Series & Sequence

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Defn: Sequence: An ordered set of real numbers is sequence.

i.e if $u_1, u_2, u_3, \ldots, u_n, \ldots$ is ordered set of real number the seq^n is denoted by $\{u_n\}$. where u_n is n^{th} term of seq^n.

Examples of seq^n:

(1) $\{2, 4, 6, 8, \ldots\} = \{a_n\}_{n=1}^{\infty}$ where $a_n = 2n$

(2) $\{-1, 1, -1, 1, \ldots\} = \{a_n\}_{n=1}^{\infty}$ where $a_n = (-1)^n$

Defn: Series: It is an infinite sum of ~~numbers~~ terms of a sequence.

It is denoted by $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

e.g:

(1) $2+4+6+8+\cdots = \sum_{n=1}^{\infty} 2n$

(2) $(-1)+1+(-1)+1+\cdots = \sum_{n=1}^{\infty} (-1)^n$

* Convergence of a sequence:

A sequence $\{a_n\}$ is called convergent if it's term reaches to a number l as n become larger & larger (i.e $n \to \infty$) otherwise it is called divergent sequence.

eg (1) Consider the sequence $\{a_n\}$ where $a_n = \frac{2}{n}$

i.e. $\{a_n\} : \{2, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \dots\}$

One can observe that as n become larger the terms of sequence approches to zero. Therefore, it is convergent.

(2) $\{a_n\} = \{2, 4, 6, 8, \dots\}$ where $a_n = 2n$

One can observe that as n become larger & larger the term of sequence also get larger & larger, it doesn't approche to any number. Therefore it is divergent.

* Results:

(1) $\lim_{n\to\infty} (n)^{1/n} = 1$

(6) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

(2) $\lim_{n\to\infty} x^{1/n} = 1$, $x > 0$, x is fixed no.

(3) $\lim_{n\to\infty} x^n = 0$, $|x| < 1$, $\lim_{n\to\infty} x^n = \infty$ if $x > 1$

(4) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ (any x)

(5) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

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* Increasing & decreasing sequence

A seq^n is said to be increasing if $U_{n+1} > U_n$ for each value of n & decreasing if $U_{n+1} < U_n$ for each value of n

~~ex~~

e.g $a_n = \frac{3n+1}{n+1}$ increasing

$a_n = \frac{n}{2^n}$ decreasing

* Convergence of a seq^n:

If The seq^n $\{U_n\}$ has a finite limit i.e $\lim_{n \to \infty} U_n$ is finite, the seq^n is said to be Convergent

If $\lim_{n\to\infty} U_n$ is infinite, seq^n is said to be divergent

If ~~the~~ limit of $\{U_n\}$ is not unique then ~~is~~. seq^n is said to be oscillatory.

e.g $\lim_{n\to\infty} \frac{n^2+1}{2n^2-1} = \lim_{n\to\infty} \frac{n^2(1+1/n^2)}{n^2(2-1/n^2)} = \lim_{n\to\infty} \frac{1+1/n^2}{2-1/n^2}$

$= \frac{1}{2}$

$\therefore \left\{\frac{n^2+1}{2n^2-1}\right\}$ is Convergent.

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* Geometric series :

Consider the geometric series

$$\sum_{n=1}^{\infty} u_n = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots , \quad \text{where } u_n = ar^{n-1}$$

Take partial sum (sum of n^{th} term)

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{if } r < 1$$

$$S_n = \frac{a(r^n - 1)}{r-1} \quad \text{if } r > 1$$

If $|r| < 1$ $\therefore \lim_{n\to\infty} \frac{a}{1-r}$ is finite

$\therefore$ Series Convergent

If $|r| > 1$ $\therefore \lim_{n\to\infty} \cancel{\frac{a}{1-r}}\ S_n = \infty$

$\therefore$ Series divergent

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e.g $\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}$

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} 5 \cdot \frac{1}{3^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{5}{3}\left(\frac{1}{3}\right)^{n-1}$$

Since, Here $a_1 = \frac{2}{3}$ & $r_1 = \frac{2}{3}$

$$\therefore S_1 = \frac{2/3}{1 - 2/3} = 2$$

& $a_2 = 5/3$ & $r_2 = 1/3$

$$\therefore S_2 = \frac{5/3}{1 - 1/3} = \frac{5}{2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} = 2 + \frac{5}{2} = \frac{9}{2}$$

* P - series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

if $p > 1$ Series Convergent

if $p \leq 1$ Series divergent.

* n^{th} term test for divergence :

Consider a seq^n $\{u_n\}$.

if Σu_n - Convergent $\Rightarrow \lim_{n\to\infty} u_n = 0$

i.e if $\lim_{n\to\infty} u_n \neq 0 \Rightarrow \Sigma u_n$ is not Convergent

eg $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$

sol^n: $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+1/n}}$

$= 1 \neq 0$

$\therefore \sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$ is diverge.

* Integral test :

Consider series $\sum_{n=1}^{\infty} a_n$

let $f(n) = a_n$

i.e $f(x) = a_x$

if $\int_1^{\infty} f(x)\,dx$ is Converge $\Rightarrow \Sigma a_n$ Converge

if $\int_1^{\infty} f(x)\,dx$ is diverge $\Rightarrow \Sigma a_n$ diverge

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e.g $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \cdots$

Here $a_n = \frac{1}{n^2+1}$

i.e $f(x) = \frac{1}{x^2+1}$

Now, $\int_1^{\infty} \frac{1}{x^2+1}\,dx = \left[\tan^{-1} x\right]_1^{\infty}$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is Converge

* Comparison Test :

If Σu_n & Σv_n are series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite & non-zero)

then both series Converge or diverge together

Note :- The Test series Σv_n is known as auxiliary series & it is Obtain by

$v_n = \dfrac{\text{highest power term in numerator in } u_n}{\text{highest power term in denominator in } u_n}$

Generally $\Sigma \frac{1}{n^p}$ will pick up as auxiliary series.

Ex:- ~~Te~~ Test the Convergence of the Series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

Solⁿ: Let $u_n = \frac{\sqrt{n}}{n^2+1} = \frac{n^{1/2}}{n^2+1}$

$$= \frac{1}{n^{3/2}(1+1/n^2)}$$

let $v_n = \frac{1}{n^{3/2}}$

Since, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1$ (finite & non-zero)

Also, $\Sigma v_n = \sum \frac{1}{n^{3/2}}$ is Convergent as

~~$p>1$~~ $p = 3/2 > 1$.

Hence, by Comparison test Σu_n is also Convergent

Ex:- $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$

Solⁿ: $u_n = \frac{1}{3^n+1}$ $v_n = \frac{1}{3^n}$

$\Sigma v_n = \Sigma \frac{1}{3^n}$ is Convergent as G.S

$r = \frac{1}{3}$. $\therefore \Sigma u_n$ Convergent

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* Ratio test:

If ΣU_n is a positive term series &

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$ then

ΣU_n is Convergent if $l < 1$

ΣU_n is divergent if $l > 1$

Test fails if $l = 1$

or If ΣU_n is a positive term series &

$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = l$ then

ΣU_n is Convergent if $l > 1$

ΣU_n is divergent if $l < 1$

Test fails if $l = 1$

Ex:-1 $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

Sol^n:- let $U_n = \frac{2^n+5}{3^n}$

$U_{n+1} = \frac{2^{n+1}+5}{3^{n+1}} = \frac{2\cdot 2^n+5}{3\cdot 3^n}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2\,2^n+5}{3\cdot 3^n} \times \frac{3^n}{2^n+5}$$

$$= \lim_{n\to\infty} \frac{2^n(2+5/2^n)}{3\cdot 2^n(1+5/2^n)}$$

$$= \lim_{n\to\infty} \frac{2+5/2^n}{3(1+5/2^n)}$$

$$= \frac{2}{3} \qquad \left(\because \lim_{n\to\infty} x^n = \infty,\ x>1\right)$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{2}{3} < 1$$

$\therefore$ Σu_n is Convergent.

Ex:-2 $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n!}$

Solⁿ: let $u_n = \dfrac{a^n}{n!}$, $u_{n+1} = \dfrac{a^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{a^{n+1}}{(n+1)!} \times \frac{n!}{a^n}$$

$$= \lim_{n\to\infty} \frac{a}{(n+1)}$$

$$= 0 < 1$$

$\therefore$ Σu_n is Convergent.

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* Cauchy's Root Test:

If Σu_n is positive term series &
if
$$\lim_{n\to\infty} (u_n)^{1/n} = l \quad \text{then}$$

Σu_n is Convergent if $l < 1$

Σu_n is divergent if $l > 1$

Test fails if $l = 1$

Note: If there is $\{f(n)\}^{g(n)}$ type given then apply root test

Result: $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

e.g: $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

Sol^n: Let $u_n = \left(\frac{n}{n+1}\right)^{n^2}$

$$\therefore \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$

$= \frac{1}{e} < 1 \quad \left(\because \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \right)$

$\therefore$ Σu_n is Convergent.

* **Alternating Series:**

An infinite series with alternate positive & negative terms is called alternating series.

* **Leibnitz's test for Alternating series:**

An Alternating series $\Sigma u_n = \Sigma (-1)^{n-1} v_n$ is Convergent if

(i) each $|u_{n+1}| < |u_n|$ / $|u_n| > |u_{n+1}|$

(ii) $\lim_{n\to\infty} |u_n| = 0$.

eg check the Convergence of the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

Sol:- let $u_n = (-1)^{n-1} \frac{1}{\sqrt{n}}$

$|u_n| = \frac{1}{\sqrt{n}}$

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The given series is an alternating series.

(i) $|u_n| - |u_{n+1}| = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$

$= \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}\sqrt{n+1}} > 0 \quad \forall\ n \in \mathbb{N}$

$\therefore |u_n| > |u_{n+1}|$

(ii) $\lim_{n \to \infty} |u_n| = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

Hence, by Leibnitz's test, the series is Convergent.