<u>Successive Differentiation</u> :

Introduction: Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives.

❖ Common notations of higher order Derivatives of $y = f(x)$

1st Derivative: $f'(x)$ **or** y' **or** y_1 **or** $\frac{dy}{dx}$ **or** Dy

2nd Derivative: $f''(x)$ **or** y'' **or** y_2 **or** $\frac{d^2y}{dx^2}$ **or** D^2y

3rd Derivative: $f'''(x)$ **or** y''' **or** y_3 **or** $\frac{d^3y}{dx^3}$ **or** D^3y

$\vdots$

n^{th} **Derivatives:** $f^{(n)}(x)$ **or** $y^{(n)}$ **or** y_n **or** $\frac{d^ny}{dx^n}$ **or** D^ny.

❖ n^{th} derivatives of some standard Functions :

1) n^{th} **Derivative of** $y = e^{ax}$.

$\Rightarrow$ Let $y = e^{ax}$ then

$$y_1 = a\,e^{ax}$$
$$y_2 = a^2 e^{ax}$$
$$y_3 = a^3 e^{ax}$$
$$\vdots$$
$$y_n = a^n\,e^{ax}\,.$$

2) n^{th} **Derivative of** $y = a^{bx}$.

$\Rightarrow$ Let $y = a^{bx}$ then

$$y_1 = b\,a^{bx}\,\log a$$
$$y_2 = b^2\,a^{bx}\,(\log a)^2$$
$$y_3 = b^3 a^{bx}\,(\log a)^3$$
$$\vdots$$
$$y_n = b^n\,a^{bx}\,(\log a)^n\,.$$

3) n^{th} **Derivative of** $y = (ax + b)^m$ **,** m **is a positive integer greater than** n.

$\Rightarrow$ Let $y = (ax + b)^m$ then

$$y_1 = ma\,(ax + b)^{m-1}$$
$$y_2 = m(m-1)a^2\,(ax + b)^{m-2}$$
$$y_3 = m(m-1)(m-2)a^3\,(ax + b)^{m-3}$$
$$\vdots$$
$$y_n = m(m-1)(m-2)\ldots\ldots(m-(n-1))a^n\,(ax + b)^{m-n}$$
$$= m(m-1)(m-2)\ldots\ldots(m-n+1)\,a^n\,(ax + b)^{m-n}$$

$$= \frac{m!}{(m-n)!}\ a^n\ (ax+b)^{m-n}.$$

Case (i): If $\boldsymbol{m}$ is a Positive integer and $\boldsymbol{m = n}$, then $y_n = n!\ a^n$.

Case (ii): If $\boldsymbol{m}$ is a Positive integer and $\boldsymbol{m < n}$, then $y_n = 0$.

Case(iii): If $\boldsymbol{m = -1}$ **,** i.e.$y = \frac{1}{ax+b}$ then $\mathrm{y_n} = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$.

4) $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{y = \log(ax+b)}$.

$\Longrightarrow$ Let $y = \log(ax+b)$ then

$$y_1 = \frac{a}{ax+b}$$

$$y_2 = -\frac{a^2}{(ax+b)^2}$$

$$y_3 = \frac{2\ a^3}{(ax+b)^3} = \frac{2!\ a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1}\frac{(n-1)!\, a^n}{(ax+b)^n}\ .$$

5) $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{y = \sin(ax+b)}$

$\Rightarrow$ Let $y = \sin(ax+b)$ then

$$y_1 = a\ \cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\ \cos\left(ax+b+\frac{\pi}{2}\right) = a^2\ \sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = a^3\ \cos\left(ax+b+\frac{2\pi}{2}\right) = a^3\ \sin\left(ax+b+\frac{3\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right).$$

Similarly if $\boldsymbol{y = \cos(ax+b)}$ then

$$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right).$$

6) $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{y = e^{ax}\sin(bx+c)}$.

$\Rightarrow$ Let $y = e^{ax}\sin(bx+c)$ then

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$

$$= e^{ax}(a\sin(bx+c) + b\ \cos(bx+c))$$

Putting $a = \mathrm{r}\cos\alpha$, $b = r\ \sin\alpha$ we get,

$$y_1 = e^{ax}(\mathrm{r}\cos\alpha\ \sin(bx+c) + r\ \sin\alpha\cos(bx+c))$$

$$= r\, e^{ax}(\cos\alpha\ \sin(bx+c) + \sin\alpha\cos(bx+c))$$

$$= re^{ax}\ \sin(bx+c+\alpha)$$

Similarly $y_2 = r^2 e^{ax}\sin(bx+c+2\alpha)$

$$y_3 = r^3 e^{ax}\sin(bx+c+3\alpha)$$

$$\vdots$$

$$y_n = r^n e^{ax}\sin(bx+c+n\alpha).$$

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right).$$

Where $r = \sqrt{a^2+b^2}$ and $\tan\alpha = \frac{b}{a}$.

Similarly n^{th} Derivative of $y = e^{ax}\cos(bx+c)$ is,

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\cos\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right).$$

❖ Summary :

Function	n^{th} Derivative
$y = e^{ax}$	$y_n = a^n\, e^{ax}$
$y = a^{bx}$	$y_n = b^n\, a^{bx}\,(\log a)^n$
$y = (ax+b)^m$	$y_n = \begin{cases} \frac{m!}{(m-n)!}\ a^n\,(ax+b)^{m-n}, & m>0, m>n \\ 0, & m>0,\quad m<n \\ n!\ a^n, & m>0,\quad m=n \\ \frac{(-1)^n n!\ a^n}{(ax+b)^{n+1}}, & m=-1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1}\frac{(n-1)!\ a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$
$y = \cos(ax+b)$	$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$
$y = e^{ax}\, sin(bx+c)$	$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right)$
$y = e^{ax}\, cos(bx+c)$	$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\cos\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right)$

Example-1: Find the n^{th} derivative of the function $y = \frac{1}{1-5x+6x^2}$.

Solution : Here $y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$

$$\therefore \quad \frac{1}{(2x-1)(3x-1)} = \frac{A}{2x-1} + \frac{B}{3x-1} \quad \text{................}(1)$$

$$\therefore \; 1 = A(3x-1) + B(2x-1)$$

If $x = {}^{1}\!/_{2}$ then $A = 2$

If $x = {}^{1}\!/_{3}$ then $B = -3.$

∴ From equation (1) , we get

$y = \frac{2}{2x-1} - \frac{3}{3x-1}$ So the n^{th} derivative of the given function is,

$$y_n = \frac{2\,(-1)^n\, n!\, 2^n}{(2x-1)^{n+1}} - \frac{3\,(-1)^n\, n!\, 3^n}{(3x-1)^{n+1}}$$

$$= (-1)^n\, n! \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right].$$

Example-2 : Find the n^{th} derivative of the function $y = \frac{2x-1}{(x^2-5x+6)}$.

Solution : Here $y = \frac{2x-1}{(x^2-5x+6)} = \frac{2x-1}{(x-2)(x-3)}$.

Let $y = \frac{2x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$.

$$\therefore \; 2x - 1 = A(x-3) + B\,(x-2).$$

If $x = 2$, then $A = -3.$

If $x = 3,$ then $B = 5.$

$$\therefore \; y = \frac{5}{x-3} - \frac{3}{x-2}$$

$$\therefore y_n = \frac{5\,(-1)^n\, n!}{(x-3)^{n+1}} - \frac{3\,(-1)^n\, n!}{(x-2)^{n+1}}.$$

Example-3 : Find the n^{th} derivative of the function $y = \frac{x^4}{x^2-3x+2}$.

Solution : Here $y = \frac{x^4}{x^2-3x+2} = \frac{x^4}{(x-2)(x-1)}$

$$y = \frac{x^4}{(x-2)(x-1)} = x^2 + 3x + 7 + \frac{15x-14}{(x-2)(x-1)}$$

$$= x^2 + 3x + 7 + \frac{16}{x-2} - \frac{1}{x-1}$$

$$\therefore y_n = 0 + \frac{16(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}}.$$

Example-4 : Find the n^{th} derivative of the function $y = \sin 6x\; \cos 4x$.

Solution : Here $y = \sin 6x\; \cos 4x = \frac{1}{2}\,(\sin 10x + \sin 2x)\;\; (\because s + s = 2\,s\,c)$

$$\therefore \; y_n = \frac{1}{2}\left(10^n \sin\left(10x + \frac{n\pi}{2}\right) + 2^n\, \sin\left(2x + \frac{n\pi}{2}\right)\right).$$

Example-5 : Find the n^{th} derivative of the function $y = sin^4 x$.

Solution : Here $y = sin^4 x = (sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2$

$$= \frac{1}{4}\,(1 - 2\cos 2x + cos^2 2x) = \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{8}\,(3 - 4\cos 2x + \cos 4x)$$

$$\therefore y_n = \frac{1}{8}\left(0 - 4\,2^n\,\cos\left(2x + \frac{n\pi}{2}\right) + 4^n\,\cos\left(4x + \frac{n\pi}{2}\right)\right).$$

Example-6 : Find the n^{th} derivative of the function $y = e^{2x}\,\cos 2x\,\cos x$.

Solution : Here $y = e^{2x}\,\cos 2x\,\cos x = \frac{1}{2}\,e^{2x}(\,2\cos 2x\,\cos x)$

$$= \frac{1}{2}\,e^{2x}(\cos 3x + \cos x)\;\;(\because c + c = 2\,c\,c)$$

$$= \frac{1}{2}\,(e^{2x}\,\cos 3x + e^{2x}\,\cos x)$$

$$= \frac{1}{2}\left[(13)^{\frac{n}{2}}\,e^{2x}\,\cos\left(3x + n\tan^{-1}\frac{3}{2}\right) + (5)^{\frac{n}{2}}\,e^{2x}\,\cos\left(x + n\tan^{-1}\frac{1}{2}\right)\right].$$

Problems :

1. If $x = \sin t$, $y = \sin pt$, prove that $(1 - x^2)y_2 - xy_1 + p^2y = 0$.
2. Find the n^{th} derivatives of the following functions **:**

(i) $y = cos\,x\,cos\,2x\,cos\,3x$
(ii) $y = e^{2x}\,cos^2 x\,\sin x$
(iii) $y = \frac{x}{(x-1)\,(2x+3)}$
(iv) $y = e^{-x}sin^2 x$
(v) $y = \frac{x^2-4x+1}{x^3+2x^2-x-2}$
(vi) $y = sin^2 x\,cos^3\,x$
(vii) $y = e^{-x}\,sin^2\,x$
(viii) $y = \log(ax + b)(cx + d)$
(ix) $y = cos^6 x.$

LEIBNITZ'S THEOREM :

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u\,v\,)_n = u_n\,v + n_{C_1}\,u_{n-1}\,v_1 + n_{C_2}\,u_{n-2}\,v_2 + \cdots\cdots + n_{C_r}\,u_{n-r}\,v_r + \cdots + u\,v_n.$$

Where u_r and v_r represents r^{th} derivatives of u and v respectively.

Example-1 Find the n^{th} derivative of $x\log x$.

Solution : Let $u = \log x$ and $v = x$.

Then $u_n = (-1)^{n-1} \frac{(n-1)!}{x^n}$, $u_{n-1} = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$ and $v_1 = 1,\ v_2 = 0$.

$\therefore$ Using Leibnitz's theorem , we have

$$(u\,v)_n = u_n\, v + n_{C_1}\, u_{n-1}\, v_1 + n_{C_2}\, u_{n-2}\, v_2 + \cdots\cdots + n_{C_r}\, u_{n-r}\, v_r + \cdots + u\, v_n$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1} \frac{(n-1)!}{x^n}\, x + n\,(-1)^{n-2} \frac{(n-2)!}{x^{n-1}}\ 1 + 0$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1} \frac{(n-1)(n-2)!}{x^{n-1}} + n\ (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

$$= -(-1)^{n-2} \frac{(n-1)(n-2)!}{x^{n-1}} + n\ (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} [-(n-1) + n]$$

$$= = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}\ 1.$$

Example-2 Find the n^{th} derivative of $x^2 e^{3x}\ \sin 4x$.

Solution : Let $u = e^{3x} \sin 4x$ and $v = x^2$.

Then $u_n = (25)^{\frac{n}{2}}\, e^{3x}\ \sin\left(4x + n\ \tan^{-1}\frac{4}{3}\right) = 5^n e^{3x}\ \sin\left(4x + n\ \tan^{-1}\frac{4}{3}\right)$,

$u_{n-1} = 5^{n-1} e^{3x}\ \sin\left(4x + (n-1)\ \tan^{-1}\frac{4}{3}\right)$ and

$v_1 = 2x, v_2 = 2, v_3 = 0$.

$\therefore$ Using Leibnitz's theorem , we have

$$(u\,v)_n = u_n\, v + n_{C_1}\, u_{n-1}\, v_1 + n_{C_2}\, u_{n-2}\, v_2 + \cdots\cdots + n_{C_r}\, u_{n-r}\, v_r + \cdots + u\, v_n$$

$$\Rightarrow (x^2 e^{3x}\ \sin 4x)_n = 5^n e^{3x}\ \sin\left(4x + n\ \tan^{-1}\frac{4}{3}\right) x^2$$

$$+ n\, 5^{n-1} e^{3x} \sin\left(4x + (n-1)\ \tan^{-1}\frac{4}{3}\right)\ (2x)$$

$$+ \frac{n(n-1)}{2} 5^{n-2} e^{3x}\ \sin\left(4x + (n-2)\ \tan^{-1}\frac{4}{3}\right) 2 + 0 .$$

$$= e^{3x}\, 5^n [x^2 \sin\left(4x + n\ \tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5} \sin\left(4x + (n-1)\ \tan^{-1}\frac{4}{3}\right)$$

$$+ \frac{n(n-1)}{25} \sin\left(4x + (n-2)\ \tan^{-1}\frac{4}{3}\right)] .$$

Example-3 Find the n^{th} derivative of $e^x\ (2x+3)^3$.

Solution : Let $u = e^x$ and $v = (2x+3)^3$.

Then $u_n = e^x$, for all integer values of n, and

$v_1 = 6(2x+3)^2$, $v_2 = 24(2x+3), v_3 = 48, v_4 = 0$.

$\therefore$ Using Leibnitz's theorem , we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots\cdots + n_{C_r} u_{n-r} v_r + \cdots + u\,v_n$$

$$\Rightarrow (e^x (2x+3)^3)_n = e^x(2x+3)^3 + n\,e^x 6(2x+3)^2 + \frac{n(n-1)}{2} e^x 24(2x+3)$$

$$+\frac{n(n-1)(n-2)}{6} e^x 48 + 0$$

$$= e^x\{(2x+3)^3 + 6n\,(2x+3)^2 + 12n\,(n-1)(2x+3) + 8n(n-1)(n-2)\}.$$

Example-4 If $y = (\sin^{-1} x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$

Solution : Here If $y = (\sin^{-1} x)^2$ then differentiating with respect to x we get,

$$y_1 = \frac{2(\sin^{-1} x)}{\sqrt{1-x^2}} \text{ or } (1-x^2){y_1}^2 = 4(\sin^{-1} x)^2 = 4y$$

Again differentiating , we get

$$(1-x^2)2\,y_1\,y_2 - 2x{y_1}^2 = 4\,y_1 \text{ or } (1-x^2)\,y_2 - xy_1 - 2 = 0$$

Differentiating it n times by Leibnitz's theorem,

$$(1-x^2)y_{n+2} + n\,(-2x)\,y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - [xy_{n+1} + n\,y_n] = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nx\,y_{n+1} - n(n-1)y_n - x\,y_{n+1} - n\,y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2y_n = 0.$$

Which is the required result.

Problems :

1. If $y^{1/m} + y^{-1/m} = 2\,x$, prove that
   $(x^2-1)y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)y_n = 0.$
2. If $y = e^{m\cos^{-1} x}$, prove that (i) $(1-x^2)y_2 - xy_1 = m^2y$
   (ii) $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - (n^2+m^2) = 0.$
3. If $y = \tan^{-1} x$, prove that $(1+x^2)y_{n+1} + 2nxy_n + n\,(n-1)y_{n-1} = 0.$
4. Find the nth derivative of the following functions **:**
   (i) $x^2 \log 3x$
   (ii) $x^2 \cos x$
   (iii) $x^2 e^x$

<u>Indeterminate Forms and L'Hôpital's (L'Hospital's) Rule:</u>

<u>Indeterminate Form 0/0:</u>

If the continuous functions $f(x)$ and $g\,(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces **0/0**, a meaningless expression, which we cannot evaluate. We use **0/0** as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such $\frac{\infty}{\infty}, \infty \cdot \mathbf{0}$, $\infty - \infty$, $\mathbf{0^0}, \infty^{\mathbf{0}}, \mathbf{1}^{\infty}$.

<u>L'Hospital's rule:</u> If f and g are differentiable functions on an open interval I containing a and suppose that $f(a) = g(a) = 0$, $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} ,$$

assuming that the limit on the right side of this equation exists.

Example-1: Find $\lim_{x \to 0} \frac{3x - \sin x}{x}$.

Solution : Here $\lim_{x \to 0} \frac{3x - \sin x}{x}$ $\left(0/_0 - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{3 - \cos x}{1} = \frac{3-1}{1} = 2 \ .$$

Example-2: Find $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

Solution : Here $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ $\left(0/_0 - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \left(0/_0 - form\right)$$

Again using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{6x} = \frac{1}{6}\left(\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right).$$

Example-3: Find $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$.

Solution : Here $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$ $\left(0/_0 - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2} .$$

Example-4: Find $\lim_{x \to 1} \frac{x - x^x}{1 + \log x - x}$.

Solution : Here $\lim_{x \to 1} \frac{x - x^x}{1 + \log x - x}$ $(0/0 - form)$

So using L'Hospital rule, we get

$$= \lim_{x \to 1} \frac{1 - x^x(1 + \log x)}{\frac{1}{x} - 1} \quad (0/0 - form)$$

Again using L'Hospital rule, we get

$$= \lim_{x \to 1} \frac{-x^x(1+\log x)^2 - x^{x-1}}{\left(-\frac{1}{x^2}\right)} = 2.$$

Problems :

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$ (Ans: 1/3)
2. $\lim_{x \to \pi/2} \frac{\log(\sin x)}{(\pi - 2x)^2}$ (Ans: -1/8)
3. $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$ (Ans: -e/2)
4. $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$ (Ans: 1/2)
5. $\lim_{x \to y} \frac{x^y - y^x}{x^x - y^y}$ (Ans: $\frac{1-\log y}{1+\log y}$)
6. $\lim_{x \to 1} \frac{x\log x - (x-1)}{(x-1)\ \log x}$ (Ans: $1/2$)
7. $\lim_{x \to 1/2} \frac{cos^2\, \pi x}{e^{2x} - 2xe}$ (Ans: $\frac{\pi^2}{2e}$)
8. $\lim_{x \to \pi/2} \frac{2x - \pi}{\cos x}$ (Ans: -2)

9. If $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$ is finite, then find the value of a and hence the value of limit.

   (Ans: $= -2$, limit = -1)

10. Find the value of a, b and c such that $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2.$

    (Ans: $= 1, b = 2, c = 1$)

<u>Indeterminate Form ∞/∞</u> :

If f and g are differentiable functions on an open interval I containing a and suppose that $f(a) = g(a) = \infty$, $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} ,$$

assuming that the limit on the right side of this equation exists.

Example-1: Find $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1+\tan x}$.

Solution : Here $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1+\tan x}$ $\left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{\sec x \, \tan x}{sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \sin x = 1.$$

Example-2: Find $\lim_{x \to \infty} \frac{e^x}{x^2}$.

Solution : Here $\lim_{x \to \infty} \frac{e^x}{x^2} \quad \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{e^x}{2x} \left(\frac{\infty}{\infty} - form\right)$$

Again using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{e^x}{2} = \infty.$$

Example-3: Find $\lim_{x \to \infty} \frac{\log x}{2\sqrt{x}}$.

Solution : Here $\lim_{x \to \infty} \frac{\log x}{2\sqrt{x}} \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{1/x}{2/2\sqrt{x}} = \lim_{x \to \infty} \sqrt{x}/x = \lim_{x \to \infty} 1/\sqrt{x} = 0.$$

Example-4: Find $\lim_{x \to \frac{\pi}{2}} \frac{\log(x - \pi/2)}{\tan x}$.

Solution : Here $\lim_{x \to \frac{\pi}{2}} \frac{\log(x - \pi/2)}{\tan x} \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{1}{sec^2 x \, (x - \pi/2)} = \lim_{x \to \frac{\pi}{2}} \frac{cos^2 x}{(x - \pi/2)} \left(\frac{0}{0} - form\right)$$

Again using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{-2 \sin x \, \cos x}{1} = 0.$$

Problems :

Evaluate the following limits:

1. $\lim_{x \to a} \frac{\log(x-a)}{\log(e^x - e^a)}$ (Ans:1)
2. $\lim_{x \to 0} \frac{\log x^2}{\cot x^2}$ (Ans: 0)
3. $\lim_{x \to 0} \frac{\log(\sin 2x)}{\log(\sin x)}$ (Ans: 1)
4. $\lim_{x \to \infty} \frac{x^3 + 3x^2}{7x^3 - 4x}$ (Ans: 1/ 7)
5. $\lim_{x \to 0} \log_{\tan x} \tan 2x$ (Ans: 1)

6. Prove that $\lim\limits_{x \to 1} \frac{\log(1-x)}{\cot \pi x} = 0$.

Indeterminate Form $(0 \cdot \infty)$ or $(\infty - \infty)$:

If f and g are differentiable functions on an open interval I containing a and suppose that $\lim\limits_{x \to a} f(x) = 0$, $\lim\limits_{x \to a} g(x) = \infty$ then $\lim\limits_{x \to a} f(x)g(x)$ is in $0 \cdot \infty$ form. We write given function as $\lim\limits_{x \to a} \frac{f(x)}{1/g(x)}$ or $\lim\limits_{x \to a} \frac{g(x)}{1/f(x)}$ so it is in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form respectively, which can be solved using L'Hospital's rule.

To evaluate the limits of the type $\lim\limits_{x \to a} [f(x) - g(x)]$, when $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = \infty$, we reduce the expression in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking LCM or by rearranging the terms and then apply L'Hospital's rule.

Example-1: Find $\lim\limits_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution : Here $\lim\limits_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ $(\infty - \infty\ form)$

$$\therefore \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \left(\frac{0}{0} - form\right)$$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \left(\frac{0}{0} - form\right)$$

Again using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = 0.$$

Example-2: Find $\lim\limits_{x \to \infty} \left(a^{1/x} - 1\right)x$.

Solution : Here $\lim\limits_{x \to \infty} \left(a^{1/x} - 1\right)x$ $(0 \cdot \infty - form)$

$$\therefore \lim_{x \to \infty} \frac{\left(a^{1/x} - 1\right)}{\frac{1}{x}} \left(\frac{0}{0} - form\right)$$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{a^{1/x}\left(-\frac{1}{x^2}\right) \log a}{\left(-\frac{1}{x^2}\right)} = \log a\ .$$

Example-3: Find $\lim\limits_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right]$.

Solution : Here $\lim\limits_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right]$ $(\infty - \infty\ form)$

$$\therefore \lim_{x \to 2} \left[\frac{\log(x-1) - (x-2)}{(x-2)\log(x-1)}\right] \left(\frac{0}{0} - form\right)$$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 2} \left[\frac{\frac{1}{x-1}-1}{log(x-1)+\frac{(x-2)}{x-1}}\right] = \lim_{x \to 2} \frac{1-(x-1)}{(x-2)+(x-1)\,log(x-1)} \left(\frac{0}{0} - form\right)$$

Again using L'Hospital rule, we get

$$= \lim_{x \to 2} \frac{-1}{1+\frac{x-1}{x-1}+log(x-1)} = \frac{1}{2}.$$

Example-4: Find $\lim_{x \to 1} (x^2 - 1)\ tan\left(\frac{\pi x}{2}\right)$.

Solution : Here $\lim_{x \to 1} (x^2 - 1)\ tan\left(\frac{\pi x}{2}\right)\ (0 \cdot \infty - form)$

$$\therefore \lim_{x \to 1} \frac{(x^2-1)}{\cot\left(\frac{\pi x}{2}\right)}\ (0/0 - form)$$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 1} \frac{2x}{-cosec^2\left(\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)} = \frac{2}{-\left(\frac{\pi}{2}\right)} = -\frac{4}{\pi}\ .$$

Problems :

Evaluate the following limits:

1. $\lim_{x \to 1} (1-x)\, tan\left(\frac{\pi x}{2}\right)$ (Ans: $2/\pi$)
2. $\lim_{x \to 0} \frac{1}{x}(1 - x\cot x)$ (Ans: 0)
3. $\lim_{x \to \infty} \left(x + 1/2\right)\ log\left(\frac{2x+1}{2x}\right)$ (Ans: $1/2$)
4. $\lim_{x \to a} log\left(2 - \frac{x}{a}\right) cot(x-a)$ (Ans : $-\left(\frac{1}{a}\right)$)
5. $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x}\right)$ (Ans: $1/2$)
6. Prove that $\lim_{x \to \infty} x^2\, e^{-x} = 0.$
7. Prove that $\lim_{x \to 1} (1 + \sec \pi x) \tan \pi x = 0$
8. Prove that $\lim_{x \to \frac{\pi}{2}} \left(\tan x - \frac{2x \sec x}{\pi}\right) = \frac{2}{\pi}.$
9. If $\lim_{x \to 0} \left(\frac{a \cot x}{x} + \frac{b}{x^2}\right) = \frac{1}{3}$ then find a and b .

<u>Indeterminate Forms $1^\infty, \infty^0, 0^0$:</u>

To evaluate the limits of the type $\lim_{x \to a} [f(x)]^{g(x)}$, which takes any one of the indeterminate forms 1^∞ , ∞^0 , 0^0 for $f(x) > 0$, we proceed as follows:

Let $\quad l = \lim\limits_{x \to a} [f(x)]^{g(x)} \quad , f(x) > 0$

Applying log function on both the sides we get, $L = \log l = \lim\limits_{x \to a} [g(x) \log f(x)]$ which takes the form $0 \cdot \infty$ which can be solved using L'Hospital rule.

So $\lim\limits_{x \to a} [f(x)]^{g(x)} = \lim\limits_{x \to a} e^{g(x) \log f(x)} = e^L$.

Example-1: Find $\lim\limits_{x \to o^+} (1+x)^{1/x}$.

Solution : Here $\lim\limits_{x \to o^+} (1+x)^{1/x} \quad (1^\infty - form)$

$$\therefore \quad L = \lim_{x \to o^+} \frac{1}{x} \log(1+x) \quad (0 \cdot \infty - form)$$

$$= \lim_{x \to o^+} \frac{\log(1+x)}{x} \quad (0/0 - form)$$

$\therefore$ Using L'Hospital rule, we get

$$= \lim_{x \to o^+} \frac{1}{1+x} = 1$$

Therefore, $\lim\limits_{x \to o^+} (1+x)^{1/x} = \lim\limits_{x \to o^+} e^{\frac{1}{x}\log(1+x)} = e^1 = e.$

Example-2: Find $\lim\limits_{x \to \infty} x^{1/x}$.

Solution : Here $\lim\limits_{x \to \infty} x^{1/x} \quad (\infty^0 - form)$

$$\therefore \quad L = \lim_{x \to \infty} \frac{1}{x} \log x \quad (0 \cdot \infty - form)$$

$$= \lim_{x \to \infty} \frac{\log x}{x} \quad (\infty/\infty - form)$$

$\therefore$ Using L'Hospital rule, we get

$$= \lim_{x \to \infty} \frac{1}{x} = 0$$

Therefore, $\lim\limits_{x \to \infty} x^{1/x} = \lim\limits_{x \to \infty} e^{\frac{1}{x}\log x} = e^0 = 1.$

Example-3: Find $\lim\limits_{x \to o^+} x^x$.

Solution : Here $\lim\limits_{x \to o^+} x^x \; (0^0 - form)$

$$\therefore \quad L = \lim_{x \to o^+} x \log x \quad (0 \cdot \infty - form)$$

$$= \lim_{x \to o^+} \frac{\log x}{\frac{1}{x}} \quad (\infty/\infty - form)$$

$\therefore$ Using L'Hospital rule, we get

$$= \lim_{x \to o^+} \frac{1/x}{\left(-\frac{1}{x^2}\right)} = \lim_{x \to o^+} -x = 0$$

Therefore, $\lim_{x \to o^+} x^x = \lim_{x \to o^+} e^{x \log x} = e^0 = 1.$

Problems :

Evaluate the following limits:

1. $\lim_{x \to 0} (a^x + x)^{1/x}$ (Ans: ae)
2. $\lim_{x \to 0} (e^{3x} - 5x)^{1/x}$ (Ans: e^{-2})
3. $\lim_{x \to 0} \left(\frac{e^x + e^{2x} + e^{3x}}{3}\right)^{1/x}$ (Ans: e^2)
4. $\lim_{x \to \infty} \left(\frac{ax+1}{ax-1}\right)^x$ (Ans: $e^{2/a}$)
5. $\lim_{x \to 1} (1 - x^2)^{1/\log(1-x)}$ (Ans: e)
6. $\lim_{x \to 0} (\cos x)^{\cot x}$ (Ans: 1)
7. $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$ (Ans: $e^{1/3}$)
8. $\lim_{x \to 0} (\cot x)^{\sin x}$ (Ans: 1)
9. $\lim_{x \to 0} \left(\frac{1}{x}\right)^{1-\cos x}$ (Ans: 1)
10. $\lim_{x \to \frac{\pi}{2}} (\cos x)^{\frac{\pi}{2}-x}$ (Ans: 1)
11. $\lim_{x \to 0} (\sin x)^x$ (Ans: 1)
12. $\lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$ (Ans: $\left((abc)^{1/3}\right)$
13. $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$ (Ans:1)