2.9 Derive the equation $V_C = V\,[1 - e^{-t/RC}]$ for charging of a capacitor.

Ans. Figure 2.11 shows a capacitor of capacitance C connected in series with a noninductive resistor R and the combination is connected to a d.c. supply of V volts through a switch S. For charging the switch must be closed in position 'a'. At the instant of closing the switch S, there is no charge on the capacitor and therefore no potential difference across it. As a result, the whole of the applied voltage must momentarily act across a non-inductive resistor R. As a result, initial value of charging current becomes equal to $\frac{V}{R}$ which is maximum possible. The charging current gradually decreases from its maximum value till it finally becomes zero when the potential difference across the capacitor plates becomes equal and opposite to the supply voltage V.

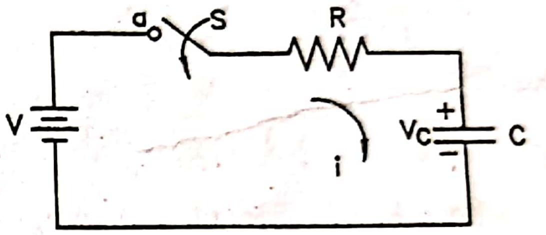

FIG. 2.11

After closing the switch, the capacitor starts charging and voltage across it increases gradually.

$V = V_R + V_C$

Let at any instant during charging,

v_C = potential difference across the capacitor

i = charging current

q = charge on the capacitor

According to Kirchhoff's second law,

Applied voltage = p.d. across R + p.d. across C

$$\therefore \quad V = iR + v_C \quad \text{... (i)}$$

$$= C\frac{dv_c}{dt} \times R + v_c \qquad \left[\because i = \frac{dq}{dt} = \frac{d}{dt}(Cv_C) = C\frac{dv_C}{dt}\right]$$

$$= v_C + RC\frac{dv_C}{dt}$$

KVL

$$\boxed{V - v_C = RC\frac{dv_C}{dt}}$$

$$\therefore \quad \frac{dv_C}{V - v_C} = \frac{1}{RC}dt$$

Multiplying both the sides by negative sign

$$\therefore \quad \frac{-dv_C}{V - v_C} = \frac{-1}{RC}dt$$

Integrating on both sides

$$\int \frac{-d\,v_c}{V - v_c} = \int \frac{-1}{RC}dt$$

$$\therefore \quad \log_e (V - v_c) = \frac{-t}{RC} + K \quad \text{... (ii)}$$

where K is a constant of integration.

Its value can be determined from the initial conditions. At the instant of closing the switch, the initial conditions are,

$$t = 0 \quad \text{and} \quad v_C = 0$$

Substituting the initial conditions in eqn. (ii), we get

$$\log_e (V - 0) = 0 + K$$

$$\therefore \boxed{K = \log_e V}$$

Substituting the value of K in eqn. (ii), we get,

$$\log_e (V - v_c) = -\frac{t}{RC} + \log_e V$$

$$\therefore \quad \log_e (V - v_c) - \log_e V = -\frac{t}{RC}$$

$$\therefore \quad \log_e \left(\frac{V - v_c}{V}\right) = -\frac{t}{RC}$$

$$\frac{V - v_c}{V} = e^{-t/RC}$$

$V - Ve^{-t/RC} = v_c$

$$V - v_c = V\,e^{-t/RC}$$

$$\boxed{v_c = V\,[1 - e^{-t/RC}]}$$

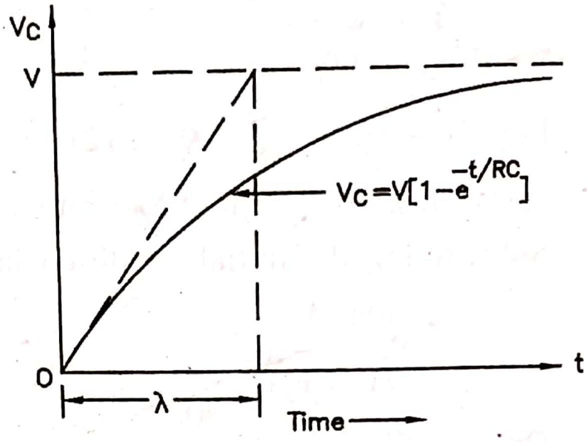

FIG. 2.12

The above equation gives an expression showing the variation of voltage across the capacitor with respect to time during the charging process of the capacitor. It is represented graphically in Fig. 2.12. The variation of voltage across the capacitor follows an 'exponential law'.