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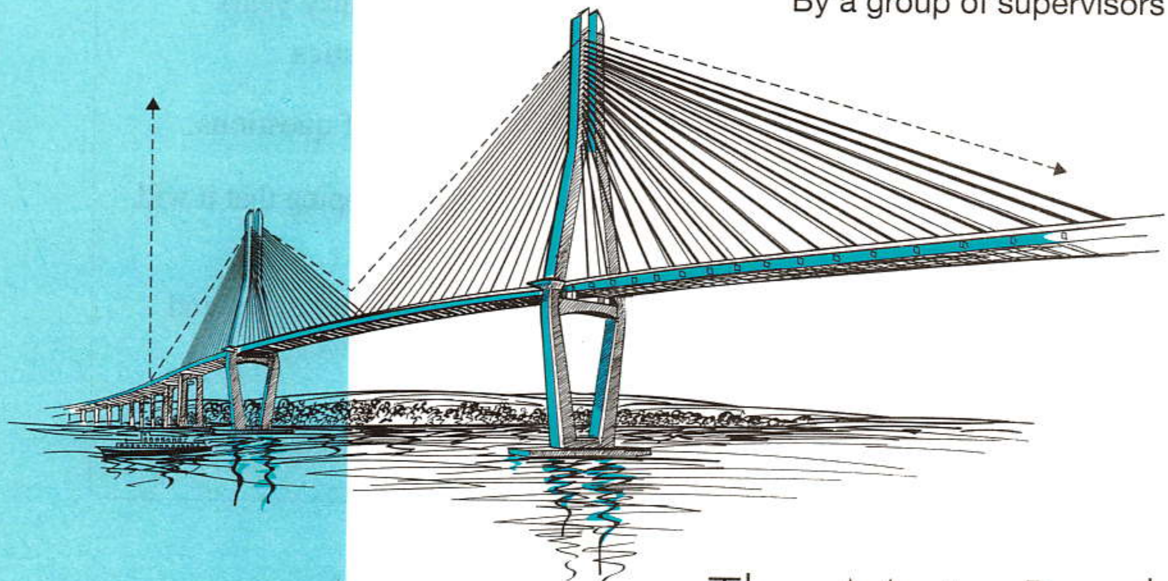

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Preface

Thanks to God who helped us to introduce one of our famous series **"El Moasser"** in mathematics.

We introduce this book to our colleagues.
We also introduce it to our students to help them study mathematics.

In fact, this book is the outcome of more than thirty years experience in the field of teaching mathematics.

This book will make students aware of all types of questions.

We would like to know your opinions about the book hoping that it will win your admiration.

We will be grateful if you send us your recommendations and your comments.

the authors

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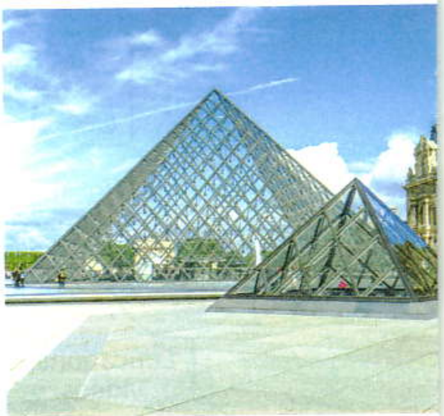


Revision on vectors

The quantities which we deal with in our life are divided into two kinds of quantities :

(1) Scalar quantities : They are the quantities which are completely determined if we knew their magnitude only as a real number

As : length , mass , time , temperature degree , volume , distance.

(2) Vector quantity : It is a quantity determined by a real number (the magnitude of this quantity) besides the direction.

i.e. The vector quantity is determined completely if we knew its magnitude and its direction.

- **Directed line segment :**

  It is a straight line segment having an initial point and an ending point , with direction defined from the initial point to the ending point.

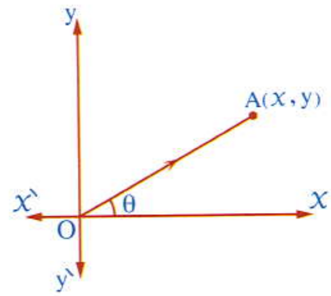
- The norm of the directed line segment $\left(\overrightarrow{AB}\right)$:
  the norm of $\overrightarrow{AB}$ is the length of $\overline{AB}$ and it is denoted by $\|\overrightarrow{AB}\|$
- The two directed line segments are equivalent if they have :

  – The same length (the norm) and the same direction.
- $\overrightarrow{AB} \neq \overrightarrow{BA}$ (they have opposite directions)
- $\overrightarrow{AB} = -\overrightarrow{BA}$
- $\|\overrightarrow{AB}\| = \|\overrightarrow{BA}\|$

- The position vector of a given point (A) with respect to the origin point (O) it is the directed line segment $\overline{OA}$, it is denoted by $\vec{A}$

For example :

In the opposite figure :



If $\overrightarrow{OA}$ is the position vector of the point (A) = (X , y) then :

* $\|\vec{A}\|$ = the length of $\overline{OA} = \sqrt{X^2 + y^2}$
* If $\|\vec{A}\| = 1$ length unit (unity)
  , then $\vec{A}$ is called the unit vector.
* $\vec{i} = (1 , 0)$
  , $\vec{j} = (0 , 1)$ are two unit vectors
  (called two basic unit vectors) in the two directions of the two coordinate axes.
* $\vec{O} = (0 , 0)$ it is zero vector which has no direction and it is denoted by $\vec{O}$
* $\vec{A} = (X , y)$ is called the cartesian form of the vector $\vec{A}$
* $\vec{A} = X\vec{i} + y\,\vec{j}$ expresses the vector $\vec{A}$ in terms of the two basic unit vectors.
* $\vec{A} = \left(\|\vec{A}\| , \theta\right)$ is called the polar form of the vector $\vec{A}$
* θ is the measure of the angle made by the vector $\overrightarrow{OA}$ with the positive direction of X-axis , it is called the polar angle.
* $X = \|\vec{A}\| \cos\theta$, then $\cos\theta = \frac{X}{\|\vec{A}\|}$
* $y = \|\vec{A}\| \sin\theta$, then $\sin\theta = \frac{y}{\|\vec{A}\|}$

- If $\vec{A} = (X_1 , y_1)$, $\vec{B} = (X_2 , y_2)$, then :

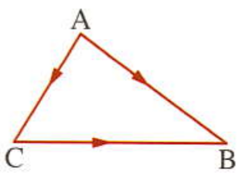
* $\vec{A} = \vec{B}$ if and only if $X_1 = X_2 , y_1 = y_2$
  $\vec{A} \pm \vec{B} = (X_1 \pm X_2 , y_1 \pm y_2)$
* $\overrightarrow{AB} = \vec{B} - \vec{A} = (X_2 - X_1 , y_2 - y_1)$
* $K\vec{A} = K(X , y) = (KX , Ky)$
* $\vec{A} \,//\, K\vec{A}$ with regarding that :

If $K > 0$ then → $\vec{A}$ and $K\vec{A}$ have the same direction

If $K < 0$ then → $\vec{A}$ and $K\vec{A}$ have opposite direction

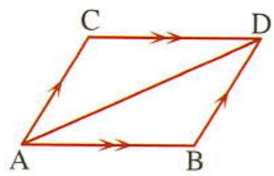
where $k \neq 0$

- **Adding and subtracting vectors geometrically :**



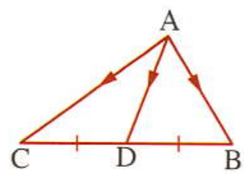
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{O}$



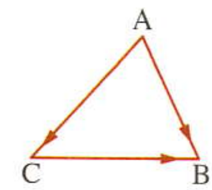
$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$



$\overrightarrow{AB} + \overrightarrow{AC} = 2\,\overrightarrow{AD}$

$\overrightarrow{BD} + \overrightarrow{CD} = \overrightarrow{O}$



$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$

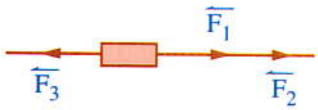
- **Physical applications :**

The resultant force $\overrightarrow{R}$

- The resultant of a set of forces acting on a body is operated as the operation of adding vectors

i.e. The resultant force

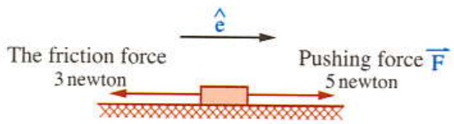
$\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} + \ldots$



For example :

If we defined a unit vector $\hat{e}$ in the direction of the motion of the body then :

In the case of motion of the body on a rough plane



The resultant force

$\overrightarrow{R} = 5\,\hat{e} + (-3\,\hat{e}) = 2\,\hat{e}$

i.e. The magnitude of the resultant = 2 newton

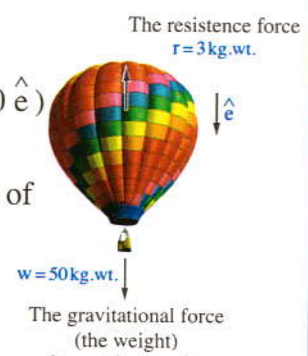
- The direction of the resultant is in the direction of the motion of the body.

In the case of vertical motion

The resultant force $= 50\,\hat{e} + (-30\,\hat{e})$

$= 20\,\hat{e}$

i.e. The magnitude of the resultant = 20 Kg.wt.



- The direction of resultant is in the direction of the weight

- If the two forces have the same magnitude and the same line of action but in two opposite directions then the resultant $\overrightarrow{R} = \overrightarrow{O}$
- If the resultant of a set of concurrent forces $= \overrightarrow{O}$ this means the set of forces are in equilibrium.

Example 1

(1) Write the vector $\overrightarrow{A} = \left(3 , -\sqrt{3}\right)$ in the polar form.

(2) Write in terms of the two basic unit vectors the vector $\overrightarrow{A}$ whose norm = 10 length unit and act in the direction of Western North.

Solution

(1) $\because \|\overrightarrow{A}\| = \sqrt{9+3} = 2\sqrt{3}$

$\therefore \cos\theta = \frac{x}{\|\overrightarrow{A}\|} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} > 0$

$\sin\theta = \frac{y}{\|\overrightarrow{A}\|} = \frac{-\sqrt{3}}{2\sqrt{3}} = -\frac{1}{2} < 0$

$\therefore \theta$ lies in the 4^{th} quadrant

$\therefore \theta = 360 - 30 = 330°$

$\therefore \overrightarrow{A} = \left(2\sqrt{3} , 330°\right)$

(2) $\because \|\overrightarrow{A}\| = 10 \quad , \quad \theta = 135°$

$\therefore x = \|\overrightarrow{A}\| \cos\theta = 10\cos 135° = -5\sqrt{2}$

$y = \|\overrightarrow{A}\| \sin\theta = 10 \sin 135° = 5\sqrt{2}$

$\therefore \overrightarrow{A} = \left(-5\sqrt{2} , 5\sqrt{2}\right)$

$\therefore \overrightarrow{A} = -5\sqrt{2}\,\overrightarrow{i} + 5\sqrt{2}\,\overrightarrow{j}$

Example 2

If the forces $\overrightarrow{F_1} = 2\overrightarrow{i} + 3\overrightarrow{j}$, $\overrightarrow{F_2} = a\overrightarrow{i} + \overrightarrow{j}$, $\overrightarrow{F_3} = 5\overrightarrow{i} + b\overrightarrow{j}$ act on a particle

Find the values of a and b if these forces :

(1) Their resultant $= 5\overrightarrow{i} - 2\overrightarrow{j}$

(2) Are in equilibrium.

Solution

The resultant $= \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \left(2\overrightarrow{i} + 3\overrightarrow{j}\right) + \left(a\overrightarrow{i} + \overrightarrow{j}\right) + \left(5\overrightarrow{i} + b\overrightarrow{j}\right)$

$= (2 + a + 5)\overrightarrow{i} + (3 + 1 + b)\overrightarrow{j}$

(1) $\because$ The resultant $= 5\overrightarrow{i} - 2\overrightarrow{j}$

$\therefore (7 + a)\overrightarrow{i} + (4 + b)\overrightarrow{j} = 5\overrightarrow{i} - 2\overrightarrow{j}$

$\therefore 7 + a = 5$

$\therefore a = -2$

$\therefore 4 + b = -2$

$\therefore b = -6$

(2) $\because$ The forces are in equilibrium

$\therefore \overrightarrow{R} = \overrightarrow{O}$

$\therefore (7 + a)\overrightarrow{i} + (4 + b)\overrightarrow{j} = \overrightarrow{O}$

$\therefore a = -7 \quad , \quad b = -4$

Unit One

Statics

Lesson **1** Forces - Resultant of two forces meeting at a point.

Lesson **2** Forces resolution into two components.

Lesson **3** The resultant of coplanar forces meeting at a point.

Lesson **4** Equilibrium of a rigid body under the effect of two forces / three forces meeting at a point.
(The triangle of forces rule - Lami's rule).

Lesson **5** Follow : The equilibrium
(Meeting lines of action of three equilibrium forces).

Lesson

1

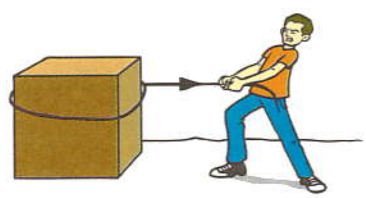
Forces - Resultant of two forces meeting at a point

The force

"The force is defined as the effect of a natural body upon another one" by pushing, attraction, pressure or repulsion". The natural body is a body consisting of material (mass) and volume not equal to zero.

The natural bodies can be classified into two kinds :

- Rigid bodies (solid bodies) : They are the bodies whose shapes do not change whatever the forces which are acting on them as solid metals, rocks, ...



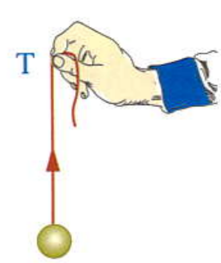
- Deformable bodies : They are the bodies whose shapes can be disfigured as strings, liquids, gases, rubber and clay and our study in this unit will be continued to rigid bodies only.

Kinds of forces

There are different kinds of forces , as :

(1) Tension force (T) :

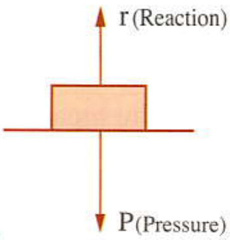
As the force in the string (or the rope) when carrying a body at it.



(Tension in the string)

2 Pressure force (P) :

As the force that appears when a body stabilized on a surface.



3 Reaction force (r) :

As the reaction of a smooth surface on a body stabilized on it.

4 Attraction forces and repulsion forces :

As the forces which formed between magnetic poles , electric charges and astronomical objects.

5 Gravitational forces (weights) :

If we let a body in the air , then it will drop down towards the Earth because the attraction force of the Earth attracts any body towards it.

This force is called gravitational force or weights.

- Notice that : the weight (W) = the body mass × acceleration of gravity = m × g

Expressing force

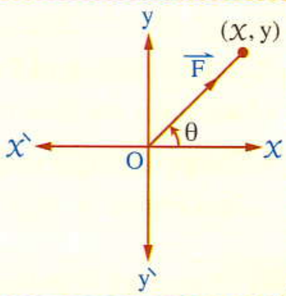
The force is a vector quantity so it can be represented by the same way as the vectors.

***i.e.* The force can be expressed as follow :**

(1) $\vec{F} = (x , y) \Rightarrow$ the cartesian form.

(2) $\vec{F} = x\,\vec{i} + y\,\vec{j} \Rightarrow$ in terms of the fundamental unit vectors.

(3) $\vec{F} = \left(\|\vec{F}\| , \theta\right) \Rightarrow$ the polar form.



Determination of the force

The force is a vector which passes through a fixed point.

i.e. It acts in a given straight line.

***i.e.* The force is determined by :**

(1) The magnitude of the force.

(2) The direction of the force.

(3) The point of action of the force.

For example :

The football player kicks the ball by a determined force (magnitude of the force) in a determined direction (direction of the force) in a certain point on the surface of the ball (Point of action of the force)

UNIT 1

The magnitude of the force

1 The measurement units :

* The magnitude of the force (The numerical value of the force) is measured by units which are called weight units.

As : gram weight (gm.wt.) , kilogram weight (kg.wt.)

, where $1 \text{ kg.wt.} = 1000 \text{ gm.wt.} = 10^3 \text{ gm.wt.}$

* There are other units to measure the magnitude of force (they are called absolute units)

As : The dyne , the newton :

, where $1 \text{ newton} = 100\,000 \text{ dyne} = 10^5 \text{ dyne}$

* The weight units connect with the absolute units by the relation :

$1 \text{ kg.wt.} = 9.8 \text{ newton}$, $1 \text{ gm.wt.} = 980 \text{ dyne}$

(unless something else mentioned)

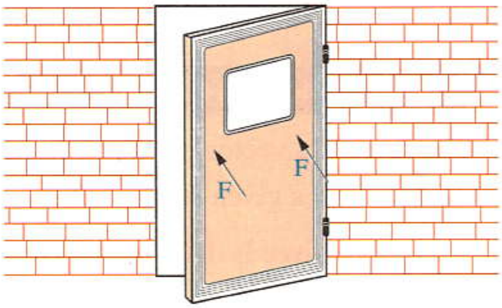
2 The direction of the force :

It is the direction of the vector which represents this force and it is determined by the measure of the polar angle of the force vector in the case of the coplanar forces in the same plane.

- The polar angle is the positive directed angle which the vector makes with the positive direction of X-axis.

3 The point of action of the force :

The action of the force is determined by its point of action. If you try to open the door of a room or close it with a force near of the line of hinges , you will find difficult to rotate it.
As you are far from the line of hinges as the difficulty becomes less. As shown in the figure.



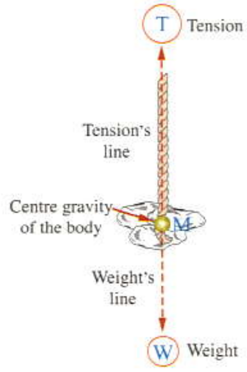
The line of action of the force

The line of action of the force is the line passing through the point of action parallel to the direction of the force.

For example :

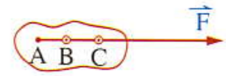
– The tension line in a string is the string itself.

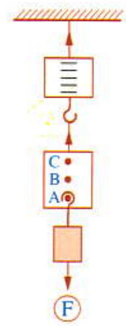
– The line of action of the weight of a body is the vertical line passing through the center gravity of the body.



Displacing (or translation) of the point of action of the force (force penetration)

If the force $\overrightarrow{F}$ acts on a rigid body and A is the point of its action , then we can displace this point to another point on the body "B" or "C" or .... on the line of action of the force without changing in its influence on the body.





i.e. Any point lying on the line of action of a force can be considered as a point of action of this force.

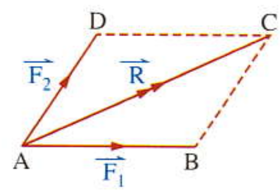
Resultant of two forces meeting at a point

The resultant of two or more forces is a single force has the same effect as the two or more forces.

Finding the resultant of two forces meeting at a point (geometrically)

This method depends on the parallelogram rule to add two forces :

If two forces ($\overrightarrow{F_1}$, $\overrightarrow{F_2}$) meeting at a point are represented in magnitude and direction by two sides of a parallelogram meeting at this point , then their resultant ($\overrightarrow{R}$) is represented in magnitude and direction by the diagonal of the parallelogram which starts from the same point.



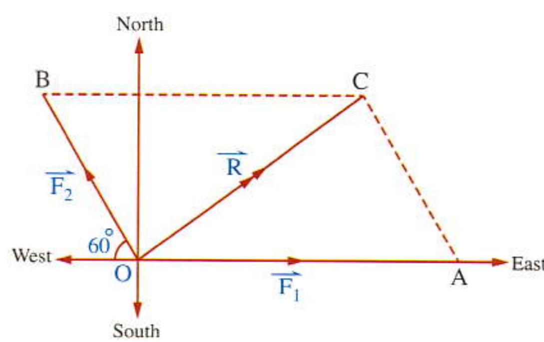
i.e. $\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2}$

Example 1

$\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ are two forces acting on the point O from a solid body , where F_1 = 500 newton and acts on the direction of East , F_2 = 300 newton and acts in the direction 60° North of west , find their resultant graphically.

Solution

* We use the drawing scale one cm. per 100 newton
* Draw $\overrightarrow{OA}$ to represent $\overrightarrow{F_1}$ and $\overrightarrow{OB}$ to represent $\overrightarrow{F_2}$ where $\| \overrightarrow{OA} \| = 5$ cm. , $\| \overrightarrow{OB} \| = 3$ cm.
* Then complete the parallelogram OACB , then $\overrightarrow{OC}$ represents the resultant $\overrightarrow{R}$

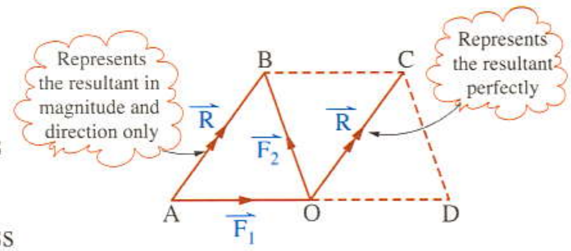


* By measuring, we find that $\|\overrightarrow{OC}\| = 4.4$ cm. approximately, $m(\angle AOC) = 37^\circ$

$\therefore$ $\overrightarrow{R}$ acts at O and its magnitude = 440 newton in the direction 37° North of east approximately.

Remark

If $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ act at the point O and if they are represented by the two vectors $\overrightarrow{AO}$ and $\overrightarrow{OB}$ as in the opposite figure, then according to the rule of addition of two vectors, $\overrightarrow{AB}$ represents the resultant of these two vectors. But the line of action of the resultant of $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ must pass through O

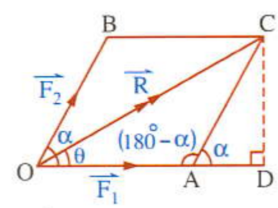


Therefore, we draw from O the directed line segment $\overrightarrow{OC}$ equivalent to $\overrightarrow{AB}$ which represents the resultant of these two forces perfectly.

Finding the resultant of two forces meeting at a point analytically

Let the two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ meet at O and α is the measure of the angle between the directions of the two forces.

If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$, then $\overrightarrow{OC}$ represents the resultant $\overrightarrow{R}$



Let θ be the measure of the angle between the resultant $\overrightarrow{R}$ and the force $\overrightarrow{F_1}$, then from our study of the cosine law in trigonometry, we can get the resultant of the two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ in magnitude and direction from the following relations :

$$R = \sqrt{(F_1)^2 + (F_2)^2 + 2F_1F_2\cos\alpha} \quad , \quad \tan\theta = \frac{F_2 \sin\alpha}{F_1 + F_2\cos\alpha}$$

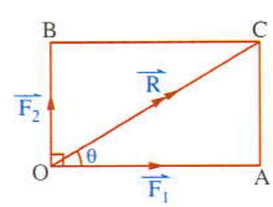
where F_1, F_2 and R are the magnitudes of $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ and $\overrightarrow{R}$

Special cases

1 If the two forces are perpendicular (*i.e.* $\alpha = 90^\circ$) :

$\therefore \cos\alpha = 0$, $\sin\alpha = 1$

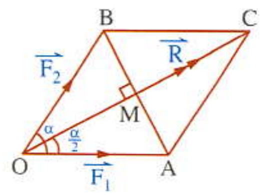
Substituting in the two previous relations, we get that :



$$R = \sqrt{F_1^2 + F_2^2} \quad , \quad \tan\theta = \frac{F_2}{F_1}$$

2 If the two forces are equal in magnitude (*i.e.* $F_1 = F_2 = F$) :

In this case , the parallelogram OACB exchanges to a rhombus , then :



$$R = OC = 2\,OM = 2\,OA \cos\frac{\alpha}{2}$$
$$= 2\,F \cos\frac{\alpha}{2}$$

i.e. $R = 2\,F\cos\frac{\alpha}{2}$, $\theta = \frac{\alpha}{2}$ (where $\overrightarrow{R}$ bisects the angle between the two forces)

Notice that : $\alpha = 120°$, So $R = F$

3 If the two forces have the same line of action and the same direction (*i.e.* $\alpha = 0°$) :

$\therefore \cos\alpha = 1$

Substituting :

$$\therefore R = \sqrt{F_1^{\,2} + F_2^{\,2} + 2F_1F_2 \times 1} = \sqrt{(F_1 + F_2)^2} = F_1 + F_2$$

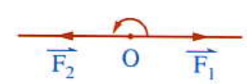
i.e. $R = F_1 + F_2$

and the direction of the resultant is the same direction of the line of action of the two forces.

∗ In this case , R is called the greatest or the maximum value of the resultant.

4 If the two forces have the same line of action but in opposite directions (*i.e.* $\alpha = 180°$) :

$\therefore \cos\alpha = -1$



Substituting :

$$\therefore R = \sqrt{F_1^{\,2} + F_2^{\,2} + 2\,F_1F_2 \times (-1)} = \sqrt{F_1^{\,2} + F_2^{\,2} - 2\,F_1F_2} = \sqrt{(F_1 - F_2)^2} = |\,F_1 - F_2\,|$$

i.e. $R = |\,F_1 - F_2\,|$

and the direction of the resultant is the direction of the greater force in magnitude.

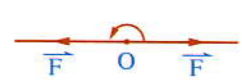
∗ In this case , R is called the smallest or the minimum value of the resultant.

5 If the two forces are equal in magnitude and have the same line of action but in opposite directions :

In this case : $F_1 = F_2 = F$, $\alpha = 180°$

$\therefore \cos\alpha = -1$ $\qquad\qquad \therefore R = \sqrt{F^2 + F^2 - 2\,F^2} = 0$

$\therefore$ $R = \text{zero}$

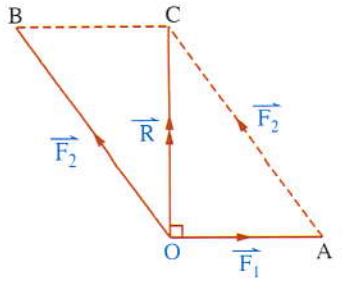


i.e. The resultant is zero vector.

6 If the resultant is perpendicular to the first force (*i.e.* $\theta = 90°$) :

$\because \theta = 90°$

$\therefore \boxed{R^2 = F_2^2 - F_1^2}$ (Pythagoras' theorem)



, $\cot \theta = 0$ $\qquad \therefore \dfrac{F_1 + F_2 \cos \alpha}{F_2 \sin \alpha} = 0$

$\therefore \boxed{F_1 + F_2 \cos \alpha = 0}$

$\therefore \cos \alpha = \dfrac{-F_1}{F_2}$

$\therefore \alpha$ is an obtuse angle , $F_1 < F_2$

i.e. When the resultant is perpendicular to one of the two forces it is perpendicular to the smallest force.

Example 2

Two forces of magnitudes 5 newton and 3 netwon act at a point and include an angle of measure 60°, find their resultant in magnitude and direction analytically.

Solution

$\because R = \sqrt{F_1^{\ 2} + F_2^{\ 2} + 2\, F_1 F_2 \cos \alpha}$

$\therefore R = \sqrt{25 + 9 + 2 \times 5 \times 3 \times \cos 60°} = 7$ newton.

$\because \tan \theta = \dfrac{F_2 \sin \alpha}{F_1 + F_2 \cos \alpha}$ $\qquad \therefore \tan \theta = \dfrac{3 \sin 60^\circ}{5 + 3 \cos 60^\circ} = \dfrac{3\sqrt{3}}{13}$

$\therefore \theta \simeq 21° \ 47'$

$\therefore$ The magnitude of $\overrightarrow{R}$ is 7 newton and include an angle of measure $21° \ 47'$ with the first force.

Example 3

Two perpendicular forces act at a point such that $F_1 = 6$ newton and $F_2 = 2.5$ newton. Find their resultant in magnitude and find its direction.

Solution

$\because R = \sqrt{F_1^{\ 2} + F_2^{\ 2}}$ $\qquad \therefore R = \sqrt{(6)^2 + (2.5)^2} = 6.5$ newton.

$\because \tan \theta = \dfrac{F_2}{F_1}$ $\qquad \therefore \tan \theta = \dfrac{2.5}{6} = \dfrac{5}{12}$ $\qquad \therefore \theta = 22° \ 37'$

$\therefore$ The magnitude of $\overrightarrow{R}$ is 6.5 newton and include an angle of measure $22° \ 37'$ with the first force.

Example 4

Two forces of magnitudes 50 newton and 100 newton act at a point. Their resultant is perpendicular to the first force. Find the measure of the angle included between the two forces and the magnitude of the resultant.

Solution

$F_1 = 50$ newton , $F_2 = 100$ newton.

$\because$ The resultant is perpendicular to the first force.

$\therefore F_1 + F_2 \cos \alpha = 0$ $\qquad \therefore 50 + 100 \cos \alpha = 0$

$\therefore \cos \alpha = \frac{-50}{100} = -\frac{1}{2}$ $\qquad \therefore \alpha = 120°$

$\because R = \sqrt{(F_1)^2 + (F_2)^2 + 2(F_1)(F_2)\cos\alpha}$

$\therefore R = \sqrt{(50)^2 + (100)^2 + 2 \times 50 \times 100 \cos 120°} = 50\sqrt{3}$ newton.

Another solution :

Let $\overrightarrow{OA}$ represents the force whose magnitude is 50 newton ,

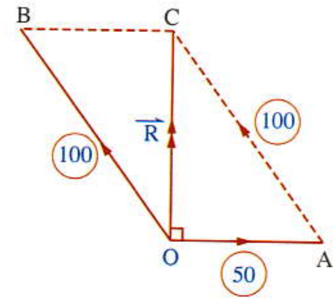
$\overrightarrow{OB}$ represents the force whose magnitude is 100 newton.

, $\because$ the resultant is perpendicular to the first force

$\therefore \Delta$ OAC is right-angled triangle at O

$\therefore \cos A = \frac{OA}{AC} = \frac{50}{100} = \frac{1}{2}$

$\therefore m(\angle A) = 60°$



$\therefore m(\angle AOB) = 120°$ and it is the measure of the angle between the two forces.

, $\because R^2 = (100)^2 - (50)^2$ $\qquad \therefore R = \sqrt{(100)^2 - (50)^2} = 50\sqrt{3}$ newton.

Example 5

Two forces act at a point. The greatest value of their resultant = 32 kg.wt. and the smallest value of their resultant is 12 kg.wt. Find the magnitude of each of them , then find the magnitude of their resultant if the measure of the included angle between them is 60°

Solution

Let the great force = F_1 and the small force = F_2

$\therefore F_1 + F_2 = 32$ (1) , $F_1 - F_2 = 12$ (2)

From (1) and (2) : $F_1 = 22$ kg.wt. and $F_2 = 10$ kg.wt.

If $\alpha = 60°$, then $R = \sqrt{(22)^2 + (10)^2 + 2 \times 22 \times 10 \cos 60°} = 2\sqrt{201}$ kg.wt.

Example 6

Two forces are equal in magnitude. The magnitude of their resultant is $70\sqrt{3}$ newton and the measure of the angle between them is 60°. Find the magnitude of each of the two forces.

Solution

$\because$ The two forces are equal in magnitude.

$\therefore R = 2F\cos\frac{\alpha}{2}$ $\qquad \therefore 70\sqrt{3} = 2F\cos 30°$ $\qquad \therefore F = 70$ newton.

$\therefore$ The two forces are 70 netwon and 70 netwon.

Example 7

Two forces of magnitude 6 and F kg.wt. act at a particle such that the measure of the angle between them is 135°
Find the magnitude of their resultant if the line of action of the resultant inclines by an angle of measure 45° with the force F

Solution

$$\because \tan\theta = \frac{F_1 \sin\alpha}{F_2 + F_1 \cos\alpha}$$

, where θ is the measure of the angle between the resultant and the force F

$\therefore \tan 45° = \dfrac{6\sin 135^\circ}{F + 6\cos 135^\circ}$ $\qquad \therefore 1 = \dfrac{3\sqrt{2}}{F - 3\sqrt{2}}$

$\therefore F - 3\sqrt{2} = 3\sqrt{2}$ $\qquad \therefore F = 6\sqrt{2}$ kg.wt.

$$, \because R = \sqrt{(F_1)^2 + (F_2)^2 + 2F_1F_2\cos\alpha}$$

$$\therefore R = \sqrt{(6)^2 + (6\sqrt{2})^2 + 2\times 6\times 6\sqrt{2}\cos 135°} = 6 \text{ kg.wt.}$$

Example 8

Two forces acting at a point. If their magnitudes are 4 F and 3 F , find the measure of the angle between them if the magnitude of the resultant is $\sqrt{13}$ F

Solution

$$\because R^2 = (F_1)^2 + (F_2)^2 + 2(F_1)(F_2)\cos\alpha$$

$$\therefore \left(\sqrt{13}\,F\right)^2 = (4F)^2 + (3F)^2 + 2\times 4F\times 3F\times\cos\alpha$$

$\therefore 13F^2 = 16F^2 + 9F^2 + 24F^2\cos\alpha$ $\qquad \therefore -12F^2 = 24F^2\cos\alpha$

$\therefore \cos\alpha = \dfrac{-12F^2}{24F^2} = -\dfrac{1}{2}$ $\qquad \therefore \alpha = 120°$

Example 9

Two forces of magnitude 7 kg.wt. and F kg.wt. act at a particle and the measure of the included angle between their directions is 120°

If the magnitude of their resultant is $7\sqrt{3}$ kg.wt.

Find the value of F and the measure of the angle which the resultant makes with the first force.

Solution

$\because R^2 = F_1^2 + F_2^2 + 2\,F_1F_2\cos\alpha$ $\therefore \left(7\sqrt{3}\right)^2 = (7)^2 + F^2 + 2\times 7\times F\cos 120°$

$\therefore 147 = 49 + F^2 - 7\,F$ $\therefore F^2 - 7\,F - 98 = 0$

$\therefore (F - 14)\,(F + 7) = 0$ $\therefore F = 14$ kg.wt.

$\because \tan\theta = \dfrac{F_2\sin\alpha}{F_1 + F_2\cos\alpha}$ $\therefore \tan\theta = \dfrac{14\sin 120°}{7 + 14\cos 120°} = \dfrac{7\sqrt{3}}{\text{zero}}$ «undefined»

$\therefore \cot\theta = \text{zero}$ $\therefore \theta = 90°$

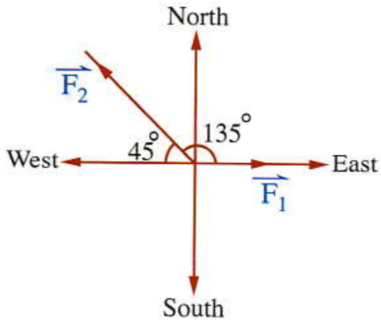
i.e. The resultant is perpendicular to the first force.

Example 10

Two forces of magnitudes 5 and $5\sqrt{2}$ kg.wt. act at a point.

The first towards East , the second is towards Western North. Prove that the magnitude of the resultant = the magnitude of the first force and find the measure of the angle which the resultant makes with each of the two forces.

Solution



$F_1 = 5$ kg.wt. , $F_2 = 5\sqrt{2}$ kg.wt.

From the figure $\alpha = 135°$

$\because R = \sqrt{(F_1)^2 + (F_2)^2 + 2F_1F_2\cos\alpha}$

$\therefore R = \sqrt{25 + 50 + 2\times 5\times 5\sqrt{2}\times\cos 135°}$

$\therefore R = 5$ kg.wt. $= F_1$, $\because \tan\theta = \dfrac{F_2\sin\alpha}{F_1 + F_2\cos\alpha}$

$\therefore \tan\theta = \dfrac{5\sqrt{2}\sin 135°}{5 + 5\sqrt{2}\cos 135°} = \dfrac{5}{\text{zero}}$ «undefined»

$\therefore \cot\theta = \text{zero}$ $\therefore \theta = 90°$

$\therefore \overrightarrow{R}$ is perpendicular to $\overrightarrow{F_1}$

i.e. Towards North and makes an angle of measure $135° - 90° = 45°$ with $\overrightarrow{F_2}$

Example 11

Two equal forces intersect at a point and the magnitude of their resultant equals 8 newton , if one of them is reversed , then the magnitude of their resultant equals 6 newtons. Find the magnitude of each force.

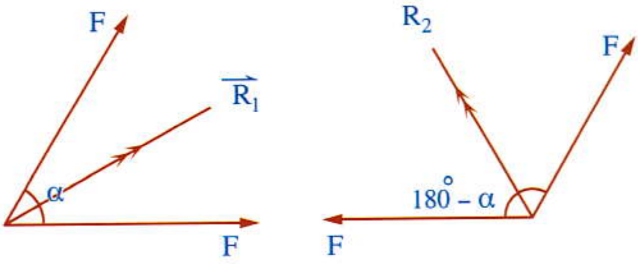
Solution

$R_1 = 2\,F \cos\frac{\alpha}{2} = 8$

$\therefore F \cos\frac{\alpha}{2} = 4 \qquad (1)$

$, R_2 = 2\,F \cos\left(\frac{180 - \alpha}{2}\right) = 6$

$\therefore F \sin\frac{\alpha}{2} = 3 \qquad (2)$



(First case) (Second case)

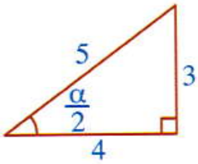
By squaring the two equations (1) , (2) and add :

$\therefore F^2 \cos^2\frac{\alpha}{2} + F^2 \sin^2\frac{\alpha}{2} = 16 + 9 \qquad \therefore F^2\left(\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}\right) = 25$

$\therefore F^2 = 25 \qquad \therefore F = 5$ newton

$\therefore$ The magnitude of each force 5 , 5 newton

- **Notice that the two equations can be solved as the following :**



Dividing equation (2) by equations (1) :

$\therefore \tan\frac{\alpha}{2} = \frac{3}{4} \qquad \therefore \sin\frac{\alpha}{2} = \frac{3}{5}$

, by substituting in equation (2) :

$\therefore F \times \frac{3}{5} = 3 \qquad \therefore F = 5$ newton.

$\therefore$ The magnitude of each force is 5 newton.

Another solution (Geometrically) :

$\because$ The two forces are equal.

$\therefore R_1 , R_2$ bisect the angle between the two forces.

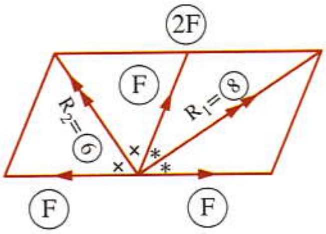
$\therefore R_1 \perp R_2$

$\therefore (R_1)^2 + (R_2^2) = (2\,F)^2$

$\therefore 64 + 36 = 4\,F^2$

$\therefore F^2 = 25 \qquad \therefore F = 5$

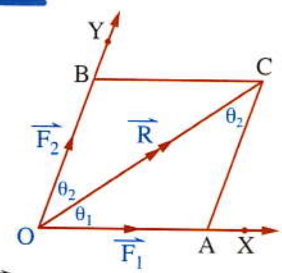
$\therefore$ The magnitude of the two forces are 5 , 5 newton.



Lesson 2

Forces resolution into two components

Resolution of a known force into two known directions



Suppose that the force $\overrightarrow{R}$ acts at a point O and it is required to resolve $\overrightarrow{R}$ into two components $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$. Let θ_1 and θ_2 be the measure of angles of inclination of $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ to the direction of $\overrightarrow{R}$

Therefore , we draw using a drawing scale the vector $\overrightarrow{OC}$ to represent the force $\overrightarrow{R}$, then we draw from O the two rays $\overrightarrow{OX}$ and $\overrightarrow{OY}$ making two angles θ_1 and θ_2 with $\overrightarrow{OC}$ and in different sides of it.

Then we draw from C two rays one is parallel to $\overrightarrow{OX}$ and the other is parallel to $\overrightarrow{OY}$ to get the parallelogram OACB as in the shown figure , thus the vector $\overrightarrow{OA}$ represents the component $\overrightarrow{F_1}$ and $\overrightarrow{OB}$ represents the component $\overrightarrow{F_2}$ and the vector $\overrightarrow{AC}$ represents $\overrightarrow{F_2}$ also

By using the sine rule on Δ OAC , where m ($\angle$ ACO) = θ_2

and sin ($\angle$ OAC) = sin [180° – ($\theta_1 + \theta_2$)] = sin ($\theta_1 + \theta_2$)

$$\therefore \quad \frac{F_1}{\sin \theta_2} = \frac{F_2}{\sin \theta_1} = \frac{R}{\sin(\theta_1 + \theta_2)}$$

i.e.

F_1 (the magnitude of the component of $\overrightarrow{R}$, which inclines by θ_1 on $\overrightarrow{R}$) $= \dfrac{R \sin \theta_2}{\sin(\theta_1 + \theta_2)}$

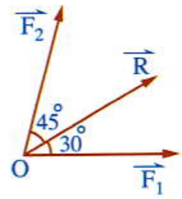
And

F_2 (the magnitude of the component of $\overrightarrow{R}$, which inclines by θ_2 on $\overrightarrow{R}$) $= \dfrac{R \sin \theta_1}{\sin(\theta_1 + \theta_2)}$

Example 1

Resolve the force of magnitude 20 newton into two components one of them inclined on the given force with an angle of measures 30° and the other force inclined by an angle of measure 45° on the other side of the force , then approximate the answer to the nearest one decimal.

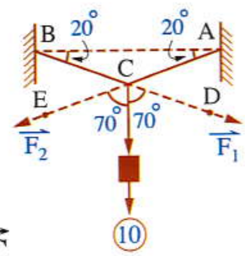
Solution



$$F_1 = \frac{R \sin \theta_2}{\sin(\theta_1 + \theta_2)} = \frac{20 \sin 45°}{\sin 75°} \simeq 14.6 \text{ newton.}$$

$$F_2 = \frac{R \sin \theta_1}{\sin(\theta_1 + \theta_2)} = \frac{20 \sin 30°}{\sin 75°} \simeq 10.4 \text{ newton.}$$

Example 2



In the opposite figure :

A light of weight 10 Newton is suspended by two strings $\overline{AC}$, $\overline{BC}$ fixed in two horizontal points with equal two angles , the measure of each of them is 20°

(1) Resolve the weight of the light in each of the two directions $\overrightarrow{AC}$, $\overrightarrow{BC}$

(2) What happen , if the magnitude of the components of the weight in the two directions of the strings , if its angle decreased with horizontal less than 20 , and what do you deduce to the magnitude of the component of the weight , when the string becomes horizontal.

Solution

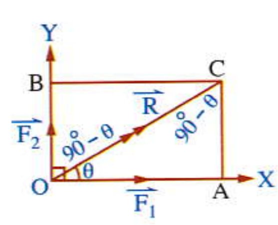
(1) The weight (10 newton) acts vertically downwards , and from the figure :

$$\frac{W_1}{\sin 70°} = \frac{W_2}{\sin 70°} = \frac{10}{\sin 140°}$$

$$\therefore W_1 = W_2 = \frac{10 \sin 70°}{\sin 140°} \simeq 15 \text{ newton.}$$

(2) If the measure of the angle decreased with horizontal less than 20° , then the magnitude of the component will increase to become unlimited when the strings are horizontal.

Resolution of the force into two perpendicular directions



Let the force $\overrightarrow{R}$ acts at the point O and we want to resolve this force into two perpendicular forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ such that $\overrightarrow{F_1}$ inclines by θ on the direction of $\overrightarrow{R}$

In this case , the parallelogram become a rectangle.

Applying the sine rule on Δ OAC we get :

$$\frac{F_1}{\sin(90^\circ - \theta)} = \frac{F_2}{\sin\theta} = \frac{R}{\sin 90^\circ}$$

$$\frac{F_1}{\cos\theta} = \frac{F_2}{\sin\theta} = \frac{R}{1} = R$$ Thus , $F_1 = R\cos\theta$, $F_2 = R\sin\theta$

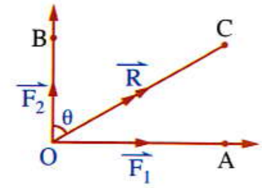
$\therefore$ F_1 (the magnitude of the component in the given direction) $= R\cos\theta$, and F_2 (the magnitude of the component in the perpendicular direction to the given direction) $= R\sin\theta$

The component $\overrightarrow{F_1}$ sometimes is called the projection of $\overrightarrow{R}$ in the direction of $\overrightarrow{OA}$ and the component $\overrightarrow{F_2}$ is called the projection of $\overrightarrow{R}$ in the direction of $\overrightarrow{OB}$

Remarks

(1) The magnitude of the component adjacent to the given angle = R cos (this angle) , the magnitude of the other perpendicular component to the previous component = R sin (this angle)

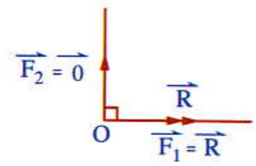
In the opposite figure :



If the component $\overrightarrow{F_2}$ inclines on the direction of $\overrightarrow{R}$ by an angle of measure θ , then

$F_2 = R\cos\theta$, $F_1 = R\sin\theta$

(2) The component of $\overrightarrow{R}$ in the same direction of $\overrightarrow{R}$ = The same force $\overrightarrow{R}$ and its component in the perpendicular direction to its direction $= \overrightarrow{0}$



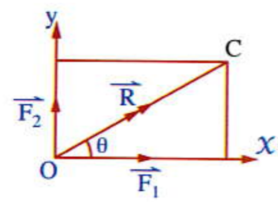
Because in this case , the measure of the angle between $\overrightarrow{R}$ and the first component = zero , then the magnitude of the first component $= R\cos 0^\circ = R \times 1 = R$

and the magnitude of the perpendicular component on the previous component $= R\sin 0^\circ = R \times 0 =$ zero

(3) If $\overrightarrow{i}$ and $\overrightarrow{j}$ are two perpendicular unit vectors in the directions $\overrightarrow{OX}$ and $\overrightarrow{OY}$ where O is the origin point.

Then $\overrightarrow{F_1} = (R\cos\theta)\,\overrightarrow{i}$, $\overrightarrow{F_2} = (R\sin\theta)\,\overrightarrow{j}$

$\therefore \overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} = (R\cos\theta)\,\overrightarrow{i} + (R\sin\theta)\,\overrightarrow{j}$

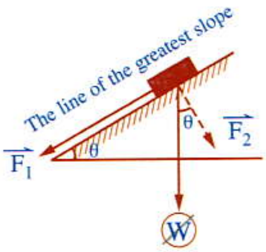


(4) If $\overrightarrow{F} = (F , \theta)$, then $\overrightarrow{F} = F \cos\theta \, \overrightarrow{i} + F \sin\theta \, \overrightarrow{j}$

(5) If $\theta \in \left]0 , \frac{\pi}{2}\right[$, then the magnitude of the two components $(R \cos\theta)$, $(R \sin\theta)$ is less than the magnitude of the force (R) because $\theta \in \left]0 , \frac{\pi}{2}\right[$

, thus $0 < \sin\theta < 1$, $0 < \cos\theta < 1$

(6) If a body of weight (w) is placed on a smooth inclined plane with the horizontal by an angle (θ), then we can resolve the weight (w) which acts vertically downwards into two components.



* F_1 (The magnitude of the component in the direction of the line of the greatest slope)

$= w \sin\theta$

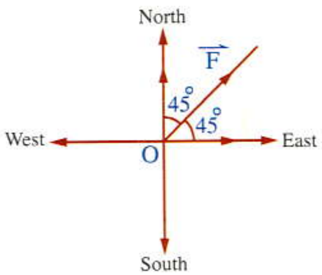
* F_2 (The magnitude of the component in the perpendicular direction on the plane)

$= w \cos\theta$

Example 3

Resolve the force of magnitude $8\sqrt{2}$ newton which acts at the point O in the direction of Eastern North into two components. One of them in the East direction and the other in the North direction.

Solution



$\because$ The two components inclined on the direction of the force by angles of measures 45° and 45° , then they are perpendicular.

$\therefore$ The magnitude of the component which is in the East direction $= F \cos 45° = 8\sqrt{2} \times \frac{1}{\sqrt{2}} = 8$ newton.

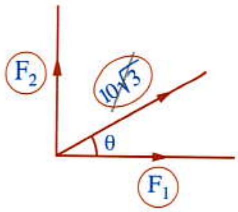
The magnitude of the component in North direction

$= F \sin 45° = 8\sqrt{2} \times \frac{1}{\sqrt{2}} = 8$ newton.

Example 4

A force of magnitude $10\sqrt{3}$ kg.wt. was resolved into two perpendicular components , one of them is of magnitude 15 kg.wt. Find the magnitude of the other component.

Solution



Let the direction of the given component $(\overrightarrow{F_1})$ inclines on the direction of the force by an angle of measure θ

$\therefore$ The magnitude of this component $\overrightarrow{F_1} = 10\sqrt{3}\cos\theta$

$\therefore 15 = 10\sqrt{3}\cos\theta \qquad \therefore \cos\theta = \frac{15}{10\sqrt{3}} = \frac{\sqrt{3}}{2} \qquad \therefore \theta = 30°$

The magnitude of the other component $F_2 = 10\sqrt{3}\sin 30° = 10\sqrt{3} \times \frac{1}{2} = 5\sqrt{3}$ kg.wt.

Another solution :

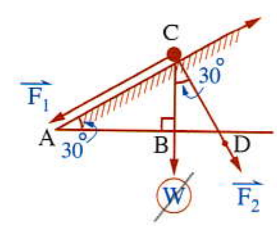
$\because R = \sqrt{F_1^2 + F_2^2} \qquad \therefore \left(10\sqrt{3}\right)^2 = (15)^2 + F_2^2$

$\therefore F_2^2 = 75 \qquad \therefore F_2 = 5\sqrt{3}$ kg. wt.

Example 5

A body of weight 50 newton is placed on a smooth inclined plane by 30° with the horizontal. Find the two components of the weight in the direction of the line of the greatest slope and the direction perpendicular to it.

Solution



From the figure , we notice that : m $(\angle BCD) = (\angle BAC) = 30°$

$\therefore F_1$ = (the magnitude of the component in the direction of the line of the greatest slope)

$= W\sin 30° = 50 \times \frac{1}{2} = 25$ newton.

, F_2 = (the magnitude of the component in the perpendicular direction to the plane)

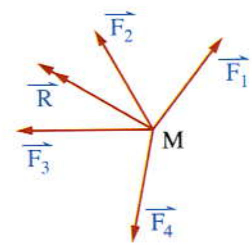
$= W\cos 30° = 50 \times \frac{\sqrt{3}}{2} = 25\sqrt{3}$ newton.

Lesson

3

The resultant of coplanar forces meeting at a point

1 The geometrical method

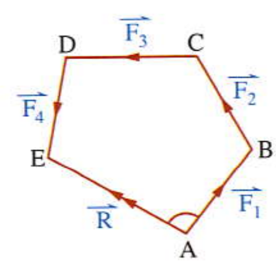


Suppose that the system of coplanar forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$, $\overrightarrow{F_4}$ acts at point M as in the opposite figure.

To find the resultant of this forces :

* Use an appropriate drawing scale.
* From any point as A draw the vector $\overrightarrow{AB}$ to represent $\overrightarrow{F_1}$ (in magnitude and direction)
* From point B draw the vector $\overrightarrow{BC}$ to represent $\overrightarrow{F_2}$
* From point C draw the vector $\overrightarrow{CD}$ to represent $\overrightarrow{F_3}$
* At last , from point D draw the vector $\overrightarrow{DE}$ to represent $\overrightarrow{F_4}$

  Match the first point (A) to the last point (E) to be the vector $\overrightarrow{AE}$ which represent the resultant ($\overrightarrow{R}$) in magnitude and direction , where $\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} + \overrightarrow{F_4}$



* Find the length of $\overline{AE}$ and m ($\angle$ EAB) assume it is the included angle between the resultant and the first force and using the drawing scale we can find the magnitude and the direction of $\overrightarrow{R}$
* Then the resultant of the set of forces is a force of magnitude R and acts at point M in direction $\overrightarrow{AE}$

Notice that :

The vector $\overrightarrow{AE}$ which represent $\overrightarrow{R}$ has an opposite direction to the other directions of vectors which represent the forces and the polygon ABCDE is called "The force polygon"

Remark

If the first and last points are congruent in the force polygon , then $(\overrightarrow{R}) = \overrightarrow{O}$ and the set of forces are equilibrium.

i.e. The adjusted and sufficient condition to equilibrium a set of concurrent forces is a representing of these forces geometrically by the sides of a closed polygon taken in the same direction.

2 The analytical method

Suppose that the system of coplanar forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$, ... , $\overrightarrow{F_n}$ meet at the point O and the point O is the origin point of a coplanar cartesian axis.

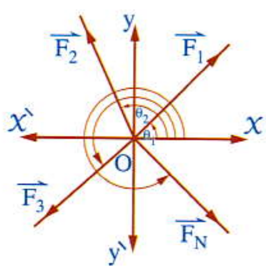
and θ_1 , θ_2 , θ_3 , ... , θ_n are the polar angles of the forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, F_3 , ... , $\overrightarrow{F_n}$ respectively , let $\vec{i}$ and $\vec{j}$ be the fundamental unit vectors in the two directions $\overrightarrow{OX}$ and $\overrightarrow{OY}$, then :

$\overrightarrow{F_1} = (F_1 , \theta_1) = F_1 \cos \theta_1 \vec{i} + F_1 \sin \theta_1 \vec{j}$,

$\overrightarrow{F_2} = (F_2 , \theta_2) = F_2 \cos \theta_2 \vec{i} + F_2 \sin \theta_2 \vec{j}$,

$\overrightarrow{F_3} = (F_3 , \theta_3) = F_3 \cos \theta_3 \vec{i} + F_3 \sin \theta_3 \vec{j}$

and so on till : $\overrightarrow{F_n} = (F_n , \theta_n) = F_n \cos \theta_n \vec{i} + F_n \sin \theta_n \vec{j}$,



$\because \overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} + \cdots + \overrightarrow{F_n}$

, then by adding we get :

$$\overrightarrow{R} = (F_1 \cos \theta_1 \vec{i} + F_1 \sin \theta_1 \vec{j}) + (F_2 \cos \theta_2 \vec{i} + F_2 \sin \theta_2 \vec{j})$$

$$+ (F_3 \cos \theta_3 \vec{i} + F_3 \sin \theta_3 \vec{j}) + \cdots + (F_n \cos \theta_n \vec{i} + F_n \sin \theta_n \vec{j})$$

$$= (F_1 \cos \theta_1 + F_2 \cos \theta_2 + F_3 \cos \theta_3 + \cdots + F_n \cos \theta_n) \vec{i}$$

$$+ (F_1 \sin \theta_1 + F_2 \sin \theta_2 + F_3 \sin \theta_3 + \cdots + F_n \sin \theta_n) \vec{j}$$

i.e. $$\overrightarrow{R} = \left(\sum_{r=1}^{n} F_r \cos \theta_r \right) \vec{i} + \left(\sum_{r=1}^{n} F_r \sin \theta_r \right) \vec{j}$$

The expression $\left(\sum_{r=1}^{n} F_r \cos \theta_r \right)$ is called the algebraic sum of the components in the direction $\overrightarrow{OX}$ and is denoted by X and the expression $\left(\sum_{r=1}^{n} F_r \sin \theta_r \right)$ is called the algebraic sum of the components in the direction $\overrightarrow{OY}$ and is denoted by Y Hence , we can write the previous equation in the form : $\overrightarrow{R} = X \vec{i} + Y \vec{j}$

And let R be the magnitude of $\overrightarrow{R}$ and α is the measure of the polar angle of the resultant $\overrightarrow{R}$, then :

$R = \sqrt{X^2 + Y^2}$ and $\tan \alpha = \frac{Y}{X}$ where $\overrightarrow{R} = (R , \alpha)$

Remarks

(1) Notice the difference between X and $\overrightarrow{i}$:

- X is the algebraic sum of the components of forces in the direction of $\overrightarrow{OX}$
- $\overrightarrow{i}$ is the fundamental unit vector in the direction of $\overrightarrow{OX}$

(2) If $X = \text{zero}$, then $\overrightarrow{R} = Y\overrightarrow{j}$

and $\theta = 90°$, if $\overrightarrow{R}$ in the direction $\overrightarrow{OY}$

, $\theta = 270°$, if $\overrightarrow{R}$ in the direction $\overrightarrow{OY'}$

(3) If $Y = \text{zero}$, then $\overrightarrow{R} = X\overrightarrow{i}$

and $\theta = 0°$, if $\overrightarrow{R}$ in the direction $\overrightarrow{OX}$

, $\theta = 180°$, if $\overrightarrow{R}$ in the direction $\overrightarrow{OX'}$

(4) If $X = \text{zero}$ and $Y = \text{zero}$, then $\overrightarrow{R} = \overrightarrow{O}$

In this case , the set of coplanar concurrent forces are in equilibrium.

(5) To determine the direction of the resultant, consider that :

x	y	quad.	θ
+	+	1^{st}	measure of the acute angle
–	+	2^{nd}	180° – measure of the acute angle
–	–	3^{rd}	180° + measure of the acute angle
+	–	4^{th}	360° – measure of the acute angle

(6) The resultant of set of forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$

is $\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$, and if $\overrightarrow{R} = \overrightarrow{O}$, then the set of forces are equilibrium

For Example :

If $\overrightarrow{F_1} = 5\overrightarrow{i} + 2\overrightarrow{j}$, $\overrightarrow{F_2} = -6\overrightarrow{i} + 3\overrightarrow{j}$

and $\overrightarrow{F_3} = \overrightarrow{i} - 5\overrightarrow{j}$, then $\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \overrightarrow{O}$

$\therefore$ The forces are in equilibrium.

Example 1

If the forces $\overrightarrow{F_1} = 5\,\overrightarrow{i} - 4\,\overrightarrow{j}$, $\overrightarrow{F_2} = -6\,\overrightarrow{i} + a\,\overrightarrow{j}$ and $\overrightarrow{F_3} = b\,\overrightarrow{i} + 7\,\overrightarrow{j}$ are meeting at a point and are in equilibrium. Find the value of each of : a and b

Solution

$\because$ The forces are in equilibrium $\qquad \therefore \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \overrightarrow{O}$

$\therefore (5\,\overrightarrow{i} - 4\,\overrightarrow{j}) + (-6\,\overrightarrow{i} + a\,\overrightarrow{j}) + (b\,\overrightarrow{i} + 7\,\overrightarrow{j}) = \overrightarrow{O} \qquad \therefore (-1 + b)\,\overrightarrow{i} + (3 + a)\,\overrightarrow{j} = \overrightarrow{O}$

$\therefore -1 + b = 0 \qquad \therefore b = 1$

$, 3 + a = 0 \qquad \therefore a = -3$

Example 2

In each of the following three figures , a set of forces meeting at a point O and their magnitudes are in newton unit.

Determine the magnitude and the direction of the resultant of each of them.

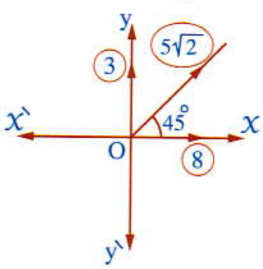


Fig. (1)

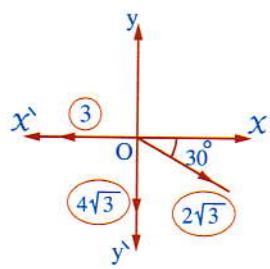


Fig. (2)

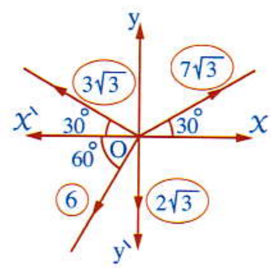


Fig. (3)

Solution

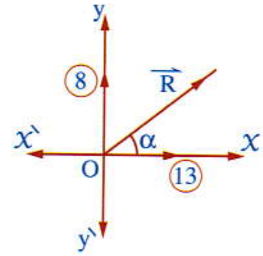
In Fig. (1) : The three forces whose magnitudes are 8 , $5\sqrt{2}$ and 3 newton and their polar angles are of measures 0° , 45° and 90° respectively.

- The algebraic sum of the components in the direction of $\overrightarrow{OX}$ is

$$X = 8\cos 0^\circ + 5\sqrt{2}\cos 45^\circ + 3\cos 90^\circ$$
$$= 8 \times 1 + 5\sqrt{2} \times \frac{1}{\sqrt{2}} + 3 \times \text{zero} = 8 + 5 + 0 = 13 \text{ newton.}$$

- The algebraic sum of the components in the direction of $\overrightarrow{OY}$ is

$$Y = 8\sin 0^\circ + 5\sqrt{2}\sin 45^\circ + 3\sin 90^\circ$$
$$= 8 \times \text{zero} + 5\sqrt{2} \times \frac{1}{\sqrt{2}} + 3 \times 1 = \text{zero} + 5 + 3 = 8 \text{ newton.}$$



$\therefore \overrightarrow{R} = 13\,\overrightarrow{i} + 8\,\overrightarrow{j}$, then $R = \sqrt{X^2 + Y^2} = \sqrt{169 + 64} = \sqrt{233}$

$\therefore R \simeq 15.264$ newton , $\tan\alpha = \frac{Y}{X} = \frac{8}{13}$

$\because X > 0 , Y > 0$

$\therefore \overrightarrow{R}$ lies in the first quadrant , using the calculator. $\qquad \therefore \alpha \simeq 31^\circ\, 36'$

In Fig. (2) : The three forces whose magnitudes are 3 , $4\sqrt{3}$ and $2\sqrt{3}$ newton and their polar angles are of measures 180° , 270° and 330° respectively.

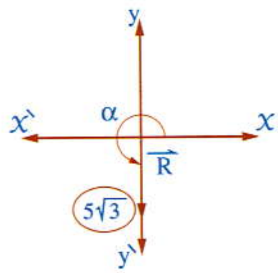
$$X = 3\cos 180° + 4\sqrt{3}\cos 270° + 2\sqrt{3}\cos 330°$$

$$= 3 \times (-1) + 4\sqrt{3} \times 0 + 2\sqrt{3} \times \frac{\sqrt{3}}{2} = -3 + 0 + 3 = \text{zero}$$

$$Y = 3\sin 180° + 4\sqrt{3}\sin 270° + 2\sqrt{3}\sin 330°$$

$$= 3 \times 0 + 4\sqrt{3} \times (-1) + 2\sqrt{3} \times \left(-\frac{1}{2}\right)$$

$$= 0 - 4\sqrt{3} - \sqrt{3} = -5\sqrt{3} \text{ newton.}$$



$\therefore \vec{R} = -5\sqrt{3}\ \vec{j}$

, then $R = 5\sqrt{3}$ newton and $\alpha = 270°$

In Fig. (3) : Four forces of magnitudes $7\sqrt{3}$, $3\sqrt{3}$, 6 and $2\sqrt{3}$ newton and their polar angles are of measures 30° , 150° , 240° and 270° respectively.

$$X = 7\sqrt{3}\cos 30° + 3\sqrt{3}\cos 150° + 6\cos 240° + 2\sqrt{3}\cos 270°$$

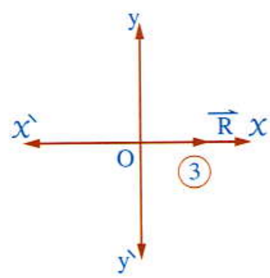
$$= 7\sqrt{3} \times \frac{\sqrt{3}}{2} + 3\sqrt{3} \times \left(-\frac{\sqrt{3}}{2}\right) + 6 \times \left(-\frac{1}{2}\right) + 2\sqrt{3} \times 0$$

$$= 10.5 - 4.5 - 3 + 0 = 3 \text{ newton ,}$$

$$Y = 7\sqrt{3}\sin 30° + 3\sqrt{3}\sin 150° + 6\sin 240° + 2\sqrt{3}\sin 270°$$

$$= 7\sqrt{3} \times \frac{1}{2} + 3\sqrt{3} \times \frac{1}{2} + 6 \times \left(-\frac{\sqrt{3}}{2}\right) + 2\sqrt{3} \times (-1)$$

$$= \frac{7\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} - 3\sqrt{3} - 2\sqrt{3} = \text{zero}$$



$\therefore \vec{R} = 3\,\vec{i}$, then R = 3 newton , α = zero

Another solution for the figure (3) :

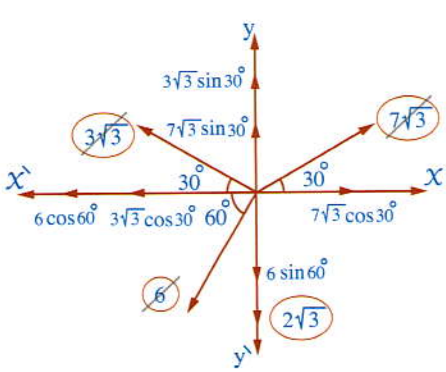
Using the analyzing of the forces into two perpendicular directions :

$$\therefore X = 7\sqrt{3}\cos 30° - 3\sqrt{3}\cos 30° - 6\cos 60°$$

$$= 7\sqrt{3} \times \frac{\sqrt{3}}{2} - 3\sqrt{3} \times \frac{\sqrt{3}}{2} - 6 \times \frac{1}{2} = 3 \text{ newton.}$$

$$, Y = 3\sqrt{3}\sin 30° + 7\sqrt{3}\sin 30° - 6\sin 60° - 2\sqrt{3}$$

$$= 3\sqrt{3} \times \frac{1}{2} + 7\sqrt{3} \times \frac{1}{2} - 6 \times \frac{\sqrt{3}}{2} - 2\sqrt{3} = 0$$



$$\therefore R = \sqrt{(3)^2 + (0)^2} = 3 \text{ newton.}$$

$$, \tan\alpha = \frac{Y}{X} = \frac{0}{3} = 0 \qquad \therefore \alpha = 0°$$

Example 3

Five coplanar forces meeting at a point , their magnitudes are 12 , 9 , $5\sqrt{2}$, $7\sqrt{2}$ and 7 kg.wt. , act in the directions : East , North , Western North , Western South and South respectively. Prove that the set of these forces are in equilibrium.

Solution

$\because$ The forces are $(12 , 0°)$, $(9 , 90°)$, $(5\sqrt{2} , 135°)$, $(7\sqrt{2} , 225°)$, $(7 , 270°)$

$\therefore X = 12\cos 0° + 9\cos 90° + 5\sqrt{2}\cos 135° + 7\sqrt{2}\cos 225° + 7\cos 270°$

$= 12 \times 1 + \text{zero} + 5\sqrt{2} \times \left(-\frac{1}{\sqrt{2}}\right) + 7\sqrt{2} \times \left(-\frac{1}{\sqrt{2}}\right) + \text{zero}$

$= 12 - 5 - 7 = \text{zero}$

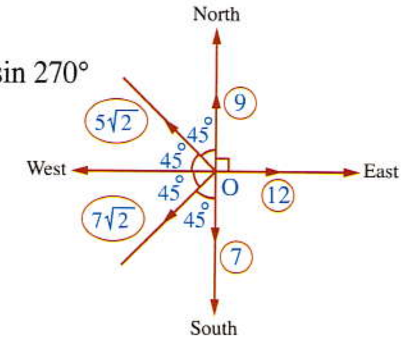
, $Y = 12\sin 0° + 9\sin 90° + 5\sqrt{2}\sin 135° + 7\sqrt{2}\sin 225° + 7\sin 270°$

$= \text{zero} + 9 + 5\sqrt{2} \times \frac{1}{\sqrt{2}} + 7\sqrt{2} \times \left(-\frac{1}{\sqrt{2}}\right) + 7 \times -1$

$= 9 + 5 - 7 - 7 = \text{zero}$

$\therefore X = \text{zero}$, $Y = \text{zero}$ $\qquad \therefore \overrightarrow{R} = \overrightarrow{O}$

$\therefore$ The set of forces are in equilibrium.



Example 4

Four coplanar forces meeting at a point and their magnitudes are F , 2 F , $3\sqrt{3}$ F and 4 F kg.wt. The measure of the angle between the first and second forces is 60° and between the second and the third is 90° and between the third and the fourth is 150° Find the magnitude and the direction of $\overrightarrow{R}$

Solution

Let $\overrightarrow{OX}$ is the direction of the first force

, then the forces in polar form are $(F , 0°)$, $(2F , 60°)$, $(3\sqrt{3}F , 150°)$, $(4F , 300°)$ respectively.

$\therefore X = F\cos 0° + 2F\cos 60° + 3\sqrt{3}F\cos 150° + 4F\cos 300°$

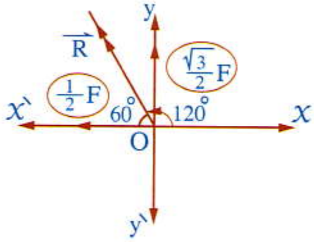
$= F \times 1 + 2F \times \frac{1}{2} + 3\sqrt{3}F \times \left(-\frac{\sqrt{3}}{2}\right) + 4F \times \frac{1}{2}$

$= F + F - \frac{9}{2}F + 2F = -\frac{1}{2}F$,

$Y = F\sin 0° + 2F\sin 60° + 3\sqrt{3}F\sin 150° + 4F\sin 300°$

$= F \times 0 + 2F \times \frac{\sqrt{3}}{2} + 3\sqrt{3}F \times \frac{1}{2} + 4F \times \left(-\frac{\sqrt{3}}{2}\right)$

$= 0 + \sqrt{3}F + \frac{3\sqrt{3}}{2}F - 2\sqrt{3}F = \frac{\sqrt{3}}{2}F$



$\vec{R} = -\frac{1}{2} F \vec{i} + \frac{\sqrt{3}}{2} F \vec{j}$ $\qquad \therefore R = \sqrt{\frac{1}{4} F^2 + \frac{3}{4} F^2} = \sqrt{F^2} = F$

$\therefore R = F$, $\tan \alpha = \frac{Y}{X} = \frac{\sqrt{3} F}{2} \times -\frac{2}{F} = -\sqrt{3}$

$\because X < 0$, $Y > 0$ $\qquad \therefore \alpha = 180° - 60° = 120°$

i.e. The resultant magnitude is F and its direction between 2^{nd} and 3^{rd} forces making an angle of measure 30° with the 3^{rd} force.

* Try to solve this example using the analyzing of the forces into two perpendicular directions.

Example 5

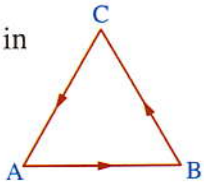
Three forces of magnitudes 2 F , 4 F , 6 F act at a point in directions parallel to the sides of an equilateral triangle in the same cyclic order. Find the magnitude and the direction of the resultant.

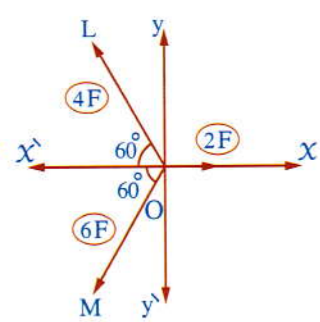
Solution

Let the forces act at the point O in the directions $\overrightarrow{OX}$, $\overrightarrow{OL}$, $\overrightarrow{OM}$ which are parallel to the directions $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ in the equilateral triangle ABC

, then the forces in the polar form are :

(2 F , 0°) , (4 F , 120°) , (6 F , 240°)





$\therefore X = 2 F \cos 0° + 4 F \cos 120° + 6 F \cos 240°$

$= 2 F \times 1 + 4 F \times \left(-\frac{1}{2}\right) + 6 F \times \left(-\frac{1}{2}\right) = -3 F$,

$Y = 2 F \sin 0° + 4 F \sin 120° + 6 F \sin 240°$

$= 2 F \times 0 + 4 F \times \left(\frac{\sqrt{3}}{2}\right) + 6 F \times \left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3} F$

$\therefore \vec{R} = -3 F \vec{i} - \sqrt{3} F \vec{j}$

$\therefore R = \sqrt{X^2 + y^2} = \sqrt{(-3 F)^2 + \left(-\sqrt{3} F\right)^2} = \sqrt{12 F^2} = 2\sqrt{3} F$,

$\tan \alpha = \frac{Y}{X} = \frac{-\sqrt{3} F}{-3 F} = \frac{1}{\sqrt{3}}$

, $\because$ X and Y are negative , then $\alpha = 210°$

i.e. Resultant magnitude is $2\sqrt{3}$ F and its direction between the two forces of magnitudes 6 F , 4 F making an angle of measure 30° with the force 6 F

* Try to solve this example using the resolution of the forces into two perpendicular directions.

Example 6

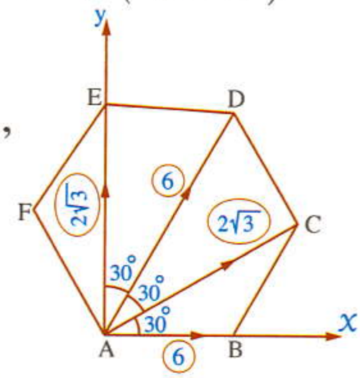
ABCDEF is a regular hexagon. Forces of magnitudes 6 , $2\sqrt{3}$, 6 , $2\sqrt{3}$ newton act along $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ and $\overrightarrow{AE}$ respectively.

Find the magnitude and the direction of the resultant of these forces.

Solution

Suppose $\overrightarrow{OX}$ is the direction of the first force.

Then the polar form of the forces are $(6 , 0°)$, $\left(2\sqrt{3} , 30°\right)$, $(6 , 60°)$, $\left(2\sqrt{3} , 90°\right)$



$\therefore X = 6 \cos 0° + 2\sqrt{3} \cos 30° + 6 \cos 60° + 2\sqrt{3} \cos 90°$

$= 6 \times 1 + 2\sqrt{3} \times \left(\frac{\sqrt{3}}{2}\right) + 6 \times \frac{1}{2} + 2\sqrt{3} \times 0 = 12$ newton ,

$Y = 6 \sin 0° + 2\sqrt{3} \sin 30° + 6 \sin 60° + 2\sqrt{3} \sin 90°$

$= 6 \times 0 + 2\sqrt{3} \times \frac{1}{2} + 6 \times \frac{\sqrt{3}}{2} + 2\sqrt{3} \times 1 = 6\sqrt{3}$ newton.

$\therefore \vec{R} = 12\,\vec{i} + 6\sqrt{3}\,\vec{j}$ $\qquad \therefore R = \sqrt{X^2 + Y^2}$

$R = \sqrt{(12)^2 + \left(6\sqrt{3}\right)^2} = 6\sqrt{7}$ newton , $\tan \alpha = \frac{Y}{X} = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$

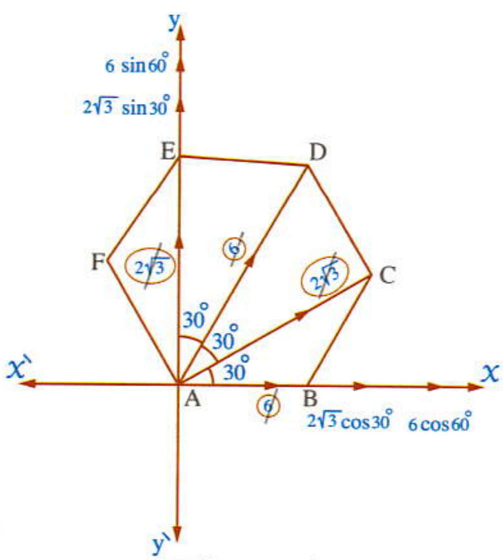
, $\because X > 0$, $Y > 0$ $\qquad \therefore \alpha = 40° \, 53' \, 36''$

i.e. The resultant magnitude is $6\sqrt{7}$ N.

and its direction between $\overrightarrow{AC}$ and $\overrightarrow{AD}$ making an angle of measure $10° \, 53' \, 36''$ with $\overrightarrow{AC}$

Another solution :

Using the resolution of the forces into two perpendicular directions :



$\therefore X = 6 \cos 60° + 2\sqrt{3} \cos 30° + 6$

$= 6 \times \frac{1}{2} + 2\sqrt{3} \times \frac{\sqrt{3}}{2} + 6 = 12$ newton.

, $Y = 6 \sin 60° + 2\sqrt{3} \sin 30° + 2\sqrt{3}$

$= 6 \times \frac{\sqrt{3}}{2} + 2\sqrt{3} \times \frac{1}{2} + 2\sqrt{3} = 6\sqrt{3}$ newton.

$\therefore R = \sqrt{(12)^2 + \left(6\sqrt{3}\right)^2} = 6\sqrt{7}$ newton.

, $\tan \alpha = \frac{Y}{X} = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$ $\qquad \therefore \alpha = 40° \, 53' \, 36''$

i.e. The resultant magnitude is $6\sqrt{7}$ N. and its direction between $\overrightarrow{AC}$ and $\overrightarrow{AD}$ making an angle of measure $10° \, 53' \, 36''$ with $\overrightarrow{AC}$

Example 7

Four coplanar forces meeting at a point their magnitudes are F_1 , $6\sqrt{2}$, $8\sqrt{2}$, F_2 gm.wt. The first force in the East direction , the second force is in the direction of Eastern North , the third is in the direction of Western North and the fourth acts in South direction.

If their resultant is 7 gm.wt. in magnitude and acts in the East direction , find the value of F_1 and F_2

Solution

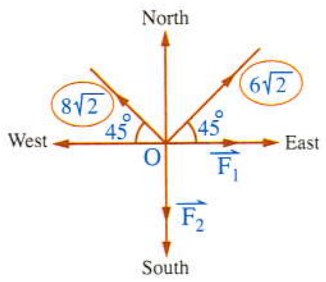
The magnitude of the resultant is 7 gm. wt. and acts towards East

$\therefore X = 7$ and $Y =$ zero

$\therefore F_1 \cos 0° + 6\sqrt{2} \cos 45° + 8\sqrt{2} \cos 135° + F_2 \cos 270° = 7$

$\therefore F_1 \times 1 + 6\sqrt{2} \times \frac{1}{\sqrt{2}} + 8\sqrt{2} \times -\frac{1}{\sqrt{2}} + F_2 \times 0 = 7$

$\therefore F_1 + 6 - 8 + 0 = 7$ $\qquad \therefore F_1 = 9$ gm. wt.



$F_1 \sin 0° + 6\sqrt{2} \sin 45° + 8\sqrt{2} \sin 135° + F_2 \sin 270° = 0$

$\therefore F_1 \times 0 + 6\sqrt{2} \times \frac{1}{\sqrt{2}} + 8\sqrt{2} \times \frac{1}{\sqrt{2}} + F_2 \times (-1) = 0$

$\therefore 6 + 8 - F_2 = 0$ $\qquad \therefore F_2 = 14$ gm. wt.

* Try to solve this example using the resolution of the forces into two perpendicular directions.

Example 8

Five coplanar forces meeting at a point their magnitudes are F , 9 , $5\sqrt{2}$, $7\sqrt{2}$, K (kg.wt.) The measure of the angle between the first force and the second force is 90° , between the second and the third is 45° , between the third and the fourth is 90° and between the fourth and the fifth 45° If the system of forces is in equilibrium , find the value of F and K

Solution

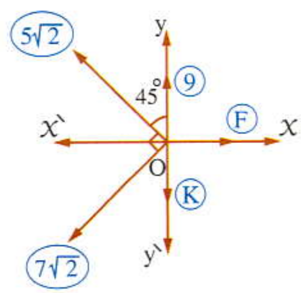
Let $\overrightarrow{OX}$ is the direction of the first force

, then the forces in the polar form are :

$(F , 0°)$, $(9 , 90°)$, $(5\sqrt{2} , 135°)$, $(7\sqrt{2} , 225°)$, $(K , 270°)$

, $\because$ the forces are in equilibrium

$\therefore X = Y = 0$



$\because X = F\cos 0° + 9\cos 90° + 5\sqrt{2}\cos 135° + 7\sqrt{2}\cos 225° + K\cos 270°$

$\therefore F + 9\times 0 + 5\sqrt{2}\times -\frac{1}{\sqrt{2}} + 7\sqrt{2}\times -\frac{1}{\sqrt{2}} + K\times 0 = 0$ $\therefore F = 12$

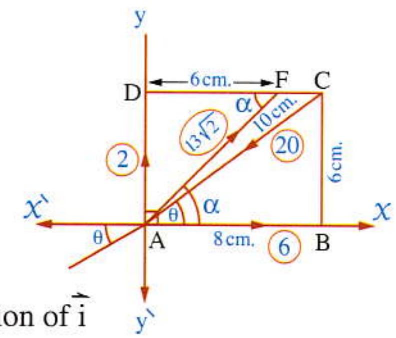
$\because Y = F\sin 0° + 9\sin 90° + 5\sqrt{2}\sin 135° + 7\sqrt{2}\sin 225° + K\sin 270°$

$\therefore 12\times 0 + 9\times 1 + 5\sqrt{2}\times\frac{1}{\sqrt{2}} + 7\sqrt{2}\times -\frac{1}{\sqrt{2}} + K\times -1 = \text{zero}$ $\therefore K = 7$

Example 9

ABCD is a rectangle in which AB = 8 cm. , BC = 6 cm. , $F\in\overline{CD}$ where FD = 6 cm. The forces of magnitudes 6 , 20 , $13\sqrt{2}$ and 2 newton act along $\overrightarrow{AB}$, $\overrightarrow{CA}$, $\overrightarrow{AF}$, $\overrightarrow{AD}$ repectively. Find the magnitude and the direction of the resultant of these forces.

Solution



In Δ ABC : $\because (AC)^2 = 6^2 + 8^2 = 100$ $\therefore$ AC = 10 cm.

$\therefore \sin\theta = \frac{6}{10} = \frac{3}{5}$, $\cos\theta = \frac{8}{10} = \frac{4}{5}$

$\because \Delta$ AFD is an isosceles triangle

$\therefore m(\angle AFD) = 45°$ *i.e.* $\alpha = 45°$

Suppose $\overrightarrow{AB}$ is the direction of the first force and in the direction of $\vec{i}$

$\therefore$ The measures of the polar angles of the forces are zero° , $180° + \theta$, α , $90°$ respectively

$\therefore X = 6\cos 0° + 20\cos(180° + \theta) + 13\sqrt{2}\cos\alpha + 2\cos 90°$

$= 6\times\cos 0° + 20(-\cos\theta) + 13\sqrt{2}\times\cos 45° + 2\cos 90°$

$= 6\times 1 - 20\times\frac{4}{5} + 13\sqrt{2}\times\frac{1}{\sqrt{2}} + 2\times 0 = 6 - 16 + 13 = 3$ newton.

$, Y = 6\times\sin 0° + 20\sin(180° + \theta) + 13\sqrt{2}\sin\alpha + 2\sin 90°$

$= 6\times\sin 0° - 20\sin\theta + 13\sqrt{2}\sin 45° + 2\sin 90°$

$= 6\times 0 - 20\times\frac{3}{5} + 13\sqrt{2}\times\frac{1}{\sqrt{2}} + 2\times 1 = 3$ newton.

$\therefore R = \sqrt{X^2 + Y^2} = \sqrt{(3)^2 + (3)^2} = 3\sqrt{2}$ newton ,

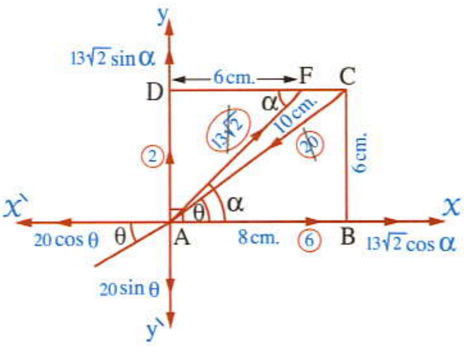
$\tan\ell = \frac{Y}{X} = \frac{3}{3} = 1$ where ℓ is the measure of the polar angle of $\vec{R}$ in this example

$\because X > 0 , Y > 0$ $\therefore \ell = 45°$

i.e. $\vec{R}$ is in the direction of $\overrightarrow{AF}$

Another solution :

Using the analyzing of the forces into two perpendicular directions :



From Pythagoras' theorm :

AC = 10 cm.

$\therefore \sin\theta = \frac{6}{10} = \frac{3}{5}$, $\cos\theta = \frac{8}{10} = \frac{4}{5}$

, $\because \Delta AFD$ is an isosceles triangle

$\therefore m(\angle \alpha) = 45^\circ$

$\therefore X = 13\sqrt{2}\cos\alpha + 6 - 20\cos\theta$

$= 13\sqrt{2} \times \frac{1}{\sqrt{2}} + 6 - 20 \times \frac{4}{5} = 3$ newton.

, $Y = 13\sqrt{2}\sin\alpha + 2 - 20\sin\theta$

$= 13\sqrt{2} \times \frac{1}{\sqrt{2}} + 2 - 20 \times \frac{3}{5} = 3$ newton.

$\therefore R = \sqrt{(3)^2 + (3)^2} = 3\sqrt{2}$ newton. , $\tan\ell = \frac{3}{3} = 1$

, $\because X > 0$, $Y > 0$ $\therefore \ell = 45^\circ$

i.e. The resultant in direction of $\overrightarrow{AF}$

Lesson 4

Equilibrium of a rigid body under the effect of two forces / three forces meeting at a point
(The triangle of forces rule - Lami's rule)

First Equilibrium of a rigid body under the action of two forces :

The conditions of equilibrium of a rigid body under the action of two forces

The rigid body is in equilibrium under the action of two forces only , if the two forces :

(1) Are equal in magnitude.

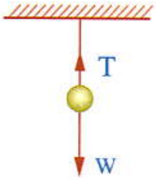
(2) Are opposite in direction.

(3) Their lines of action are on the same straight line.

✶ Examples on the equilibrium of a body under the action of two forces :

(1) A body suspended by a light string :

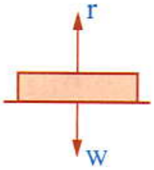
If a weight (W) is suspended by a light string. It balances under the action of two forces which are : weight (W) acting vertically downwards and the tension in the string (T) acting vertically upwards therefore : T = W



(2) A body of weight W placed on a horizontal smooth plane :

If a body of weight (W) is placed on a smooth horizontal plane.

It balances under the action of two forces which are : weight (W) acting vertically downwards and the reaction of the horizontal smooth plane (r) acting vertically upwards as shown in the figure.

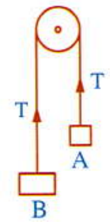


, we deduce that : r = W

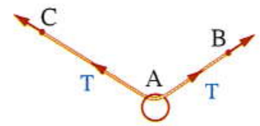
Remarks

(1) If a rigid body is acted upon by two forces equal in magnitude , opposite in direction and their lines of action are on the same straight line , they have no effect on the body neither in case of rest nor motion.

(2) **In the opposite figure :** If a string passes over a smooth pulley. and two bodies A and B are suspended from its two terminals such that the string is tensioned , then the two tensions in the two terminals of the string are equal in magnitude.

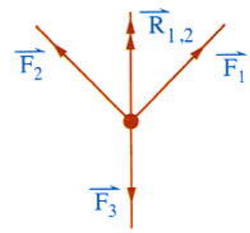


(3) **In the opposite figure :** If a string passes through a smooth ring to be suspended freely in it , then the tensions in each of the two branches of the string $\overline{AB}$, $\overline{AC}$ are equal in magnitude.



Second Equilibrium of a rigid body under the action of three forces acting at a point :

If three coplanar forces as $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ and $\overrightarrow{F_3}$ are acting at a point and they are in equilibrium as shown in the figure , and if $\overrightarrow{R_{1,2}}$ is the resultant of the two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$, then the two forces , $\overrightarrow{R_{1,2}}$ and $\overrightarrow{F_3}$ are balanced. Then from the conditions of the equilibrium of two forces , we deduce that $\overrightarrow{R_{1,2}}$ and $\overrightarrow{F_3}$ are equal in magnitude , opposite in direction and they have the same line of action.



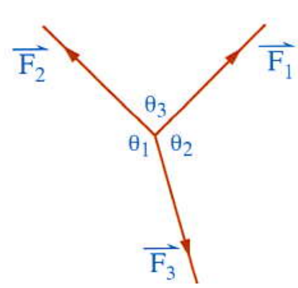
Generally : If three forces acting at a point are in equilibrium , then the resultant of any two forces of them is equal in magnitude to the third force and acts in the opposite direction of it and they have the same line of action.

Example 1

$\overrightarrow{F_1}$, $\overrightarrow{F_2}$ and $\overrightarrow{F_3}$ are three coplanar forces meeting at a point , their magnitudes are 12 , $12\sqrt{3}$ and 24 newton respectively. If these forces are balanced , find the measures of the angles among the three lines of action of the three forces.

Solution

Suppose the measure of the angle between the lines of action of $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ be θ_3



$\because$ The three forces are balanced.

$\therefore R_{1,2} = F_3 = 24$ and act in the opposite direction.

$\because (R_{1,2})^2 = F_1^2 + F_2^2 + 2\,F_1\,F_2 \cos\theta_3$

$\therefore (24)^2 = (12)^2 + \left(12\sqrt{3}\right)^2 + 2 \times 12 \times 12\sqrt{3} \cos \theta_3$

$\therefore 576 = 144 + 432 + 288\sqrt{3} \cos \theta_3$

$\therefore \cos \theta_3 = \text{zero} \qquad \therefore \theta_3 = 90°$

Similarly , suppose the measure of the angle between the lines of action of $\overrightarrow{F_2}$ and $\overrightarrow{F_3}$ is θ_1 ,

$\because$ the three forces are balanced.

$\therefore R_{2,3} = F_1 = 12$ and act in the opposite direction.

$\therefore (R_{2,3})^2 = F_2^2 + F_3^2 + 2\, F_2\, F_3 \cos \theta_1$

$\therefore 144 = 432 + 576 + 2 \times 12\sqrt{3} \times 24 \cos \theta_1 \qquad \therefore \cos \theta_1 = -\frac{\sqrt{3}}{2}$

$\therefore \theta_1 = 150°$

$\therefore \theta_2$ (the measure of the angle between the lines of action of $\overrightarrow{F_1}$ and $\overrightarrow{F_3}$)

$= 360° - (90° + 150°) = 120°$

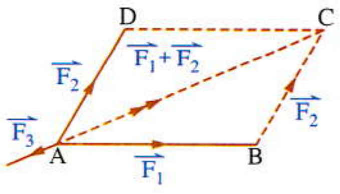
* We know that , the adjusted and sufficient condition to equilibrium of a rigid body under acting of a set of concurrent forces is a representing of these forces geometrically by the sides of a closed polygon , then we can deduce the following rule.

Rule (1)

If three forces are acting at a point and can be represented by the sides of a triangle taken in the same cyclic order , then the forces are in equilibrium.

In the opposite figure :

If $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ and $\overrightarrow{F_3}$ are three coplanar forces meeting at A and the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ represent these forces in magnitude and direction.



$\because \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\therefore \overrightarrow{AC}$ represents the resultant of the two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$

but $\overrightarrow{CA}$ represents $\overrightarrow{F_3}$,

$\because \overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{O}$

$\therefore \overrightarrow{F_3}$ equals in magnitude and opposite in direction to the resultant of $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$

i.e. $\overrightarrow{F_3}$ is balanced with the resultant of the two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$

$\therefore$ The three forces are in equilibrium

Remark

The three coplanar non collinear forces acting at a point. In order to be in equilibrium , their magnitude should be formed to be lengths of sides of a triangle.

i.e. The greatest magnitude of these forces should be less than the sum of the other two magnitudes of the other two forces because in any triangle , the longest side should be less than the sum of two lengths of the other two sides.

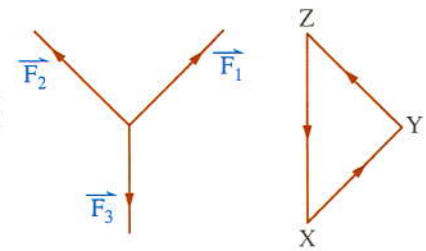
For example :

The three forces whose magnitudes are 3 , 4 and 9 force unit cannot be in equilibrium because the numbers 3 , 4 and 9 cannot be lengths of sides of any triangle because $9 > 3 + 4$
but the forces whose magnitudes are 4 , 7 , 8 could be in equilibrium, but we can not say that they are in equilibrium because that is depending on their magnitudes and their directions also.

- Three forces act at a point equilibrate if the magnitude of the greatest force equals the sum of the magnitudes of the other two forces in case they act on the same straight line.

Rule (2) The triangle of forces rule

If a rigid body is in equilibrium under the action of three forces acting at a point and a triangle is drawn whose sides are parallel to the lines of action of the forces and taken in the same cyclic order , then the lengths of the sides of the triangle are proportional to the magnitudes of the corresponding forces.



And if we symbolized to the forces' magnitude by F_1 , F_2 and F_3 , and Δ ABC was the triangle whose sides are parallel to the lines of action of three forces then $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ represents the forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ and $\overrightarrow{F_3}$ respectively in the magnitude and the direction where the body is in equilibrium , then $\boxed{\frac{F_1}{AB} = \frac{F_2}{BC} = \frac{F_3}{AC}}$ and Δ ABC is called "the triangle of forces".

It is noted that , we can draw an infinite number of similar triangles each of them is called the triangle of forces.

Example 2

Three forces of magnitudes F_1 , F_2 and 96 kg.wt. act on a particle. They are represented by the directed line segments $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ respectively in Δ ABC , where AB = 8 cm. , BC = 10 cm. and CA = 12 cm. Find the value of each of F_1 and F_2

Solution

$\because$ The forces are represented by the directed line segments of a triangle taken in one direction.

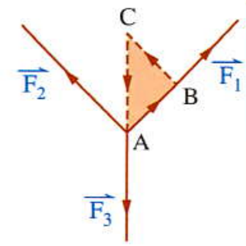
$\therefore$ The three forces are balanced , then using the triangle of forces rule we get :

$$\frac{F_1}{AB} = \frac{F_2}{BC} = \frac{96}{CA} \qquad \therefore \frac{F_1}{8} = \frac{F_2}{10} = \frac{96}{12}$$

$\therefore F_1 = 64$ kg. wt. , $F_2 = 80$ kg.wt.

Important remarks :

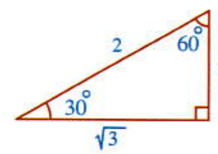
(1) It is possible to draw the triangle of forces , such that two of its sides are on the line of action of two forces and the third side is parallel to the line of action of the third force.



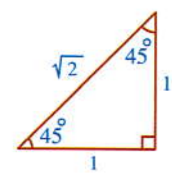
As in the opposite figure :

Δ ABC is the triangle of forces.

(2) * If the triangle of forces of three equilibrium forces is a right-angled triangle and it has an angle of measure $30°$, then the ratio among its side lengths is $1 : 2 : \sqrt{3}$

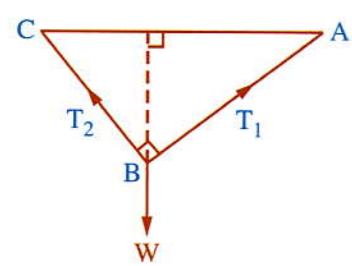


* If the triangle of forces is isosceles right-angled triangle , then the ratio among its side lengths is $1 : 1 : \sqrt{2}$



Enrich knowledge

If a triangle is drawn in which its sides are perpendicular to the directions of the equilibrium , then the ratio between each force and the length of the side perpendicular to it is equal.



In the opposite figure :

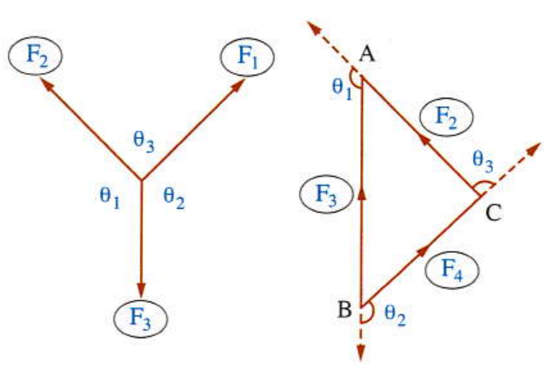
$\overrightarrow{W} \perp \overline{AC}$, $\overrightarrow{T_1} \perp \overline{BC}$, $\overrightarrow{T_2} \perp \overline{AB}$ $\qquad \therefore \dfrac{W}{AC} = \dfrac{T_1}{BC} = \dfrac{T_2}{AB}$

This rule is called perpendicular of forces triangle.

Rule (3) Lami's rule

If three coplanar forces meeting at a point and acting up on a particle are in equilibrium , then the magnitude of each force is proportional to the sine of the angle between the two other forces.

If the symbols of the magnitudes of the forces are F_1 , F_2 and F_3 , and θ_1 , θ_2 and θ_3 are the measures of the opposite angles for them respectively as shown in the opposite figure :
Then Δ BCA is the triangle of forces



$$\therefore \frac{F_1}{BC} = \frac{F_2}{CA} = \frac{F_3}{AB} \qquad (1)$$

and from sin law :

$$\therefore \frac{BC}{\sin(180 - \theta_1)} = \frac{CA}{\sin(180 - \theta_2)} = \frac{AB}{\sin(180 - \theta_3)}$$

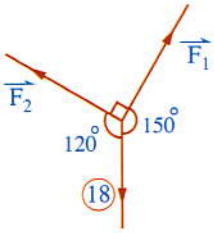
$$i.e. \frac{BC}{\sin\theta_1} = \frac{CA}{\sin\theta_2} = \frac{AB}{\sin\theta_3} \qquad (2)$$

From (1) , (2) we deduce that : $\boxed{\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}}$

Example 3

Three coplanar forces of magnitudes F_1 , F_2 and 18 newton meeting at a particle in balance. If the measure of the angle between the line of action of 1^{st} and 2^{nd} forces is 90° and between the 2^{nd} and the 3^{rd} is 120° Find the value of each of F_1 and F_2

Solution



The measure of the angle between the 1^{st} and 3^{rd} forces = 360° – (90° + 120°) = 150°

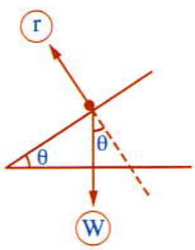
Due to Lami's rule we get :

$$\because \frac{F_1}{\sin 120°} = \frac{F_2}{\sin 150°} = \frac{F_3}{\sin 90°} \qquad \therefore \frac{F_1}{\frac{\sqrt{3}}{2}} = \frac{F_2}{\frac{1}{2}} = \frac{18}{1}$$

$$\therefore F_1 = 18 \times \frac{\sqrt{3}}{2} = 9\sqrt{3} \text{ newton}$$

$$, F_2 = 18 \times \frac{1}{2} = 9 \text{ newton.}$$

Equilibrium of a body placed on a smooth inclined plane



If a body of weight (W) is placed on a smooth inclined plane which inclines by an angle of measure θ with the horizontal , then the body will be under the action of two forces :

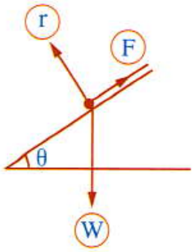
(1) The weight force ($\overrightarrow{W}$) acting vertically downwards.

(2) The reaction force ($\overrightarrow{r}$) of the inclined plane and it acts in direction perpendicular to the plane.

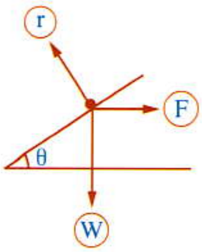
These two forces cannot be in equilibrium because they have two different lines of action. Therefore , in order to be in equilibrium , a third force must act on the body.

It may be in one of the following forms :

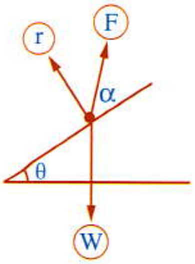
(1) The force is in the direction of the line of the greatest slope upwards.



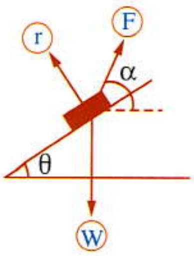
(2) The force acts horizontally.



(3) The force inclines by α with the plane upwards.



(4) The force inclines by α with the horizontal upward.



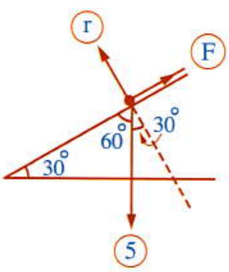
Remark

The reaction of the smooth plane $(\overset{\frown}{r})$ is perpendicular to the plane.

Example 4

A body of weight 5 kg.wt. is placed on a smooth plane inclined to the horizontal at an angle of measure 30°. It is pulled up the plane under the action of a force of magnitude F which its line of action coincides the line of the greatest slope up the plane. Find F and the reaction of the plane.

Solution



The body is in equilibrium under the action of forces of magnitudes F , r and 5 kg.wt. as shown in the figure.

and the measure of the angle between the two lines of action of the two first forces = 90°

and between the 2^{nd} and the 3^{rd} = 180° – 30° = 150°

and between the 3^{rd} and the 1^{st} = 90° + 30° = 120°

Applying Lami's rule we get :

$$\frac{F}{\sin 150°} = \frac{r}{\sin 120°} = \frac{5}{\sin 90°} \qquad \textit{i.e.}\ \frac{F}{\frac{1}{2}} = \frac{r}{\frac{\sqrt{3}}{2}} = \frac{5}{1}$$

$$\therefore F = 5 \times \frac{1}{2} = 2\frac{1}{2} \text{ kg.wt.} \qquad , r = 5 \times \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2} \text{ kg.wt.}$$

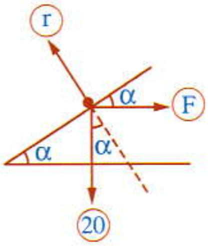
Example 5

A body of weight 20 kg.wt. is placed on a smooth plane inclined to the horizontal with an angle of measure α where $\cos\alpha = \frac{4}{5}$

The body is kept in equilibrium by a horizontal force of magnitude F

Find F and the reaction of the plane.

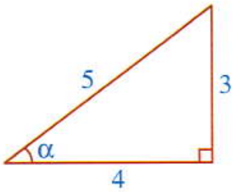
Solution



$\because$ The weight is in equilibrium under the action of forces of magnitudes F , r and 20 kg.wt. therefore , the measure of the angle between the 1st and 2nd forces = $90° + \alpha$ and between the 2nd and 3rd = $180° - \alpha$ and between the 3rd and the first = $90°$

Applying Lami's rule we get :

$$\frac{F}{\sin(180° - \alpha)} = \frac{r}{\sin 90°} = \frac{20}{\sin(90° + \alpha)}$$



i.e. $\frac{F}{\sin\alpha} = \frac{r}{1} = \frac{20}{\cos\alpha}$

, $\because \cos\alpha = \frac{4}{5}$ $\therefore \sin\alpha = \frac{3}{5}$

$\therefore \frac{F}{\frac{3}{5}} = \frac{r}{1} = \frac{20}{\frac{4}{5}}$ $\therefore F = 20 \times \frac{3}{5} \times \frac{5}{4} = 15$ kg.wt.

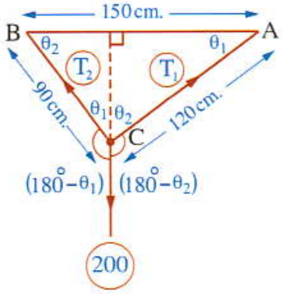
, $r = 20 \times \frac{5}{4} = 25$ kg.wt.

General examples on the equilibrium of three coplanar concurrent forces

Example 6

A weight of 200 gm.wt. is suspended by two strings of lengths 90 cm. and 120 cm. fixed in two horizontal points , the distance between them is 150 cm. Find the value of the tension in each of the two strings in case of equilibrium.

Solution



$\because (90)^2 + (120)^2 = (150)^2$

$\therefore \Delta ABC$ is right-angled at C , from the figure we get :

$\sin\theta_1 = \frac{90}{150} = \frac{3}{5}$, $\sin\theta_2 = \frac{120}{150} = \frac{4}{5}$

Using Lami's rule :

$\therefore \dfrac{T_1}{\sin(180^\circ - \theta_1)} = \dfrac{T_2}{\sin(180^\circ - \theta_2)} = \dfrac{200}{\sin 90^\circ}$

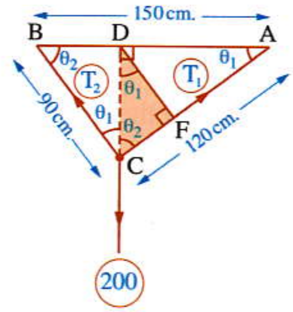
$\therefore \dfrac{T_1}{\sin\theta_1} = \dfrac{T_2}{\sin\theta_2} = \dfrac{200}{\sin 90^\circ}$ $\qquad \therefore \dfrac{T_1}{\frac{3}{5}} = \dfrac{T_2}{\frac{4}{5}} = \dfrac{200}{1}$

$\therefore T_1 = \dfrac{200}{1} \times \dfrac{3}{5} = 120$ gm.wt. , $T_2 = \dfrac{200}{1} \times \dfrac{4}{5} = 160$ gm.wt.

Another solution :

By using the triangle of forces rule : Draw $\overline{DF} \parallel \overline{CB}$

, then Δ DFC is the triangle of forces

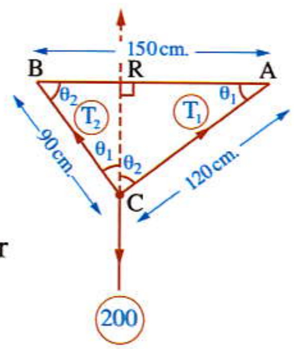


$\therefore \dfrac{T_1}{CF} = \dfrac{T_2}{FD} = \dfrac{200}{DC}$

$\therefore T_1 = 200 \times \dfrac{CF}{DC} = 200 \sin\theta_1 = 200 \times \dfrac{3}{5} = 120$ gm.wt.

, $T_2 = 200 \times \dfrac{FD}{DC} = 200 \sin\theta_2 = 200 \times \dfrac{4}{5} = 160$ gm.wt.

Third solution : (By resolution) :



$\because$ The three forces are in equilibrium.

$\therefore$ The resultant of the two tensions = The third force and in the opposite direction

$\therefore \overrightarrow{T_1}$ and $\overrightarrow{T_2}$ are the two components of $\overrightarrow{R}$ and they are perpendicular

$\therefore T_1 = R\cos\theta_2 = 200 \times \dfrac{3}{5} = 120$ gm.wt. ,

$T_2 = R\sin\theta_2 = 200 \times \dfrac{4}{5} = 160$ gm.wt.

Fourth solution : (By perpendicular of forces triangle) :

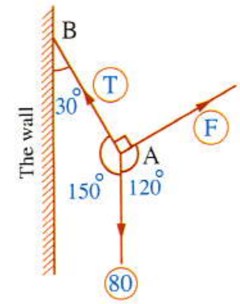
$\because \Delta$ ABC is perpendicular of forces triangle.

$\therefore \dfrac{T_1}{90} = \dfrac{200}{150} = \dfrac{T_2}{120}$ $\qquad \therefore T_1 = 120$ gm. wt. , $T_2 = 160$ gm. wt.

Example 7

A weight of 80 gm.wt. is suspended by a string fixed in a vertical wall. The weight is pulled by a force perpendicular to the string till it becomes in equilibrium when it is inclined on the wall by an angle of measure 30° , find in case of equilibrium , the magnitude of the force and the tension in the string.

Solution

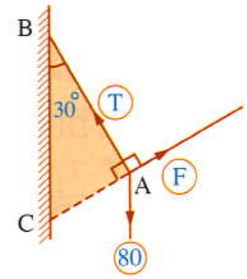


Due to Lami's rule we get : $\frac{F}{\sin 150°} = \frac{T}{\sin 120°} = \frac{80}{\sin 90°}$

$\therefore \frac{F}{\frac{1}{2}} = \frac{T}{\frac{\sqrt{3}}{2}} = 80$

$\therefore F = 80 \times \frac{1}{2} = 40$ gm.wt. , $T = 80 \times \frac{\sqrt{3}}{2} = 40\sqrt{3}$ gm.wt.

Another solution : (By triangle of forces) :



Draw the line of action of $\overrightarrow{F}$ to meet the wall at C

, then Δ CAB is the triangle of forces.

Due to the triangle of forces rule we get : $\frac{F}{CA} = \frac{T}{AB} = \frac{80}{BC}$

In Δ CAB : $\therefore$ CA : AB : BC = $1 : \sqrt{3} : 2$

$\therefore \frac{F}{1} = \frac{T}{\sqrt{3}} = \frac{80}{2}$ $\qquad \therefore F = 40$ gm.wt. , $T = 40\sqrt{3}$ gm.wt.

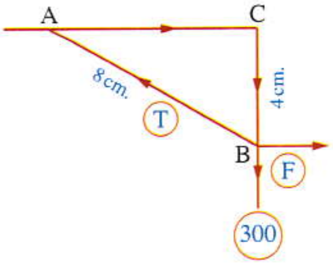
Example 8

A light string $\overline{AB}$ of length 8 cm. Its terminal A is fixed at a point. A weight of 300 gm.wt. , is suspended at the other terminal B. Find the magnitude of the needed force to keep the weight in equilibrium at a distance of 4 cm. From the horizontal line passing through A , also find the tension in the string in each of the two cases.

(1) If the force is horizontal. (2) If the direction of the force is perpendicular to $\overrightarrow{AB}$

Solution

The first case :



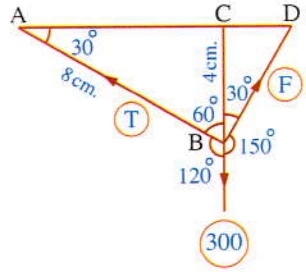
If the force is horizontal , we can take Δ ABC as the triangle of forces.

$\therefore \frac{T}{AB} = \frac{F}{AC} = \frac{300}{BC}$

, $\because AC = \sqrt{(8)^2 - (4)^2} = 4\sqrt{3}$ cm. $\qquad \therefore \frac{T}{8} = \frac{F}{4\sqrt{3}} = \frac{300}{4}$

$\therefore T = 600$ gm.wt. , $F = 300\sqrt{3}$ gm.wt.

The second case :



If the direction of the force is perpendicular to $\overrightarrow{AB}$

Due to Lami's rule :

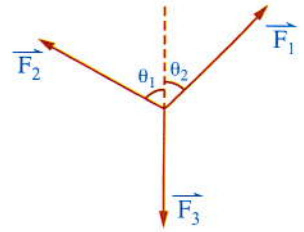
$\therefore \frac{T}{\sin 150°} = \frac{F}{\sin 120°} = \frac{300}{\sin 90°}$ $\qquad \therefore T = \frac{300 \sin 150°}{\sin 90°}$

$\therefore T = 150$ gm.wt. , $F = \frac{300 \sin 120°}{\sin 90°} = 150\sqrt{3}$ gm.wt.

Remark

If the line of action of one force of three equilibrium forces is extended to divide the angle between the two lines of action of the other two forces into two angles , then we can apply Lami's rule as follows :

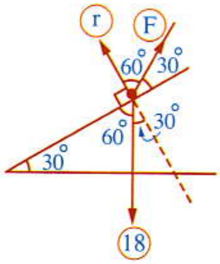
$$\frac{F_1}{\sin \theta_1} = \frac{F_2}{\sin \theta_2} = \frac{F_3}{\sin (\theta_1 + \theta_2)}$$



Example 9

A body of weight 18 kg.wt. is placed on a smooth plane inclined to the horizontal by an angle of measure 30°. It is pulled upwards under the action of a force (F) inclines with the line of greatest slope of the plane by an angle of measure 30°. Find the magnitude of this force and the reaction of the plane.

Solution



The body is in equilibrium under the action of the three forces of magnitudes F , r and 18 kg.wt. where :

The measure of the angle between the 1^{st} and 2^{nd} forces is 60°

and between the 2^{nd} and the 3^{rd} is 150°

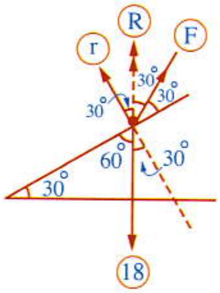
and between the 3^{rd} and the 1^{st} is 30° + 90° + 30° = 150° also.

Applying Lami's rule we get :

$$\frac{F}{\sin 150°} = \frac{r}{\sin 150°} = \frac{18}{\sin 60°} \qquad \textit{i.e.} \ \frac{F}{\frac{1}{2}} = \frac{r}{\frac{1}{2}} = \frac{18}{\frac{\sqrt{3}}{2}}$$

$$\therefore F = r = 18 \times \frac{1}{2} \div \frac{\sqrt{3}}{2} = 6\sqrt{3} \text{ kg.wt.}$$

Another solution :



$\because$ The line of action of the weight $\overrightarrow{W}$ which is the line of action of the resultant of the two forces $\overrightarrow{F}$ and $\overrightarrow{r}$ and it bisects the angle between them.

F = r ,

$\because R = 2 F \cos \frac{\alpha}{2}$ $\qquad \therefore 18 = 2 F \times \cos \frac{60°}{2}$

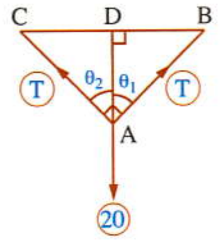
$\therefore 18 = 2 F \cos 30°$ $\qquad \therefore 18 = 2 F \times \frac{\sqrt{3}}{2}$

$\therefore F = \frac{18}{\sqrt{3}} = 6\sqrt{3}$ kg.wt. $\qquad \therefore F = r = 6\sqrt{3}$ kg.wt.

Example 10

A light string is fastened from its terminals at two points B and C such that $\overline{BC}$ is equilibrium horizontal. A small smooth ring of weight 20 gm.wt. slides on the string till the angle between the two branches of the string in equilibrium becomes 90° in measure. Prove that the lengths of the two branches of the string are equal , then find the value of the tension in each of them.

Solution



$\because$ The ring is smooth.

$\therefore$ The values of tension in the two branches of the string are equal.

i.e. Tension in the branch $\overline{AB}$ = tension in the branch $\overline{AC}$ = T

Using Lami's rule we get :

$\frac{T}{\sin \theta_2} = \frac{T}{\sin \theta_1} = \frac{20}{\sin 90°}$ $\qquad \therefore \sin \theta_1 = \sin \theta_2$

$\therefore \theta_1 = \theta_2 = \frac{90°}{2} = 45°$ $\qquad \therefore \frac{T}{\sin 45°} = \frac{20}{\sin 90°}$

$\therefore T = 10\sqrt{2}$

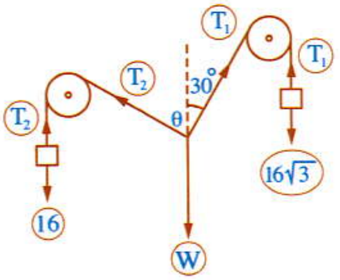
, $\because \overline{AD} \perp \overline{BC}$, $\theta_1 = \theta_2 = 45°$ $\qquad \therefore AB = AC$

$\therefore$ The lengths of the two branches of the string are equal.

Example 11

A body of weight (W) newton is suspended by two strings. The first inclines on the vertical by an angle of measure 30° and passes over a fixed smooth pulley and carries at its free end a weight $16\sqrt{3}$ newton. The second string inclines on the vertical by an angle of measure θ and passes over another fixed smooth pulley and carries at its terminal a body of weight 16 newton. Find in equilibrium case the value of W and the value of θ

Solution



Using Lami's rule :

$\therefore \frac{T_1}{\sin \theta} = \frac{T_2}{\sin 30°} = \frac{W}{\sin(\theta + 30°)}$

$\therefore \frac{16\sqrt{3}}{\sin \theta} = \frac{16}{\sin 30°} = \frac{W}{\sin(\theta + 30°)}$

$\therefore \sin \theta = \frac{16\sqrt{3} \sin 30°}{16} = \frac{\sqrt{3}}{2}$ $\qquad \therefore \theta = 60°$

$\therefore \frac{16}{\sin 30°} = \frac{W}{\sin(60° + 30°)}$ $\qquad \therefore W = 32$ newton.

Lesson 5

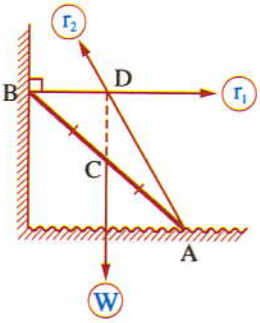
Follow : The equilibrium

(Meeting lines of action of three equilibrium forces)

Rule (4)

If a rigid body is in equilibrium under the action of three coplanar non parallel forces , then the lines of action of these forces meet at a point.

For example , in the opposite figure :



If a uniform rod of weight (W) is in equilibrium on a smooth vertical wall and on rough horizontal ground then :

(1) The weight of the rod acts vertically downwards at its midpoint. (centre of gravity)

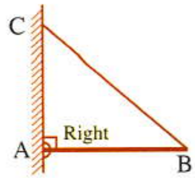
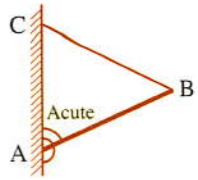
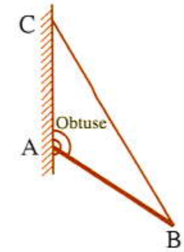
(2) The reaction of the smooth vertically wall (r_1) which is perpendicular to the wall in direction of $\overrightarrow{BD}$

(3) The reaction of the rough ground (r_2) with unknown direction , and to determine its direction , we draw $\overrightarrow{AD}$ passes through the point D (The point of intersection between $\overrightarrow{w}$, $\overrightarrow{r_1}$)

Remarks

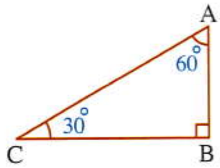
(1) The weight of the uniform sphere acts at its geometric centre (centre of gravity)

(2) If $\overline{AB}$ is a rod , its end A is attached to a hinge fixed at a vertical wall and the other end B is attached to a string fixed at the point C which lies above A exactly and we notice that :

$(BC)^2 = (AB)^2 + (AC)^2$, then $\angle$ BAC is right-angled.	$(BC)^2 < (AB)^2 + (AC)^2$, then $\angle$ BAC is acute.	$(BC)^2 > (AB)^2 + (AC)^2$, then $\angle$ BAC is obtuse.
		

(3) If Δ ABC is a $30° - 60°$ triangle

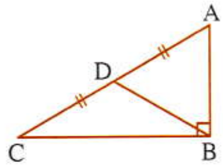
$\therefore AB = \frac{1}{2} AC , BC = \frac{\sqrt{3}}{2} AC$



(4) If Δ ABC is right-angled at B ,

$\overline{BD}$ is a median of it

, then $BD = \frac{1}{2} AC$

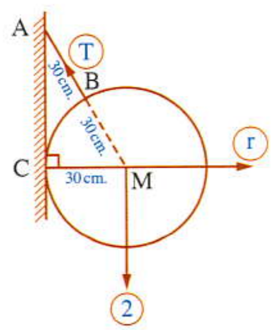


Example 1

A metallic sphere of weight 2 kg.wt. and of radius length 30 cm. is suspended at a point B on its surface by a string of length 30 cm. , its other end A is fixed at a point in a vertical wall to be in equilibrium as it rests on the wall. Find the magnitude of the tension in the string and the magnitude of the reaction of the wall.

Solution

The sphere is at rest under the action of three forces :



(1) The weight of the sphere whose magnitude is 2 kg.wt. acts vertically downwards at its centre M

(2) The reaction of the wall of magnitude r , acts at the point of touch of the sphere with the wall (C) in the direction perpendicular to the wall , hence it passes through the centre of the sphere M

(3) The tension in the string of magnitude T acts in the direction of $\overrightarrow{BA}$

∵ The lines of action of the weight force and the reaction force meet at M

∴ The line action of the tension force in the string should pass through the point M

i.e. $\overleftrightarrow{AB}$ passes through the point M , then Δ MAC is the triangle of forces where

MA = MB + BA = 60 cm. , CM = 30 cm.

∴ Δ AMC is a right-angled triangle of (30° – 60°) angles.

∴ $AC = 30\sqrt{3}$ cm. ∴ $\frac{T}{60} = \frac{r}{30} = \frac{2}{30\sqrt{3}}$

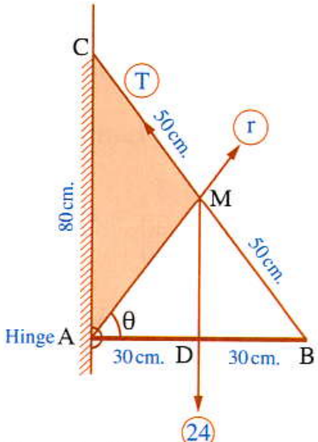
∴ $T = \frac{4\sqrt{3}}{3}$ kg.wt. , $r = \frac{2\sqrt{3}}{3}$ kg.wt.

Drill Try to solve the previous problem by Lami's rule.

Example 2

$\overline{AB}$ is a uniform rod of length 60 cm. , and its weight 24 kg.wt. , acting at (D) the midpoint of $\overline{AB}$, the end A of the rod is attached to a hinge fixed at a vertical wall. The other end B is attached to a light string , its other end is fixed at the point C on a vertical wall above A at a distance 80 cm. from A. If the rod is in equilibrium horizontally. Find the magnitude of the tension in the string and the magnitude and direction of the reaction of the hinge at A

Solution



From Δ ABC : $BC = \sqrt{(60)^2 + (80)^2} = 100$ cm.

The rod is in equilibrium under the effect of three forces :

(1) Its weight 24 kg.wt. , acts vertically downwards at the point D (the midpoint of $\overline{AB}$)

(2) The tension in the string (T) acts in the direction of $\overrightarrow{BC}$

(3) The reaction of the hinge at A (its magnitude = r)

∵ The two lines of action of the weight and the tension meet at the point M

∴ The line of action of the reaction of the hinge passes through the point M also ,

∵ D is the midpoint of $\overline{AB}$, $\overline{MD} \parallel \overline{AC}$

∴ M is the midpoint of $\overline{BC}$

$\therefore \overline{AM}$ is a median of ΔABC which is right at A

$\therefore AM = \frac{1}{2} BC = 50$ cm.

$\because \Delta AMC$ is the triangle of forces $\quad \therefore \frac{T}{MC} = \frac{r}{AM} = \frac{24}{CA}$

i.e. $\frac{T}{50} = \frac{r}{50} = \frac{24}{80}$ $\quad \therefore r = T = 24 \times \frac{50}{80} = 15$ kg.wt. ,

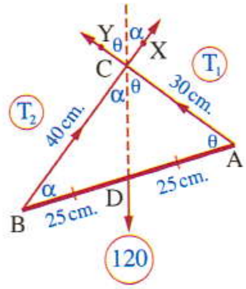
From ΔAMD : $\tan \theta = \frac{40}{30} = \frac{4}{3}$ $\quad \therefore \theta \simeq 53° \grave{8}$

$\therefore$ The reaction of the hinge at A inclines to the rod with an angle of measure $53° \grave{8}$

Example 3

A uniform rod of length 50 cm. and weight 120 gm. wt. , is suspended at its two ends freely by two strings , their other two ends are fixed at one point.
If the lengths of the two strings are 30 cm. and 40 cm. respectively.
Find the magnitude of the tension in each of the two strings.

Solution



The rod is in equilibrium under the effect of three forces :
their magnitudes are T_1 , T_2 and 120 gm.wt , where $\overrightarrow{AC}$ is the line of action of T_1 and $\overrightarrow{BC}$ is the line of action of T_2
they are meeting at the point C

$\therefore$ The action line of the weight of the rod should pass through C also

$\because (AB)^2 = 2500$, $(AC)^2 + (CB)^2 = 900 + 1600 = 2500$ $\quad \therefore m(\angle ACB) = 90°$

$\because$ D is the midpoint of the hypotenuse $\overline{AB}$ $\quad \therefore DC = DA = DB$

$\therefore m(\angle ACD) = m(\angle DAC)$, $m(\angle DCB) = m(\angle DBC)$

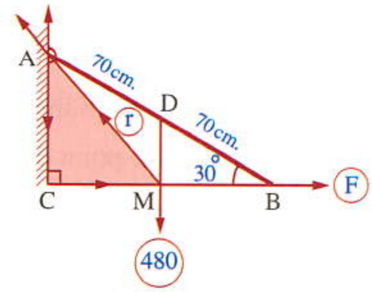
Applying Lami's rule we get : $\frac{T_1}{\sin(\alpha)} = \frac{T_2}{\sin(\theta)} = \frac{120}{\sin 90°}$

$\therefore \frac{T_1}{\frac{30}{50}} = \frac{T_2}{\frac{40}{50}} = 120$ $\quad \therefore T_1 = 120 \times \frac{30}{50} = 72$ gm.wt. , $T_2 = 120 \times \frac{40}{50} = 96$ gm.wt.

Example 4

$\overline{AB}$ is a uniform rod of length 140 cm. and weight 480 gm.wt. , its end A is attached to a hinge fixed on a vertical wall. A force $\overrightarrow{F}$ acts horizontally at the other end B to make the rod at rest at a position in which the rod inclines to the horizontal at an angle of measure 30°
Find the magnitude of $\overrightarrow{F}$, and the magnitude and the direction of the reaction of the hinge at A

Solution



The rod is at rest under the effect of three forces :

(1) Its weight 480 gm.wt. acts vertically downwards at the point D (the midpoint of $\overline{AB}$)

(2) The horizontal force $\vec{F}$ at B

(3) The reaction of the hinge at A of magnitude r

$\because$ The lines of action of the forces of weight and the horizontal force are meeting at the point M (*i.e.* in the direction of $\overrightarrow{MA}$)

$\therefore$ The line of action of the reaction of the hinge should pass through the point M also

$\therefore$ Δ CMA is the triangle of forces.

$\therefore \frac{F}{CM} = \frac{r}{MA} = \frac{480}{AC}$ $\qquad \because m(\angle ABC) = 30°$

$\therefore AC = \frac{1}{2} AB = 70$ cm. , $BC = \frac{\sqrt{3}}{2} AB = 70\sqrt{3}$ cm.

$\because$ D is the midpoint of $\overline{AB}$, $\overrightarrow{DM} \,//\, \overline{AC}$

$\therefore$ M is the midpoint of $\overline{BC}$ $\qquad \therefore MC = 35\sqrt{3}$ cm.

$\therefore AM = \sqrt{(70)^2 + \left(35\sqrt{3}\right)^2} = 35\sqrt{7}$ cm. $\qquad \therefore \frac{F}{35\sqrt{3}} = \frac{r}{35\sqrt{7}} = \frac{480}{70}$

$\therefore F = 240\sqrt{3}$ gm.wt. , $r = 240\sqrt{7}$ gm.wt.

, $\because$ Δ AMC is right-angled triangle at C $\qquad \therefore \tan(\angle AMC) = \frac{AC}{MC} = \frac{70}{35\sqrt{3}} = \frac{2}{\sqrt{3}}$

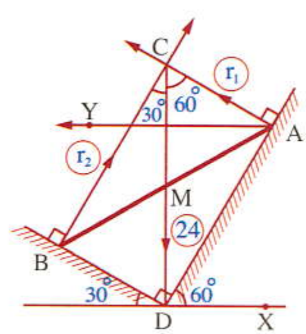
$\therefore m(\angle AMC) \simeq 49° \, 6'$

$\therefore$ The reaction of the hinge makes an angle of measure $49° \, 6'$ with the horizontal.

Example 5

A uniform rod rests with its two ends on two smooth planes incline to the horizontal at two angles of measures 60° and 30°. Find the measure of the angle which the rod makes with the horizontal at the equilibrium position and if the weight of the rod equals 24 newton. Determine the magnitudes of the two reactions of the two planes.

Solution



The rod is in equilibrium under the action of three forces :

(1) The weight 24 newton acts vertically downwards at the point M (the midpoint of $\overline{AB}$)

(2) The reaction of the first plane r_1

(3) The reaction of the second plane r_2

$\because$ The lines of action of the two forces of reactions meet at the point C

$\therefore$ The line of action of the weight of the rod passes through the same point "C" either.

If D is the point of meeting the two planes , then $\angle A$, $\angle D$ and $\angle B$ are right-angles

$\therefore$ The figure ACBD is a rectangle.

If M is the midpoint of $\overline{AB}$

$\therefore$ M is the point of intersection of the two diagonals of the rectangle.

$\therefore$ $\overline{CD}$ is a diagonal in the rectangle passing through M

, $\because$ $\overline{CD}$ is vertical $\quad \therefore m(\angle CDX) = 90°$

$\therefore m(\angle MDA) = 30°$

, $\because MD = MA \quad \therefore m(\angle DAM) = 30°$

$\because m(\angle YAD) = 60° \quad \therefore m(\angle YAM) = 30°$

$\therefore$ The rod makes an angle of measure 30° with the horizontal.

From Δ ADC : $\therefore m(\angle ACD) = 60°$

Applying Lami's rule we get : $\therefore \dfrac{r_1}{\sin 150°} = \dfrac{r_2}{\sin 120°} = \dfrac{24}{\sin 90°}$

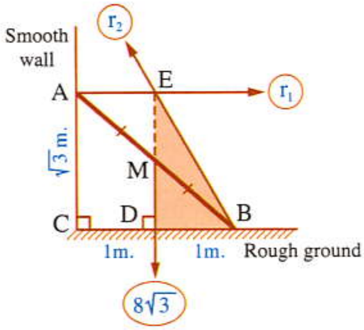
$\therefore r_1 = 12$ newton , $r_2 = 12\sqrt{3}$ newton.

Example 6

$\overline{AB}$ is a uniform ladder of weight $8\sqrt{3}$ kg.wt. , its upper end A rests on a smooth vertical wall and its lower end B rests on rough horizontal ground such that the upper end is far from the surface of the ground by $\sqrt{3}$ metre and the lower end is far from the wall by 2 metre. Find the magnitude of pressure on each of the wall and the ground in case of equilibrium.

Solution

The ladder is in equilibrium under the effect of three forces :



(1) The weight of the ladder of magnitude $8\sqrt{3}$ kg.wt. , acts vertically downwards at the midpoint of the ladder (M)

(2) The reaction of the smooth vertical wall of magnitude r_1 which is perpendicular to the wall at A

(3) The reaction of the rough ground of magnitude r_2

$\because$ The lines of action of the two forces of the weight and the reaction of the wall meet at the point E

$\therefore$ The line of action of the reaction of the ground must pass through the point E

, then Δ DBE is the triangle of forces where :

$DE = AC = \sqrt{3}$ metre. , $BD = \frac{1}{2} BC = 1$ metre

, $BE = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = 2$ metre

Applying the triangle of forces rule we get :

$\frac{r_1}{DB} = \frac{r_2}{BE} = \frac{8\sqrt{3}}{ED}$ $\therefore \frac{r_1}{1} = \frac{r_2}{2} = \frac{8\sqrt{3}}{\sqrt{3}}$

$\therefore r_1 = 8$ kg.wt. , $r_2 = 16$ kg.wt.

$\therefore$ The pressure on the wall = 8 kg.wt. , the pressure on the ground = 16 kg.wt.

Remark

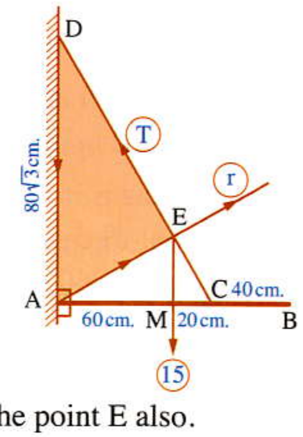
The pressure of the two ends of the ladder on the floor and the wall equals in magnitude the reactions of the floor and the wall on the two ends of the ladder.

Example 7

$\overline{AB}$ is a uniform rod of length 120 cm. and weight 15 kg.wt. , its end A is attached to a hinge fixed at a point on a vertical wall.
The rod is kept in equilibrium horizontally by attaching it at a point C on it where AC = 80 cm. by a string. The other end of the string is fixed at a point D on the vertical wall above A and at a distance $80\sqrt{3}$ cm. from it. Calculate the magnitude of each of the tension in the string and the reaction of the hinge.

Solution

The rod is kept in equilibrium under the effect of three forces :



(1) Its weight 15 kg.wt. , acts vertically downwards at M (the midpoint of $\overline{AB}$)

(2) The tension force in the string (T)

(3) The reaction of the hinge (r)

$\because$ The lines of action of the weight and the tension are meeting at the point E

$\therefore$ The line of action of the reaction of the hinge passes through the point E also.

$\therefore$ Δ AED is the triangle of forces

, $\because CD = \sqrt{(80)^2 + \left(80\sqrt{3}\right)^2} = 160$ cm.

$\because \Delta CME \sim \Delta CAD$: $\therefore \frac{CM}{CA} = \frac{CE}{CD} = \frac{ME}{AD}$

$\therefore \frac{20}{80} = \frac{CE}{160} = \frac{ME}{80\sqrt{3}}$ $\therefore CE = 40$ cm. , $ME = 20\sqrt{3}$, $ED = 120$ cm.

$\therefore AE = \sqrt{(60)^2 + \left(20\sqrt{3}\right)^2} = 40\sqrt{3}$ cm.

Applying the triangle of forces rule : $\therefore \frac{r}{40\sqrt{3}} = \frac{T}{120} = \frac{15}{80\sqrt{3}}$

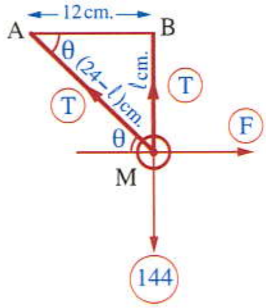
$\therefore r = 7.5$ kg.wt. $\therefore T = 7.5\sqrt{3}$ kg.wt.

Example 8

A string of length 24 cm. , is fixed from its ends at two pins A and B in a horizontal line , the distance between them is 12 cm. the string passes inside a smooth ring to be suspended in it , its weight is 144 dyne , then the ring is pulled horizontally by a force $\overrightarrow{F}$ till it becomes down B directly.

Find the magnitude of tension in each of the two branches of the string , find also the magnitude of $\overrightarrow{F}$

Solution



Supposing that BM = ℓ cm.

$\therefore MA = (24 - \ell)$ cm. , $\because m(\angle ABM) = 90°$

$\therefore (24 - \ell)^2 = \ell^2 + (12)^2$ $\therefore 576 - 48\ell + \ell^2 = \ell^2 + 144$

$\therefore \ell = 9$ cm. $\therefore MB = 9$ cm. , $AM = 15$ cm.

Let $m(\angle MAB) = \theta$

$\therefore \cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$

$\because$ The ring is smooth.

$\therefore$ The tension in the two branches $\overline{MB}$ and $\overline{MA}$ are equal in magnitude.

$\therefore$ The ring is in equilibrium under the effect of four forces which are :

(F , 0°) , (T , 90°) , (T , 180° – θ) , (144 , 270°)

Notice that

The ring is in equilibrium under the effect of four forces , so it will be solved by using the method you studied in lesson 3 X = 0 , Y = 0

, $\because X = 0$

$\therefore F\cos 0° + T\cos 90° + T\cos(180° - \theta) + 144\cos 270° = \text{zero}$

$\therefore F \times 1 + T \times 0 + T \times -\cos\theta + 144 \times 0 = \text{zero}$

$\therefore F - \frac{4}{5}T = 0$ (1)

, $\because Y = 0$

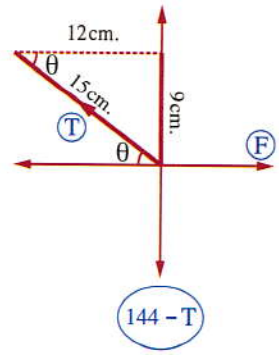
$\therefore F\sin 0° + T\sin 90° + T\sin(180° - \theta) + 144 \times \sin 270° = \text{zero}$

$\therefore F \times 0 + T \times 1 + T\sin\theta + 144 \times -1 = \text{zero}$ $\therefore T + \frac{3}{5}T - 144 = \text{zero}$

$\therefore T\left(1 + \frac{3}{5}\right) = 144$ $\therefore T = 90$ dyne.

Substituting in (1) : $\therefore F = \frac{4}{5} \times 90 = 72$ dyne.

Another solution :



$\because$ The weight and the vertical tension on the same straight line

$\therefore$ Their resultant can be calculated as $(144 - T)$

Using lami's rule :

$\therefore \dfrac{F}{\sin(90° + \theta)} = \dfrac{T}{\sin 90°} = \dfrac{144 - T}{\sin(180° - \theta)}$

$\therefore \dfrac{F}{\cos\theta} = \dfrac{T}{1} = \dfrac{144 - T}{\sin\theta}$

$\therefore \dfrac{F}{\left(\frac{4}{5}\right)} = T = \dfrac{144 - T}{\left(\frac{3}{5}\right)}$

$\therefore \dfrac{3}{5}\,T = 144 - T$

$\therefore \dfrac{8}{5}\,T = 144$

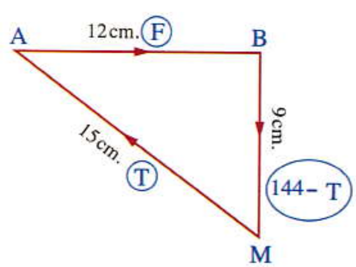
$\therefore T = 90$ dyne , $F = 90 \times \dfrac{4}{5} = 72$ dyne

Third solution :

$\because$ The weight and the vertical tension on the same straight line

$\therefore$ Their resultant can be calculated as $(144 - T)$

and the Δ ABM becomes the triangle of forces



$\therefore \dfrac{144 - T}{9} = \dfrac{T}{15} = \dfrac{F}{12}$

$\therefore \dfrac{144 - T}{3} = \dfrac{T}{5}$

$\therefore 3\,T = 720 - 5\,T$

$\therefore 8\,T = 720$

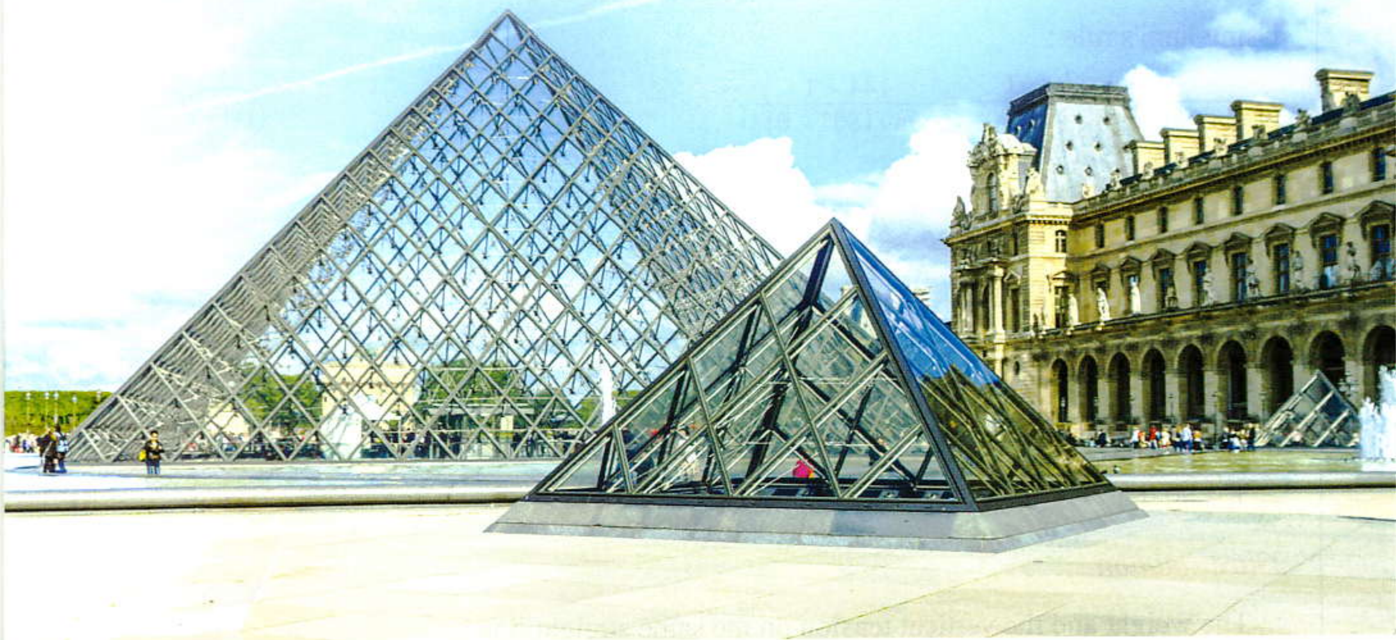
$\therefore T = 90$ dyne

$\therefore \dfrac{90}{15} = \dfrac{F}{12}$

$\therefore F = 72$ dyne

Unit Two

Geometry and Measurement



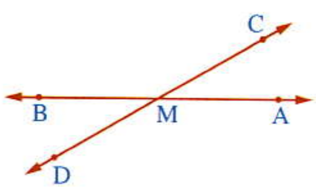
Lesson 1

The straight lines and the planes in the space

Geometrical concepts and axioms

1 The straight line :

Is an infinite set of points , and we can determine it exactly if we know any two different points on it.



For example :

In the opposite figure :

The two points "A , B" passing through one and only one straight line which is $\overleftrightarrow{AB}$ while the two points "C , D" passing through another straight line which is $\overleftrightarrow{CD}$

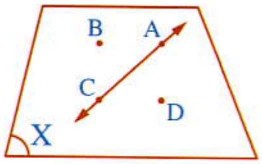
i.e. The straight line is determined by two different points on it.

Remark

$\overline{AB} \subset \overrightarrow{AB} \subset \overleftrightarrow{AB}$

2 The plane :

Is an infinite set of points represents a surface with no ends where any straight line passing through two points on it lies completely on that surface and we denote it by capital letters as X , Y or ……… and we can denote it using at least three non-collinear points on the plane as : ABC



i.e. The plane is determined by three distinct non-collinear points.

UNIT 2

Remarks

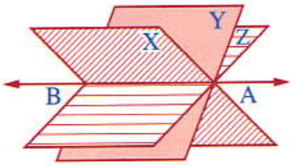
(1) The geometrical shapes as the triangle , the square , the circle and ... is an infinite set of points and these shapes are called planed geometrical shapes because each of them is a subset of its plane.

(2) Where the plane extends into infinity in all directions , we can represent it using a planed geometrical shape on it as a square or a circle or a parallelogram or ...

3 The space :

Is an infinite set of points , it contains all the geometric figures and planes , surfaces and solids while the solids as the sphere and the cylinder and the cube , ... etc. are a set of infinite points but we can't contain it in one plane , but we can contain it in the space and the faces of these solids formed from some planed parts as the cube or non-planed as the sphere.

Remarks



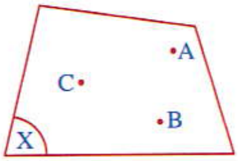
- Any point in the space passing through it an infinite number of the straight lines.
- Any point in the space passing through it an infinite number of the planes.
- Any two points in the space passing through them one and only one straight line.
- Any two points in the space passing through them an infinite number of the planes.

Determination of the plane in the space

The plane is determined in each of the following cases :

1 Three distinct non-collinear points :

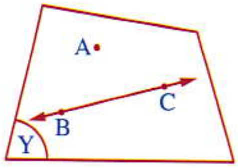
In the opposite figure :



The points A , B , C are non-collinear so that we can determine the plane (X) or ABC from that we can deduce :

There is one and only one plane which passes through three non-collinear points.

2 A straight line and a point not belonging to it :



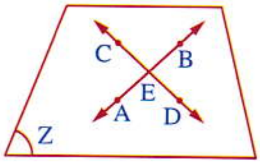
In the opposite figure :

$A \notin \overleftrightarrow{BC}$, then the point A and

the straight line $\overleftrightarrow{BC}$

determine the plane (Y) or ABC

3 Two intersecting straight lines :



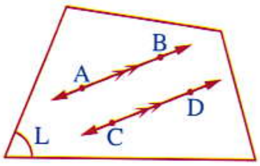
In the opposite figure :

$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{E\}$

, then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$

determine the plane (Z)

4 Two parallel and non-coincident straight lines :



In the opposite figure :

$\overleftrightarrow{AB} \,//\, \overleftrightarrow{CD}$, $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \varnothing$

, then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$

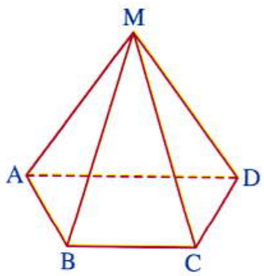
determine the plane (L)

Example 1

In the opposite figure :

If $M \notin$ the plane ABCD

Find :



(1) Four straight lines passing through the point (A)

(2) Three planes passing through the point (A)

(3) The straight lines passing through the two points A and B together.

(4) Two planes each of them passing through the two points A and B together.

(5) Four planes passing through the point (M)

(6) The number of the planes which determine the solid in the figure.

Solution

(1) $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{AM}$

(3) $\overleftrightarrow{AB}$

(5) MAB , MBC , MCD , MAD

(2) ABCD , ABM , ADM

(4) ABCD , ABM

(6) Five planes.

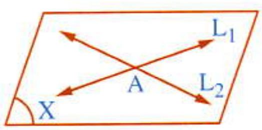
UNIT 2

Relative positions of lines and planes in the space

1 The relative positions of two different straight lines in the space :

Intersecting straight lines

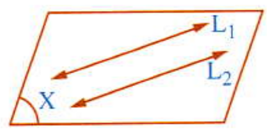
Two straight lines lie on the same plane and have one point in common.



- L_1 , L_2 are intersecting.
- $L_1 \cap L_2 = \{A\}$
- They lie in one plane.

Parallel straight lines

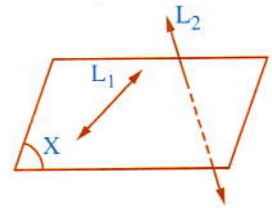
Two straight lines lie on the same plane and doesn't have any common point.



- $L_1 // L_2$
- $L_1 \cap L_2 = \varnothing$
- They lie in one plane.

Skew straight lines

Two straight lines can't be lie in one plane.



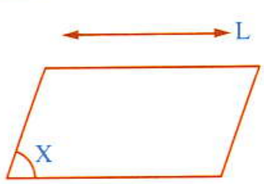
- L_1 , L_2 are skew.
- $L_1 \cap L_2 = \varnothing$
- They do not lie in one plane.

Notice that

The two skew straight lines are not parallel and not intersecting because they are not in one plane.

2 The relative positions for the straight line and the plane in the space :

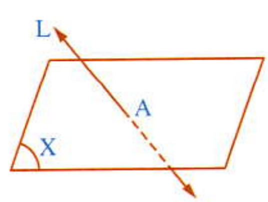
The straight line is parallel to the plane



- The straight line L // the plane X

i.e. $L \cap X = \varnothing$

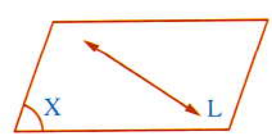
The straight line intersects the plane



- The straight line L intersects the plane X in one point.

i.e. $L \cap X = \{A\}$

The straight line lies completely in the plane



- The straight line L lies completely in the plane X ($L \subset X$)

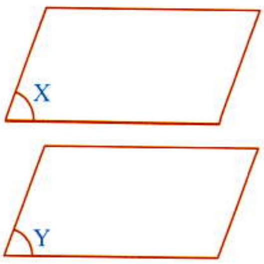
i.e. $L \cap X = L$

Notice that

If a straight line intersects a plane in more than one point , then the straight line lies completely in the plane.

3 The relative positions for two different planes in the space :

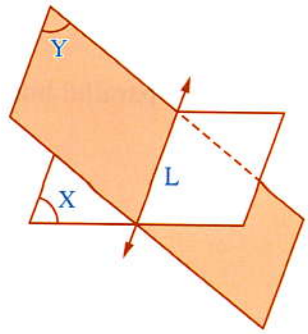
Two parallel planes



The plane X // the plane Y

i.e. $X \cap Y = \varnothing$

Two intersecting planes



The two planes intersecting at a straight line L

i.e. $X \cap Y = L$

Two coincident planes

The two planes are in common in all points (Coincide)

i.e. $X \cap Y = X = Y$

Remarks

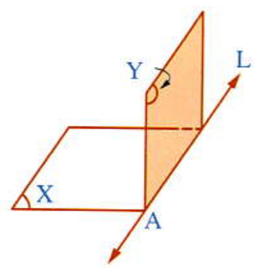
(1) If there are two planes have a common point , then they have in a common a straight line passing through this point.

In the opposite figure :

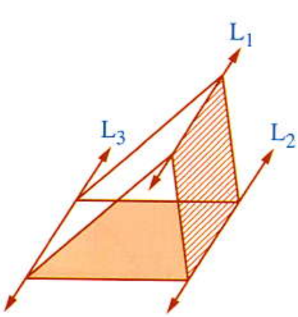
The two planes X and Y have a common point (A) then :

the two planes X and Y have a common straight line (L)

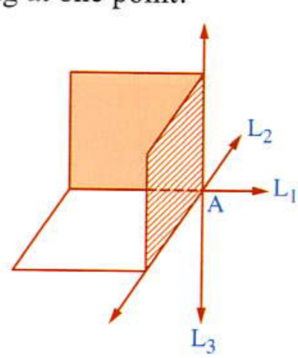
i.e. $X \cap Y = L$ where $A \in L$



(2) If there are three planes are intersected "each two with each other" , then their intersected straight lines will be parallel or intersecting at one point.



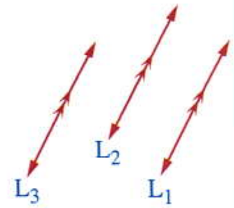
$L_1 // L_2 // L_3$



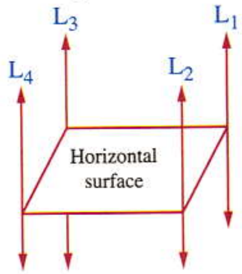
$L_1 \cap L_2 \cap L_3 = \{A\}$

(3) If two straight lines are parallel to a third in the space , then they all are parallel.

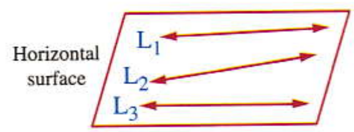
i.e. If $L_1 // L_3$, $L_2 // L_3$, then $L_1 // L_2$



(4) All vertical straight lines in the space are parallel but not all horizontal straight lines in the space are parallel.

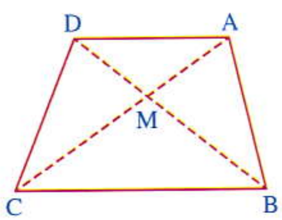


All the vertical straight lines L_1 , L_2 , L_3 and L_4 are parallel.

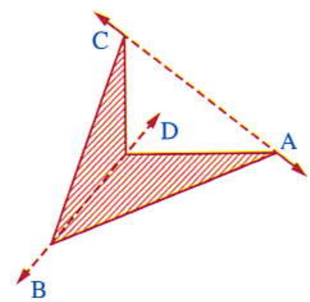


The horizontal straight lines L_1 , L_2 and L_3 are not parallel.

(5) If the straight lines contain the diagonals of a quadrilateral intersected at a point , then all its sides lie in one plane.



The sides of the quadrilateral ABCD lie in one plane , because $\overleftrightarrow{AC} \cap \overleftrightarrow{BD} = \{M\}$

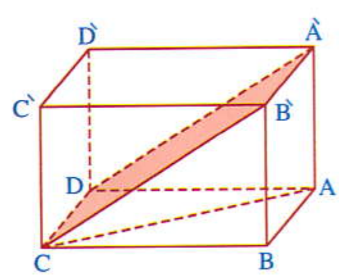


The sides of the quadrilateral ABCD doesn't lie in one plane , because $\overleftrightarrow{AC} \cap \overleftrightarrow{BD} = \varnothing$ $\left(\overleftrightarrow{AC} , \overleftrightarrow{BD} \text{ are skew}\right)$

Example 2

In the opposite figure :

ABCD $A'B'C'D'$ is a cuboid , complete the following :



(1) $\overleftrightarrow{BC}$ // the plane ....

(2) $\overleftrightarrow{AB}$ and .... are skew.

(3) The plane AB $B'A'$ // the plane ....

(4) The plane $ABB'A' \cap$ the plane $A'B'CD = \cdots\cdots$

(5) The plane $ABC \cap$ the plane $A'B'CD = \cdots\cdots$

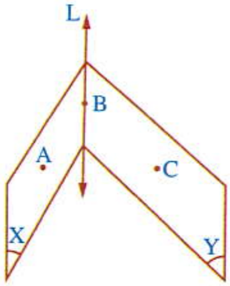
(6) The plane $AA'D'D \cap$ the plane $ABB'A' \cap$ the plane $ABCD = \cdots\cdots$

Solution

(1) $AA'D'D$ or $A'B'C'D'$ (2) $\overleftrightarrow{DD'}$ or $\overleftrightarrow{A'D'}$ (Find other answers) (3) $DCC'D'$

(4) $\overleftrightarrow{AB}$ (5) $\overleftrightarrow{DC}$ (6) $\{A\}$

Example 3



In the opposite figure :

The plane $X \cap$ the plane Y = the straight line L

$A \in X$, $C \in Y$, $B \in L$

Choose the correct answer from those given :

(1) The plane $ABC \cap$ the plane $X = \cdots\cdots$ ($\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\{B\}$)

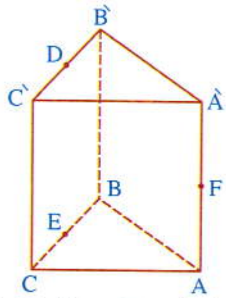
(2) The plane $ABC \cap$ the plane $Y = \cdots\cdots$ ($\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\{B\}$)

(3) The plane $ABC \cap$ the plane $X \cap$ the plane $Y = \cdots\cdots$ ($\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\{B\}$)

Solution

(1) $\overleftrightarrow{AB}$ (2) $\overleftrightarrow{BC}$ (3) $\{B\}$

Example 4



In the opposite figure :

If the plane ABC // the plane $A'B'C'$, D and E are the midpoints of $\overline{B'C'}$ and $\overline{BC}$ respectively , $F \in \overline{AA'}$

1 Find four planes passing through the point (A)

2 Choose the correct answer from those given :

(1) A $\cdots\cdots$ the plane DFE ($\in$, $\notin$, $\subset$, $\not\subset$)

(2) $\overleftrightarrow{AA'}$ $\cdots\cdots$ the plane DFE ($\in$, $\notin$, $\subset$, $\not\subset$)

(3) $\overleftrightarrow{AA'}$, $\overleftrightarrow{BC}$ are two $\cdots\cdots$ straight lines.(parallel , skew , intersecting , perpendicular)

(4) The plane $DFE \cap$ The plane $A'B'C' = \cdots\cdots$ ($\overleftrightarrow{A'D}$, $\overleftrightarrow{AE}$, $\overleftrightarrow{DE}$, $\overleftrightarrow{FE}$)

(5) The plane $DEF \cap$ The plane $ABB'A' = \cdots\cdots$ ($\overleftrightarrow{AA'}$, $\overleftrightarrow{AE}$, $\overleftrightarrow{A'D}$, $\overleftrightarrow{DE}$)

(6) The plane $AEA' \cap$ The plane $B'BCC' = \cdots\cdots$ ($\overleftrightarrow{DE}$, $\overleftrightarrow{AE}$, $\overleftrightarrow{A'D}$, $\overleftrightarrow{AA'}$)

Solution

1 The planes are : $AA'B'B$, $AA'C'C$, $AA'E$, ABC

2 (1) $\in$ (2) $\subset$ (3) skew

(4) $\overleftrightarrow{A'D}$ (5) $\overleftrightarrow{AA'}$ (6) $\overleftrightarrow{DE}$

Lesson

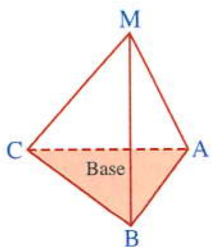
2

The pyramid

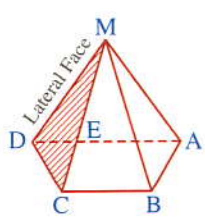
The definition of the pyramid

Is a solid has one base as a polygon and all its other faces are triangles with a common vertex and the pyramid is called a triangular , quadrilateral, pentagonal or .... according to the number of sides of the polygon of its base.

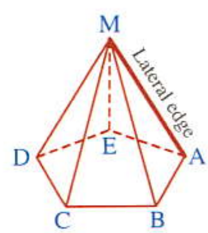
For example :



triangular pyramid,
its base is a triangle

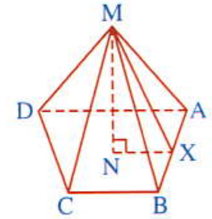


quadrilateral pyramid,
its base is a quadrilateral



pentagonal pyramid,
its base is a pentagon

By using the opposite figure : we can explain some concepts of the pyramid :



- MABCD is a quadrilateral pyramid.
  its lateral faces are the triangles
  MAB, MBC, MCD, MAD, and its
  base is a polygon ABCD
- The lateral faces of the pyramid : are always triangles but the base could be a triangle or a quadrilateral or a pentagon or ....
- The vertex of the pyramid : Is the common point for all lateral faces of the pyramid.
  In the figure : The point "M" is the vertex of the pyramid MABCD

- The lateral edge of the pyramid : Is a line segment joining between the vertex of the pyramid and any vertex of its base vertices. as ($\overline{MA}$, $\overline{MB}$, $\overline{MC}$, $\overline{MD}$, as in the figure)
- The height of the pyramid : Is the distance between the vertex of the pyramid and its base surface.

  i.e. Is the length of the perpendicular from the vertex of the pyramid to its base surface.

  (MN is the height of the pyramid as in the figure)
- The slant height of the pyramid : Is the distance between the vertex of the pyramid and one of its base sides.

  i.e. Is the length of the perpendicular line segment from the vertex to one of the base sides of the pyramid.

  ($\overline{MX}$ is a slant height of the pyramid MABCD where $\overline{MX} \perp \overline{AB}$)

Remarks

- The perpendicular straight line to a plane is perpendicular to any straight line in that plane , then the perpendicular straight line to the base of the pyramid is perpendicular to any straight line in it.
- The regular polygon is a polygon in which its sides are equal in length and its angles are equal in measure.
- The geometrical centre of any regular polygon is the centre of inscribed circle or the circumcircle for it.
- The geometrical centre of the parallelogram and its special cases is the point of intersection of the diagonals.
- The geometrical centre of the triangle is the point of intersection of its medians.

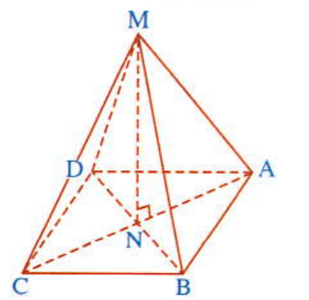
Special cases of the pyramid

(1) The right pyramid :

The pyramid is right if the position of perpendicular from the vertex of the pyramid to the base is passing through its geometrical centre.

For example :

- In the pyramid MABCD as in the figure : If N is the geometrical centre of the base ABCD and $\overline{MN} \perp$ the plane of the base ABCD , then the pyramid MABCD is called a right pyramid.

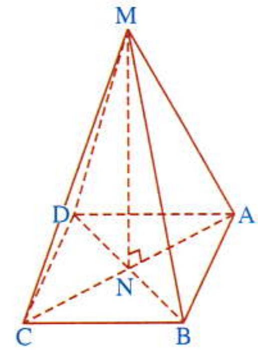


(2) The regular pyramid :

Is the pyramid in which its base is a regular polygon whose centre is the position of the perpendicular from the vertex of the pyramid to the base.
i.e. A right pyramid of a base as a regular polygon.

For example :

In the pyramid MABCD as in the figure :
if N is the geometrical centre of the regular base ABCD "square" and $\overline{MN} \perp$ the plane of the base , then the pyramid MABCD is a regular pyramid.



Properties of the regular pyramid :

(1) Its lateral edges are equal in length.
(2) Its slant heights are equal in length.
(3) Its lateral faces are congruent isosceles triangles.

Remarks

- Every regular pyramid is a right pyramid but not vice verse.
- Not necessary that the lateral edges of the right pyramid are equal in length.
- Not necessary that the slant heights of the right pyramid are equal in length.
- The regular triangular pyramid is called a triangular pyramid of regular faces if its all faces are equilateral triangles and any one of them is its base.

Example 1

MABCD is a regular quadrilateral pyramid , the length of its base side is 12 cm. and its height length equals 8 cm. Find the length of its slant height.

Solution

Let X is a midpoint of $\overline{AB}$

$\because$ MABCD is a regular quadrilateral pyramid

$\therefore MA = MB$ $\qquad \therefore \overline{MX} \perp \overline{AB}$

$\therefore \overline{MX}$ is the slant height of the pyramid

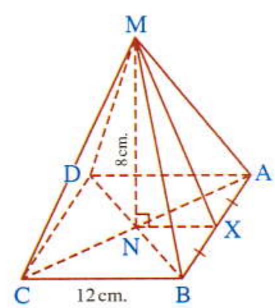
In ΔABC : X is a midpoint of $\overline{AB}$, N is a midpoint of $\overline{AC}$

$\therefore XN = \frac{1}{2} BC$ $\qquad \therefore XN = 6$ cm.

$\because \overline{MN} \perp$ The plane ABCD $\qquad \therefore \overline{MN} \perp \overline{XN}$

$\therefore \Delta MXN$ is right at N

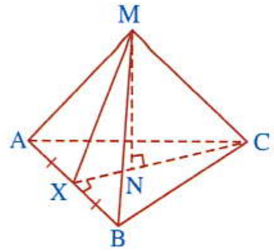
$\therefore (XM)^2 = (XN)^2 + (MN)^2$ $\qquad \therefore (XM)^2 = 36 + 64 = 100$ $\qquad \therefore XM = 10$ cm.



Example 2

MABC is a regular triangular pyramid with base Δ ABC, the length of its base length is 6 cm. and the length of its height is 4 cm. Find the length of its edge and its slant height.

Solution



Let X is the midpoint of $\overline{AB}$

$\because$ MABC is a regular triangular pyramid

$\therefore \Delta$ ABC is an equilateral triangle

, $\because$ X is the midpoint of $\overline{AB}$ $\quad\therefore \overline{CX} \perp \overline{AB}$

$\therefore \Delta$ BXC is a right-angled triangle at X

$\therefore (XC)^2 = (BC)^2 - (BX)^2 = 36 - 9 = 27$

$\therefore XC = \sqrt{27} = 3\sqrt{3}$ cm.

, $\because$ N is the centre of Δ ABC

$\therefore$ N is the point of intersection of the medians of Δ ABC

$\therefore NX : NC = 1 : 2$ $\quad\therefore NX = \sqrt{3}$ cm. , $NC = 2\sqrt{3}$ cm.

$\because \overline{MN} \perp$ The plane ABC $\quad\therefore \overline{MN} \perp \overline{XC}$

$\therefore \Delta$ MNC is right-angled triangle at N

$\therefore (MC)^2 = (MN)^2 + (NC)^2 = 16 + 12 = 28$ $\quad\therefore MC = \sqrt{28} = 2\sqrt{7}$ cm.

$\therefore$ The length of the edge of the pyramid $= 2\sqrt{7}$ cm.

, $\because \Delta$ MNC is right-angled triangle at N

$\therefore (MX)^2 = (MN)^2 + (NX)^2 = 16 + 3 = 19$

$\therefore MX = \sqrt{19}$ cm.

, $\because$ X is a midpoint of $\overline{AB}$ $\quad\therefore \overline{MX} \perp \overline{AB}$

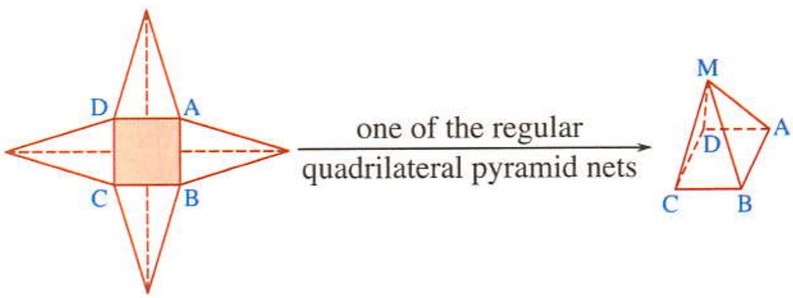
$\therefore \overline{MX}$ is the slant height of the pyramid $= \sqrt{19}$ cm.

UNIT 2

Solids net

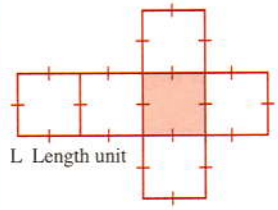
The solids net is used to make the solids by tracking the shape of the solid on the surface of the plane and folding this plane to form the solid.

The pyramid net :

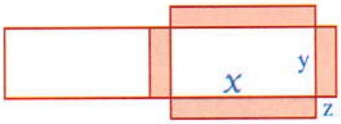


Remember

(1) One of the cube nets



(2) One of the cuboid nets



From the net of the regular quadrilateral pyramid we note that :

(1) It has 5 faces , four of them are lateral faces and one face to the base.

(2) It has 8 edges , four of them are lateral edges.

(3) It has 5 vertices , one of them (M) is called the vertex of the pyramid.

Enrich your knowledge

Euler relation : For any solid in which its base as a polygon , then :

(Number of its faces + number of its vertices = number of its edges + 2)

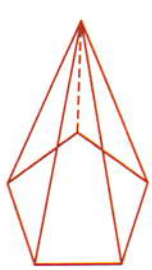
For example : In the pentagonal pyramid : number of its faces = 6 faces ,

number of its vertices = 6 vertices , number of its edges = 10 edges)

i.e. Number of its faces + number of its vertices = 6 + 6 = 12

, number of its edges + 2 = 10 + 2 = 12

∴ Number of faces + number of vertices = number of edges + 2

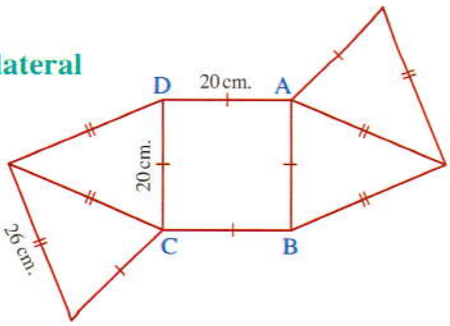


Example 3

The opposite net represents a net of a regular quadrilateral pyramid. Find :

(1) Height of the pyramid.

(2) The slant height of the pyramid.



Solution

The net is representing a regular quadrilateral pyramid its base ABCD is a square , its vertex M , and its height $\overline{MN}$, where N is the point of intersection of the diagonals of the base.

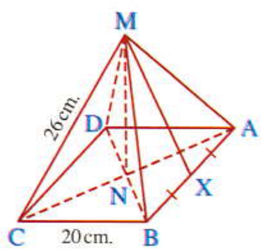
$\because$ ABCD is a square

$\therefore$ The length of its diagonal = its side length $\times \sqrt{2}$

$\therefore AC = 20 \times \sqrt{2} = 20\sqrt{2}$ cm.

$\therefore AN = 10\sqrt{2}$ cm.

$\because$ MABCD is a right quadrilateral pyramid



$\therefore \overline{MN} \perp$ The plane of the base ABCD $\qquad \therefore \overline{MN} \perp \overline{AN}$

$\therefore \Delta$ ANM in which m $(\angle ANM) = 90^\circ$

$\therefore (MN)^2 = (AM)^2 - (AN)^2 = (26)^2 - \left(10\sqrt{2}\right)^2 = 476$

$\therefore MN = \sqrt{476} = 2\sqrt{119}$ cm.

$\therefore$ The height of the pyramid $= 2\sqrt{119}$ cm.

Let X is a midpoint of $\overline{AB}$ $\qquad \therefore AX = 10$ cm.

, $\because MA = MB$ $\qquad \therefore \overline{MX} \perp \overline{AB}$

$\therefore \Delta$ AXM in which m $(\angle AXM) = 90^\circ$

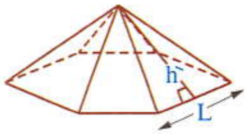
$\therefore (MX)^2 = (AM)^2 - (AX)^2 = (26)^2 - (10)^2 = 576 \qquad \therefore MX = \sqrt{576} = 24$ cm.

$\therefore$ The slant height of the pyramid = 24 cm.

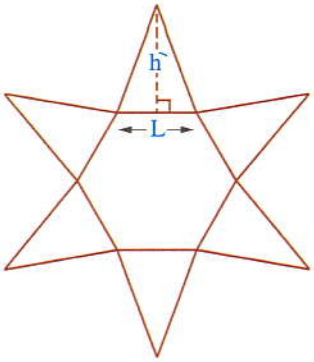
The lateral area of regular pyramid - the total area of pyramid - the volume of pyramid :

* The lateral area of the pyramid = the sum of areas of the lateral faces
* The lateral area of the regular pyramid = $\frac{1}{2}$ base perimeter × slant height
* The total surface area of the pyramid = lateral area + area of the base
* The volume of the pyramid = $\frac{1}{3}$ base area × height

Finding the lateral area of regular pyramid



If the side length of the base in regular pyramid is L and the number of its base sides is n and its lateral height h̀ , then from the net of this pyramid we find it has n congruent faces each one is an isosceles triangle and the area of each triangle = $\frac{1}{2} \times L \times \grave{h}$



$\therefore$ The lateral area of the regular pyramid

$$= \frac{1}{2} L \times \grave{h} \times n ,$$

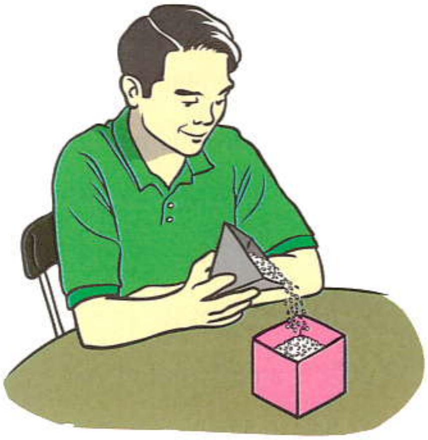
, $\because$ perimeter of the base = n × L

$\therefore$ The lateral area of the regular pyramid

$= \frac{1}{2} \times$ base perimeter × slant height

Finding the volume of the pyramid

Experiment activity



* Bring a hollow vessel in the shape of a right prism , and another one in the shape of a right pyramid where be their bases are congruent and they have the same height as in the opposite figure.
* Fill the pyramid vessel with **grains** of rice or sand then put it in the prism.
* Repeat this process three times and you will note that : The prism will filled completely with the **grains** and that means :

Volume of the pyramid = $\frac{1}{3}$ volume of the prism has the same base and height.

$\because$ The volume of the prism = the base area × height

$\therefore$ Volume of the pyramid = $\frac{1}{3}$ × base area × height

Remarks

(1) In the triangular pyramid of regular faces :
double the square of its edge length = 3 times the square of its height.

i.e. $2\,L^2 = 3\,h^2$ Where L = edge length , h = the height

(2) The total surface area to the triangular pyramid of regular faces $= L^2\sqrt{3}$ where L is the length of edge.

(3) The volume of the triangular pyramid of regular faces $= \frac{\sqrt{2}}{12} L^3$ where L is the edge length.

Example 4

A regular quadrilateral pyramid the length of its base diagonal is $60\sqrt{2}$ cm. and its slant height is 50 cm. Find :

(1) The height of the pyramid.

(2) L.S.A and T.S.A of the pyramid.

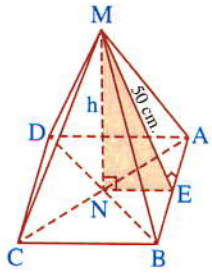
(3) Volume of the pyramid.

Solution

* Let MABCD is a regular quadrilateral pyramid , the diagonals of its base intersected at N

, the length of the base side $= \frac{60\sqrt{2}}{\sqrt{2}} = 60$ cm.

, E is the midpoint of $\overline{AB}$



(1) $\because$ The quadrilateral pyramid is regular.

$\therefore$ Its base as a square shape.

, $\overline{MN} \perp$ the plane ABCD $\qquad \therefore \overline{MN} \perp \overline{EN}$

$\therefore \Delta$ MEN is right angled at N ,

$\because$ E is the midpoint of $\overline{AB}$, N is the midpoint of $\overline{AC}$

$\therefore EN = \frac{1}{2} BC = 30$ cm. $\qquad \therefore h = \sqrt{50^2 - 30^2} = 40$ cm.

(2) The L.S.A of the pyramid $= \frac{1}{2} \times$ base perimeter $\times$ slant height

$= \frac{1}{2} \times (60 \times 4) \times 50 = 6000 \text{ cm}^2.$

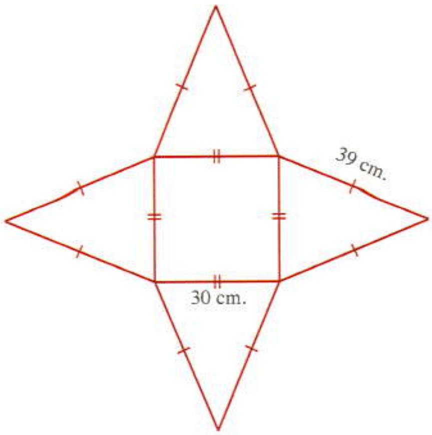
$\because$ The area of the base $= 60 \times 60 = 3600 \text{ cm}^2.$

$\therefore$ The T.S.A of the pyramid = L.S.A + the base area $= 6000 + 3600 = 9600 \text{ cm}^2.$

(3) Volume of the pyramid $= \frac{1}{3}$ base area $\times$ height $= \frac{1}{3} \times 3600 \times 40 = 48000 \text{ cm}^3.$

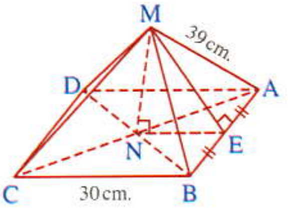
Example 5

Use the opposite net to describe the formed solid , then find its total surface area and its volume.



Solution

The net represents a regular quadrilateral pyramid , its base as a square of side length = 30 cm. and the length of its lateral edge = 39 cm.



, let the pyramid MABCD , N is the point of intersection of the diagonals of the base , E is the midpoint of $\overline{AB}$

, $\because$ the lateral face MAB is an isosceles triangle.

$\therefore$ $\overline{ME}$ is slant height.

, AE = 15 cm.

$\therefore$ Δ AEM which is right - angled at E :

$\therefore$ $ME = \sqrt{(MA)^2 - (AE)^2} = \sqrt{(39)^2 - (15)^2} = 36$ cm.

$\because$ $\overline{MN} \perp$ the plane ABCD $\therefore$ $\overline{MN} \perp \overline{EN}$

, $\because$ $EN = \frac{1}{2} BC = 15$ cm.

$\therefore$ In Δ MEN which is right - angled at N :

$MN = \sqrt{(ME)^2 - (EN)^2} = \sqrt{(36)^2 - (15)^2} = 3\sqrt{119}$ cm.

$\therefore$ The L.S.A. of the pyramid $= \frac{1}{2} \times$ base perimeter $\times$ slant height

$= \frac{1}{2} \times (30 \times 4) \times 36 = 2160$ cm².

, area of the base $= 30 \times 30 = 900$ cm².

$\therefore$ The T.S.A. of the pyramid = L.S.A + the base area

$= 2160 + 900 = 3060$ cm².

, $\because$ the volume of the pyramid $= \frac{1}{3}$ base area $\times$ height

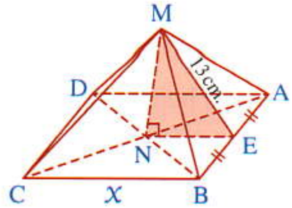
$= \frac{1}{3} \times 900 \times 3\sqrt{119} = 900\sqrt{119}$ cm³.

Example 6

MABCD is a regular quadrilateral pyramid , its total surface area = 360 cm². and its slant height = 13 cm. Find the length of its base edge , then find its volume.

Solution

Let the edge length of the squared base = X cm.



, $\because$ the T.S.A of the pyramid = 360 cm².

$\therefore$ The base area + L.S.A = 360

$\therefore X \times X + \frac{1}{2} \times 4X \times 13 = 360$

$\therefore X^2 + 26X - 360 = 0$

$\therefore (X + 36)(X - 10) = 0$

$\therefore X = -36$ (refused) or $X = 10$ $\therefore$ The edge length of the pyramid base = 10 cm.

$\because$ EN = $\frac{1}{2}$ BC = 5 cm. , $\because$ Δ MEN is right-angled at N

$\therefore$ MN = $\sqrt{13^2 - 5^2} = 12$ cm.

$\therefore$ Volume of the pyramid = $\frac{1}{3}$ base area $\times$ height = $\frac{1}{3} \times (10)^2 \times 12 = 400$ cm³.

Example 7

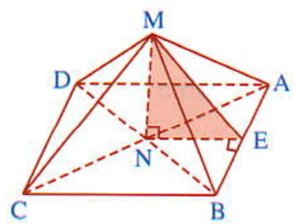
A regular quadrilateral pyramid of volume 48 cm³. and the length of its base edge = 6 cm. , find its total surface area.

Solution

Let MABCD is a regular quadrilateral pyramid

, N is the intersection point of its base diagonal

, E is the midpoint of $\overline{AB}$



$\because$ Volume of the pyramid = 48 cm³.

$\therefore \frac{1}{3} \times$ the base area $\times$ the height = 48

$\therefore \frac{1}{3} \times 6 \times 6 \times h = 48$, $\therefore$ h = 4 cm.

$\therefore$ The height of the pyramid = MN = 4 cm. , $\because$ EN = $\frac{1}{2}$ BC = 3 cm.,

$\because$ Δ MEN is right-angled at N $\therefore$ ME = $\sqrt{4^2 + 3^2} = 5$ cm.

$\therefore$ The T.S.A of the pyramid = L.S.A. + the base area

$= \frac{1}{2}$ base perimeter $\times$ slant height + base area

$= \frac{1}{2} \times (4 \times 6) \times 5 + 6 \times 6 = 96$ cm².

Example 8

MABC is triangular pyramid of regular faces , the length of each edge of its edges equals $8\sqrt{3}$ cm. Find :

(1) The slant height of the pyramid.

(2) Height of the pyramid.

(3) T.S.A of the pyramid.

(4) Volume of the pyramid.

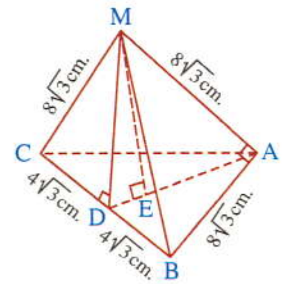
Solution

$\because$ The triangular pyramid is a regular faces.

$\therefore$ Each face is an equilateral triangle.

$\therefore$ The slant height of the pyramid

$= MD = AD = 8\sqrt{3}\sin 60° = 12$ cm.



, $\because$ E is the point of intersection of the medians of Δ ABC

$\therefore AE = \frac{2}{3} AD = \frac{2}{3} \times 12 = 8$ cm.

, $\because \overline{ME} \perp$ the plane ABC

$\therefore \overline{ME} \perp \overline{AD}$

$\therefore \Delta$ MAE is right-angled at E

$\therefore ME = \sqrt{(8\sqrt{3})^2 - (8)^2} = 8\sqrt{2}$ cm.

$\therefore$ The height of the pyramid $= 8\sqrt{2}$ cm.

, $\because$ L.S.A. of the pyramid

$= \frac{1}{2}$ base perimeter $\times$ slant height

$= \frac{1}{2} \times (3 \times 8\sqrt{3}) \times 12 = 144\sqrt{3}$ cm^2.

Notice that

$\because$ The triangular pyramid is a regular faces.

$\therefore 2L^2 = 3h^2$

$\therefore 2 \times (8\sqrt{3})^2 = 3h^2$

$\therefore h = 8\sqrt{2}$

$\therefore$ Height of the pyramid $= 8\sqrt{2}$ cm.

$\because$ Area of the base $= \frac{1}{2} \times 8\sqrt{3} \times 8\sqrt{3}\sin 60° = 48\sqrt{3}$ cm^2.

$\therefore$ T.S.A. of the pyramid $= 144\sqrt{3} + 48\sqrt{3} = 192\sqrt{3}$ cm^2.

, $\because$ volume of the pyramid $= \frac{1}{3}$ base area $\times$ height

$= \frac{1}{3} \times 48\sqrt{3} \times 8\sqrt{2} = 128\sqrt{6}$ cm^3.

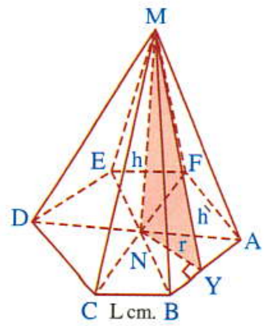
Example 9

A regular hexagonal pyramid in which the sum of areas of its lateral faces is seven times its base area.

Prove that : The volume of the pyramid $= 8r^3$

Where (r) is the radius of the inscribed circle of the base.

Solution



Let the edge length of the base of the pyramid = L cm.

, height of the pyramid = h , and its slant height = $\grave{h}$

$\because$ The sum of areas of its lateral faces = 7 × base area

$\therefore \frac{1}{2}$ × base perimeter × slant height = 7 × base area

$\therefore \frac{1}{2} \times 6 L \times \grave{h} = 7 \times \frac{1}{2} \times L \times r \times 6$

$\therefore 3 L \grave{h} = 21 Lr$ $\therefore \boxed{\grave{h} = 7 r}$

, $\because \overline{MN} \perp$ the plane ABCDEF

$\therefore \overline{MN} \perp \overline{NY}$

$\therefore \Delta$ MNY is right-angled at N

$\therefore h = \sqrt{\grave{h}^2 - r^2} = \sqrt{(7r)^2 - r^2} = 4\sqrt{3}\, r$

, $\because$ volume of the pyramid = $\frac{1}{3}$ × base area × height

$= \frac{1}{3} \times \frac{1}{2} \times L \times r \times 6 \times 4\sqrt{3}\, r = 4\sqrt{3}\, L\, r^2$

, $\because r = L \sin 60°$ $\therefore L = \frac{2}{\sqrt{3}} r$

$\therefore$ Volume of the pyramid $= 4\sqrt{3} \times \frac{2}{\sqrt{3}} r \times r^2 = 8 r^3$

Example 10

MABC is a triangular pyramid its vertex M is at distance $4\sqrt{5}$ cm. from the base ABC where AB = 7 cm. , BC = 8 cm. , AC = 9 cm. Find the volume of the pyramid.

Solution

$\because$ The perimeter of Δ ABC

$= 7 + 8 + 9 = 24$ cm.

$\therefore$ Half the perimeter = 12 cm.

$\therefore$ The area of Δ ABC

$= \sqrt{12 (12 - 7)(12 - 8)(12 - 9)}$

$= 12\sqrt{5}$ cm^2.

Remember that

The area of $\Delta ABC = \sqrt{S (S - AB)(S - BC)(S - AC)}$
where : S is half the perimeter of Δ ABC

$\therefore$ The volume of the pyramid = $\frac{1}{3}$ × base area × height

$= \frac{1}{3} \times 12\sqrt{5} \times 4\sqrt{5} = 80$ cm^3.

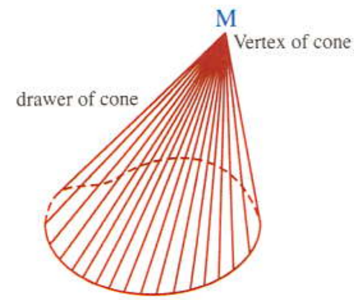
Lesson

3

The cone

The definition of the cone

Is a solid has only one base as a closed curve and one vertex , and its lateral surface formed from line segments drawn from its vertex to its curved base and each of them is called drawer of the cone.

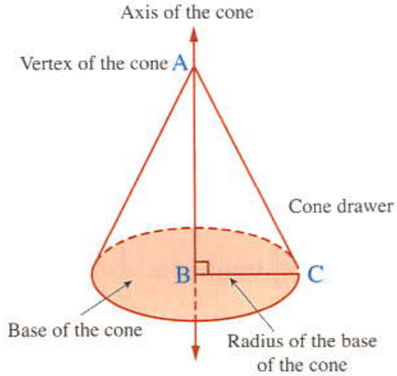


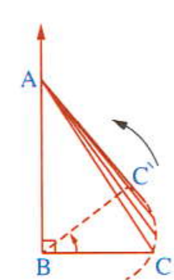
The right circular cone

Is the solid formed from the rotation of a right-angled triangle with complete rotation about one of its right sides as an axis or is the space formed from the folding of a circular sector where their two radii coincide on each other.

In the opposite figure :

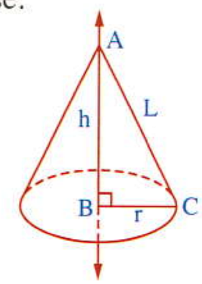
Δ ABC is right angled triangle at B , if it is rotated about the axis $\overleftrightarrow{AB}$ with a full turn , the formed solid is called right circular cone , and the point A is called vertex of the cone , $\overline{AC}$ is the drawer of the cone , $\overleftrightarrow{AB}$ is axis of the cone , surface of circle B is the base of the cone.





Properties of the right circular cone :

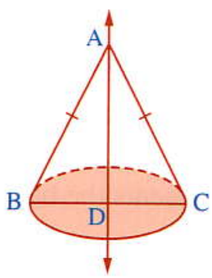
(1) The axis of the right circular cone is perpendicular to the plane of the base.
i.e. $\overleftrightarrow{AB} \perp$ the plane of circle B

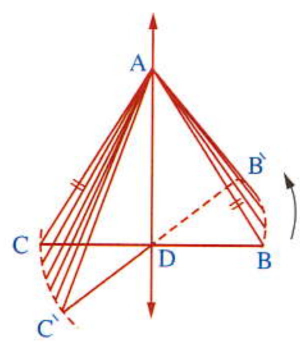


(2) The height of the right circular cone is the length of the line segment joining between the vertex of the cone and the centre of its base and its length is always less than the length of the drawer of the cone.
If the length of $\overline{AB}$ = h length unit , the length of $\overline{AC}$ = L length unit.
Then the height of the cone (h) = $\sqrt{L^2 - r^2}$, then : $h < L$

Remark

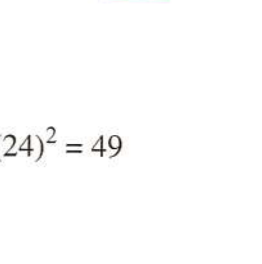
The right circular cone can be formed by the rotation of an isosceles triangle about its axis of symmetry by a half turn.
In the opposite figure :
If Δ ABC is isosceles in which AB = AC , $\overleftrightarrow{AD}$ is the axis of symmetry of Δ ABC and the triangle ABC is rotated about $\overleftrightarrow{AD}$ by a half turn , then the formed solid is a right circular cone its base is the circle D , and its drawer is $\overline{AB}$ or $\overline{AC}$, its height is $\overline{AD}$ and its vertex is the point A





Example 1

A right circular cone , the length of its drawer is 25 cm. and its height is 24 cm. Find the perimeter and the area of the base of the cone. $\left(\pi = \frac{22}{7}\right)$



Solution

$\because \overleftrightarrow{AB} \perp$ the plane of circle B $\qquad \therefore \overline{AB} \perp \overline{BC}$

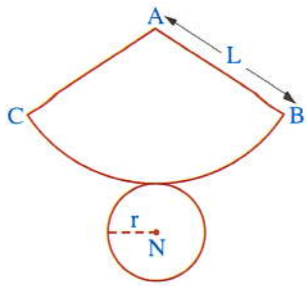
$\therefore m(\angle ABC) = 90°$ $\qquad \therefore (BC)^2 = (AC)^2 - (AB)^2 = (25)^2 - (24)^2 = 49$

$\therefore$ BC = 7 cm. $\qquad \therefore$ r (the radius of the base) = 7 cm.

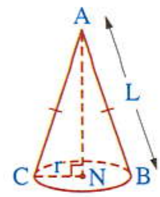
$\therefore$ The perimeter of the base $= 2\pi r = 2 \times \frac{22}{7} \times 7 = 44$ cm.

, the area of the base $= \pi r^2 = \frac{22}{7} \times 49 = 154\ cm^2$.

The right cone net :

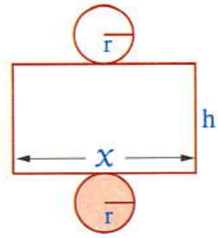


the right cone net →



Remember

One of the right circular cylinder nets



From the net of the right cone we note that :

(1) AB = AC = L , where L is the length of drawer of the cone.

(2) The circular sector ABC represents the lateral surface of the cone and the length of $\overset{\frown}{BC}$ = perimeter of the circle N = $2\pi r$

(3) Surface of the circle N represents the base of the cone.

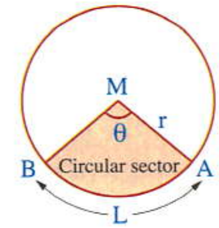
Remember that

The circular sector is a part of the surface of a circle bounded by two radii and an arc of the circle.

* Area of the circular sector = $\frac{1}{2}$ L r

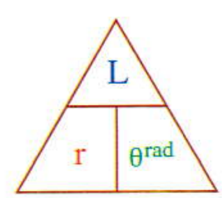
(where L is the length of the arc of the sector)

* Area of the circular sector = $\frac{1}{2}\theta^{rad} r^2$ (where θ^{rad} is the radian measure of the sector angle)



* Area of the circular sector = $\frac{x^\circ}{360^\circ} \times \pi r^2 = \frac{x^\circ}{360^\circ} \times$ area of the circle

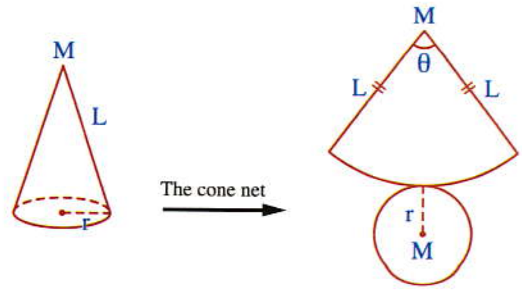
(where x° is degree measure of the sector angle)



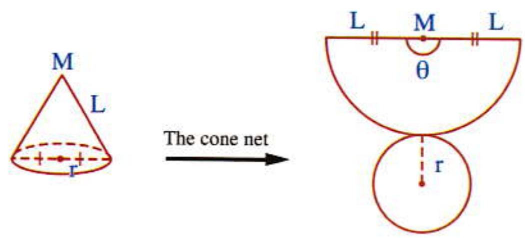
* Perimeter of the sector = 2 r + L length unit.

Important remarks

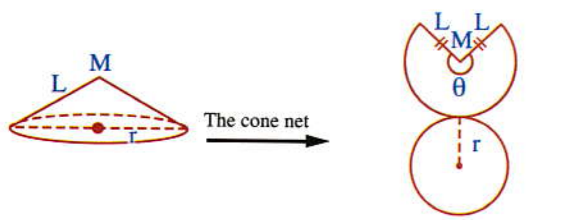
(1) If $L > 2\,r$, then the cone net as shown , $0° < \theta < 180°$



(2) If $L = 2\,r$, then the cone net as shown , $\theta = 180°$



(3) If $L < 2\,r$, then the cone net as shown , $180° < \theta < 360°$

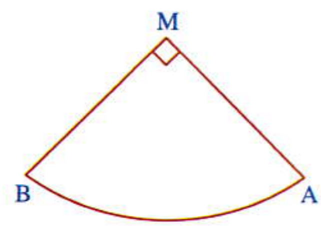


Example 2

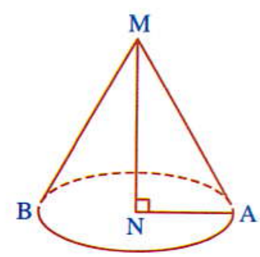
In the opposite figure :

A piece of paper as a circular sector , the area of its surface = $25\,\pi$ cm^2.

and the measure of its central angle equals 90° folded to touch $\overline{MA}$ and $\overline{MB}$ and formed a cone. Find the height of the cone to nearest tenth.



Solution



$\because$ Area of the sector = $\frac{1}{2}\,\theta^{rad}\,r^2$

$\therefore \frac{1}{2}\,\theta^{rad}\,r^2 = 25\,\pi$

$\therefore \frac{1}{2} \times \frac{\pi}{2} \times r^2 = 25\,\pi$ $\qquad \therefore r^2 = 100$

$\therefore r = 10$ cm. $\qquad \therefore MA = 10$ cm.

, $\because$ area of the sector = $\frac{1}{2}\,L\,r$

$\therefore \frac{1}{2} \times L \times 10 = 25\,\pi$ $\qquad \therefore L = 5\,\pi$ cm.

$\therefore$ The length of $\overset{\frown}{AB} = 5\pi$ cm.

$\therefore$ The circumference of the circle N = 5π cm.

$\therefore 2\pi r_N = 5\pi$ $\qquad\qquad$ $\therefore r_N = 2.5$ cm.

$\therefore \Delta$ ANM in which m ($\angle$ ANM) = 90° , NA = 2.5 cm.

, MA = 10 cm.

$\therefore MN = \sqrt{(MA)^2 - (NA)^2} = \sqrt{(10)^2 - (2.5)^2} \simeq 9.7$ cm.

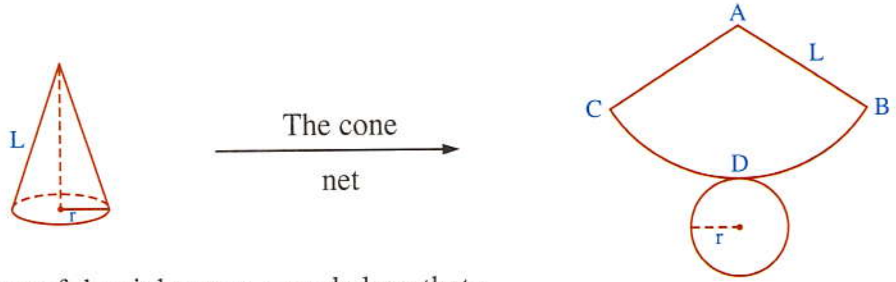
$\therefore$ Height of the cone = 9.7 cm.

The lateral area - total area - volume of a right cone :

If (r) is the radius of the cone base , (L) is the cone drawer , (h) is the height , then :

- The lateral surface area (L.S.A.) of the right cone = $\pi L r$
- The total surface area (T.S.A.) of the right cone = $\pi r (L + r)$
- Volume of the right cone = $\frac{1}{3}\pi r^2 h$

Finding the lateral surface area and total surface area of the right cone



From the net of the right cone , we deduce that :

The lateral surface area of the right cone = the area of sector ABC

$= \frac{1}{2} \times$ length of $\overset{\frown}{BC} \times AB$

$= \frac{1}{2} \times$ perimeter of the cone base $\times$ AB

$= \frac{1}{2} \times 2\pi r \times L$

$= \pi L r$

$\therefore$ The lateral surface area of the right cone = $\pi L r$

, $\because$ the total surface area of the right cone = the lateral surface area + the base area

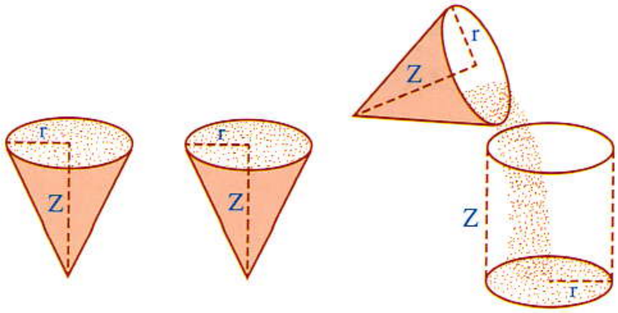
$= \pi L r + \pi r^2$

$\therefore$ The total surface area of the right cone = $\pi r (L + r)$

Finding the volume of the right cone

Experiment activity

* Bring a hollow vessel as a right circular cylinder and another one as a right circular cone where be their bases are congruent and they have the same height as in the opposite figure.



* Fill the cone vessel with grains of rice or sand then empties it in the cylinder vessel.
* Repeat this process three times and you will note that : the cylinder will filled completely with the grains and that means :

The volume of the cone = $\frac{1}{3}$ volume of the cylinder has the same base and height

, $\because$ the volume of the cylinder = base area × height

$\therefore$ The volume of the right cone = $\frac{1}{3}$ base area × height

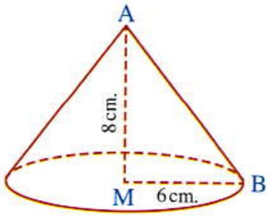
$= \frac{1}{3} \pi r^2 h$

Example 3

A right circular cone , the length of its base diameter is 12 cm. and its height is 8 cm. , find :

(1) The L.S.A. (2) The T.S.A. (3) The volume.

Solution



$\because \overline{AM} \perp$ the circle plane

$\therefore \overline{AM} \perp \overline{MB}$

$\therefore \Delta$ MAB is right-angled at M

, $\because r = \frac{12}{2} = 6$ cm.

$\therefore AB = \sqrt{(8)^2 + (6)^2} = 10$ cm. $\therefore L = 10$ cm.

$\therefore$ The lateral surface area (L.S.A.) = $\pi r L = \pi \times 6 \times 10 = 60 \pi$ cm^2.

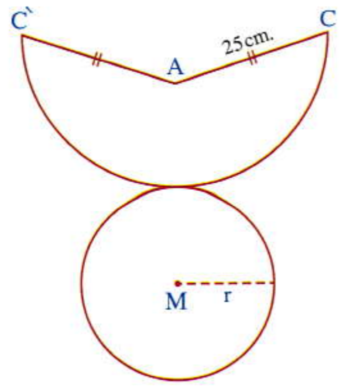
, area of the base = $\pi r^2 = 36 \pi$ cm^2.

$\therefore$ The total surface area (T.S.A) = $60 \pi + 36 \pi = 96 \pi$ cm^2.

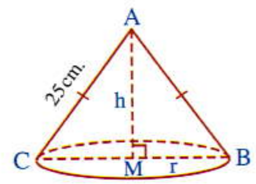
, volume of the cone = $\frac{1}{3} \pi r^2 h = \frac{1}{3} \times \pi \times 36 \times 8 = 96 \pi$ cm^3.

Example 4

Use the opposite net to describe the formed solid and if the length of the arc $\overset{\frown}{CC'} = 30\pi$ cm. Find the volume of this solid and its total surface area.



Solution



The net represents a right cone

, $\because$ the length of $\overset{\frown}{CC'} = 30\pi$

$\therefore 2\pi r = 30\pi$ $\qquad\qquad$ $\therefore r = 15$ cm.

, $\because \Delta ABM$ is right-angled at M

$\therefore h = \sqrt{25^2 - 15^2} = 20$ cm.

$\therefore$ The volume $= \frac{1}{3}$ base area $\times$ height

$= \frac{1}{3} \times \pi \times (15)^2 \times 20 = 1500\pi$ cm^3.

, the total surface area $= \pi r (L + r) = \pi \times 15 \times (15 + 25)$

$= 600\pi$ cm^2.

Example 5

A flask in the shape of a cone of capacity 6.16 litres. and height 30 cm. Find the length of the radius of its base. $\left(\pi \simeq \frac{22}{7}\right)$

Solution

$\because$ The capacity of the flask = 6.16 litres.

$\therefore$ Volume of the right cone $= 6.16 \times 1000$ cm^3.

$\therefore \frac{1}{3} \times \frac{22}{7} \times r^2 \times 30 = 6160$

$\therefore r = 14$ cm.

Remember that

1 litre = 1000 millilitre

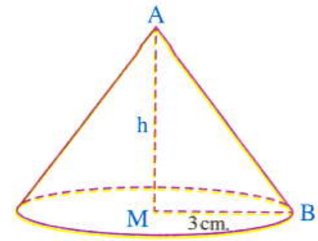
= 1000 cm^3 = 1 dm^3

Example 6

A pure gold alloy as a right cone of radius length 3 cm. and lateral surface area = 15 π cm². Find the gold density if the mass of the alloy = 727 gm. "π = 3.14"

Solution

$\because$ The L.S.A. of the cone = 15 π



$\therefore \pi L r = 15\pi$

$\therefore \pi \times L \times 3 = 15\pi$

$\therefore L = 5$ cm.

$\because \Delta ABM$ is right-angled at M

$\therefore h = AM = \sqrt{5^2 - 3^2} = 4$ cm.

$\because$ Volume of the cone = $\frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \pi \times 3^2 \times 4 = 12\pi = 37.68$ cm³

$\therefore$ The density = $\frac{\text{mass}}{\text{volume}} = \frac{727}{37.68} \simeq 19.3$ gm./cm³

Remember that

The density = $\frac{\text{mass}}{\text{volume}}$

Example 7

A regular octagonal pyramid of silver , its base side length is 6 cm. and its height 30 cm. , melted and convert into a circular right cone whose length of its base radius is 9 cm. if 10% of the silver was missed through the melting process. Find the height of the cone to nearest one decimal place.

Solution

$\because$ The area of the regular octagon = $\frac{8}{4} x^2 \cot \frac{\pi}{8}$

$= 2 \times (6)^2 \cot 22° 30' \simeq 173.82$ cm²

$\therefore$ Volume of the pyramid = $\frac{1}{3}$ base area × height

$= \frac{1}{3} \times 173.82 \times 30 = 1738.2$ cm³

$\therefore$ Volume of the silver in the cone

$= \frac{90}{100} \times 1738.2 \simeq 1564.4$ cm³

$\therefore \frac{1}{3}\pi \times (9)^2 \times h \simeq 1564.4$

$\therefore h \simeq 18.4$ cm.

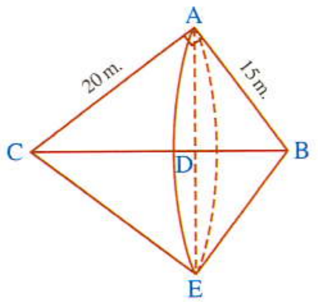
Remember that

Area of the regular polygon whose number of sides = n , and the length of its side x equals $\frac{n}{4} x^2 \cot \frac{\pi}{n}$

Example 8

Δ ABC is right-angled at A , AB = 15 m. , AC = 20 m. , if the triangle is rotated a complete rotation around $\overline{BC}$, describe the formed solid , then find the cost of painting this solid with a material resistant to erosion , if the cost of one square metre = 10 pounds and find the volume of this solid. " $\pi = \frac{22}{7}$ "

Solution



The solid will be as a two right cones with the same base.

From the figure :

Δ ABC is right angled at A , $\overline{AD} \perp \overline{BC}$

$\therefore$ BC $= \sqrt{(15)^2 + (20)^2} = 25$ metres.

, AD $= \frac{15 \times 20}{25} = 12$ metres.

, BD $= \sqrt{(15)^2 - (12)^2} = 9$ metres.

, CD $= 25 - 9 = 16$ metres.

According to the first cone whose vertex is B :

L = 15 m. , r = 12 m. , h = 9 m.

$\therefore$ The L.S.A. $= \pi$ L r $= \pi \times 15 \times 12 = 180\,\pi$ m^2.

, the volume $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times (12)^2 \times 9 = 432\,\pi$ m^3.

According to the second cone whose vertex is C :

L = 20 m. , r = 12 m. , h = 16 m.

$\therefore$ The L.S.A. $= \pi$ L r $= \pi \times 20 \times 12 = 240\,\pi$ m^2.

, the volume $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times (12)^2 \times 16 = 768\,\pi$ m^3.

$\therefore$ The total painting area = the sum of the lateral surface areas of the two cones

$= 180\,\pi + 240\,\pi = 420\,\pi = 420 \times \frac{22}{7} = 1320$ m^2.

$\therefore$ The cost of the painting $= 1320 \times 10 = 13200$ L.E.

, volume of the solid = sum of volumes of the two cones $= 432\,\pi + 768\,\pi = 1200\,\pi$

$= 1200 \times \frac{22}{7} = 3771\frac{3}{7}$ m^3.

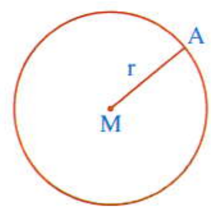
Lesson

4

The circle

Definition of the circle

It is the set of points of the plane which are at a constant distance from a fixed point in the same plane.

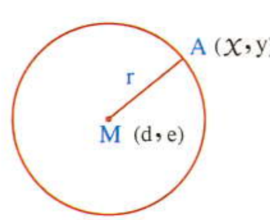


- The fixed point is called "the centre of the circle". (M)
- The constant distance is called "the radius length of the circle". (r)
- The circle is usually denoted by (C)

, where $C = \{A : MA = r , r > 0\}$

First Equation of the circle (In terms of its centre coordinates and radius length)

If $A = (x , y)$ is a point on the circle whose centre $M (d , e)$, and the length of its radius $= r$, in the perpendicular cartesian coordinates using "the distance between two points" rule we find :



$$\sqrt{(x-d)^2+(y-e)^2} = r$$

i.e. $(x-d)^2+(y-e)^2 = r^2$ "Equation of the circle"

Remarks

(1) If the centre of the circle is the origin point $(0, 0)$, then the equation of the circle is :

$$x^2 + y^2 = r^2$$

(2) The position of the point (x_1, y_1) in respect to the circle

$C : (x - d)^2 + (y - e)^2 = r^2$

* If $(x_1 - d)^2 + (y_1 - e)^2 = r^2$, then the point lies on the circle.
* If $(x_1 - d)^2 + (y_1 - e)^2 > r^2$, then the point lies outside the circle.
* If $(x_1 - d)^2 + (y_1 - e)^2 < r^2$, then the point lies inside the circle.

(3) Two circles are congruent if the lengths of their radii are equal.

For example : If the circle equation of C_1 is : $x^2 + y^2 = 49$

, the circle equation of C_2 is $(x - 3)^2 + (y - 4)^2 = 49$

, then $r_1 = r_2 = \sqrt{49} = 7$ length unit, then the two circles are congruent, and the circle C_2 is the image of the circle C_1 by translation $(3, 4)$

Where the image of point (x, y) by translation (a, b) is : $(x + a, y + b)$

Second The general form of the circle equation

The general form of the circle equation is :

$$x^2 + y^2 + 2Lx + 2Ky + C = 0$$

Where the centre (M) $= (-L, -K) = (-\frac{1}{2}$ coefficient of $x, -\frac{1}{2}$ coefficient of y)

, $r = \sqrt{L^2 + K^2 - C}$, $L^2 + K^2 - C > 0$

For example : The circle whose equation is : $x^2 + y^2 + 8x - 4y - 16 = 0$

its centre $= (-\frac{1}{2}$ coefficient of $x, -\frac{1}{2}$ coefficient of y$) = (-4, 2)$

, $r = \sqrt{L^2 + k^2 - c} = \sqrt{16 + 4 - (-16)} = 6$ length unit.

• We can deduce the general form of the circle equation as follows :

We know that : The circle whose centre (d , e) , the length of its radius = r is :

$(x - d)^2 + (y - e)^2 = r^2$

i.e. $x^2 + y^2 - 2dx - 2ey + d^2 + e^2 - r^2 = 0$

Let M (D , K) = (– L , – K) $\therefore x^2 + y^2 + 2Lx + 2Ky + L^2 + K^2 - r^2 = 0$

, ∵ L , K and r are constants. $\therefore L^2 + K^2 - r^2 = C$ (constant)

∴ The general form of the circle equation is :

$$x^2 + y^2 + 2Lx + 2Ky + C = 0$$

Remarks

(1) The general form of the circle equation $x^2 + y^2 + 2Lx + 2Ky + C = 0$ is :

* An equation of 2^{nd} degree in x , y
* Free of the term xy *i.e.* Coefficient of xy = zero
* Coefficient of x^2 = coefficient of $y^2 = 1$

(2) To be the equation of the 2^{nd} degree in x , y represents a circle , it must satisfy the three conditions in the previous remark and $L^2 + K^2 - C > 0$

(3) To identify the centre or the radius length of a circle using the general form must be the coefficient of x^2 = the coefficient of $y^2 = 1$, so we need to divide by this coefficient if is not equals 1

Special cases

1 Equation of the circle passing through the origin point :

$$x^2 + y^2 + 2Lx + 2Ky = 0$$ The equation is free of the absolute term *i.e.* (C = 0)

2 Equation of the circle whose centre lies on x-axis :

$$x^2 + y^2 + 2Lx + C = 0$$ The equation is free of the term containing y *i.e.* (K = 0)

3 Equation of the circle whose centre lies on y-axis :

$$x^2 + y^2 + 2Ky + C = 0$$ The equation is free of the term containing x *i.e.* (L = 0)

4 Equation of the circle touching x-axis :

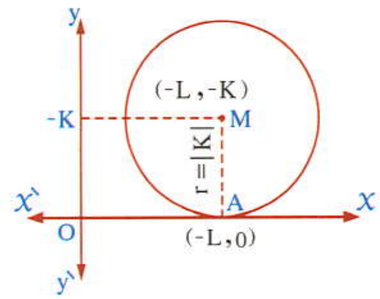
If the circle whose centre $(-L , -K)$ touches x-axis then :

the point of tangency A is : $(-L , 0)$ and $r = |K|$

$\therefore C = L^2 + K^2 - r^2 = L^2 + K^2 - K^2 = L^2$

Then the equation of the circle becomes :

$$x^2 + y^2 + 2Lx + 2Ky + L^2 = 0$$



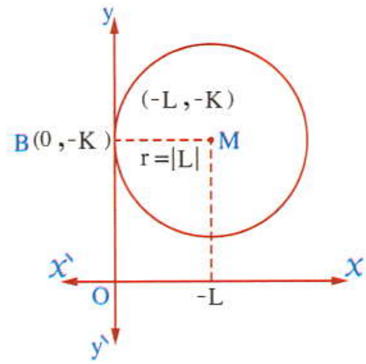
5 Equation of the circle touching y-axis :

If the circle whose centre $(-L , -K)$ touches y-axis , then the point of tangency B is $(0 , -K)$ and $r = |L|$

$\therefore C = L^2 + K^2 - r^2 = L^2 + K^2 - L^2 = K^2$

, then the equation of the circle becomes :

$$x^2 + y^2 + 2Lx + 2Ky + K^2 = 0$$



6 Equation of the circle touching the two coordinates :

If the circle whose centre is $(-L , -K)$ touches the two coordinates axes then $r = |L| = |K|$

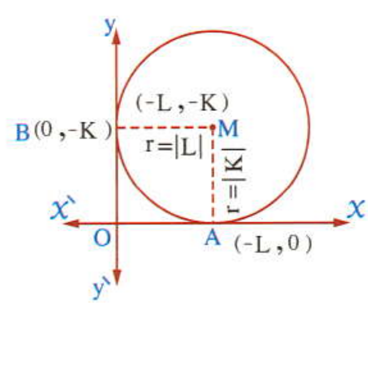
$\therefore C = L^2 + K^2 - r^2 = r^2 + r^2 - r^2 = r^2$

$\therefore \quad C = L^2 = K^2 = r^2$

and the equation of the circle becomes :

$$x^2 + y^2 + 2Lx + 2Ky + C = 0$$

where $|L| = |K| = r , C = L^2 = K^2 = r^2$



Remember that :

(1) The position of a straight line with respect to a circle (D) whose centre (M) and let $\overrightarrow{MC} \perp L$ and intersects it at C

* If $MC < r$, then L is a secant to the circle at two points.
* If $MC = r$, then L is a tangent to the circle.
* If $MC > r$, then L is outside the circle and doesn't intersect it at any point.

(2) If M , N are two circles of radii r_1 , r_2 respectively (where $r_1 > r_2$)

If the two circles M and N	Then
(1) Distant	$MN > r_1 + r_2$
(2) Touching externally	$MN = r_1 + r_2$
(3) Intersecting	$r_1 - r_2 < MN < r_1 + r_2$
(4) Touching internally	$MN = r_1 - r_2$
(5) One inside the other	$MN < r_1 - r_2$
(6) Concentric	MN = zero

(3) The tangent to a circle is perpendicular to the radius drawn from the point of tangency.

(4) The two tangents which are drawn from the two endpoints of a diameter of a circle are parallel.

(5) The two tangent-segments drawn to a circle from a point outside it are equal in length.

(6) If $A = (X_1 , y_1)$, $B = (X_2 , y_2)$ then the midpoint of $\overline{AB} = \left(\frac{X_1 + X_2}{2} , \frac{y_1 + y_2}{2}\right)$

(7) The equation of the straight line passing through (X_1 , y_1)

and its slope (m) is : $\boxed{\frac{y - y_1}{X - X_1} = m}$

(8) The length of the perpendicular from the point (X_1 , y_1) on the straight line whose equation : $aX + by + C = 0$ equals $\frac{| aX_1 + by_1 + c |}{\sqrt{a^2 + b^2}}$

Example 1

Find the general form of the equation of the circle whose centre is (– 2 , 3) and its radius length is 5 length units.

Solution

The equation of the circle is : $(X + 2)^2 + (y - 3)^2 = (5)^2$

$\therefore X^2 + y^2 + 4X - 6y - 12 = 0$ "After simplify"

Another solution :

$\because L = 2$, $K = -3$, $C = L^2 + K^2 - r^2 = (2)^2 + (-3)^2 - (5)^2 = -12$

$\therefore$ The general form of the equation of the circle is $X^2 + y^2 + 4X - 6y - 12 = 0$

"The same form we obtained before"

Example 2

Find the equation of the circle whose centre is the origin point and its diameter length = $6\sqrt{2}$ length unit , then prove that the circle passing through the point $(\sqrt{2}, -4)$

Solution

The equation of the circle is : $x^2 + y^2 = \left(3\sqrt{2}\right)^2$

, then $x^2 + y^2 = 18$

by substitute by the point $\left(\sqrt{2}, -4\right)$

$\therefore$ L.H.S. $= \left(\sqrt{2}\right)^2 + (-4)^2 = 18 =$ R.H.S.

$\therefore$ The point $\left(\sqrt{2}, -4\right) \in$ the circle.

Example 3

Find the equation of the circle whose centre M = (3 , – 2) and passing through the point A = (– 1 , 1)

Solution

$r = MA = \sqrt{(3+1)^2 + (-2-1)^2} = 5$ length unit.

$\therefore$ The equation of the circle is : $(x-3)^2 + (y+2)^2 = 25$

Example 4

Find the equation of the circle whose diameter $\overline{AB}$ where A = (4 , – 1) , B = (– 2 , 1)

Solution

$\because$ The centre of the circle M is the midpoint of $\overline{AB}$

$\therefore M = \left(\frac{4-2}{2}, \frac{-1+1}{2}\right) = (1, 0)$

, $\because r = MA = \sqrt{(4-1)^2 + (-1-0)^2} = \sqrt{10}$ length unit.

$\therefore$ The equation of the circle is : $(x-1)^2 + (y-0)^2 = (\sqrt{10})^2$

, then : $(x-1)^2 + y^2 = 10$

Example 5

Find the centre and the length of the radius for each of the following circles :

(1) $x^2 + y^2 - 2x + 4y - 4 = 0$

(2) $x^2 + y^2 - 4y - 9 = 0$

(3) $7x^2 + 7y^2 + 42x - 14y + 28 = 0$

Solution

(1) $\because x^2 + y^2 - 2x + 4y - 4 = 0$ $\therefore L = -1 , K = 2 , C = -4$

$\therefore$ The centre $= (-L , -K) = (1 , -2)$

$, r = \sqrt{L^2 + K^2 - C} = \sqrt{(-1)^2 + (2)^2 - (-4)} = 3$ length unit.

(2) $\because x^2 + y^2 - 4y - 9 = 0$

$\therefore L =$ zero , $K = -2$, $C = -9$

$\therefore$ The centre $= (-L , -K) = (0 , 2)$

$, r = \sqrt{L^2 + K^2 - C} = \sqrt{(0)^2 + (-2)^2 - (-9)} = \sqrt{13}$ length unit.

Notice that

$L = 0$ because the coefficient of $x = 0$

(3) By dividing by 7 to make the coefficient of $x^2 =$ the coefficient of $y^2 = 1$

$\therefore$ The equation will be : $x^2 + y^2 + 6x - 2y + 4 = 0$

$\therefore L = 3$, $K = -1$, $C = 4$

$\therefore$ The centre $= (-L , -K) = (-3 , 1)$

$, r = \sqrt{L^2 + K^2 - C} = \sqrt{(3)^2 + (-1)^2 - 4} = \sqrt{6}$ length unit.

Example 6

Find the equation of the circle whose centre (3 , – 4) and touches x-axis

Solution

$\because L = -3$, $K = 4$, $\because$ the circle touches x-axis

$\therefore r = |K|$, $C = L^2$

$\therefore r = 4$ length unit , $C = 9$ "We can find C using the relation : $C = L^2 + K^2 - r^2$"

$\therefore$ Equation of the circle is : $x^2 + y^2 - 6x + 8y + 9 = 0$

Example 7

Find the equation of the circle whose the length of its radius is 5 units and touches y-axis at the point (0 , 3)

Solution

$\because$ The circle touches y-axis at the point $(0 , 3)$

$\therefore$ The centre $= (-L , 3)$, $r = |L|$ length unit. *i.e.* $|L| = 5$

$\therefore L = \pm 5$, $C = K^2 = 9$

$\therefore$ There are two circles one of them has a centre $(-5 , 3)$ and its equation :
$x^2 + y^2 + 10x - 6y + 9 = 0$ and the other has a centre $(5 , 3)$ and its equation :
$x^2 + y^2 - 10x - 6y + 9 = 0$

Example 8

Find the equation of the circle which touches the two coordinate axes , and its centre is the point (– 4 , 4)

Solution

$\because$ The circle touches the two coordinate axes

$\therefore$ $C = L^2 = K^2 = 16$

$\therefore$ The equation is : $x^2 + y^2 + 8x - 8y + 16 = 0$

Example 9

Determine which of the following equations represent a circle :

(1) $x^2 + 3y^2 - 2x + 4y + 5 = 0$

(2) $2x^2 - xy + 2y^2 + 5x - y - 2 = 0$

(3) $x^2 + y^2 + 7x - y + 8 = 0$

(4) $2x^2 + 2y^2 - 6x + 4y + 9 = 0$

(5) $x^2 + y^2 - 16x + 12y + 100 = 0$

(6) $x^2 + 6x - 8y - 7 = 0$

Solution

(1) $\because$ The coefficient of $x^2 \neq$ the coefficient of y^2

$\therefore$ The equation doesn't represent a circle.

(2) $\because$ The equation has a term contains xy

$\therefore$ The equation doesn't represent a circle.

(3) The coefficient of $x^2 =$ the coefficient of y^2 and the equation is free of a term contains xy

$\therefore$ The equation may represent a circle.

$\because$ $2L = 7$, $2K = -1$ $\qquad$ $\therefore$ $L = \frac{7}{2}$, $K = \frac{-1}{2} =$, $C = 8$

$\therefore$ $L^2 + K^2 - C = \frac{49}{4} + \frac{1}{4} - 8 = \frac{9}{2} > 0$

$\because$ The equation is represent a circle whose centre $\left(\frac{-7}{2}, \frac{1}{2}\right)$

, $r = \sqrt{\frac{9}{2}} = \frac{3\sqrt{2}}{2}$ length unit.

(4) $\because$ The coefficient of x^2 = The coefficient of y^2 and the equation is free of a term contains xy

$\therefore$ The equation may represent a circle.

Multiply by $\frac{1}{2}$ to make the coefficient of x^2 = the coefficient of $y^2 = 1$

$\therefore x^2 + y^2 - 3x + 2y + \frac{9}{2} = 0$

$\because 2L = -3 \quad , \quad 2k = 2$

$\therefore L = \frac{-3}{2} \quad , \quad k = 1 \quad , \quad c = \frac{9}{2}$

$\therefore L^2 + K^2 - C = \frac{9}{4} + 1 - \frac{9}{2} = \frac{-5}{4} < 0$

$\therefore$ The equation doesn't represent a circle.

(5) $\because$ The coefficient of x^2 = the coefficient of y^2 , and the equation is free of a term contains xy

$\therefore$ The equation may represent a circle.

$\therefore 2L = -16 \quad , \quad 2k = 12 \qquad \therefore L = -8 \quad , \quad k = 6 \quad , \quad c = 100$

$\therefore L^2 + K^2 - C = 64 + 36 - 100 = \text{zero}$

$\therefore$ The equation doesn't represent a circle.

(6) $\because$ The equation is free of the term y^2

$\therefore$ The equation doesn't represent a circle.

Example 10

Find the equation of the circle whose radius length = 3 units , and the two equations of the straight lines carrying two diameters are $x + y = 2$, $2x - y = 7$

Solution

The centre of the circle is the point of intersection of the two straight lines :

$x + y = 2 \quad (1) \qquad , \qquad 2x - y = 7 \quad (2)$

by adding $\quad \therefore 3x = 9 \quad \therefore x = 3$

by substitute $\quad \therefore y = -1$

$\therefore$ The centre is the point $(3 , -1)$

$\therefore L = -3 \quad , \quad K = 1 \quad , \quad C = L^2 + K^2 - r^2 = 9 + 1 - 9 = 1$

$\therefore$ Equation of the circle is : $x^2 + y^2 - 6x + 2y + 1 = 0$

Example 11

A circle whose centre M $(-2 , 7)$, and the length of its radius r = 5 units , state which of the following points lies on the circle , inside the circle and outside the circle.

$A = (-1 , 3) \quad , \quad B = (0 , -5) \quad , \quad C = (2 , 4)$

Solution

$\because$ Equation of the circle is : $(x+2)^2 + (y-7)^2 = 25$

by substitute by the points A , B and C in the L.H.S. of the equation :

$\because (-1+2)^2 + (3-7)^2 = 17 < r^2$ $\therefore$ Point A $(-1 , 3)$ lies inside the circle.

, $\because (0+2)^2 + (-5-7)^2 = 148 > r^2$ $\therefore$ Point B $(0 , -5)$ lies outside the circle.

, $\because (2+2)^2 + (4-7)^2 = 25 = r^2$ $\therefore$ Point C $(2 , 4)$ lies on the circle.

Example 12

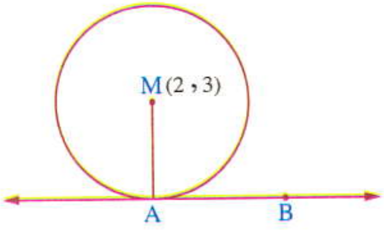
Find the equation of the circle whose centre M = (2 , 3) and the straight line $3x + 4y + 2 = 0$ is a tangent at the point A

Solution

$\because \overline{MA}$ is a radius , $\overleftrightarrow{AB}$ is tangent to the circle.

$\therefore \overline{MA} \perp \overleftrightarrow{AB}$

$\therefore MA = \dfrac{|3 \times 2 + 4 \times 3 + 2|}{\sqrt{3^2 + 4^2}} = 4$ length unit.



$\therefore r = 4$ length unit.

$\therefore$ Equation of the circle is :

$(x-2)^2 + (y-3)^2 = 16$

Example 13

Determine the position of the circle $C_1 : (x-3)^2 + (y-2)^2 = 4$ with respect to the circle $C_2 : x^2 + y^2 + 2x + 2y + 1 = 0$

Solution

$\because C_1 : (x-3)^2 + (y-2)^2 = 4$

$\therefore$ The centre $M_1 = (3 , 2)$, $r_1 = \sqrt{4} = 2$ length unit.

, $C_2 : x^2 + y^2 + 2x + 2y + 1 = 0$

The centre $M_2 = (-1 , -1)$, $r_2 = \sqrt{1+1-1} = 1$ length unit.

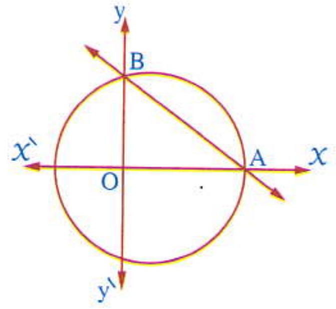
$\therefore r_1 + r_2 = 2 + 1 = 3$ length unit.

, $M_1 M_2 = \sqrt{(3+1)^2 + (2+1)^2} = 5$ length unit.

$\therefore M_1 M_2 > r_1 + r_2$ $\therefore$ The two circles are distant.

Example 14

In the opposite figure :
If the equation of $\overleftrightarrow{AB}$
is : $6X + 8y - 48 = 0$ intersects.
The two coordinate axes at A and B ,
Find the equation of the circle passing through the points A , O and B



Solution

$\because m(\angle AOB) = 90°$

$\therefore \overline{AB}$ is a diameter of the circle $\because$ The equation of $\overleftrightarrow{AB}$ is $6X + 8y = 48$

i.e. $\frac{X}{8} + \frac{y}{6} = 1$

$\therefore$ The straight line cuts X-axis at the point $A = (8 , 0)$

, cuts y-axis at the point $B = (0 , 6)$

Let M be the centre of the circle.

$\therefore$ M is the midpoint of $\overline{AB} = \left(\frac{8+0}{2} , \frac{0+6}{2}\right) = (4 , 3)$

, $AB = \sqrt{8^2 + 6^2} = 10$ length unit.

$\therefore r = 5$ length unit.

$\therefore$ Equation of the circle is $(X-4)^2 + (y-3)^2 = 25$

Example 15

Find the area of the equilateral triangle whose vertices passing through the circle :
$X^2 + y^2 + 6X - 2y - 15 = 0$
"Where each length unit in the plane represents 4 cm."

Solution

$L = 3 , K = -1 , C = -15$

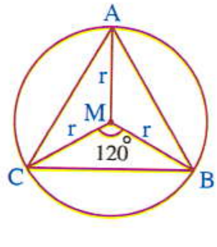
$\therefore r = \sqrt{L^2 + k^2 - C} = \sqrt{9 + 1 + 15} = 5$ length unit , M is the centre of the circumcircle of ΔABC

If ΔABC is equilateral by drawing

$\overline{MA} , \overline{MB} , \overline{MC}$, then :

$m(\angle BMC) = \frac{360°}{3} = 120°$, then :

Area of $\Delta ABC = 3 \times$ area of ΔMBC



$= 3 \times \frac{1}{2} MB \times MC \times \sin(\angle BMC)$

$= \frac{3}{2} r^2 \sin 120°$

$= \frac{3}{2} \times 25 \sin 60° = \frac{3}{2} \times 25 \times \frac{\sqrt{3}}{2} = \frac{75\sqrt{3}}{4}$ square unit.

$\because$ Each length unit in the plane represent 4 cm.

$\therefore$ The square unit represent $4^2 = 16$ cm^2.

$\therefore$ Area of $\Delta ABC = \frac{75\sqrt{3}}{4} \times 16 = 300\sqrt{3}$ cm^2.

Remark

If the number of sides of a regular polygon = n sides , the length of the radius to the circle passing through its vertices = r , then

Area of the regular polygon $= \frac{n}{2} r^2 \sin\left(\frac{360°}{n}\right)$

For example :

The regular hexagon polygon whose drawn inside a circle of radius length 8 cm. , its area equals :

$\frac{6}{2} \times (8)^2 \times \sin\left(\frac{360°}{6}\right)$

$= 3 \times 64 \times \sin 60°$

$= 3 \times 64 \times \frac{\sqrt{3}}{2} = 96\sqrt{3}$ square unit.

Example 16

Find the cartesian equation of the circle passing through the points $A = (6 , 3)$, $B = (2 , 3)$ and $C = (4 , 1)$, then determine its centre and length of its radius.

Solution

Let the equation is : $x^2 + y^2 + 2Lx + 2Ky + C = 0$

$\because$ The points A , B and C lies on the circle.

$\therefore 36 + 9 + 12L + 6k + c = 0$ *i.e.* $12L + 6K + C = -45$ (1)

$, 4 + 9 + 4L + 6K + C = 0$ *i.e.* $4L + 6K + C = -13$ (2)

$, 16 + 1 + 8L + 2K + C = 0$ *i.e.* $8L + 2K + C = -17$ (3)

by subtracting (2) from (1)

$\therefore 8L = -32$ $\therefore L = -4$

and by subtracting (3) from (1)

$\therefore 4L + 4K = -28$ $\qquad$ $\therefore L + K = -7$

$\therefore -4 + K = -7$ $\qquad$ $\therefore K = -3$

by substitute in (3)

$\therefore -32 - 6 + c = -17$ $\qquad$ $\therefore C = 21$

$\therefore$ The equation is : $x^2 + y^2 - 8x - 6y + 21 = 0$

, where the centre = (4 , 3)

, $r = \sqrt{16 + 9 - 21} = \sqrt{4} = 2$ length unit.

Example 17

Find the equation of the circle whose touches x-axis and passing through the points (– 1 , 2) , (– 3 , 4)

Solution

$\because$ The circle touches x-axis $\qquad$ $\therefore r = |K|, C = L^2$

Let the equation of the circle is :

$x^2 + y^2 + 2Lx + 2Ky + L^2 = 0$

$\therefore$ The circle passing through the point (–1 , 2) then :

$1 + 4 - 2L + 4K + L^2 = 0$ $\qquad$ $\therefore L^2 - 2L + 4K = -5$ $\qquad$ (1)

$\because$ The circle passing through the point (–3 , 4) then :

$9 + 16 - 6L + 8K + L^2 = 0$ $\qquad$ $\therefore L^2 - 6L + 8K = -25$ $\qquad$ (2)

by multiplying the equation (1) × 2 then subtracting from the equation (2) :

$\therefore L^2 - 2L = -15$

$\therefore L^2 + 2L - 15 = 0$

$(L - 3)(L + 5) = 0$

$\therefore L = 3$ or $L = -5$

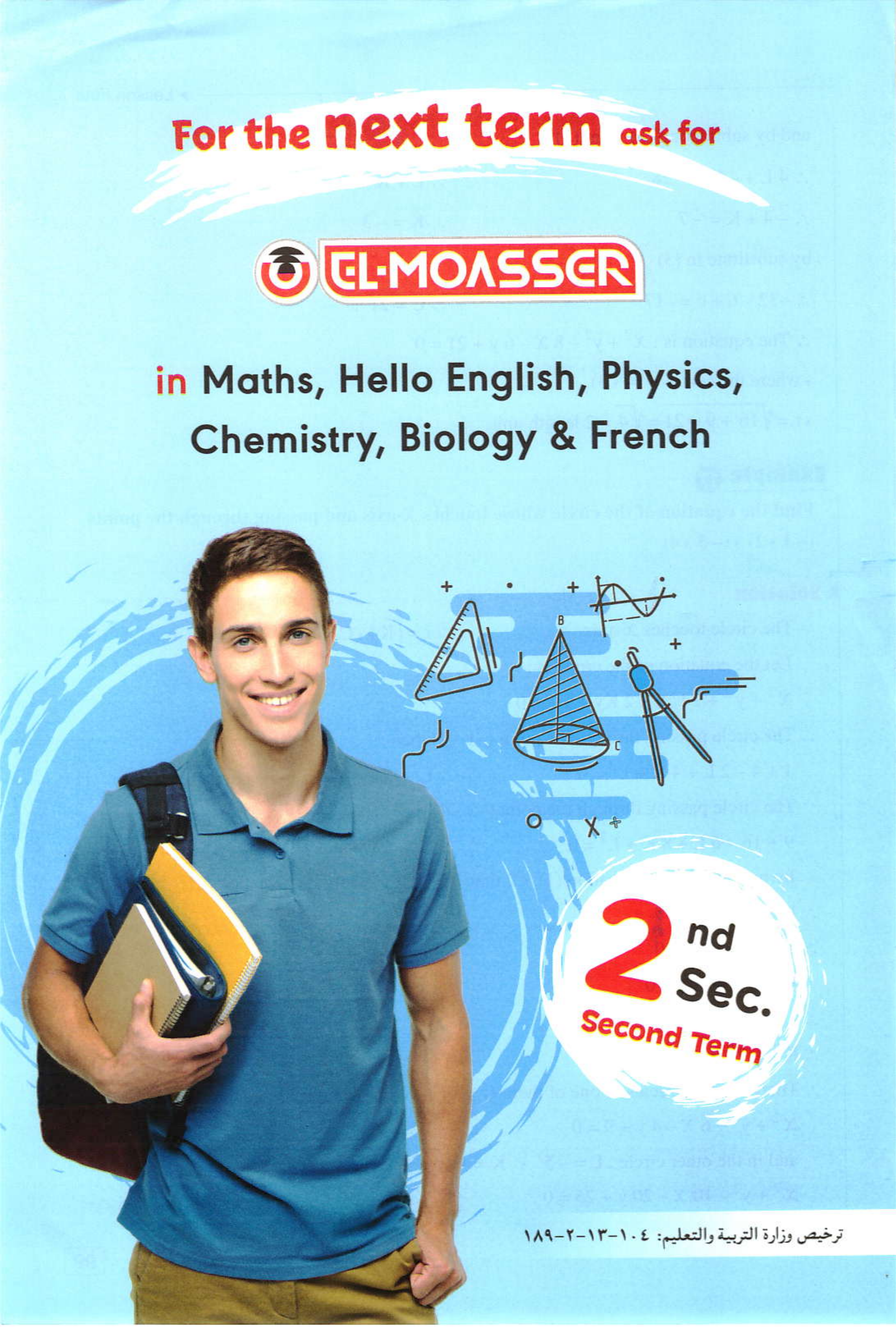
$\therefore K = -2$ or $K = -10$

$\therefore$ There are two circles in one of them $L = 3$, $K = -2$ and its equation is :

$x^2 + y^2 + 6x - 4y + 9 = 0$

and in the other circle : $L = -5$, $K = -10$ and its equation is :

$x^2 + y^2 - 10x - 20y + 25 = 0$



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