Unit-4

* Topic Covered:
1) Reduction Formula
2) Beta & gamma function
3) Improper integral of first & second kind.

* Some usefull properties of Integration:

1) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3) For $a < c < b$, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

6) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even fu^n

$= 0$, if $f(x)$ is odd fu^n

What is odd fu^n & what is even function?

If $f(-x) = f(x)$ then $f(x)$ is even

If $f(-x) = -f(x)$ then $f(x)$ is odd.

Page: Date: / /

* Reduction Formulae:

1) $\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{(n-1)(n-3)\cdots 1}{n(n-2)(n-4)\cdots 2} \cdot \frac{\pi}{2}$

if n is even +ve integer

$= \frac{(n-1)(n-3)\cdots 2}{n(n-2)(n-4)\cdots 1}$

if n is odd +ve integer

Examples:

(1) $\int_0^{\frac{\pi}{2}} \sin^7\theta\, d\theta$

$\sin^7 2\theta$

take $2\theta = t$

Sol^n:- W.K.T. $\int_0^{\frac{\pi}{2}} \sin^7\theta\, d\theta = \frac{(n-1)(n-3)\cdots 1 \text{ or } 2}{n(n-2)(n-4)\cdots 2 \text{ or } 1}$, n – odd

Here $n=7$ which is odd

$\therefore \int_0^{\frac{\pi}{2}} \sin^7\theta\, d\theta = \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)(7-6)}$

$= \frac{\overset{2}{\cancel{6}}\times 4\times 2}{7\times 5\times \cancel{3}\times 1} = \frac{16}{35}$

(2) $\int_0^{\frac{\pi}{2}} \cos^8\theta\, d\theta$.

Sol^n:- Here $n=8$ (even)

$\therefore \int_0^{\frac{\pi}{2}} \cos^8\theta\, d\theta = \frac{(8-1)(8-3)(8-5)(8-7)}{8(8-2)(8-4)(8-6)} \cdot \frac{\pi}{2}$

$$= \frac{7\times 5\times \cancel{3}\times 1}{8\times \cancel{6}_2\times 4\times 2}\times\frac{\pi}{2} = \frac{35}{128}\cdot\frac{\pi}{2}$$

(3) Evaluate $\int_0^{\pi} (1+\cos\theta)^4\, d\theta$

Sol:- $\int_0^{\pi} (1+\cos\theta)^4\, d\theta = \int_0^{\pi} (2\cos^2\theta/_2)^4\, d\theta$

$$= 2^4 \int_0^{\pi} \cos^8\theta/_2\, d\theta$$

Now, take $\frac{\theta}{2} = t \Rightarrow \frac{d\theta}{2} = dt \Rightarrow d\theta = 2\,dt$.

Also, $\theta\to 0 \Rightarrow t\to 0$ & $\theta\to\pi \Rightarrow t\to\frac{\pi}{2}$

$\therefore \int_0^{\pi/2} 2^4 \cos^8 t\; 2dt = 2^5 \int_0^{\pi/2} \cos^8 t\, dt$

Here $n=8$ (even)

$$\therefore \int_0^{\pi} (1+\cos\theta)^4\, d\theta = 2^5\, \frac{(8-1)(8-3)(8-5)(8-7)}{8(8-2)(8-4)(8-6)}\cdot\frac{\pi}{2}$$

$$= \frac{\cancel{2^5}\times 7\times 5\times \cancel{3}\times 1}{\cancel{8}\times \cancel{6}_2\times \cancel{4}\times 2}\times\frac{\pi}{2}$$

$$= \frac{35\pi}{8}$$

(4) $\int_0^{\pi} (1-\cos\theta)^3\, d\theta$ Ans $\left(=\frac{5\pi}{2}\right)$

Page :
Date : / /

(5) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7\theta \, d\theta = 0$ $\quad (\because \sin$ is odd $fu^n)$

* $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$

$$= \frac{[(m-1)(m-3)\cdots 1][(n-1)(n-3)\cdots 1]}{(m+n)(m+n-2)(m+n-4)\cdots 2} \cdot \frac{\pi}{2}$$

if $m \,\&\, n$ = even

$$= \frac{[(m-1)(m-3)\cdots 2][(n-1)(n-3)\cdots (2)]}{(m+n)(m+n-2)(m+n-4)\cdots 1} \cdot 1$$

if m = odd, n = even
n = odd, m = even
both $m \,\&\, n$ are odd

Ex:-1 Evaluate $\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^6\theta \, d\theta$

Sol^n:- Here $m = 2$ & $n = 6$ both are even

$$\therefore \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^6\theta \, d\theta = \frac{(2-1)(6-1)(6-3)(6-5)}{8(8-2)(8-4)(8-6)} \cdot \frac{\pi}{2}$$

$$= \frac{1 \times 5 \times 3 \times 1}{8 \times 6 \times 4 \times 2} \cdot \frac{\pi}{2}$$

$$= \boxed{\frac{15\pi}{768}}$$

Ex:-2 $\int_0^{\pi/6} \sin^2 6\theta \cdot \cos^6 3\theta \, d\theta$

Sol:- Let $3\theta = t \Rightarrow 3\,d\theta = dt$

When $\theta = 0 \Rightarrow t = 0$

$\theta = \frac{\pi}{6} \Rightarrow t = \frac{\pi}{2}$

$$\therefore \int_0^{\pi/6} \sin^2 6\theta \cdot \cos^6 3\theta \, d\theta = \int_0^{\pi/2} \sin^2 2t \cdot \cos^6 t \, \frac{dt}{3}$$

$$= \int_0^{\pi/2} \cos^6 t \cdot (2\sin t \cdot \cos t)^2 \, \frac{dt}{3}$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2 t \cdot \cos^8 t \, dt$$

Here $m = 2$, $n = 8$ both are even

$$\therefore \int_0^{\pi/6} \sin^2 6\theta \cos^6 3\theta \, d\theta = \frac{4}{3} \frac{(2-1)(8-1)(8-3)(8-5)(8-7)}{(10)(10-2)(10-4)(10-6)(10-8)} \cdot \frac{\pi}{2}$$

$$= \frac{\cancel{4}}{\cancel{3}} \times \frac{1 \times 7 \times \cancel{5} \times \cancel{3} \times 1}{{}_2\cancel{10} \times 8 \times 6 \times \cancel{4} \times 2} \times \frac{\pi}{2}$$

$$= \boxed{\frac{7\pi}{384}}$$

Ex:-3 $\int_0^{\pi} \sin^2\theta \, (1+\cos\theta)^4 \, d\theta$ Ans. $\frac{21\pi}{16}$

Ex:-4 $\int_0^{\pi/4} \cos^3 2x \cdot \sin^4 4x \, dx$

Sol:- Let $2x = t \Rightarrow 2\,dx = dt$

When $x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{4} \Rightarrow t = \frac{\pi}{2}$

$$\therefore \int_0^{\pi/2} \cos^3 t \cdot \sin^4 2t \cdot \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{\pi/2} (2\sin t \cdot \cos t)^4 \cos^3 t \; dt$$

$$= \frac{1}{2} \int_0^{\pi/2} 2^4 \cdot \sin^4 t \cdot \cos^7 t \; dt$$

$$= 2^3 \int_0^{\pi/2} \sin^4 t \cdot \cos^7 t \; dt$$

Here $m = 4$ & $n = 7$ we get,

$$= 2^3 \times \frac{\cancel{3} \times 1 \times \cancel{6} \times 4 \times 2}{11 \times \cancel{9} \times 7 \times 5 \times 3 \times 1}$$

$$= \boxed{\frac{128}{1155}}$$

* Gamma function:

Let $n > 0$ be any +ve real number then the gamma function of n is denoted by Γn & defined as

$$\boxed{\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} \, dx}$$

(1) P.T. $\Gamma n = (n-1) \Gamma n-1$

Sol^n:- L.H.S $= \Gamma n$

$$= \int_0^{\infty} x^{n-1} e^{-x} \, dx$$

$$= \left[x^{n-1} (-e^{-x}) \right]_0^{\infty} - \int_0^{\infty} (n-1) x^{n-2} (-e^{-x}) \, dx$$

$$= [0-0] + (n-1)\int_0^\infty x^{n-2} e^{-x}\, dx$$

$$= (n-1)\int_0^\infty x^{(n-1)-1} e^{-x}\, dx$$

$$= \boxed{(n-1)\Gamma_{n-1}}$$

$= R.H.S$

(2) P.T. $\Gamma_n = (n-1)!\ \forall\ n \in \mathbb{N}$ or
State & prove relation betn gamma fun & factorial fun

Soln. W.K.T $\Gamma_n = (n-1)\Gamma_{n-1}$

$\Gamma_{n-1} = (n-2)\Gamma_{n-2}$

$\Gamma_{n-2} = (n-3)\Gamma_{n-3}$

$\vdots$

$\Gamma_2 = 1(\Gamma_1)$

Thus, for $n \in \mathbb{N}$

$\Gamma_n = (n-1)(n-2)\cdots 1\,\Gamma_1$ (II)

$\Gamma_n = (n-1)!\ \Gamma_1$

Now, $\Gamma_n = \int_0^\infty x^{n-1} e^{-x}\, dx$

$\Gamma_1 = \int_0^\infty x^0 e^{-x}\, dx = \int_0^\infty e^{-x}\, dx$

$= [-e^{-x}]_0^\infty$

$= -e^{-\infty} + e^0$

$\boxed{\Gamma_1 = 1}$

By eq^n (1) we get,

$$\boxed{\Gamma n = (n-1)!}$$

* Beta function :-

Let m, n be any two positive real number then Beta function of m & n is denoted by $\beta(m,n)$ & defined as,

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx}$$

(1) P.T. $\beta(m,n) = \beta(n,m)$

Solⁿ:- W.K.T $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Also,

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

$$= \int_0^1 (1-x)^{m-1} (1-1+x)^{n-1} \, dx$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} \, dx$$

$$= \beta(n,m)$$

(2) $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

Sol^n: W.K.T. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$

let $x = \sin^2\theta$ then $dx = 2\sin\theta\cos\theta \, d\theta$

Also, $x = 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore \beta(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\cos^2\theta)^{n-1} 2\sin\theta\cos\theta \, d\theta$

$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta . \cos\theta \, d\theta$

$$\boxed{\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta}$$

Hence, proved

(3) $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta$

W.k.t $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

let $x = \cos^2\theta$ then $dx = -2\cos\theta\sin\theta$

Also, $x = 0 \Rightarrow \theta = \frac{\pi}{2}$

$x = 1 \Rightarrow \theta = 0$

$\therefore \beta(m,n) = \int_{\frac{\pi}{2}}^0 (\cos^2\theta)^{m-1}(1-\cos^2\theta)^{n-1}(-2\sin\theta\cos\theta) \, d\theta$

$= -\int_{\pi/}^0 \cos^{2m-2}\theta \sin^{2n-1}\theta \sin\theta\cos\theta \, d\theta$

Page:
Date: / /

$$\boxed{\beta(m,n) = \int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta}$$

Hence, proved

(9) $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Solⁿ:- w.k.t. $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ —— (1)

Let $2m-1 = p$ & $2n-1 = q$

$\Rightarrow m = \frac{p+1}{2}$ & $n = \frac{q+1}{2}$

From (1)

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Remark:- Relation betⁿ. beta & gamma function

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$

Scanned with CamScanner

Que! P.T. $\Gamma\frac{1}{2} = \sqrt{\pi}$

Proof: W.K.T. $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$ $(m,n>0)$ —— (1)

Also, $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ —— (2)

From (1) & (2) we get,

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Take $m=n=\frac{1}{2}$ we get,

$$\frac{\Gamma\frac{1}{2} \, \Gamma\frac{1}{2}}{\Gamma 1} = 2\int_0^{\frac{\pi}{2}} \sin^{0}\theta \cos^{0}\theta \, d\theta$$

$$(\Gamma\tfrac{1}{2})^2 = 2\int_0^{\frac{\pi}{2}} d\theta = 2\left[\theta\right]_0^{\frac{\pi}{2}} \quad (\because \Gamma 1 = 1)$$

$$\therefore (\Gamma\tfrac{1}{2})^2 = \cancel{2}\times\frac{\pi}{\cancel{2}} = \pi$$

$$\therefore \boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

Hence, proved.

Page :
Date : / /

Ex:- Evaluate the following:

(1) $\int_0^\infty e^{-x^2}\,dx$

Sol^n: let $x^2 = t \Rightarrow 2x\,dx = dt$
$\Rightarrow dx = \frac{1}{2\sqrt{t}}\,dt$

Also, $x=0 \Rightarrow t=0$
$x=\infty \Rightarrow t=\infty$

$\therefore$ given integral $\int_0^\infty e^{-t}\frac{1}{2\sqrt{t}}\,dt$

$$= \frac{1}{2}\int_0^\infty t^{-1/2} e^{-t}\,dt$$

$$= \frac{1}{2}\int_0^\infty t^{1/2-1} e^{-t}\,dt$$

$$= \frac{1}{2}\Gamma\frac{1}{2} \qquad \left(\because \Gamma n = \int_0^\infty x^{n-1}e^{-x}\,dx\right)$$

$$= \boxed{\frac{\sqrt{\pi}}{2}} \qquad \left(\because \Gamma\frac{1}{2} = \sqrt{\pi}\right)$$

(2) $\int_0^{\pi/2} \sin^7\theta \cos^9\theta\,d\theta$

Sol^n: We.k.t $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$\therefore \int_0^{\pi/2} \sin^7\theta\cos^9\theta\,d\theta = \frac{1}{2}\beta\left(\frac{7+1}{2}, \frac{9+1}{2}\right)$$

$$= \frac{1}{2}\beta(4,5)$$

$$= \frac{1}{2}\frac{\Gamma 4\,\Gamma 5}{\Gamma 9} \qquad \left(\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right)$$

$$= \frac{1}{2}\frac{(3!)(4!)}{8!} = \frac{1}{2}\frac{\times \cancel{6}}{8\times 7\times \cancel{6}\times 5} = \boxed{\frac{1}{560}}$$

(3) $\int_0^{\pi/2} \sin^5\theta \cos^2\theta \, d\theta$ (Ans: $\frac{8}{105}$)

(4) $\int_0^{\pi/2} \sin^6\theta \cos^4\theta \, d\theta$ (Ans:- $\frac{3\pi}{512}$)

(5) $\int_0^1 x^5 (1-x^3)^{10} dx$

Sol^n: Let $x^3 = t \Rightarrow 3x^2 dx = dt$

$x = t^{1/3}$

$\Rightarrow dx = \frac{1}{3t^{2/3}} dt = \frac{1}{3} t^{-2/3} dt$

if $x=0 \Rightarrow t=0$

$x=1 \Rightarrow t=1$

$$\therefore \int_0^1 t^{5/3} (1-t)^{10} \cdot \frac{1}{3} t^{-2/3} dt$$

$$= \frac{1}{3}\int_0^1 t (1-t)^{10} dt$$

Here $m-1=1 \Rightarrow m=2$

$n-1=10 \Rightarrow n=11$

$$\therefore = \frac{1}{3}\beta(2,11) = \frac{1}{3}\frac{\Gamma 2\,\Gamma 11}{\Gamma 13} = \frac{1}{3}\frac{1!\,10!}{12!}$$

$$= \frac{1}{3}\times\frac{1}{12\times 11} = \frac{1}{396}$$

(6) $\displaystyle\int_0^2 x^7(16-x^4)^{10}\,dx$

Solⁿ: Let $x^4 = 16t$

$\Rightarrow x = 2t^{1/4} \qquad \Rightarrow dx = \frac{1}{2} t^{-3/4}\,dt$

Also, $x = 0 \Rightarrow t = 0$

$x = 2 \Rightarrow t = 1$

$$\therefore \int_0^1 2^7 t^{7/4} (16-16t)^{10} \cdot \frac{1}{2} t^{-3/4}\,dt$$

$$= \frac{2^7 \times 16^{10}}{2} \int_0^1 t(1-t)^{10}\,dt$$

$$= 2^{46} \int_0^1 t(1-t)^{10}\,dt$$

$$\begin{pmatrix} m-1=1 \Rightarrow m=2 \\ n-1=10 \Rightarrow n=11 \end{pmatrix}$$

$$= 2^{46}\,\beta(2,11)$$

$$= 2^{46}\,\frac{1!\,10!}{12!} = \frac{2^{46}}{12 \times 11} = \boxed{\frac{2^{44}}{33}}$$

* Improper Integrals:

* Improper integral of First Kind:

It is definite integral in which one or both limits of integration are infinite.

e.g. $\int_{0}^{\infty} e^{-x}\,dx$, $\int_{-\infty}^{0} e^{-x}\,dx$, $\int_{-\infty}^{\infty} e^{-x}\,dx$

These integrals are evaluated as follows:

1) $\int_{a}^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_{a}^{b} f(x)\,dx$

2) $\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\,dx$

3) $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{0} f(x)\,dx + \lim_{b \to \infty} \int_{0}^{b} f(x)\,dx$

The improper integral is said to Converge when the limit in R.H.S of 1, 2 & 3 are finite. otherwise it is said to diverge.

Page :
Date : / /

Ex:-(1) Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$

Sol^n: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

$$= \lim_{b\to\infty} \left[\frac{-1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{b} + 1\right]$$

$$= 0 + 1 = 1$$

$\therefore \int_1^\infty \frac{1}{x^2}\,dx = 1$, it is Converge

Ex:-2 Evaluate: $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

Sol^n:- $\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b (x)^{-1/2}\,dx$

$$= \lim_{b\to\infty} \left[\frac{x^{1/2}}{1/2}\right]_1^b$$

$$= \lim_{b\to\infty} \left[2\sqrt{b} - 2\right]$$

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \infty$$

$\therefore$ it is diverge.

Ex:- 3 $\int_{-\infty}^{0} e^{2x}\,dx$

Sol:- $\int_{-\infty}^{0} e^{2x}\,dx = \lim_{a\to-\infty} \int_{a}^{0} e^{2x}\,dx$

$$= \lim_{a\to-\infty} \left[\frac{e^{2x}}{2}\right]_{a}^{0}$$

$$= \lim_{a\to-\infty} \left[\frac{1}{2} - \frac{e^{2a}}{2}\right]$$

$$= \frac{1}{2} - \frac{e^{-\infty}}{2}$$

$$\int_{-\infty}^{0} e^{2x}\,dx = \frac{1}{2}$$

Ex:- 4 $\int_{-\infty}^{\infty} e^{x}\,dx$

Sol:- $\int_{-\infty}^{\infty} e^{x}\,dx = \lim_{a\to-\infty} \int_{a}^{0} e^{x}\,dx + \lim_{b\to\infty} \int_{0}^{b} e^{x}\,dx$

$$= \lim_{a\to-\infty} \left[e^{x}\right]_{a}^{0} + \lim_{b\to\infty} \left[e^{x}\right]_{0}^{b}$$

$$= \lim_{a\to\infty} \left[e^{0} - e^{a}\right] + \lim_{b\to\infty} \left[e^{b} - e^{0}\right]$$

$$= (1-0) + (e^{\infty} - 1)$$

$$= \infty$$

Page :
Date : / /

Ex:-5 Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Sol^n:- $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{1+x^2} dx + \lim_{b \to \infty} \int_{0}^{b} \frac{1}{1+x^2} dx$

$$= \lim_{a \to -\infty} \left[\tan^{-1}x\right]_a^0 + \lim_{b \to \infty} \left[\tan^{-1}x\right]_0^b$$

$$= \lim_{a \to -\infty} \left[\tan^{-1}0 - \tan^{-1}a\right] + \lim_{b \to \infty} \left[\tan^{-1}b - \tan^{-1}0\right]$$

$$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi$$

* Improper integral of second kind :

It is definite integral in which integrand (function) become infinite (not define) at one or more points within or at the end points of the interval of integration.

e.g $\int_0^1 \frac{1}{x} dx$ is an improper integral of second kind because $\frac{1}{x}$ is not defined at $x = 0$.

Scanned with CamScanner

These integrals are evaluated as follows :

(1) $f(x)$ is not defined at $x = a$

$$\int_a^b f(x)\,dx = \lim_{c \to a} \int_c^b f(x)\,dx$$

(2) $f(x)$ is not defined at $x = b$

$$\therefore \int_a^b f(x)\,dx = \lim_{c \to b} \int_a^c f(x)\,dx$$

(3) $f(x)$ is not defined at $x = a$ & $x = b$

$$\therefore \int_a^b f(x)\,dx = \lim_{c_1 \to a} \int_{c_1}^{0} f(x)\,dx + \lim_{c_2 \to b} \int_0^b f(x)\,dx$$

The improper integral is said to converge when the limit in RHS of 1, 2 & 3 exist. otherwise it is said to diverge.

Ex:- Evaluate $\int_0^1 \frac{1}{x^2}\,dx$

Sol:- $\cancel{\int_0^1 \frac{1}{x^2}\,dx} =$

The integrand $\frac{1}{x^2}$ is not define at $x = 0$

$$\therefore \int_0^1 \frac{1}{x^2}\,dx = \lim_{c \to 0} \int_c^1 \frac{1}{x^2}\,dx$$

$$= \lim_{c \to 0} \left[-\frac{1}{x}\right]_c^1$$

$$= \lim_{c \to 0} \left[-1 + \frac{1}{c} \right]$$

$$\int_0^1 \frac{1}{x^2}\, dx = \infty$$

$\therefore$ It is diverge.

Ex:-2 Evaluate $\int_0^3 \frac{1}{\sqrt{3-x}}\, dx$

Sol^n! The integrand $\frac{1}{\sqrt{3-x}}$ is not define at $x=3$

$$\therefore \int_0^3 \frac{1}{\sqrt{3-x}}\, dx = \lim_{c \to 3} \int_0^c \frac{1}{\sqrt{3-x}}\, dx$$

$$= \lim_{c \to 3} \left[-2\sqrt{3-x} \right]_0^c$$

$$= \lim_{c \to 3} \left[-2\sqrt{3-c} + 2\sqrt{3} \right]$$

$$= 2\sqrt{3}$$

Ex:-3 Check the Convergence of $\int_0^1 \frac{dx}{1-x}$. If Convergent the evaluate the limit.

Sol^n:- The integrand $\frac{1}{1-x}$ is not define at $x=1$

$$\therefore \int_0^1 \frac{1}{1-x}\,dx = \lim_{c\to 1} \int_0^c \frac{1}{1-x}\,dx$$

$$= \lim_{c\to 1} \left[-\log(1-x)\right]_0^c$$

$$= \lim_{c\to 1} \left[-\log(1-c) + \log(1)\right]$$

$$= -\log(0)$$

$$= \infty$$

$\therefore$ it is divergent.

<u>Ex:- 4</u> Evaluate $\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

<u>Sol^n:-</u> The integrand $\frac{1}{\sqrt{9-x^2}}$ is not defined at $x=3$

$$\therefore \int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx = \lim_{c\to 3} \int_0^c \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \lim_{c\to 3} \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^c$$

$$= \lim_{c\to 3} \left[\sin^{-1}\left(\frac{c}{3}\right) - \sin^{-1}(0)\right]$$

$$= \sin^{-1}(1) - 0$$

$$= \pi/2$$ $\therefore$ it is Convergent.