(1)

Taylor's series:-

(1) $f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x)$

$+ - - -$

(2) Taylor's series in powers of $x-a$

$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a)$

$+ \frac{(x-a)^3}{3!} f'''(a) + - - -$

(It is expansion of $f(x)$ about $x=a$)

ex: Expand $\sin x$ in powers of $x - \frac{\pi}{6}$

$f(x) = \sin x$, $\quad x - a = x - \frac{\pi}{6}$

$a = \pi/6$

$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a)$

$+ \frac{(x-a)^3}{3!} f'''(a) + - - -$

$f(x) = f(\frac{\pi}{6}) + (x - \frac{\pi}{6}) f'(\frac{\pi}{6}) + \frac{(x-\frac{\pi}{6})^2}{2!} f''(\frac{\pi}{6})$

$+ \frac{(x-\frac{\pi}{6})^3}{3!} f'''(\frac{\pi}{6}) + - - -$

—— (1)

$f(x) = \sin x \quad \therefore f(\frac{\pi}{6}) = \sin\frac{\pi}{6} = \frac{1}{2}$

$f'(x) = \cos x \quad \therefore f'(\frac{\pi}{6}) = \cos\frac{\pi}{6}$

$= \sqrt{3}/2$

$f''(x) = -\sin x \therefore f''(\frac{\pi}{6}) = -\sin\frac{\pi}{6}$

$= -\frac{1}{2}$

$f'''(x) = -\cos x \therefore f'''(\frac{\pi}{6}) = -\cos\frac{\pi}{6}$

$= -\sqrt{3}/2$

From ①

$$f(x) = \frac{1}{2} + (x - \frac{\pi}{6})(\frac{\sqrt{3}}{2}) + \frac{(x-\frac{\pi}{6})^2}{2!}(-\frac{1}{2}) + \frac{(x-\frac{\pi}{6})^3}{3!}(-\frac{\sqrt{3}}{2}) + \cdots$$

ex Find the Taylor's series expansion of $\frac{1}{x}$ in powers of $x-2$

$f(x) = \frac{1}{x}$, $\quad x-2 = x-a$

$-2 = -a$

$\therefore a = 2$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots \quad ①$$

$f(x) = \frac{1}{x} \therefore f(2) = \frac{1}{2}$

$f'(x) = -\frac{1}{x^2} \therefore f'(2) = -\frac{1}{2^2} = -\frac{1}{4}$

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$$f''(x) = \frac{-(-2)}{x^3} = \frac{2}{x^3}$$

$$f''(2) = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$$

$$f'''(x) = \frac{2(-3)}{x^4} = -\frac{6}{x^4}$$

$$f'''(2) = -\frac{6}{2^4} = -\frac{6}{16} = -\frac{3}{8}$$

From (1)

$$f(x) = \frac{1}{2} + (x-2)\left(-\frac{1}{4}\right) + \frac{(x-2)^2}{2!}\left(\frac{1}{4}\right) + \frac{(x-2)^3}{3!}\left(-\frac{3}{8}\right) + \cdots$$

ex Find the approximate value of $\sqrt{26}$ correct up to four ~~decim~~ decimal place by using Tayor's series.

Let $f(x) = \sqrt{x} = x^{1/2}$

$x+h = 26$

$x+h = 25+1 \quad \therefore x = 25, h = 1$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$f(26) = f(25) + (1) f'(25) + \frac{1^2}{2\times 1} f''(25) + \cdots \quad (1)$$

$f(x) = \sqrt{x} \quad \therefore f(25) = \sqrt{25} = 5$

(4)

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \therefore f'(25) = \frac{1}{2\sqrt{25}}$$

$$f'(x) = \frac{1}{2 \cdot x^{1/2}} \qquad = \frac{1}{2(5)} = \frac{1}{10}$$

$$f''(x) = \frac{(-1/2)(1)}{2 \cdot x^{3/2}} = -\frac{1}{4} \cdot \frac{1}{x^{3/2}}$$

$$f''(25) = -\frac{1}{4} \cdot \frac{1}{(25)^{3/2}}$$

$$= -\frac{1}{4} \cdot \frac{1}{(5^2)^{3/2}}$$

$$= -\frac{1}{4} \cdot \frac{1}{5^3} = -\frac{1}{4} \cdot \frac{1}{125}$$

$$= -\frac{1}{500}$$

from (1)

from (1)

$$f(26) = 5 + \frac{1}{10} + \frac{1}{4}\left(-\frac{1}{500}\right)$$

$$= 5 + 0.1 - 0.0005$$

$$= 5.1 - 0.0005$$

$$= 5.0995$$

Ex: find approximate value of $\sqrt[3]{63}$ using Taylor's series

$f(x) = \sqrt[3]{x} = x^{1/3}$ (5)

$x+h = 63$

$x+h = 64-1$

$x = 64, \quad h = -1$

$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \cdots$

$f(63) = f(64) + (-1)\, f'(64) + \frac{(-1)^2}{2!} f''(64) + \cdots$ —— (1)

$f(x) = \sqrt[3]{x} = x^{1/3}$

$f(64) = \sqrt[3]{64} = (64)^{1/3} = (4^3)^{1/3} = 4$

$f'(x) = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3} x^{-2/3}$

$f'(64) = \frac{1}{3} (64)^{-2/3}$

$= \frac{1}{3} (4^{\cancel{3}})^{-2/\cancel{3}}$

$= \frac{1}{3} \cdot 4^{-2} = \frac{1}{3} \cdot \frac{1}{4^2}$

$= \frac{1}{3} \cdot \frac{1}{16} = \frac{1}{48} = 0.02083$

$f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right) x^{-2/3-1}$

$= -\frac{2}{9} x^{-5/3}$

$f''(64) = -\frac{2}{9} (64)^{-5/3}$

$= -\frac{2}{9} (4^{\cancel{3}})^{-5/\cancel{3}}$

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$= -\frac{2}{9} \cdot 4^{-5}$

$= -\frac{2}{9} \cdot \frac{1}{4^5}$

$= -\frac{2}{9} \cdot \frac{1}{1024}$

$= -\frac{1}{9} \cdot \frac{1}{512}$

$= -\frac{1}{4608}$

$= -0.000217$

From ①

$f(63) = 4 - 0.02083 + \frac{1}{2}(-0.000217) + \ldots$

$= 4 - 0.02083 - 0.0001085 + \ldots$

$= 4 - 0.0209$

$= 3.9791$

~~maclau~~ maclaurin's series:-

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$

i.e expansion of $f(x)$ in powers of x

ex: Find the maclaurin's series of $\sin x$.

$\rightarrow f(x) = \sin x$

maclaurin's series is,

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$ ——①

$f(x) = \sin x \quad \therefore f(0) = \sin 0 = 0$

$f'(x) = \cos x \quad \therefore f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \quad \therefore f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \quad \therefore f'''(0) = -\cos 0 = -1$

From ①

$$f(x) = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \cdots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \cdots$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

~~similarly~~

Note:-

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Ex: Find expansion of $\log(1+x)$ in powers of x

$f(x) = \log(1+x)$

Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad \text{— ①}$$

$f(x) = \log(1+x) \quad \therefore f(0) = \log(1+0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x} \quad \therefore f'(0) = \frac{1}{1+0} = 1$

$f''(x) = -\frac{1}{(1+x)^2} \quad \therefore f''(0) = -\frac{1}{(1+0)^2} = -1$

$f'''(x) = -\frac{(-2)}{(1+x)^3} \quad \therefore f'''(0) = \frac{2}{(1+0)^3} = 2$

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from (1)

$$f(x) = 0 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(2) + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2\times1} + \frac{x^3}{3\times2\times1}\times2 + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Note:

(1) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

(2) using maclaurin's series we can find

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \cdots$$

~~sombre~~

Ex: Find the maclaurin's series of $e^x \sin 2x$

$f(x) = e^x \sin 2x$

maclaurin's series is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots \quad (1)$$

$f(x) = e^x \sin 2x \therefore f(0) = e^0 \sin 0 = 0$

$f'(x) = e^x \sin 2x + e^x \cos 2x \cdot 2$

$f'(x) = e^x \sin 2x + 2e^x \cos 2x$

$= f(x) + 2e^x \cos 2x$

⑨

$f'(0) = f(0) + 2e^0\cos 0$

$f'(0) = 0 + 2(1)(1) = 2$

$f''(x) = f'(x) + 2[e^x\cos 2x + e^x(-\sin 2x)\cdot 2]$

$f''(x) = f'(x) + 2e^x\cos 2x - 4\underline{e^x\sin 2x}$

$f''(x) = f'(x) + \cancel{2e^x - 4f(x)}\, 2e^x\cos 2x \cancel{-f(x)} - 4f(x)$

$f''(0) = f'(0) + 2e^0 - 4f(0)$

$= 2 + 2(1) - 4(0) = 4$

$f'''(x) = f'(x) + 2e^x\cos 2x + 2e^x(-\sin 2x)\cdot 2 - 4f'(x)$

$= f'(x) + 2e^x\cos 2x - 4e^x\sin 2x - 4f'(x)$

$= -3f'(x) + 2e^x\cos 2x - 4e^x\sin 2x$

$f'''(0) = -3f'(0) + 2e^0\cos 0 - 4e^0\sin 0$

$= -3(2) + 2(1)(1) - 0$

$= -6 + 2 = -4$

from ①

$f(x) = 0 + x(2) + 4\frac{x^2}{2!} + (-4)\frac{x^3}{3!} + \text{---}$

$= 2x + \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}\times 1} - \overset{2}{\cancel{4}}\frac{x^3}{3\times\cancel{2}\times 1} + \text{---}$

$= 2x + 2x^2 - \frac{2}{3}x^3 + \text{---}$