Unit-4

Complex Analysis

2 Marks Questions.

Ques 1 Write Cauchy-Riemann equations in the polar coordinates.

Ans Cauchy Riemann equations in the polar coordinates are

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}.$$

Ques 2 Find the limit of $\lim\limits_{z\to\infty} \frac{z}{2-iz}$.

Ans

$$\lim_{z\to\infty} \frac{z}{2-iz} = \lim_{z\to\infty} \frac{z}{z(-i+2/z)}$$

$$= \lim_{z\to\infty} \frac{1}{-i+\frac{2}{z}}$$

$$= \frac{-1 * i}{i * i}$$

$$= i$$

$$\boxed{\lim_{z\to\infty} \frac{z}{2-iz} = i.}$$

Ques 3 Check the continuity of $f(z) = \bar{z}$ about origin.

Ans

$$f(z) = \bar{z}$$

$$= x - iy.$$

$$f(0) = 0 - i\cdot 0$$
$$= 0$$

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} x - iy$$
$$= 0.$$

since $\lim_{z\to 0} f(z) = f(0)$

$f(z)$ is continuous at origin.

Ques 4 Using Cauchy Riemann equation show that $f(z) = \sin z$ is analytic.

Ans

$$f(z) = \sin z$$
$$= \sin(x+iy)$$
$$= \sin x \cos iy + \cos x \sin iy.$$
$$= \sin x \cosh y + i \cos x \sinh y$$

$u(x,y) = \sin x \cosh y \qquad v(x,y) = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y \qquad \frac{\partial v}{\partial y} = \cos x \cosh y$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are all continuous functions.

Also, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Thus, $f(z) = \sin z$ is analytic.

Ques 5 Check whether the function $u(r,\theta) = r^2\cos 2\theta$ is harmonic or not.

Ans A function $f(r,\theta)$ is harmonic if it satisfies.

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0.$$

$$u(r,\theta) = r^2\cos 2\theta$$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta \qquad \frac{\partial^2 u}{\partial r^2} = 2\cos 2\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta \qquad \frac{\partial^2 u}{\partial \theta^2} = -4r^2\cos 2\theta.$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 2\cos 2\theta + \frac{2r\cos 2\theta}{2} - \frac{4r^2\cos 2\theta}{r^2}$$

$$= 2\cos 2\theta + 2\cos 2\theta - 4\cos 2\theta$$

$$= 4\cos 2\theta - 4\cos 2\theta$$

$$= 0$$

$\Rightarrow u$ is a harmonic function.

6 Marks Questions.

Ques 1 Show that the function $f(z) = |z|^2$ is differentiable at the origin and nowhere else.

Ans $f(z) = |z|^2$ where $z = x + iy$

$$f(z) = x^2 + y^2$$

$u(x,y) = x^2 + y^2$ $\qquad$ $v(x,y) = 0$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 2y$ $\qquad$ $\frac{\partial v}{\partial y} = 0$.

If $f(z)$ is differentiable, then CR equations are to be satisfied.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\qquad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow 2x = 0$ $\qquad$ $\Rightarrow 2y = 0$

$\Rightarrow x = 0$ $\qquad$ $\Rightarrow y = 0$

CR equations are satisfied only when $x = 0$ and $y = 0$.

CR equations are satisfied only at origin.

$\Rightarrow f(z)$ is differentiable only at origin.

Ques2 Find the constants a, b, c such that the function

$$f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2) \text{ is analytic.}$$

Ans For $f(z)$ to be analytic, it should satisfy CR equation.

$$f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$$

$u(x,y) = -x^2 + xy + y^2$ $\qquad$ $v(x,y) = ax^2 + bxy + cy^2$

$\frac{\partial u}{\partial x} = -2x + y$ $\qquad$ $\frac{\partial v}{\partial x} = 2ax + by$

$\frac{\partial u}{\partial y} = 2y + x$ $\qquad$ $\frac{\partial v}{\partial y} = 2cy + bx$

CR equations :- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Rightarrow -2x + y = 2cy + bx$

$\Rightarrow b = -2 \qquad 2c = 1 \Rightarrow c = \frac{1}{2}$

also $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow 2y + x = -(by + 2ax)$

$\Rightarrow b = -2 \qquad -2a = 1 \Rightarrow a = -\frac{1}{2}$

So $a = -\frac{1}{2}$, $b = -2$ and $c = \frac{1}{2}$.

Ques 3 Prove that $u(x,y) = e^x \cos y$ is a harmonic function. Find its complex conjugate.

Ans $u(x,y) = e^x \cos y$

A function $u(x,y)$ is said to be harmonic if it satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\frac{\partial u}{\partial x} = e^x \cos y$ $\qquad$ $\frac{\partial^2 u}{\partial x^2} = e^x \cos y$ $\qquad$ $\frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial^2 u}{\partial y^2} = -e^x \cos y$

So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\Rightarrow$ u is a harmonic function.

A function $v(x,y)$ is said to be the complex conjugate of $u(x,y)$ if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$e^x \cos y = \frac{\partial v}{\partial y} \quad \text{(I)} \quad \text{and} \quad \frac{\partial v}{\partial x} = e^x \sin y \quad \text{(II)}$$

Integrating (I) wrt y.

$$v = \int e^x \cos y \, dy + f(x)$$

$$v = e^x \sin y + f(x)$$

$$\frac{\partial v}{\partial x} = e^x \sin y + f'(x) \quad \text{(III)}$$

From (II) and (III), $f'(x) = 0$

$\Rightarrow f(x) = c.$

So $\boxed{v = e^x \sin y + c}$

Ques 4 Using the Cauchy Riemann equations, show that

(i) $f(z) = |z|^2$ is not analytic at any point.

Ans $f(z) = |z|^2 = (x^2 + y^2).$

Here $u(x,y) = x^2+y^2$ $\qquad v(x,y) = 0$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ only at origin.

So the function $f(z)$ is analytic only at the origin.

(ii) $f(z) = \frac{1}{z}, z \neq 0$ is analytic at all points except at origin.

Ans

$$f(z) = \frac{1}{z} = \frac{1}{(x+iy)} * \frac{(x-iy)}{(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$u(x,y) = \frac{x}{x^2+y^2} \qquad v(x,y) = \frac{-y}{x^2+y^2}$.

$\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{+2xy}{(x^2+y^2)^2}$

$= \frac{y^2-x^2}{(x^2+y^2)^2} \qquad = \frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{-x * 2y}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1)+2y^2}{(x^2+y^2)^2}$

$= -\frac{2xy}{(x^2+y^2)^2} \qquad = \frac{y^2-x^2}{(x^2+y^2)^2}$.

Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

But $f(z)$ is not defined at 0.

So $f(z)$ is analytic at all points except origin.

10 Marks Questions.

Ques 1 Check the continuity of the function $f(z)$ about the origin, where

(i)
$$f(z) = \begin{cases} \dfrac{\operatorname{Im}(z)}{|z|} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Ans
$$z = x + iy$$
$$|z| = \sqrt{x^2+y^2} \qquad \operatorname{Im}(z) = y$$

$$f(z) = \begin{cases} \dfrac{y}{\sqrt{x^2+y^2}} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} f(x,y)$$
$$= \lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}}$$

Put $y = mx$
$$= \lim_{x\to 0} \frac{mx}{\sqrt{x^2+m^2x^2}}$$
$$= \lim_{x\to 0} \frac{mx}{x\sqrt{1+m^2}}$$
$$= \frac{m}{\sqrt{1+m^2}}$$

Thus, the limit is dependent upon m.

Hence, limit does not exist at origin

⇒ The function f(z) is not continuous at origin.

(ii) $f(z) = \begin{cases} \dfrac{Re(z^2)}{|z|^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$

$z = x + iy$

$z^2 = (x^2 - y^2) + 2ixy$

$Re(z^2) = x^2 - y^2$

$|z|^2 = x^2 + y^2$

$f(z) = \dfrac{x^2 - y^2}{x^2 + y^2}$

$\lim\limits_{z \to 0} f(z) = \lim\limits_{(x,y) \to (0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

Put $y = mx$

$= \lim\limits_{x \to 0} \dfrac{x^2 - m^2x^2}{x^2 + m^2x^2}$

$= \dfrac{1 - m^2}{1 + m^2}$

Thus the limit is dependent upon m.

Hence, limit does not exist at origin.

The function f(z) is not continuous at origin.

Ques 2 Prove that the function $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0. \end{cases}$

satisfies Cauchy Riemann equation, yet it is not analytic about origin.

Ans

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \qquad z \neq 0$$

$$= \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2+y^2} \qquad z \neq 0$$

$$u = \frac{x^3 - y^3}{x^2+y^2} \qquad v = \frac{x^3+y^3}{x^2+y^2}$$

At origin

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1.$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \to 0} \frac{-k - 0}{k} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(h,0) - v(0,0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k \to 0} \frac{k - 0}{k} = 1$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence CR equations are satisfied.

For analyticity about origin we find $f'(0)$.

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}.$$

$$= \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}.$$

Put $y = mx$

$$= \lim_{x \to 0} \frac{(x^3 - m^3x^3) + i(x^3 + m^3x^3)}{(x^2 + m^2x^2)(x + imx)}$$

$$= \frac{1 - m^3 + i(1 + m^3)}{(1 + m^2)(1 + im)}.$$

Limit is dependent upon m. Hence, it doesn't exist.

$\Rightarrow f'(0)$ is not defined.

$\Rightarrow f(z)$ is not analytic about origin.

Ques 3 Prove that the function $u = x^4 - 6x^2y^2 + y^4$ is harmonic

Find the analytic function $f(z) = u(x,y) + iv(x,y)$.

Ans $u = x^4 - 6x^2y^2 + y^4$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 \qquad \frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$

$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3 \qquad \frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2 + 12y^2 - 12x^2$$

$$= 0.$$

Since, u satisfies Laplace's equation.

$\Rightarrow$ u is a harmonic function.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial y} = 4x^3 - 12xy^2 \quad \text{— (A)} \qquad \frac{\partial v}{\partial x} = +12x^2y - 4y^3 \quad \text{— (B)}$$

Integrate wrt y on both sides.

$$v = 4x^3y - \frac{12xy^3}{3} + f(x)$$

$$\Rightarrow v = 4x^3y - 4xy^3 + f(x)$$

$$\frac{\partial v}{\partial x} = 12x^2y - 4y^3 + f'(x) \quad \text{— (C)}$$

and from (B) & (C) $f'(x) = 0$

$$f(x) = c$$

So $v(x,y) = 4x^3y - 4xy^3 + c$

So, $\boxed{f(z) = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3) + c.}$

Ques 4 Determine the analytic function by Milne Thomson method

(ii) whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$.

Ans

$$u = e^{2x}(x\cos 2y - y\sin 2y)$$

$$\frac{\partial u}{\partial x} = e^{2x}(\cos 2y) + (x\cos 2y - y\sin 2y)\cdot 2e^{2x}$$

$$= e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y).$$

$$\phi_1(x,y) = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y).$$

$$\phi_1(z,0) = e^{2z}(1+2z)$$

$$\frac{\partial u}{\partial y} = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y).$$

$$\phi_2(x,y) = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y).$$

$$\phi_2(z,0) = 0.$$

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c.$$

$$= \int e^{2z}(1+2z)\,dz$$

$$= \int e^{2z}\,dz + 2\int z\cdot e^{2z}\,dz$$

$$= \frac{e^{2z}}{2} + 2\left(\frac{ze^{2z}}{2} - \frac{e^{2z}}{4}\right) + c.$$

$$= \cancel{\frac{e^{2z}}{2}} + ze^{2z} - \cancel{\frac{e^{2z}}{2}} + c.$$

$$\boxed{f(z) = ze^{2z} + C}$$

(ii) whose imaginary part is $e^{-x}(x\cos y + y\sin y)$.

Ans $\quad v = e^{-x}(x\cos y + y\sin y)$.

$$\frac{\partial v}{\partial x} = e^{-x}(\cos y) + (x\cos y + y\sin y)(-e^{-x}).$$

$$= e^{-x}(\cos y - x\cos y - y\sin y).$$

$$\psi_2(x,y) = e^{-x}(\cos y - x\cos y - y\sin y).$$

$$\psi_2(z,0) = e^{-z}$$

$$\frac{\partial v}{\partial y} = e^{-x}(-x\sin y + y\cos y + \sin y).$$

$$\psi_1(x,y) = e^{-x}(-x\sin y + y\cos y + \sin y).$$

$$\psi_1(z,0) = 0.$$

$$f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + C.$$

$$= i\int e^{-z}\,dz + C$$

$$\boxed{f(z) = -ie^{-z} + C}$$

Ques 5 Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

i) If $u(x,y) = e^{-2xy} \sin(x^2-y^2)$.

Construct the corresponding analytic function in terms of z.

Ans

$$u(x,y) = e^{-2xy} \sin(x^2-y^2)$$

$$\frac{\partial u}{\partial x} = e^{-2xy} * \cos(x^2-y^2) * 2x + \sin(x^2-y^2) * -2y e^{-2xy}$$

$$= 2e^{-2xy} \left[x \cos(x^2-y^2) - y \sin(x^2-y^2)\right]$$

$$\frac{\partial u}{\partial y} = e^{-2xy} \cos(x^2-y^2)(-2y) + \sin(x^2-y^2)(-2x) e^{-2xy}$$

$$= 2e^{-2xy} \left[-y \cos(x^2-y^2) - x \sin(x^2-y^2)\right]$$

$$\phi_1(x,y) = 2e^{-2xy} \left[x \cos(x^2-y^2) - y \sin(x^2-y^2)\right]$$

$$\phi_2(x,y) = 2e^{-2xy} \left[-y \cos(x^2-y^2) - x \sin(x^2-y^2)\right]$$

$$\phi_1(z,0) = 2z \cos z^2$$

$$\phi_2(z,0) = -2z \sin(z^2).$$

$$f(z) = \int \phi_1(z,0)\, dz - i \int \phi_2(z,0)\, dz$$

$$= \int 2z \cos z^2\, dz + i \int 2z \sin z^2\, dz$$

$$z^2 = p \quad \Rightarrow 2z\, dz = dp.$$

$$= \int \cos p \, dp + i \int \sin p \, dp$$

$$= +\sin p + (-i \cos p) + C$$

$$= +\sin z^2 + (-i \cos z^2) + C$$

$$= -i \left(\cos z^2 + i \sin z^2 \right) + C$$

$$= -i e^{iz^2} + C.$$

(iii) If $v(x,y) = \log(x^2+y^2) + x - 2y$.

$$v(x,y) = \log(x^2+y^2) + x - 2y$$

$$\frac{\partial v}{\partial x} = \frac{2x}{2(x^2+y^2)} + 1$$

$$\frac{\partial v}{\partial y} = \frac{2y}{2(x^2+y^2)} - 2.$$

$$\psi_2(x,y) = \frac{x}{x^2+y^2} + 1 \qquad \psi_2(z,0) = \frac{1}{z} + 1$$

$$\psi_1(x,y) = \frac{2y}{2(x^2+y^2)} - 2 \qquad \psi_1(z,0) = -2$$

$$f(z) = \int \psi_1(z,0)\, dz + i \int \psi_2(z,0)\, dz + C.$$

$$= \int \left(\frac{1}{z} + 1\right) dz + i \int -2\, dz + C$$

$$= \int -2\, dz + i \int \left(\frac{1}{z} + 1\right) dz$$

(ii) If $v(x,y) = \log(x^2+y^2) + x - 2y$.

$$v(x,y) = \log(x^2+y^2) + x - 2y$$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1 = \phi_2(x,y)$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2 = \phi_1(x,y)$$

$$\phi_1(z,0) = -2$$

$$\phi_2(z,0) = \frac{2}{z} + 1$$

$$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$$

$$= \int -2dz + i\int \frac{2}{z} + 1\,dz + c$$

$$= -2z + i\left(2\log z + z\right) + c$$

$$\boxed{f(z) = (i-2)z + 2i\log z + c}$$