Successive Differentiation

① $\boxed{y = e^{mx}}$

n^{th} derivative $y_n = m^n e^{mx}$

Ex: ① $y = e^{2x}$ $\therefore y_n = 2^n \cdot e^{2x}$

② $y = e^{-3x}$ $\therefore y_n = (-3)^n \cdot e^{-3x}$

③ $y = e^{-x}$ $\therefore$ ~~y_n~~ $=$

$\therefore y = e^{(-1)x}$ $\therefore$ $y_n = (-1)^n \cdot e^{(-1)x}$

④ $y = e^{4x}$ find y_5. $y = e^{4x}$ $\therefore y_5 = 4^5 \cdot e^{4x}$

② $\boxed{y = a^{mx}}$

n^{th} derivative $y_n = m^n (\log a)^n a^{mx}$

Ex: ① $y = 3^{5x}$ find n^{th} derivative

$y_n = 5^n (\log 3)^n \cdot 3^{5x}$

② $y = 5^{-x}$ $\therefore y = 5^{(-1)x}$

~~$y_n = (-1)^n (\log 5)$~~

$y_n = (-1)^n (\log 5)^n 5^{(-1)x}$

③ $y = 4^{2x}$ find y_5

$y_5 = 2^5 (\log 4)^5 \cdot 4^{2x}$

③

(3) $y = (ax+b)^m,\ a \neq 0,\ m \in R$

$$y_n = \frac{a^n m!}{(m-n)!}(ax+b)^{m-n},\ \text{if } n<m$$

$$= a^n \cdot n! \quad \text{if } m=n$$

$$= 0, \quad \text{if } n>m$$

Ex: If $y = (3x+5)^7$ find y_5, y_7, y_8

$$y = (\underset{\uparrow a}{3}x+5)^{7 \leftarrow m}$$

$$\underset{\uparrow n}{y_5} = \frac{3^5 \cdot 7!}{(7-5)!}(3x+5)^{7-5}$$

$$= \frac{3^5 \cdot 7!}{2!}(3x+5)^2$$

$$\underset{\uparrow n}{y_7} = 3^7 \cdot 7! \quad (\text{Here } m=n)$$

$$\underset{\uparrow n}{y_8} = 0 \quad (\text{Here } n>m)$$

Ex: If $y = (-x+8)^9$ find y_n, $(n<9)$

$$y = ((\underset{\uparrow a}{-1})x+8)^{9 \leftarrow m}$$

$$y_n = \frac{(-1)^n \, 9!}{(9-n)!}((-1)x+8)^{9-n}$$

Ex: $y = (2x-3)^6$ find y_5, y_6, y_7

$$y = (\underset{a}{2}x + (\underset{b}{-3}))^{6 \leftarrow m}$$

$$\underset{n}{y_5} = \frac{2^5 \cdot 6!}{(6-5)!}(2x+(-3))^{6-5}$$

$$= \frac{32 \cdot 6!}{1!}(2x+(-3))^1$$

(4) $y = \dfrac{1}{ax+b}$

$$y_n = \frac{a^n (-1)^n \cdot n!}{(ax+b)^{n+1}}$$

Ex: (i) $y = \dfrac{1}{2x+3}$ find y_n

$$\cancel{y_n} \quad y = \frac{1}{\underset{a}{2}x+3}$$

$$y_n = \frac{2^n (-1)^n \cdot n!}{(2x+3)^{n+1}}$$

(ii) $y = \dfrac{1}{x+5}$ find y_8

$$y = \frac{1}{\cancel{(1)}(1)x+5}$$

$$y_8 = \frac{(1)^8 (-1)^8 \cdot 8!}{(x+5)^9}$$

$$y_8 = \frac{8!}{(x+5)^9}$$

(4)

(ii) $y = \dfrac{1}{6-5x}$ find y_n

$$y = \frac{1}{-5x+6}$$
$\uparrow$ a

$$y_n = \frac{(-5)^n \cdot (-1)^n \cdot n!}{(-5x+6)^{n+1}}$$

(5) $y = \log(ax+b)$

$$y_n = \frac{a^n (-1)^{n-1} \cdot (n-1)!}{(ax+b)^n}$$

Ex: (i) $y = \log(5x+6)$ find y_n

$y = \log(5x+6)$
$\uparrow$ a

$$y_n = \frac{5^n (-1)^{n-1} \cdot (n-1)!}{(5x+6)^n}$$

(ii) $y = \log(-x+9)$ find y_5

$$y = \log(-x+9)$$
$$= \log((-1)x+9)$$

$$y_5 = \frac{(-1)^5 \cdot (-1)^{5-1} \cdot (5-1)!}{(-x+9)^5}$$

$$= \frac{-1 \cdot (-1)^4 \cdot 4!}{(-x+9)^5}$$

$$= -1 \cdot \frac{4 \times 3 \times 2 \times 1}{(-x+9)^5} = \frac{-24}{(-x+9)^5}$$

6 (i) $y = \sin(ax+b)$

$$y_n = a^n \sin(ax+b+n\tfrac{\pi}{2})$$

(ii) $y = \cos(ax+b)$

$$y_n = a^n \cos(ax+b+n\tfrac{\pi}{2})$$

Ex: ~~y = sin (2i~~

$y = \sin(2x-3)$ find y_n

~~y_n~~ $y = \sin(\underset{\overset{\uparrow}{a}}{2}x+(-3))$

$$y_n = 2^n \sin(2x+(-3)+n\tfrac{\pi}{2})$$

Ex: $y = \cos(3x+4)$ find y_4

$y = \cos(\underset{\overset{\uparrow}{a}}{3}x+4)$

$$y_4 = 3^4 \cos(3x+4+\cancel{4}^{2} \cdot \tfrac{\pi}{\cancel{2}})$$

$$y_4 = 3^4 \cos(3x+4+2\pi)$$

~~ex~~ ~~y=~~

Note:- (6)

$2\sin x\cos y = \sin(x+y)+\sin(x-y)$

$2\cos x\sin y = \sin(x+y)-\sin(x-y)$

$2\cos x\cos y = \cos(x+y)+\cos(x-y)$

$-2\sin x\sin y = \cos(x+y)-\cos(x-y)$

$$\boxed{\begin{array}{l} 2sc = s+s,\ 2cs = s-s,\ 2cc = c+c \\ -2ss = c-c \end{array}}$$

Ex: $y=\sin 3x\cos 5x$ find y_n

$y=\sin 3x\cos 5x$

$y=\frac{1}{2}(2\cos 5x\sin 3x)$

$=\frac{1}{2}[\sin(5x+3x)-\sin(5x-3x)]$

$y=\frac{1}{2}[\sin 8x-\sin 2x]$

$y=\frac{1}{2}\sin 8x-\frac{1}{2}\sin 2x$

$y_n=\frac{1}{2}(\sin 8x)_n-\frac{1}{2}(\sin 2x)_n$

$y_n = \cancel{\frac{1}{2}8\sin(8x+n\pi}$

$y_n = 8^n\frac{1}{2}\sin(8x+n\frac{\pi}{2})$

$\quad -2^n\frac{1}{2}\sin(2x+n\frac{\pi}{2})$

Ex: $y=\cos 5x\cos 3x$ find y_4

$y=\frac{1}{2}(2\cos 5x\cos 3x)$

$=\frac{1}{2}[\cos(5x+3x) + \cos(5x-3x)]$

$y = \frac{1}{2}[\cos 8x + \cos 2x]$

$y = \frac{1}{2}\cos 8x + \frac{1}{2}\cos 2x$

$y_4 = \frac{1}{2}(\cos 8x)_4 + \frac{1}{2}(\cos 2x)_4$

$y_4 = \frac{1}{2} \cdot 8^4 \cos\left(8x + \cancel{4}^{2}\frac{\pi}{\cancel{2}}\right)$

$+ \frac{1}{2} \cdot 2^4 \cos\left(2x + \cancel{4}^{2}\frac{\pi}{\cancel{2}}\right)$

Ex: $y = \sin^2 3x$ find y_n

$y = \sin^2 3x$

$y = \sin 3x \sin 3x$

$y = -\frac{1}{2}[-2\sin 3x \sin 3x]$

$y = -\frac{1}{2}[\cos(3x+3x) - \cos(3x-3x)]$

$y = -\frac{1}{2}[\cos 6x - \cos 0]$

$y = -\frac{1}{2}[\cos 6x - 1]$

$y = -\frac{1}{2}\cos 6x + \frac{1}{2}$

$y_n = (\frac{1}{2})(\cos 6x)_n + (\frac{1}{2})_n$

$y_n = -\frac{1}{2} \cdot 6^n \cos(6x + n\frac{\pi}{2}) + 0$

$y_n = -\frac{1}{2} \cdot 6^n \cos(6x + n\frac{\pi}{2})$

Ex: $y = \cos^2 5x$ find y_5

$$y = \cos^2 5x$$

$$y = \cos 5x \cos 5x$$

$$y = \frac{1}{2}[2\cos 5x \cos 5x]$$

$$y = \frac{1}{2}[\cos(5x+5x) + \cos(5x-5x)]$$

$$y = \frac{1}{2}[\cos 10x + \cos 0]$$

$$y = \frac{1}{2}[\cos 10x + 1]$$

$$y = \frac{1}{2}\cos 10x + \frac{1}{2}$$

$$y_5 = \frac{1}{2}(\cos 10x)_5 + \left(\frac{1}{2}\right)_5$$

$$= \frac{1}{2}\cdot 10^5 \cos\left(10x + 5\frac{\pi}{2}\right) + 0$$

$$y_5 = \frac{1}{2}\cdot 10^5 \cos\left(10x + 5\frac{\pi}{2}\right)$$

(7) (i) $y = e^{ax}\sin(bx+c)$

$$y_n = (a^2+b^2)^{n/2} e^{ax} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

(ii) $y = e^{ax}\cos(bx+c)$

$$y_n = (a^2+b^2)^{n/2} e^{ax} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Ex: $y = e^{3x}\sin(2x+1)$ find y_n

$$y = e^{\overset{a}{3}x}\sin(\underset{b}{2}x+1)$$

$$y_n = (3^2+2^2)^{n/2} e^{3x}\sin\left(2x+1+n\tan^{-1}\left(\tfrac{2}{3}\right)\right)$$

$$y_n = (9+4)^{n/2} e^{3x}\sin\left(2x+1+n\tan^{-1}\left(\tfrac{2}{3}\right)\right)$$

Ex: $y = e^{-2x}\cos(3x+5)$ find y_4

$$y = e^{\overset{a}{-2}x}\cos(\overset{b}{3}x+5)$$

$$y_4 = \left((-2)^2+3^2\right)^{4/2} e^{-2x}\cos\left(3x+5+4\tan^{-1}\left(\tfrac{3}{-2}\right)\right)$$

$$y_4 = (4+9)^2 e^{-2x}\cos\left(3x+5+4\tan^{-1}(-3/2)\right)$$

$$= (13)^2 e^{-2x}\cos\left(3x+5-4\tan^{-1}(3/2)\right)$$

$$= 169\, e^{-2x}\cos\left(3x+5-4\tan^{-1}(3/2)\right)$$

Ex: $y = e^{x}\sin 2x\cos x$ find y_3

$$y = \frac{1}{2}\left[e^{x}\, 2\sin 2x\cos x\right]$$

$$y = \frac{1}{2}\left[e^{x}\left(\sin(2x+x)+\sin(2x-x)\right)\right]$$

$$y = \frac{1}{2}\left[e^{x}(\sin 3x+\sin x)\right]$$

$$y = \frac{1}{2} e^{1x} \sin 3x + \frac{1}{2} e^{1x} \sin x$$

$$y_3 = \frac{1}{2}\left(e^{1x} \sin 3x\right)_3 + \frac{1}{2}\left(e^{1x} \sin 1x\right)_3$$

$$y_3 = \frac{1}{2}(1^2 + 3^2)^{3/2} e^{1x} \sin\left(3x + 3\tan^{-1}\left(\frac{3}{1}\right)\right)$$

$$+ \frac{1}{2}(1^2 + 1^2)^{3/2} e^{1x} \sin\left(1x + 3\tan^{-1}\left(\frac{1}{1}\right)\right)$$

$$y_3 = \frac{1}{2}(1+9)^{3/2} e^{x} \sin\left(3x + 3\tan^{-1}(3)\right)$$

$$+ \frac{1}{2}(1+1)^{3/2} e^{x} \sin\left(x + 3\tan^{-1}(1)\right)$$

$$y_3 = \frac{1}{2} 10^{3/2} e^{x} \sin\left(3x + 3\tan^{-1}(3)\right)$$

$$+ \frac{1}{2} 2^{3/2} e^{x} \sin\left(x + 3\frac{\pi}{4}\right)$$

Ex: Partial Fraction:

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x+a)(x+b)(x+c)}$$

$$\frac{p(x)}{(x+a)(x+b)(x+c)} = \frac{p(-a)}{(x+a)(-a+b)(-a+c)}$$

$$+ \frac{p(-b)}{(-b+a)(x+b)(-b+c)}$$

$$+ \frac{p(-c)}{(-c+a)(-c+b)(x+c)}$$

Ex: If $y = \frac{x+3}{(x-5)(x+4)}$ find y_n

$$y = \frac{x+3}{(x-5)(x+4)}$$

$$y = \frac{(5+3)}{(x-5)(5+4)} + \frac{(-4+3)}{(-4-5)(x+4)}$$

$$y = \frac{8}{(x-5)(9)} + \frac{(-1)}{(-9)(x+4)}$$

$$= \frac{8}{9}\,\frac{1}{x-5} + \frac{1}{9}\,\frac{1}{x+4}$$

$$y_n = \frac{8}{9}\left[\frac{1}{1x-5}\right]_n + \frac{1}{9}\left[\frac{1}{1x+4}\right]_n$$

$$y_n = \frac{8}{9}\,\frac{1^n\cdot(-1)^{n}\,n!}{(x-5)^{n+1}} + \frac{1}{9}\,\frac{1^n\cdot(-1)^{n}\,n!}{(x+4)^{n+1}}$$

$$y_n = \frac{8}{9}\,\frac{(-1)^{n}\cdot n!}{(x-5)^{n+1}} + \frac{1}{9}\,\frac{(-1)^{n}\,n!}{(x+4)^{n+1}}$$

Ex If $y = \frac{2x+3}{x^2-7x+12}$ find y_3

$$y = \frac{2x+3}{(x-4)(x-3)}$$

$$y = \frac{2(4)+3}{(x-4)(4-3)} + \frac{2(3)+3}{(3-4)(x-3)}$$

x^2 12

$-4 + -3 = -7$

$x^2-7x+12$

$= (x-4)(x-3)$

$$y = \frac{11}{(x-4)(1)} + \frac{9}{(-1)(x-3)}$$

$$y_3 = 11\cdot\left[\frac{1}{1x-4}\right]_3 - 9\left[\frac{1}{1x-3}\right]_3$$

$$y_n = \cancel{11\cdot 1^n(-1)}$$

$$y_3 = 11\cdot\frac{1^3\cdot(-1)^3\,(3)!}{(x-4)^{3+1}} - 9\cdot\frac{1(-1)^3\cdot(3)!}{(x-3)^{3+1}}$$

$$\therefore y_3 = 11\cdot\frac{(-1)^3\,3!}{(x-4)^4} - 9\,\frac{(-1)^3\,3!}{(x-3)^4}$$

$$= -\frac{11\cdot 3\times 2\times 1}{(x-4)^4} - \frac{9(-1)\,3\times 2\times 1}{(x-3)^4}$$

$$= -\frac{66}{(x-4)^4} + \frac{54}{(x-3)^4}$$

Leibniz's rule:-

If $y = uv$, u and v are functions of x.

then n^{th} derivative

$$y_n = (uv)_n$$

$$y_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \cdots + {}^nC_n\, u v_n$$

Note: ${}^nC_0$ is denoted by $\binom{n}{0}$

${}^nC_1$ " " " $\binom{n}{1}$

${}^nC_2$ " " " $\binom{n}{2}$

$nC_0 = 1, \; nC_1 = n, \; nC_n = 1$ (13)

$$nC_3 = \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}$$

$$\cancel{nC_2 = \frac{n\cdot(n-1)}{2!}}$$

$$nC_4 = \frac{n(n-1)(n-2)(n-3)}{4\cdot 3\cdot 2\cdot 1}$$

Ex: If $y = x^2 \log 3x$ find n^{th} derivative

$$y = x^2 \log 3x$$

$$y_n = (x^2 \log 3x)_n$$

$(x^2)_1 = 2x$

$(x^2)_2 = 2$

$(x^2)_3 = 0$

$$y_n = (\log 3x \cdot x^2)_n$$

$$y_n = nC_0 (\log 3x)_n (x^2) + nC_1 (\log 3x)_{n-1} (x^2)_1 + nC_2 (\log 3x)_{n-2} (x^2)_2 + nC_3 (\log 3x)_{n-3} (x^2)_3$$

$$y_n = 1 \cdot \frac{3^n (-1)^{n-1} (n-1)!}{(3x)^n} x^2 + n \cdot \frac{3^{n-1} (-1)^{(n-1)-1} ((n-1)-1)!}{(3x)^{n-1}} (2x) + \frac{n(n-1)}{2\cdot 1} \frac{3^{n-2} \cdot (-1)^{(n-2)-1} ((n-2)-1)!}{(3x)^{n-2}} (2)$$

$+ 0$

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$$= \frac{\cancel{3^n} (-1)^{n-1} (n-1)!}{\cancel{3^n} x^n} x^2 + n \frac{\cancel{3^{n-1}} (-1)^{n-2} (n-2)!}{\cancel{3^{n-1}} x^{n-1}} (2x)$$

$$+ \frac{n(n-1)}{\cancel{2}} \frac{\cancel{3^{n-2}} \cdot (-1)^{n-3} \cdot (n-3)!}{\cancel{3^{n-2}} x^{n-2}} (\cancel{2})$$

$\underline{\text{Ex}}$: Find n^{th} derivative of $x^2 \cos x$

$y = x^2 \cos x$

$y_n = (x^2 \cos x)_n$

$y_n = (\cos x \cdot x^2)_n$

$$\boxed{\begin{aligned} v &= x^2 \\ v_1 &= 2x \\ v_2 &= 2 \\ v_3 &= 0 \end{aligned}}$$

$y_n = nC_0 (\cos x)_n x^2 + nC_1 (\cos x)_{n-1} (x^2)_1$

$+ nC_2 (\cos x)_{n-2} (x^2)_2$

$y_n = 1\, 1^n \cos(x + n\frac{\pi}{2}) x^2 + n\, 1^{n-1} \cos(x + (n-1)\frac{\pi}{2}) \cdot (2x)$

$+ \frac{n(n-1)}{\cancel{2}} 1^{n-2} \cos(x + (n-2)\frac{\pi}{2}) (\cancel{2})$

$y_n = \cos(x + n\frac{\pi}{2}) x^2 +$

$n \cos(x + (n-1)\frac{\pi}{2}) 2x$

$+ n(n-1) \cos(x + (n-2)\frac{\pi}{2}) 2$

Ex: If $y = \tan^{-1}x$ P.T. (15)

$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

$y = \tan^{-1}x$

Taking derivative

$y_1 = \frac{1}{1+x^2}$

$\therefore y_1(1+x^2) = 1$

Taking n^{th} derivative

$[\underset{u}{y_1}(\underset{v}{1+x^2})]_n = (1)_n$

$v = 1+x^2$
$v_1 = 2x$
$v_2 = 2$
$v_3 = 0$

$nc_0(y_1)_n(1+x^2) + nc_1(y_1)_{n-1}(1+x^2)_1$
$+ nc_2(y_1)_{n-2}(1+x^2)_2 = 0$

$y_{n+1}(1+x^2) + n\, y_{1+n-1}(0+2x)$
$+ \frac{n(n-1)}{2\times1} y_{1+n-2}(2) = 0$

$y_{n+1}(1+x^2) + 2nxy_n + n(n-1)y_{n-1} = 0$

Ex If $y = e^{m\cos^{-1}x}$ P.T.

(i) $(1-x^2)y_2 - xy_1 = m^2y$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

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$y = e^{m\cos^{-1}x}$

$y_1 = m\left(-\frac{1}{\sqrt{1-x^2}}\right) \cdot e^{m\cos^{-1}x}$

$y_1 = -\frac{m}{\sqrt{1-x^2}} \cdot y$

$y_1(\sqrt{1-x^2}) = -my$

~~Tak~~ squaring on both sides

$[y_1(\sqrt{1-x^2})]^2 = [-my]^2$

$y_1^2(\sqrt{1-x^2})^2 = m^2y^2$

$y_1^2(1-x^2) = m^2y^2$

Taking derivative

$2y_1y_2(1-x^2) + y_1^2(0-2x) = m^2(2y \cdot y_1)$

$2y_1y_2(1-x^2) - 2xy_1^2 = m^2 2yy_1$

$\cancel{2y_1}[y_2(1-x^2) - xy_1] = \cancel{2y_1}(m^2y)$

$y_2(1-x^2) - xy_1 = m^2y$

Taking n^{th} derivative

$[\underset{u}{y_2}\underset{v}{(1-x^2)}]_n - (\underset{v}{x}\underset{u}{y_1})_n = (m^2y)_n$

$[\underset{u}{y_2}\underset{v}{(1-x^2)}]_n - [\underset{u}{y_1}\underset{v}{x}]_n = m^2y_n$

$(1-x^2)_1 = -2x$

$(1-x^2)_2 = -2$

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$$^{n}C_0 (y_2)_n (1-x^2) + {}^{n}C_1 (y_2)_{n-1} (1-x^2)_1$$
$$+ {}^{n}C_2 (y_2)_{n-2} (1-x^2)_2$$
$$- [{}^{n}C_0 (y_1)_n (x) + {}^{n}C_1 (y_1)_{n-1} (x)_1]$$
$$= m^2 y_n$$

$v = x$
$v_1 = 1$
$v_2 = 0$

$$1 \cdot y_{n+2} (1-x^2) + n\, y_{2+n-1} (0-2x)$$
$$+ \frac{n(n-1)}{2 \cdot 1} y_{2+n-2} (-2)$$
$$- [y_{n+1} x + n\, y_{1+n-1} (1)] = m^2 y_n$$

$$(1-x^2) y_{n+2} - 2nx\, y_{n+1} - \frac{n(n-1)}{\cancel{2}} y_n (\cancel{2})$$
$$- x y_{n+1} - n y_n = m^2 y_n$$

$$(1-x^2) y_{n+2} - 2nx\, \underline{\underline{y_{n+1}}} - x \underline{\underline{y_{n+1}}}$$
$$- n(n-1) y_n - n y_n - m^2 y_n = 0$$

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1}$$
$$- [n(n-1) + n + m^2] y_n = 0$$

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1}$$
$$- [n^2 - \cancel{n} + \cancel{n} + m^2] y_n = 0$$