Unit – III

Short Answer Questions

1. Evaluate $\int_0^3 \int_x^{4x-x^2} y \, dy \, dx$.

Ans

$$\int_0^3 \int_x^{4x-x^2} y \, dy \, dx = \int_0^3 \left(\frac{y^2}{2}\right)_x^{4x-x^2} dx$$

$$= \int_0^3 \frac{1}{2} (4x-x^2)^2 - (x)^2 \, dx$$

$$= \frac{1}{2} \int_0^3 16x^2 + x^4 - 8x^3 - x^2 \, dx$$

$$= \frac{1}{2} \int_0^3 15x^2 + x^4 - 8x^3 \, dx$$

$$= \frac{1}{2} \left(\frac{15x^3}{3} + \frac{x^5}{5} - \frac{8x^4}{4} \right)_0^3$$

$$= \frac{1}{2} \left(\frac{\overset{9}{\cancel{27}} * 15}{\cancel{3}} + \frac{27*9}{5} - 2*81 \right)$$

$$= \frac{1}{2} \left(135 + \frac{243}{5} - 162 \right)$$

$$= \frac{1}{2} \left(\frac{243}{5} - 27 \right)$$

$$= \frac{1}{2} \left(\frac{243 - 135}{5} \right)$$

$$= \frac{108}{10}$$

$$\boxed{\int_0^3 \int_x^{4x-x^2} y \, dy \, dx = \frac{54}{5} \quad \text{Ans.}}$$

Ques 2. Evaluate $\int_0^a \int_0^x \frac{x\, dy\, dx}{x^2+y^2}$

Ans $\int_0^a \int_0^x \frac{x\, dy\, dx}{x^2+y^2} = \int_0^a \left[\frac{x}{x} * \tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$

$= \int_0^a \tan^{-1}(1) - \tan^{-1}(0)\, dx$

$= \frac{\pi}{4} \int_0^a dx$

$= \frac{\pi}{4}\left(x\right)_0^a$

$$\boxed{\int_0^a \int_0^x \frac{x\, dy\, dx}{x^2+y^2} = \frac{\pi a}{4}}$$

Ques 3. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx.$

$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left(z\right)_0^{1-x-y} dy\, dx$

$= \int_0^1 \int_0^{1-x} (1-x-y)\, dy\, dx$

$= \int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx$

$= \int_0^1 (1-x)^2 - \frac{(1-x)^2}{2}\, dx$

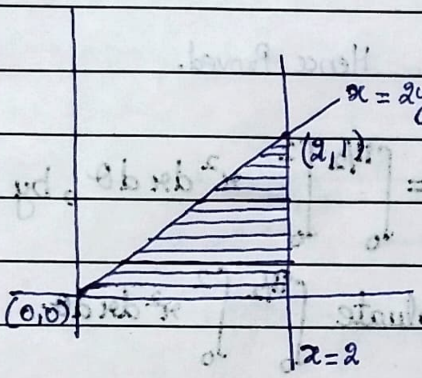
$= \int_0^1 \frac{(1-x)^2}{2}\, dx$

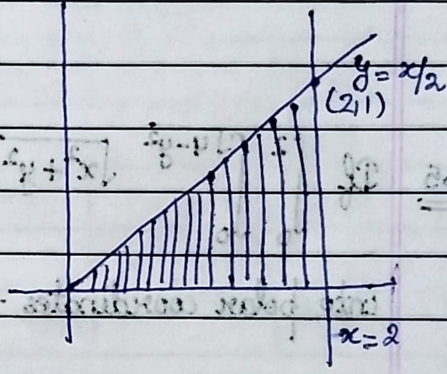
$= \left(\frac{(1-x)^3}{-6}\right)_0^1$

$$\boxed{\int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\,dy\,dx = \frac{1}{6}}$$

Ques 4 Prove that $\int_0^1\int_{2y}^{2} dx\,dy = \int_0^2\int_0^{x/2} dy\,dx$, by changing the order of integration.

Ans





When we change order of integration $\int_0^1\int_{2y}^{2} dx\,dy$ will become $\int_0^2\int_0^{x/2} dy\,dx$ and this can be very well understood with the help of diagrams above.

Solving LHS $\int_0^1\int_{2y}^{2} dx\,dy = \int_0^1 (2-2y)\,dy$

$$= \left(2y - y^2\right)_0^1$$

$$= 1.$$

Solving RHS $\int_0^2\int_0^{x/2} dy\,dx = \int_0^2 \left(y\right)_0^{x/2} dx.$

$$= \int_0^2 \frac{x}{2}\,dx$$

$$= \left(\frac{x^2}{4}\right)_0^2$$

$$= 1.$$

LHS = RHS

Hence Proved.

Ques5 If $\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta$, by changing into polar coordinates. Then evaluate $\int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta$.

Ans $\int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta = \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_0^2 d\theta$

$$= \frac{8}{3}\int_0^{\pi/2} d\theta$$

$$= \frac{8}{3} * \left(\theta\right)_0^{\pi/2}$$

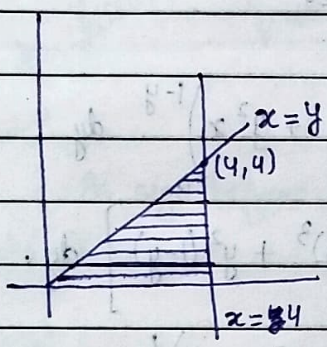
$$= \frac{8\pi}{6}$$

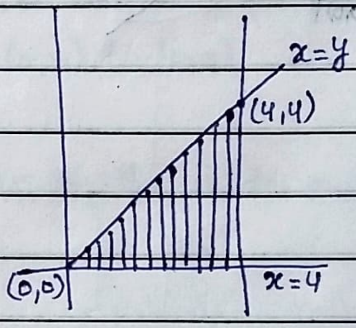
$$\boxed{\int_0^{\pi/2}\int_0^2 r^2\,dr\,d\theta = \frac{4\pi}{3}}$$

Long Answer Questions

Ques 1 Change the order of integration in $\int_0^4 \int_y^4 \frac{x\,dx}{x^2+y^2}$ and evaluate the same.

Ans First we change order of integration.





After changing order of integration

$$\int_0^4 \int_y^4 \frac{x\,dx\,dy}{x^2+y^2} \text{ becomes } \int_{x=0}^{4} \int_{y=0}^{x} \frac{x\,dy\,dx}{x^2+y^2}$$

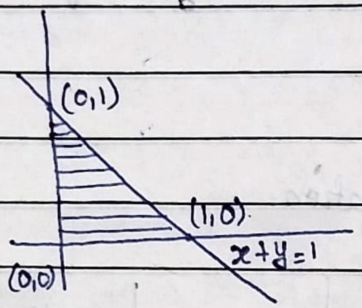
$$\int_{x=0}^{4} \int_{y=0}^{x} \frac{x\,dy\,dx}{x^2+y^2} = \int_{x=0}^{4} \frac{x}{x}\left(\tan^{-1}\frac{y}{x}\right)_0^x dx$$

$$= \int_{x=0}^{4} \frac{\pi}{4}\,dx$$

$$= \frac{\pi}{4}\left(x\right)_0^4$$

$$\boxed{\int_0^4 \int_0^x \frac{x\,dy\,dx}{x^2+y^2} = \pi}$$

Ques2 Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \leq 1$.



$$\iint_R (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1-y} (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{1} \left(\frac{x^3}{3} + y^2 x\right)_0^{1-y} dy$$

$$= \int_{y=0}^{1} \left[\frac{(1-y)^3}{3} + y^2(1-y)\right] dy$$

$$= \left(\frac{(1-y)^4}{-12} + \frac{y^3}{3} - \frac{y^4}{4}\right)_0^1$$

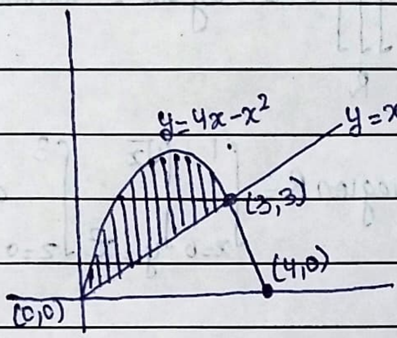
$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{4-3+1}{12}$$

$$\boxed{\iint_R (x^2+y^2)\,dx\,dy = \frac{1}{6}}$$

Ques 3: Find the area lying between the parabola $y = 4x - x^2$ and above the line $y = x$.

Ans



$4x - x^2 = 0$

$x(4-x) = 0$

$\Rightarrow x = 0, \quad x = 4.$

So vertices of parabola are $(0,0)$ & $(4,0)$

As sign before x^2 is $-$ve. So, this is downward parabola.

Point of Intersection $\quad 4x - x^2 = x$

$\Rightarrow 3x = x^2$

$\Rightarrow x = 0, 3.$

Area of region $R = \iint_R 1 \cdot dy\, dx$

$$= \int_{x=0}^{3} \int_{y=x}^{4x-x^2} 1 \, dy \, dx$$

$$= \int_{x=0}^{3} \left(y \right)_{x}^{4x-x^2} dx$$

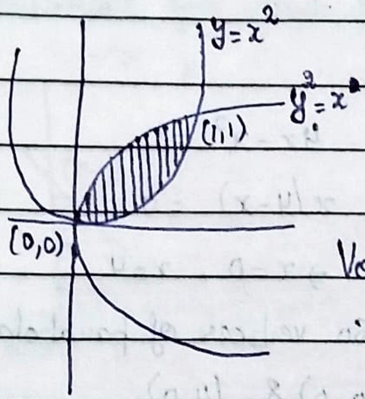
$$= \int_{0}^{3} 4x - x^2 - x \, dx$$

$$= \left(\frac{3x^2}{2} - \frac{x^3}{3} \right)_0^3 = \frac{27}{6} = \frac{9}{2}.$$

$$\boxed{\text{Area of region} = \frac{9}{2}}$$

Ques 4 - Find the volume of region bounded by the surface $y = x^2$, $x = y^2$ and the planes $z = 0$, $z = 3$.

Ans



$$\iiint_R 1 \, dz \, dy \, dx = \text{Volume of region } R.$$

$$\text{Volume of region } R = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} \int_{z=0}^{3} dz \, dy \, dx$$

$$\int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} \left(z \right)_0^3 dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 3 \, dy \, dx$$

$$= 3 \int_{x=0}^{1} \sqrt{x} - x^2 \, dx$$

$$= 3 \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right)_0^1$$

$$= 3 \left(\frac{2}{3} - \frac{1}{3} \right)$$

$$\boxed{\text{Volume} = 1.}$$ **Ans**

E - Long Answer Questions

Ques1 Evaluate the following integration

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx.$$

Ans :-

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}} \right) \right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \sin^{-1}(1) - \sin^{-1}(0)\, dy\, dx$$

$$= \frac{\pi}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx.$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x \right]_0^1$$

$$= \frac{\pi}{2} * \frac{1}{2} * \frac{\pi}{2}$$

$$= \frac{\pi^2}{8}.$$

$$\boxed{\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx = \frac{\pi^2}{8}.}$$

(b) $\iiint_R (x+y+z)\,dz\,dy\,dx$ where R: $0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3$.

$$\int_{x=0}^{1}\int_{y=1}^{2}\int_{z=2}^{3}(x+y+z)\,dz\,dy\,dx = \int_{x=0}^{1}\int_{y=1}^{2}\left[(x+y)z + \frac{z^2}{2}\right]_{z=2}^{3} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=1}^{2} (x+y) + \frac{5}{2}\,dy\,dx$$

$$= \int_{x=0}^{1}\left[\left(x+\frac{5}{2}\right)y + \frac{y^2}{2}\right]_{1}^{2} dx$$

$$= \int_{x=0}^{1} x+\frac{5}{2} + \frac{3}{2}\,dx$$

$$= \int_{x=0}^{1}(x+4)\,dx$$

$$= \left(\frac{x^2}{2} + 4x\right)_0^1$$

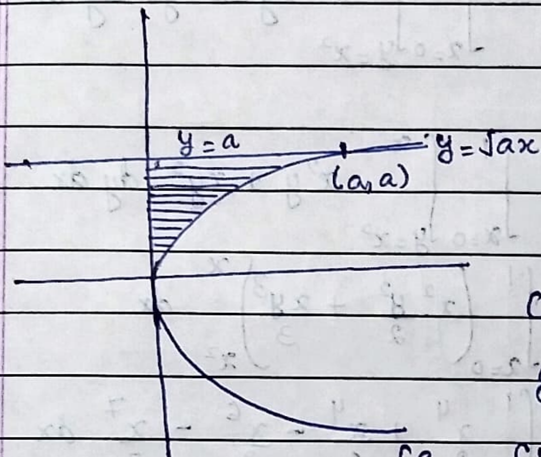
$$= 4 + \frac{1}{2}$$

$$\boxed{\int_{x=0}^{1}\int_{y=1}^{2}\int_{z=2}^{3}(x+y+z)\,dz\,dy\,dx = \frac{9}{2}}$$

Ques2 Change the order of integration and hence evaluate:

$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2x^2}}\, dy\, dx.$$

Ans



$y = \sqrt{ax}$

$\Rightarrow y^2 = ax.$

Changing order of integration, we get.

$$I = \int_{y=0}^{a} \int_{x=0}^{y^2/a} \frac{y^2}{\sqrt{y^4 - a^2x^2}}\, dx\, dy.$$

$$= \int_{y=0}^{a} \frac{y^2}{\cancel{y^2}} \left[\sin^{-1} \frac{ax}{y^2}\right]_0^{y^2/a} \frac{1}{a}\, dy$$

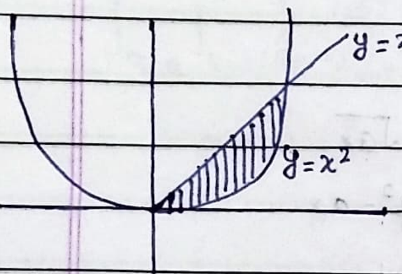
$$= \int_{y=0}^{a} \frac{y^2}{a} \left(\frac{\pi}{2}\right) dy.$$

$$= \frac{\pi}{2} \cdot \frac{1}{a} \left(\frac{y^3}{3}\right)_0^a$$

$$= \frac{\pi a^3}{6a}$$

$$\boxed{I = \frac{\pi a^2}{6} \text{ Ans.}}$$

Ques3 Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$.



$$\int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} x^2y + xy^2\,dy\,dx$$

$$= \int_{x=0}^{1}\left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)_{x^2}^{x} dx$$

$$= \int_0^1 \frac{x^4}{2} + \frac{x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\,dx$$

$$= \left(\frac{5x^5}{6\cdot 5} - \frac{x^7}{14} - \frac{x^8}{24}\right)_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{28-12-7}{2\cdot 7\cdot 3\cdot 4}$$

$$= \frac{9}{56\cdot 3}$$

$$\boxed{\iint_R xy(x+y)\,dy\,dx = \frac{3}{56}}$$

Ques 4 Change into polar coordinates and evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\,dx$.

Hence, show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Ans $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\,dx = I$

Changing to polar coordinates $x = r\cos\theta$
$y = r\sin\theta$

$$x^2 + y^2 = r^2$$

$$dy\,dx = r\,dr\,d\theta$$

$$\int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\,dx$$

$r^2 = p$

$2r\,dr = dp$

$$\int_0^{\pi/2} \int_0^\infty \frac{e^{-p}}{2} dp\,d\theta = \int_0^{\pi/2} \frac{1}{2} \left(-e^{-p}\right)_0^\infty d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 1\,d\theta$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

$$I = \int_0^\infty \int_0^\infty e^{-x^2-y^2} dy\,dx = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy.$$

If we take $\int_0^\infty e^{-x^2} dx = I_1$

then $I = I_1 \cdot I_1 = \frac{\pi}{4} \Rightarrow I_1^2 = \left(\frac{\pi}{4}\right)$.

So, $I_1 = \frac{\sqrt{\pi}}{2}$

$$\boxed{\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}$$

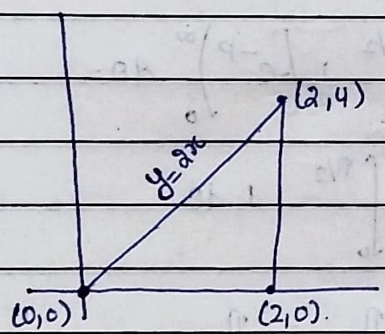
Hence Proved.

Ques 5 A triangular thin plate with vertices (0,0), (2,0) and (2,4) has density $\rho = 1+x+y$. Then find:

a) the mass of plate.

b) the position of its center of gravity.

Ans Mass $= \iint_A \rho \, dx\, dy$.



$$M = \int_{x=0}^{2}\int_{y=0}^{2x} (1+x+y)\,dy\,dx$$

$$M = \int_{x=0}^{2} \left[(1+x)y + \frac{y^2}{2}\right]_0^{2x} dx$$

$$= \int_{x=0}^{2} 2x(1+x) + \frac{4x^2}{2}\,dx$$

$$= \int_0^2 2x + 2x^2 + 2x^2\,dx$$

$$= \left(x^2 + \frac{4x^3}{3}\right)_0^2$$

$$= 4 + \frac{32}{3} = \frac{44}{3}$$

$$\boxed{\text{Mass} = \frac{44}{3}.}$$

(b) Center of gravity.

$$\bar{x} = \frac{\iint_A x\,e\,dx\,dy}{\iint_A e\,dx\,dy} \qquad \bar{y} = \frac{\iint_A y\,e\,dx\,dy}{\iint_A e\,dx\,dy}$$

$$\iint_A x\,e\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{2x} x(1+x+y)\,dy\,dx$$

$$= \int_{x=0}^{2}\left[x(1+x)y + \frac{xy^2}{2}\right]_0^{2x} dx$$

$$= \int_0^2 2x^2(1+x) + \frac{x}{2}\cdot 4x^2\,dx.$$

$$= \left(\frac{2x^3}{3} + \frac{2x^4}{4} + \frac{2x^4}{4}\right)_0^2$$

$$= \left(\frac{2\cdot 8}{3} + \frac{\cancel{2}\cdot 16}{\cancel{4}_2} + \frac{\cancel{2}\,16}{\cancel{4}_2}\right)$$

$$= \frac{16}{3} + 16$$

$$= \frac{64}{3}$$

$$\bar{x} = \frac{64/3}{44/3} = \frac{64}{44} = \frac{16}{11}.$$

$$\bar{y} = \frac{\iint_A y\,\ell\,dy\,dx}{44/3}$$

$$\iint_A y(1+x+y)\,dy\,dx = \int_{x=0}^{2}\int_{y=0}^{2x} (1+x)y + y^2\,dxy\,dx$$

$$= \int_0^2 \left((1+x)\frac{y^2}{2} + \frac{y^3}{3}\right)_0^{2x} dx$$

$$= \int_0^2 (1+x)\cdot 2x^2 + \frac{8x^3}{3}\,dx$$

$$= \int_0^2 2x^2 + 2x^3 + \frac{8x^3}{3}\,dx$$

$$= \left(\frac{2x^3}{3} + \frac{14\cdot x^4}{12}\right)_0^2$$

$$= \frac{16}{3} + \frac{14}{\cancel{12}\,3}\cdot\cancel{16}\,4$$

$$= \frac{16}{3} + \frac{56}{3} = \frac{72}{3}$$

$$\bar{y} = \frac{72/3}{44/3}$$

$$= \frac{72}{44} = \frac{18}{11}.$$

$$g \to \left(\frac{16}{11}, \frac{18}{11}\right).$$

$$\boxed{\text{Center of gravity} = \left(\frac{16}{11}, \frac{18}{11}\right)}$$