19

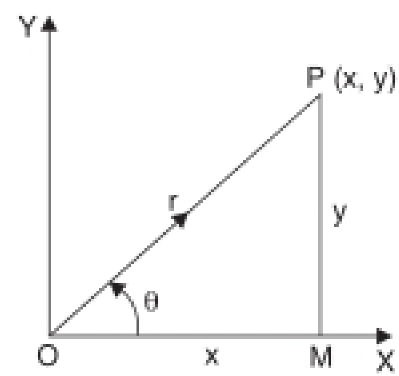
Functions of a Complex Variable

19.1. INTRODUCTION

A complex number z is an ordered pair (x, y) of real numbers and is written as

$$z = x + iy, \quad \text{where } i = \sqrt{-1}.$$

The real numbers x and y are called the real and imaginary parts of z. In the Argand's diagram, the complex number z is represented by the point $P(x, y)$. If (r, θ) are the polar coordinates of P, then $r = \sqrt{x^2 + y^2}$ is called the modulus of z and is denoted by $| z |$. Also $\theta = \tan^{-1} \dfrac{y}{x}$ is called the argument of z and is denoted by arg. z. Every non-zero complex number z can be expressed as



$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

If $z = x + iy$, then the complex number $x - iy$ is called the conjugate of the complex number z and is denoted by $\bar{z}$.

Clearly, $\quad | \bar{z} | = | z |, | z |^2 = z\bar{z},$

$$\text{Re}(z) = \frac{z + \bar{z}}{2}, \quad \text{Im}(z) = \frac{z - \bar{z}}{2i}.$$

19.2. FUNCTION OF A COMPLEX VARIABLE

If x and y are real variables, then $z = x + iy$ is called a **complex variable.** If corresponding to each value of a complex variable $z(= x + iy)$ in a given region R, there correspond one or more values of another complex variable w $(= u + iv)$, then w is called a function of the complex variable z and is denoted by

$$w = f(z) = u + iv$$

For example, if $\quad w = z^2, \quad$ where $z = x + iy$ and $w = f(z) = u + iv$

then $\quad u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

$\Rightarrow \quad u = x^2 - y^2 \quad$ and $\quad v = 2xy$

Thus u and v, the real and imaginary parts of w, are functions of the real variables x and y.

$\therefore \quad w = f(z) = u(x, y) + iv(x, y)$

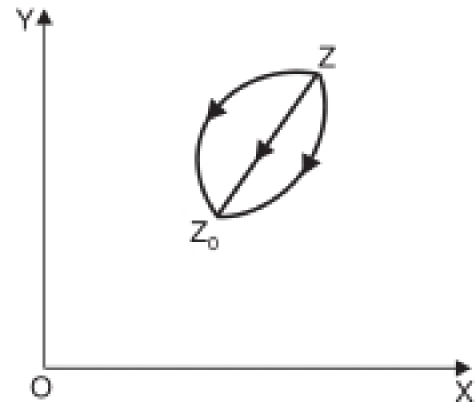
If to each value of z, there corresponds one and only one value of w, then w is called a *single-valued function* of z. If to each value of z, there correspond more than one values of w, then w is called a *multi-valued function of z*.

To represent $w = f(z)$ graphically, we take two Argand diagrams : one to represent the point z and the other to represent w. The former diagram is called the *xoy*-plane or the z-plane and the latter *uov*-plane or the w-plane.

19.3. LIMIT OF f(z)

A function $f(z)$ tends to the limit l as z tends to z_0 *along any path,* if to each positive arbitrary number ε, however small, there corresponds a positive number δ, such that

$$| f(z) - l | < \varepsilon \quad \text{whenever} \quad 0 < | z - z_0 | < \delta$$



Note. In real variables, $x \to x_0$ implies that x approaches x_0 along the number line, either from left or from right. In complex variables, $z \to z_0$ implies that z approaches z_0 along any path, straight or curved, since the two points representing z and z_0 in a complex plane can be joined by an infinite number of curves.

19.4. CONTINUITY OF f(z)

A single-valued function $f(z)$ is said to be continuous at a point $z = z_0$ if $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$.

A function $f(z)$ is said to be continuous in a region R of the z-plane if it is continuous at every point of the region.

19.5. DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z(= x + iy)$, then the derivative or differential co-efficient of $w = f(z)$ is defined as

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z}$$

provided the limit exists, independent of the manner in which $\delta z \to 0$.

19.6. ANALYTIC FUNCTION

If a single-valued function $f(z)$ possesses a unique derivative at every point of a region R, then $f(z)$ is called an **analytic function** or a **regular function** or a **holomorphic function** of z in R.

A point where the function ceases to be analytic is called a **singular point.**

19.7. NECESSARY AND SUFFICIENT CONDITIONS FOR f(z) TO BE ANALYTIC

The necessary and sufficient conditions for the function

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region R, are

(*i*) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *are continuous functions of x and y in the region R.*

(*ii*) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The conditions in (*ii*) are known as **Cauchy-Riemann equations** or briefly **C-R equations.**

Proof. (*a*) **Necessary Condition.** Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region R, then $\dfrac{dw}{dz} = f'(z)$ exists uniquely at every point of that region.

Let δx and δy be the increments in x and y respectively. Let δu, δv and δz be the corresponding increments in u, v and z respectively. Then,

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \underset{\delta z \to 0}{\text{Lt}} \frac{(u + \delta u) + i(v + \delta v) - (u + iv)}{\delta z}$$

$$= \underset{\delta z \to 0}{\text{Lt}} \left(\frac{\delta u}{\delta z} + i \frac{\delta v}{\delta z} \right) \quad \text{...(1)}$$

Since the function $w = f(z)$ is analytic in the region R, the limit (1) must exist independent of the manner in which $\delta z \to 0$, *i.e.*, along whichever path δx and $\delta y \to 0$.

First, let $\delta z \to 0$ along a line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$.

[since $z = x + iy$, $z + \delta z = (x + \delta x) + i(y + \delta y)$ and $\delta z = \delta x + i\delta y$]

$\therefore$ From (1), $$f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{...(2)}$$

Now, let $\delta z \to 0$ along a line parallel to y-axis so that $\delta x = 0$ and $\delta z = i\,\delta y$.

$\therefore$ From (1), $$f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{\delta u}{i\delta y} + i \frac{\delta v}{i\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad \text{...(3)} \quad \left| \because \frac{1}{i} = -i \right.$$

From (2) and (3), we have $\dfrac{\partial u}{\partial x} + i \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i \dfrac{\partial u}{\partial y}$

Equating the real and imaginary parts, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Hence the necessary condition for f(z) to be analytic is that the C-R equations must be satisfied.

(*b*) **Sufficient Condition.** Let $f(z) = u + iv$ be a single-valued function possessing partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ at each point of a region R and satisfying C-R equations.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We shall show that $f(z)$ is analytic, *i.e.*, $f'(z)$ exists at every point of the region R.

By Taylor's theorem for functions of two variables, we have, on omitting second and higher degree terms of δx and δy.

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= \left[u(x, y) + \left(\frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right) \right] + i \left[v(x, y) + \left(\frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y \right) \right]$$

$$= [u(x, y) + iv(x, y)] + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

or
$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y \qquad | \text{ Using C-R equations}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) i\delta y \qquad | \because \; -1 = i^2$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta z \qquad | \because \; \delta x + i\delta y = \delta z$$

$$\Rightarrow \qquad \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \qquad f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Thus $f'(z)$ exists, because $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$ exist.

Hence $f(z)$ is analytic.

Note 1. The real and imaginary parts of an analytic function are called **conjugate functions.** Thus, if $f(z) = u(x, y) + iv(x, y)$ is an analytic function, then $u(x, y)$ and $v(x, y)$ are conjugate functions. The relation between two conjugate functions is given by C-R equations.

Note 2. When a function $f(z)$ is known to be analytic, it can be differentiated in the ordinary way as if z is a real variable.

Thus, $\quad f(z) = z^2 \;\Rightarrow\; f'(z) = 2z$

$f(z) = \sin z \;\Rightarrow\; f'(z) = \cos z$ etc.

19.8. CAUCHY-RIEMANN EQUATIONS IN POLAR COORDINATES

Let (r, θ) be the polar coordinates of the point whose cartesian coordinates are (x, y), then

$$x = r\cos\theta,\; y = r\sin\theta,$$
$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$\therefore \qquad u + iv = f(z) = f(re^{i\theta}) \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. r, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \,.\, e^{i\theta} \qquad \text{...(2)}$$

Differentiating (1) partially w.r.t. θ, we have

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \,.\, ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \qquad | \text{ Using (2)}$$

$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

or $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$ which is the polar form of C-R equations.

19.9. HARMONIC FUNCTIONS

Any solution of the Laplace's equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called a harmonic function.

Let $f(z) = u + iv$ be analytic in some region of the z-plane, then u and v satisfy C-R equations.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{...(1)}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{...(2)}$$

Differentiating (1) partially w.r.t. x and (2) w.r.t. y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{...(3)}$$

and $$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad \text{...(4)}$$

Assuming $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ and adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(5)}$$

Now, differentiating (1) partially w.r.t. y and (2) w.r.t. x, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \qquad \text{...(6)}$$

and $$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \qquad \text{...(7)}$$

Assuming $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ and subtracting (7) from (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{...(8)}$$

Equations (5) and (8) show that the real and imaginary parts u and v of an analytic function satisfy the Laplace's equation.

Hence u and v are known as harmonic functions.

19.10. ORTHOGONAL SYSTEM

Every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.

Consider the two families of curves

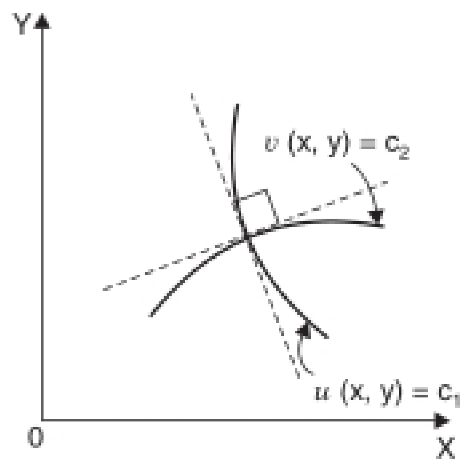
$$u(x, y) = c_1 \quad \ldots(1)$$

and

$$v(x, y) = c_2 \quad \ldots(2)$$

Differentiating (1) w.r.t. x, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \text{ (say)}$$



Similarly, from (2), we get

$$\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2 \text{ (say)}$$

$$\therefore \quad m_1 m_2 = \frac{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}} \quad \ldots(3)$$

Since $f(z)$ is analytic, u and v satisfy C-R equations

i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \quad \text{From (3),} \quad m_1 m_2 = \frac{\frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial x}}{-\frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y}} = -1$$

Thus the product of the slopes of the curves (1) and (2) is – 1. Hence the curves intersect at right angles, *i.e.,* they form an orthogonal system.

19.11. APPLICATION OF ANALYTIC FUNCTIONS TO FLOW PROBLEMS

Since the real and imaginary parts of an analytic function satisfy the Laplace's equation in two variables, these conjugate functions provide solutions to a number of field and flow problems.

For example, consider the two dimensional irrotational motion of an incompressible fluid, in planes parallel to xy-plane.

Let V be the velocity of a fluid particle, then it can be expressed as

$$V = v_x \hat{i} + v_y \hat{j} \quad \ldots(1)$$

Since the motion is irrotational, there exists a scalar function $\phi(x, y)$, such that

$$V = \nabla\phi(x, y) = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} \quad \ldots(2)$$

From (1) and (2), we have

$$v_x = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v_y = \frac{\partial \phi}{\partial y} \quad \ldots(3)$$

The scalar function $\phi(x, y)$, which gives the velocity components, is called the **velocity potential function** or simply the **velocity potential.**

Also the fluid being incompressible, div V = 0

$$\Rightarrow \qquad \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)(v_x\hat{i} + v_y\hat{j}) = 0$$

$$\Rightarrow \qquad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \ldots(4)$$

Substituting the values of v_x and v_y from (3) in (4), we get

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = 0 \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Thus the function ϕ is harmonic and can be treated as real part of an analytic function

$$w = f(z) = \phi(x, y) + i\,\psi(x, y)$$

For interpretation of conjugate function $\psi(x, y)$, the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx} = -\frac{\dfrac{\partial \psi}{\partial x}}{\dfrac{\partial \psi}{\partial y}} = \frac{\dfrac{\partial \phi}{\partial y}}{\dfrac{\partial \phi}{\partial x}} \qquad \text{[By C-R equations]}$$

$$= \frac{v_y}{v_x} \qquad \text{[By (3)]}$$

This shows that the resultant velocity $\sqrt{v_x^2 + v_y^2}$ of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$ *i.e.*, the fluid particles move along this curve. Such curves are known as *stream lines* and $\psi(x, y)$ is called the *stream function*. The curves represented by $\phi(x, y) = c$ are called *equipotential lines*.

Since $\phi(x, y)$ and $\psi(x, y)$ are conjugate functions of analytic function $w = f(z)$, the equipotential lines $\phi(x, y) = c$ and the stream lines $\phi(x, y) = c'$, intersect each other orthogonally.

Now,
$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$= \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} \qquad \text{[By C-R equations]}$$

$$= v_x - iv_y \qquad \text{[By (3)]}$$

$\therefore$ The magnitude of resultant velocity $= \left|\dfrac{dw}{dz}\right| = \sqrt{v_x^2 + v_y^2}$

The function $w = f(z)$ which fully represents the flow pattern is called the *complex potential*.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are called *equipotential lines* and *lines of force* respectively. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

ILLUSTRATIVE EXAMPLES

Example 1. *Find p such that the function f(z) expressed in polar coordinates as $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$ is analytic.*

Sol. Let $f(z) = u + iv$, then $u = r^2 \cos 2\theta$, $v = r^2 \sin p\theta$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta, \quad \frac{\partial v}{\partial r} = 2r \sin p\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta, \quad \frac{\partial v}{\partial \theta} = pr^2 \cos p\theta$$

For $f(z)$ to be analytic, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

$\therefore$ $\quad 2r \cos 2\theta = pr \cos p\theta$ and $2r \sin p\theta = 2r \sin 2\theta$

Both these equations are satisfied if $p = 2$.

Note. For a function $f(z)$ to be analytic, the first order partial derivatives of u and v must be continuous in addition to C-R equations.

Example 2. *Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied.*

Sol. Let $\quad f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$, then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

Clearly, $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Hence C-R equations are satisfied at the origin.

Now $\quad f'(0) = \underset{z \to 0}{\text{Lt}} \dfrac{f(z) - f(0)}{z} = \underset{z \to 0}{\text{Lt}} \dfrac{\sqrt{|xy|} - 0}{x + iy}$

If $z \to 0$ along the line $y = mx$, we get

$$f'(0) = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|m|}}{1 + im}$$

Now this limit is not unique since it depends on m. Therefore, $f'(0)$ does not exist.

Hence the function $f(z)$ is not regular at the origin.

Example 3. *Prove that the function f(z) defined by*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, \; z \neq 0 \text{ and } f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

Sol. Here, $$f(z) = \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2}, \; z \neq 0$$

Let $$f(z) = u + iv = \frac{x^3 - y^3}{x^2 + y^2} + i\,\frac{x^3 + y^3}{x^2 + y^2},$$

then $$u = \frac{x^3 - y^3}{x^2 + y^2}, \; v = \frac{x^3 + y^3}{x^2 + y^2}$$

Since $z \neq 0 \;\Rightarrow\; x \neq 0, y \neq 0$

$\therefore$ u and v are rational functions of x and y with non-zero denominators. Thus, u, v and hence $f(z)$ *are continuous functions when* $z \neq 0$. To test them for continuity at $z = 0$, on changing u, v to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, we get

$$u = r(\cos^3\theta - \sin^3\theta) \text{ and } v = r(\cos^3\theta + \sin^3\theta)$$

When $z \to 0$, $r \to 0$

$$\therefore \quad \underset{z \to 0}{\text{Lt}}\, u = \underset{r \to 0}{\text{Lt}}\, r(\cos^3\theta - \sin^3\theta) = 0$$

Similarly, $$\underset{z \to 0}{\text{Lt}}\, v = 0$$

$$\therefore \quad \underset{r \to 0}{\text{Lt}}\, f(z) = 0 = f(0)$$

$\Rightarrow$ $f(z)$ is continuous at $z = 0$.

Hence $f(z)$ is continuous for all values of z.

At the origin $(0, 0)$, we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}}\, \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}}\, \frac{x - 0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}}\, \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}}\, \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}}\, \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}}\, \frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}}\, \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}}\, \frac{y - 0}{y} = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at the origin.

Now, $$f'(0) = \underset{z \to 0}{\text{Lt}}\, \frac{f(z) - f(0)}{z} = \underset{z \to 0}{\text{Lt}}\, \frac{(x^3 - y^3) + i(x^3 + y^3) - 0}{(x^2 + y^2)(x + iy)}$$

Let $z \to 0$ along the line $y = x$, then

$$f'(0) = \underset{x \to 0}{\text{Lt}}\, \frac{0 + 2ix^3}{2x^3(1 + i)} = \frac{i}{1 + i} = \frac{i(1 - i)}{2} = \frac{1 + i}{2} \qquad \text{...(1)}$$

Also, let $z \to 0$ along the x-axis (*i.e.*, $y = 0$), then

$$f'(0) = \underset{x \to 0}{\text{Lt}} \frac{x^3 + ix^3}{x^3} = 1 + i \tag{2}$$

Since the limits (1) and (2) are different, $f'(0)$ does not exist.

Example 4. *Prove that the function sinh z is analytic and find its derivative.*

Sol. Here $f(z) = u + iv = \sinh z = \sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y$

$\therefore$ $u = \sinh x \cos y$ and $v = \cosh x \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y,\ \frac{\partial u}{\partial y} = -\sinh x \sin y$$

$$\frac{\partial v}{\partial x} = \sinh x \sin y,\ \frac{\partial v}{\partial y} = \cosh x \cos y$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus C-R equations are satisfied.

Since $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous functions, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R equations.

Hence $f(z)$ is analytic everywhere.

Now $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cosh x \cos y + i \sinh x \sin y = \cosh (x + iy) = \cosh z.$

Example 5. *Determine the analytic function whose real part is $e^{2x} (x \cos 2y - y \sin 2y)$.*

Sol. Let $f(z) = u + iv$ be the analytic function,

where $u = e^{2x} (x \cos 2y - y \sin 2y)$

$$\therefore \quad \frac{\partial u}{\partial x} = 2e^{2x} (x \cos 2y - y \sin 2y) + e^{2x} \cos 2y$$

$$= e^{2x} (2x \cos 2y - 2y \sin 2y + \cos 2y) \tag{1}$$

$$\frac{\partial u}{\partial y} = e^{2x} (-2x \sin 2y - \sin 2y - 2y \cos 2y)$$

$$= -e^{2x}(2x \sin 2y + \sin 2y + 2y \cos 2y) \tag{2}$$

Since $f(z)$ is analytic, u and v must satisfy C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{2x} (2x \cos 2y - 2y \sin 2y + \cos 2y)$$

Integrating w.r.t. y, treating x as constant, we get

$$v = e^{2x}\left[2x \cdot \frac{\sin 2y}{2} - \left\{2y\left(-\frac{\cos 2y}{2}\right) - (2)\left(-\frac{\sin 2y}{4}\right)\right\} + \frac{\sin 2y}{2}\right] + \phi(x)$$

$$= e^{2x} (x \sin 2y + y \cos 2y) + \phi(x) \tag{3}$$

where $\phi(x)$ is an arbitrary function of x.

$$\therefore \qquad \frac{\partial v}{\partial x} = 2e^{2x}\,(x \sin 2y + y \cos 2y) + e^{2x}\,(\sin 2y) + \phi'(x)$$

$$= e^{2x}\,(2x \sin 2y + \sin 2y + 2y \cos 2y) + \phi'(x)$$

But $\qquad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = e^{2x}\,(2x \sin 2y + \sin 2y + 2y \cos 2y)$ [From (2)]

$\therefore \qquad \phi'(x) = 0$

$\Rightarrow \qquad \phi(x) = c$, an arbitrary constant.

$\therefore \qquad$ From (3), $v = e^{2x}\,(x \sin 2y + y \cos 2y) + c$

$$f(z) = u + iv = e^{2x}\,(x \cos 2y - y \sin 2y) + ie^{2x}\,(x \sin 2y + y \cos 2y) + ic$$

$$= e^{2x}\,[(x + iy) \cos 2y + i(x + iy) \sin 2y] + c'$$

$$= (x + iy)\, e^{2x}\,(\cos 2y + i \sin 2y) + c'$$

$$= ze^{2z} \,.\, e^{2iy} + c' = ze^{2(x+iy)} + c' = ze^{2z} + c'.$$

(Milne Thomson's Method)

This method determines the analytic function $f(z)$ when u or v is given.

Since $\qquad z = x + iy,\ \bar{z} = x - iy \quad$ so that $\quad x = \dfrac{z + \bar{z}}{2},\ \ y = \dfrac{z - \bar{z}}{2i}$

Let $\qquad f(z) = u(x, y) + iv\,(x, y) \qquad (1)$

$$= u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Considering this as an identity in the two independent variables $z, \bar{z}$ and putting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv\,(z, 0)$$

which is the same as (1) if we replace x by z and y by 0.

Now $\qquad f(z) = u + iv$

$\Rightarrow \qquad f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ [C-R equations]

$$= e^{2x}\,(2x \cos 2y - 2y \sin 2y + \cos 2y) + ie^{2x}\,(2x \sin 2y + \sin 2y + 2y \cos 2y)$$

On replacing x by z and y by 0 on R.H.S., we get $f'(z) = e^{2z}\,(2z + 1)$

Integrating w.r.t. z, we have $\quad f(z) = (2z + 1)\dfrac{e^{2z}}{2} - 2\,.\,\dfrac{e^{2z}}{4} + c = ze^{2z} + c.$

Example 6. *Determine the analytic function* $w = u + iv$, *if* $v = \log\,(x^2 + y^2) + x - 2y$.

Sol. Here $\qquad v = \log\,(x^2 + y^2) + x - 2y$

$$\therefore \qquad \frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} - 2$$

Since $\quad w = u + iv$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \text{[C-R equations]}$$

$$= \left(\frac{2y}{x^2+y^2} - 2\right) + i\left(\frac{2x}{x^2+y^2} + 1\right)$$

Replacing x by z and y by 0, we get $\dfrac{dw}{dz} = -2 + i\left(\dfrac{2}{z} + 1\right) = (i-2) + \dfrac{2i}{z}$

Integrating w.r.t.z, we have $\quad w = (i-2)z + 2i \log z + c.$

Example 7. *Show that the function $u = e^{-2xy} \sin (x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z.*

Sol. Here $\quad u = e^{-2xy} \sin (x^2 - y^2)$

$$\therefore \quad \frac{\partial u}{\partial x} = -2y\, e^{-2xy} \sin (x^2 - y^2) + 2x\, e^{-2xy} \cos (x^2 - y^2)$$

$$\frac{\partial^2 u}{\partial x^2} = 4y^2 e^{-2xy} \sin (x^2 - y^2) - 4xy\, e^{-2xy} \cos (x^2 - y^2) + 2e^{-2xy} \cos (x^2 - y^2)$$
$$- 4xy\, e^{-2xy} \cos (x^2 - y^2) - 4x^2 e^{-2xy} \sin (x^2 - y^2) \qquad (1)$$

$$\frac{\partial u}{\partial y} = -2x\, e^{-2xy} \sin (x^2 - y^2) - 2y\, e^{-2xy} \cos (x^2 - y^2)$$

$$\frac{\partial^2 u}{\partial y^2} = 4x^2 e^{-2xy} \sin (x^2 - y^2) + 4xy\, e^{-2xy} \cos (x^2 - y^2) - 2e^{-2xy} \cos (x^2 - y^2)$$
$$+ 4xy\, e^{-2xy} \cos (x^2 - y^2) - 4y^2 e^{-2xy} \sin (x^2 - y^2) \qquad (2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{which proves that } u \text{ is harmonic.}$$

Now, let $\quad f(z) = u + iv$

then $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[C-R equations]}$$

$$= [-2y\, e^{-2xy} \sin (x^2 - y^2) + 2x\, e^{-2xy} \cos (x^2 - y^2)]$$
$$+ i\, [2x\, e^{-2xy} \sin (x^2 - y^2) + 2y\, e^{-2xy} \cos (x^2 - y^2)]$$

Replacing x by z and y by 0, we get

$$f'(z) = 2z \cos z^2 + 2iz \sin z^2 = 2z (\cos z^2 + i \sin z^2) = 2z e^{iz^2}$$

Integrating w.r.t. z, we have

$$f(z) = -i\, e^{iz^2} + \text{C} \quad (\text{C} = \text{A} + i\text{B is a complex constant})$$

which expresses $u + iv$ as an analytic function of z.

Since $$u + iv = -i e^{iz^2} + \text{C} = -i e^{i(x+iy)^2} + \text{C}$$

$$= -i e^{i(x^2-y^2+2ixy)} + \text{C} = -ie^{-2xy} \,.\, e^{i(x^2-y^2)} + \text{C}$$
$$= -ie^{-2xy} [\cos (x^2 - y^2) + i \sin (x^2 - y^2)] + \text{C}$$

$$= e^{-2xy}\sin(x^2 - y^2) + i[-e^{-2xy}\cos(x^2 - y^2)] + C$$

$$\therefore \quad v = -e^{-2xy}\cos(x^2 - y^2) + B.$$

Example 8. *An electrostatic field in the xy-plane is given by the potential function* $\phi = 3x^2y - y^3$, *find the stream function.*

Sol. Let $\psi(x, y)$ be a stream function.

By C-R equations $$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -3x^2 + 3y^2 \quad ...(1)$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 6xy \quad ...(2)$$

Integrating (1) w.r.t. x, treating y as constant, we get

$$\psi = -x^3 + 3xy^2 + F(y)$$

so that $$\frac{\partial \psi}{\partial y} = 6xy + F'(y) \quad ...(3)$$

From (2) and (3), $6xy + F'(y) = 6xy$ or $F'(y) = 0$ $\therefore$ $F(y) = c$

Hence $$\psi = -x^3 + 3xy^2 + c.$$

Example 9. *If* $u - v = (x - y)(x^2 + 4xy + y^2)$ *and* $f(z) = u + iv$ *is an analytic function of* $z = x + iy$, *find* $f(z)$ *in terms of* z.

Sol. Here $$f(z) = u + iv$$

$$\therefore \quad i f(z) = iu + v$$

Adding $$(1 + i) f(z) = (u - v) + i(u + v)$$

Let $(1 + i) f(z) = F(z)$, $u - v = U$, $u + v = V$, then

$$F(z) = U + iV$$

Now $$U = u - v = (x - y)(x^2 + 4xy + y^2)$$

$$\Rightarrow \quad \frac{\partial U}{\partial x} = x^2 + 4xy + y^2 + (x - y)(2x + 4y) = 3x^2 + 6xy - 3y^2$$

and $$\frac{\partial U}{\partial y} = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y) = 3x^2 - 6xy - 3y^2$$

$$\therefore \quad F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} \quad \text{[C-R equations]}$$

$$= (3x^2 + 6xy - 3y^2) - i(3x^2 - 6xy - 3y^2)$$

Replacing x by z and y by 0, we get

$$F'(z) = 3z^2 - 3iz^2 = 3(1 - i)z^2$$

Integrating both sides,

$$F(z) = (1 - i)z^3 + c$$

$$\Rightarrow \quad (1 + i) f(z) = (1 - i)z^3 + c$$

$$f(z) = \frac{1 - i}{1 + i} z^3 + \frac{c}{1 + i} = \frac{(1 - i)^2}{1 - i^2} z^3 + \frac{c}{1 + i}$$

Hence $$f(z) = -iz^3 + A.$$

Example 10. *If f(z) is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\ |f'(z)|^2.$$

Sol. Let $f(z) = u + iv$ so that $|f(z)| = \sqrt{u^2+v^2}$

or $$|f(z)|^2 = u^2 + v^2 = \phi(x, y) \text{ (say)}$$

$$\therefore \qquad \frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly, $$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

Adding, we get

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 + v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right] \qquad ...(1)$$

Since $f(z) = u + iv$ is a regular function of z, u and v satisfy C-R equations and Laplace's equation.

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$$

$\therefore$ From (1), we get

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2} = 2\left[0+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+0+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] \qquad ...(2)$$

Now $$f(z) = u + iv$$

$$\therefore \qquad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{and} \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

From (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = 4\ |f'(z)|^2 \quad \text{or} \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\ |f'(z)|^2.$$

TEST YOUR KNOWLEDGE

1. Determine a, b, c, d so that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.
2. Determine p such that the function $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{px}{y}$ is an analytic function.

3. Show that the polar form of Cauchy-Riemann equations are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Deduce that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$

4. If $f(z) = \frac{x^3 y(y - ix)}{x^6 + y^2}, z \neq 0, f(0) = 0$, prove that $\frac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

5. Show that the function $f(z)$ defined by $f(z) = \frac{x^3 y^5 (x + iy)}{x^6 + y^{10}}, z \neq 0, f(0) = 0$, is not analytic at the origin even though it satisfies Cauchy-Riemann equations at the origin.

6. Determine which of the following functions are analytic :

(*i*) e^z (*ii*) $\sin z$ (*iii*) $\cosh z$

(*iv*) $\frac{1}{z}$ (*v*) $\frac{x + iy}{x^2 + y^2}$ (*vi*) $\frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x}$.

7. Show that $|z|^2$ is not analytic at any point.

8. Show that $u + iv = \frac{x - iy}{x - iy + a}$ where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.

9. Determine the analytic function whose real part is

(*i*) $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ (*ii*) $\log\sqrt{x^2 + y^2}$ (*iii*) $e^x(x\cos y - y\sin y)$

(*iv*) $\cos x \cosh y$ (*v*) $e^{-x}(x\sin y - y\cos y)$ (*vi*) $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.

10. Find the regular function whose imaginary part is

(*i*) $\frac{x - y}{x^2 + y^2}$ (*ii*) $\cos x \cosh y$ (*iii*) $\sinh x \cos y$.

11. Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z.

12. An electrostatic field in the xy-plane is given by the potential function $\phi = x^2 - y^2$, find the stream function.

13. If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2 + y^2}$, determine the function ϕ.

14. If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.

15. In a two dimensional fluid flow, the stream function is $\psi = \tan^{-1}\left(\frac{y}{x}\right)$, find the velocity potential ϕ.

16. If $f(z) = u + iv$ is an analytic function, find $f(z)$ if

(*i*) $u - v = e^x(\cos y - \sin y)$ (*ii*) $u + v = \frac{x}{x^2 + y^2}$, when $f(1) = 1$

(*iii*) $u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}}$, when $f\left(\frac{\pi}{2}\right) = 0$.

17. If $f(z)$ is an analytic function of z, prove that

(i) $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)\log\,|\,f'(z)\,|=0$ (ii) $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|\,\mathrm{R}\,f(z)\,|^2=2\,|\,f'(z)\,|^2.$

Answers

1. $a=2, b=-1, c=-1, d=2$ **2.** $p=-1$

6. (i), (ii), (iii), (iv) except when $z=0$, (v) except when $z=0$

9. (i) z^3+3z^2+1+c (ii) $\log z+c$ (iii) $z\,e^z+c$
(iv) $\cos z+c$ (v) $ize^{-z}+6i$ (vi) $\cot z+c$

10. (i) $\dfrac{1+i}{z}+c$ (ii) $\sin x\sinh y+c$ (iii) $-\cosh x\sin y+c$

11. $v=x^2-y^2+2xy-2y-3x+6,\ f(z)=(1+i)z^2-(2+3i)z+6i$

12. $\psi=2xy+c$ **13.** $-2xy+\dfrac{x}{x^2+y^2}+c$ **14.** $2\tan^{-1}\left(\dfrac{y}{x}\right)$, $2\log z+c$

15. $\dfrac{1}{2}\log(x^2+y^2)+c$

16. (i) e^z+c (ii) $\dfrac{1}{1-i}\left(\dfrac{1}{z}-1\right)$ (iii) $\dfrac{1}{2}\left(1-\cot\dfrac{z}{2}\right)$.

19.12. TRANSFORMATION OR MAPPING

We know that the real function $y=f(x)$ can be represented graphically by a curve in the xy-plane. Also, the real function $z=f(x,y)$ can be represented by a surface in three dimensional space. However, this method of graphical representation fails in the case of complex functions because a complex function $w=f(z)$ *i.e.*, $u+iv=f(x+iy)$ involves four real variables, two independent variables x, y and two dependent variables u, v. Thus a four dimensional region is required to represent it graphically in the cartesian fashion. As it is not possible, we choose, two complex planes and call them z-plane and w-plane. In the z-plane, we plot the point $z=x+iy$ and in the w-plane, we plot the corresponding point $w=u+iv$. Thus the function $w=f(z)$ defines a correspondence between points of these two planes. If the point z describes some curve C in the z-plane, the point w will move along a corresponding curve C′ in the w-plane, since to each (x, y) there corresponds a point (u, v). *The function $w=f(z)$ thus defines a mapping or transformation of the z-plane into the w-plane.*

For example, consider the transformation $w=z+(1-i)$. Let us determine the region D′ of the w-plane corresponding to the rectangular region D in the z-plane bounded by $x=0$, $y=0$, $x=1$ and $y=2$.

Since $w=z+(1-i)$, we have

$$u+iv=(x+iy)+(1-i)=(x+1)+i(y-1)$$

Thus $u=x+1$ and $v=y-1$

Hence the lines $x=0$, $y=0$, $x=1$ and $y=2$ in the z-plane are mapped onto the lines $u=1$, $v=-1$, $u=2$ and $v=1$ in the w-plane. The regions D and D′ are shown shaded in the figure.