Infinite sequence and series

➢ Sequence and series

Definition: An ordered set $\{a_1, a_2, a_3, \ldots\}$ of real numbers is called a sequence of real number OR An infinite sequence of numbers is a function from N to R

- **EXAMPLE** 1) $2,4,6,8,10,12, \ldots, 2n, \ldots$

  Sol: Here $F: N \to R$

  Where $F(n) = 2n = a_n$

  2) $a_n = \sqrt{n}$,

  3) $b_n = (-1)^{(n+1)}\frac{1}{n}$,

  4) $c_n = \frac{n-1}{n}$

➢ Convergence And Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases

EX: $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$

Whose terms approach 0 to n gets large OR we can say an converge to 0 as n→ ∞

On the other hand a sequence like $\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ has only two values 1 and -1 which never converging to a single value

- **Definitions: Convergence, Divergence, Limit**

  Let $\{a_n\}$ be a sequence of real numbers .The sequence $\{a_n\}$ converges to the number L if for every positive number ϵ there exist an integer $|a_n - L| < \epsilon$ whenever $n \geq N$
  If no such number L exist then we say that $\{a_n\}$ diverges.
  If $\{a_n\}$ converges to L ,We write $\lim_{n \to \infty} a_n = L$
  L is called limit of $\{a_n\}$

- **Theorem**: Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers.

  If $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$. Then

  1) $\lim_{n \to \infty} (a_n \pm b_n) = A \pm B$ (sum/difference rule)
  2) $\lim_{n \to \infty} (a_N \cdot b_n) = A \cdot B$(product rule)
  3) $\lim_{n \to \infty} (K \cdot b_n) = K \cdot B$(constant rule)
  4) $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ (quotient rule)

- **EXAMPLE: Solve using above theorem**

  1) $\lim_{n \to \infty} \left(-\frac{1}{n}\right)$

SOLUTION: $\lim_{n \to \infty} (-\frac{1}{n}) = \lim_{n \to \infty} (-1)\left(\frac{1}{n}\right)$

$=(-1)\lim_{n \to \infty} \frac{1}{n}$ (rule 4)

$=(-1)(0)$ $\left(\lim_{n \to \infty} \frac{1}{n} = 0\right)$

Hence, $\lim_{n \to \infty} \left(-\frac{1}{n}\right) = 0$

- **MONOTONIC SEQUANCE:**

  i. $\{a_n\}$ is said to be monotonically increasing if $a_{n+1} \geq a_n$for all n. And strictly increasing if $a_{n+1} > a_n$,for all n.

  ii. $\{a_n\}$ is said to be monotonically decreasing if $a_{n+1} \leq a_n$for all n. And strictly decreasing if $a_{n+1} < a_n$,for all n.

- **BOUNDED SEQUENCE:**

  A sequence $\{a_n\}$ is said to be bounded if there exist numbers m and M such that $m < a_n < M$,for all n.

- **THE SANDWICH THEOREM FOR SEQUANCE** :

  Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequence of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N , And if $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$,Then $\lim_{n \to \infty} b_n = L$.

  - **EXAMPLES :**

    1. $\lim_{n \to \infty} \left(\frac{cosn}{n}\right)$

Solution: We know that

$$\Rightarrow\ \boldsymbol{-1 \le cosn \le 1}$$

$$\Rightarrow -\frac{\boldsymbol{1}}{\boldsymbol{n}} \le \frac{\boldsymbol{cosn}}{\boldsymbol{n}} \le \frac{\boldsymbol{1}}{\boldsymbol{n}}$$

$$\Rightarrow \lim_{n \to \infty}\left(-\frac{1}{n}\right) \le \lim_{n \to \infty}\left(\frac{cosn}{n}\right) \le \lim_{n \to \infty}\left(\frac{1}{n}\right)$$

But we know that $\lim_{n \to \infty}\left(\frac{1}{n}\right) = 0.$

Thus $\lim_{n \to \infty}\left(\frac{cosn}{n}\right) = 0$.

- **THEOREM :**

The following sin sequences converges to the limit listed below

1) $\boldsymbol{\lim_{n \to \infty} \frac{ln(n)}{n} = 0}$
2) $\boldsymbol{\lim_{n \to \infty} (n)^{\frac{1}{n}} = 1}$
3) $\boldsymbol{\lim_{n \to \infty} x^{\frac{1}{n}} = 1}$ $\qquad (\boldsymbol{For\ x} > 0)$
4) $\boldsymbol{\lim_{n \to \infty} x^{n} = 0}$ $\qquad (\boldsymbol{For}\ |\boldsymbol{x}| < 1)$
5) $\boldsymbol{\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{n} = e^{x}}$ $\qquad (\boldsymbol{Any\ x})$
6) $\boldsymbol{\lim_{n \to \infty} \frac{x^{n}}{n!} = 0}$ $\qquad (\boldsymbol{Any\ x})$

- **NOTE : In formulas (3) to (6) ,x remains fixed as $n \to \infty$**
- **EXAMPLES :**

1) $\left(\frac{\ln(n^2)}{n}\right)$

SOLUTION: Here $\boldsymbol{\lim_{n \to \infty}\left(\frac{\ln(n^2)}{n}\right) = 2 \cdot \lim_{n \to \infty}\left(\frac{\ln(n)}{n}\right)}$

$$\boldsymbol{= 2 \cdot 0} \qquad \textbf{(By (1))}$$

$$= 0$$

2) $\left(-\frac{1}{2}\right)^{n}$

SOLUTION: $\boldsymbol{\lim_{n \to \infty}\left(-\frac{1}{2}\right)^{n} = 0}$ $\qquad (\boldsymbol{By}\ (4)\ \boldsymbol{where\ x} = \left(\frac{1}{2}\right))$

- **<u>NON-DECREASING SEQUENCE</u> :**

  A sequence $\{a_n\}$ with the property that $\{a_n\} \leq \{a_{n+1}\}$ for all n is called non-decreasing sequence.

  i.e. $a_{n+1} - a_n \geq 0$, $a_n > 0$ for all n

  - **<u>EXAMPLES</u>** :

    1) $\frac{\mathbf{n^2}}{\mathbf{e^n}}$

    SOLUTION: $\boldsymbol{a_n = \frac{n^2}{e^n} \Rightarrow a_{n+1} = \frac{(n+1)^2}{e^{n+1}}}$

    Now,

$$\boldsymbol{a_{n+1} - a_n = \frac{(n+1)^2}{e^{n+1}} - \frac{n^2}{e^n}}$$

$$\boldsymbol{= \frac{(n+1)^2 - e \cdot n^2}{e^{n+1}}}$$

$$\boldsymbol{= \frac{n^2 + 2 \cdot n + 1 - e \cdot n^2}{e^{n+1}}}$$

$$\boldsymbol{= \frac{(1-e) \cdot n^2 + 2 \cdot n + 1}{e^{n+1}}}$$

$$\boldsymbol{= \frac{(1-e) \cdot n^2}{e^{n+1}} + \frac{(2 \cdot n)}{e^{n+1}} + \frac{1}{e^{n+1}}}$$

$$> 0$$

$$\boldsymbol{\therefore a_n \; is \; non - decreasing \; sequence}$$

    2) $\boldsymbol{\frac{3^n}{n}}$

    SOLUTION: $\boldsymbol{a_n = \frac{3^n}{n}}$ **and** $\boldsymbol{a_{n+1} = \frac{3^{n+1}}{n+1}}$

    Now,

$$\boldsymbol{a_{n+1} - a_n = \frac{3^{n+1}}{n+1} - \frac{3^n}{n}}$$

$$\boldsymbol{= 3^n \cdot \left(\frac{3^{n+1}}{n+1} - \frac{3^n}{n}\right)}$$

$$\boldsymbol{= 3^n \cdot \left(\frac{3}{n+1} - \frac{1}{n}\right)}$$

$$= \mathbf{3^n} \cdot \left(\frac{(\mathbf{3 \cdot n - n - 1})}{\mathbf{n \cdot (n+1)}}\right)$$

$$= \mathbf{3^n} \cdot \frac{\mathbf{2 \cdot n - 1}}{\mathbf{n \cdot (n+1)}}$$

$$> 0$$

$\therefore \boldsymbol{a_{n+1}} - \boldsymbol{a_n} > 0 \ \Rightarrow \boldsymbol{a_{n+1}} > \boldsymbol{a_n}$

$\therefore \boldsymbol{a_n}$ **is non-decreasing sequence**

Now, If $\boldsymbol{a_n} = \frac{\boldsymbol{n}}{\mathbf{3^n}} \Rightarrow \boldsymbol{a_{n+1}} - \boldsymbol{a_n} < 0$

$\Rightarrow \boldsymbol{a_{n+1}} < \boldsymbol{a_n}$

$\therefore \boldsymbol{a_n}$ **is decreasing sequence**

<u>SERIES:</u>

- **<u>DEFINATION</u>:** Let $\{a_n\}$ be any sequence of real numbers. An infinite series is the sum of an infinite sequence of number $a_1 + a_2 + a_3 + \cdots + a_{n+\cdots}$

  It is denoted by $\sum_{n=1}^{\infty} a_n$
- The sum of first n terms,

  $S_n = a_1 + a_2 + \cdots + a_n$ is called n^{th}partial sum.

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$.$$
$$.$$
$$.$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a\_n.$$

<u>EXAMPLES:</u> 1) $\sum_{n=1}^{\infty} \frac{1}{n}$

2)$\sum_{n=1}^{\infty} \frac{n+1}{n}$

3) $\sum_{n=1}^{\infty} \frac{2 \cdot n}{n+1}$

- **<u>DEFINATION</u>: <u>CONVERGENCE OF A SERIES:</u>**

  If the sequence of partial sum converges to the limit$L$, we say that series converges and that its sum is$L$.

  We write $a_1 + a_2 + a_3 + \cdots + a_n = \sum_{n=1}^{\infty} a_n = L$

  If the sequence of partial sums of the series does not converge, we say that the series diverges.

<u>EXAMPLES:</u>1) Discuss the convergence of $\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}}$

SOLUTION: We are given $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

Here $a_n = \frac{1}{2^{n-1}}$

$$S_1 = a_1 = 1$$

$$S_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2} = 2 - \frac{1}{2}$$

$$S_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 2 - \frac{1}{8}$$

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$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$

$\therefore S_n = 2 - \frac{1}{2^{n-1}}$

Now $\lim_{n \to \infty} S_n = 2 - \lim_{n \to \infty} \frac{1}{2^{n-1}}$

$= 2$

So $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges to 2.

Hence $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2$.

2) $\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)}}$

SOLUTION: Here $a_n = \frac{\mathbf{1}}{\boldsymbol{n \cdot (n+1)}} = \frac{1}{n} - \frac{1}{n+1}$

So $S_1 = a_1 = 1 - \frac{1}{2}$

$$S_2 = a_1 + a_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = a_1 + a_2 + a_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

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$S_n = \frac{1}{n} - \frac{1}{n-1}$

$\Rightarrow \lim_{n \to \infty} S_n = 1$

Thus $\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)} = 1$

- **GEOMETRIC SERIES:**

Geometric series are series of the form

$$a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots + a \cdot r^n + \cdots \quad = \sum_{n=1}^{\infty} a \cdot r^{n-1}$$

Where a and r (ratio) are fixed real numbers and $a \neq 0$.

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EXAMPLES: 1) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$, $r = \frac{1}{2}$

2) $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1}$, $r = -\frac{1}{3}$

- **RESULTS:** Let $\sum_{n=1}^{\infty} a \cdot r^{n-1}$ be a geometric series .
  1) If $|r| < 1$then $\sum_{n=1}^{\infty} a \cdot r^{n-1}$ converges to $\frac{a}{1-r}$

     $\therefore \sum_{n=1}^{\infty} a \cdot r^{n-1} = \frac{a}{1-r}$ if$|r| < 1$.
  2) If $|r| \geq 1$then the series diverges.
- **EXAMPLS: SOLVE THE FOLLOWING**

1) $\sum_{n=1}^{\infty} \frac{1}{9} \cdot \left(\frac{1}{3}\right)^{n-1}$

SOLUTION: $\sum_{n=1}^{\infty} \frac{1}{9} \cdot \left(\frac{1}{3}\right)^{n-1} = \frac{1}{9} \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$

$$= \frac{1}{9} \cdot \frac{1}{1-\frac{1}{3}} = \frac{1}{6}$$

2) $\sum_{n=1}^{\infty} \frac{1}{9} \cdot \left(\frac{1}{3}\right)^{n-1}$

SOLUTION: Here $a = \frac{2}{5}$, $r = \left(-\frac{1}{3}\right)$

$$\therefore \frac{a}{1-r} = \frac{\frac{2}{5}}{1-\left(-\frac{1}{3}\right)} = \frac{\frac{2}{5}}{1+\left(\frac{1}{3}\right)} = \frac{3}{10} \qquad By\ result\ (1)$$

- **THEOREM:** If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$.
- **THE n^{th} TERM TEST FOR DIVERGENCE:**

Let $\sum_{n=1}^{\infty} a_n$ be a series. If $\lim_{n\to\infty} a_n$ does not exist or is different from zero then $\sum_{n=1}^{\infty} a_n$ diverges.

- **EXAMPLES: SOLVE THE FOLLOWING**

1) $\sum_{n=1}^{\infty} n^2$

SOLUTION: Here $a_n = n^2$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 \to \infty$$

$\therefore$By n^{th} term test $\sum_{n=1}^{\infty} n^2$ diverges

2) $\sum_{n=1}^{\infty} \frac{n+1}{n}$

SOLUTION: Let $a_n = \frac{n+1}{n}$

$$\therefore \to \lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty}(n+1)}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$
$$= 1$$

That is $\frac{\lim_{n\to\infty}(n+1)}{n} \to 1\ (\neq 0)$

Thus $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges.

3) $\sum_{n=1}^{\infty} (-1)^{n+1}$

SOLUTION: Let $a_n = (-1)^{n+1}$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1} \to (-1)^{\infty} \text{ (not exist)}$$

Thus $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges.

- **EXAMPLE: TEST THE CONVERGENCE OF $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$**

SOLUTION: Here $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$ is given

Note that the given series is a geometric series with ratio $r = \left(-\frac{2}{3}\right)$

Now $|r| = \left|-\frac{2}{3}\right| < 1$

.$\therefore$The given series is convergent and its sum is

$$\frac{a}{1-r} = \frac{5}{1-\left(-\frac{2}{3}\right)} = \frac{5}{1+\frac{2}{3}} = 3$$

➢ **<u>THEOREM:</u>** If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

I. $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$ (sum rule)

II. $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$ (difference rule)

III. $\sum k \cdot a_n = k \cdot \sum a_n = k \cdot A$ (any number) (constant multiple rule)

- **<u>EXAMPLS:</u>**

1) **Check the convergence of the series $\sum_{n=1}^{\infty} \frac{4^n+5^n}{6^n}$**

SOLUTION: Here $a_n = \frac{4^n+5^n}{6^n} = \left(\frac{4}{6}\right)^n + \left(\frac{5}{6}\right)^n$

$= a_n + b_n$ (say)

Now $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{4}{6}\right)^n$ is geometric series and it is convergent.

Also $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$ is convergent geometric series.

Hence $\sum_{n=1}^{\infty} \frac{4^n+5^n}{6^n}$ is convergent.

($\because \sum(a_n + b_n) = \sum a_n + \sum b_n$ is convergent)

- **<u>EXAMPLES:</u> FIND THE SUMS OF THE FOLLOWING SERIES**

a. $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$

Solution: Here $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}}\right)$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \qquad \text{(By difference rule)}$$

$$= \frac{1}{1-\left(\frac{1}{2}\right)} - \frac{1}{1-\left(\frac{1}{6}\right)}$$

(Geometric series with $a = 1$and$r = \frac{1}{2}, \frac{1}{4}$)

$$= 2 - \frac{6}{5} = \frac{4}{5}$$

Thus $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \frac{4}{5}$

b. $\sum_{n=1}^{\infty} \frac{4}{2^n}$

SOLUTION: Here $\sum_{n=1}^{\infty} \frac{4}{2^n} = 4 \cdot \sum_{n=1}^{\infty} \frac{1}{2^n}$ (constant multiple rule)

$$= 4 \cdot \left(\frac{1}{1-\left(\frac{1}{2}\right)}\right)$$

(Geometric series with $a = 1$&$r = \frac{1}{2}$)

$= 8$

Exercise:

1. **Check the convergence of the following**

   I. $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ **Ans:** divergent

   II. $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$ **Ans:** divergent

   III. $\sum_{n=1}^{\infty} n \tan(\frac{1}{n})$ **Ans:** divergent

P Series:

The p-series is

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \text{ is}$$

1) Convergent if $p > 1$ and
2) Divergent if $p \leq 1$.

- **DEFINATION: HARMONIC SERIES**

The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ is called the harmonic series is divergent. (By above example case $p = 1$)

- **COMPARISON TEST:**

1) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$ be two series with no negative terms

a) If $\sum_{n=1}^{\infty} c_n$ is a convergent series with $0 \leq a_n \leq c_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} c_n$.

b) If $\sum_{n=1}^{\infty} c_n$ is divergent series with $0 \leq c_n \leq a_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} c_n$.

- **REMARK:**

  1. Compare the given series $\sum a_n$ with auxiliary series $\sum c_n$ whose convergence or divergence should already be known to us.

  2. Generally $\sum \frac{1}{n^p}$ will be useful as an auxiliary series.

- **EXAMPLES:**

  1. Prove that $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ converges.

SOLUTION: Let $a_n = \frac{1}{3^n+1}$

Take $c_n = \frac{1}{3^n}$

By the geometric series, we know that the geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$converges

Since $\frac{1}{3^n+1} < \frac{1}{3^n}$ for $n \geq 1$.

$\therefore$By comparison test $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$is also converges.

2. Prove that $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-1}$diverges.

SOLUTION: Let $a_n = \frac{1}{2\sqrt{n}-1}$ and $c_n = 1/2\sqrt{n}$ observation that

$\frac{1}{2\sqrt{n}-1} > \frac{1}{2\sqrt{n}}$ for $n \geq 1$.

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2}$ is diverges.

$\therefore$By comparison test given series $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-1}$is also diverges.

1) THE LIMIT COMPARISON TEST:

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$.

1. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$and $\sum b_n$both converge or both diverge.
2. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$converges, then $\sum a_n$converges.
3. If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$diverges, then $\sum a_n$diverges.

- **EXAMPLES:**

1. Test the convergence of the series $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots$

SOLUTION: Here $a_n = \frac{1}{n\cdot(n+1)} = \frac{1}{n^2+n}$

Take $b_n = \frac{1}{n^2}$ then

$= \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2}{(n^2+n)}$

$$= \lim_{n \to \infty} \frac{n^2}{n^2 \cdot \left(1 + \frac{1}{n}\right)}$$

$$= \lim_{n \to \infty} \frac{1}{(1+\frac{1}{n})} = 1 \neq 0 \text{ (Finite)}$$

$$\therefore \lim_{n \to \infty} \frac{a_n}{b_n} = 1$$

Thus by comparison test both $\sum a_n$ and $\sum b_n$ converges or diverges

But $\sum b_n = \sum \frac{1}{n^2}$ is convergent

Hence $\sum a_n$ is also convergent.

2. Test the convergence of the series $1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

SOLUTION: Let $a_n = \frac{1}{2^n - 1}$

Take $b_n = \frac{1}{2^n}$

Now $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2^n}{(2^n - 1)}$

$$= \lim_{n \to \infty} \frac{1}{1 - \left(\frac{1}{2^n}\right)} = 1$$

$\therefore$By comparison test both $\sum a_n$ and $\sum b_n$ converges or diverges

But $\sum b_n = \sum \frac{1}{2^n}$convergent

Hence $\sum a_n$ is convergent.

Exercise:

1) Show that $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots$is convergent

2) Check the convergence of the following

I. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ **Ans:** convergent

II. $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ **Ans:** convergent

III. $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ **Ans:** convergent

IV. $\sum_{n=1}^{\infty} \frac{\log n}{\sqrt{n}}$ **Ans:** divergent

3) Check the convergence of the following

I. $\sum_{n=1}^{\infty} \frac{3n^2+5n}{7+n^4}$ **Ans:** convergent

II. $\sum_{n=1}^{\infty}\frac{1}{5^n-1}$ **Ans:** convergent

III. $\sum_{n=1}^{\infty}\frac{7}{7n-2}$ **Ans:** divergent

IV. $\sum_{n=1}^{\infty}\frac{\sqrt{n}-1}{n^2+1}$ **Ans:** convergent

V. $\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}$ **Ans:** divergent

VI. $\sum_{n=1}^{\infty}\frac{2n+1}{n^2+n+1}$ **Ans:** divergent

- **THE D ALEMBERT'S RATIO TEST AND CAUCHY'S ROOT TEST:**
- **THE RATIO TEST:**

Let $\sum a_n$ be a series of positive terms and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=k$

Then the series $\sum a_n$ is

(i) convergent if $k < 1$

(ii) divergent if $k > 1$

(iii) Test is incomplete if k=1

- **EXAMPLES: INVESTIGATE THE CONVERGENCE OF THE FOLLOWING SERIES**

1) $\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$

SOLUTION: Here $a_n=\frac{2^n+5}{3^n}$

Now $\frac{a_{n+1}}{a_n}=\frac{2^n+5}{3^n}\cdot\frac{3^n}{2^n+5}=\frac{1}{3}\frac{2^{n+1}+5}{2^n+5}$

$$=\frac{1}{3}\frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}}$$

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1}{3}\left(\frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}}\right)=\frac{1}{3}(2)=\frac{2}{3}=k$$

Here $k=\frac{2}{3}<1$

$\therefore$By D Alembert's ratio test, the given series is converge

NOTE: It does not mean that $\frac{2}{3}$ is the sum of the series.

The fact,

$$\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}=\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n+\sum_{n=0}^{\infty}\left(\frac{5}{3}\right)^n$$
$$=\frac{1}{1-\left(\frac{2}{3}\right)}+\frac{5}{1-\left(\frac{1}{3}\right)}=21/2 \qquad (\therefore \text{Geometric series})$$

i.e. $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n} = \frac{21}{2}$

2) $\boldsymbol{\sum_{n=0}^{\infty} \frac{2^n}{n!}}$

SOLUTION: Here $a_n = \frac{2^n}{n!}$

Now $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = \frac{2}{n+1}$

$$\therefore \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2}{n+1} = 0 = k(say)$$

∴By Ratio test, the series converges as $k < 1$.

- **<u>CAUCHY ROOT-TEST:</u>**

Let $\sum a_n$be a series with $a_n \geq 0$ for $n \geq N$, and

Suppose that $\lim_{n \to \infty} (a_n)^{\frac{1}{n}} = L$.

i. If $0 \leq L < 1$, then the series converge.
ii. If $L > 1$, then the series diverge or L is infinite
iii. If $L = 1$, then test fails

- **EXAMPLES: Which of the following series are convergent and which are divergent**

1) $\boldsymbol{\sum_{n=0}^{\infty} \frac{n^2}{2^n}}$

SOLUTION: Here $a_n = \frac{n^2}{2^n}$

$$\therefore (a_n)^{\frac{1}{n}} = \left(\frac{n^2}{2^n}\right)^{\frac{1}{3}} = \frac{n^{\frac{2}{n}}}{2}$$

$$\therefore \lim_{n \to \infty} (a_n)^{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^{\frac{2}{n}}}{2} = \frac{1}{2} < 1$$

So by Root test the series is convergent.

2) $\boldsymbol{\sum_{n=0}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n}$

SOLUTION: Here $a_n = \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

$$\therefore \lim_{n \to \infty} (a_n)^{\frac{1}{n}} = \lim_{n \to \infty} \left(\frac{1}{n} - 1/n^2\right)^{\frac{1}{n}} \to \infty$$

$\therefore$By Root-test the series is diverge.

Exercise:

1) Check the convergence of the following

I. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\cdots$ **Ans:** convergent

II. $1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\cdots$ **Ans:** convergent

III. $\sum_{n=0}^{\infty}\frac{2^n+5}{5^n}$ **Ans:** convergent

IV. $\sum_{n=0}^{\infty}\frac{n^3+2}{2^n+2}$ **Ans:** convergent

V. $\sum_{n=0}^{\infty}\frac{2^n-1}{3^n}$ **Ans:** convergent

VI. $\sum_{n=1}^{\infty}\frac{2^n}{n^2+1}$ **Ans:** divergent

2) Check the convergence of the following

I. $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\cdots$ **Ans:** convergent

II. $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\cdots, x>0$ **Ans:** convergent

III. $\sum_{n=1}^{\infty}\left(\frac{n+1}{n+2}\right)^n x^n \ ; x>0$

Ans: $x\geq 1 \Rightarrow divergent\ , 0\leq x<1 \Rightarrow convergent$

IV. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ **Ans:** divergent

➢ **ALTERNATIVE SERIES:**

A Series with alternate positive and negative terms is known as an alternative series.

➢ **LEIBNITZ'S TEST FOR ALTERNATIVE SERIES:**

The given alternative series $\sum_{n=0}^{\infty}(-1)^{n+1}u_n = u_1-u_2+u_3-u_4+\cdots$
Converges if all three of the following conditions are satisfied:

1) The $u_n{}'s$ are all positive.
2) $u_n \geq u_{n+1}$ for all $n\geq N$, for some integer N.
3) $u_n \to 0$.

➢ **ABSOLUTE AND CONDITIONAL CONVERGENCE:**

➢ **DEFINATION:**

- **ABSOLUTELY CONVERGENT**

  A Series $\sum a_n$converges absolutely if the corresponding series of absolute values, $\sum|a_n|$, converges.

- **Conditionally convergent:**

A Series$\sum a_n$ that converges but$\sum|a_n|$ does not converge it's called conditionally converge.

- **<u>THE ABSOLUTE CONVERGENCE TEST:</u>**

If $\sum|a_n|$converges, then $\sum a_n$converges.

- **<u>EXAMPLES:</u>** SOLVE THE FOLLOWING EXAMPLE

1) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n^2}\right)$

SOLUTION:

$\sum_{n=0}^{\infty}(-1)^{n+1}(\frac{1}{n^2}) = 1 - 1/4 + 1/9 - 1/16 + \cdots$, The corresponding series of absolute value is the convergent series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

The original series converges because it converge absolutely.

2) $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$

SOLUTION:

$$\sum_{n=1}^{\infty}\frac{\sin n}{n^2} = \frac{|sin\ 1|}{1} + \frac{|sin 2|}{4} + \cdots,$$

Which converge by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$because $|\sin n| \leq 1$for every n.

The original series converges absolutely, therefore it converges.

- **<u>CONDITIONALLY CONVERGENT SERIES:</u>**

Discuss convergence of given $\sum a_n$using above any applicable test. Derive new series as $\sum b_n = \sum|a_n|$ from given series. Discuss convergence of $\sum b_n$using above any applicable test. If $\sum a_n$is convergent and $\sum b_n$is divergent then given series is conditionally convergent.

- **NOTE:** Infinite series is not absolute convergent and conditionally convergent together.
- **<u>EXAMPLES:</u>** SOLVE THE FOLLOWING EXAMPLE

1. Test the series for absolute or conditional convergent

$\frac{2}{3} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{3}\right) - \left(\frac{5}{6}\right)\left(\frac{1}{4}\right) + \cdots$

SOLUTION: Let $u_n = (-1)^{n-1}(\frac{n+1}{n+2}\frac{1}{n})$

$$\sum_{n=1}^{\infty}|u_n| = \frac{2}{3} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{3}\right) - \left(\frac{5}{6}\right)\left(\frac{1}{4}\right) + \cdots$$

$$|u_n| = \frac{n+1}{n+2}\frac{1}{n}$$

Let $v_n = \frac{1}{n}$

$$\lim_{n \to \infty} \frac{|u_n|}{v_n} = \lim_{n \to \infty} \frac{n+1}{n+2} = \lim_{n \to \infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 1$$

And $\sum v_n = \sum \frac{1}{n}$ is divergent as p=1

By comparision test, $\sum |u_n|$ is also divergent.

Hence, the series is not absolutely convergent

To check the conditional convergence, applying Leibnitz's test,

(i) $|u_n| - |u_{n+1}| = \frac{n+1}{n(n+2)} - \frac{n+2}{(n+1)(n+3)}$

$= \frac{n^2+3n+3}{n(n+1)(n+2)(n+3)} > 0$ for all n

$|u_n| > |u_{n+1}|$

(ii) $\lim_{n \to \infty} |u_n| = \lim_{n \to \infty} \frac{n+1}{n(n+2)}$

$= \lim_{n \to \infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 0$

By lebnitz's test, $\sum u_n$ is convergent.

The series $\sum u_n$ is convergent and $\sum |u_n|$ is divergent.

Hence, the series is conditionally convergent.

Exercise:

1) Check the absolute converge of the following

i. $\mathbf{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots}$

Ans: convergent but not absolute convergent

ii. $\mathbf{1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} - \cdots}$

Ans: Absolute convergent

iii. $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \cdots$

Ans: Absolute convergent

iv. $1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$

Ans: Absolute convergent

v. $\sum \frac{(-1)^{n-1}}{n^p}$ $(p > 1)$ **Ans:** Absolute convergent

2) Check the conditionally converge of the following

(i) $\mathbf{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots}$

Ans: conditionally convergent

(ii) $\mathbf{1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\cdots}$

Ans: conditionally convergent

(iii)$-\frac{1}{1^p}+\frac{1}{2^p}-\frac{1}{3^p}+\frac{1}{4^p}-\frac{1}{5^p}+\cdots$

Ans: $p>1 \Longrightarrow Absolute\ convergent,$

$p \leq 1 \Longrightarrow conditionally\ convergent$

(iv)$\frac{1}{ln2}-\frac{1}{ln3}+\frac{1}{ln4}-\frac{1}{ln5}+\cdots$

Ans: conditionally convergent

(v) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ **Ans:** conditionally convergent