①

Homogeneous function:-
Any function ~~f~~ $f(x,y)$ is said to be of homogeneous if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ degree of homogeneous function is n

$$\boxed{\begin{matrix} x \to \lambda x \\ y \to \lambda y \end{matrix}}$$

Ex check whether the following functions are homogeneous or not

① $f(x,y) = x^2 + 2xy + y^2$

$\Rightarrow$ replace x by λx and y by λy

$$\therefore f(\lambda x, \lambda y) = (\lambda x)^2 + 2(\lambda x)(\lambda y) + (\lambda y)^2$$

$$f(\lambda x, \lambda y) = \lambda^2 x^2 + 2\lambda^2 xy + \lambda^2 y^2$$

$$= \lambda^2 (x^2 + 2xy + y^2)$$

$$f(\lambda x, \lambda y) = \lambda^2 f(x,y)$$

$\therefore$ given function is homogeneous with degree 2

(2)

(2) $f(x,y) = \dfrac{x^3+y^3}{x+y}$

$x \to \lambda x,\ y \to \lambda y$

$f(\lambda x, \lambda y) = \dfrac{(\lambda x)^3 + (\lambda y)^3}{\lambda x + \lambda y}$

$= \dfrac{\lambda^3 x^3 + \lambda^3 y^3}{\lambda (x+y)}$

$= \dfrac{\lambda^3 (x^3+y^3)}{\lambda (x+y)}$

$= \lambda^2 \dfrac{(x^3+y^3)}{(x+y)}$

$= \lambda^2 f(x,y)$

$\therefore$ given function is homogeneous with degree 2

(3) $f(x,y) = \dfrac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}$

$x \to \lambda x,\ y \to \lambda y$

$f(\lambda x, \lambda y) = \dfrac{(\lambda x)^{\frac{1}{4}} + (\lambda y)^{\frac{1}{4}}}{(\lambda x)^{\frac{1}{6}} + (\lambda y)^{\frac{1}{6}}}$

$= \dfrac{\lambda^{\frac{1}{4}} x^{1/4} + \lambda^{\frac{1}{4}} y^{1/4}}{\lambda^{1/6} \cdot x^{1/6} + \lambda^{1/6} \cdot y^{1/6}}$

$= \dfrac{\lambda^{1/4}(x^{1/4} + y^{1/4})}{\lambda^{1/6}(x^{1/6} + y^{1/6})}$

③

$$= \lambda^{\frac{1}{4}-\frac{1}{6}} \frac{(x^{\frac{1}{4}} + y^{\frac{1}{4}})}{(x^{1/6} + y^{1/6})}$$

$4 \to 2 \times 2$
$6 \to 2 \times 3$
LCM $= 2 \times 2 \times 3$
$= 12$

$$= \lambda^{\frac{3-2}{12}} \frac{(x^{\frac{1}{4}} + y^{1/4})}{(x^{\frac{1}{6}} + y^{1/6})}$$

$$= \lambda^{\frac{1}{12}} f(x, y)$$

$\therefore f(x, y)$ is homogeneous of degree $\frac{1}{12}$

④ $u(x, y) = \log\left(\frac{x^7 + y^7}{x+y}\right)$

$x \to \lambda x$
$y \to \lambda y$

$$u(\lambda x, \lambda y) = \log\left(\frac{(\lambda x)^7 + (\lambda y)^7}{\lambda x + \lambda y}\right)$$

$$= \log\left(\frac{\lambda^7 x^7 + \lambda^7 y^7}{\lambda x + \lambda y}\right)$$

$$= \log\left(\frac{\lambda^7 (x^7 + y^7)}{\lambda (x+y)}\right)$$

$$= \log\left(\lambda^6 \frac{(x^7 + y^7)}{(x+y)}\right)$$

$$\neq \lambda^n u(x, y)$$

$\therefore u(x, y)$ is not homogeneous

(4)

(5)

$$u(x,y) = \operatorname{cosec}^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{x-y}\right)$$

$x \to \lambda x$, $y \to \lambda y$

$$u(\lambda x, \lambda y) = \operatorname{cosec}^{-1}\left(\frac{\sqrt{\lambda x}-\sqrt{\lambda y}}{\lambda x - \lambda y}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{\lambda}\sqrt{x} - \sqrt{\lambda}\cdot\sqrt{y}}{\lambda(x-y)}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{\lambda}(\sqrt{x}-\sqrt{y})}{\lambda(x-y)}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{\lambda^{\frac{1}{2}}(\sqrt{x}-\sqrt{y})}{\lambda(x-y)}\right)$$

$$= \operatorname{cosec}^{-1}\left(\lambda^{\frac{1}{2}-1}\left(\frac{\sqrt{x}-\sqrt{y}}{x-y}\right)\right)$$

$$= \operatorname{cosec}^{-1}\left(\lambda^{-1/2}\left(\frac{\sqrt{x}-\sqrt{y}}{x-y}\right)\right)$$

$$\neq \lambda^n u(x,y)$$

$\therefore$ $u(x,y)$ is not homogeneous

(6) $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

$x \to \lambda x$ $y \to \lambda y$ $z \to \lambda z$

$$u(\lambda x, \lambda y, \lambda z) = [(\lambda x)^2 + (\lambda y)^2 + (\lambda z)^2]^{-\frac{1}{2}}$$

$$= [\lambda^2 x^2 + \lambda^2 y^2 + \lambda^2 z^2]^{-1/2}$$

$$= [\lambda^2(x^2+y^2+z^2)]^{-1/2}$$

$$= (\lambda^2)^{-1/2}(x^2+y^2+z^2)^{-1/2}$$

$$= \lambda^{-1} u(x,y,z)$$

(3)

$u(x y z)$ is homogeneous with degree -1

Euler's Theorem on homogeneous functions:

If u is homogeneous function of degree n in variables x and y then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

Note: (1) If u is homogeneous function of degree n in variables x and y then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

(2) If u is homogeneous function of degree n in variables x, y and z then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu$$

Ex Verify euler's theorem for

(1) $u = \sin^{-1}\left(\frac{x}{y}\right)$ ~~$+ \tan^{-1}\left(\frac{y}{x}\right)$~~

~~$u = \sin^{-1}\left(\frac{x}{y}\right)$~~

$u(x y) = \sin^{-1}\left(\frac{x}{y}\right)$

⑥ $x \to \lambda x, \quad y \to \lambda y$

$$u(\lambda x, \lambda y) = \sin^{-1}\left(\frac{\lambda x}{\lambda y}\right)$$

$$= \sin^{-1}\left(\frac{x}{y}\right)$$

$$= \lambda^0 \sin^{-1}\left(\frac{x}{y}\right)$$

$$= \lambda^0 u(x,y)$$

$\therefore$ $u(x,y)$ is homogeneous with degree 0 i.e $n=0$.

$\therefore$ Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \quad \text{—— (1)}$$

Now $u = \sin^{-1}\left(\frac{x}{y}\right)$

$\frac{d}{dx}\sin^{-1}x$

7

Now $u = \sin^{-1}\left(\frac{x}{y}\right)$

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \frac{1}{y}$$

$$= \frac{1}{\frac{\sqrt{y^2-x^2}}{y}} \cdot \frac{1}{y} = \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

$x \cdot \frac{1}{y}$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot x\left(-\frac{1}{y^2}\right)$$

$$= -\frac{1}{\frac{\sqrt{y^2-x^2}}{\sqrt{y^2}}} \cdot \frac{x}{y^2}$$

$$= -\frac{1}{\frac{\sqrt{y^2-x^2}}{y}} \cdot \frac{x}{y^2}$$

$$= -\frac{\cancel{y}}{\sqrt{y^2-x^2}} \cdot \frac{x}{y^{\cancel{2}}}$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y\sqrt{y^2-x^2}}$$

~~LHS~~ From ①

LHS of Euler's theorem

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x\left(\frac{1}{\sqrt{y^2-x^2}}\right) + y\left(\frac{-x}{\cancel{y}\sqrt{y^2-x^2}}\right)$$

$$= \frac{x}{\sqrt{y^2-x^2}} - \frac{x}{\sqrt{y^2-x^2}} = 0 = RHS$$

thus LHS = RHS of euler's theorem

Hence euler's theorem is verified.

② $u = x^2 + 2xy + y^2$

$$u(x, y) = x^2 + 2xy + y^2$$

$$u(\lambda x, \lambda y) = (\lambda x)^2 + 2(\lambda x)(\lambda y) + (\lambda y)^2$$

$$= \lambda^2 x^2 + 2\lambda^2 xy + \lambda^2 y^2$$

$$= \lambda^2 (x^2 + 2xy + y^2)$$

$$= \lambda^2 u(x, y)$$

$\therefore$ u is homogeneous function of degree 2 i.e $n = 2$

$\therefore$ euler's thm $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \quad \text{—— ①}$$

Now $u = x^2 + 2xy + y^2$

$$\frac{\partial u}{\partial x} = 2x + 2(1)y + 0$$
$$= 2x + 2y$$

$$\frac{\partial u}{\partial y} = 0 + 2x(1) + 2y$$
$$= 2x + 2y$$

From ①
LHS of euler's theorem

$$= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x(2x+2y) + y(2x+2y)$$
$$= 2x^2 + 2xy + 2xy + 2y^2$$
$$= 2x^2 + 4xy + 2y^2$$
$$= 2(x^2 + 2xy + y^2)$$
$$= 2u = RHS$$

thus LHS = RHS

$\therefore$ euler's theorem is verified

Ex Use euler's theorem ~~~~ for

① If $u = \dfrac{y^3 - x^3}{y^2 + x^2}$ find

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ and

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$

first we find the degree of homogeneous function

(10)

$$u(x,y) = \frac{y^3 - x^3}{y^2 + x^2}$$

$$u(\lambda x, \lambda y) = \frac{(\lambda y)^3 - (\lambda x)^3}{(\lambda y)^2 + (\lambda x)^2}$$

$$= \frac{\lambda^3 y^3 - \lambda^3 x^3}{\lambda^2 y^2 + \lambda^2 x^2}$$

$$= \frac{\lambda^3 (y^3 - x^3)}{\lambda^2 (y^2 + x^2)}$$

$$= \lambda \frac{(y^3 - x^3)}{y^2 + x^2}$$

$$= \lambda^0 u(x,y)$$

u is homogeneous function of degree 0 i.e $n = 0$

from euler's theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0u$$

$$\therefore \boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0}$$

and from

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0(0-1)u$$
$$= 0$$

$$\therefore x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$$

2) $f(x,y) = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$

find (i) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$

(ii) $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}$

$f(\cancel{\lambda x, \lambda y}) =$

first we check $f(x,y)$ is homogeneous and if so then we find its degree

$x \to \lambda x, \quad y \to \lambda y$

$f(\lambda x, \lambda y) = (\lambda x)^4 (\lambda y)^2 \sin^{-1}\left(\frac{\lambda y}{\lambda x}\right)$

$= \lambda^4 \cdot x^4 \lambda^2 \cdot y^2 \sin^{-1}\left(\frac{y}{x}\right)$

$= \lambda^6 x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right)$

$= \lambda^6 f$

$= \lambda^6 f(x,y)$

$\therefore f(x,y)$ is homogeneous ~~of~~ with degree 6 i.e $n = 6$

$\therefore$ from Euler's theorem

$\cancel{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =}$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x,y)$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 6 f(x,y)$

Now from

$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}$ ~~= n(n-1)f~~

$= n(n-1) f(x,y)$

(12)

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}$$

$$=6(6-1)f(x,y)$$

$$=6(5)f(x,y)$$

$$=30f(x,y)$$

$$\therefore x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=30f(x,y)$$

modified Euler's theorem:-

$u(x,y)$ is non homogeneous but ~~z=f(u)~~ In $z=f(u)$, z is homogeneous ~~then~~ with degree n then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Note:- $u(x,y)$ is non homogeneous but in $z=f(u)$, z is homogeneous with degree n then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$=g(u)\left[g'(u)-1\right]$$

where $g(u)=n\frac{f(u)}{f'(u)}$

Ex: If $u=\log\left(\frac{x^4+y^4}{x+y}\right)$ show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$$

$$u(x,y) = \log_e\left(\frac{x^4+y^4}{x+y}\right)$$

$$u(\lambda x, \lambda y) = \log_e\left(\frac{(\lambda x)^4 + (\lambda y)^4}{\lambda x + \lambda y}\right)$$

$$= \log_e\left(\frac{\lambda^4 x^4 + \lambda^4 y^4}{\lambda(x+y)}\right)$$

$$= \log_e\left(\frac{\lambda^4(x^4+y^4)}{\lambda(x+y)}\right)$$

$$= \log\left(\lambda^3\left(\frac{x^4+y^4}{x+y}\right)\right)$$

$$\neq \lambda^n u(x,y)$$

$\therefore$ u is non homogeneous

From

$$u = \log_e\left(\frac{x^4+y^4}{x+y}\right)$$

$$\boxed{\begin{array}{c}\log_y x = z \\ \Leftrightarrow x = y^z\end{array}}$$

$$e^u = \frac{x^4+y^4}{x+y}$$

i.e $$\frac{x^4+y^4}{x+y} = e^u$$

$$\boxed{z = f(u)}$$

$$z(x,y) = \frac{x^4+y^4}{x+y}, \quad f(u) = e^u$$

$$z(\lambda x, \lambda y) = \frac{(\lambda x)^4 + (\lambda y)^4}{\lambda x + \lambda y}$$

$$z(\lambda x, \lambda y) = \frac{\lambda^4(x^4+y^4)}{\lambda(x+y)}$$ (14)

$$= \lambda^3\left(\frac{x^4+y^4}{x+y}\right)$$

$$= \lambda^3 z(x,y)$$

$\therefore$ z is homogoneous of degree 3 i.e $n=3$

From modified Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{e^u}{e^u}$$

$$\boxed{\begin{aligned} f(u) &= e^u \\ f'(u) &= e^u \end{aligned}}$$

$$= 3$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

Ex: $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

$$= 2\sin u\cos 3u$$

$$u(x,y) = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$u(\lambda x, \lambda y) = \tan^{-1}\left(\frac{(\lambda x)^3+(\lambda y)^3}{\lambda x - \lambda y}\right)$$

$$= \tan^{-1}\left(\frac{\lambda^3x^3+\lambda^3y^3}{\lambda(x-y)}\right)$$

$$= \tan^{-1}\left(\frac{\lambda^3(x^3+y^3)}{\lambda(x-y)}\right)$$

$$= \tan^{-1}\left(\lambda^2\left(\frac{x^3+y^3}{x-y}\right)\right)$$

$$\neq \lambda^n u(x,y)$$

$\therefore$ u is non homogeneous

from

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$\tan u = \frac{x^3+y^3}{x-y}$$

i.e $\dfrac{x^3+y^3}{x-y} = \tan u$

$$z(x,y) = \frac{x^3+y^3}{x-y}, \quad f(u) = \tan u \qquad \boxed{z = f(u)}$$

$$z(\lambda x, \lambda y) = \frac{(\lambda x)^3 + (\lambda y)^3}{\lambda x + \lambda y}$$

$$= \frac{\lambda^3(x^3+y^3)}{\lambda(x+y)}$$

$$= \lambda^2\left(\frac{x^3+y^3}{x+y}\right)$$

$$= \lambda^2 z(x,y)$$

z is homogeneous of degree 2 i.e $n=2$

$$\tan u = \frac{\sin u}{\cos u}$$
$$\frac{1}{\sec^2 u} = \cos^2 u$$

from
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

Now $f(u) = \tan u$

$\therefore f'(u) = \sec^2 u$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u}$$
$$= 2 \cdot \frac{\sin u}{\cancel{\cos u}} \cdot \cos^{\cancel{2}} u$$
$$= 2\sin u \cos u$$
$$= \sin 2u$$

and
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

Now
$$g(u) = \frac{n.f(u)}{f'(u)}$$
$$= 2\frac{\tan u}{\sec^2 u} = 2\frac{\sin u}{\cos u} \cdot \cos^2 u$$
$$= 2\sin u \cos u$$
$$= \sin 2u$$

$\therefore g(u) = \sin 2u$

$g'(u) = \cos 2u \cdot 2$

$= 2\cos 2u$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u[2\cos 2u - 1]$$

$$= 2\cos 2u \sin 2u - \sin 2u$$

$$= 2\sin 2u \cos 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$= 2\cos\left(\frac{4u+2u}{2}\right)\sin\left(\frac{4u-2u}{2}\right)$$

$$\boxed{2CS = S - S}$$

$$= 2\cos\left(\frac{6u}{2}\right)\sin\left(\frac{2u}{2}\right)$$

$$= 2\cos 3u \sin u$$

$$= 2\sin u \cos 3u$$

Ex If $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$ show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$$

$$u(x,y) = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$$

$$u(\lambda x, \lambda y) = \sec^{-1}\left(\frac{(\lambda x)^3 - (\lambda y)^3}{(\lambda x) + (\lambda y)}\right)$$

$$= \sec^{-1}\left(\frac{\lambda^3 x^3 - \lambda^3 y^3}{\lambda x + \lambda y}\right)$$

$$= \sec^{-1}\left(\frac{\lambda^3(x^3-y^3)}{\lambda(x+y)}\right)$$

$$= \sec^{-1}\left(\lambda^2\left(\frac{x^3-y^3}{x+y}\right)\right)$$

(18) $\neq \lambda^n u(x,y)$

$\therefore u = \sec^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ is not homogeneous

$\therefore \sec u = \frac{x^3+y^3}{x-y}$

i.e $\frac{x^3+y^3}{x-y} = \sec u$ $\boxed{z=f(u)}$

$z = \frac{x^3+y^3}{x-y}$, $f(u) = \sec u$

$z(x,y) = \frac{x^3+y^3}{x-y}$

$$z(\lambda x, \lambda y) = \frac{(\lambda x)^3 + (\lambda y)^3}{\lambda x - \lambda y}$$

$$= \frac{\lambda^3(x^3+y^3)}{\lambda(x-y)}$$

$$= \lambda^2\left(\frac{x^3+y^3}{x-y}\right)$$

$$= \lambda^2 z(x,y)$$

$\therefore z$ is homogeneous with degree $n=2$

From modified Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$f(u) = \sec u$
$\therefore f'(u) = \sec u \tan u$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\cancel{\sec u}}{\cancel{\sec u}\tan u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cdot\frac{1}{\tan u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$$

(19)

Ex If $u = \sin^{-1}\left(\frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{1/6} + y^{1/6}}\right)$ prove

that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{1}{144} \tan u \left[\tan^2 u - 11\right]$$

$$u(x\, y) = \sin^{-1}\left(\frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{1/6} + y^{1/6}}\right)$$

$$u(\lambda x, \lambda y) = \sin^{-1}\left(\frac{(\lambda x)^{\frac{1}{4}} + (\lambda y)^{\frac{1}{4}}}{(\lambda x)^{1/6} + (\lambda y)^{1/6}}\right)$$

$$= \sin^{-1}\left(\frac{\lambda^{1/4} \cdot x^{1/4} + \lambda^{1/4} \cdot y^{1/4}}{\lambda^{1/6} \cdot x^{1/6} + \lambda^{1/6} \cdot y^{1/6}}\right)$$

$$= \sin^{-1}\left(\frac{\lambda^{1/4}\,(x^{1/4} + y^{1/4})}{\lambda^{1/6}\,(x^{1/6} + y^{1/6})}\right)$$

$$\boxed{\begin{aligned} &\tfrac{1}{4} - \tfrac{1}{6} \\ &= \tfrac{3-2}{12} \\ &= \tfrac{1}{12} \end{aligned}}$$

$$= \sin^{-1}\left(\lambda^{\frac{1}{4} - \frac{1}{6}}\left(\frac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)\right)$$

$$= \sin^{-1} \cdot \left(\lambda^{\frac{1}{12}}\left(\frac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)\right)$$

$$\neq \lambda^n u(x, y)$$

(20)

$\therefore u = \sin^{-1}\left(\frac{x^{\frac{1}{4}} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)$ is not homogeneous

$\therefore \sin u = \frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{1/6} + y^{1/6}}$

i.e $\frac{x^{\frac{1}{4}} + y^{1/4}}{x^{1/6} + y^{1/6}} = \sin u$

$z(x,y) = \frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{1/6} + y^{1/6}}$, $f(u) = \sin u$

$$z(\lambda x, \lambda y) = \frac{(\lambda x)^{\frac{1}{4}} + (\lambda y)^{\frac{1}{4}}}{(\lambda x)^{\frac{1}{6}} + (\lambda y)^{1/6}}$$

$$= \frac{\lambda^{1/4} \cdot x^{1/4} + \lambda^{1/4} \cdot y^{1/4}}{\lambda^{1/6} x^{1/6} + \lambda^{1/6} \cdot y^{1/6}}$$

$$= \frac{\lambda^{1/4}(x^{1/4} + y^{1/4})}{\lambda^{1/6}(x^{1/6} + y^{1/6})}$$

$$\boxed{\frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}}$$

$$= \lambda^{\frac{1}{4} - \frac{1}{6}}\left(\frac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)$$

$$= \lambda^{\frac{1}{12}} \cdot z(x,y)$$

$\therefore z$ is homogeneous with degree $n = \frac{1}{12}$

(21)

From modified euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1] \quad \text{(1)}$$

$$g(u)=\frac{n\,f(u)}{f'(u)}=\frac{1}{12}\cdot\frac{\sin u}{\cos u}$$

$$\boxed{\begin{aligned} f(u)&=\sin u\\ f'(u)&=\cos u\end{aligned}}$$

$$g(u)=\frac{1}{12}\tan u$$

$$\therefore g'(u)=\frac{1}{12}\sec^2 u$$

~~From (1)~~ LHS of (1)

$$\therefore g(u)[g'(u)-1]=\frac{1}{12}\tan u\left[\frac{\sec^2 u}{12}-1\right]$$

$$\boxed{1+\tan^2 u=\sec^2 u}$$

$$=\frac{1}{12}\tan u\left[\frac{1}{12}(1+\tan^2 u)-1\right]$$

$$=\frac{1}{12}\tan u\left[\frac{1}{12}+\frac{1}{12}\tan^2 u-1\right]$$

$$=\frac{1}{12}\tan u\left[\frac{1}{12}\tan^2 u+\frac{1}{12}-1\right]$$

$$=\frac{1}{12}\tan u\left[\frac{1}{12}\tan^2 u-\frac{11}{12}\right]$$

$$=\frac{1}{12}\tan u\left[\frac{\tan^2 u-11}{12}\right]$$

$$=\frac{1}{144}\tan u\left[\tan^2 u-11\right]$$