Gamma Functions:

The integral $\int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0$ is called gamma function. It is denoted by

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0$$

Properties of Gamma function:

1. $\Gamma(n+1) = \begin{cases} n\Gamma(n) \\ n! \quad n \text{ is a positive integer} \end{cases}$
2. $\displaystyle\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$
3. $\displaystyle 2\int_0^\infty e^{-x^2} x^{2n-1} dx = \Gamma(n)$
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Examples:

1. Evaluate $\displaystyle\int_{-\infty}^{\infty} e^{-k^2 x^2} dx$

Solution: Let

$$I = \int_{-\infty}^{\infty} e^{-k^2 x^2} dx$$

$$= 2\int_0^\infty e^{-k^2 x^2} dx$$

Let, $k^2 x^2 = u \quad \therefore 2k^2 x dx = du$

$$dx = \frac{1}{2k^2 x} du = \frac{1}{2k} u^{-\frac{1}{2}} du$$

$$x : 0 \to \infty \Rightarrow u : 0 \to \infty$$

$$\therefore I = 2\int_0^\infty e^{-u}\ \frac{1}{2k} u^{-\frac{1}{2}} du$$

$$= \frac{1}{k}\int_0^\infty e^{-u} u^{\frac{1}{2}-1} du$$

$$= \frac{1}{k}\Gamma\frac{1}{2}$$

$$= \frac{1}{k}\sqrt{\pi}$$

2. Evaluate $\int_0^\infty e^{-x^2} x^5 dx$

Solution: Take $x^2 = u$

$$\therefore 2xdx = du$$

$$\therefore x : 0 \to \infty \Rightarrow u : 0 \to \infty$$

$$\therefore I = \int_0^\infty e^{-x^2} x^4 dx$$

$$\therefore I = \int_0^\infty e^{-u} u^2 du$$

$$= \frac{1}{2}\int^\infty e^{-u} u^{3-1} du$$

$$= \frac{1}{2}\Gamma 3$$

$$= 1$$

3. Evaluate $\int_0^1 x^4 e^{-x^4} dx$

Solution: Take $x^4 = u$

$$\therefore 4x^3 dx = du$$

$$\therefore x^3 dx = \frac{1}{4} du$$

Also $x : 0 \to \infty \Rightarrow u : 0 \to \infty$

$$\therefore I = \int_0^\infty x\, e^{-x^4}\, x^3 dx$$

$$= \int_0^\infty u^{\frac{1}{4}} e^{-u}\, \frac{1}{4} du$$

$$= \frac{1}{4}\int_0^\infty e^{-u} u^{\frac{5}{4}-1} du$$

$$= \frac{1}{4}\Gamma\frac{5}{4}$$

$$= \frac{1}{4}\,\frac{1}{4}\Gamma\frac{1}{4}$$

Practice Examples:

1. Evaluate $\int_0^1 x^m (\log x)^n\, dx$
2. Evaluate $\int_0^\infty \frac{x^5}{5^x}\, dx$
3. Evaluate $\int_0^1 \frac{1}{\sqrt{x \log\left(\frac{1}{x}\right)}}\, dx$
4. Evaluate $\int_0^\infty 5^{-3x^2}\, dx$
5. Evaluate $\int_0^\infty 5^{-4x^2}\, dx$

Beta Functions:

The integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx \quad m>0, n>0$ is called Beta function. It is denoted by

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \quad m>0, n>0$$

Properties of Beta function:

1. $B(m,n)=B(n,m)$
2. $B(m,n)=2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, dx$
3. Relation between Beta and Gamma function: $B(m,n)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$
4. Duplication Formula or Legendre's formula:
   $\Gamma(m)\,\Gamma\!\left(m+\dfrac{1}{2}\right)=\dfrac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}}$

Examples:

1. Prove that $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta=\dfrac{1}{2}B\left(\dfrac{m+1}{2},\dfrac{n+1}{2}\right)$

   Solution: By definition

   $$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \quad m>0, n>0$$

   Take

   $x=\sin^2\theta \;\; \therefore dx=2\sin\theta\cos\theta d\theta$

   $x:0\to 1 \Rightarrow \theta:0\to\dfrac{\pi}{2}$

$$\therefore B(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\ 2\sin\theta\cos\theta\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\ d\theta$$

$$\therefore \int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\ d\theta = \frac{1}{2}B(m,n)$$

Taking $2m-1 = p \quad and \quad 2n-1 = q$ we get

$$\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\ d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$ **Or** $$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\ d\theta = \frac{1}{2}B\left(\frac{m+1}{2},\frac{n+1}{2}\right).$$

2. Prove that $n\ B(m+1,n) = n\ B(m,n+1)$

Solution:

$$n\ B(m+1,n) = n \times \frac{\Gamma(m+1)\ \Gamma(n)}{\Gamma(m+n+1)}$$

$$= \frac{m\Gamma(m) \times n\Gamma(n)}{\Gamma(m+n+1)}$$

$$= m \times \frac{\Gamma(m)\ \Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= mB\ (m,n+1)$$

3. Evaluate $\int_0^m x^m(\mathrm{m}-\mathrm{x})^n\,dx$

Solution: take

$x = mu \quad \therefore dx = mdu$

$x : 0 \to 1 \Rightarrow u : 0 \to 1$

$$\therefore I = \int_0^m (mu)^m (m - mu)^n m \, du$$

$$= m^{m+n+1} \int_0^1 u^m \, (1-u)^n \, du$$

$$= m^{m+n+1} \int_0^1 u^{(m+1)-1} \, (1-u)^{(n+1)-1} \, du$$

$$= m^{m+n+1} B(m+1, n+1)$$

Practice Examples:

1. Evaluate $\int_0^1 x^5 (1-x^3)^{10} \, dx$ in terms of beta function.
2. Evaluate $\int_0^2 x^4 (8-x^3)^{-\frac{1}{3}} \, dx$
3. Evaluate $\int_0^\infty \frac{x^8 (1-x^6)}{(1+x)^{24}} \, dx$
4. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta} \, d\theta \times \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin\theta}} \, d\theta$
5. Evaluate $\int_0^{\frac{\pi}{2}} \sin^2\theta (1+\cos\theta)^4 \, d\theta$ by using gamma function.

Reduction Formulae:

Useful Properties:

1. $\int_a^b f(x) \, dx = \int_a^b f(\mathrm{t}) \, dt$
2. $\int_a^b f(x) \, dx = -\int_b^a f(\mathrm{t}) \, dt$
3. For $a < c < b$, $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$
4. $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$

5. $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$

6. $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx; & \text{if } f(x) \text{ is an even function} \\ 0; & \text{if } f(x) \text{ is an odd function} \end{cases}$

 # Reduction Formula for $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ **and** $\int_0^{\frac{\pi}{2}} \cos^n x\,dx \quad (n \in N,\ n > 1)$

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{(n-1)(n-3)\ldots\ \ 2 \text{ or } 1}{n(n-2)\ldots\ \ 2 \text{ or } 1} \times K \text{ where } K = \begin{cases} \frac{\pi}{2}; & n \text{ is even} \\ 1; & n \text{ is odd} \end{cases}$$

 # Reduction Formula for $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx \quad (m, n \in N, \quad m, n > 1)$

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)\ldots\ \ 2 \text{ or } 1][(n-1)(n-3)\ldots\ \ 2 \text{ or } 1]}{(m+n)(m+n-2)\ldots\ \ 2 \text{ or } 1} \times K$$

$$\text{Where } K = \begin{cases} \frac{\pi}{2}; & m \text{ and } n \text{ both are even} \\ 1; & \text{otherwise} \end{cases}$$

Examples:

1. Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 \theta\,d\theta$

Solution: $I = \int_0^{\frac{\pi}{2}} \sin^6 \theta\,d\theta$

$$= \frac{5\times3\times1}{6\times4\times2}\times\frac{\pi}{2}$$
$$= \frac{5\pi}{32}$$

2. Evaluate $\int_0^{\frac{\pi}{8}} \cos^3 4\theta \, d\theta$

Solution: Let $4\theta = x \quad \therefore d\theta = \frac{1}{4}dx$

Also, $\theta : 0 \to \frac{\pi}{8} \Rightarrow x : 0 \to \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \cos^3 x \times \frac{1}{4}dx$$
$$= \frac{1}{4}\times\frac{2}{3\times1}\times1$$
$$= \frac{1}{6}$$

3. Evaluate $\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta d\theta$

Solution: let $I = \int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta d\theta$

$$= \frac{[6\times4\times2][4\times2]}{[12\times10\times8\times6\times4\times2]}\times1$$

$$= \frac{1}{120}$$

Practice Examples:

1. Evaluate $\int_0^{\frac{\pi}{6}} \sin^2 6\theta \cos^6 3\theta \, d\theta$
2. Evaluate $\int_0^{\pi} \theta \sin^8\theta \cos^6\theta \, d\theta$

3. Evaluate $\int_{0}^{\pi} \sin^2\theta\,(1+\cos\theta)^4\ d\theta$

4. $\int_{0}^{\infty} \frac{x^2}{(1+x^2)^8}\ dx$

5. Evaluate $\int_{0}^{4} x^3\sqrt{4x-x^2}\ dx$

6. Evaluate $\int_{0}^{2a} x^3(2ax-x^2)^{\frac{3}{2}}\ dx$

Improper Integral of First Kind

In the definite integral $\int_a^b f(x)\,dx$, if either a or b or both a and b are infinite, then $\int_a^b f(x)\,dx$ is called an improper integral of the first kind.

For Example, $\int_1^{\infty}\frac{dx}{x}$, $\int_{-\infty}^{1} e^{2x}\,dx$, $\int_{-\infty}^{\infty}\frac{dx}{x^2+3x+1}$ are improper integrals of the first kind.

CASE:I

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx, (t>a)$$

CASE:II

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_{t}^{b} f(x)\,dx,\ (b>t)$$

CASE:III

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t_1\to-\infty}\int_{t_1}^{b} f(x)\,dx + \lim_{t_2\to-\infty}\int_{b}^{t_2} f(x)\,dx,$$ where b is any real number

The improper integrals in CASE I, CASE II and in CASE III said to be convergent if the limits on the right hand side exist finitely and is said to be divergent if limit is $\pm\infty$.

Improper Integral of Second Kind

In the integral $\int_{a}^{b} f(x)\,dx$, if both a and b are finite but the integrand (that is $f(x)$) is infinite in $a \le x \le b$ then $\int_{a}^{b} f(x)\,dx$ is called an improper integral of the second kind.

For example, $\int_{0}^{1}\frac{dx}{x^2}, \int_{0}^{3}\frac{dx}{3-x}, \int_{0}^{4}\frac{dx}{x(x-2)}$ are improper integrals of the second kind.(as the integrand are infinite at x=0, x=3 and x=0,2 respectively)

CASE:I

If $f(x)$ becomes infinite at $x=b$ only, we define $\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+}\int_a^{b-\varepsilon} f(x)\,dx$. The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

CASE:II

If $f(x)$ becomes infinite at $x=a$ only, we define $\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+}\int_{a+\varepsilon}^{b} f(x)\,dx$. The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely, otherwise it is said to be divergent.

CASE:III

If $f(x)$ becomes infinite at $x=c$ only where $a<c<b$, , we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{\varepsilon_1\to 0+}\int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2\to 0+}\int_{c+\varepsilon_2}^{b} f(x)\,dx$$

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if both the limit on the right hand side exists finitely and independent of each other, otherwise it is said to be divergent.

Evaluation of Improper Integral of First Kind

Determine if the following integral converges or diverges. If the integral converges determine its value.

Problem: $\int_0^{\infty} \frac{e^x}{e^{2x}+3}\,dx$

Solution:

Improper because one of the limit of integral is infinite (Type I).

Let's do a u-substitution first.

Let $u = e^x$, then $du = e^x dx$.

When $x = 0, u = 1$ and when $x \to \infty, u \to \infty$:

$$\int_0^{\infty} \frac{e^x}{e^{2x}+3}\,dx = \int_0^{\infty} \frac{e^x}{\left(e^x\right)^2+3}\,dx = \int_1^{\infty} \frac{1}{u^2+3}\,du = \lim_{t\to\infty} \int_1^{t} \frac{1}{u^2+3}\,du$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Bigg|_1^t = \lim_{t\to\infty} \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{\pi}{2} - \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

Problems:

$(i)\ \int_0^{\infty} \frac{1}{\sqrt[4]{1+x}}\,dx$ $(ii)\ \int_{-\infty}^{0} 2^x\,dx$ $(iii)\ \int_{-\infty}^{\infty} \left(y^3 - 3y^2\right) dy$

$(iv)\ \int_{-\infty}^{\infty} \cos\left(\pi t\right) dt$ $(v)\ \int_{-\infty}^{-1} \frac{dx}{x^2+4x+5}$

Ans: (i) Diverges (ii) Converges, $\frac{1}{\ln 2}$(iii) Diverges

(iv) Diverges(v) converges, $\frac{3\pi}{4}$

Evaluation of Improper Integral of Second Kind

Determine if the following integral converges or diverges. If the integral converges determine its value.

Problem: $\int_0^4 \frac{x}{x^2-9}\,dx$

Solution: Improper because integrand $\frac{x}{x^2-9}$ becomes infinite at $x=3$. 3 lies between the range of integration. (Type II).

We split up the integral at $x=3$.

$$\int_0^4 \frac{x}{x^2-9}\,dx = \int_0^3 \frac{x}{x^2-9}\,dx + \int_3^4 \frac{x}{x^2-9}\,dx$$

$$\int_0^4 \frac{x}{x^2-9}\,dx = \lim_{t\to 3^-}\int_0^t \frac{x}{x^2-9}\,dx + \lim_{t\to 3^+}\int_t^4 \frac{x}{x^2-9}\,dx$$

$$\int_0^4 \frac{x}{x^2-9}\,dx = \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|x^2-9\right|\right)\Bigg|_0^t + \lim_{t\to 3^+}\left(\frac{1}{2}\ln\left|x^2-9\right|\right)\Bigg|_t^4$$

$$= \lim_{t\to 3^-}\left(\frac{1}{2}\ln\left|t^2-9\right| - \frac{1}{2}\ln(9)\right)\Bigg| + \lim_{t\to 3^+}\left(\frac{1}{2}\ln(7) - \frac{1}{2}\ln\left|t^2-9\right|\right)$$

$$= \left[-\infty - \frac{1}{2}\ln(9)\right] + \left[\frac{1}{2}\ln(7) + \infty\right]$$

This indicates that the First integral tends to $-\infty$ whereas Second integral tends to ∞.

Therefore, each of these integrals is divergent. This in turn means that the integral diverges.

Problems:

$(i)\ \int_0^5 \frac{1}{\sqrt[3]{2-w}}\,dw \quad (ii)\ \int_1^2 \frac{1}{\sqrt{2-x}}\,dx \quad (iii)\ \int_1^4 \frac{1}{x^2+x-6}\,dx$

$(iv)\ \int_{-2}^2 \frac{dx}{x^3} \quad (v)\ \int_{-1}^4 \frac{x}{x^2-9}\,dx$

Ans: (i) Converges, $-\frac{3}{2}\left(\sqrt[3]{9}-\sqrt[3]{4}\right)$ (ii) Converges, 2

(iii) Diverges (iv) Diverges (v) Diverges