PARTIAL DERIVATIVES

We say that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \mathrm{L}.$$

- **Continuity of a Function of Two Variables**

  A function $f(x,y)$ is said to be *continuous* at the point $(x_0,\ y_0)$ if
  1. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists;
  2. $f(x_0,\ y_0)$ is defined;
  3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,\ y_0)$

Example-1. Find the limit

$$\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}.$$

Solution. Observe that the point (1, 2) does not cause division by zero or other domain issues. So,

$$\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2} = \frac{5(1)^2(2)}{(1)^2+(2)^2} = 2.$$

Example-2. Show that

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not continuous at the origin.

Solution. Let us apply different path approach. We check the limit along different paths $y = mx, x \neq 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{2x(mx)}{x^2+(mx)^2} = \frac{2m}{1+m^2}.$$

This limit changes with m. Therefore $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence the function $f(x,y)$ is not continuous at the origin.

Exercise-1. Find the limits:

(a) $lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y-y^3+5xy}$; Ans: -3

(b) $lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$; Ans: 0

(c) $lim_{(x,y)\to(\pi/2,0)} \frac{cos\, y+1}{y-sin\, x}$; Ans: -2

(d) $lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$; Ans: 1

(e) $lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$; Ans: 0

Exercise-2. By considering different paths of approach, show that the function in below examples have no limit as $(x, y) \to (0,0)$.

(a) $f(x,y) = \frac{xy}{|xy|}$; Ans: Different limits along $y = mx$

(b) $g(x,y) = \frac{x-y}{x+y}$; Ans: Different limits along $y = mx$

(c) $f(x,y) = \frac{x^4-y^2}{x^4+y^2}$; Ans: Different limits along $y = mx^2$

(d) $h(x,y) = \frac{2xy}{3x^2+y^2}$; Ans: Different limits along $y = mx$

Exercise-3. Check whether the following functions are continuous or not at $(0,0)$.

(a) $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2}; if\ (x,y) \neq (0,0) \\ 1 \qquad ; if\ (x,y) \neq (0,0) \end{cases}$ Ans: continuous

(b) $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}; if\ (x,y) \neq (0,0) \\ 0 \qquad ; if\ (x,y) \neq (0,0) \end{cases}$; Ans: Not continuous

(c) $f(x,y) = \begin{cases} \frac{2x^2y}{x^3+y^3}, if(x,y) \neq (0,0) \\ 0 \qquad , if(x,y) = (0,0) \end{cases}$; Ans: Not continuous

- **Partial Derivative**

  When the function involves two or more independent variables, like $u = f(x,y)$ or $u = f(x,y,z)$, then the derivative of u with respect to any one of the independent variables, treating all other variables as constant is referred as *partial derivative* of u with respect to that variable.

- **Mathematical Form**

  The partial derivative of $u = f(x,y)$ w. r. t. x at a point (x_0, y_0) is denoted by $\frac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$ or $\frac{\partial u}{\partial x}(x_0, y_0)$ or $u_x(x_0, y_0)$ and is defined as

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

  provided the limit exists.

  The partial derivative of $u = f(x,y)$ w. r. t. y at a point (x_0, y_0) is denoted by

$\frac{\partial f}{\partial y}(x_0, y_0)$ or $f_y(x_0, y_0)$ or $\frac{\partial u}{\partial y}(x_0, y_0)$ or $u_y(x_0, y_0)$ and is defined as

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0,y_0+h) - f(x_0, y_0)}{h}$$

provided the limit exists.

- **Higher order Partial Derivatives**

  For a function $f(x,y)$ the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are themselves are functions of x and y, so we can take partial derivatives of them as

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\partial x}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \qquad f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\partial y}$$

  Higher order partial derivatives (e.g. f_{xxy}) can also be calculated. Using the subscript notation, the order of differentiation is from left to right.

Example-1. Let $f(x,y) = 3x^2 + e^{-xy^2}$. Find f_x, f_y.

Solution. $f_x(x,y) = 6x - y^2e^{-xy^2}$ and $f_y(x,y) = -2xye^{-xy^2}$.

Example-2. Let $f(x,y) = ycos(xy)$. Find f_x, f_y.

Solution. $f_x(x,y) = -y^2\sin(xy)$ and $f_y(x,y) = cos(xy) - xysin(xy)$.

Example-3. Let $f(x,y) = x^2 - 4xy^3$. Find $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

Solution.

$$f_x(x,y) = 2x - 4y^3 \qquad f_y(x,y) = -12xy^2$$

$$f_{xx}(x,y) = 2 \quad f_{xy}(x,y) = f_{yx}(x,y) = -12y^2 \quad f_{yy}(x,y) = -24xy.$$

Example-4. If $z = x + y^x$, prove that $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$.

Solution. Here

$$z = x + y^x$$

Differentiating z partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = 1 + y^x\log y.$$

Differentiating z partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = xy^{x-1}.$$

Differentiating $\frac{\partial z}{\partial y}$ partially w. r. t. x, we get

$$\frac{\partial^2 z}{\partial x \partial y} = y^{x-1} \cdot 1 + xy^{x-1} \log y = y^{x-1}(1 + x \log y) \dots\dots\dots\dots\dots (1)$$

Differentiating $\frac{\partial z}{\partial x}$ partially w. r. t. y, we get

$$\frac{\partial^2 z}{\partial y \partial x} = y^x \cdot \frac{1}{y} + \log y \cdot xy^{x-1} = y^{x-1}(1 + x \log y) \dots\dots\dots\dots\dots (2)$$

By (1) and (2),

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Example-5. If $u = \frac{x^2+y^2}{x+y}$, then prove that $\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Solution. Here,

$$u = \frac{x^2 + y^2}{x + y}$$

$$\frac{\partial u}{\partial x} = \frac{(x+y)2x - (x^2+y^2)1}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x+y)2y - (x^2+y^2)1}{(x+y)^2} = \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

Now,

$$LHS = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = \left(\frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}\right)^2$$

$$= \left(\frac{2(x^2 - y^2)}{(x+y)^2}\right)^2 = 4\left(\frac{x-y}{x+y}\right)^2$$

$$RHS. = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 4\left(1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}\right)$$

$$= 4\left(\frac{(x+y)^2 - x^2 - 2xy + y^2 - y^2 - 2xy - x^2}{(x+y)^2}\right) = 4\left(\frac{x-y}{x+y}\right)^2$$

Thus

$$LHS = RHS$$

Example-6. If $u = \log(x^3 + y^3 - x^2y - xy^2)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}.$$

Solution. Here

$$u = \log(x^3 + y^3 - x^2y - xy^2)$$

$$= \log(x^3 - x^2y + y^3 - xy^2)$$

$$= \log[x^2(x - y) - y^2(x - y)]$$

$$= \log[(x - y)(x^2 - y^2)]$$

$$= \log[(x + y)(x - y)^2]$$

$$= \log(x + y) + 2\log(x - y).$$

Differentiating u w. r. t. x partially,

$$\frac{\partial u}{\partial x} = \frac{1}{x + y} + \frac{2}{x - y}.$$

Differentiating $\frac{\partial u}{\partial x}$ w. r. t. x partially,

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x + y)^2} - \frac{2}{(x - y)^2} \ldots\ldots\ldots (1)$$

Differentiating u w. r. t. y partially,

$$\frac{\partial u}{\partial y} = \frac{1}{x + y} - \frac{2}{x - y}.$$

Differentiating $\frac{\partial u}{\partial y}$ w. r. t. y partially,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x + y)^2} - \frac{2}{(x - y)^2} \ldots\ldots\ldots (2)$$

Differentiating $\frac{\partial u}{\partial y}$ w. r. t. x partially,

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{(x + y)^2} + \frac{2}{(x - y)^2} \ldots\ldots\ldots (3)$$

By (1), (2) and (3),

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x + y)^2}.$$

Example-7. If $x = r\cos\theta, y = r\sin\theta$, show that $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1.$

Solution. Here

$$x = r\cos\theta, y = r\sin\theta$$

$$\Rightarrow\ x^2 + y^2 = r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating (1) w.r.t. x partially

$$2x = 2r\frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiating (1) w.r.t. y partially

$$2y = 2r\frac{\partial r}{\partial y} \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

Hence

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1.$$

Exercise-1. Find all the first and second order partial derivatives of the following functions. Hence, verify Mixed Derivative Theorem (Clairaut's Theorem).

(a) $f(x, y) = \ln(2x + 3y)$

(b) $f(x, y) = xy + \frac{e^y}{y^2+1}$

(c) $f(x, y) = tan^{-1}\left({}^{y}/_{x}\right)$

(d) $g(x, y) = x^2y + cosy + ysinx$

Exercise-2. Evaluate the following partial derivatives.

(a) $z = \sqrt{x^2 + 4y^2}$; $Find$ ${}^{\partial z}/_{\partial x}$ (1,2) and ${}^{\partial z}/_{\partial y}$ (1,2) ; Ans: $\frac{1}{\sqrt{17}}, \frac{8}{\sqrt{17}}$

(b) $w = x^2ycosz$; $Find$ ${}^{\partial w}/_{\partial x}$, ${}^{\partial w}/_{\partial y}$, ${}^{\partial w}/_{\partial z}$ *at (2, 1, 0)*

Exercise-3. Find the indicated higher order partial derivatives.

(a) $f(x, t) = x^2e^{-ct}$; f_{ttt}, f_{txx}

(b) $f(x, y, z) = \cos(4x + 3y + 2z)$; f_{xyz}, f_{yzz}

(c) $f(x, y, z) = 1 - 2xy^2z + x^2yz + 3z$; f_{yxyz}

Exercise-4. Use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

(a) $f(x, y) = 1 - x + y - 3x^2y$, ${}^{\partial f}/_{\partial x}$ and ${}^{\partial f}/_{\partial y}$ $at\ (1,2)$; Ans: $-13, -2$

(b) $w(x,y,z) = x^2yz^2$, ${\partial w}/{\partial z}$ at (1,2,3) ; Ans: 12

Exercise-5. Show that each of the following functions satisfies a Laplace equation.

(a) $u = ln\sqrt{x^2 + y^2}$
(b) $u = e^{-2y}cos2x$
(c) $u = (x^2 + y^2 + z^2)^{-1/2}$
(d) $u = e^{-x}cosy - e^{-y}cosx$

Exercise-6. If resistors R_1, R_2 and R_3 ohms are connected in parallel to make an R-ohm resistors, the value of R can be found from the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. Find the value of ${\partial R}/{\partial R_2}$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms. ; Book[1], Ans: ${\partial R}/{\partial R_2} = \frac{1}{9}$

Exercise-7. Find partial derivatives with constrained variables:

(a) If $x = rcos\theta, y = rsin\theta$, prove that $(i) \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}, r\frac{\partial \theta}{\partial x} = \frac{1}{r}\frac{\partial x}{\partial \theta}$; $(ii) \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$; $(iii) \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

(b) If $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$, find (i) $\left(\frac{\partial w}{\partial y}\right)_z$ (ii) $\left(\frac{\partial w}{\partial y}\right)_x$ (iii) $\left(\frac{\partial w}{\partial y}\right)_y$;

Ans: (i) 0, (ii) $1 + 2z$, (iii) $1 + 2z$

(c) If $x^2 = au + bv, y^2 = au + bv$, prove that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$

- **Chain Rule**
  - Let $z = f(u)$, where u is again a function of two variables x and y, i.e., $u = u(x,y)$. Then
  $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}$$
  - Let $z = f(x,y)$, where x and y are again functions of t, i.e., $x = x(t), y = y(t)$. Then
  $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
  Here $\frac{dz}{dt}$ is called the *total derivative* of z.
  - Let $= f(x,y)$, where x and y are again functions of two variables s and t, i.e., $x = x(s,t), y = y(s,t)$. Then
  $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example-1. For $z = xe^{xy}$, $x = t^2, y = t^{-1}$, compute $\frac{dz}{dt}$.

Solution. Using chain rule, we have

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=(e^{xy}+xye^{xy})(2t)+x^2e^{xy}(-t^{-2}).$$

Putting the values of x and y in terms of t, we get

$$\frac{dz}{dt}=(2t+t^2)e^t.$$

Example-2. Let $z=e^{x^2y}$, where $x(u,v)=\sqrt{uv}$ and $y(u,v)=\frac{1}{v}$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Solution. Using chain rule.

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\left(2xye^{x^2y}\right)\left(\frac{\sqrt{v}}{2\sqrt{u}}\right)+\left(x^2e^{x^2y}\right)(0)$$

$$=2\sqrt{uv}\cdot\frac{1}{v}e^{uv\cdot\frac{1}{v}}\cdot\frac{\sqrt{v}}{2\sqrt{u}}+uv\cdot e^{uv\cdot\frac{1}{v}}(0)=e^u.$$

Also,

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\left(2xye^{x^2y}\right)\left(\frac{\sqrt{u}}{2\sqrt{v}}\right)+\left(x^2e^{x^2y}\right)\left(-\frac{1}{v^2}\right)$$

$$=2\sqrt{uv}\cdot\frac{1}{v}e^{uv\cdot\frac{1}{v}}\cdot\frac{\sqrt{u}}{2\sqrt{v}}+uv\cdot e^{uv\cdot\frac{1}{v}}\left(-\frac{1}{v^2}\right)=\frac{u}{v}e^u-\frac{u}{v}e^u=0.$$

Example-3. If u is a function of x and y and x and y are functions of r and θ given by $x=e^r\cos\theta, y=e^r\sin\theta$, then show that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right]$$

Solution. Here $u=f(x,y),\ x=e^r\cos\theta, y=e^r\sin\theta.$

Now

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}e^r\cos\theta+\frac{\partial u}{\partial y}e^r\sin\theta=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\ \ldots\ldots\ldots\ldots\ldots(1)$$

Also

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial u}{\partial x}(-e^r\sin\theta)+\frac{\partial u}{\partial y}e^r\cos\theta=-y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}\ \ldots\ldots\ldots\ldots\ldots(2)$$

By equations (1) and (2), we get

$$\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2=x^2\left(\frac{\partial u}{\partial x}\right)^2+2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+y^2\left(\frac{\partial u}{\partial y}\right)^2+y^2\left(\frac{\partial u}{\partial x}\right)^2-2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+x^2\left(\frac{\partial u}{\partial y}\right)^2$$

$$=(x^2+y^2)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=e^{2r}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right].$$

Thus

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=e^{-2r}\left[\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial u}{\partial \theta}\right)^2\right].$$

Example-4. If $u=f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.

Solution. Let $=\frac{x}{y},s=\frac{y}{z},t=\frac{z}{x}$. Then $u=f(r,s,t)$. Using chain rule,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}=\frac{\partial u}{\partial r}\left(\frac{1}{y}\right)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)=\frac{1}{y}\frac{\partial u}{\partial r}--\frac{z}{x^2}\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}=\frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right)+\frac{\partial u}{\partial s}\left(\frac{1}{z}\right)+\frac{\partial u}{\partial t}(0)=-\frac{x}{y^2}\frac{\partial u}{\partial r}+\frac{1}{z}\frac{\partial u}{\partial s}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right)+\frac{\partial u}{\partial t}\left(\frac{1}{x}\right)=-\frac{y}{z^2}\frac{\partial u}{\partial s}+\frac{1}{x}\frac{\partial u}{\partial t}$$

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{x}{y}\frac{\partial u}{\partial r}-\frac{z}{x}\frac{\partial u}{\partial t}-\frac{x}{y}\frac{\partial u}{\partial r}+\frac{y}{z}\frac{\partial u}{\partial s}-\frac{y}{z}\frac{\partial u}{\partial s}+\frac{z}{x}\frac{\partial u}{\partial t}=0.$$

Exercise-1. Evaluate $\frac{dw}{dt}$ at the given value of t by using chain Rule.

(a) $w=x^2+y^2, x=cost, y=sint, t=\pi;$ Ans: 0
(b) $w=z-sinxy, x=t, y=\ln t, \ z=e^{t-1}, t=1$ Ans: 0
(c) $w=\ln(x^2+y^2+z^2),\ x=cost, y=sint, z=4\sqrt{t}, t=3$ Ans: $=\frac{16}{49}$

Exercise-2. Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ if

(a) $w=xy+yz+zx, x=u+v, y=u-v, z=uv;$ at $(u,v)=(1/2,\ 1)$; Ans: $\frac{\partial w}{\partial u}=3,\quad \frac{\partial w}{\partial v}=-\frac{3}{2}$.
(b) $w=\ln(x^2+y^2+z^2),\ x=ue^v sinv,\ y=ue^v cosv, z=ue^v$, at $(u,v)=(-2,\ 0)$; Ans: $\frac{\partial w}{\partial u}=-1,\quad \frac{\partial w}{\partial v}=2$.
(c) $w=e^{xyz}, x=3u+v, y=3u-v, z=u^2v$; Ans: $\frac{\partial w}{\partial u}=e^{xyz}(3yz+3xz+2xyuv)$, $\frac{\partial w}{\partial v}=e^{xyz}(yz-xz+xyu^2)$,

Exercise-3. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ if

(a) $z=\tan^{-1}\left(\frac{x}{y}\right)$, $x=ucosv,\ y=usinv\ ,(u,v)=(1.3,\pi/6)$; Ans: $\frac{\partial z}{\partial u}=0$, $\frac{\partial z}{\partial v}=-1$.

(b) $z=e^{x^2y}, x=\sqrt{uv}, y=\frac{1}{v}$; Ans: $\frac{\partial z}{\partial u}=e^{u},\ \frac{\partial z}{\partial v}=\ 0$

Exercise-4. If $z=\sin^{-1}(x-y), x=3t, y=4t^3$, show that $\frac{dz}{dt}=\frac{3}{\sqrt{1-t^2}}$.

Exercise-5. If $u=u(y-z,z-x,\ x-y)$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.

Exercise-6. If $u=u\left(\frac{y-x}{xy},\frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0$.

Exercise-7. If $f\ =xsiny+e^{x}cosy, x=t^2+1,\ y=t^2$, then find the value of $\frac{\partial f}{\partial t}$ at t=0; Ans:0

Exercise-8. Find $\frac{dz}{dt}$ if

(a) z= u^2+v^2 and $u=at^2, v=2at$; Ans: $4a^2t(t^2+2)$.

(b) z= $x^2y+xy^2, x=2+t^4, y=1-t^3$; Ans: $4(2xy+y^2)t^3-3(x^2+2xy)t^2$

Exercise-9. If $u=f(x^2+2yz,\ y^2+2zx)$ then prove that

$$(y^2-zx)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}=0.$$

- **Homogeneous Function**
  A function f of two independent variables x and y is said to be *homogeneous* of degree n if for real number t we have
  $$f(tx,ty)=t^nf(x,y)$$
- **Euler's Theorem on Homogeneous Functions**
  If z is a smooth homogeneous function of x and y of degree n, then
  $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$
- **Corollary-1**
  If z is a smooth homogeneous function of x and y of degree n, then
  $$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$
- **Corollary-2**
  If z is a homogeneous function of x and y of degree n and $z=f(u)$, then
  $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$
- **Corollary-3**
  If z is a homogeneous function of x and y of degree n and $z=f(u)$, then
  $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

where $g(u) = n\frac{f(u)}{f'(u)}$.

Example-1. If $u(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2}$, then find the value of

(a) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ (b) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Solution. Here

$$u(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2} = u(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{1}{x^2}\log\left(\frac{x}{y}\right).$$

Replacing x by tx and y by ty, we get

$$(tx,ty) = t^{-2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{x^2}\log\left(\frac{x}{y}\right)\right]$$

Thus u is a homogeneous function of degree -2 in x and y. Hence by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u$$

and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (-2)(-2-1)u = 6u.$$

Example-2. If $u = tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -2sin^3 u cos\, u.$$

Solution. Here

$$u = tan^{-1}\left(\frac{x^2+y^2}{x+y}\right) \Rightarrow \tan u = \frac{x^2+y^2}{x+y} = f(u) \ \ (say).$$

Replacing x by tx and y by ty, we can see that $f(u) = \tan u$ is a homogeneous function of degree 1 in x and y. Hence by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = g(u) = 1\frac{\tan u}{sec^2 u} = \frac{1}{2}\sin 2u.$$

and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

$$= \frac{1}{2}\sin 2u(\cos 2u - 1) = \sin u \cos u(-2sin^2 u) = -2sin^3 u cos\, u.$$

Exercise-1. Check whether the following functions are homogeneous or not. If yes, find its degree 'n'.

(a) $f(x,y) = \frac{x^3+y^3}{x+y}$,

(b) $f(x,y) = \frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}$

(c) $u(x,y) = log\left(\frac{x^7+y^7}{x+y+z}\right)$

(d) $u(x,y) = \text{cosec}^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{x-y}\right)$

(e) $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Exercise-2. Verify Euler's theorem for the function $u = sin^{-1}\left(\frac{x}{y}\right) + tan^{-1}\left(\frac{y}{x}\right)$.

Exercise-3. Use Euler's theorem to solve the following problems:

1. If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = tanu.$
2. If $u = \log\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$
3. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$; show that (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin2u$

   (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2sinucos3u$.;
4. If $v = \frac{x^3y}{x^3+y^3}$, show that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3v.$
5. If $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2cotu.$
6. If $u = \sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$, prove that

   $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{144}\ tan\, u[tan^2 u - 11]$,
7. If $u = \frac{y^3 - x^3}{y^2+x^2}$ find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. Ans:0.

- **Implicit Function**

Theorem: Suppose that $F(x,y)$ is differentiable and that the equation $F(x,y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$, we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Theorem: Suppose that $F(x,y,z)$ is differentiable and that the equation $F(x,y,z) = 0$ defines z is a differentiable function of x and y. Then at any point where $F_z \neq 0$, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example-1. Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

Solution. Take $F(x, y) = y^2 - x^2 - \sin xy$

Here $F_x = \frac{\partial F}{\partial x} = -2x - ycosxy$

$F_y = \frac{\partial F}{\partial y} = 2y - xcosxy$

Therefore, $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - ycosxy}{2y - xcosxy}$

$= \frac{2x + ycosxy}{2y - xcosxy}$

Example-2. Find $\frac{dy}{dx}$ when $(cosx)^y = (siny)^x$.

Solution. Here $(cosx)^y = (siny)^x$

Taking logarithm function on both sides, we get

$ylog(cosx) = xlog(siny)$

Let $F(x,y) = ylog(cosx) - xlog(siny)$

$F_x = \frac{\partial F}{\partial x} = y\frac{1}{cosx}(-sinx) - \log(siny)$

$= -ytanx - \log(siny)$

$F_y = \frac{\partial F}{\partial y} = log(cosx) - x\frac{1}{siny}(cosy)$

$= log(cosx) - xcoty$

Therefore, $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-ytanx - \log(siny)}{log(cosx) - xcoty}$

$= \frac{ytanx + \log(siny)}{log(cosx) - xcoty}$

Example-3. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xyz = \cos(x + y + z)$.

Solution. Take $F(x, y, z) = xyz - \cos(x + y + z)$

Here $F_x = \frac{\partial F}{\partial x} = yz + \sin(x + y + z) \cdot 1$

$F_y = \frac{\partial F}{\partial y} = xz + \sin(x + y + z) \cdot 1$

$$F_z = \frac{\partial F}{\partial z} = xy + \sin(x+y+z)\cdot 1$$

Therefore, $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}$$

Exercise-1. Find the value of $\frac{dy}{dx}$ at the given point.

(a) $xe^y + \sin xy + y - log2 = 0$, (0,log2), Ans: $-(2+ln2)$.
(b) $y^3 + y^2 - 5y - x^2 + 4 = 0$, Ans: $\frac{-2x}{3y^2+2y-5}$.
(c) $\sqrt{xy} = 1 + x^2y$, Ans: $\frac{4(xy)^{3/2}-y}{x-2x^2\sqrt{xy}}$.
(d) $x^y = y^x$, Ans: $\frac{y\,(y-xlogy)}{\mathrm{x}(\mathrm{x}-\mathrm{y}\log x)}$.
(e) $e^{xy} + ye^y = 1$, Ans: $\frac{ye^{xy}}{xe^{xy}+ye^y+y}$.

Exercise-2. Find the value of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the given point.

(a) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1 = 0, (2,3,6)$, Ans: $\frac{\partial z}{\partial x} = -9, \frac{\partial z}{\partial y} = -4$.
(b) $xe^y + ye^z + 2lnx - 2 - 3ln2 = 0, (1, \ln 2, \ln 3)$, Ans: $\frac{\partial z}{\partial x} = -\frac{4}{3\,ln2}$, $\frac{\partial z}{\partial y} = -\frac{5}{3\,ln2}$.
(c) $x^2 - 3yz^2 + xyz - 2 = 0$, Ans: $\frac{\partial z}{\partial x} = \frac{2x+yz}{6yz-xy}, \frac{\partial z}{\partial y} = -\frac{xz-3z^2}{6yz-xy}$.
(d) $ye^x - 5sin3z = 3z$, Ans: $\frac{\partial z}{\partial x} = \frac{ye^x}{15cos3z+3}, \frac{\partial z}{\partial y} = \frac{e^x}{15cos3z+3}$.

- **Jacobians**

Definition: The Jacobian of the transformation $x = g(u,v),\ y = h(u,v)$ is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Similarly, the Jacobian of the transformation $x = f(u,v,w),\ y = g(u,v,w),\ z = h(u,v,w)$ is

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- **Properties of Jacobians:**
  1. If $J = \frac{\partial(x,y)}{\partial(u,v)}$ and $J' = \frac{\partial(u,v)}{\partial(x,y)}$ then $JJ' = 1$.
  2. If x, y are the function of r, s where r, s are function of u, v then $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(u,v)}$.

Example-1. Find the Jacobian for the transformation $x = r\cos\theta, y = r\sin\theta$.

Solution. $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Example-2. If $u = x + 3y^2 - z^3,\ v = 4x^2yz,\ w = 2z^2 - xy$, evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1, -1,0).

Solution. $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$

$$\left.\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|_{(1,-1,0)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 4(-1+6) = 20$$

Example-3. If $u = x^2 - y^2, v = 2xy$ and $x = r\cos\theta, y = r\sin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Solution. We have $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$

Since $u = x^2 - y^2, v = 2xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2)$$

Also $x = r\cos\theta, y = r\sin\theta$,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$= r(cos^2\theta + sin^2\theta)$$

$$= r$$

Hence $\frac{\partial(u,v)}{\partial(r,\theta)} = 4(x^2 + y^2) \cdot r$

$$= 4(r^2 cos^2\theta + r^2 sin^2\theta) \cdot r$$

$$= 4r^3$$

Exercise-1. If $x = rsin\theta cos\phi, y = rsin\theta sin\phi, z = rcos\theta$, find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$. Ans: $r^2 \sin\theta$.

Exercise-2. If $x = u(1 - v),\ y = uv$ *then* evaluate $J = \frac{\partial(x,y)}{\partial(u,v)}$ and $J' = \frac{\partial(u,v)}{\partial(x,y)}$ and hence verify that $J' = 1$, Ans: $J = u,\ J' = \frac{1}{u}$.

Exercise-3. If $u = xyz, v = x^2 + y^2 + z^2, \mathrm{w} = \mathrm{x} + \mathrm{y} + \mathrm{z}$, find $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$,

Ans: $[-2(x - y)(y - z)(z - x)]^{-1}$.

Exercise-4. If $x + y = 2e^{\theta} cos\,\phi\ , x - y = 2i\, e^{\theta} sin\ \phi$ then verify that $JJ' = 1$, where $J = \frac{\partial(x,y)}{\partial(\theta,\phi)}$ and $J' = \frac{\partial(\theta,\phi)}{\partial(x,y)}$.

Exercise-5. If $u = \frac{(2x-y)}{2}, v = \frac{y}{2}, \mathrm{w} = \frac{\mathrm{z}}{3}$, find $J(u, v, w)$. Ans:6

Exercise-6. Solve the system $u = 3x + 2y, v = x + 4y$, for x and y in terms of u and v. Then find the value of the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. Ans: $x = \frac{1}{5}(2u - v),\ y = \frac{1}{10}(3v - u),\ \frac{1}{10}$

- **Directional Derivative and Gradient**
  - **Scalar point function.** A function $\phi(x, y, z)$ is called a scalar point function if it associate a scalar at every point in space. The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of scalar point function.
  - **Vector Point function.** If a function $\mathrm{F}(x, y, z)$ defines a vector at every point of a region, then F(x,y,z) is called a vector point function. The velocity of a moving fluid, gravitational force are the example of vector point function.
  - **Vector Differential Operator Del.** The vector differential operator del (or nabla) is denoted by ∇. It is defined as

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

- **Gradient.** The gradient vector (gradient) of a scalar point function $f(x, y, z)$ is denoted by $\nabla f(grad\ f)$ and it is defined by

$$grad\ f = \nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

The $grad\ f$ is a vector normal to the surface $f(x, y, z) =$ constant and it has a magnitude equal to the rate of change of $f(x, y, z)$ along this normal.

Example-1. Find the gradient of $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + tan^{-1}(xz)$ at the point (1,1, 1).

Solution. Using definition of gradient

$$\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \left[-6xz + \frac{1}{1+x^2z^2} \cdot z\right]\hat{\imath} + [-6yz]\hat{\jmath} + \left[6z^2 - 3(x^2 + y^2) + \frac{1}{1+x^2z^2}\right]\hat{k}$$

$$\Rightarrow (\nabla f)_{(1,1,1)} = \frac{-11}{2}\hat{\imath} - 6\hat{\jmath} + \frac{1}{2}\hat{k}$$

- **Direction Derivative.** The directional derivative of a function $f(x, y, z)$ at a point $P(x, y, z)$ in the direction of vector $\bar{a}$ is given by

$$D_{\bar{a}}f = \left(\frac{df}{ds}\right)_{\bar{a},\ P} = (\nabla f)_P \cdot \hat{a}$$

Remark.

1. The function f increases most rapidly in the direction of the gradient vector ∇f or in the direction of $\frac{\nabla f}{|\nabla f|}$ at point P. The derivative in this direction is magnitude of ∇f $(i.e. |\nabla f|)$.
2. The function f decreases most rapidly in the direction of the gradient vector $-\nabla f$ or $-\frac{\nabla f}{|\nabla f|}$ at point P. The derivative in this direction is $-|\nabla f|$.

Example-1. Find the derivative of $f(x, y) = x^2 sin2y$ at the point $\left(1, \frac{\pi}{2}\right)$ in the direction of $\bar{v} = 3\hat{\imath} - 4\hat{\jmath}$.

Solution. The unit vector $\hat{v}$ is obtained by dividing $\bar{v}$ by its length

$$\hat{v} = \frac{\bar{v}}{|\bar{v}|} = \frac{3\hat{\imath} - 4\hat{\jmath}}{\sqrt{3^2 + (-4)^2}} = \frac{3\hat{\imath} - 4\hat{\jmath}}{5} = \frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}$$

The gradient of f at $\left(1, \frac{\pi}{2}\right)$ is

$$\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y}$$

$$= \hat{\imath}(2x sin2y) + \hat{\jmath}(2x^2 cos2y)$$

$$\Rightarrow (\nabla f)_{\left(1,\ \frac{\pi}{2}\right)} = 0\hat{\imath} + 2\hat{\jmath} = 2\hat{\jmath}$$

The derivative of f in the direction of the vector $\bar{v}$ at the point P is given by

$$D_{\bar{v}} f = \left(\frac{df}{ds}\right)_{\bar{v},\ P} = (\nabla f)_P \cdot \hat{v}$$

$$= (2\hat{\jmath})\left(\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}\right)$$

$$= \frac{8}{5}$$

Example-2. Find the direction in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$,

(a) Increases most rapidly at the point $(1,1)$

(b) Decreases most rapidly at the point $(1,1)$.

(c) What are rates of change in these direction?

Solution. The gradient of $f(x,y)$ is

$$\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y}$$

$$= \frac{2x}{2}\hat{\imath} + \frac{2y}{2}\hat{\jmath}$$

$$(\nabla f)_{(1,\ 1)} = \hat{\imath} + \hat{\jmath}$$

(a) The function f increases most rapidly in the direction of ∇f at $(1,1)$.

$$u = \frac{\nabla f}{|\nabla f|} = \frac{\hat{\imath} + \hat{\jmath}}{\sqrt{1^2 + 1^2}} = \frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}}$$

The rate of change in this direction is $|\nabla f|$

$$|\nabla f| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

(b) The function f decreases most rapidly in the direction,

$$-u = -\frac{\nabla f}{|\nabla f|} = -\frac{\hat{\imath} + \hat{\jmath}}{\sqrt{1^2 + 1^2}} = -\frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}}$$

The rate of change in this direction is $-|\nabla f|$

$$-|\nabla f| = -\sqrt{1^2 + 1^2} = -\sqrt{2}$$

Example-3. The temperature at any point in space is given by $T = xy + yz + zx$. Determine the derivative of T in the direction of the vector $3\hat{\imath} - 4\hat{k}$ at the point (1, 1, 1).

Solution. Let $\bar{a} = 3\hat{\imath} - 4\hat{k}$. Then

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{3\hat{\imath} - 4\hat{k}}{\sqrt{3^2 + (-4)^2}} = \frac{3\hat{\imath} - 4\hat{k}}{5}.$$

Also, $\nabla T = \hat{\imath}\frac{\partial T}{\partial x} + \hat{\jmath}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$

$$= (y + z)\hat{\imath} + (x + z)\hat{\jmath} + (x + y)\hat{k}$$

⇨ $(\nabla T)_{(1,1,1)} = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

The derivative of T in the direction of the vector $\bar{a}$ at the point P is given by

⇨ $D_{\bar{a}}T = \frac{dT}{ds} = (\nabla T)_P \cdot \hat{a} = (2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) \cdot \left(\frac{3\hat{\imath} - 4\hat{k}}{5}\right) = -\frac{2}{5}$.

Exercise-1. Find gradient of a function at the given point.

(a) $g(x, y) = \frac{x^2}{2} - \frac{y^2}{2}$, $(\sqrt{2}, 1)$, Ans: $\sqrt{2}\,\hat{\imath} - \hat{\jmath}$.
(b) $f(x, y) = (x^2 + xy)^3$, $(-1, -1)$, Ans: $-36\hat{\imath} - 12\hat{\jmath}$.
(c) $\varphi(x, y, z) = \ln(x^2 + y^2 + z^2)$, Ans: $\frac{2(x\hat{\imath} + y\,\hat{\jmath} + z\hat{k})}{x^2 + y^2 + z^2}$.
(d) $f(x, y, z) = e^{x+y}\cos z + (y + 1)sin^{-1}x$, $(0, 0, \pi/6)$, Ans: $\left(\frac{\sqrt{3} + 2}{2}\right)\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath} - \frac{1}{2}\hat{k}$.
(e) $\varphi(x, y, z) = 3x^2y - y^3z^2$, $(1, -2, -1)$, Ans: $-12\hat{\imath} - 9\hat{\jmath} + 16\hat{k}$.

Exercise-2. Find the derivative of the function at P_0 in the direction of given vector.

(a) $f(x, y) = x^2 sin2y$, $P_0(1, \pi/2)$, $\bar{A} = 3\hat{\imath} - 4\hat{\jmath}$, Ans: 8/5.
(b) $f(x, y) = tan^{-1}(y/x)$, $P_0(-2, 2)$, $\bar{v} = -\hat{\imath} - \hat{\jmath}$, Ans: $72/\sqrt{14}$.
(c) $g(x, y, z) = 3e^x\cos(yz)$, $P_0(0, 0, 0)$, $\bar{A} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$, Ans: 2.
(d) $h(x, y, z) = \cos(xy) + e^{yz} + \ln(zx)$, $P_0\left(1, 0, \frac{1}{2}\right)$, $\bar{A} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$, Ans: 2
(e) $f(x, y, z) = (x^2 + y^2 + z^2)^{3/2}$, $P_0(1,\ 1, 2)$, $\bar{A} = 2\hat{\jmath} - \hat{k}$, Ans: $9/2\sqrt{5}$.
(f) $f(x, y, z) = x^2y - yz^3 + z$, $P_0(1, -2, 0)$, $\bar{v} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$, Ans: - 3.

Exercise-3. Find the direction in which the function increases and decreases most rapidly at P_0. Then find derivative of the function in these directions.

(a) $f(x, y) = x^2 + xy + y^2, P_0(-1, 1)$, Ans: $u = -\frac{\hat{\imath}}{\sqrt{2}} + \frac{\hat{\jmath}}{\sqrt{2}}, (D_u f)_{P_0} = \sqrt{2}$
$-u = \frac{\hat{\imath}}{\sqrt{2}} - \frac{\hat{\jmath}}{\sqrt{2}}, (D_{-u} f)_{P_0} = -\sqrt{2}$.

(b) $f(x,y,z) = \left(\frac{x}{y}\right) - yz,\ P_0(4,1,1)$, Ans: $u = \frac{\hat{\imath}}{3\sqrt{3}} - \frac{5\hat{\jmath}}{3\sqrt{3}} - \frac{\hat{k}}{3\sqrt{3}}, (D_u f)_{P_0} = 3\sqrt{3}$,
$-u = -\frac{\hat{\imath}}{3\sqrt{3}} + \frac{5\hat{\jmath}}{3\sqrt{3}} + \frac{\hat{k}}{3\sqrt{3}}, (D_{-u} f)_{P_0} = -3\sqrt{3}$.

Exercise-4. Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1,1,-2)$? What is the maximum rate of increase? Ans: $(\nabla \mathrm{T})_{(1,1,-2)} = \frac{5}{8}(-\hat{\imath} - 2\hat{\jmath} + 6\hat{k})$, $\left|(\nabla \mathrm{T})_{(1,1,-2)}\right| = \frac{5\sqrt{41}}{8}$

- **Tangent plane and Normal line**

Let the equation of the surface be $f(x, y, z) = 0$. The equation of tangent plane at $P(x_1, y_1, z_1)$ to the surface is

$$(x - x_1)\left(\frac{\partial f}{\partial x}\right)_P + (y - y_1)\left(\frac{\partial f}{\partial y}\right)_P + (z - z_1)\left(\frac{\partial f}{\partial z}\right)_P = 0$$

The equations of normal line are

$$\frac{(x - x_1)}{\left(\frac{\partial f}{\partial x}\right)_P} = \frac{(y - y_1)}{\left(\frac{\partial f}{\partial y}\right)_P} = \frac{(z - z_1)}{\left(\frac{\partial f}{\partial z}\right)_P}$$

Example-1. Find the equations of the tangent plane and normal line at the point (-2, 1, -3) to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Solution. Let $f(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3$

$f_x(x, y, z) = \frac{x}{2}$ $\qquad f_x(-2,1,-3) = -1$

$f_y(x, y, z) = 2y$ $\qquad f_y(-2,1,-3) = 2$

$f_z(x, y, z) = \frac{2z}{9}$ $\qquad f_z(-2,1,-3) = -\frac{2}{3}$

Hence, the equation of the tangent plane at (-2, 1, -3) is

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0$$

$$3x - 6y + 2z = -18$$

The equations of normal line are

$$\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-2/3}$$

Example-2. Find the plane tangent to the surface $z = 1 - \frac{1}{10}(x^2 + 4y^2)$, at $\left(1, 1, , \frac{1}{2}\right)$.

Solution. Let $f(x, y, z) = z - 1 + \frac{1}{10}(x^2 + 4y^2)$,

$f_x(x, y, z) = \frac{x}{5}$ $\quad$ $f_x\left(1, 1, , \frac{1}{2}\right) = \frac{1}{5}$

$f_y(x, y, z) = \frac{4}{5}y$ $\quad$ $f_y\left(1, 1, \frac{1}{2}\right) = \frac{4}{5}$

$f_z(x, y, z) = 1$ $\quad$ $f_z\left(1, 1, , \frac{1}{2}\right) = 1$

Hence, the equation of the tangent plane at $(1, 1, 1/2)$ is

$$\frac{1}{5}(x-1) + \frac{4}{5}(y-1) + 1\left(z - \frac{1}{2}\right) = 0$$

$$\frac{1}{5}x + \frac{4}{5}y + z - \frac{3}{2} = 0$$

$$2x + 8y + 10z = 15$$

Exercise-1. Find equations for the (i) tangent plane and (ii) normal line at the point P_0 on the surface:

(a) $cos\pi x - x^2y + e^{xz} + yz = 4,\ P_0(0,1,2)$, Ans: 2x+2y+z-4=0, x=2t, y=1+2t, z=2+t.

(b) $x^2 + y^2 - 2xy - x + 3y - z = -4,\ P_0(2,-3,18)$, Ans: (i) 9x-$7y - z = 21$, (ii) $x = 2 + 9t,\ y = -3 - 7t,\ z = 18 - t$.

(c) $z^2 = 4(1 + x^2 + y^2),\ P_0(2,2,6)$, Ans: (i)$4x + 4y - 3z = -2$, (ii) $x = 2 + 4t$, $y = 2 + 4t,\ z = 6 - 3t$.

(d) $z + 1 = xe^y \cos z, P_0\,(1,0,0)$, Ans: (i) $x + y - z = 1$, (ii) $x = 1 + t,\ y = t,\ z = -t$.

Exercise-2. Find an equation for the plane that is tangent to the given surface at the given point.

(a) $z = \ln(x^2 + y^2)$, (1,0,0), Ans: 2x - $z = 2$.

(b) $z = \sqrt{y - x}$, (1,2,1), Ans: x – y + 2$z = 1$.

(c) $z = e^{-(x^2+y^2)}$, (0,0,1), Ans: $z = 1$.

(d) z = $4x^3y^2 + 2y$, $(1, -2, 12)$, Ans: $48x - 14y - z = 64$.

(e) $z = e^{3y} sin3x$, $(\pi/6, 0, 1)$, Ans: $3y - z = -1$.

- **Maxima and Minima of Functions of Two Variables**

- **First Derivative Test for Local Extreme Values:** If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain and if the first order partial derivative exist then $\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0$ and $\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0$.

- **Critical point:** An interior point of the domain of a function $f(x,y)$ where both first order partial derivatives are zero or where one or both of the first order partial derivative do not exist is a critical point of f.

- **Second Derivative Test for Local Extreme Values:** Suppose that $f(x,y)$ and its first and second derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$.

  At (a,b) let $r = f_{xx},\ s = f_{xy},\ t = f_{yy.}$

i. f has a local maximum at (a,b) if $rt - s^2 > 0$ and $r < 0$.
ii. f has a local minimum at (a,b) if $rt - s^2 > 0$ and $r > 0$.
iii. f has a saddle point at (a,b) if $rt - s^2 < 0$.
iv. The test is inconclusive at (a,b) if $rt - s^2 = 0$

Example 1. Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Solution : Here $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

For stationary values,

$$\frac{\partial f}{\partial x} = 0$$

$$\therefore 3x^2 - 3 = 0$$

$$x = \pm 1$$

And

$$\frac{\partial f}{\partial y} = 0$$

$$\therefore 3y^2 - 12 = 0$$

$$x = \pm 2$$

Thus stationary points are $(1,2), (1,-2), (-1,2), (-1,-2)$;

$$r = \frac{\partial^2 f}{\partial x^2} = 6x\,, \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0\ , \quad t = \frac{\partial^2 f}{\partial y^2} = 6y$$

(x,y)	$rt-s^2$	r	Max/Min/Saddle Point
$(1,2)$	72 > 0 r > 0	6	local minimum
$(1,-2)$	$-72<0$	6	neither max nor min
$(-1,2)$	$-72<0$	-6	neither max nor min
$(-1,-2)$	72 > 0	-6	local maximum

Example 2: Find the maximum and minimum values of $2(x^2-y^2)-x^4+y^4$.

Solution: Let $f(x,y) = 2(x^2-y^2)-x^4+y^4$

$$\frac{\partial f}{\partial x} = 4x-4x^3$$

$$\frac{\partial f}{\partial y} = -4y+4y^3$$

$$r = \frac{\partial^2 f}{\partial x^2} = 4-12x^2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = -4+12y^2$$

For maxima or minima,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$\therefore 4x-4x^3 = 0$$

$$\therefore 4x\,(1-x^2) = 0$$

$$\therefore x = 0 \quad or \quad x = \pm 1$$

Also

$$-4y+4y^3 = 0$$

$$-4y\,(1-y^2) = 0$$

$$\therefore y = 0 \quad or \quad y = \pm 1$$

The likely points where $f(x,y)$ has maxima or minima are

$$(0,0),(0,\pm 1),(\pm 1,0)$$

The results for these points are in the following table:

Points	$rt - s^2$	r	Max/Min/Saddle Point
(0,0)	-16 < 0	0	Saddle Point
(0,1)	32 > 0	4 > 0	Minimum Value -1
$(0,-1)$	32 > 0	4 > 0	Minimum Value -1
(1,0)	32 > 0	-8 < 0	Maximum Value 1
$(-1,0)$	32 > 0	-8 < 0	Maximum Value 1

Example 3. (Solving a volume problem with a constraint) A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume.

Solution: Let x, y and z represent the length, width and height of the rectangular box, respectively. Then the girth is $2x + 2z$. We want to maximize the volume $V = xyz$ of the box satisfying $x + 2y + 2z = 108$ (the largest box accepted by the delivery company). Thus, we can write the volume of the box as a function of two variables.

$$V(y,z) = (108 - 2y - 2z)yz$$

$$V(y,z) = 108yz - 2y^2z - 2yz^2$$

Setting the first partial derivatives equal to zero,

$$V_y(y,z) = 108z - 4yz - 2z^2 = (108 - 4y - 2z)z = 0$$

$$V_z(y,z) = 108y - 2y^2 - 4yz = (108 - 2y - 4z)y = 0$$

The critical points are $(0,0), (0,54), (54,0)$ *and* $(18,18)$. The volume is zero at$(0,0), (0,54), (54,0)$, which are not maximum values. At the point$(18,18)$, we apply the second derivative test.

$$r = V_{yy} = -4z, \qquad t = V_{zz} = -4y, \qquad s = V_{yz} = 108 - 4y - 4z$$

Then $\qquad rt - s^2 = V_{yy}V_{zz} - {V_{yz}}^2 = 16yz - 16(27 - y - z)^2$

Thus $\qquad r = V_{yy}(18,18) = -4(18) < 0$

And at point $(18, 18)$; $\quad rt - s^2 = V_{yy}V_{zz} - {V_{yz}}^2 = 16(18)(18) - 16(-9)^2 > 0$

At point (18,18), the function has maximum volume. The dimensions of the package are

$$x = 108 - 2(18) - 2(18) = 36\ in.$$

$$y = 18\ in. \quad \text{and} \quad z = 18\ in.$$

The maximum volume is $V = (36)(18)(18) = 11{,}664\ in.^3$

Exercise-3.

Find all the local maxima, local minima and saddle points (if exist) of the below functions.

(a) $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$; Ans: local min at (-3,3). $f(-3,3) = -5$

(b) $f(x,y) = 2xy - x^2 - 2y^2 + 3x + 4$; Ans: local max at $\left(3,\frac{3}{2}\right)$. $f\left(3,\frac{3}{2}\right) = \frac{17}{2}$

(c) $f(x,y) = x^2 + xy + 3x + 2y + 5$; Ans: critical point $(-2,1)$ is saddle point

(d) $f(x,y) = x^3 - 2xy - y^3 + 6$; Ans: critical points: $(0,0)$, $\left(-\frac{2}{3},\frac{2}{3}\right)$; saddle point is $(0,0)$, local maximum is $f\left(-\frac{2}{3},\frac{2}{3}\right) = \frac{170}{27}$

(e) $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$; Ans: critical points: $(0,0)$, $(1,-1)$; saddle point is $(1,-1)$, local minimum is $f(0,0) = 0$

(f) $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$; Ans: critical points (0,0), (0,2),(-2,0), (-2,2); saddle points are $(0,0)$, $(-2,2)$; local min is $f(0,2) = -12$, local max is $f(-2,0) = -4$

(g) $f(x,y) = 4xy - x^4 - y^4$; Ans: critical points: $(0,0), (1,1), (-1,-1)$; saddle point is $(0,0)$; local min is $f(0,2) = -12$, local max is $f(1,1) = f(-1,-1) = 2$