Practice questions

Q1 change the order of integration in the integral $I = \int_{-a}^{a} \int_{0}^{\sqrt{(a^2-y^2)}} f(x,y)\, dx\, dy$

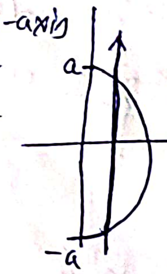
— dx dy | dy dx

$0 \le x \le \sqrt{a^2-y^2} \Rightarrow x = 0$ - y-axis

$-a \le y \le a$

$x^2 = a^2 - y^2$

$x^2 + y^2 = a^2$



$$\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2+x^2}} f(x,y)\, dy\, dx$$

$x^2 + y^2 = a^2$

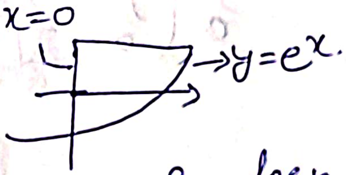
$y^2 = a^2 - x^2 \Rightarrow y = \pm\sqrt{a^2-x^2}$

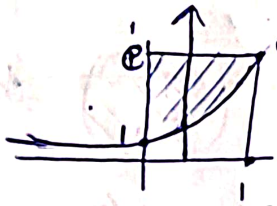
Q2 Evaluate $\int_0^1 \int_{e^x}^{e} \frac{dy\, dx}{\log y}$ by changing the order of integration

$e^x \le y \le e$

$0 \le x \le 1$

$x = 0$





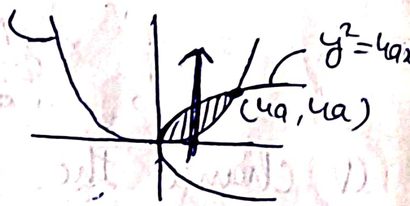
| dy dx.

$$\int_1^e \int_0^{\log y} \frac{1}{\log y}\, dx\, dy = \int_1^e \left[\frac{1}{\log y} x\right]_0^{\log y} dy = \int_1^e 1\, dy = e - 1$$

Q3 Change the order of integration & hence evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$

Sol^n $0 \le x \le 4a$ $\frac{x^2}{4a} \le y \le 2\sqrt{ax}$

$x^2 = 4ay$



$$\int_0^{4a} \int_{y^2/4a}^{\sqrt{4ay}} dx\, dy$$

$x^2 = 4ay$ $y^2 = 4ax$

point of intersection

$$= \int_0^{4a} \left(\sqrt{4ay} - y^2/4a\right) dy$$

$x^2 = 4a \cdot 2\sqrt{ax}$

$x^2 = 8a^{3/2}\sqrt{x}$

$x^{2-1/2} = 8a^{3/2} = (2a^{1/2})^3$

$x^{3/2} = (2a^{1/2})^3 \Rightarrow x^{1/2} = 2a^{1/2}$ squaring

$x = 4a$.

$$= \sqrt{4a}\, \frac{y^{3/2}}{3} \cdot 2 - \frac{1}{4a} \frac{y^3}{3} \Big|_0^{4a}$$

$$= \sqrt{4a}(4a)^{3/2} \cdot \frac{2}{3} - \frac{1}{4a} \frac{(4a)^3}{3} - 0.$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

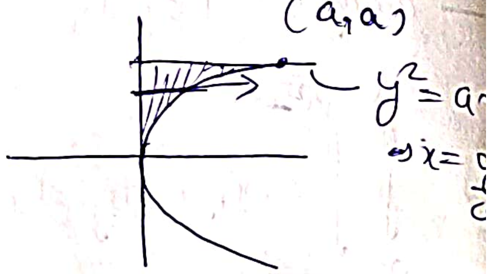
$y^2 = 16a^2 \Rightarrow y = \pm 4a$

$x^2 = 4ay$.

$16a^2 = 16a$ satisfies only the $+4a$

Q. Change the order of integration and hence evaluate

$$I = \int_0^a \int_{\sqrt{ax}}^{a} \frac{y^2 \, dy\, dx}{\sqrt{y^4 - a^2x^2}}$$



$0 \le x \le a \qquad \sqrt{ax} \le y \le a$

$ax = y^2 \qquad y = a$

at $y = a$
$y^2 = ax$
$\Rightarrow \boxed{a = x}$

$$\int_0^a \int_0^{y^2/a} \frac{y^2}{\sqrt{y^4 - a^2x^2}} \cdot dx\, dy = \int_0^a \left(\int_0^{y^2/a} \frac{y^2/a}{\sqrt{y^4/a^2 - x^2}}\, dx \right) dy$$

y - constant.

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}$$

$$= \int_0^a \frac{y^2}{a} \sin^{-1}\left(\frac{xa}{y^2}\right)\Big|_0^{y^2/a} dy$$

$$= \int_0^a \frac{y^2}{a}\left(\sin^{-1}\left(\frac{y^2 a}{a y^2}\right) - \sin^{-1}(0)\right) dy$$

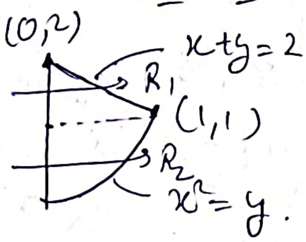
$$= \int_0^a \frac{y^2 \pi}{a\, 2} - 0 \, dy = \frac{\pi}{2a} \frac{y^3}{3}\Big|_0^a = \frac{\pi a^2}{6}$$

Q. Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy \, dy\, dx$ & evaluate.

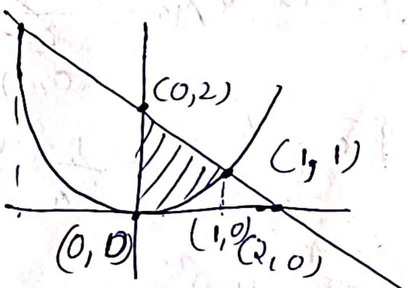
$0 \le x \le 1 \qquad x^2 \le y \le 2 - x$

1 dy dx

– dx dy



$y + x = 2$
$x^2 = y$.
$x^2 = 2 - x$
$x^2 + x - 2 = 0$
$\Rightarrow x = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2.$



$$\int_1^2 \int_0^{2-y} xy \, dx\, dy + \int_0^1 \int_0^{\sqrt{y}} xy \, dx\, dy$$

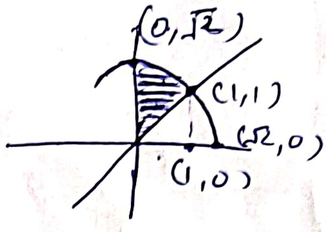
$$= \frac{5}{24} + \frac{1}{6} = 3/8$$

Practice questions

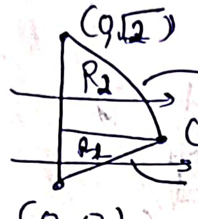
Q① Change the order of integration in $I = \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx$ & evaluate it.

$0 \le x \le 1 \qquad x \le y \le \sqrt{2-x^2}$

$$\iint_R \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy \qquad y = x \qquad y^2+x^2=2$$



$R = R_1 + R_2$



$x^2+y^2=2$

$y = x$

$$= \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

$x^2+y^2=t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$+ \int_1^{\sqrt2} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^1 \left(\int_0^{y} \frac{1}{\sqrt t}\frac{dt}{2}\right) dy + \int_1^{\sqrt2} \left(\int_0^{\sqrt{2-y^2}} \frac{1}{\sqrt t}\frac{dt}{2}\right) dy$$

$$= \int_0^1 \frac{2}{2}(x^2+y^2)^{1/2}\Big|_0^y dy + \int_1^{\sqrt2} (x^2+y^2)^{1/2}\Big|_0^{\sqrt{2-y^2}}$$

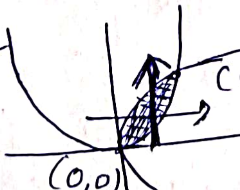
$$= \int_0^1 \left((y^2+y^2)^{1/2} - y\right) dy + \int_1^{\sqrt2} \left((2-y^2+y^2)^{1/2} - y\right) dy$$

$$= \sqrt2\,\frac{y^2}{2}\Big|_0^1 - \frac{y^2}{2}\Big|_0^1 + \sqrt2\,y\Big|_1^{\sqrt2} - \frac{y^2}{2}\Big|_1^{\sqrt2} = \frac{\sqrt2}{2} - \frac12 + \sqrt2\sqrt2 - \frac22 = \frac{\sqrt2+2-2-1}{2}$$

$$= \frac{\sqrt2-1}{2}$$

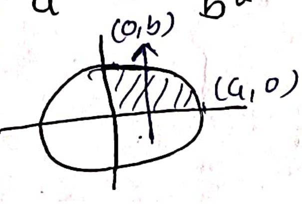
Q Find the area between $y^2 = 4ax$ & $x^2 = 4ay$.

$$\int_0^{4a}\int_{x^2/4a}^{\sqrt{4ax}} dy\,dx = \frac{16a^2}{3}$$



Q② Find the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx \qquad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \rightarrow y = \pm\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)} = \frac{b}{a}\sqrt{a^2-x^2}$$



$$\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[\frac{a}{2}\cancel{\sqrt{a^2-a^2}}^{\,0} + \frac{a^2}{2}\sin^{-1}(1) - 0 - \frac{a^2}{2}\sin^{-1}0\right]$$

$$= \frac{b}{a}\left[\frac{a^2\pi}{4}\right] = \frac{ba}{4}\pi$$

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,|J|\,r\,dr\,d\theta.$$ $\boxed{\text{Cartesian to Polar}}$

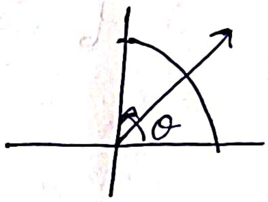
$x = r\cos\theta$
$y = r\sin\theta$

$$J = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Q1 Find ~~the polar~~ $\iint x^2+y^2\,dx\,dy$ bounded by quarter circle $x^2+y^2=1$ in the first quadrant.

$$\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$

$$\int_0^{\pi/2}\int_0^1 (r^2\cos^2\theta + r^2\sin^2\theta)\cdot r\,dr\,d\theta = \frac{\pi}{8}.$$

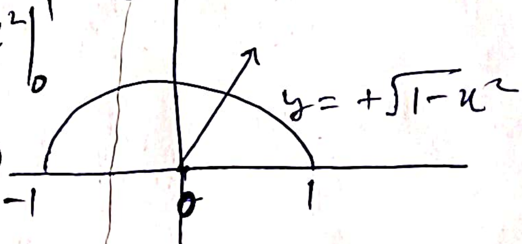


Q2 $\iint_R e^{x^2+y^2}\,dy\,dx$ R is the semicircular region bounded by the x-axis & the curve $y=\sqrt{1-x^2}$

$$\int_0^{\pi}\int_0^1 e^{r^2}\cdot r\,dr\,d\theta = \int_0^{\pi}\int_0^1 \frac{e^{t}}{2}\,dt = \int_0^{\pi}\frac{e^{t}}{2}\Big|_0^1$$

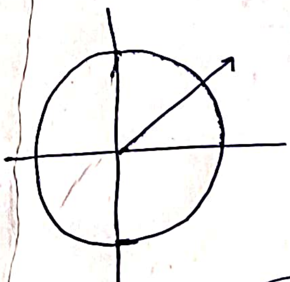
$$= \frac{1}{2}\int_0^{\pi}(e^1-1)\,d\theta = \frac{\pi}{2}(e-1)$$

$r^2 = t \Rightarrow 2r\,dr = dt$
$r\,dr = \frac{1}{2}dt$

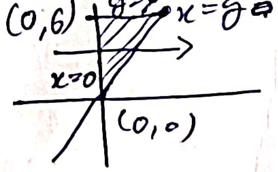


Q3 $$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$$
$$= \pi a^2$$

$y^2+x^2=a^2$



Q4 $$\int_0^6\int_0^y x\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r\cos\theta\cdot r\,dr\,d\theta$$



$y = 6$
$r\sin\theta = 6$
$r = \frac{6}{\sin\theta}$
$= 6\csc\theta$

$x = y.$
~~$x = r\sin\theta$~~

$$= \int_{\pi/4}^{\pi/2} \frac{r^3}{3}\Big|_0^{6\csc\theta} \cos\theta\,d\theta = \int_{\pi/4}^{\pi/2} \frac{72\csc^3\theta\cos\theta\,d\theta}{\cancel{3}} = 72\int_{\pi/4}^{\pi/2}\csc^2\theta\cot\theta\,d\theta.$$

$\cot\theta = t$
$-\csc^2\theta\,d\theta = dt$

$$= 72\int_{\pi/4}^{\pi/2} -t\,dt = \frac{72}{2}t^2\Big|_{\pi/4}^{\pi/2}$$

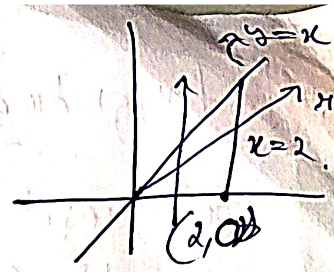
$$= -36\left[\cot^2\frac{\pi}{2} - \cot^2\frac{\pi}{4}\right] = -36[0-1] = 36$$

Q5 $\int_0^2 \int_0^x y\, dy\, dx = \int_0^{\pi/4} \int_0^{2\sec\theta} r\sin\theta \cdot r \cdot dr\, d\theta$

$x = 2$

$r\cos\theta = 2$

$r = 2\sec\theta$



$\frac{r^3}{3}\Big|_0^{2\sec\theta}$

$\frac{8}{3}\sec^3\theta$

$= \frac{8}{3}\int_0^{\pi/4} \sec^2\theta \tan\theta \cdot d\theta = \frac{8}{3}\int_0^{\pi/4} t\, dt$

$\tan\theta = t$

$\sec^2\theta\, d\theta = dt$

$= \frac{8}{3}\left[\frac{t^2}{2}\right]_0^{\pi/4} = \frac{4}{3}\left[\tan^2\frac{\pi}{4} - \tan^2 0\right] = \frac{4}{3}[1-0] = 4/3.$

Unit 6 Lect 2

Area using Double integral

double integral over the region R is $\iint_R f(x,y)\,dx\,dy$

if $f(x,y)=1$, then the integral $A=\iint_R dx\,dy$ is called area of the region R.

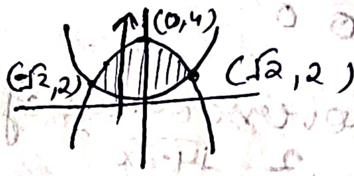
area of the bounded region in the plane.

Q1 Find the area bounded by the curves $y=x^2$, $y=4-x^2$

Soln $y=x^2$ $\quad y=4-x^2$

$4-x^2=x^2 \Rightarrow 2x^2=4 \Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}$

$y=2,2$



$R=\begin{Bmatrix} -\sqrt{2}\le x\le\sqrt{2} \\ x^2\le y\le 4-x^2\end{Bmatrix}$ $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2}^{4-x^2} dy\,dx$

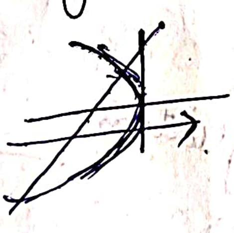
$$=\int_{-\sqrt{2}}^{\sqrt{2}}(4-x^2-x^2)\,dx$$

$\displaystyle\int_0\int_{4-x^2}^{4-x^2} dx\,dy$

$$=4x-\frac{2}{3}x^3\Big|_{-\sqrt{2}}^{\sqrt{2}}=\frac{16\sqrt{2}}{3}$$

(2) area bounded by $y-x-2=0$, $y^2=-x$, ~~reverse order of integration~~

$$\int_{-2}^{1}\int_{y-2}^{-y^2} dx\,dy$$

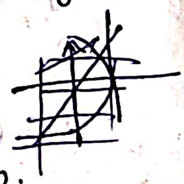


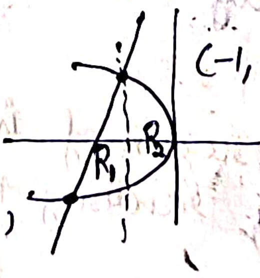
$y-2\le x\le -y^2$

$-2\le y\le 1$

Reverse the order of integration.

$$\iint dy\,dx$$





$y=\pm\sqrt{-x}$

$y=x+2$

$R=R_1+R_2$

$$\iint_R dy\,dx=\iint_{R_1} dy\,dx+\iint_{R_2} dy\,dx$$

$$\int_{-4}^{-1}\int_{-\sqrt{-x}}^{x+2} dy\,dx+\int_{-1}^{0}\int_{-\sqrt{-x}}^{\sqrt{-x}} dy\,dx = 9/2$$

$x=-1$

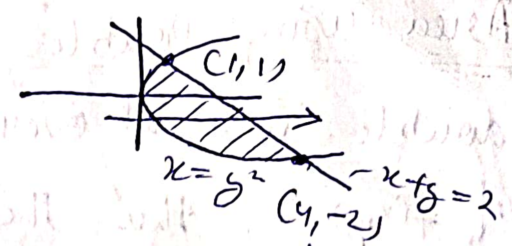
$\pm\sqrt{-(-1)}$

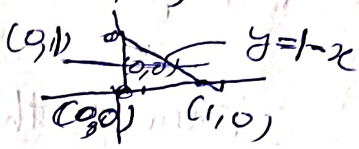
$=\pm 1$ $\quad +1$ $\quad -1$

Q) Find the area bounded by the curves

$x=y^2$, $x+y-2=0$

$$\int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy = 9/2$$

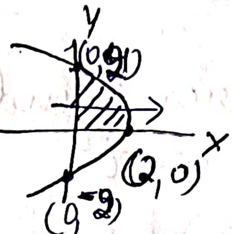




$0 < y < 1-x$
$0 < x < 1$

Q) bdd by $y^2 = 4-2x$; $x \geq 0$, $y \geq 0$

$$\int_0^2\int_0^{\frac{4-y^2}{2}} dx\,dy = 8/3$$



$x, \leq y+x \leq 1$

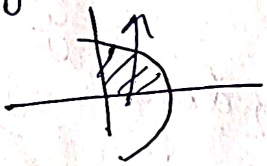
$y^2 = 4-2x$

$0 = 4-2x$

$\boxed{x=2}$

Reverse order of integration

$$\int_0^2\int_0^{\sqrt{4-2x}} dy\,dx$$

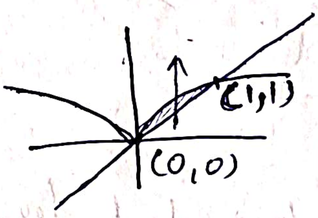


Q) Area bdd by $x^2=y^3$, $x=y$

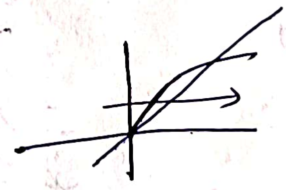
$y^2-y^3=0$

$y^2(1-y)=0$ $y=0$, $y=1$

$$\int_0^1\int_x^{x^{2/3}} dy\,dx = \frac{1}{10}$$



Reverse order of integration



$$\int_0^1\int_{y^{3/2}}^{y} dx\,dy$$

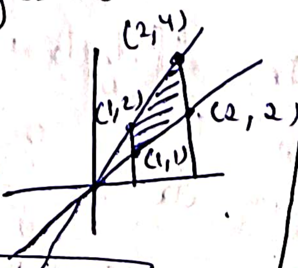
Q) Sketch the region of integration for integral $I = \int_{x=1}^{2}\int_{y=x}^{2x} \frac{x}{y}\,dy\,dx$. Hence solve it by reversing the order of integration.

$3/2 \ln 2$

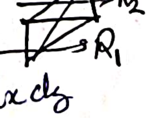
$$\int_1^2 x \ln y \Big|_x^{2x} dx = \int_1^2 x(\ln 2x - \ln x)\,dx$$

$$= \int_1^2 x \ln 2\,dx = \ln 2 \left.\frac{x^2}{2}\right|_1^2 = \frac{3}{2}\ln 2.$$

$\boxed{\ln a - \ln b = \ln \frac{a}{b}}$



Reverse



$$\int_1^2\int_1^{y} \frac{x}{y}\,dx\,dy + \int_2^4\int_{y/2}^{2} \frac{x}{y}\,dx\,dy$$

$= 3/2 \ln 2$.

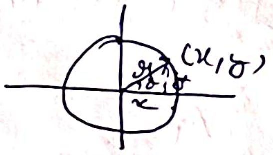
Change of variable

Let the region of integration be a circular one $x^2+y^2 \le r^2$

To change from cartesian to polar coordinates

Put $x = r\cos\theta$, $y = r\sin\theta$

On sq. & adding. $x^2+y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$



$\frac{y}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}(y/x)$

$$\text{Jacobian} = |J| = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r$$

Change of double integrals from cartesian to polar system

Cartesian Coordinates

$$\iint_R f(x,y)\,dx\,dy$$

Polar Coordinates

$$\iint_R f(r,\theta)\,|J|\,dr\,d\theta$$

~~If the~~ $\Rightarrow$ $$\iint_R f(x,y)\,dx\,dy = \iint_R f(r,\theta)\,r\,dr\,d\theta$$

Q Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ $\quad R: 0 \le y \le \sqrt{1-x^2}, \quad 0 \le x \le 1$

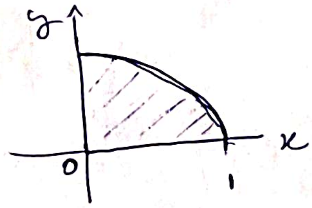
Sol^n. $I = \iint_R (x^2+y^2)\,dx\,dy = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} dx$

$$= \int_0^1 \left[x^2\sqrt{1-x^2} + \frac{(\sqrt{1-x^2})^3}{3}\right] dx$$

It is quite difficult to evaluate its integral using cartesian coordinates

now let $x = r\cos\theta$
$y = r\sin\theta$ $\quad r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}(y/x)$

$|J| = r$.



$0 \le x \le 1$
$0 \le y \le \sqrt{1-x^2}$.

$0 \le r \le 1$
$0 \le \theta \le \pi/2$

$$\iint_R f(x,y)\,dx\,dy$$

$$= \iint_R f(r,\theta)\,|J|\,dr\,d\theta$$

$$= \iint_R \underbrace{f(r,\theta)}_{r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \int_0^{\pi/2} \frac{r^4}{4}\Big|_0^1 d\theta = \frac{\pi}{8}$$

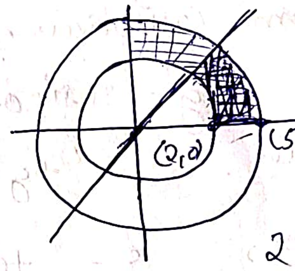
Q Evaluate $\iint_R e^{(x^2+y^2)}\,dx\,dy$ R: $x^2+y^2 \geq 4$ $x^2+y^2 = r^2$

$x^2+y^2 \leq 25$ $r_1 = 2$

$y \leq x$, $x \geq 0$, $y \geq 0$ $r_2 = 5$

$x = r\cos\theta$ $y = r\sin\theta$

$x^2+y^2 = r^2$ & $|J| = r$



$$\iint_R f(r,\theta)|J|\,dr\,d\theta = \int_0^{\pi/4}\int_2^5 e^{r^2}\cdot r\,dr\,d\theta$$

$2 \leq r \leq 5$

$0 \leq \theta \leq \pi/4$

$$= \int_0^{\pi/4} d\theta \int_2^5 e^{r^2}\cdot r\,dr$$ $r^2 = t$

$2r\,dr = dt$

$r\,dr = dt/2$

$$= [\theta]_0^{\pi/4} \cdot \left[\frac{e^{r^2}}{2}\right]_2^5 = \frac{\pi}{4}\,\frac{[e^{25} - e^4]}{2}$$

Volume as a double integral

① Cartesian coordinates

$$z = f(x, y)$$

$$V = \iint_R z \, dx \, dy.$$

② cylindrical co-ordinates

$$z = f(r, \theta)$$

$$V = \iint_R z \cdot r \cdot dr \, d\theta$$

Area by double integral

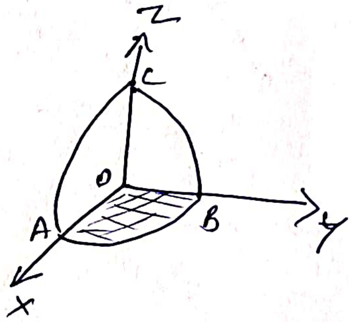
① Cartesian

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy \, dx.$$

② Polar co-ordinates

$$A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r \cdot dr \cdot d\theta.$$

Q1 Find the vol. of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



Sol^n Vol: OABC in the +ve octant lies between the ellipsoid $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

The ellipsoid cuts the plane XOY in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$$

The Region OAB above which the vol. OABC lies is bounded by $x = 0$ $x = a$ and $y = 0$, $y = b\sqrt{1 - \frac{x^2}{a^2}}$

$\therefore$ the required vol of the ellipsoid $= 8\int_0^a \int_0^{b\sqrt{1 - x^2/a^2}} z \, dy \, dx$

$$= 8\int_0^a \int_0^{b\sqrt{1 - x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dy \, dx$$

$$= 8\int_0^a \int_0^t \frac{c}{b}\sqrt{t^2 - y^2} \, dy \, dx \qquad b\sqrt{1 - \frac{x^2}{a^2}} = t$$

$$= 8\int_0^a \int_0^t \frac{c}{b}\left[\frac{y\sqrt{t^2 - y^2}}{2} + \frac{t^2}{2}\sin^{-1}\frac{y}{t}\right]_0^t dx = \frac{4c}{b}\int_0^a t^2 \sin^{-1} 1 \, dx = \frac{2\pi c}{b}\int_0^a t^2 dx$$

$$= \frac{2\pi c}{b}\int_0^a b^2\left(1 - \frac{x^2}{a^2}\right) dx = 2\pi bc\left[x - \frac{x^3}{3a^2}\right]_0^a = 2\pi bc\left[a - \frac{a}{3}\right] = \frac{4}{3}\pi abc$$