<u>Successive Differentiation</u> :

Introduction: Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are utmost importance in scientific and engineering applications.

Let $f(x)$ be a differentiable function and let its successive derivatives be denoted by $f'(x), f''(x), f'''(x), \dots\dots.., f^{(n)}(x)$.

❖ Common notations of higher order Derivatives of $y = f(x)$

1st Derivative: $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy

2nd Derivative: $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y

3rd Derivative: $f'''(x)$ or y''' or y_3 or $\frac{d^3y}{dx^3}$ or D^3y

$\vdots$

n^{th} Derivatives: $f^{(n)}(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny.

❖ n^{th} derivatives of some standard Functions :

1) n^{th} Derivative of $y = e^{ax}$.

$\Rightarrow$ Let $y = e^{ax}$ then

$$y_1 = a\, e^{ax}$$
$$y_2 = a^2 e^{ax}$$
$$y_3 = a^3 e^{ax}$$
$$\vdots$$
$$y_n = a^n\, e^{ax}\ .$$

2) n^{th} Derivative of $y = a^{bx}$.

$\Rightarrow$ Let $y = a^{bx}$ then

$$y_1 = b\, a^{bx}\ \log a$$
$$y_2 = b^2\ a^{bx}\, (\log a)^2$$
$$y_3 = b^3 a^{bx}\, (\log a)^3$$
$$\vdots$$
$$y_n = b^n\ a^{bx}\, (\log a)^n\ .$$

3) n^{th} Derivative of $y = (ax + b)^m$, m is a positive integer greater than n.

$\Rightarrow$ Let $y = (ax + b)^m$ then

$$y_1 = ma\, (ax + b)^{m-1}$$
$$y_2 = m(m-1)a^2\, (ax + b)^{m-2}$$
$$y_3 = m(m-1)(m-2)a^3\, (ax + b)^{m-3}$$

$$\vdots$$

$$y_n = m(m-1)(m-2)\ldots\ldots(m-(n-1))a^n\,(ax+b)^{m-n}$$

$$= m(m-1)(m-2)\ldots\ldots(m-n+1)\,a^n\,(ax+b)^{m-n}$$

$$= \frac{m!}{(m-n)!}\,a^n\,(ax+b)^{m-n}.$$

Case (i): If $\boldsymbol{m}$ is a Positive integer and $\boldsymbol{m} = \boldsymbol{n}$, then $y_n = n!\; a^n$.

Case (ii): If $\boldsymbol{m}$ is a Positive integer and $\boldsymbol{m} < \boldsymbol{n}$, then $y_n = 0$.

Case(iii): If $\boldsymbol{m} = \boldsymbol{-1}$ **,** i.e.$y = \frac{1}{ax+b}$ then $y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$.

4) $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{y = \log(ax+b)}$.

$\Longrightarrow$ Let $y = \log(ax+b)$ then

$$y_1 = \frac{a}{ax+b}$$

$$y_2 = -\frac{a^2}{(ax+b)^2}$$

$$y_3 = \frac{2\,a^3}{(ax+b)^3} = \frac{2!\;a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}.$$

5) $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{y = \sin(ax+b)}$

$\Rightarrow$ Let $y = \sin(ax+b)$ then

$$y_1 = a\,\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\,\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\,\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = a^3\,\cos\left(ax+b+\frac{2\pi}{2}\right) = a^3\,\sin\left(ax+b+\frac{3\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right).$$

Similarly if $\boldsymbol{y = \cos(ax+b)}$ then

$$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right).$$

6) $\boldsymbol{n^{th}}$ Derivative of $\boldsymbol{y = e^{ax}\sin(bx+c)}$.

$\Rightarrow$ Let $y = e^{ax}\sin(bx+c)$ then

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$

$$= e^{ax}(a\sin(bx+c) + b\,\cos(bx+c))$$

Putting $a = \mathrm{r}\cos\alpha\,,\ \ b = r\ \sin\alpha$ we get,

$$y_1 = e^{ax}(\mathrm{r}\cos\alpha\ \sin(bx+c) + \ r\ \sin\alpha\cos(bx+c))$$

$$= r\ e^{ax}(\cos\alpha\ \sin(bx+c) + \sin\alpha\cos(bx+c))$$

$$= re^{ax}\ \sin(bx+c+\alpha)$$

Similarly $y_2 = r^2 e^{ax}\sin(bx+c+2\alpha)$

$$y_3 = r^3 e^{ax}\sin(bx+c+3\alpha)$$

$$\vdots$$

$$y_n = r^n e^{ax}\sin(bx+c+n\alpha).$$

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right).$$

Where $r = \sqrt{a^2+b^2}$ and $\tan\alpha = \frac{b}{a}$.

Similarly n^{th} Derivative of $y = e^{ax}\cos(bx+c)$ is,

$$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\cos\left(bx+c+n\ \tan^{-1}\frac{b}{a}\right).$$

❖ Summary :

Function	n^{th} Derivative
$y = e^{ax}$	$y_n = a^n\, e^{ax}$
$y = a^{bx}$	$y_n = b^n\, a^{bx}\,(\log a)^n$
$y = (ax+b)^m$	$y_n = \begin{cases} \dfrac{m!}{(m-n)!}\ a^n\ (ax+b)^{m-n}, & m>0, m>n \\ 0\ , & m>0,\quad m<n \\ n!\ a^n\,, & m>0,\quad m=n \\ \dfrac{(-1)^n n!\ a^n}{(ax+b)^{n+1}}\ , & m=-1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1}\dfrac{(n-1)!\ a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n\sin\left(ax+b+\dfrac{n\pi}{2}\right)$
$y = \cos(ax+b)$	$y_n = a^n\cos\left(ax+b+\dfrac{n\pi}{2}\right)$
$y = e^{ax}\, sin(bx+c)$	$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\sin\left(bx+c+n\ \tan^{-1}\dfrac{b}{a}\right)$
$y = e^{ax}\, cos(bx+c)$	$y_n = (a^2+b^2)^{\frac{n}{2}} e^{ax}\cos\left(bx+c+n\ \tan^{-1}\dfrac{b}{a}\right)$

Example-1: Find the n^{th} derivative of the function $y = \frac{1}{1-5x+6x^2}$.

Solution : Here $y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$

$$\therefore \frac{1}{(2x-1)(3x-1)} = \frac{A}{2x-1} + \frac{B}{3x-1} \quad \text{..............}(1)$$

$\therefore 1 = A(3x-1) + B(2x-1)$

If $x = {}^1/_2$ then $A = 2$

If $x = {}^1/_3$ then $B = -3$.

$\therefore$ From equation (1) , we get

$y = \frac{2}{2x-1} - \frac{3}{3x-1}$ So the n^{th} derivative of the given function is,

$$y_n = \frac{2\,(-1)^n\, n!\, 2^n}{(2x-1)^{n+1}} - \frac{3\,(-1)^n\, n!\, 3^n}{(3x-1)^{n+1}}$$

$$= (-1)^n\, n! \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right].$$

Example-2 : Find the n^{th} derivative of the function $y = \frac{2x-1}{(x^2-5x+6)}$.

Solution : Here $y = \frac{2x-1}{(x^2-5x+6)} = \frac{2x-1}{(x-2)(x-3)}$.

Let $y = \frac{2x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$.

$\therefore 2x - 1 = A(x-3) + B\,(x-2)$.

If $x = 2$, then $A = -3$.

If $x = 3$, then $B = 5$.

$$\therefore y = \frac{5}{x-3} - \frac{3}{x-2}$$

$$\therefore y_n = \frac{5\,(-1)^n\, n!}{(x-3)^{n+1}} - \frac{3\,(-1)^n\, n!}{(x-2)^{n+1}}.$$

Example-3 : Find the n^{th} derivative of the function $y = \frac{x^4}{x^2-3x+2}$.

Solution : Here $y = \frac{x^4}{x^2-3x+2} = \frac{x^4}{(x-2)(x-1)}$

$$y = \frac{x^4}{(x-2)(x-1)} = x^2 + 3x + 7 + \frac{15x-14}{(x-2)(x-1)}$$

$$= x^2 + 3x + 7 + \frac{16}{x-2} - \frac{1}{x-1}$$

$$\therefore y_n = 0 + \frac{16(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}}.$$

Example-4 : Find the n^{th} derivative of the function $y = \sin 6x\ \cos 4x$.

Solution : Here $y = \sin 6x\ \cos 4x = \frac{1}{2}\,(\sin 10x + \sin 2x)$ $(\because s + s = 2\,s\,c)$

$$\therefore y_n = \frac{1}{2}\left(10^n \sin\left(10x + \frac{n\pi}{2}\right) + 2^n\ \sin\left(2x + \frac{n\pi}{2}\right)\right).$$

Example-5 : Find the n^{th} derivative of the function $y = sin^4 x$.

Solution : Here $y = sin^4 x = (sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2$

$$= \frac{1}{4}\,(1 - 2\cos 2x + cos^2 2x) = \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{8}\,(3 - 4\cos 2x + \cos 4x)$$

$$\therefore y_n = \frac{1}{8}\left(0 - 4\ 2^n\ \cos\left(2x + \frac{n\pi}{2}\right) + \ 4^n\ \cos\left(4x + \frac{n\pi}{2}\right)\right).$$

Example-6 : Find the n^{th} derivative of the function $y = e^{2x}\ \cos 2x\ \cos x$.

Solution : Here $y = e^{2x}\ \cos 2x\ \cos x = \frac{1}{2}\ e^{2x}(\ 2\cos 2x\ \cos x)$

$$= \frac{1}{2}\ e^{2x}(\cos 3x + \cos x)\ \ (\because c + c = 2\ c\ c)$$

$$= \frac{1}{2}\ (e^{2x}\ \cos 3x + \ e^{2x}\ \cos x)$$

$$= \frac{1}{2}\left[(13)^{\frac{n}{2}}\ e^{2x}\ \cos\left(3x + n\tan^{-1}\frac{3}{2}\right) + \ (5)^{\frac{n}{2}} e^{2x}\ \cos\left(x + n\tan^{-1}\frac{1}{2}\right)\right].$$

Example-7 : If $y = e^{ax}\ \sin bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)\ y = 0$.

Solution : Here $y = e^{ax}\ \sin bx$ then $y_1 = \ e^{ax}\ b\cos bx + ae^{ax}\ \sin bx$

$$\therefore y_1 = e^{ax}\ b\ \cos bx + a\ y.$$

$\Longrightarrow y_1 - ay = e^{ax}\ b\ \cos bx$, again differentiating with respect to x we get,

$$y_2 - ay_1 = ae^{ax}\ b\ \cos bx - \ b^2 e^{ax}\ \sin bx$$

$$\Longrightarrow y_2 - ay_1 = a(\ y_1 - ay) - b^2\ y = a\ y_1 - a^2\ y - b^2 y$$

$$\therefore \therefore y_2 - 2ay_1 + (a^2 + b^2)y = 0\ .$$

Example-8 : If $x = a(\cos t + t\ \sin t)$, $y = a(\sin t - t\ \cos t)$, find $\frac{d^2y}{dx^2}$.

Solution : We have $\frac{dx}{dt} = a(-\sin t + \sin t + t\ \cos t) = a\ t\ \cos t$

and $\frac{dy}{dt} = a(\cos t - \cos t + t\ \sin t) = a\ t\ \sin t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{a\ t\ sin\, t}{a\ t\ cos\, t} = \ \tan t\ , \frac{d^2y}{dx^2} = \ \frac{d}{dt}(\tan t)\ \frac{dt}{dx}$$

$$\therefore\ \ \frac{d^2y}{dx^2} = \ sec^2\, t\ \frac{1}{at\ \cos t} = \ \frac{sec^3\, t}{at}.$$

Problems :

1. If $a\,x^2 + 2hxy + b\,y^2 = 1$, prove that $\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$.

2. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x\,\frac{dy}{dx} + y\,cos^2\,x = 0$.

3. Find $\frac{d^2y}{dx^2}$, when $x = a\,cos^3\theta\,, y = b\,sin^3\theta$.

4. If $x = \sin t\,, y = \sin pt$, prove that $(1-x^2)y_2 - xy_1 + p^2y = 0$.

5. If $x = 2\cos t - \cos 2t,\; y = 2\sin t - \sin 2t$, find the value of $\frac{d^2y}{dx^2}$ when $t = {}^{\pi}/_{2}$.

6. If $x^3 + y^3 = 3axy$ then prove that $\frac{d^2y}{dx^2} = -\frac{2\,a^2\,xy}{(y^2-ax)^3}$.

7. If $y = \tan^{-1}(\sin hx)$, prove that $\frac{d^2y}{dx^2} + \tan y\left(\frac{dy}{dx}\right)^2 = 0$.

8. If $y = e^{-kt}\cos(lt+c)$, show that $\frac{d^2y}{dt^2} + 2k\,\frac{dy}{dt} + n^2y = 0$, where $n^2 = k^2 + l^2$.

9. Find the n^{th} derivatives of the following functions **:**

(i) $y = cos\,x\,cos\,2x\,cos\,3x$
(ii) $y = e^{2x}\,cos^2x\,\sin x$
(iii) $y = \frac{x}{(x-1)\,(2x+3)}$
(iv) $y = e^{-x}sin^2x$
(v) $y = \frac{x^2-4x+1}{x^3+2x^2-x-2}$
(vi) $y = sin^2x\,cos^3\,x$
(vii) $y = e^{-x}\,sin^3\,x$
(viii) $y = \log(ax+b)(cx+d)$
(ix) $y = cos^6x$.

LEIBNITZ'S THEOREM :

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u\,v)_n = u_n\,v + n_{C_1}\,u_{n-1}\,v_1 + n_{C_2}\,u_{n-2}\,v_2 + \cdots\cdots + n_{C_r}\,u_{n-r}\,v_r + \cdots + u\,v_n.$$

Where u_r and v_r represents r^{th} derivatives of u and v respectively.

Example-1 Find the n^{th} derivative of $x \log x$.

Solution : Let $u = \log x$ and $v = x$.

Then $u_n = (-1)^{n-1}\frac{(n-1)!}{x^n}$, $u_{n-1} = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$ and $v_1 = 1,\; v_2 = 0.$

∴ Using Leibnitz's theorem , we have

$$(u\,v)_n = u_n\,v + n_{C_1}\,u_{n-1}\,v_1 + n_{C_2}\,u_{n-2}\,v_2 + \cdots\cdots + n_{C_r}\,u_{n-r}\,v_r + \cdots + u\,v_n$$

$$\Rightarrow (x\log x)_n = (-1)^{n-1}\frac{(n-1)!}{x^n}\,x + n\,(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}\;1 + 0$$

$$\Rightarrow (x\log x)_n = (-1)^{n-1}\frac{(n-1)(n-2)!}{x^{n-1}} + n\;(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

$$= -(-1)^{n-2}\frac{(n-1)(n-2)!}{x^{n-1}} + \;n\;(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}\,[-(n-1)+n]$$

$$= = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}\;1.$$

Example-2 Find the n^{th} derivative of $x^2 e^{3x}\,\sin 4x$.

Solution : Let $u = e^{3x}\sin 4x$ and $v = x^2$.

Then $u_n = (25)^{\frac{n}{2}}\,e^{3x}\,\sin\left(4x + n\,\tan^{-1}\frac{4}{3}\right) = 5^n e^{3x}\,\sin\left(4x + n\,\tan^{-1}\frac{4}{3}\right)$,

$u_{n-1} = 5^{n-1}e^{3x}\,\sin\left(4x + (n-1)\,\tan^{-1}\frac{4}{3}\right)$ and

$v_1 = 2x, v_2 = 2, v_3 = 0.$

∴ Using Leibnitz's theorem , we have

$$(u\,v)_n = u_n\,v + n_{C_1}\,u_{n-1}\,v_1 + n_{C_2}\,u_{n-2}\,v_2 + \cdots\cdots + n_{C_r}\,u_{n-r}\,v_r + \cdots + u\,v_n$$

$$\Rightarrow (x^2 e^{3x}\,\sin 4x)_n = 5^n e^{3x}\,\sin\left(4x + n\,\tan^{-1}\frac{4}{3}\right)\,x^2$$

$$+\,n\,5^{n-1}e^{3x}\sin\left(4x + (n-1)\,\tan^{-1}\frac{4}{3}\right)\;(2x)$$

$$+\;\frac{n(n-1)}{2}5^{n-2}e^{3x}\,\sin\left(4x + (n-2)\,\tan^{-1}\frac{4}{3}\right)\,2 + \;0\,.$$

$$= e^{3x}\,5^n[x^2\sin\left(4x + n\,\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\,\tan^{-1}\frac{4}{3}\right)$$

$$+\,\frac{n(n-1)}{25}\sin\left(4x + (n-2)\,\tan^{-1}\frac{4}{3}\right)]\;.$$

Example-3 Find the n^{th} derivative of $e^x\,(2x+3)^3$.

Solution : Let $u = e^x$ and $v = (2x+3)^3$.

Then $u_n = e^x$, for all integer values of n, and

$v_1 = 6(2x+3)^2\,, v_2 = 24(2x+3), v_3 = 48, v_4 = 0.$

$\therefore$ Using Leibnitz's theorem , we have

$$(u\,v)_n = u_n\,v + n_{C_1}\,u_{n-1}\,v_1 + n_{C_2}\,u_{n-2}\,v_2 + \cdots\cdots + n_{C_r}\,u_{n-r}\,v_r + \cdots + u\,v_n$$

$$\Rightarrow (e^x\,(2x+3)^3)_n = e^x(2x+3)^3 + n\,e^x 6(2x+3)^2 + \frac{n(n-1)}{2}\,e^x 24(2x+3)$$

$$+\frac{n(n-1)(n-2)}{6}e^x 48\ + 0$$

$$= e^x\{\,(2x+3)^3 + 6n\,(2x+3)^2 + 12n\,(n-1)(2x+3) + 8n(n-1)(n-2)\}.$$

Example-4 If $y = (\sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$.

Solution : Here If $y = (\sin^{-1}x)^2$ then differentiating with respect to x we get,

$$y_1 = \frac{2(\sin^{-1}x)}{\sqrt{1-x^2}} \text{ or } (1-x^2){y_1}^2 = 4(\sin^{-1}x)^2 = 4y$$

Again differentiating , we get

$$(1-x^2)2\,y_1\,y_2 - 2x{y_1}^2 = 4\,y_1 \text{ or } (1-x^2)\,y_2 - xy_1 - 2 = 0$$

Differentiating it n times by Leibnitz's theorem,

$$(1-x^2)y_{n+2} + n\,(-2x)\,y_{n+1} + \frac{n(n-1)}{2}\,(-2)y_n - [xy_{n+1} + n\,y_n] = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nx\,y_{n+1} - n(n-1)y_n - x\,y_{n+1} - n\ \ y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2y_n = 0.$$

Which is the required result.

Problems :

1. If $y^{1/m} + y^{-1/m} = 2\,x$, prove that
   $(x^2-1)y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)y_n = 0.$
2. If $y = e^{m\,\cos^{-1}x}$, prove that (i) $(1-x^2)y_2 - xy_1 = m^2y$
   (ii) $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - (n^2+m^2) = 0.$
3. If $y = \tan^{-1}x$, prove that $(1+x^2)y_{n+1} + 2nxy_n + n\,(n-1)y_{n-1} = 0.$
4. Find the nth derivative of the following functions **:**
   (i) $x^2\log 3x$
   (ii) $x^2\cos x$
   (iii) x^2e^x

Indeterminate Forms and L'Hôpital's (L'Hospital's) Rule:

John (Johann) Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or infinity. The rule is known today as **L'Hôpital's Rule**, after Guillaume de L'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of the rule for their calculation.

Indeterminate Form 0/0:

If the continuous functions $f(x)$ and $g\ (x)$ are both zero at $x\ =\ a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x\ =\ a$. The substitution produces **0/0**, a meaningless expression, which we cannot evaluate. We use **0/0** as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such $\frac{\infty}{\infty}, \infty\ \cdot \mathbf{0}$, $\infty - \infty$, $\mathbf{0^0}, \infty^{\mathbf{0}}, \mathbf{1}^{\infty}$ which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic manipulations.

L'Hospital's rule: If f and g are differentiable functions on an open interval I containing a and suppose that $f(a) = g(a) = 0$, $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists or again if it is $(0/0)$ form repeat the same process until we get the finite limit.

Example-1: Find $\lim_{x \to 0} \frac{3x - \sin x}{x}$.

Solution : Here $\lim_{x \to 0} \frac{3x - \sin x}{x}$ $\left({}^{0}/_{0} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{3 - \cos x}{1} = \frac{3-1}{1} = 2\ .$$

Example-2: Find $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

Solution : Here $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ $\left({}^{0}/_{0} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \left({}^{0}/_{0} - form\right)$$

Again using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{6\,x} = \frac{1}{6} \left(\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right).$$

Example-3: Find $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$.

Solution : Here $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$ $\left(0/_0 - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}.$$

Example-4: Find $\lim_{x \to 1} \frac{x-x^x}{1+\log x - x}$.

Solution : Here $\lim_{x \to 1} \frac{x-x^x}{1+\log x - x}$ $(0/0 - form)$

So using L'Hospital rule, we get

$$= \lim_{x \to 1} \frac{1-x^x(1+\log x)}{\frac{1}{x}-1} \quad (0/0 - form)$$

Again using L'Hospital rule, we get

$$= \lim_{x \to 1} \frac{-x^x(1+\log x)^2 - x^{x-1}}{\left(-\frac{1}{x^2}\right)} = 2.$$

Problems :

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$ (Ans: 1/3)
2. $\lim_{x \to \pi/2} \frac{\log(\sin x)}{(\pi-2x)^2}$ (Ans: -1/8)
3. $\lim_{x \to 0} \frac{(1+x)^{1/x}-e}{x}$ (Ans: -e/2)
4. $\lim_{x \to 0} \frac{e^x-1-x}{x^2}$ (Ans: 1/2)
5. $\lim_{x \to y} \frac{x^y-y^x}{x^x-y^y}$ (Ans: $\frac{1-\log y}{1+\log y}$)
6. $\lim_{x \to 1} \frac{x\log x-(x-1)}{(x-1)\log x}$ (Ans: $1/2$)
7. $\lim_{x \to 1/2} \frac{cos^2 \pi x}{e^{2x}-2xe}$ (Ans: $\frac{\pi^2}{2e}$)
8. $\lim_{x \to \pi/2} \frac{2x-\pi}{\cos x}$ (Ans: -2)
9. If $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$ is finite, then find the value of a and hence the value of limit.

   (Ans: $= -2$, limit = -1)

10. Find the value of a, b and c such that $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2.$

(Ans: $= 1, b = 2, c = 1$)

<u>Indeterminate Form ∞/∞</u> :

If f and g are differentiable functions on an open interval I containing a and suppose that $f(a) = g(a) = \infty$, $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Example-1: Find $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1+\tan x}$.

Solution : Here $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1+\tan x} \quad \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{\sec x \tan x}{sec^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \sin x = 1.$$

Example-2: Find $\lim_{x \to \infty} \frac{e^x}{x^2}$.

Solution : Here $\lim_{x \to \infty} \frac{e^x}{x^2} \quad \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{e^x}{2x} \left(\frac{\infty}{\infty} - form\right)$$

Again using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{e^x}{2} = \infty.$$

Example-3: Find $\lim_{x \to \infty} \frac{\log x}{2\sqrt{x}}$.

Solution : Here $\lim_{x \to \infty} \frac{\log x}{2\sqrt{x}} \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{1/x}{2/_{2\sqrt{x}}} = \lim_{x \to \infty} \sqrt{x}/_x = \lim_{x \to \infty} 1/_{\sqrt{x}} = 0.$$

Example-4: Find $\lim_{x \to \frac{\pi}{2}} \frac{\log(x - \pi/2)}{\tan x}$.

Solution : Here $\lim_{x \to \frac{\pi}{2}} \frac{\log(x - \pi/2)}{\tan x} \quad \left(\frac{\infty}{\infty} - form\right)$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{1}{sec^2 x \ (x - \pi/2)} = \lim_{x \to \frac{\pi}{2}} \frac{cos^2 x}{(x - \pi/2)} \left(\frac{0}{0} - form\right)$$

Again using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{-2 \sin x \cos x}{1} = 0.$$

Problems :

Evaluate the following limits:

1. $\lim_{x \to a} \frac{\log(x-a)}{\log(e^x - e^a)}$ (Ans:1)
2. $\lim_{x \to 0} \frac{\log x^2}{\cot x^2}$ (Ans: 0)
3. $\lim_{x \to 0} \frac{\log(\sin 2x)}{\log(\sin x)}$ (Ans: 1)
4. $\lim_{x \to \infty} \frac{x^3+3x^2}{7x^3-4x}$ (Ans: 1/ 7)
5. $\lim_{x \to 0} \log_{\tan x} \tan 2x$ (Ans: 1)
6. Prove that $\lim_{x \to 1} \frac{\log(1-x)}{\cot \pi x} = 0$.

<u>Indeterminate Form $(0 \cdot \infty)$ or $(\infty - \infty)$</u> :

If f and g are differentiable functions on an open interval I containing a and suppose that $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = \infty$ then $\lim_{x \to a} f(x)g(x)$ is in $0 \cdot \infty$ form. We write given function as $\lim_{x \to a} \frac{f(x)}{1/g(x)}$ or $\lim_{x \to a} \frac{g(x)}{1/f(x)}$ so it is in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form respectively, which can be solved using L'Hospital's rule.

To evaluate the limits of the type $\lim_{x \to a} [f(x) - g(x)]$, when $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, we reduce the expression in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking LCM or by rearranging the terms and then apply L'Hospital's rule.

Example-1: Find $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution : Here $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ $(\infty - \infty\ form)$

$$\therefore \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \left(\frac{0}{0} - form\right)$$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \left(\frac{0}{0} - form\right)$$

Again using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = 0.$$

Example-2: Find $\lim_{x \to \infty} \left(a^{1/x} - 1\right) x.$

Solution : Here $\lim_{x \to \infty} \left(a^{1/x} - 1\right) x \; (\, 0 \cdot \infty - form)$

$$\therefore \lim_{x \to \infty} \frac{\left(a^{1/x} - 1\right)}{\frac{1}{x}} \quad \left(\frac{0}{0} - form\right)$$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to \infty} \frac{a^{1/x}\left(-\frac{1}{x^2}\right)\log a}{\left(-\frac{1}{x^2}\right)} = \log a \; .$$

Example-3: Find $\lim_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right].$

Solution : Here $\lim_{x \to 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)}\right] (\infty - \infty \; form)$

$$\therefore \lim_{x \to 2} \left[\frac{\log(x-1) - (x-2)}{(x-2)\,\log(x-1)}\right] \quad \left(\frac{0}{0} - form\right)$$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 2} \left[\frac{\frac{1}{x-1} - 1}{log(x-1) + \frac{(x-2)}{x-1}}\right] = \lim_{x \to 2} \frac{1-(x-1)}{(x-2)+(x-1)\,log(x-1)} \quad \left(\frac{0}{0} - form\right)$$

Again using L'Hospital rule, we get

$$= \lim_{x \to 2} \frac{-1}{1 + \frac{x-1}{x-1} + log(x-1)} = \frac{1}{2}.$$

Example-4: Find $\lim_{x \to 1} (x^2 - 1)\; tan\left(\frac{\pi x}{2}\right).$

Solution : Here $\lim_{x \to 1} (x^2 - 1)\; tan\left(\frac{\pi x}{2}\right) \; (0 \cdot \infty - form)$

$$\therefore \lim_{x \to 1} \frac{(x^2-1)}{\cot\left(\frac{\pi x}{2}\right)} \quad (0/0 - form)$$

So using L'Hospital rule, we get

$$\Rightarrow \lim_{x \to 1} \frac{2x}{-cosec^2\left(\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)} = \frac{2}{-\left(\frac{\pi}{2}\right)} = -\frac{4}{\pi} \; .$$

Problems :

Evaluate the following limits:

1. $\lim_{x \to 1} (1 - x)\, tan\left(\frac{\pi x}{2}\right)$ (Ans: $2/\pi$)
2. $\lim_{x \to 0} \frac{1}{x}(1 - x\, cot\, x)$ (Ans: 0)
3. $\lim_{x \to \infty} \left(x + 1/2\right)\, log\left(\frac{2x+1}{2x}\right)$ (Ans: $1/2$)
4. $\lim_{x \to a} log\left(2 - \frac{x}{a}\right) cot(x - a)$ (Ans : $-\left(\frac{1}{a}\right)$)

5. $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x}\right)$ (Ans: $1/2$)

6. Prove that $\lim_{x \to \infty} x^2 e^{-x} = 0.$

7. Prove that $\lim_{x \to 1} (1 + \sec \pi x) \tan \pi x = 0$

8. Prove that $\lim_{x \to \frac{\pi}{2}} \left(\tan x - \frac{2x \sec x}{\pi}\right) = \frac{2}{\pi}.$

9. If $\lim_{x \to 0} \left(\frac{a \cot x}{x} + \frac{b}{x^2}\right) = \frac{1}{3}$ then find a and b .

Indeterminate Forms $1^{\infty}, \infty^0, 0^0$:

To evaluate the limits of the type $\lim_{x \to a} [f(x)]^{g(x)}$, which takes any one of the indeterminate forms $1^{\infty}, \infty^0, 0^0$ for $f(x) > 0$, we proceed as follows:

Let $l = \lim_{x \to a} [f(x)]^{g(x)} \quad , f(x) > 0$

Applying log function on both the sides we get, $L = \log l = \lim_{x \to a} [g(x) \log f(x)]$ which takes the form $0 \cdot \infty$ which can be solved using L'Hospital rule.

So $\lim_{x \to a} [f(x)]^{g(x)} = \lim_{x \to a} e^{g(x) \log f(x)} = e^L$.

Example-1: Find $\lim_{x \to o^+} (1 + x)^{1/x}$.

Solution : Here $\lim_{x \to o^+} (1 + x)^{1/x} \quad (1^{\infty} - form)$

$$\therefore \quad L = \lim_{x \to o^+} \frac{1}{x} \log(1 + x) \quad (0 \cdot \infty - form)$$

$$= \lim_{x \to o^+} \frac{\log(1+x)}{x} \quad (0/0 - form)$$

$\therefore$ Using L'Hospital rule, we get

$$= \lim_{x \to o^+} \frac{1}{1+x} = 1$$

Therefore, $\lim_{x \to o^+} (1 + x)^{1/x} = \lim_{x \to o^+} e^{\frac{1}{x} \log(1+x)} = e^1 = e.$

Example-2: Find $\lim_{x \to \infty} x^{1/x}$.

Solution : Here $\lim_{x \to \infty} x^{1/x} \quad (\infty^0 - form)$

$$\therefore \quad L = \lim_{x \to \infty} \frac{1}{x} \log x \quad (0 \cdot \infty - form)$$

$$= \lim_{x \to \infty} \frac{\log x}{x} \quad (\infty/\infty - form)$$

∴ Using L'Hospital rule, we get

$$= \lim_{x \to \infty} \frac{1}{x} = 0$$

Therefore, $\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{\frac{1}{x}\log x} = e^0 = 1.$

Example-3: Find $\lim_{x \to o^+} x^x$.

Solution : Here $\lim_{x \to o^+} x^x \ (0^0 - form)$

$$\therefore \quad L = \lim_{x \to o^+} x \log x \ (0 \cdot \infty - form)$$

$$= \lim_{x \to o^+} \frac{\log x}{\frac{1}{x}} \ (\infty/\infty - form)$$

∴ Using L'Hospital rule, we get

$$= \lim_{x \to o^+} \frac{1/x}{\left(-\frac{1}{x^2}\right)} = \lim_{x \to o^+} -x = 0$$

Therefore, $\lim_{x \to o^+} x^x = \lim_{x \to o^+} e^{x \log x} = e^0 = 1.$

Problems :

Evaluate the following limits:

1. $\lim_{x \to 0} (a^x + x)^{1/x}$ (Ans: ae)
2. $\lim_{x \to 0} (e^{3x} - 5x)^{1/x}$ (Ans: e^{-2})
3. $\lim_{x \to 0} \left(\frac{e^x + e^{2x} + e^{3x}}{3}\right)^{1/x}$ (Ans: e^2)
4. $\lim_{x \to \infty} \left(\frac{ax+1}{ax-1}\right)^x$ (Ans: $e^{2/a}$)
5. $\lim_{x \to 1} (1 - x^2)^{1/\log(1-x)}$ (Ans: e)
6. $\lim_{x \to 0} (\cos x)^{\cot x}$ (Ans: 1)
7. $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$ (Ans: $e^{1/3}$)
8. $\lim_{x \to 0} (\cot x)^{\sin x}$ (Ans: 1)
9. $\lim_{x \to 0} \left(\frac{1}{x}\right)^{1-\cos x}$ (Ans: 1)
10. $\lim_{x \to \frac{\pi}{2}} (\cos x)^{\frac{\pi}{2} - x}$ (Ans: 1)
11. $\lim_{x \to 0} (\sin x)^x$ (Ans: 1)
12. $\lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$ (Ans: $\left((abc)^{1/3}\right)$

13. $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$ (Ans:1)

Expansions of functions :

In this section we shall discuss expansions of functions into infinite series with the help of Maclaurin's theorem and Taylor's theorem.

Power Series :

An infinite series of the form

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots\cdots + a_n(x-a)^n + \cdots\cdots$$

is called a power series in $(x-a)$, where the a_i's are constants.

If $a = 0$ then a power series can be written as

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots + a_n x^n + \cdots\cdots$$

Taylor's and Maclaurin's Series :

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by** f at $x = a$ is ,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$$

If $a = 0$ then the **Maclaurin's series** of $\boldsymbol{f}$ is the Taylor's series generated by f at $x = 0$ is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots\cdots$$

Remark :

Let $f(x)$ be a differentiable function of order n at a point $x = a$ then its Taylor's series expansion is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$$

In above expansion if we substitute $x - a = h$ then

$$f(a+h) = f(a) + f'(a)\, h + \frac{f''(a)}{2!} h^2 + \cdots\cdots (*)$$

If $h = x$ in above formula we get,

$$f(a+x) = f(a) + f'(a)\, x + \frac{f''(a)}{2!} x^2 + \cdots\cdots$$

If $a = x$ in (*) formula we get,

$$f(x+h) = f(x) + f'(x)\, h + \frac{f''(x)}{2!} h^2 + \cdots\cdots$$

Which is the **alternative form** (another form) of the **Taylor's series.**

Maclaurin's expansion of some standard functions :

1. $f(x) = e^x$.

Here $f(x) = e^x$, $a = 0$ so $f(0) = 1$

$f'(x) = e^x$, $f'(0) = 1$

$f''(x) = e^x$, $f''(0) = 1$

$f'''(x) = e^x$, $f'''(0) = 1$

$f^{IV}(x) = e^x$, $f^{IV}(0) = 1$

$\vdots$ $\vdots$

$\vdots$ $\vdots$

So using Maclaurin's series expansion we get,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots\cdots$$

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \cdots\cdots + \frac{x^n}{n!} + \cdots\cdots$$

If we replace by x by $-x$ in above series expansion , we get

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\cdots + (-1)^n \frac{x^n}{n!} + \cdots\cdots$$

2. $f(x) = \sin x$, $\therefore f(0) = 0$

$f'(x) = \cos x$, $\therefore f'(0) = 1$

$f''(x) = -\sin x$ $\therefore f''(0) = 0$

$f'''(x) = -\cos x$ $\therefore f'''(0) = -1$

$f^{IV}(x) = \sin x$ $\therefore f^{IV}(0) = 0$

$\vdots$ $\vdots$

$\vdots$ $\vdots$

So using Maclaurin's series expansion we get,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots\cdots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \cdots\cdots\cdots\cdots$$

3. $f(x) = \cos x$, $\therefore f(0) = 1$

$f'(x) = -\sin x$, $\therefore f'(0) = 0$

$f''(x) = -\cos x$, $\therefore f''(0) = -1$

$f'''(x) = \sin x$ $\therefore f'''(0) = 0$

$f^{IV}(x) = \cos x$ $\therefore f^{IV}(0) = 1$

$\vdots$ $\vdots$

$\vdots$ $\vdots$

So using Maclaurin's series expansion we get,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{IV}(0) + \cdots\cdots$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots\cdots\cdots$$

4. $f(x) = \log(1+x)$, $\therefore f(0) = 0$

$f'(x) = \frac{1}{1+x}$, $\therefore f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$, $\therefore f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$, $\therefore f'''(0) = 2 = 2!$

$f^{IV}(x) = -\frac{6}{(1+x)^4}$, $\therefore f^{IV}(0) = -6 = -3!$

$\vdots$ $\vdots$

$\vdots$ $\vdots$

So using Maclaurin's series expansion we get,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{IV}(0) + \cdots\cdots$$

$$\therefore\ log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots\cdots\cdots$$

If we replace x by $-x$ in above series expansion, we get

$$f(x) = \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots\cdots\cdots$$

5. $f(x) = sinhx = \frac{e^x - e^{-x}}{2}$

So using standard expansions of e^x and e^{-x}, we get

$f(x) = sinhx$

$$= \frac{1}{2}\left\{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots\right]\right\}$$

$$= \frac{1}{2}\left\{2x + 2\frac{x^3}{3!} + \cdots\cdots\cdots\right\} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots + \frac{x^{(2n+1)}}{(2n+1)!} + \cdots\cdots$$

Also $f(x) = coshx = \frac{e^x + e^{-x}}{2}$

$$= \frac{1}{2}\left\{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots + \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots\right]\right\}$$

$$= \frac{1}{2}\left\{2 + 2\frac{x^2}{2!} + \cdots\cdots\right\} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\cdots + \frac{x^{2n}}{(2n)!} + \cdots\cdots$$

Summary :

Function	**Maclaurin's Series expansion**
$y = e^x$	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$
$y = e^{-x}$	$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots$

$y = \sin x$	$x - \frac{x^3}{3!} + \cdots\cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$
$y = \cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$
$y = \log(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots\cdots\cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots$
$y = \log(1-x)$	$-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\cdots\cdots\cdots - \frac{x^n}{n} + \cdots$
$y = sinhx$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots + \frac{x^{(2n+1)}}{(2n+1)!} + \cdots\cdots$
$y = coshx$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\cdots + \frac{x^{2n}}{(2n)!} + \cdots\cdots$

Example:1 Find the Maclaurin's series expansion of $\tan x$.

Solution: Let $y = f(x) = tanx$ $\quad\quad \therefore y(0) = f(0) = 0$

$y_1 = f'(x) = sec^2x = 1 + tan^2x \quad\quad \therefore y_1(0) = f'(0) = 1$

$= 1 + y^2$

$y_2 = f''(x) = 2yy_1 \quad\quad \therefore y_2(0) = f''(0) = 2(0)(1) = 0$

$y_3 = f'''^{(x)} = 2{y_1}^2 + 2yy_2 \quad\quad \therefore y_3(0) = f'''(0) = 2(1) + 2 \cdot 0 =$

2

$y_4 = f^{IV}(x) = 6y_1y_2 + 2yy_3 \quad\quad \therefore y_4(0) = f^{IV}(0) = 0$

$y_5 = f^V(x) = 6{y_2}^2 + 8\, y_1y_3 + 2yy_4 \quad\quad \therefore y_5(0) = f^V(0) = 16$

$\vdots \quad\quad \vdots$

$\vdots \quad\quad \vdots$

Now, by Maclaurin's series expansion, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots\cdots$$

$$\therefore \tan x = x + \frac{x^3}{3!}\ (2) + \frac{x^5}{5!}\ (16) + \cdots\cdots = x + \frac{x^3}{3} + \frac{2}{15}\ x^5 + \cdots\cdots$$

Example:2 Expand $\sec x$ in powers of x up to x^4 by Maclaurin's series.

Solution: Let $y = f(x) = \sec x \quad\quad \therefore y(0) = f(0) = 1$

$y_1 = \sec x \tan x = y \tan x \quad\quad \therefore y_1(0) = f'(0) = 0$

$y_2 = y_1 \tan x + y\, sec^2x = y_1 \tan x + y^3\,, \quad\quad \therefore y_2(0) = f''(0) \ = 1$

$y_3 = y_2 \tan x + 2y_1\, sec^2x + 2ysec^2x\ \tan x$

$= y_2 \tan x + 2y_1\, y^2 + 2y^3\ \tan x\ , \quad\quad \therefore y_3(0) = f'''(0)\ =0$

$y_4 = 3y_2y^2 + y_3\ \tan x + 4y\, {y_1}^2\ + 6y^2\, y_1\ \tan x + 2\, y^5\,,\ \therefore y_4(0) = \ 5$

$\vdots \quad\quad \vdots$

$\vdots \quad\quad \vdots$

Now, by Maclaurin's series expansion, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots\cdots$$

$$\therefore \ \sec x = 1 + \frac{x^2}{2!} + 5\frac{x^4}{4!} + \cdots\cdots\cdots$$

Example:3 Find the Maclaurin's series expansion of the function $y = \log(1 + \sin x)$.

Solution: Here $y = f(x) = \log(1 + \sin x)$ $\quad \therefore f(0) = 0$

$y_1 = f'(x) = \frac{\cos x}{1+\sin x}$ $\quad \therefore f'(0) = 1$

$y_2 = f''(x) = \frac{(1+\sin x)\ (-\sin x) - cos^2 x}{(1+\sin x)^2}$

$y_2 = \ -\frac{1}{1+\sin x}$ $\quad \therefore f''(0) =$ -1

⋮ ⋮

⋮ ⋮

Now, by Maclaurin's series expansion, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots\cdots$$

$$\log(1 + \sin x) = x - \frac{x^2}{2!} + \cdots\cdots\cdots$$

Example:4 Arrange the following Polynomial in powers of x using Maclaurin's series $f(x) = 5 + (x + 3) + 7(x + 3)^2$.

Solution: Here $f(x) = 5 + (x + 3) + 7(x + 3)^2$, $\quad \therefore f(0) = 71$

$f'(x) = 1 + 14(x + 3),$ $\quad \therefore f'(0) = 43$

$f''(x) = 14$, $\quad \therefore f''(0) = 14$

$f'''(x) = 0$, $\quad \therefore f'''(0) = 0$

Now, by Maclaurin's series expansion, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots\cdots$$

$$\therefore \ 5 + (x + 3) + 7(x + 3)^2 = 71 + 43x + 7x^2 \ .$$

Example:5 Expand $\log x$ in powers of $(x - 1)$ up to three power and hence evaluate $\log 1.1$ correct to four decimal places.

Solution: Here $(x) = \log x$, $\quad a = 1$ $\quad \therefore \ f(1) = 0$

$f'(x) = \frac{1}{x}$, $\quad \therefore f'\ (1) = 1$

$f''(x) = -\frac{1}{x^2},$ $\quad \therefore \ f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$, $\quad \therefore f'''(1) = 2$

⋮ ⋮

⋮ ⋮

Now, by Taylor's series expansion, we have

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!} f''(1) + \cdots\cdots$$

$$\therefore \log x = (x-1)\,1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}2 + \cdots\cdots$$

$$\therefore \log x = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3} + \cdots\cdots$$

Now taking $x = 1.1$, we get

$\therefore \log 1.1 = 0.09533.$

Example:6 Expand $2x^3 + 7x^2 + 1$ in powers of $(x-3)$ by using Taylor's series expansion.

Solution: Here $(x) = 2x^3 + 7x^2 + 1$, $a = 3$, $f(3) = 118$

$f'(x) = 6x^2 + 14x$, $f'(3) = 96$

$f''(x) = 12x + 14$, $f''(3) = 50$

$f'''(x) = 12$, $f'''(3) = 12$

$f^{IV}(x) = 0$, $f^{IV}(3) = 0$

Now, by Taylor's series expansion, we have

$$f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!} f''(3) + \cdots\cdots$$

$$\therefore 2x^3 + 7x^2 + 1 = 118 + (x-3)\,96 + \frac{(x-3)^2}{2!}\,50 + \frac{(x-3)^3}{3!}\,12 + 0$$

$$= 118 + 96(x-3) + 25(x-3)^2 + 2\,(\mathrm{x}-3)^3.$$

Example:7 Find the Taylor's series expansion of $\sin\left(x + \frac{\pi}{4}\right)$ in powers of x and hence find the value of $\sin 46°$.

Solution: Here let $f(x) = \sin x$ and $h = \frac{\pi}{4}$

So using another form of Taylor's series we have

$$f(x+h) = f(h) + f'(h)\,x + \frac{f''(h)}{2!}\,x^2 + \cdots\cdots \qquad \ldots\ldots\ldots(1)$$

Now $f(x) = \sin x$, $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x$, $f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x$, $f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and so on

Hence from equation (1), we have

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + x\left(\frac{1}{\sqrt{2}}\right) + \frac{x^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots\cdots$$

Consider $x = 1° = 0.01745$,

$\therefore \sin\left(1° + \frac{\pi}{4}\right) = \sin 46° = 0.71934.$

Problem set :

1. Find the Maclaurin's series expansion of the following functions:
   (i) $y = \frac{e^x}{e^x+1}$
   (ii) $y = \tan^{-1} x$
   (iii) $y = \log(\sec x)$
   (iv) $y = \sin^{-1} x$
   (v) $y = e^x \sin 2x$
   (vi) $y = 17 + 6(x+2) + 3(x+2)^3 + (x+2)^4$

2. Calculate the approximate value of $\tan 50°$ using Taylor's series expansion.

3. Find the Taylor's series expansion of $\log\cos x$ about the point $\pi/3$. Hence find approximate value of $\log\cos 61°$.
4. Expand $\log\sin x$ in powers of $(x-2)$.
5. Find the approximate value of $\sqrt[3]{28}$ correct up to four decimal places by using Taylor's series expansion.

6. Expand $3x^3 - 2x^2 + x - 4$ in a series of powers of $(x+2)$.
7. Calculate the value of $\sqrt{9.12}$ using Taylor's series expansion.
8. Expand $\sin x$ in a series of powers of $\left(x - \frac{\pi}{6}\right)$.
9. Find the Taylor's series expansion of $\frac{1}{x}$ in powers of $(x-2)$.