41. If the first partial derivatives are continuous throughout an open region R, then by Eq. (3) in this section of the text, $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$, where $\epsilon_1,\ \epsilon_2 \to 0$ as $\Delta x,\ \Delta y \to 0$. Then as $(x,y) \to (x_0,y_0)$, $\Delta x \to 0$ and $\Delta y \to 0 \Rightarrow \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \Rightarrow$ f is continuous at every point (x_0,y_0) in R.

42. Yes, since f_{xx}, f_{yy}, f_{xy}, and f_{yx} are all continuous on R, use the same reasoning as in Exercise 41 with $f_x(x,y) = f_x(x_0,y_0) + f_{xx}(x_0,y_0)\,\Delta x + f_{xy}(x_0,y_0)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$ and $f_y(x,y) = f_y(x_0,y_0) + f_{yx}(x_0,y_0)\,\Delta x + f_{yy}(x_0,y_0)\,\Delta y + \widehat{\epsilon}_1\Delta x + \widehat{\epsilon}_2\Delta y$. Then $\lim\limits_{(x,y)\to(x_0,y_0)} f_x(x,y) = f_x(x_0,y_0)$ and $\lim\limits_{(x,y)\to(x_0,y_0)} f_y(x,y) = f_y(x_0,y_0)$.

12.5 THE CHAIN RULE

1. (a) $\frac{\partial w}{\partial x} = 2x,\ \frac{\partial w}{\partial y} = 2y,\ \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t \Rightarrow \frac{dw}{dt} = -2x \sin t + 2y \cos t = -2 \cos t \sin t + 2 \sin t \cos t$ $= 0$; $w = x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow \frac{dw}{dt} = 0$

   (b) $\frac{dw}{dt}(\pi) = 0$

2. (a) $\frac{\partial w}{\partial x} = 2x,\ \frac{\partial w}{\partial y} = 2y,\ \frac{dx}{dt} = -\sin t + \cos t,\ \frac{dy}{dt} = -\sin t - \cos t \Rightarrow \frac{dw}{dt}$
   $= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t)$
   $= 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t) = \left(2 \cos^2 t - 2 \sin^2 t\right) - \left(2 \cos^2 t - 2 \sin^2 t\right)$
   $= 0$; $w = x^2 + y^2 = (\cos t + \sin t)^2 + (\cos t - \sin t)^2 = 2 \cos^2 t + 2 \sin^2 t = 2 \Rightarrow \frac{dw}{dt} = 0$

   (b) $\frac{dw}{dt}(0) = 0$

3. (a) $\frac{\partial w}{\partial x} = \frac{1}{z},\ \frac{\partial w}{\partial y} = \frac{1}{z},\ \frac{\partial w}{\partial z} = \frac{-(x+y)}{z^2},\ \frac{dx}{dt} = -2 \cos t \sin t,\ \frac{dy}{dt} = 2 \sin t \cos t,\ \frac{dz}{dt} = -\frac{1}{t^2}$
   $\Rightarrow \frac{dw}{dt} = -\frac{2}{z} \cos t \sin t + \frac{2}{z} \sin t \cos t + \frac{x+y}{z^2 t^2} = \frac{\cos^2 t + \sin^2 t}{\left(\frac{1}{t^2}\right)(t^2)} = 1$; $w = \frac{x}{z} + \frac{y}{z} = \frac{\cos^2 t}{\left(\frac{1}{t}\right)} + \frac{\sin^2 t}{\left(\frac{1}{t}\right)} = t \Rightarrow \frac{dw}{dt} = 1$

   (b) $\frac{dw}{dt}(3) = 1$

4. (a) $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+z^2},\ \frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2},\ \frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2},\ \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t,\ \frac{dz}{dt} = 2t^{-1/2}$
   $\Rightarrow \frac{dw}{dt} = \frac{-2x \sin t}{x^2+y^2+z^2} + \frac{2y \cos t}{x^2+y^2+z^2} + \frac{4zt^{-1/2}}{x^2+y^2+z^2} = \frac{-2 \cos t \sin t + 2 \sin t \cos t + 4\left(4t^{-1/2}\right)t^{-1/2}}{\cos^2 t + \sin^2 t + 16t}$
   $= \frac{16}{1+16t}$; $w = \ln\left(x^2+y^2+z^2\right) = \ln\left(\cos^2 t + \sin^2 t + 16t\right) = \ln(1+16t) \Rightarrow \frac{dw}{dt} = \frac{16}{1+16t}$

   (b) $\frac{dw}{dt}(3) = \frac{16}{49}$

5. (a) $\frac{\partial w}{\partial x} = 2ye^x,\ \frac{\partial w}{\partial y} = 2e^x,\ \frac{\partial w}{\partial z} = -\frac{1}{z},\ \frac{dx}{dt} = \frac{2t}{t^2+1},\ \frac{dy}{dt} = \frac{1}{t^2+1},\ \frac{dz}{dt} = e^t \Rightarrow \frac{dw}{dt} = \frac{4yte^x}{t^2+1} + \frac{2e^x}{t^2+1} - \frac{e^t}{z}$
   $= \frac{(4t)\left(\tan^{-1} t\right)\left(t^2+1\right)}{t^2+1} + \frac{2\left(t^2+1\right)}{t^2+1} - \frac{e^t}{e^t} = 4t \tan^{-1} t + 1$; $w = 2ye^x - \ln z = \left(2 \tan^{-1} t\right)\left(t^2+1\right) - t$

$\Rightarrow \frac{dw}{dt} = \left(\frac{2}{t^2+1}\right)(t^2+1) + (2\tan^{-1} t)(2t) - 1 = 4t\tan^{-1} t + 1$

(b) $\frac{dw}{dt}(1) = (4)(1)\left(\frac{\pi}{4}\right) + 1 = \pi + 1$

6. (a) $\frac{\partial w}{\partial x} = -y\cos xy,\ \frac{\partial w}{\partial y} = -x\cos xy,\ \frac{\partial w}{\partial z} = 1,\ \frac{dx}{dt} = 1,\ \frac{dy}{dt} = \frac{1}{t},\ \frac{dz}{dt} = e^{t-1} \Rightarrow \frac{dw}{dt} = -y\cos xy - \frac{x\cos xy}{t} + e^{t-1}$

$= -(\ln t)[\cos(t\ln t)] - \frac{t\cos(t\ln t)}{t} + e^{t-1} = -(\ln t)[\cos(t\ln t)] - \cos(t\ln t) + e^{t-1}$; $w = z - \sin xy$

$= e^{t-1} - \sin(t\ln t) \Rightarrow \frac{dw}{dt} = e^{t-1} - [\cos(t\ln t)]\left[\ln t + t\left(\frac{1}{t}\right)\right] = e^{t-1} - (1+\ln t)\cos(t\ln t)$

(b) $\frac{dw}{dt}(1) = 1 - (1+0)(1) = 0$

7. (a) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (4e^x\ln y)\left(\frac{\cos\theta}{r\cos\theta}\right) + \left(\frac{4e^x}{y}\right)(\sin\theta) = \frac{4e^x\ln y}{r} + \frac{4e^x\sin\theta}{y}$

$= \frac{4(r\cos\theta)\ln(r\sin\theta)}{r} + \frac{4(r\cos\theta)(\sin\theta)}{r\sin\theta} = (4\cos\theta)\ln(r\sin\theta) + 4\cos\theta$;

$\frac{\partial z}{\partial\theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial\theta} = (4e^x\ln y)\left(\frac{-r\sin\theta}{r\cos\theta}\right) + \left(\frac{4e^x}{y}\right)(r\cos\theta) = -(4e^x\ln y)(\tan\theta) + \frac{4e^x r\cos\theta}{y}$

$= [-4(r\cos\theta)\ln(r\sin\theta)](\tan\theta) + \frac{4(r\cos\theta)(r\cos\theta)}{r\sin\theta} = (-4r\sin\theta)\ln(r\sin\theta) + \frac{4r\cos^2\theta}{\sin\theta}$;

$z = 4e^x\ln y = 4(r\cos\theta)\ln(r\sin\theta) \Rightarrow \frac{\partial z}{\partial r} = (4\cos\theta)\ln(r\sin\theta) + 4(r\cos\theta)\left(\frac{\sin\theta}{r\sin\theta}\right)$

$= (4\cos\theta)\ln(r\sin\theta) + 4\cos\theta$; also $\frac{\partial z}{\partial\theta} = (-4r\sin\theta)\ln(r\sin\theta) + 4(r\cos\theta)\left(\frac{r\cos\theta}{r\sin\theta}\right)$

$= (-4r\sin\theta)\ln(r\sin\theta) + \frac{4r\cos^2\theta}{\sin\theta}$

(b) At $\left(2, \frac{\pi}{4}\right)$: $\frac{\partial z}{\partial r} = 4\cos\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + 4\cos\frac{\pi}{4} = 2\sqrt{2}\ln\sqrt{2} + 2\sqrt{2} = \sqrt{2}(\ln 2 + 2)$;

$\frac{\partial z}{\partial\theta} = (-4)(2)\sin\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + \frac{(4)(2)\left(\cos^2\frac{\pi}{4}\right)}{\left(\sin\frac{\pi}{4}\right)} = -4\sqrt{2}\ln\sqrt{2} + 4\sqrt{2} = -2\sqrt{2}\ln 2 + 4\sqrt{2}$

8. (a) $\frac{\partial z}{\partial r} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right]\cos\theta + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right]\sin\theta = \frac{y\cos\theta}{x^2+y^2} - \frac{x\sin\theta}{x^2+y^2} = \frac{(r\sin\theta)(\cos\theta) - (r\cos\theta)(\sin\theta)}{r^2} = 0$;

$\frac{\partial z}{\partial\theta} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right](-r\sin\theta) + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2 + 1}\right]r\cos\theta = -\frac{yr\sin\theta}{x^2+y^2} - \frac{xr\cos\theta}{x^2+y^2} = \frac{-(r\sin\theta)(r\sin\theta) - (r\cos\theta)(r\cos\theta)}{r^2}$

$= -\sin^2\theta - \cos^2\theta = -1$; $z = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}(\cot\theta) \Rightarrow \frac{\partial z}{\partial r} = 0$ and $\frac{\partial z}{\partial\theta} = \left(\frac{1}{1+\cot^2\theta}\right)(-\csc^2\theta)$

$= \frac{-1}{\sin^2\theta + \cos^2\theta} = -1$

(b) At $\left(1.3, \frac{\pi}{6}\right)$: $\frac{\partial z}{\partial r} = 0$ and $\frac{\partial z}{\partial\theta} = -1$

9. (a) $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = (y+z)(1) + (x+z)(1) + (y+x)(v) = x + y + 2z + v(y+x)$

$= (u+v) + (u-v) + 2uv + v(2u) = 2u + 4uv$; $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$

$= (y+z)(1) + (x+z)(-1) + (y+x)(u) = y - x + (y+x)u = -2v + (2u)u = -2v + 2u^2;$
$w = xy + yz + xz = \left(u^2 - v^2\right) + \left(u^2v - uv^2\right) + \left(u^2v + uv^2\right) = u^2 - v^2 + 2u^2v \Rightarrow \frac{\partial w}{\partial u} = 2u + 4uv$ and
$\frac{\partial w}{\partial v} = -2v + 2u^2$

(b) At $\left(\frac{1}{2}, 1\right)$: $\frac{\partial w}{\partial u} = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1) = 3$ and $\frac{\partial w}{\partial v} = -2(1) + 2\left(\frac{1}{2}\right)^2 = -\frac{3}{2}$

10. (a) $\frac{\partial w}{\partial u} = \left(\frac{2x}{x^2+y^2+z^2}\right)\left(e^v \sin u + ue^v \cos u\right) + \left(\frac{2y}{x^2+y^2+z^2}\right)\left(e^v \cos u - ue^v \sin u\right) + \left(\frac{2z}{x^2+y^2+z^2}\right)\left(e^v\right)$

$= \left(\frac{2ue^v \sin u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)\left(e^v \sin u + ue^v \cos u\right)$

$+ \left(\frac{2ue^v \cos u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)\left(e^v \cos u - ue^v \sin u\right)$

$+ \left(\frac{2ue^v}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)\left(e^v\right) = \frac{2}{u};$

$\frac{\partial w}{\partial v} = \left(\frac{2x}{x^2+y^2+z^2}\right)\left(ue^v \sin u\right) + \left(\frac{2y}{x^2+y^2+z^2}\right)\left(ue^v \cos u\right) + \left(\frac{2z}{x^2+y^2+z^2}\right)\left(ue^v\right)$

$= \left(\frac{2ue^v \sin u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)\left(ue^v \sin u\right)$

$+ \left(\frac{2ue^v \cos u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)\left(ue^v \cos u\right)$

$+ \left(\frac{2ue^v}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)\left(ue^v\right) = 2;$ $w = \ln\left(u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}\right) = \ln\left(2u^2e^{2v}\right)$

$= \ln 2 + 2\ln u + 2v \Rightarrow \frac{\partial w}{\partial u} = \frac{2}{u}$ and $\frac{\partial w}{\partial v} = 2$

(b) At $(-2, 0)$: $\frac{\partial w}{\partial u} = \frac{2}{-2} = -1$ and $\frac{\partial w}{\partial v} = 2$

11. (a) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{q-r} + \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r+r-p+p-q}{(q-r)^2} = 0;$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = \frac{1}{q-r} - \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r-r+p+p-q}{(q-r)^2} = \frac{2p-2r}{(q-r)^2}$

$= \frac{(2x+2y+2z)-(2x+2y-2z)}{(2z-2y)^2} = \frac{z}{(z-y)^2};$ $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$

$= \frac{1}{q-r} + \frac{r-p}{(q-r)^2} - \frac{p-q}{(q-r)^2} = \frac{q-r+r-p-p+q}{(q-r)^2} = \frac{2q-2p}{(q-r)^2} = \frac{-4y}{(2z-2y)^2} = -\frac{y}{(z-y)^2};$

$u = \frac{p-q}{q-r} = \frac{2y}{2z-2y} = \frac{y}{z-y} \Rightarrow \frac{\partial u}{\partial x} = 0,$ $\frac{\partial u}{\partial y} = \frac{(z-y)-y(-1)}{(z-y)^2} = \frac{z}{(z-y)^2},$ and $\frac{\partial u}{\partial z} = \frac{(z-y)(0)-y(1)}{(z-y)^2}$

$= -\frac{y}{(z-y)^2}$

(b) At $\left(\sqrt{3}, 2, 1\right)$: $\frac{\partial u}{\partial x} = 0,$ $\frac{\partial u}{\partial y} = \frac{1}{(1-2)^2} = 1,$ and $\frac{\partial u}{\partial z} = \frac{-2}{(1-2)^2} = -2$

12. (a) $\frac{\partial u}{\partial x} = \frac{e^{qr}}{\sqrt{1-p^2}}(\cos x) + \left(re^{qr}\sin^{-1}p\right)(0) + \left(qe^{qr}\sin^{-1}p\right)(0) = \frac{e^{qr}\cos x}{\sqrt{1-p^2}} = \frac{e^{z\ln y}\cos x}{\sqrt{1-\sin^2 x}} = y^z$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$;

$\frac{\partial u}{\partial y} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + \left(re^{qr}\sin^{-1}p\right)\left(\frac{z^2}{y}\right) + \left(qe^{qr}\sin^{-1}p\right)(0) = \frac{z^2\, re^{qr}\sin^{-1}p}{y} = \frac{z^2\left(\frac{1}{z}\right)y^z x}{y} = xzy^{z-1}$;

$\frac{\partial u}{\partial z} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + \left(re^{qr}\sin^{-1}p\right)(2z\ln y) + \left(qe^{qr}\sin^{-1}p\right)\left(-\frac{1}{z^2}\right) = \left(2zre^{qr}\sin^{-1}p\right)(\ln y) - \frac{qe^{qr}\sin^{-1}p}{z^2}$

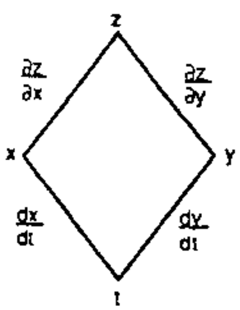
$= (2z)\left(\frac{1}{z}\right)\left(y^z x\ln y\right) - \frac{\left(z^2\ln y\right)\left(y^z\right)x}{z^2} = xy^z\ln y$; $u = e^{z\ln y}\sin^{-1}(\sin x) = xy^z$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2} \Rightarrow \frac{\partial u}{\partial x} = y^z$,

$\frac{\partial u}{\partial y} = xzy^{z-1}$, and $\frac{\partial u}{\partial z} = \; = xy^z\ln y$ from direct calculations

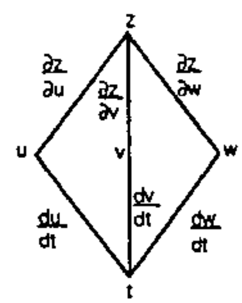
(b) At $\left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2}\right)$: $\frac{\partial u}{\partial x} = \left(\frac{1}{2}\right)^{-1/2} = \sqrt{2}$, $\frac{\partial u}{\partial y} = \left(\frac{\pi}{4}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{(-1/2)-1} = -\frac{\pi\sqrt{2}}{4}$, $\frac{\partial u}{\partial z} = \left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^{-1/2}\ln\left(\frac{1}{2}\right)$

$= -\frac{\pi\sqrt{2}\ln 2}{4}$

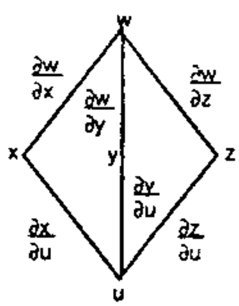
13. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$



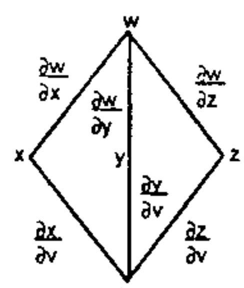
14. $\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial x}{\partial w}\frac{dw}{dt}$



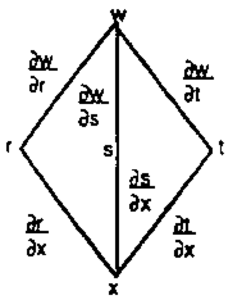
15. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$



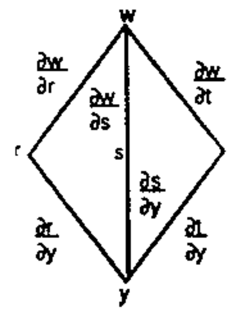
$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$



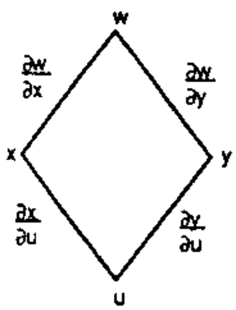
16. $\frac{\partial w}{\partial x}=\frac{\partial w}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial w}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$



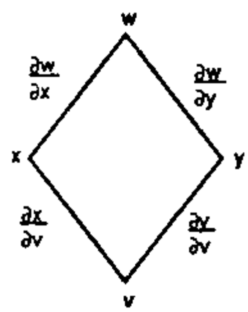
$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial w}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$



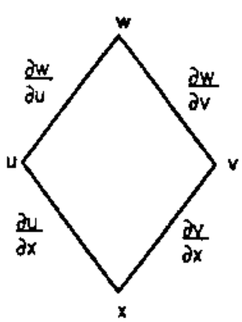
17. $\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$



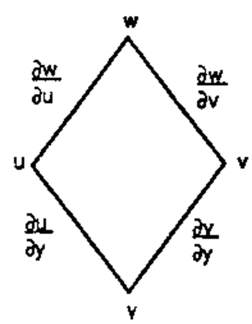
$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$



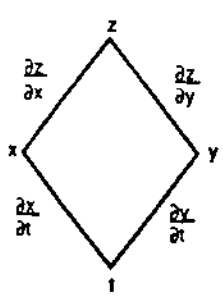
18. $\frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$



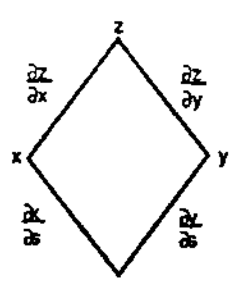
$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$



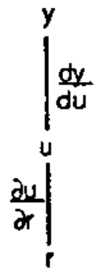
19. $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$



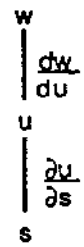
$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$



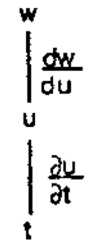
20. $\frac{\partial y}{\partial r} = \frac{dy}{du}\frac{\partial u}{\partial r}$



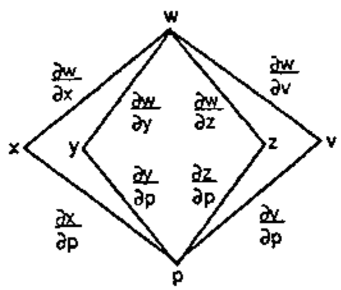
21. $\frac{\partial w}{\partial s} = \frac{dw}{du}\frac{\partial u}{\partial s}$



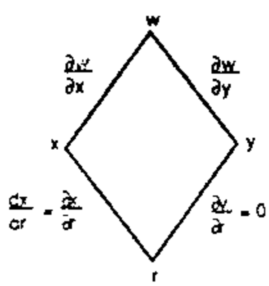
$\frac{\partial w}{\partial t} = \frac{dw}{du}\frac{\partial u}{\partial t}$



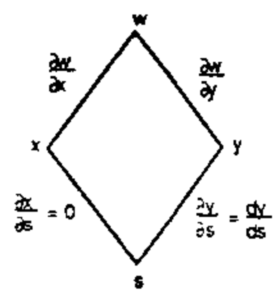
22. $\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$



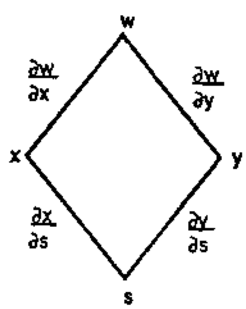
23. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{dx}{dr} + \frac{\partial w}{\partial y}\frac{dy}{dr} = \frac{\partial w}{\partial x}\frac{dx}{dr}$ since $\frac{dy}{dr} = 0$



$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} = \frac{\partial w}{\partial y}\frac{dy}{ds}$ since $\frac{dx}{ds} = 0$



24. $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$



25. Let $F(x,y) = x^3 - 2y^2 + xy = 0 \Rightarrow F_x(x,y) = 3x^2 + y$ and $F_y(x,y) = -4y + x \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+y}{(-4y+x)}$ $\Rightarrow \frac{dy}{dx}(1,1) = \frac{4}{3}$

26. Let $F(x,y) = xy + y^2 - 3x - 3 = 0 \Rightarrow F_x(x,y) = y - 3$ and $F_y(x,y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y}$
$\Rightarrow \frac{dy}{dx}(-1,1) = 2$

27. Let $F(x,y) = x^2 + xy + y^2 - 7 = 0 \Rightarrow F_x(x,y) = 2x + y$ and $F_y(x,y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y}$
$\Rightarrow \frac{dy}{dx}(1,2) = -\frac{4}{5}$

28. Let $F(x,y) = xe^y + \sin xy + y - \ln 2 = 0 \Rightarrow F_x(x,y) = e^y + y\cos xy$ and $F_y(x,y) = xe^y + x\sin xy + 1$
$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + y\cos xy}{xe^y + x\sin xy + 1} \Rightarrow \frac{dy}{dx}(0,\ln 2) = -(2 + \ln 2)$

29. Let $F(x,y,z) = z^3 - xy + yz + y^3 - 2 = 0 \Rightarrow F_x(x,y,z) = -y,\ F_y(x,y,z) = -x + z + 3y^2,\ F_z(x,y,z) = 3z^2 + y$
$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_y} = -\frac{-y}{3x^2+y} = \frac{y}{3z^2+y} \Rightarrow \frac{\partial z}{\partial x}(1,1,1) = \frac{1}{4};\ \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x+z+3y^2}{3z^2+y} = \frac{x-z-3y^2}{3z^2+y}$
$\Rightarrow \frac{\partial z}{\partial y}(1,1,1) = -\frac{3}{4}$

30. Let $F(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0 \Rightarrow F_x(x,y,z) = -\frac{1}{x^2},\ F_y(x,y,z) = -\frac{1}{y^2},\ F_z(x,y,z) = -\frac{1}{z^2}$
$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{x^2} \Rightarrow \frac{\partial z}{\partial x}(2,3,6) = -9;\ \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\left(-\frac{1}{y^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{y^2} \Rightarrow \frac{\partial z}{\partial y}(2,3,6) = -4$

31. Let $F(x,y,z) = \sin(x+y) + \sin(y+z) + \sin(z+z) = 0 \Rightarrow F_x(x,y,z) = \cos(x+y) + \cos(x+z)$,
$F_y(x,y,z) = \cos(x+y) + \cos(y+z),\ F_z(x,y,z) = \cos(y+z) + \cos(x+z) \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$
$= -\frac{\cos(x+y) + \cos(x+z)}{\cos(y+z) + \cos(x+z)} \Rightarrow \frac{\partial z}{\partial x}(\pi,\pi,\pi) = -1;\ \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x+y) + \cos(y+z)}{\cos(y+z) + \cos(x+z)} \Rightarrow \frac{\partial z}{\partial y}(\pi,\pi,\pi) = -1$

32. Let $F(x,y,z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0 \Rightarrow F_x(x,y,z) = e^y + \frac{2}{x},\ F_y(x,y,z) = xe^y + e^z,\ F_z(x,y,z) = ye^z$
$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(e^y + \frac{2}{x}\right)}{ye^z} \Rightarrow \frac{\partial z}{\partial x}(1,\ln 2,\ln 3) = -\frac{4}{3\ln 2};\ \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xe^y + e^z}{ye^z} \Rightarrow \frac{\partial z}{\partial y}(1,\ln 2,\ln 3) = -\frac{5}{3\ln 2}$

33. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = 2(x+y+z)(1) + 2(x+y+z)[-\sin(r+s)] + 2(x+y+z)[\cos(r+s)]$
$= 2(x+y+z)[1 - \sin(r+s) + \cos(r+s)] = 2[r - s + \cos(r+s) + \sin(r+s)][1 - \sin(r+s) + \cos(r+s)]$
$\Rightarrow \left.\frac{\partial w}{\partial r}\right|_{r=1,s=-1} = 2(3)(2) = 12$

34. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} = y\left(\frac{2v}{u}\right) + x(1) + \left(\frac{1}{z}\right)(0) = (u+v)\left(\frac{2v}{u}\right) + \frac{v^2}{u} \Rightarrow \left.\frac{\partial w}{\partial v}\right|_{u=-1,v=2} = (1)\left(\frac{4}{-1}\right) + \left(\frac{4}{-1}\right)$
$= -8$

35. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = \left(2x - \frac{y}{x^2}\right)(-2) + \left(\frac{1}{x}\right)(1) = \left[2(u - 2v + 1) - \frac{2u + v - 2}{(u - 2v + 1)^2}\right](-2) + \frac{1}{u - 2v + 1}$
$\Rightarrow \left.\frac{\partial w}{\partial v}\right|_{u=0,v=0} = -7$

36. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (y\cos xy + \sin y)(2u) + (x\cos xy + x\cos y)(v)$

$= \left[uv\cos\left(u^3v + uv^3\right) + \sin uv\right](2u) + \left[\left(u^2+v^2\right)\cos\left(u^3v+uv^3\right) + \left(u^2+v^2\right)\cos uv\right](v)$

$\Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=0,\,v=1} = 0 + (\cos 0 + \cos 0)(1) = 2$

37. $\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2}\right)e^u = \left[\frac{5}{1+\left(e^u + \ln v\right)^2}\right]e^u \Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=\ln 2,\,v=1} = \left[\frac{5}{1+(2)^2}\right](2) = 2;$

$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v} = \left(\frac{5}{1+x^2}\right)\left(\frac{1}{v}\right) = \left[\frac{5}{1+\left(e^u+\ln v\right)^2}\right]\left(\frac{1}{v}\right) \Rightarrow \left.\frac{\partial z}{\partial v}\right|_{u=\ln 2,\,v=1} = \left[\frac{5}{1+(2)^2}\right](1) = 1$

38. $\frac{\partial z}{\partial u} = \frac{dz}{dq}\frac{\partial q}{\partial u} = \left(\frac{1}{q}\right)\left(\frac{\sqrt{v+3}}{1+u^2}\right) = \left(\frac{1}{\sqrt{v+3}\,\tan^{-1}u}\right)\left(\frac{\sqrt{v+3}}{1+u^2}\right) = \frac{1}{\left(\tan^{-1}u\right)\left(1+u^2\right)}$

$\Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=1,\,v=-2} = \frac{1}{\left(\tan^{-1}1\right)\left(1+1^2\right)} = \frac{2}{\pi};\ \frac{\partial z}{\partial v} = \frac{dz}{dq}\frac{\partial q}{\partial v} = \left(\frac{1}{q}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right)$

$= \left(\frac{1}{\sqrt{v+3}\,\tan^{-1}u}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right) = \frac{1}{2(v+3)} \Rightarrow \left.\frac{\partial z}{\partial v}\right|_{u=1,\,v=-2} = \frac{1}{2}$

39. $V = IR \Rightarrow \frac{\partial V}{\partial I} = R$ and $\frac{\partial V}{\partial R} = I;\ \frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt} = R\,\frac{dI}{dt} + I\,\frac{dR}{dt} \Rightarrow -0.01$ volts/sec

$= (600 \text{ ohms})\,\frac{dI}{dt} + (0.04 \text{ amps})(0.5 \text{ ohms/sec}) \Rightarrow \frac{dI}{dt} = -0.00005$ amps/sec

40. $V = abc \Rightarrow \frac{dV}{dt} = \frac{\partial V}{\partial a}\frac{da}{dt} + \frac{\partial V}{\partial b}\frac{db}{dt} + \frac{\partial V}{\partial c}\frac{dc}{dt} = (bc)\frac{da}{dt} + (ac)\frac{db}{dt} + (ab)\frac{dc}{dt}$

$\Rightarrow \left.\frac{dV}{dt}\right|_{a=1,\,b=2,\,c=3} = (2\text{ m})(3\text{ m})(1\text{ m/sec}) + (1\text{ m})(3\text{ m})(1\text{ m/sec}) + (1\text{ m})(2\text{ m})(-3\text{ m/sec}) = 3\text{ m}^3/\text{sec}$

and the volume is increasing; $S = 2ab + 2ac + 2bc \Rightarrow \frac{dS}{dt} = \frac{\partial S}{\partial a}\frac{da}{dt} + \frac{\partial S}{\partial b}\frac{db}{dt} + \frac{\partial S}{\partial c}\frac{dc}{dt}$

$= 2(b+c)\,\frac{da}{dt} + 2(a+c)\,\frac{db}{dt} + 2(a+b)\,\frac{dc}{dt} \Rightarrow \left.\frac{dS}{dt}\right|_{a=1,\,b=2,\,c=3}$

$= 2(5\text{ m})(1\text{ m/sec}) + 2(4\text{ m})(1\text{ m/sec}) + 2(3\text{ m})(-3\text{ m/sec}) = 0\text{ m}^2/\text{sec}$ and the surface area is not changing;

$D = \sqrt{a^2+b^2+c^2} \Rightarrow \frac{dD}{dt} = \frac{\partial D}{\partial a}\frac{da}{dt} + \frac{\partial D}{\partial b}\frac{db}{dt} + \frac{\partial D}{\partial c}\frac{dc}{dt} = \frac{1}{\sqrt{a^2+b^2+c^2}}\left(a\,\frac{da}{dt} + b\,\frac{db}{dt} + c\,\frac{dc}{dt}\right) \Rightarrow \left.\frac{dD}{dt}\right|_{a=1,\,b=2,\,c=3}$

$= \left(\frac{1}{\sqrt{14}\text{ m}}\right)[(1\text{ m})(1\text{ m/sec}) + (2\text{ m})(1\text{ m/sec}) + (3\text{ m})(-3\text{ m/sec})] = -\frac{6}{\sqrt{14}}\text{ m/sec} < 0 \Rightarrow$ the diagonals are decreasing in length

41. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w},$

$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$, and

$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$