Partial Derivative

Function of one variable: $y=f(x)$

$\frac{dy}{dx} = f'(x)$ (ordinary derivative)

Function of two variables:

$z = f(x, y)$

Function of three variable:

$u = f(x, y, z)$

Partial derivative:

$z = f(x, y)$

Derivative of z w.r.t x keeping y as constant is called partial derivative of z w.r.t x and it is denoted by $\frac{\partial z}{\partial x}$

Similarly Derivative of z w.r.t y keeping x as a constant is called partial derivative of z w.r.t y it is ~~deno~~ denoted by $\frac{\partial z}{\partial y}$

e.g $\quad z = x^3 + 3xy^3 + 6xy + 5y + 9x + 6$

$$\frac{\partial z}{\partial x} = 3x^2 + 3(1)y^3 + 6(1)y + 0 + 9(1) + 0$$

$$= 3x^2 + 3y^3 + 6y + 9$$

$$\frac{\partial z}{\partial y} = 0 + 3x(3y^2) + 6x(1) + 5(1) + 0 + 0$$

$$= 9xy^2 + 6x + 5$$

Higher order partial derivatives

$$z = f(x, y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Note: $\frac{\partial f}{\partial x}$ is denoted by f_x

$$\frac{\partial f}{\partial y} \longrightarrow f_y$$

$$\frac{\partial^2 f}{\partial x^2} \Rightarrow f_{xx}$$

$$\frac{\partial^2 f}{\partial x \partial y} \Rightarrow f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} \longrightarrow f_{xy}$$

Ex. If $z = \sqrt{x^2+y^2}$ find $\frac{\partial z}{\partial x}(1,2)$ and $\frac{\partial z}{\partial y}(1,2)$

$\to$ first find $\frac{\partial z}{\partial x}$ and then put $(x,y) = (1,2)$ in $\frac{\partial z}{\partial x}$

similarly for $\frac{\partial z}{\partial y}(1,2)$

$$\boxed{z = \sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot \frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{1}{2\sqrt{x^2+y^2}}(2x+0)$$

$$\frac{\partial z}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x) = \frac{x}{\sqrt{x^2+y^2}}$$

To find $\frac{\partial z}{\partial x}(1,2)$

put $(x,y) = (1,2)$ i.e $x=1$, $y=2$

in $\frac{\partial z}{\partial x}$

$$\therefore \frac{\partial z}{\partial x}(1,2) = \frac{1}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}$$

Now $z = \sqrt{x^2+y^2}$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot \frac{\partial}{\partial y}(x^2+y^2)$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}}(0+2y) = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}$$

(4)

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

put $(x,y) = (1,2)$

i.e $x=1,\ y=2$

$$\therefore \frac{\partial z}{\partial y}(1,2) = \frac{2}{\sqrt{1^2+2^2}} = \frac{2}{\sqrt{1+4}} = \frac{2}{\sqrt{5}}$$

Ex:2 If $\omega = x^2 y \cos z$

find $\frac{\partial \omega}{\partial x}, \frac{\partial \omega}{\partial y}, \frac{\partial \omega}{\partial z}$ at $(2,1,0)$

$$\rightarrow \boxed{\omega = x^2 y \cos z}$$

$$\frac{\partial \omega}{\partial x} = 2x \cdot y \cos z$$

put $(x,y,z) = (2,1,0)$

i.e $x=2,\ y=1,\ z=0$

$$\therefore \frac{\partial \omega}{\partial x}(2,1,0) = 2(2)(1)\cos 0 = 4(1) = 4$$

Now $\frac{\partial \omega}{\partial y} = x^2(1)\cos z = x^2 \cos z$

$$\frac{\partial \omega}{\partial y}(2,1,0) = (2)^2 \cos(0) = 4(1) = 4$$

$$\frac{\partial \omega}{\partial z} = x^2 y(-\sin z) = -x^2 y \sin z$$

$$\frac{\partial \omega}{\partial z}(2,1,0) = -2^2(1)\sin 0 = -4(1)(0) = 0$$

$\boxed{\sin 0 = 0}$

(5)

Ex: $f(x,y,z) = \cos(4x+3y+2z)$

find f_{xyz}, f_{yzz}

$f_{xyz} = \frac{\partial^3 f}{\partial z \partial y \partial x}$

$f_{xyz} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)$

$f_{xyz} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)$

$= \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(-\sin(4x+3y+2z)\cdot 4\right)\right)$

$= -4\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\sin(4x+3y+2z)\right)\right)$

$= -4\frac{\partial}{\partial z}\left(\cos(4x+3y+2z)\cdot 3\right)$

$= -12\frac{\partial}{\partial z}\left(\cos(4x+3y+2z)\right)$

$= -12\left[-\sin(4x+3y+2z)\cdot 2\right]$

$= 24\sin(4x+3y+2z)$

$f_{yzz} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right)$

$= \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\left(-\sin(4x+3y+2z)\cdot 3\right)\right)$

$= -3\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\left(\sin(4x+3y+2z)\right)\right)$

$= -3\frac{\partial}{\partial z}\left(\cos(4x+3y+2z)\cdot 2\right)$

$= -6\cdot\frac{\partial}{\partial z}\left(\cos(4x+3y+2z)\right)$

$= -6\left(-\sin(4x+3y+2z)\cdot 2\right)$

$= 12\sin(4x+3y+2z)$

Laplace equations:

Let $z = f(x, y)$

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ is a Laplace eqn

Similarly If $\omega = f(x, y, z)$

then $\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} = 0$

Ex: $u = e^{-2y} \cos 2x$, show this function satisfy the Laplace equation

Here $u = u(x, y)$

Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u = e^{-2y} \cos 2x$

$\frac{\partial u}{\partial x} = e^{-2y} (-\sin 2x) \cdot 2$

$\frac{\partial u}{\partial x} = -2 e^{-2y} \sin 2x$

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -2 e^{-2y} \cos 2x \cdot (2)$

$\boxed{\frac{\partial^2 u}{\partial x^2} = -4 e^{-2y} \cos 2x}$ — (1)

$\frac{\partial u}{\partial y} = (-2) e^{-2y} \cos 2x$

$\frac{\partial u}{\partial y} = -2 e^{-2y} \cos 2x$

$\frac{\partial u}{\partial y}$

(5) (7)

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = (-2)(-2)e^{-2y}\cos 2x$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = 4e^{-2y}\cos 2x} \quad —(2)$$

from (1) and (2)

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -4e^{-2y}\cos 2x + 4e^{-2y}\cos 2x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(x) Ex: $u = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1} \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}(\cancel{2}x)$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -\left[(1)(x^2+y^2+z^2)^{-\frac{3}{2}} + x\left(-\frac{3}{\cancel{2}}\right)(x^2+y^2+z^2)^{-\frac{3}{2}-1}(\cancel{2}x)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[(x^2+y^2+z^2)^{-\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}\right]$$

$$= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{\partial^2 u}{\partial z^2} = -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$
$$-(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}$$
$$-(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-\frac{5}{2}+1}$$

$$= \cancel{-3(x^2+y^2+z^2)^{-3/2}} + \cancel{3(x^2+y^2+z^2)^{-3/2}}$$

$$= 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$ It is Laplace eqn

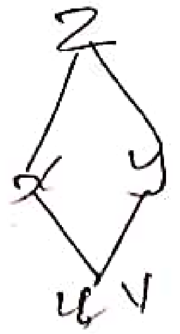
Chain Rule:

① If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$

then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

② If $z = f(x, y)$ and $x = g(u, v)$, $y = h(u, v)$

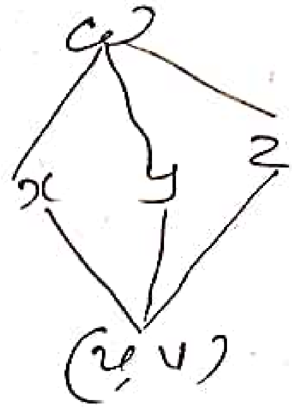
② ⑨



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

③ If $w = f(x, y, z)$
and $x = g(u, v)$
$y = h(u, v)$
$z = \phi(u, v)$



then

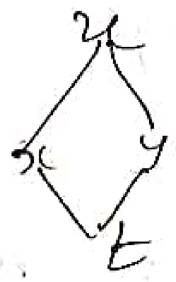
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}.$$

Ex If $u = y^2 - 4ax$ find $\frac{du}{dt}$
$x = at^2$, $y = 2at$

Here $u = u(x, y)$, $x = x(t)$
$y = y(t)$

i.e



$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{du}{dt} = (0 - 4a(1))\, a(2t) + (2y - 0)\, 2a(1)$$

$$= -4a(2at) + 2y(2a)$$

$$= -\cancel{8a^2t + 4ay}$$

$$= -8a^2t + 4ay$$

$= \cancel{-8a^2t + 4at(2at)}$

$= \cancel{-8a^2t + 8a^2}$

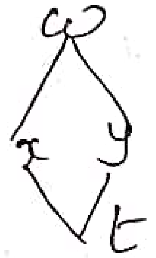
$= -8a^2t + 4a(2at) \quad (\because y = 2at \text{ given})$

$= -8a^2t + 8a^2t = 0$

Ex $w = x^2y - y^2x$, $x = \sin t$, $y = e^t$

find $\frac{dw}{dt}$ at $t = 0$

Here



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

$$\cancel{\frac{dw}{dt} = (2x)y(\cos t) +}$$

$$\frac{dw}{dt} = \left((2x)y - y^2\right)(\cos t) + (x^2 - 2y\cdot x)(e^t)$$

$$\frac{dw}{dt} = (2xy - y^2)\cos t + (x^2 - 2yx)e^t \quad \text{---} (1)$$

Now at $t = 0$

$x = \sin 0 = 0$ $\quad y = e^0 = 1$

From (1)

at $t=0$

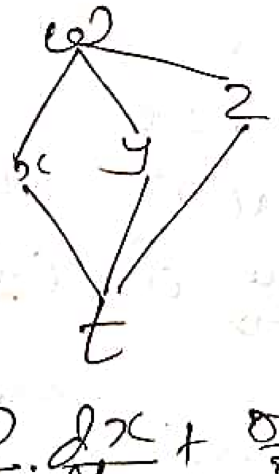
$$\frac{dw}{dt} = (2(0)(1) - 1^2)\cos 0 + (0^2 - 2(1)(0))e^0$$

$$\frac{dw}{dt} = (0-1)(1) + (0)(1)$$

$$= -1$$

~~Ex $w = z - \sin(xy)$, $x = t$, $y = \ln t$, $z = e^{1-t}$ find $\frac{dw}{dt}$ at $t=1$~~

Ex: $w = z - \sin(xy)$, $x = t$, $y = \ln t$, $z = e^{t-1}$ find $\frac{dw}{dt}$ at $t=1$.

Here 

$\boxed{\text{Here } y = \ln t \text{ i.e. } y = \log_e t}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$\frac{dw}{dt} = (0 - \cos(xy)\cdot(1)y)(1) + (0 - \cos(xy)\cdot(1)x)\frac{1}{t} + (1-0)e^{t-1}$$

$$\frac{d\omega}{dt} = -y\cos(xy) - \frac{x}{t}\cos(xy) + e^{t-1} \quad (1)$$

Now at $t = 1$

$x = t \quad \therefore x = 1$

$y = \ln t = \log_e t \quad \therefore y = \log_e 1 = 0$

$z = e^{t-1} \quad \therefore z = e^{1-1} = e^0 = 1$

From (1) at $t = 1$

$$\frac{d\omega}{dt} = -(0)\cos(0) - \frac{1}{1}\cos(0) + e^{1-1}$$

$$= 0 - 1(1) + e^0$$

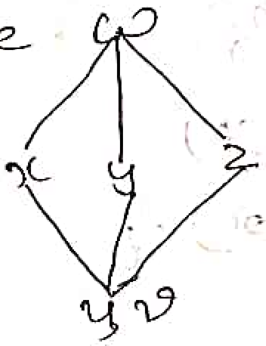
$$= -1 + 1 = 0$$

Ex If $\omega = xy + yz + zx$

$x = u + v, \quad y = u - v, \quad z = uv$

find $\frac{\partial \omega}{\partial u}$ and $\frac{\partial \omega}{\partial v}$ at $(\frac{1}{2}, 1) = (u, v)$

Here



$$\frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \omega}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial \omega}{\partial u} = (1\cdot y + 0 + z\cdot 1)(1+0)$$
$$+ (x\cdot 1 + 1\cdot z + 0)(1-0)$$
$$+ (0 + y\cdot 1 + 1\cdot x)(v)$$

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$\frac{\partial w}{\partial u} = (y+z) + (x+z) + (y+x)v$ — (1) $\left(\text{or } \frac{\partial w}{\partial u} = \cancel{2x+2y}+2z\right)$

at $(u, v) = (\frac{1}{2}, 1)$

$\frac{\partial w}{\partial u}$

$$\boxed{\begin{aligned} x &= u+v \quad \therefore x = \tfrac{1}{2}+1 = 3/2 \\ y &= u-v \quad \therefore y = \tfrac{1}{2}-1 = -1/2 \\ z &= uv \quad \therefore (\tfrac{1}{2})(1) = \tfrac{1}{2} \end{aligned}}$$

From (1)

$= (-\frac{1}{2}+\frac{1}{2}) + (\frac{3}{2}+\frac{1}{2}) + (-\frac{1}{2}+\frac{3}{2})(1)$

$\frac{\partial w}{\partial u}(\frac{1}{2}, 1) = 0 + \frac{4}{2} + \frac{2}{2} = 2+1 = 3$

Now

$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$

$= \cancel{(y+0+z)}$

$= (y\cdot 1 \cancel{+0+z\cdot 1) + (x\cdot 1+1\cdot z+0)}$

$= (y\cdot 1+0+z\cdot 1)\cdot(0+1) + (1\cdot x+1\cdot z+0)(0-1)$

$= (y\cdot 1+0+z\cdot 1)\cdot(0+1)$

$+ (0+1\cdot y+1\cdot x)(u)$

$\frac{\partial w}{\partial v} = (y+z) + (x+z)(-1) + (y+x)(u)$

at $(u, v) = (\frac{1}{2}, 1)$

$\frac{\partial w}{\partial v} = (-\frac{1}{2}+\frac{1}{2}) + (\frac{3}{2}+\frac{1}{2})(-1) + (-\frac{1}{2}+\frac{3}{2})(\frac{1}{2})$

$\cancel{\frac{\partial w}{\partial v} = 0 + \frac{4}{2}\ 0 - \frac{4}{2} + \frac{4}{2}\frac{2}{2}}$

$\frac{\partial w}{\partial v} = 0 - \frac{4}{2} + (\frac{-1+3}{2})(\frac{1}{2})$

$= -2 + \frac{2}{2}(\frac{1}{2}) = -2+\frac{1}{2} = -3/2$

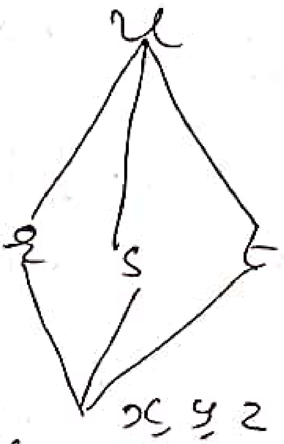
Ex: If $u = u(y-z, z-x, x-y)$ P.T.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

$u = u(y-z, z-x, x-y)$

Let $r = y-z$, $s = z-x$, $t = x-y$

$\therefore u = u(r, s, t)$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(0-1) + \frac{\partial u}{\partial t}(1-0)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \quad \text{---} (1)$$

Now $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot(1-0) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(0-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \text{---} (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0-1) + \frac{\partial u}{\partial s}(1-0) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad \text{---} (3)$$

From (1), (2) and (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}} = 0$$

EX! ~~[illegible]~~ B 15

Ex If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

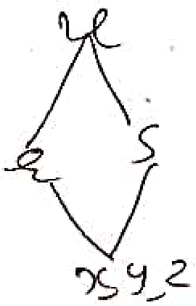
let $r = \frac{y-x}{xy} = \frac{\cancel{y}}{x\cancel{y}} - \frac{\cancel{x}}{\cancel{x}y}$

$$\boxed{\therefore r = \frac{1}{x} - \frac{1}{y}}$$

$$s = \frac{z-x}{xz} = \frac{\cancel{z}}{x\cancel{z}} - \frac{\cancel{x}}{\cancel{x}z} = \frac{1}{x} - \frac{1}{z}$$

$$\boxed{\therefore s = \frac{1}{x} - \frac{1}{z}}$$

$\therefore u = u(r, s)$ and $r = r(x,y,z)$
$s = s(x,y,z)$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(-\frac{1}{x^2} - 0\right) + \frac{\partial u}{\partial s}\left(-\frac{1}{x^2} - 0\right).$$

$$\therefore \frac{\partial u}{\partial x} = -\frac{1}{x^2}\frac{\partial u}{\partial r} - \frac{1}{x^2}\frac{\partial u}{\partial s} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(0 - \left(-\frac{1}{y^2}\right)\right) + \frac{\partial u}{\partial s}(0)$$

$$\frac{\partial u}{\partial y} = \frac{1}{y^2}\frac{\partial u}{\partial r} \quad \text{—— (2)}$$

Now $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(0 - \left(-\frac{1}{z^2}\right)\right)$$

(1-6)

$$\frac{\partial u}{\partial z} = \frac{1}{z^2}\frac{\partial u}{\partial s} \quad \text{--- (3)}$$

$$\text{LHS} = x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$$

$$= x^2\left(-\frac{1}{x^2}\frac{\partial u}{\partial r} - \frac{1}{x^2}\frac{\partial u}{\partial s}\right) + y^2\left(\frac{1}{y^2}\frac{\partial u}{\partial r}\right) + z^2\left(\frac{1}{z^2}\frac{\partial u}{\partial s}\right)$$

$$= -\frac{\cancel{x^2}}{\cancel{x^2}}\frac{\partial u}{\partial r} - \frac{\cancel{x^2}}{\cancel{x^2}}\frac{\partial u}{\partial s} + \cancel{y^2}\cdot\frac{1}{\cancel{y^2}}\frac{\partial u}{\partial r} + \cancel{z^2}\cdot\frac{1}{\cancel{z^2}}\frac{\partial u}{\partial s}$$

$$= -\cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}} = 0$$

Ex If $u = f(x^2+2yz,\ y^2+2zx)$

then P.T.

$$(y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z} = 0$$

Let $r = x^2+2yz$, $s = y^2+2zx$

$\therefore u = f(r, s)$

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(18) (17)

(x, y, z)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(2x+0) + \frac{\partial u}{\partial s}(0+2z(1))$$

$$\cancel{\frac{\partial u}{\partial x} = 2x\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial s}}$$

$$\frac{\partial u}{\partial x} = 2x\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial s} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}(0+2(1)z) + \frac{\partial u}{\partial s}(2y+0)$$

$$\frac{\partial u}{\partial y} = 2z\frac{\partial u}{\partial r} + 2y\frac{\partial u}{\partial s} \quad \text{—— (2)}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0+2y(1)) + \frac{\partial u}{\partial s}(0+2(1)x)$$

$$\frac{\partial u}{\partial z} = 2y\frac{\partial u}{\partial r} + 2x\frac{\partial u}{\partial s} \quad \text{—— (3)}$$

$$\text{LHS} = (y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z}$$

$$= (y^2-zx)\left(2x\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial s}\right) + (x^2-yz)\left(2z\frac{\partial u}{\partial r} + 2y\frac{\partial u}{\partial s}\right) + (z^2-xy)\left(2y\frac{\partial u}{\partial r} + 2x\frac{\partial u}{\partial s}\right)$$

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$= 2xy^2\frac{\partial u}{\partial \xi} - 4x^2\frac{\partial u}{\partial \xi}$

$= \cancel{2xy^2\frac{\partial u}{\partial \xi}} + \cancel{2zy^2\frac{\partial u}{\partial \eta}} + \cancel{2z^2x\frac{\partial u}{\partial \xi}} - 2xz^2\frac{\partial u}{\partial \eta}$

$+ \cancel{2xz^2\frac{\partial u}{\partial \xi}} + 2x^2y\frac{\partial u}{\partial \eta} - 2yz^2\frac{\partial u}{\partial \xi}$

$\cancel{-2y^2z\frac{\partial u}{\partial \eta}} + 2z^2y\frac{\partial u}{\partial \xi} + 2xz^2\frac{\partial u}{\partial \eta}$

$\cancel{-2xy^2\frac{\partial u}{\partial \xi}} - 2x^2y\frac{\partial u}{\partial \eta} = 0$

Implicit Differentiations:-

~~Let f(x,y,z)~~

~~Let f(x,y,z) =~~

Let $f(x,y) = c$ be a implicit function

where y is a function of x

$\boxed{\text{~~where y is function of x~~}}$

then $\dfrac{dy}{dx} = -\dfrac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

@

Ex: If $y^3+y^2-5y-x^2+4=0$ find $\frac{dy}{dx}$

Here $f(x,y)=y^3+y^2-5y-x^2+4$

$$\frac{\partial f}{\partial x}= 0+0-0-2x+0 = -2x$$

$$\frac{\partial f}{\partial y}= 3y^2+2y-5(1)-0+0$$

$$=3y^2+2y-5$$

$$\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$\frac{dy}{dx} = -\frac{(-2x)}{3y^2+2y-5} = \frac{2x}{3y^2+2y-5}$$

Ex: $\sqrt{xy}=1+x^2y$ find $\frac{dy}{dx}$

Here $\sqrt{xy}-1-x^2y=0$

$$\therefore f(x,y) = \sqrt{xy}-1-x^2y$$

$$=\sqrt{x}\cdot\sqrt{y}-1-x^2y$$

$$\frac{\partial f}{\partial x}= \frac{1}{2\sqrt{x}}\cdot\sqrt{y}-0-2xy$$

$$=\frac{\sqrt{y}}{2\sqrt{x}}-2xy$$

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$$\frac{\partial f}{\partial y} = \sqrt{x}\cdot\frac{1}{2\sqrt{y}} - 0 - x^2(1)$$

$$= \sqrt{x}\cdot\frac{1}{2\sqrt{y}} - x^2$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$\frac{dy}{dx} = -\frac{\frac{\sqrt{y}}{2\sqrt{x}} - 2xy}{\sqrt{x}\cdot\frac{1}{2\sqrt{y}} - x^2}$$

$$\frac{dy}{dx} = -\frac{\frac{\sqrt{y} - 4\sqrt{x}\cdot x\cdot y}{\cancel{2}\sqrt{x}}}{\frac{\sqrt{x} - x^2 2\sqrt{y}}{\cancel{2}\sqrt{y}}}$$

$$= -\frac{\sqrt{y}(\sqrt{y} - 4\cdot x^{\frac{1}{2}}\cdot x\cdot y)}{\sqrt{x}(\sqrt{x} - 2x^2\sqrt{y})}$$

$$= -\frac{((\sqrt{y})^2 - 4x^{3/2}\sqrt{y}\cdot y)}{\sqrt{x}\sqrt{x} - 2\sqrt{x}\cdot x^2\sqrt{y}}$$

$$= \frac{-(y - 4x^{3/2}\cdot y^{\frac{1}{2}}\cdot y)}{(\sqrt{x})^2 - 2x^2\sqrt{xy}}$$

Ex: $x^y = y^x$ find $\frac{dy}{dx}$

Here $x^y - y^x = 0$

$f(x,y) = x^y - y^x$

$$\boxed{\begin{array}{l}\frac{d}{dx}(x^n) = nx^{n-1} \\ \frac{d}{dx}(a^x) = a^x \cdot \log a\end{array}}$$

$\frac{\partial f}{\partial x} = yx^{y-1} - y^x \cdot \log y$

$\frac{\partial f}{\partial y} = x^y \cdot \log x - xy^{x-1}$

$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

$\therefore \frac{dy}{dx} = -\frac{yx^{y-1} - y^x \log y}{x^y \log x - xy^{x-1}}$

$x(y\log_e = x)$

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ex $xe^y + \sin xy + y - \log 2 = 0$
find $\frac{dy}{dx}$ at $(0, \log 2)$

$f(x,y) = xe^y + \sin xy + y - \log 2$

$\frac{\partial f}{\partial x} = (1)e^y + \cos xy \cdot y + 0 - 0$

$= e^y + y\cos xy$

$\frac{\partial f}{\partial y} = xe^y + \cos xy \cdot x + 1 - 0$

$\frac{\partial f}{\partial y} = xe^y + x\cos xy + 1$

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$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$\frac{dy}{dx} = -\frac{(e^y + y\cos xy)}{xe^y + x\cos xy + 1}$$

at $(0, \log 2)$

i.e $x = 0$, $y = \log_e 2$

$$\frac{dy}{dx} = -\frac{(e^{\log_e 2} + \log_e 2 \cos 0)}{0\cdot e^{\log_e 2} + 0\cos 0 + 1}$$

$$\boxed{x^{\log_x y} = y \quad \therefore e^{\log_e 2} = 2}$$

$$\frac{dy}{dx} = -\frac{(2 + \log_e 2(1))}{0 + 0 + 1}$$

$$\frac{dy}{dx} = -(2 + \log 2)$$