# Finite Difference Advection Fail

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*Numerical Approximation of PDEs - Summer 2024*

## 1 Simple explicit scheme for 1D advection

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### 1.1 Introduction

In this notebook we want to implement a simple advection scheme for the 1D advection equation, given by

$$\frac{\partial \rho}{\partial t} + v \cdot \frac{\partial \rho}{\partial x} = 0,$$

where $\rho$ is an advected constituent, $v \equiv 1$ is a given wind (assumed to be constant 1), and $(x, t)$ is the space-time coordinate.

In order to derive an explicit numerical scheme, we apply a forward finite difference in time and a centered finite difference in space, i.e.

$$\frac{\partial \rho}{\partial t} \approx \frac{\rho(t + \Delta t, x) - \rho(t, x)}{\Delta t}$$

and

$$\frac{\partial \rho}{\partial x} \approx \frac{\rho(t, x + \Delta x) - \rho(t, x - \Delta x)}{2\Delta x}.$$

### 1.2 Grid Function

We define a grid function by subdividing the temporal dimension into intervals $t^j = j \cdot \Delta t$ ($j = 0, \ldots, K$), where $K\Delta t = T$. Similarly, we introduce a grid for the spatial dimension by $x_i = i \cdot \Delta x$ ($i = 0, \ldots, N$).

With this the discrete form of the transport equation reads

$$\rho_i^{j+1} = \rho_i^j - \nu(\rho_{i+1}^j - \rho_{i-1}^j),$$

where $\nu = v\frac{\Delta t}{2\Delta x}$ and the indices $i$ and $j$ stand for the space and time step, resp. This scheme is called *forward-in-time-centered-in-space* (FTCS) scheme.

### 1.3 Time stepping scheme

In order to implement this scheme, we define a function that takes the grid $x$, the time-step $\Delta t$, the final time $T$, and an initial field $\rho_0$ as input parameters and returns $\rho(x, T)$.

[1]:
```python
from numpy import arange
def ftcsadvect(x,dt,T,rho0):
#-- compute spatial step size
    dx= x[1]-x[0]

    #-- initial condition; set up future time array
    u0=rho0.copy()
    u1=u0.copy()
    nu= dt/(2.*dx)

    #-- now the time loop
    for t in arange(0,T+dt,dt):
        u1[1:(len(x)-1)]= u0[1:(len(x)-1)] - nu* (u0[2:len(x)]-u0[0:(len(x)-2)])
        u1[0] = u0[0] - nu* (u0[1] - u0[len(x)-1])
        u1[len(x)-1]= u0[(len(x)-1)] - nu* (u0[0]-u0[len(x)-2])
        u0=u1.copy()

    #-- set output value
    rho = u1
    return rho
```

### 1.4 Initial Conditions

We want to generate initial conditions in their own functions. The two alternative initial conditions are given by either

$$\rho_0(x) = e^{-\frac{(20x)^2}{2}},$$

or

$$\rho_0(x) = e^{\frac{-x^2}{2\sigma^2}} \cdot \cos(Kx),$$

where $\sigma = 0.1$, and $K = \frac{\pi}{\sigma}$.

[2]:
```python
from numpy import pi, exp, cos
def init1(x):
    u0=exp(-(20*x)**2/2)
    return u0

def init2(x):
    sigm= 0.1; Ka= pi/sigm
    u0= exp(-(x**2)/(2*sigm**2)) * cos(Ka*x)
    return u0
```

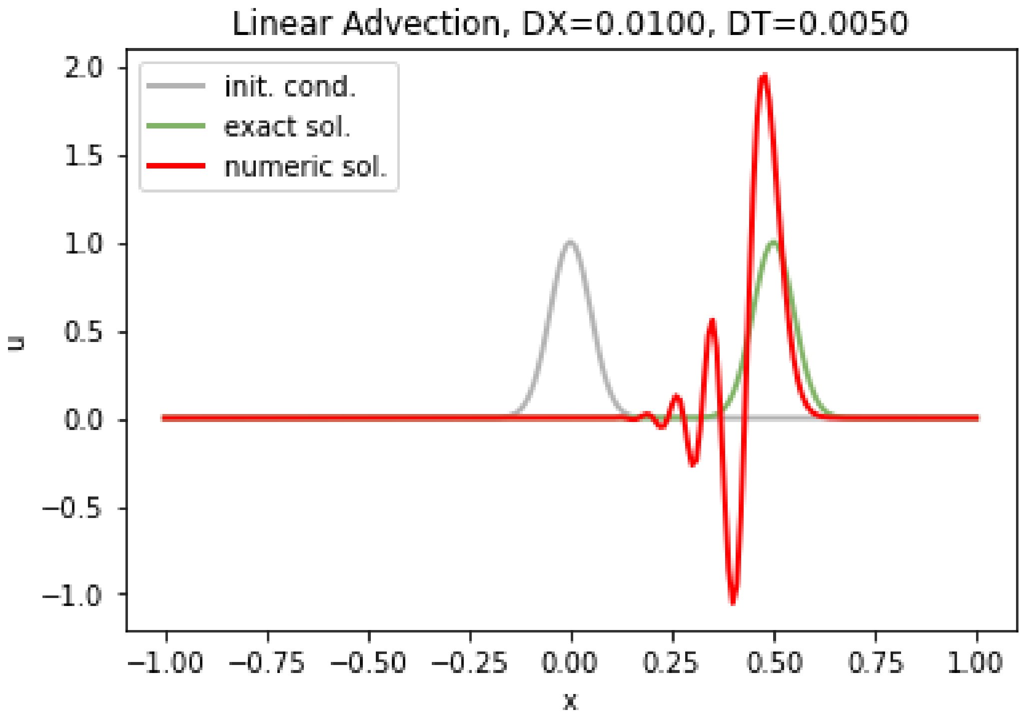
### 1.5 Main Test Program

Now that we have generated our time-stepping function, we can build a main program to test our scheme. 1. Let us set some initial values (time-step size, etc.) first. 2. Then we will set initial conditions by calling the corresponding functions. 3. Then we call the Lax-Friedrichs function defined earlier. 4. We compute an analytic solution (remember, this is just the initial solution shifted). 5. Finally, we compare our solutions.

[3]:
```python
def testlf():
    from numpy import arange, pi, exp, cos, inf
    from numpy.linalg import norm
    import matplotlib.pyplot as plt
    %matplotlib inline

    #-- initial values/settings
    dx= 1./100.                 #- spatial step size
    courant= 0.5                #- Courant No.
    dt= courant* dx             #- time step size
    Tend= .5                    #- final time
    x= arange(-1,1+dx,dx)       #- spatial grid

    #-- initial conditions
    u0 = init1(x)

    #-- Lax-Friedrichs advection scheme
    uftcs = ftcsadvect(x,dt,Tend,u0)

    #-- compute exact solution to compare with
    uexact = init1(x-Tend)

    #-- plot the result, the initial condition, and the exact solution
    fig = plt.figure(1)
    h1=plt.plot(x,u0,linewidth=2, c=[0.7, 0.7, 0.7], label='init. cond.')
    h2=plt.plot(x,uexact,linewidth=2, c=[0.5, 0.7, 0.4], label='exact sol.')
    h3=plt.plot(x,uftcs,linewidth=2, c='red', label='numeric sol.')
    plt.legend(loc='upper left')
    plt.title('Linear Advection, DX=' + '%6.4f' % (dx) + ', DT=' + '%6.4f' %␣
 ↪(dt))
    plt.xlabel('x')
    plt.ylabel('u')

    #-- compute error norms:
    infftcs= norm((uexact-uftcs),inf)/norm(uexact,inf)
    print('*** INFO: relative error in inf-norm ***')
    print('          FTCS method: ' + '%6.4f' % (infftcs))
    twoftcs= norm((uexact-uftcs))/norm(uexact)
    print('*** INFO: relative error in two-norm ***')
    print('          FTCS method: ' + '%6.4f' % (twoftcs))

    plt.show()

if __name__=="__main__":
```

```
testlf()
```

```
*** INFO: relative error in inf-norm ***
        FTCS method: 1.1871
*** INFO: relative error in two-norm ***
        FTCS method: 1.0986
```

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[ ]:
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