Curve Tracing

Generally, a curve is drawn by plotting a number of points and joining them by a smooth line.

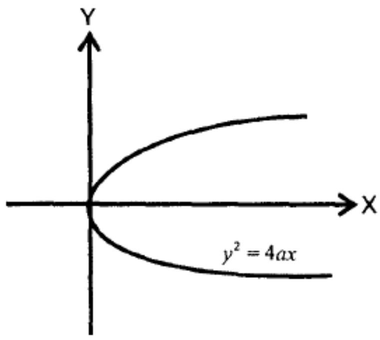
If an approximate shape of the curve is sufficient for a given purpose then it is enough to study certain important characteristics. This purpose is served by curve tracing methods.

The points to be observed for tracing of plane algebraic curves are given below.

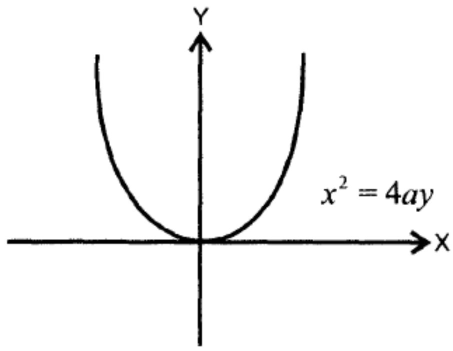
Symmetry

Whether the curve is symmetric about an axis or about other any line . if

1) $F(x,y) = F(x,-y) \Rightarrow curve\ is\ symmetric\ about\ X - axis.$
2) $F(-x,y) = F(x,y) \Rightarrow curve\ is\ symmetric\ about\ Y - axis.$
3) $F(-x,-y) = F(x,y) \Rightarrow curve\ is\ symmetric\ in\ opposite\ quadrants.$
4) $F(y,x) = F(x,y) \Rightarrow curve\ is\ symmetric\ about\ Y = X$
5) $F(-y,-x) = F(x,y) \Rightarrow curve\ is\ symmetric\ about\ Y = -X$



(i)



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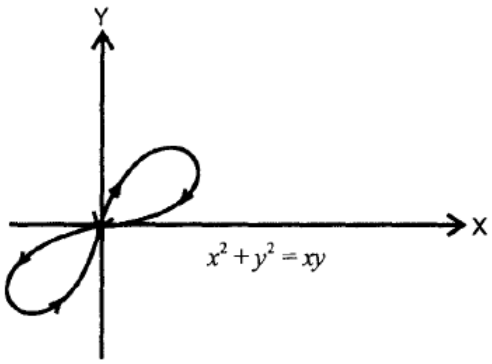

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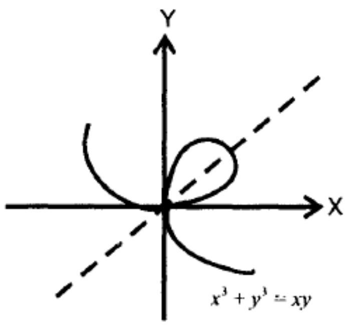
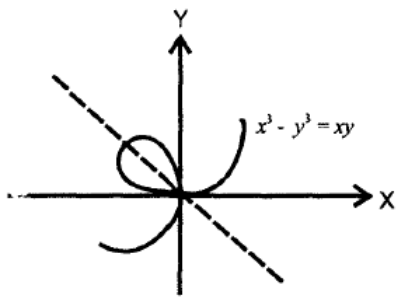


(iv)



(v)

Origin

Whether the curve is passing through the origin, if so the equations of the tangents to the curve at the origin.

Suppose $F(x, y) = 0$ is the algebraic form of the equation of the curve.

$F(0,0) = 0 \Longrightarrow$The curve is passing through origin OR If there is no constant term in $F(x, y)$ the curve passes through origin

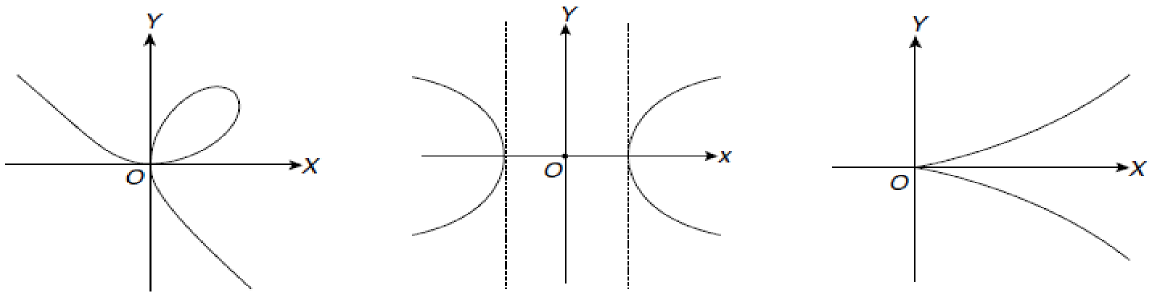
To find The equation of the tangents to the curve are obtained by equating the lowest degree terms in $F(x, y)$ to zero.

If at $O(0,0)$ the tangent s are:

1) Real and coincident then "O" is called **cusp.** **See example 1**
2) Real and different then "O" is called **node.**

e.g

a) $y^2(4 - x) = x^3$ (Equating to zero, the lowest degree term $4y^2 = 0$ so y=0 and y=0 are the tangents at the origin.) (Cusp) (See figure)
b) $x^3 + y^3 = 3axy$ (Equating to zero, the lowest degree term x y = 0 we get x = 0, y = 0 as tangents at the origin.) (Node) (See figure)



Node at origin Conjugate point at origin Cusp at origin

Point of intersection with coordinate axes:

Find the points of intersection of the curve with the axes.

Region in which the curve lies:

We can determine the region in which the curve lies by solving the equation f (x, y) = 0 for y . If y (or x) is imaginary for any range of values of x (or y) it means that the curve does not lie in that range.

Asymptotes

Finding the asymptotes.

An asymptote is a line that is at a finite distance from (0,0) and is tangential to the curve at infinity (i.e..) the curve approaches the line at infinity.

1) coefficient of the highest degree term in x equated to zero gives the equations of the asymptotes parallel to X- axis.
2) coefficient of the highest degree term in y equated to zero gives the equations of the asymptotes parallel to X- axis.

Summary of Steps:-

POSTAR:-Point of intersection **O**rigin **S**ymmetry **T**angent **A**symptote **R**egion

Example 1

Trace the curve$\boldsymbol{y^2(2a-x)=x^3}$ **(Cissoid of Diocles).**

Solution:

(1) Symmetry Given equation contains even powers of y.So, the curve is symmetric about the x-axis

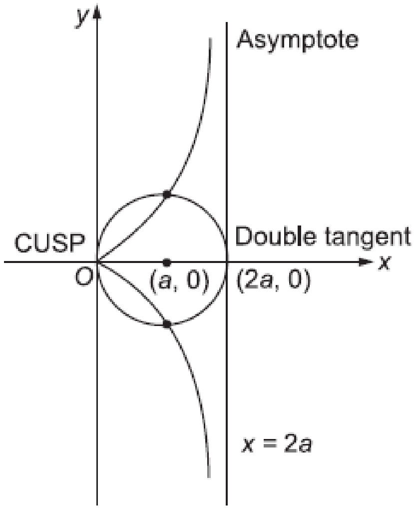
(2) Passing through the origin Given equationis satisfiedby $x = 0$; $y = 0$. So, the curve passes throughthe origin.

(3) Tangents at the origin Equating to zero the lowest degree terms,$2ay^2 = 0$ we get$y = 0, y = 0$(double root). The two tangents to the curve at the origin are real and coincident so that the origin is a **cusp.**

(4) Region $\boldsymbol{y = \frac{\sqrt{x^3}}{\sqrt{2a-x}}}$ is negative when ($\sqrt{x^3} < 0$) implies ,for negative values of x.Thus, y is imaginary for $x < 0$; so, no part of the curve lies to the left of the $Y-axis$. Similarly y is imaginary for $x > 2a$; so, no part of the curve lies to the right of the straight line $x = 2a$.

(5) Asymptotes:

Asymptotes parallel to the $Y-axis$ are obtained by equating to zero the coefficient of the highest power of y. That is $(2a-x) = 0$ Thus $x = 2a$.



$y^2(2a-x) = x^3$
(Cissoid of Diocles)

Example 2

Trace the curve $y^2 = x^2\frac{a+x}{a-x}$ **(Strophoid).**

Solution

(1) Symmetry Given equation contains only even powersof y. So, the curve is symmetrical about the $X-axis$.

(2) Origin The curve passes through the origin as no constant term or (since $x = 0, y = 0$ satisfy given equation.)

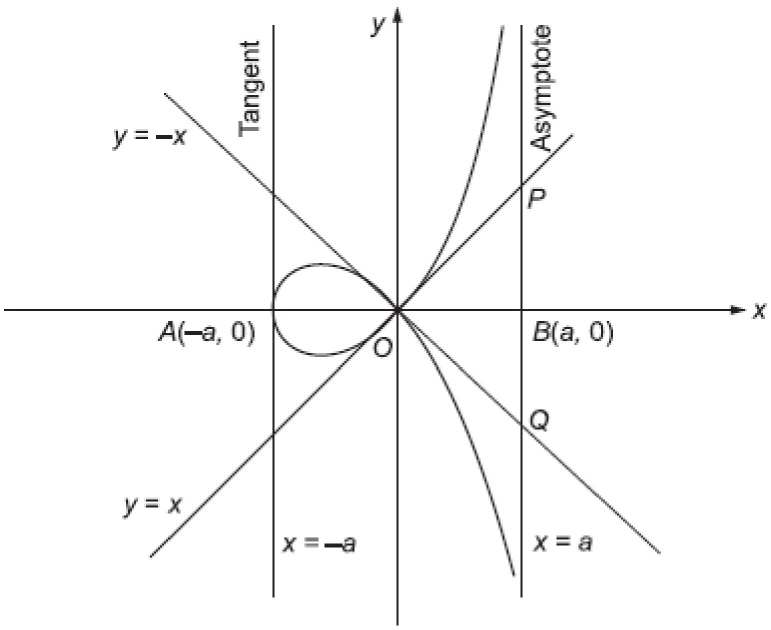
(3) Tangents at the origin Equating to zero the lowest degree terms, we get $y = \pm x$ as the tangents at the origin. These are real and different. So, the origin is a node.

(4) Intersection with the coordinate axes $y = 0$ $when$ $x = 0$ or $-a$. The curve cuts the $x - axis$ at $x = -a$ and $x = 0$.

(5) Region Given equation can be written as $y = \pm x\sqrt{\frac{a+x}{a-x}}$ When $x < -a$ or $x > a, y$ is imaginary. So no partof the curve lies outside the lines $x = -a, x = a$. As x increases from $-a$ to $\frac{-a}{2}$, y decreases from 0 to – $\frac{a}{2}$ and as x $increases$ $from$ $-\frac{a}{2}$ to $0, y$ $increases$ $from$ $\frac{-a}{2}$ to 0.

(6) Asymptotes Asymptotes parallel to the $y - axis$ are obtained by equating to zero the coefficient of the highest power of y; thus $x = a$ is an asymptote.The coefficient of the highest power of x is 1. So,there is no asymptote parallel to the $X - axis$.

The graph for the curve is given in Fig.

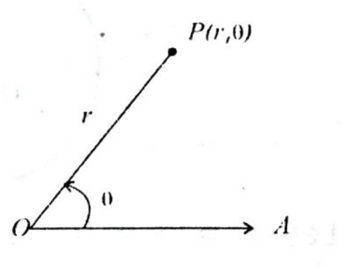


$$y^2 = x^2 \cdot \left(\frac{a+x}{a-x}\right)$$

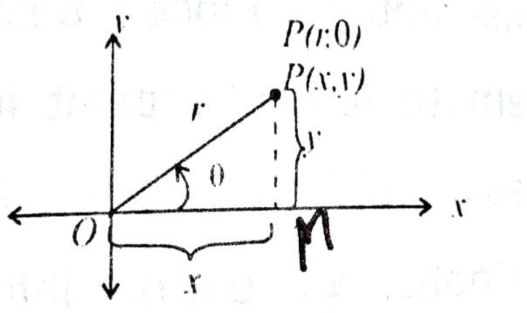
(Strophoid)

Tracing of Polar coordinates:

A point P in the plane, has polar coordinates (r, θ), where r is the distance of the point from the fixed origin O (Called the Pole) and θ is the angle between $\overrightarrow{OP}$ and initial ray $\overrightarrow{OA}$ (Called polar axis).



Relation between Polar and Cartesian coordinates:



Choose the polar axis along the positive x-axis and the pole at the origin, from right triangle PMO

$cos\theta = \frac{x}{r}$ and $sin\theta = \frac{x}{r}$

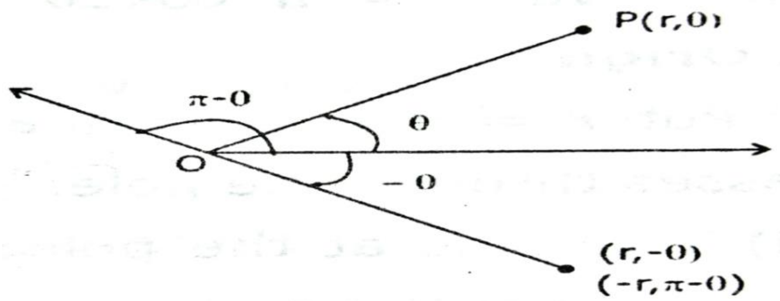
$x = rcos\theta$, $y = rsin\theta$, from these relation we have $r^2 = x^2 + y^2$ and $tan\theta = \frac{y}{x}$

Procedure for tracing polar curves:

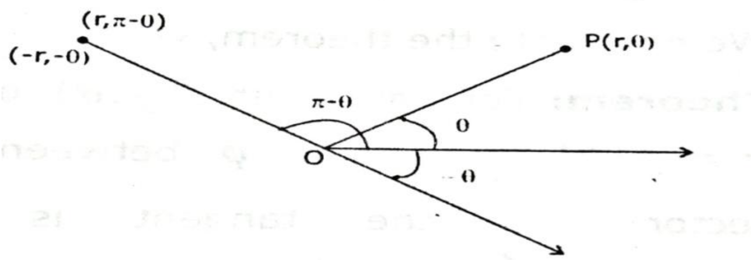
Let the equation of the curve be $f(r,\ \theta) = 0$

1.Symmetry:

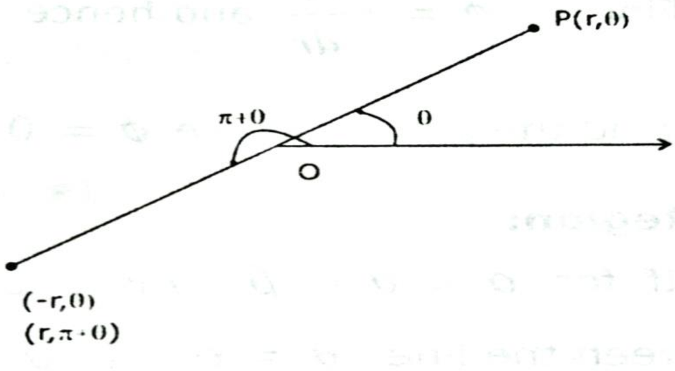
(i) Symmetry about the initial line (polar axis): If the equation of the curve remains unchanged when θ is replaced by $-\theta$ or when the equation remains unchanged on replacing r by –r and θ by $\pi - \theta$ then the curve is symmetric with respect to the initial line(polar axis).



(ii) Symmetry about the line $\theta = \frac{\pi}{2}$ (Normal axis): If the equation of the curve remains unchanged when θ is replaced by $\pi - \theta$ or when the equation remains unchanged on replacing r by -r, and θ by $-\theta$ then the curve is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (normal axis).



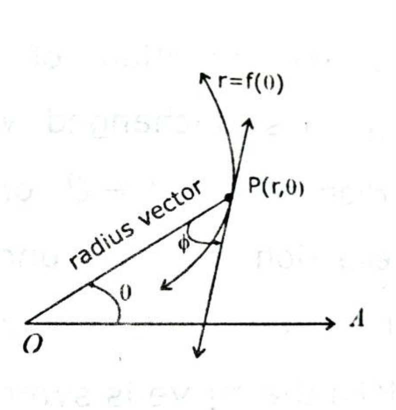
(iii) Symmetry about the pole : If the equation of the curve remains unchanged when r is replaced by -r (i.e. power of r is even) or when the equation remains unchanged on replacing θ by $\pi + \theta$ then the curve is symmetric with respect to the pole.



2. Pole(Origin): put r=0 in the equation of curve , If we get real values of θ then the curve passes through the pole, real values of θ are tangents at pole

3. **Angle between the radius vector and tangents:**

The angle $\emptyset$ between the radius vector and the tangent is given by $tan\emptyset = \frac{r}{\frac{dr}{d\theta}}$, if $tan\emptyset = 0$ tangent is coincides with the radius vector and $tan\emptyset = \infty$, tangent is perpendicular to radius vector.



(4)Tabular Values: corresponding to different values of θ find the values of r

Example:1 Trace the curve $r = a(1 + cos\theta)$ $(a > 0)$

1. symmetry : Science the equation remains unchanged when θ is replaced by $-\theta$,curve is symmetrical about the initial line .

2 pole: if r=0 $\therefore$ $a(1 + cos\theta) = 0$ $\therefore$ $cos\theta = -1$ we get $\theta = (2k + 1)\pi$, $k \in Z$,

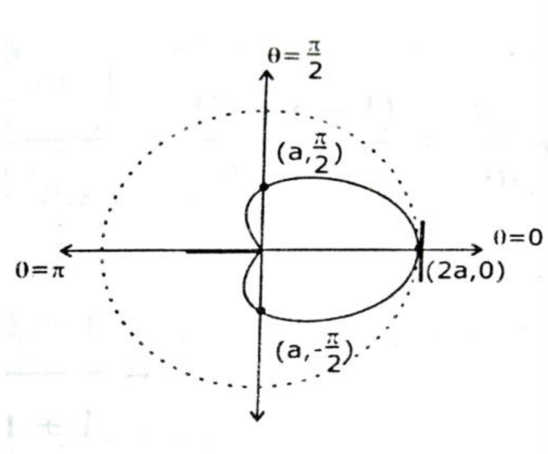
$\therefore$ curve is passes through the pole and $\theta = (2k + 1)\pi$, $k \in Z$ are tangents at pole

3 Angle between the tangent and radius vector:

$tan\emptyset = \frac{r}{\frac{dr}{d\theta}}$ $\therefore$ $tan\emptyset = \frac{a(1+cos\theta)}{-asin\theta}$ $\therefore$ $tan\emptyset = -cot\frac{\theta}{2}$

4 Tabular Values :

θ :	0	$\frac{\pi}{2}$	π
r :	2a	a	0
$tan\emptyset$:	∞	-1	0



Example:2 Trace the Curve $r^2 = a^2cos2\theta$.

1. Symmetry: Science the equation remains unchanged when θ is replaced by $-\theta$,curve is symmetrical about the initial line.

Since the equation remains unchanged when θ is replaced by $\pi - \theta$,curve is symmetrical about normal line.

Since power of r is even , curve is symmetrical about pole.

2.Pole: if r=0 $\therefore cos2\theta = 0$ $\therefore$ $\theta = (2k+1)\frac{\pi}{4}$,$k \in Z$ $\therefore$ Curve is passes through pole and $\theta = (2k+1)\frac{\pi}{4}$,$k \in Z$ are tangents at pole.

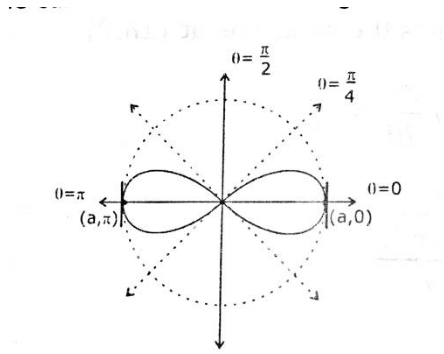
3. Angle between the tangent and radius vector:

From given equation $2r\frac{dr}{d\theta} = -2a^2 sin2\theta$

Now $tan\emptyset = \frac{r}{\frac{dr}{d\theta}}$ $\therefore tan\emptyset = \frac{r}{\frac{-a^2 sin2\theta}{r}} = -\frac{r^2}{a^2 sin2\theta} = -\frac{a^2 cos2\theta}{a^2 sin2\theta} = -cot2\theta$

4 Tabular values:

θ :	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r :	a	0	-	0	a
$tan\emptyset$:	∞	0	-	0	∞



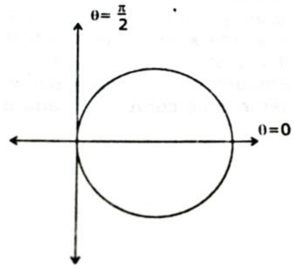
Example:3 Trace the $r = 2acos\theta$

1. Symmetry: Since the equation remain unchanged when $\theta$is replaced by $-\theta$, curve is symmetrical about initial line.

2.Pole: if r=0, $\therefore cos\theta = 0$ $\therefore \theta = (2k+1)\frac{\pi}{2}$,$k \in Z$ $\therefore$ Curve is passing through pole and

$\therefore \theta = (2k+1)\frac{\pi}{2}$,$k \in Z$ are tangents at pole.

3. Angle between the tangent and radius vector: $tan\emptyset = \frac{r}{\frac{dr}{d\theta}}$ $\therefore tan\emptyset = \frac{2acos\theta}{-2asin\theta} = -cot\theta$



4. Tabular Values:

θ:	0	$\frac{\pi}{2}$	π
r:	2a	0	-2a
$tan\emptyset$:	∞	0	∞

Exercise: Trace the following Curves:

(1) $r = a(1 - cos\theta)$ (2) $r = a(1 + sin\theta)$ (3) $r = a(1 - sin\theta)$

Ans: (1) (2) (3)