Unit - 2

Partial differentiation

* Function of two or more Variable :

The function $f(x,y)$ is called a real-valued function of two or more Variable if there are two or more independent Variable.

eg $f(x,y) = \sqrt{x^2+y^2}$

* Limit of a function of two Variables.

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

Working rule :

(1) Evaluate limits

$$\lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right) \quad \& \quad \lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right)$$

If both the limit exist & same (equal) Then

$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ exist}$$

If both the limits are unequal then limit does not exist.

(2) If $a=0$, $b=0$, Find the limit of the function taking (From) different paths Say $y=mx$ or $y=mx^n$ etc.

If all the limit Values are equal then

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ exist.}$$

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Ex:- Find $\lim_{(x,y)\to(1,2)} \frac{x^2+y}{3x+y^2}$

$$\lim_{(x,y)\to(1,2)} \frac{x^2+y}{3x+y^2} = \frac{1^2+2}{3(1)+2^2} = \frac{3}{7}.$$

Notes! If $\lim_{(x,y)\to(a,b)} f(x,y) = l$ & $\lim_{(x,y)\to(a,b)} g(x,y) = m$

(1) $\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y)$

(2) $\lim_{(x,y)\to(a,b)} [f(x,y) \cdot g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) \cdot \lim_{(x,y)\to(a,b)} g(x,y)$

(3) $\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)} = \frac{l}{m}$; $m \neq 0$.

Rule! If we find $\lim_{(x,y)\to(0,0)} f(x,y)$

(i) check $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$ & $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$

if both limit exist & equal, then proceed to next step.

(ii) Take $y=mx$ & find $\lim_{x\to 0} f(x,mx)$

& Take $y=mx^2$ & find $\lim_{x\to 0} f(x,mx)$

Examples

(1) Find $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$

Sol^n:- (i) $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x-y}{x+y}\right) = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}} = \lim_{x\to 0} 1 = 1$

(ii) $\lim_{y\to 0}\left(\lim_{x\to 0} \frac{x-y}{x+y}\right) = \lim_{y\to 0} \frac{-y}{y} = \lim_{y\to 0} -1 = -1$

Since, both the limits are different.

Hence, limit does not exist.

(2) Find limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$

Sol^n:- (i) $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{xy}{x+y}\right) = \lim_{x\to 0} \frac{0}{x} = \lim_{x\to 0} 0 = 0$

$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{xy}{x+y}\right) = \lim_{y\to 0} \frac{0}{y} = 0$

(ii) Take $y = mx$ & $x \to 0$

$\lim_{x\to 0} \frac{mx^2}{\cancel{x}(m+1)} = \lim_{x\to 0} \frac{mx}{m+1} = 0$

(iii) Take $y = mx^2$ & taking $x \to 0$

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$$\lim_{x\to 0} \frac{mx^3}{(mx+1)x} = \lim_{x\to 0} \frac{mx^2}{mx+1} = 0$$

Since, Here both the repeated limit & limits of different path are exist.

Hence, Limit is exist & it is 0.

(3) By considering different paths of approaching show that the function $f(x,y) = \frac{xy}{y^2 - x^2}$ has no limit at $(x,y) \to (0,0)$

i.e $\lim_{(x,y)\to(0,0)} \frac{xy}{y^2-x^2}$

(i) $\lim_{x\to 0}\left(\lim_{y\to 0} \frac{xy}{y^2-x^2}\right) = 0$

$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{xy}{y^2-x^2}\right) = 0$.

(ii) Take $y = mx$ & taking $x \to 0$

$$\lim_{x\to 0} \frac{x(mx)}{m^2x^2-x^2} = \lim_{x\to 0} \frac{m\cancel{x^2}}{(m^2-1)\cancel{x^2}} = \lim_{x\to 0} \frac{m}{m^2-1}$$

Since, the limit depends upon m only & m is not fixed.
Hence, limit does not exist.

(4) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^2}{x^4+y^2}$.

Sol^n: $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \dfrac{x^4-y^2}{x^4+y^2}\right) = \lim\limits_{x\to 0} \dfrac{x^4}{x^4} = 1$

$\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} \dfrac{x^4-y^2}{x^4+y^2}\right) = \lim\limits_{y\to 0} \dfrac{-y^2}{y^2} = -1$

Since both the limits are different
Hence, limit does not exist.

* Continuity :

The function $f(x,y)$ is continuous at (a,b) if the following conditions are satisfied:

(i) $f(a,b)$ exist.

(ii) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exist

(iii) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

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Example :

(1) Determine the set of points at which the given function is continuous

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol^n:- For any $(x,y) \neq (0,0)$ the function $f(x,y)$ is defined & also the limit exist & unique and the value of limit and function are to be same.
So, $f(x,y)$ is Continuous at any $(x,y) \neq (0,0)$.

Now, we ~~are~~ have to check $f(x,y)$ is Continuous at $(0,0)$ or not.
For that Find ~~f~~$f(0,0)$
$f(0,0) = 0$ (by def^n of function)

Now, we check existance of limit at $(0,0)$

(i) $$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{3x^2y}{x^2+y^2}\right) = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{3x^2y}{x^2+y^2}\right) = \lim_{y\to 0} 0 = 0$$

(ii) put $y = mx$ & taking limit $x \to 0$

$$\lim_{x\to 0}\left(\frac{3x^2mx}{x^2+m^2x^2}\right) = \lim_{x\to 0} \frac{3mx}{m^2+1} = 0$$

(iii) put $y = mx^2$ & taking limit $x \to 0$

$$\lim_{x\to 0}\left(\frac{3x^2 m x^2}{x^2+m^2x^4}\right) = \lim_{x\to 0}\frac{3mx^2}{1+m^2x^2} = 0$$

Since, all the limit among all the paths are equal and it is equal to $f(0,0)$

So, $f(x,y)$ is Continuous at $(0,0)$.

(2) check the continuity of the function at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol: (i) $\displaystyle\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{\sqrt{x^2+y^2}}\right) = \lim_{x\to 0}\frac{0}{\sqrt{x^2}} = 0$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{\sqrt{x^2+y^2}}\right) = \lim_{y\to 0}\frac{0}{\sqrt{y^2}} = 0$$

(ii) Take $y = mx$ & $x \to 0$

$$\lim_{x\to 0}\frac{x(mx)}{\sqrt{x^2+m^2x^2}} = \lim_{x\to 0}\frac{mx}{\sqrt{1+m^2}} = 0$$

(iii) Put $y = mx^2$ & taking limit $x \to 0$

$$\lim_{x\to 0}\frac{x(mx^2)}{\sqrt{x^2+m^2x^4}} = \lim_{x\to 0}\frac{mx^2}{\sqrt{1+m^2x^2}} = 0$$

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Since, limit of $f(x,y)$ among all the paths are same, the $\lim_{(x,y)\to(0,0)} f(x,y)$ exist. & is 0.

Also $f(0,0) = 0$ (by def^n).

Hence, $f(x,y)$ is Continuous at $(0,0)$.

(3) check the Continuity at origin of the f^n.

$$f(x,y) = \begin{cases} \dfrac{2x^2y}{x^3+y^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol^n Here $f(0,0) = 0$ (By def^n).

(i) $$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{2x^2y}{x^3+y^3}\right) = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} \frac{2x^2y}{x^3+y^3}\right) = 0$$

(ii) ~~first~~ put $y=mx$ & taking limit $x\to 0$

$$\lim_{x\to 0}\left(\frac{2x^2mx}{x^3+m^3x^3}\right) = \lim_{x\to 0} \frac{2m}{1+m^3}$$

Since, the limit depends only m & m is not fixed, then limit does not exist

Hence, $f(x)$ is not Continuous / discontinuous at origin

* Partial derivative :

Let $u = f(x, y)$ be a function.
Then the partial derivative of $u = f(x,y)$ w.r.t. x is the ordinary derivative of u w.r.t. x keeping y constant.

It is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x or f_x

||ly $\frac{\partial u}{\partial y}$ or $\frac{\partial f}{\partial y}$ or u_y or f_y – keeping x constant.

- $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = f_{xx} = u_{xx}$

- $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = f_{yy} = u_{yy}$

- $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = f_{yx} = u_{yx}$

- $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = f_{xy} = u_{xy}$

Note:- Standard rule for derivative of sum, difference, product and quotient are also applicable for partial derivative.

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Examples :

(1) $f(x,y) = x^2 - 4xy^3$, Find $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$

Solⁿ :- $f_x = \frac{\partial f}{\partial x} = 2x - 4y^3$

$f_y = \frac{\partial f}{\partial y} = -12xy^2$

$f_{xx} = 2$ $\quad f_{yy} = -24xy$

$f_{xy} = -12y^2$ $\quad f_{yx} = -12y^2$

Que:- Find f_{xx}, f_{yy} & f_{xy} & f_{yx} ~~of~~ the following functions. Hence, verify Mixed derivative Theorem (clairaut's Theorem)

* clairaut's Theorem → $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$
$z = f(x,y)$

(1) $z = x + y^x$

Solⁿ :- $\frac{\partial z}{\partial x} = 1 + y^x \log y$

$\frac{\partial z}{\partial y} = x y^{x-1}$

~~Now,~~ $\frac{\partial^2 z}{\partial x^2} =$ ~~$(\log y) y^x$~~ $\quad \frac{\partial^2 z}{\partial y^2} =$

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$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = y^{x-1} \cdot 1 + x y^{x-1} \log y$$

$$= y^{x-1}(1 + x\log y) \quad \text{——} (1)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = y^{x} \cdot \frac{1}{y} + \log y \cdot x y^{x-1}$$

$$= y^{x-1}[1 + x\log y] \quad \text{——} (2)$$

From (1) & (2) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

(2) $f(x,y) = \ln(2x+3y)$

Sol^n: $f_x = \frac{2}{2x+3y}$ $\qquad f_{xx} = \frac{-4}{(2x+3y)^2}$

$f_y = \frac{3}{2x+3y}$ $\qquad f_{yy} = \frac{-9}{(2x+3y)^2}$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{-6}{(2x+3y)^2}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{-6}{(2x+3y)^2}$$

Since, clearly $f_{xy} = f_{yx}$

$\therefore$ Clairaut's Theorem Verified

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(3) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

Sol^n:- $f_x = \frac{1}{1+y^2/x^2} \cdot \frac{-y}{x^2} = \frac{-y/x^2}{\cancel{x^2}+y^2/\cancel{x^2}} = \frac{-y}{x^2+y^2}$

$f_y = \frac{1}{1+y^2/x^2} \cdot 1/x = \frac{x}{x^2+y^2}$

$f_{xx} = \frac{2xy}{(x^2+y^2)^2}$ $\quad f_{yy} = \frac{-2xy}{(x^2+y^2)^2}$

$f_{xy} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$f_{yx} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

Since, Here $f_{xy} = f_{yx}$

$\therefore$ The clairaud's eq^n is satisfied

Ex:- If $x = r\cos\theta$ & $y = r\sin\theta$ then show that $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$

Sol^n:- Here $x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$ —— (1)

Diff^n eq^n (1) partially w.r.t. x we get,

$2x = 2r\frac{\partial r}{\partial x} \Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$

Diff^n eq^n (1) partially w.r.t. y we get

$$2y = 2r\frac{\partial r}{\partial y} \Rightarrow \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$$

$$L.H.S = \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2$$

$$= \frac{x^2}{r^2} + \frac{y^2}{r^2}$$

$$= \frac{x^2+y^2}{r^2}$$

$$= 1 \qquad (\because x^2+y^2 = r^2)$$

$$= R.H.S$$

Ex: If $z = \sqrt{x^2+4y^2}$. Find $\left(\frac{\partial z}{\partial x}\right)\Big|_{(1,2)}$ & $\left(\frac{\partial z}{\partial y}\right)\Big|_{(1,2)}$

Sol^n:- Here $z = \sqrt{x^2+4y^2}$

$$\frac{\partial z}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+4y^2}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+4y^2}}$$

$$\left(\frac{\partial z}{\partial x}\right)\Big|_{(1,2)} = \boxed{\frac{1}{\sqrt{17}}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2+4y^2}} \cdot 8y = \frac{4y}{\sqrt{x^2+4y^2}}$$

$$\left(\frac{\partial z}{\partial y}\right)\Big|_{(1,2)} = \boxed{\frac{8}{\sqrt{17}}}$$

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Ex:- If $f(x, y, z) = \cos(4x + 3y + 2z)$ then find f_{xyz}, f_{yzz}.

Sol^n:- $f_x = -\sin(4x+3y+2z) \cdot 4$

$f_x = -4\sin(4x+3y+2z)$

$f_{xy} = -12\cos(4x+3y+2z)$

$\boxed{f_{xyz} = 24\sin(4x+3y+2z)}$

$f_y = -3\sin(4x+3y+2z)$

$f_{yz} = -6\cos(4x+3y+2z)$

$\boxed{f_{yzz} = 12\sin(4x+3y+2z)}$

Ex:- Show that the following fu$^{n's}$ satisfies Laplace equation

Note:- Laplace eq^n : For $u = f(x, y)$

- For two variable $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

i.e $u_{xx} + u_{yy} = 0$

If $u = f(x, y, z)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

i.e $u_{xx} + u_{yy} + u_{zz} = 0$

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① $u = \ln\sqrt{x^2+y^2}$

Solⁿ:- $u_x = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x = \frac{x}{x^2+y^2}$

$u_{xx} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$u_y = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}y = \frac{y}{(x^2+y^2)}$

$u_{yy} = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

Now, $u_{xx} + u_{yy} = \frac{\cancel{y^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}}{(x^2+y^2)^2} = 0$

Hence, laplace eqⁿ satisfied.

② $u = e^{-x}\cos y - e^{-y}\cos x$

$u_x = -e^{-x}\cos y + e^{-y}\sin x$

$u_{xx} = e^{-x}\cos y + e^{-y}\sin x$

$u_y = -e^{-x}\sin y + e^{-y}\cos x$

$u_{yy} = -e^{-x}\cos y - e^{-y}\cos x$

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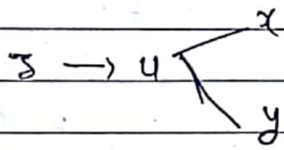
Now, $u_{xx} + u_{yy} = \cancel{e^{-x}\cos y} + \cancel{e^{-y}\sin x} - \cancel{e^{-x}\cos y} - \cancel{e^{-y}\sin x}$

$\therefore u_{xx} + u_{yy} = 0$

Hence, Laplace eq^n is satisfied

* Chain rule:

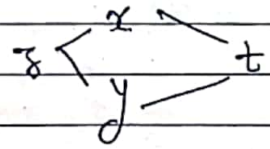
- let $z = f(u)$



  & $u = u(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} \quad \& \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y}$$

- let $z = f(x,y)$

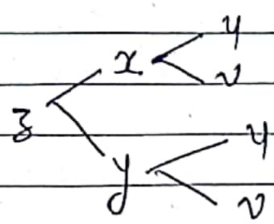


  $x = x(t)$ & $y = y(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

dz/dt is called total derivative of z.

- let $z = f(x,y)$

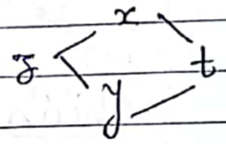


  $x = x(u,v)$ & $y = y(u,v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Ex:- For $z = x e^{xy}$, $x = t^2$, ~~get~~ $y = t^{-1}$
Compute $\frac{dz}{dt}$.



Sol^n:- Using chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= (xy e^{xy} + e^{xy}).2t + x^2 e^{xy}.(-t^{-2})$$

Now, putting the values of x & y in terms of t, we get,

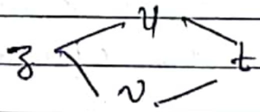
$$\frac{dz}{dt} = e^{t^2\cdot t^{-1}}\left[t^2\cdot t^{-1}+1\right]\cdot 2t + t^4 e^{t^2 t^{-1}}\cdot\left(\frac{-1}{t^2}\right)$$

$$= 2e^t t^2 + 2te^t - t^2 e^t$$

$$= t^2 e^t + 2te^t$$

$$\frac{dz}{dt} = (2t+t^2)e^t$$

Ex:- If $z = u^2+v^2$ & $u = at^2$ & $v = 2at$ then find $\frac{dz}{dt}$.



Sol^n:- By chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$$

$$= 2u \cdot 2at + 2v \cdot 2a$$

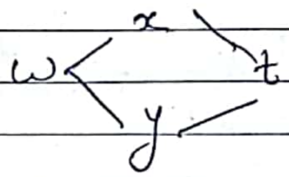
Put the values of u & v in terms of t,

$$\frac{dz}{dt} = 2at^2 \cdot 2at + 2at \cdot 2a$$

$$\frac{dz}{dt} = 4a^2 t\,(t^2+1)$$

Ex:- If $w = x^2+y^2$, $x = \cos t$, $y = \sin t$ & $t = \pi$ then evaluate $\frac{dw}{dt}$ at the given value of t.

Sol^n:- by using chain rule,



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

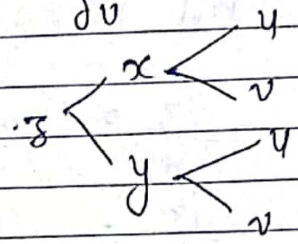
$$= 2x \cdot (-\sin t) + 2y \cdot (\cos t)$$

$$= -2 \sin t \cdot \cos t + 2 \sin t \cdot \cos t$$

$$\frac{dw}{dt} = 0 \Rightarrow \left.\frac{dw}{dt}\right|_{t=\pi} = 0$$

Ex: Let $z = e^{x^2 y}$, where $x(u,v) = \sqrt{uv}$ & $y(u,v) = \frac{1}{v}$. Find $\frac{\partial z}{\partial u}$ & $\frac{\partial z}{\partial v}$

Sol^n: Using chain rule.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \left(y e^{x^2 y} \cdot \cancel{2} x\right) \cdot \frac{\sqrt{v}}{\cancel{2}\sqrt{uv}} + x^2 e^{x^2 y} \cdot (0)$$

$$= \cancel{\sqrt{uv}} \cdot \frac{1}{v} \cdot e^{uv \cdot \frac{1}{v}} \cdot \frac{\sqrt{v}}{\cancel{\sqrt{uv}}}$$

$$\boxed{\frac{\partial z}{\partial u} = e^{u}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \left(\cancel{2} x y \, e^{x^2 y}\right)\left(\frac{\sqrt{u}}{\cancel{2}\sqrt{v}}\right) + \left(x^2 e^{x^2 y}\right)\left(\frac{-1}{v^2}\right)$$

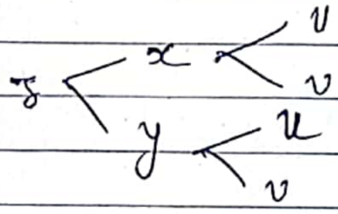
$$= \sqrt{uv} \cdot \frac{1}{v} \cdot e^{u} \frac{\sqrt{u}}{\sqrt{v}} - \frac{u\cancel{v}}{v^{\cancel{2}}} e^{u}$$

$$= \frac{u}{v} e^{u} - \frac{u}{v} e^{u}$$

$$\boxed{\frac{\partial z}{\partial v} = 0}$$

Ex:- Find $\frac{\partial z}{\partial u}$ & $\frac{\partial z}{\partial v}$, if $z = \tan^{-1}\left(\frac{x}{y}\right)$, $x = u\cos v$ & $y = u\sin v$, $(u, v) = (1.3, \pi/6)$.

Sol^n:- Using chain rule,



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \left(\frac{1}{1+x^2/y^2}\cdot\frac{1}{y}\right)(\cos v) + \left(\frac{1}{1+x^2/y^2}\cdot\frac{-x}{y^2}\right)(\sin v)$$

$$= \frac{y\cos v}{x^2+y^2} - \frac{x\sin v}{x^2+y^2}$$

$$= \frac{u\cdot\sin v\cdot\cos v}{u^2} - \frac{u\cos v\cdot\sin v}{u^2}$$

$$\boxed{\left(\frac{\partial z}{\partial u}\right)_{(1.3,\frac{\pi}{6})} = 0}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \left(\frac{1}{1+x^2/y^2}\cdot\frac{1}{y}\ \cancel{\cdot u\sin v}\right)\cdot(-u\sin v)$$

$$+ \left(\frac{1}{1+x^2/y^2}\left(-\frac{x}{y^2}\right)\right)\cdot u\cos v$$

$$= \frac{-yu\sin v}{x^2+y^2} - \frac{xu\cos v}{x^2+y^2}$$

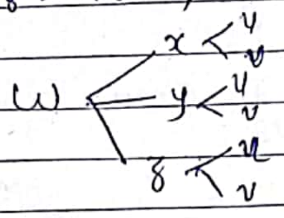
$$= \frac{-\cancel{u^2}\sin^2 v}{\cancel{u^2}} - \frac{\cancel{u^2}\cos^2 v}{\cancel{u^2}}$$

$$\boxed{\frac{\partial z}{\partial v} = -1}$$

Ex:- Find $\frac{\partial w}{\partial u}$ & $\frac{\partial w}{\partial v}$, If $w = xy + yz + zx$

$x = u+v, \quad y = u-v, \quad z = uv, \quad (u,v) = (\tfrac{1}{2}, 1)$

Sol^n:-



Using chain rule,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= (y+z)\cdot(1) + (x+z)\cdot(1) + (y+x)(v)$$

$$= u-\cancel{v} + uv + u+\cancel{v} + uv + (u+\cancel{v} + u-\cancel{v})\, v$$

$$\frac{\partial w}{\partial u} = 2u + 2uv + 2uv$$

$$\frac{\partial w}{\partial u}\Big|_{(1/2,\,1)} = 3$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= (y+z)(1) + (x+z)(-1) + (y+x)(u)$$

$$= u-v + uv - [u+v+uv] + (u-\cancel{v}+u+\cancel{v})(u)$$

$$= \cancel{u} - v + \cancel{uv} - \cancel{u} - v - \cancel{uv} + 2u^2$$

$$\frac{\partial w}{\partial v} = -2v + 2u^2 = -2 + \frac{2}{4} = -2 + \frac{1}{2} = -\frac{3}{2}$$

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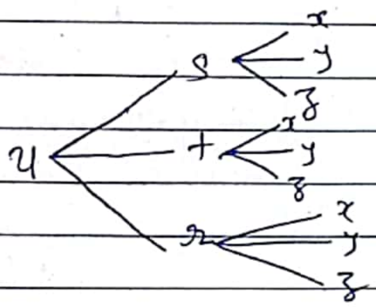
Ex:- If $u = f(y-z,\ z-x,\ x-y)$, Prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Sol^n:- By let $s = y - z$

$t = z - x$

$r = x - y$



By chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= 0\cdot\frac{\partial u}{\partial s} + (-1)\cdot\frac{\partial u}{\partial t} + 1\,\frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \text{———} \ (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$= 1\cdot\frac{\partial u}{\partial s} + 0\cdot\frac{\partial u}{\partial t} + (-1)\cdot\frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} \quad \text{———} \ (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\,\frac{\partial t}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= (-1)\,\frac{\partial u}{\partial s} + 1\,\frac{\partial u}{\partial t} + 0\cdot\frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s} \quad \text{———} \ (3)$$

From (1), (2) & (3) we get,

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0}$$

* Homogeneous Function:

A function $f(x,y)$ is said to be homogeneous of degree n if for real number t we have

$$\boxed{f(tx,ty)=t^n f(x,y)}$$

* Euler's theorem:

If u is a homogeneous fn^n of two Variables x and y of degree n then,

$$\boxed{x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=nu}$$

For three Variable.

$$\boxed{x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}+z.\frac{\partial u}{\partial z}=nu}$$

Corollary: If u is a homogeneous function of two Variables x,y of degree n then

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u}$$

Corollary: If $z = f(u)$ is a homogeneous function of x & y of degree n, then

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$$

Corollary If $z = f(u)$ is a homogeneous function of x & y of degree n, then

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\left[g'(u) - 1\right]$$

where,

$$g(u) = n \cdot \frac{f(u)}{f'(u)}$$

Blcs **Ques-** If $u(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2}$, then find the value of

(i) $x \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}$ (ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

Solⁿ:- Here $u(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2}$

$$= \frac{1}{x^2} + \frac{1}{xy} + \frac{1}{x^2} \log\left(\frac{x}{y}\right)$$

Replacing x by tx & y by ty

$$u(tx, ty) = \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{x^2} \log\left(\frac{x}{y}\right)\right]$$

$$= t^{-2}\, u(x,y)$$

Thus, u is a homogeneous function of degree -2

By Euler's theorem

$$x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = -2u$$

& $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2(-2-1)u = 6u$$

Que:- Verify Euler's theorem for the function $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Solⁿ:- Here, $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

replace x by tx & y by ty, we get,

$$u(tx, ty) = t^0 \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

Thus, u is a homogeneous fuⁿ of degree zero.

$$L.H.S = x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - x^2/y^2}} \cdot \frac{1}{y} + \frac{1}{1 + y^2/x^2} \cdot \frac{-y}{x^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - x^2/y^2}} \cdot \frac{-x}{y^2} + \frac{1}{1 + y^2/x^2} \cdot \frac{1}{x}$$

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$$\frac{\partial u}{\partial y} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

Now, $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$

$$- \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$= 0$$

$$= 0.u = nu = R.H.S$$

Hence, Euler's theorem Verified

Exercise: Use Euler's theorem to solve the following problems:

(1) If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \tan u$

Soln:- Here $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$\Rightarrow \sin u = \frac{x^2+y^2}{x+y} = f(u) \quad (\text{say})$$

$$\Rightarrow f(u) = \sin u = \frac{x^2+y^2}{x+y}$$

Replacing x by tx & y by ty in $f(u)$

$$f(u) = \frac{t^2(x^2+y^2)}{t(x+y)} = t^1 f(u)$$

Thus, $f(u) = \sin u$ is a homogeneous f^n of degree 1 in x & y.

Hence, by Euler's theorem

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)}$$

$$= 1 \cdot \frac{\sin u}{\cos u}$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u.$$

Ex:- If $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -2\sin^3 u \cos u$$

Solⁿ:- Here $u = \tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$\Rightarrow \tan u = \frac{x^2+y^2}{x+y} = f(u) \text{ (say)}$$

Replacing x by tx & y by ty, we can see that $f(u) = \tan u$ is a homogeneous function of degree 1 in x & y.
Hence, by Euler's theorem,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1] \quad \text{—— (1)}$$

where $g(u) = n \dfrac{f(u)}{f'(u)} = 1 \cdot \dfrac{\tan u}{\sec^2 u}$

$$g(u) = \frac{\sin u}{\cos u} \cdot \cos^2 u = \frac{1}{2}\, 2 \sin u \cdot \cos u =$$

$$= \frac{1}{2} \sin 2u$$

$$g'(u) = \cancel{\sin 2u} \cdot \cos 2u$$

From (1)

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{2} \sin 2u\, [\cos 2u - 1]$$

$$= \sin u \cdot \cos u\, [-2 \sin^2 u]$$

$$= -2 \sin^3 u \cdot \cos u$$

Hence proved.

* Implicit Function:

Thm: let $F(x,y)$ be a diff^{r} function & that the equation $F(x,y)=0$ defines y as a diff^{n} function of x. Then at any point where $F_y \neq 0$, we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-\partial F/\partial x}{\partial F/\partial y}$$

Thm:- let $F(x,y,z)$ is diff^{r} function & that the equation $F(x,y,z)=0$ defined z is a diff^{r} fu^{n} of x & y. Then at any point where $F_z \neq 0$, we have

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}, \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Ex:- 1 If $y^2 - x^2 - \sin xy = 0$ then find $\frac{dy}{dx}$.

Sol^{n}:- Take $F(x,y) = y^2 - x^2 - \sin xy$

$$F_x = \frac{\partial F}{\partial x} = -2x - y\cos xy$$

$$F_y = \frac{\partial F}{\partial y} = 2y - x\cos xy$$

$$\therefore \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy}$$

$$= \frac{2x + y\cos xy}{2y - x\cos xy}$$

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Ex: 2 Find $\frac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$

Here $(\cos x)^y = (\sin y)^x$

Taking log on both the sides, we get,

$$y \log \cos x = x \log \sin y$$

let, $F(x,y) = y \log(\cos x) - x \log(\sin y)$

$$F_x = \frac{y}{\cos x}\cdot(-\sin x) - \log \sin y$$

$$F_x = -y\tan x - \log \sin y$$

$$F_y = \log \cos x - \frac{x}{\sin y}\cdot \cos y$$

$$= \log \cos x - x \cot y$$

$$\therefore \frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{y\tan x + \log \sin y}{\log \cos x - x\cot y}$$

Ex: 3 Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ if $xyz = \cos(x+y+z)$.

Sol:- Take $F(x,y,z) = xyz - \cos(x+y+z)$

Here $F_x = yz + \sin(x+y+z)$

$$F_y = xz + \sin(x+y+z)$$

$$F_z = xy + \sin(x+y+z)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}$$

Ex:-4 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$ at $(2,3,6)$.

Solⁿ.- Take $F(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$

$F_x = \frac{-1}{x^2}$ $\quad F_y = \frac{-1}{y^2}$ $\quad F_z = \frac{-1}{z^2}$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-z^2}{x^2}$$

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{(2,3,6)} = \frac{-36}{4} = -9$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-z^2}{y^2} = \frac{-36}{9} = -4$$

* Jacobians:

The Jacobian of the transformation $x = g(u,v)$ & $y = h(u,v)$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

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Similarly, ~~the~~ Jacobian of the transformation $x = f(u,v,w)$, $y = g(u,v,w)$ & $z = h(u,v,w)$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Note:- If $J = \frac{\partial(x,y)}{\partial(u,v)}$ then $J' = \frac{\partial(u,v)}{\partial(x,y)}$

$$\boxed{J \cdot J' = 1}$$

Note:- x & y are fn of r,s & r,s are fn of u,v

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(u,v)}$$

Q:-1 Find the Jacobian for the transformation $x = r\cos\theta$, $y = r\sin\theta$

Sol:-

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$$

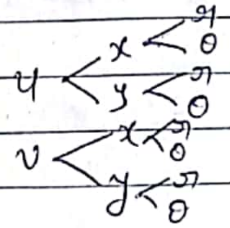
$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Ex:-2 If $u = x^2 - y^2$, $v = 2xy$ & $x = r\cos\theta$, $y = r\sin\theta$

Find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$



Soln:- $\dfrac{\partial(u,v)}{\partial(r,\theta)} = \dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(r,\theta)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4(x^2+y^2)$$

$\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$ (by previous example)

Hence, $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 4(x^2+y^2) \cdot r$

$$= 4(r^2\cos^2\theta + r^2\sin^2\theta)\, r$$

$$\boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3}$$

* Vector Differential operators Del :-
It is denoted by ∇ & defined by

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

* Gradient :- The gradient of a function $f(x,y,z)$ is denoted by ∇f (grad f) & it is defined as

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$$\text{grad } f = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Note:- The grad f is a vector normal to the surface.

eg Find gradient of $f(x,y,z) = 2z^3 - 3(x^2+y^2)z + \tan^{-1}(xz)$ at the point $(1,1,1)$.

Sol^n: Using def^n of gradient,

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\left[-6xz + \frac{z}{1+x^2z^2}\right] + \hat{j}\left[-6yz\right] + \hat{k}\left[6z^2 - 3(x^2+y^2) + \frac{x}{1+x^2z^2}\right]$$

$$(\nabla f)_{(1,1,1)} = \left[-6 + \tfrac{1}{2}\right]\hat{i} + (-6)\hat{j} + \left[6 - 6 + \tfrac{1}{2}\right]\hat{k}$$

$$= \frac{-11}{2}\hat{i} - 6\hat{j} + \frac{1}{2}\hat{k}$$

* Directional derivative:

The directional derivative of the function $f(x,y,z)$ at a point $P(x,y,z)$ in the direction of vector $\bar{a}$ is given by

$$\boxed{D_{\bar{a}}f = (\nabla f)_P \cdot \hat{a}} \quad \text{where } \hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

Step:1 Find $(\text{grad } f)_P$ / $(\nabla f)_P$

Step:2 Find $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|}$ $\left\{ \begin{array}{l} \bar{a} = a\hat{i} + b\hat{j} + c\hat{k} \\ |\bar{a}| = \sqrt{a^2+b^2+c^2} \end{array} \right\}$

Step:3 Take product of $(\nabla f)_P \cdot \hat{a}$.

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Ex:- Find the derivative of $f(x,y) = x^2 \sin 2y$ at the point $(1, \frac{\pi}{2})$ in the direction of $\bar{v} = 3\hat{i} - 4\hat{j}$.

Solⁿ:- The unit vector $\hat{v} = \frac{\bar{v}}{|\bar{v}|} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{9+16}} = \frac{3\hat{i} - 4\hat{j}}{5}$

$$\therefore \boxed{\hat{v} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}}$$

Now, The gradient of f at $(1, \frac{\pi}{2})$,

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y}$$

$$= \hat{i}(2x \sin 2y) + \hat{j}(2x^2 \cos 2y)$$

$$(\nabla f)_{(1,\frac{\pi}{2})} = (0)\hat{i} + j(-2) = -2\hat{j}$$

$$\therefore D_{\bar{v}} f = (\nabla f)_{(1,\frac{\pi}{2})} \cdot \hat{v} = (-2\hat{j})\left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right)$$

$$= \frac{8}{5}$$

Ex:- The temperature at any point in space is given by $T = xy + yz + zx$. Find the derivative of T in the direction of the vector $3\hat{i} - 4\hat{k}$ at the point $(1,1,1)$.

Solⁿ:- let $\bar{a} = 3\hat{i} - 4\hat{k}$

$$\therefore \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\nabla T = \hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$$

$$= (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$$

$$(\nabla T)_{(1,1,1)} = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$\therefore D_{\bar{a}}T = (\nabla T)_P \cdot \hat{a} = (2\hat{i}+2\hat{j}+2\hat{k}) \cdot \left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right)$

$= \frac{6}{5} - \frac{8}{5} = -\frac{2}{5}$

* Tangent plane & Normal line

- Let the equation of the surface is $f(x,y,z) = 0$.
- The eq^n of tangent plane at point $P(x_1, y_1, z_1)$ to the surface is

$$(x-x_1)\left(\frac{\partial f}{\partial x}\right)_P + (y-y_1)\cdot\left(\frac{\partial f}{\partial y}\right)_P + (z-z_1)\left(\frac{\partial f}{\partial z}\right)_P = 0.$$

- The equation of normal line are

$$\frac{(x-x_1)}{\left(\frac{\partial f}{\partial x}\right)_P} = \frac{(y-y_1)}{\left(\frac{\partial f}{\partial y}\right)_P} = \frac{(z-z_1)}{\left(\frac{\partial f}{\partial z}\right)_P}$$

Ex:- 1 Find the equation of the tangent plane & normal line at the point $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Sol^n:- Let $f(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3$

$f_x(x,y,z) = \frac{x}{2}$ $\qquad f_x(-2,1,-3) = -1$

$f_y(x,y,z) = 2y$ $\qquad f_y(-2,1,-3) = 2$

$f_z(x,y,z) = \frac{2z}{9}$ $\qquad f_z(-2,1,-3) = -\frac{2}{3}$

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Hence, the equation of tangent line at $(-2, 1, -3)$

$$(x-(-2))\cdot(-1)+(y-1)(2)+(z-(-3))\cdot\left(\frac{-2}{3}\right)=0$$

$$\Rightarrow -(x+2)+2y-2-\frac{2z}{3}-2=0$$

$$\Rightarrow 2y-x-\frac{2z}{3}=6$$

$$\Rightarrow \boxed{3x-6y+2z=-18}$$

The equation of normal line are

$$\frac{x+2}{-1}=\frac{y-1}{2}=\frac{z+3}{-2/3}$$

Ex:- 2 Find an equation for the plane that is tangent to the ~~elliop~~ surface $z=\ln(x^2+y^2)$ at the point $(1,0,0)$

Soln:- let $f(x,y,z)=z-\ln(x^2+y^2)$

$f_x(x,y,z)=\frac{-2x}{x^2+y^2}$, $f_x(1,0,0)=-2$

$f_y(x,y,z)=\frac{-2y}{x^2+y^2}$, $f_y(1,0,0)=0$

$f_z(x,y,z)=1$ $\quad f_z(1,0,0)=1$.

The eqn of tangent plane at $(1,0,0)$ is

$$(x-1)\cdot(-2)+(y-0)(0)+(z-0)(1)=0.$$

$$-2x+2+z=0$$

$$\boxed{2x-z=2}$$

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* Maxima & Minima of Two Variable Function :

* The point at which function $f(x,y)$ is either maximum or minimum is known as stationary point. The value of the function at stationary point is known as extreme (maximum or minimum) value of the function $f(x,y)$.

* The point at which both the partial derivative of fu^n are zero or one or both partial derivative does not exist is known as Critical point.

* Working Rule to determine the extreme values

1) Solve $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ Simultaneously for x & y.

2) obtain the $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

3)

Points	r	s	t	$rt - s^2$	Conclusion	Value of fu^n
(a,b)	>0			>0	minimum	
	<0			>0	maximum	
				<0	Saddle point (neither max nor min)	
				$=0$	no Conclusion	

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Ex:1 Find the extreme values of the function
$f(x,y) = x^3+y^3-3x-12y+20$.

Solⁿ:- Here $f(x,y) = x^3+y^3-3x-12y+20$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2-3=0 \Rightarrow x = \pm 1$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2-12=0 \Rightarrow y = \pm 2$

$\therefore$ stationary points are $(1,2), (1,-2), (-1,2), (-1,-2)$

$r = \frac{\partial^2 f}{\partial x^2} = 6x$, $s = \frac{\partial^2 f}{\partial x \partial y} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = 6y$

Points	r	s	t	$rt-s^2$	Conclusion	Value of fⁿ
(1,2)	6 >0	0	12	72 >0	minimum	2
(1,-2)	6	0	-12	-72 <0	neither max nor min	
(-1,2)	-6	0	12	-72 <	neither max nor min	
(-1,-2)	-6 <0	0	-12	72 >0	maximum	38

$\therefore$ minimum at $(1,2)$ & $f(1,2) = 2$

maximum at $(-1,-2)$ & $f(-1,-2) = 38$

Ex: Find the extreme values of the function $x^3+3xy^2-3x^2-3y^2+7$, if any.

Solⁿ: Here $f(x,y) = x^3+3xy^2-3x^2-3y^2+7$

Step:-1 For extreme values,

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2+3y^2-6x=0$$

$$\Rightarrow x^2+y^2-2x=0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 6xy-6y=0 \quad \text{--- (2)}$$

$$\Rightarrow 6y(x-1)=0$$

$$\Rightarrow y=0 \ \& \ x=1$$

Put $y=0$ in (1) we get,

$$x^2-2x=0 \Rightarrow x(x-2)=0$$

$$\Rightarrow x=0,\ x=2$$

$\therefore$ $(0,0)$, $(2,0)$ are stationary point

Put $x=1$ in eqⁿ (1) we get

$$1+y^2-2=0 \Rightarrow y^2-1=0$$

$$\Rightarrow \boxed{y=\pm 1}$$

$\therefore$ $(1,1)$, $(1,-1)$ are stationary point

~~step-II~~

Now, $r = \frac{\partial^2 f}{\partial x^2} = 6x - 6$

$s = \frac{\partial^2 f}{\partial x \partial y} = 6y$

$t = \frac{\partial^2 f}{\partial y^2} = 6x - 6$

Step:-3

Points	r	s	t	$rt - s^2$	Conclusion
$(0,0)$	$-6<0$	0	-6	$36>0$	maximum
$(2,0)$	$6>0$	0	6	$36>0$	minimum
$(1,1)$	0	6	0	$-6<0$	neither max nor min
$(1,-1)$	0	-6	0	$-6<0$	neither max nor min

$\therefore$ minimum at $(2,0)$ & $f(2,0) = f_{min} = 3$

$\therefore$ maximum at $(0,0)$ & $f(0,0) = f_{max} = 7$.

Also, $(0,0)$, $(2,0)$, $(1,1)$, $(1,-1)$ are Critical points.

Ex:- $f(x,y) = x^3 - 2xy - y^3 + 6$

Sol^n:- $\frac{\partial f}{\partial x} = 3x^2 - 2y = 0$

$\Rightarrow$ ~~$x(3x - 12) = 0$~~ $\quad 3x^2 = 2y$

$\Rightarrow$ ~~$x = 0 \quad x = 2/3$~~

~~$\frac{\partial f}{\partial y} = 0 \Rightarrow -2x$~~

$\frac{\partial f}{\partial y} = 0 \Rightarrow -2x - 2y^2 = 0$

$2y^2 + 2x = 0$

Ex:- $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$

Critical points $(0,0), (1,-1)$

saddle point $(1,-1)$

local minimum $f(0,0) = 0$