.

Here , $y = 0$ is the x-axis and $y = \sqrt{1 - x^2}$
i.e.$x^2 + y^2 = 1$ is the circle.
In the changed order, the strip becomes $P'Q'$, P' resting on the curve $x = 0$, Q' on the circle $x = \sqrt{1 - y^2}$ and finally the strip $P'Q'$ sliding between $y = 0$ to $y = 1$.



$$\therefore I = \int_0^1 y^2 \left(\int_0^{\sqrt{1-y^2}} dx \right) dy$$

$$= \int_0^1 y^2 \, [x]_0^{\sqrt{1-y^2}} dy$$

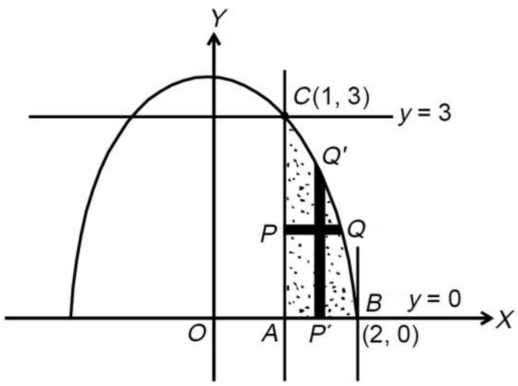
$$= \int_0^1 y^2 \, [\sqrt{1 - y^2}] dy$$

Substitute $y = \sin\theta$ so that $dy = \cos\theta d\theta$ and θ varies from $0 \; to \; \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta d\theta$$
$$= \frac{(2-1)(2-1)}{4 \times 2} \times \frac{\pi}{2}$$
$$= \frac{\pi}{16}$$

Example:2 Evaluate the integral $\int_0^3 \int_1^{\sqrt{4-y}} (x + y) dx dy$ by changing the order of integration.
Solution:

Clearly in the given form of integral, x changes as a function of y (viz.x = f(y))and y as an independent variable changes from 0 to 3.
Thus, the two curves are the straight line x = 1and the parabola, $x = \sqrt{4 - y}$and the common area under consideration is ABQCA.
For changing the order of integration, we need to convert the horizontal strip PQ to a vertical strip P’Q’over which y changes as a function of x and it slides for values of x = 1to x = 2 as shown in Fig



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$\therefore I = \int_1^2 (\int_0^{(4-x^2)} (x+y)dy)dx$

$= \int_1^2 \left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx$

$= \int_1^2 \left(x(4-x^2) + \frac{(4-x^2)^2}{2}\right) dx$

$= \int_1^2 \left(x(4-x^2) + (8 + \frac{x^4}{2} - 4x^2\right) dx$

$= \left[2x^2 - \frac{x^4}{4} + 8x + \frac{x^5}{10} - \frac{4}{3}x^3\right]_1^2$

$= 2(2^2-1^2) - \frac{2^4-1^4}{4} + 8(2-1) + \frac{2^5-1^5}{10} - \frac{4}{3}(2^3-1^3)$

$= \frac{241}{60}$

<u>EXERCISE</u>

Evaluate the following integrals by changing the order:

1. $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy \; dx \, dy$ Ans: 8 [1]
2. $\int_0^\pi \int_x^\pi \frac{siny}{y} \; dy \, dx$ Ans : 2 [1]
3. $\int_0^1 \int_y^1 x^2 e^{xy} dx \, dy$ Ans : $\frac{e-2}{2}$ [1]
4. $\int_0^{2\sqrt{\ln 3}} \int_{\frac{y}{2}}^{\sqrt{\ln 3}} e^{x^2} \; dx \, dy$ Ans : 2 [1]
5. $\int_0^{\frac{1}{16}} \int_{y^{\frac{1}{4}}}^{\frac{1}{2}} \cos(16\pi x^5) dx \, dy$ Ans : $\frac{1}{80\pi}$ [1]
6. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$ Ans. 1 [3]

<u>5.4 Double integral in POLAR COORDINATES</u>

Take r as distance of P from the origin and θ as an angle of $\overline{OP}$ with positive X-axis, then polar coordinates are $\boxed{\boldsymbol{x = r\cos\theta, y = r\sin\theta.}}$

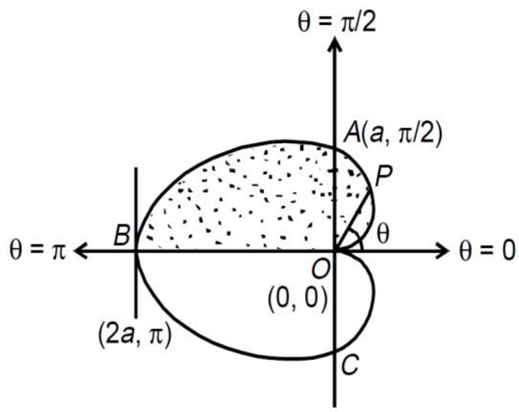
Also $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\frac{y}{x}$.

To evaluate $\iint_R \ f(r,\theta)d\theta$ we first integrate with respect to r keeping θ as a constant and then the resulting expression is integrated with respect to θ.

Example:1 Evaluate $\iint rsin\theta drd\theta$ over the cardiod $r = a(1 - cos\theta)$ above the initial line.

Solution: The region of integration under consideration is the cardiod $r = a(1 - cos\theta)$ above the initial line.

In the cardoid $r = a(1 - cos\theta), \theta = 0\ , r = 0$ $\quad \theta = \frac{\pi}{2}, r = a$ $\quad \theta = \pi, r = 2a$



As clear from the geometry along the radial strip OP, r (as a function of θ) varies from

r = 0 to r = a(1 – cos θ) and then this strip slides from $\theta = 0\ to\ \theta = \pi$ for covering the area above the initial line.

$$\therefore I = \int_0^{\pi}(\int_0^{r=a(1-cos\theta)} rdr)\ sin\theta d\theta$$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1-cos\theta)} sin\theta d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}(1 - cos\theta)^2 sin\theta d\theta$$

$$= \frac{a^2}{2}\left[\frac{(1-cos\theta)^3}{3}\right]_0^{\pi}$$

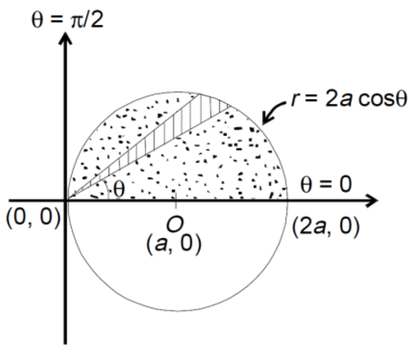
$$= \frac{a^2}{6}[(1 - cos\pi)^3 - (1 - cos0)^3]$$

$$= \frac{4a^2}{3}$$

Example:2 Evaluate $\iint_R r^2 sin\theta dr d\theta$;Where R is the semicircle $r = 2acos\theta$ above the initial line.

Solution: The region R of integration is the semi-circle $r = 2acos\theta$ above the initial line.

For the circle $r = 2acos\theta$ $\theta = 0, r = 2a$ $\theta = \frac{\pi}{2}, r = 0$



$$\therefore \iint_R r^2 sin\theta dr d\theta = \int_0^{\frac{\pi}{2}}\int_0^{2acos\theta} r^2 sin\theta dr d\theta$$
$$= \int_0^{\frac{\pi}{2}}\left(\int_0^{2acos\theta} r^2 dr\right) sin\theta d\theta$$
$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\right]_0^{2acos\theta} sin\theta d\theta$$
$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}}(2a)^3\cos^3\theta(-sin\theta)d\theta$$
$$= -\frac{8a^3}{3}\left[\frac{\cos^4\theta}{4}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{2a^3}{3}$$

Change of Cartesian Integral into polar integral

Let $\iint_R f(x, y)$dA be given any Cartesian integral. To change it into polar integral take $\boxed{\boldsymbol{x = r\cos\theta, y = r\sin\theta}}$ and value of $dx\,dy = dy\,dx = r\,dr\,d\theta$ and we get

$$\boxed{\iint_R \boldsymbol{f(x, y)dxdy} = \iint_R \boldsymbol{f(r, \theta)rdrd\theta}.}$$

Example:1 Evaluate by changing into polar coordinates $\int_0^1\int_0^1 dxdy$.

Solution:

Take $\boxed{\boldsymbol{x = r\cos\theta, y = r\sin\theta}}$

Then value of $dx\,dy = r\,dr\,d\theta$

$\int_0^1\int_0^1 dxdy = \int_0^1\int_0^1 rdrd\theta$

$$= \int_0^1 \left[\frac{r^2}{2}\right]_0^1 d\theta$$

$$= \int_0^1 \frac{1}{2} d\theta$$

$$= \frac{1}{2} [\theta]_0^1$$

$$= \frac{1}{2}$$

<u>Example</u>:2 Evaluate the integral $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2) dxdy$ by changing in polar coordinates.

<u>Solution:</u>

Take $\boxed{\boldsymbol{x = r\cos\theta, y = r\sin\theta}}$

Then value of $dx\, dy = r\, dr\, d\theta$

The parabola $y = \sqrt{\frac{x}{a}}$ implise that $y^2 = \frac{x}{a}$

So, $r^2 \sin^2\theta = \frac{r\cos\theta}{a}$ implise that $r = 0$ or $r = \frac{\cos\theta}{a\sin^2\theta}$

Limits, for the curve , $y = \frac{x}{a}$ implies that $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\frac{1}{a}$

And For the curve , $y = \sqrt{\frac{x}{a}}$ implies that $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\frac{0}{a} = \frac{\pi}{2}$

Hence, I= $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2) dxdy = \int_{\tan^{-1}\frac{1}{a}}^{\frac{\pi}{2}} \left(\int_0^{\frac{\cos\theta}{a\sin^2\theta}} r^3 dr\right) d\theta$

$$= \int_{\cot^{-1} a}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{\frac{\cos\theta}{a\sin^2\theta}} d\theta$$

$$= \frac{1}{4} \int_{\cot^{-1} a}^{\frac{\pi}{2}} \frac{\cos^4\theta}{a^4(\sin^4\theta)\text{^}2} d\theta$$

$$= \frac{1}{4a^4} \int_{\cot^{-1} a}^{\frac{\pi}{2}} \cot^4\theta(1 + \cot^2\theta) cosec^2\theta d\theta$$

Let $cot\theta = t$ then $cosec^2\theta d\theta = -dt$

Also, $\theta = \cot^{-1} a$ implies that $t = a$

$\theta = \frac{\pi}{2}$ implies that $t = 0$

$$\therefore I = \frac{1}{4a^4} \int_a^0 t^4(1 + t^2)(-dt)$$

$$= \frac{1}{4a^4} \int_0^a (t^4 + t^6)\, dt$$

$$= \frac{1}{4a^4} \left[\frac{t^5}{5} + \frac{t^7}{7}\right]_0^a$$

$$= \frac{a}{20} + \frac{a^3}{28}$$

EXERCISE

Change the following Cartesian integrals into equivalent polar integrals.

1. $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\, dy$

2. $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\, dy\, dx$

3. $\int_0^1 \int_0^x \sqrt{x^2+y^2}\, dy\, dx$

4. $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dx\, dy$

5. $\iint_R \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}\, dA$ over the region $1 \le x^2+y^2 \le e$.

Change the following Cartesian integrals into equivalent polar integrals.

1. $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\, dx\, dy$
2. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$
3. $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \left(\frac{x^2-y^2}{x^2+y^2}\right) dx\, dy$
4. $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dy\, dx$
5. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$

JACOBIAN

1: If $u = f(x,y)$ and $v = g(x,y)$ then Jacobian of u, v with respect to x, y is denoted by

$\boxed{\boldsymbol{J(u,v)\ or\ \frac{\partial(u,v)}{\partial(x,y)}}}$ and defined as $J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

2: If $u = f(x,y,z), v = g(x,y,z)\ , w = h(x,y,z)$ then

$$\boxed{\boldsymbol{J(u,v,w)} = \frac{\partial(\mathrm{u,v,w})}{\partial(\mathrm{x,y,z})} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}}$$

Properties of Jacobians

1: If $u = f(x,y)$, $v = g(x,y)$ and $\boxed{J = \frac{\partial(u,v)}{\partial(x,y)}}$ and $\boxed{J^* = \frac{\partial(x,y)}{\partial(u,v)}}$ then $\boxed{J.J^* = 1.}$

2: $\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}}$ Where u and v are functions of r and s .Also r and s are functions of x and y.

3: $\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1.}$

4: $\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(x,y,z)}{\partial(r,s,t)} \cdot \frac{\partial(r,s,t)}{\partial(x,y,z)}}$

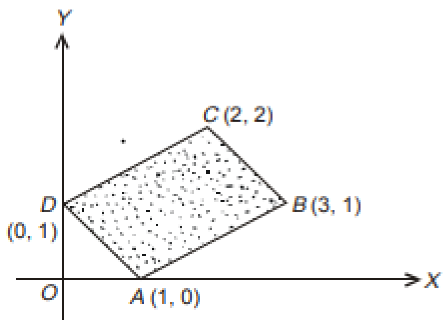
Change of variables in Double integrals by Jacobians

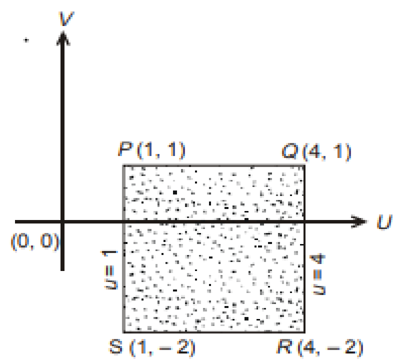
Let $\iint_R \ f(x,y)\,dx\,dy$ be given. If we take transformation $x = g(u,v)\ \ and\ y = h(u,v)$ then $\boxed{\iint_R \ f(x,y)\,dx\,dy = \iint_S \ f(g(u,v),h(u,v)).\,|J|du\,dv}$

Where J = $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ and $|J|$ means to take modulus.

Example:1 Evaluate $\iint_R \ (x+y)^2 dxdy$, where the region R is parallelogram in xy plane with vertices $(1,0)$, $(3,1)$, $(2,2)$, $(0,1)$ using the transformation $u = x + y$ and $v = x - 2y$.

Solution: R_{xy} is the region bounded by the parallelogram ABCD in the xy plane which on transformation becomes R'_{uv} i.e., the region bounded by the rectangle PQRS, as shown in the Figs.





With $u = x + y$ and $v = x - 2y$

A (1, 0) transforms to P (1, 1)
B (3, 1) transforms to Q (4, 1)
C (2, 2) transforms to R (4, – 2)
D (0, 1) transforms to S (1, – 2)

Also J = $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$

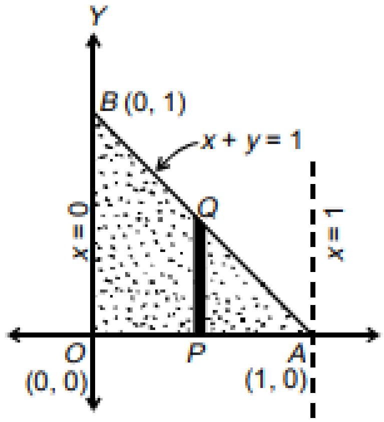
Hence, $\iint_R (x+y)^2 dxdy = \iint_R u^2 \frac{1}{3} dudv$

$$= \int_1^4 \int_{-2}^1 \frac{u^2}{3} dudv$$
$$= \int_1^4 [v]_{-2}^1 \frac{u^2}{3} du$$
$$= \frac{1}{3}(1+2)\int_1^4 u^2 du$$
$$= \frac{3}{3}\left[\frac{u^3}{3}\right]_1^4$$
$$= \frac{64}{3} - \frac{1}{3}$$
$$= \frac{63}{3}$$
$$= 21.$$

<u>Example:2</u> Using transformation $x = u + v$, $y = uv$ find $\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} dxdy.$

<u>Solution</u>: Clearly $y = f(x)$ represents curves $y = 0$ and $y = 1 - x$, and x which is an independent variable changes from $x = 0$ to $x = 1$.

Thus, the area OABO bounded between the two curves $y = 0$ and $x + y = 1$ and the two ordinates $x = 0$ and $x = 1$ is shown in Fig.



On using transformation; $x = u + v$ implies that $x = u(1 - v)$

$y = uv$ implies that $y = uv$

Now point O(0, 0) implies 0 = u(1 – v) …(1) and

0 = uv …(2)

From (2), either u = 0 or v = 0 or both zero.

From (1), we get u = 0, v = 1

Hence (x, y) = (0, 0) transforms to (u, v) = (0, 0), (0, 1)

Point A(1, 0), implies 1 = u(1 – v) …(3)

0 = uv … (4)

From (4) either u = 0 or v = 0, If v = 0 then from (3) we have u = 1, again if u = 0, equation (3) is inconsistent.

Hence, A(1, 0) transforms to (1, 0), i.e. itself.

From Point B(0, 1), we get 0 = u(1 – v) …(5) and

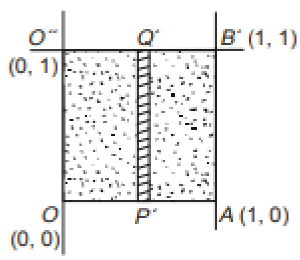
1 = vu …(6)

From (5), either u = 0 or v = 1

If u = 0, equation (6) becomes inconsistent.

If v = 1, the equation (6) gives u = 1.

Hence (0, 1) transform to (1, 1). See Fig



Also J = $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = u$

Hence, $\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}} dxdy = \int_0^1 \int_0^1 ue^v dudv$

$= \int_0^1 u(\int_0^1 e^v dv) du$

$= \int_0^1 u(e - 1) du$

$$= (e-1)\left[\frac{u^2}{2}\right]_0^1$$
$$= \frac{1}{2}(e-1)$$

EXERCISE

Q-1 Given that $x + y = u,\ y = uv$, change the variables to u, v in the integral $\iint [xy(1-x-y)]^{\frac{1}{2}}\ dx\,dy$ taken over the area of the triangle with sides $x = 0, y = 0, x + y = 1$ and hence evaluate it.

Q-2 Evaluate $\iint (x^2 - y^2)^2$dA, over the area bounded by the lines $|x| + |y| = 1$ using the transformations $x + y = u, x - y = v$.

Q-3. Evaluate $\int_0^4 \int_{y/2}^{\frac{y}{2}+1} \frac{2x-y}{2} dxdy$ by applying the transformation $u = \frac{2x-y}{2}$ and $v = \frac{y}{2}$.

Ans: 2

Q-4. Applying the transformation, evaluate $\iint_R\ (x-y)^4 e^{x+y}\ dx\ dy$, where R is the square with vertices (1,0), (2,1) , (1,2),(0,1) . Ans: $\frac{e^3-e}{5}$

Q-5.Evaluate $\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\ dy$, where R in the first quadrant in the xy-plane bounded by the hyperbolas $xy = 1,\ xy = 9$ and the lines $y = x, y = x$ using the transformation $x = \frac{u}{v}, y = uv$ with $u > 0, v > 0$. Ans: $8 + \frac{52}{3}\ln 2$

AREA USING DOUBLE INTEGRATION

1:- Area in Cartesian coordinates is defined by $\boldsymbol{A = \iint_R\ dx\,dy = \iint_R\ dy\,dx}$

Find limits of integration according to closed bounded region R.

2:- Area in polar coordinates is $\boldsymbol{A = \iint_R\ r\,dr\ d\theta = \int_\theta \int_r\ r\,dr\,d\theta}$

First find limit of r and then find of θ.

Example:1 By using Double integration, find the area bounded by the curve $y = 2 - x^2$ and y=x.

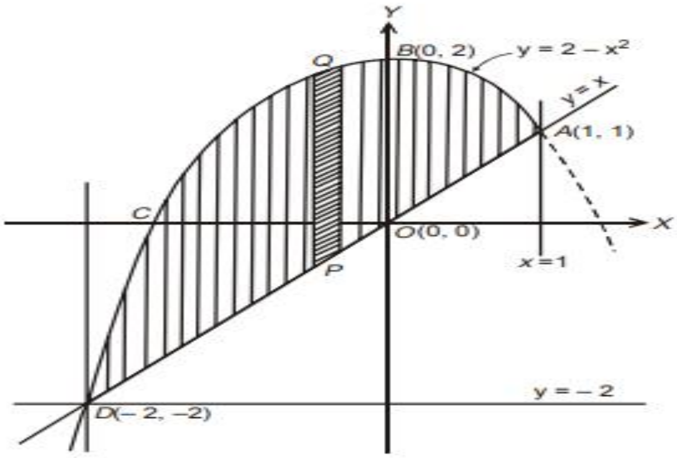
Solution: The given curve $y = 2 - x^2$ is parabola.

It passes through the points (0, 2) , (1, 1) , (2,- 2) , (-1, 1) , (-2 , -2)

The curve $y = x$ is a straight line.

It passes through the points (0, 0) , (1, 1) , (-2, -2)

the two curves intersect at (1, 1) and (–2, –2), Clearly, the area need to be required is ABCDA.



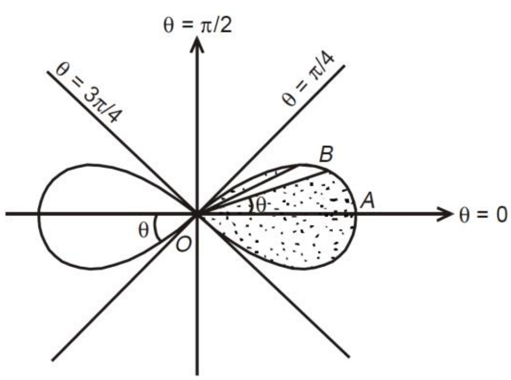
Hence, $A = \int_{-2}^{1}\int_{x}^{2-x^2} dydx$

$= \int_{-2}^{1}(2 - x^2 - x)dx$

$= \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1}$

$= \frac{9}{2}$

Example:2 Find by double integration, the area of lemniscate $r^2 = a^2cos2\theta$.
Solution: As the given curve $r^2 = a^2cos2\theta$ contains cosine terms only and hence it is Symmetric about the initial axis.
Further the curve lies wholly inside the circle $r = a,$ since the maximum value of $|cos\theta|$ is 1.
Also, no portion of the curve lies between $\theta = \frac{\pi}{4}$ to $\theta = \frac{3\pi}{4}$ and the extended axis.
See the geometry, for one loop, the curve is bounded between $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$



Hence, Area= $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{\sqrt{a^2cos2\theta}} rdrd\theta$

$= 4\int_{0}^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_{0}^{\sqrt{a^2cos2\theta}} d\theta$

$= 2a^2\int_{0}^{\frac{\pi}{4}} cos2\theta d\theta$

$$= 2a^2\left[\frac{sin2\theta}{2}\right]_0^{\frac{\pi}{4}}$$
$$= a^2$$

EXERCISE

1. Find the area bounded by y-axis, the line $y = 2x$ and the line $y = 4$.
2. Find the area bounded by the lines $y = 2 + x$, $y = 2 - x$ and the line $x = 5$.
3. Find the area bounded by the parabola $y^2 + x = 0$ and the line $y = x + 2$.
4. Find the area bounded by the parabolas $y^2 = x$, $x^2 = -8y$.
5. Find the area bounded by x-axis, the circle $x^2 + y^2 = 16$ and the line $y = x$.
6. Find the area bounded by the curves $y^2 = 4x$ and $2x - 3y + 4 = 0$.
7. Find the area bounded by the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

TRIPLE INTEGRATION

Let $f(x, y, z)$ be a continuous function defined in a closed and bounded region V in 3-dimensional space, then triple integral over the region V is denoted by $\iiint_V f(x, y, z)dV$.

It is defined by $\iiint_V f(x, y, z)dV = \lim_{\substack{n\to\infty \\ \delta V_r\to 0}} \sum_{r=1}^{n} f\left(x_r, y_r, z_r\right)\delta V_r$, $Where\ dV = dx\,dy\,dz.$

Example(1): Evaluate: $\int_0^3 \int_0^2 \int_0^1 (x + y + z)\,dx\,dy\,dz$

Solution: Integrating first w.r.t. x keeping y and z constant we get

$$\int_0^3 \int_0^2 \int_0^1 (x + y + z)\,dx\,dy\,dz = \int_0^3 \int_0^2 \left(\frac{x^2}{2} + yx + zx\right)_0^1 dy\,dz$$

$$= \int_0^3 \int_0^2 \left(\frac{1}{2} + y + z\right) dy\,dz$$

$$= \int_0^3 \left(\frac{y}{2} + \frac{y^2}{2} + zy\right)_0^2 dz$$

$$= \int_0^3 (1 + 2 + 2z)\,dz$$

$$= (3z + z^2)_0^3$$

$$= (9 + 9) = 18$$

Example(2): Evaluate $\int_{y=0}^{3} \int_{x=0}^{y} \int_{z=0}^{x} \frac{1}{x}\,dz\,dx\,dy$

Solution:

$$\int_{y=0}^{3}\int_{x=0}^{y}\int_{z=0}^{x}\frac{1}{x}\,dz\,dx\,dy = \int_0^3\int_0^y\frac{1}{x}(z)_{z=0}^{z=x}\,dx\,dy$$

$$= \int_0^3 (x)_0^y\,dy$$

$$= \int_0^3 y\,dy$$

$$= \left(\frac{y^2}{2}\right)_0^3$$

$$= \frac{9}{2}$$

<u>EXERCISE</u>

Evaluate the following triple integrals:

1. $\int_0^1\int_0^{1-z}\int_0^2 dx\,dy\,dz$
2. $\int_0^1\int_0^{1-y}\int_0^2 dx\,dy\,dz$
3. $\int_0^1\int_0^1\int_0^1(x^2+y^2+z^2)dx\,dy\,dz$
4. $\int_o^1\int_0^{\pi}\int_0^{\pi} y\sin z\;\,dx\,dy\,dz$
5. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx$
6. $\int_{ln6}^{ln7}\int_0^{ln2}\int_{ln4}^{ln5} e^{(x+y+z)}dz\,dy\,dx$
7. $\int_0^1\int_0^{x^2}\int_0^{x+y}(2x-y-z)\,dz\,dy\,dx$
8. $\int_0^1\int_0^2\int_1^2 x^2yz\,dx\,dy\,dz$
9. $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$
10. $\int_{-1}^1\int_0^z\int_{x-z}^{x+z}(x+y+z)dx\,dy\,dz$
11. $\int_0^1\int_0^{\sqrt{(1-x^2)}}\int_0^{\sqrt{(1-x^2-y^2)}} xyz\,dx\,dy\,dz$
12. $\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^a r^2 sin\theta\,dr\,d\theta\,d\emptyset$

Evaluate the following triple integrals:

1. $\iiint_R\ (x+y+z)dx\,dy\,dz$, where $0\le x\le 1,\ \ 1\le y\le 2,\ \ 2\le z\le 3$
   Ans :$\frac{9}{2}$[4]
2. $\iiint_R\ (x-2y+z)dx\,dy\,dz$, where $0\le x\le 1,\ \ 0\le y\le x^2, 2\le z\le x+y$ Ans :$\frac{8}{35}$
   [4]

3. $\iiint_R (x^2 + y^2 + z^2)dx\,dy\,dz$,where R denotes the region bounded by $x = 0, y = 0\ and\ x + y + z = a,\ a > 0$. Ans :$\frac{a^2}{25}$ [4]

4. $\iiint \frac{1}{(x+y+z+1)^3} dx\,dy\,dz$, if the region of integration is bounded by the co-ordinate planes and the plane $x + y + z = 1$. Ans :$\frac{1}{2}\log 2 - \frac{5}{16}$ [4]

5. $\iiint_s xyz\,dx\,dy\,dz$,where $S = [(x, y, z): (x^2 + y^2 + z^2) \leq 1,\ x \geq 0,\ y \geq 0, z \geq 0]$Ans :$\frac{1}{48}$ [4]

6. $\iiint_s \sqrt{x^2 + y^2}\,dx\,dy\,dz$ where S is the solid bounded by the surfaces $x^2 + y^2 = z^2, z = 0, z = 1$.Ans :$\frac{\pi}{6}$[4]

7. $\iiint x^2yz\,dx\,dy\,dz$ throughout the volume bounded by the planes $x = 0, y = 0, z = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$Ans: $\frac{a^3b^2c^2}{2520}$ [4]

8. $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}\,dx\,dy\,dz$taken throughout the volume of the sphere $x^2 + y^2 + z^2 = 1$ lying in the first octant. Ans: $\frac{\pi^2}{8}$ [4]