Unit 5 — Lect 1

Sequence & Series

Sequence It is a list of numbers $a_1, a_2, a_3 \dots a_n \dots$ in a given order. $a_1, a_2, \dots$ represents the terms of seq.

eg $1, 2, 3 \dots n, \dots$ so $a_1 = 1$ (1st term), $a_2 = 2$ (2nd term) $\dots$ $a_n = n$ (nth term)

→ $a_n = 2n+2$ $f: \mathbb{N} \to \mathbb{R}$ n can start from anywhere.

→ convergence/divergence of seq is the behaviour of the seq as $n \to \infty$

→ eg $2, 4, 6, 8 \dots$

$a_n = 2n$ $n = 1, 2 \dots$

$a_n = 2n+2$ $n = 0, 1, 2 \dots$

$a_n = 2n+4$ $n = -1, 0, 1, 2 \dots$

$\lim_{n \to \infty} 2n = \infty$ | $\lim_{n\to\infty} (2n+2) = \infty$ | $\lim_{n\to\infty} (2n+4) = \infty$

No matter from where n starts they all conv/div. same.

So begining or from where we start the seq does not matter. By default we usually ~~stra~~ start from $n=1$.

conv/div depends upon the tail of seq as $n \to \infty$.

Notation $\langle a_n \rangle$, $\{a_n\}$, $\{a_1, a_2 \dots\}$, $\{a_n\}_{n=1}^{\infty}$

eg (1) $\langle a_n \rangle = \left\langle \frac{n-1}{n} \right\rangle = 0, \frac{1}{2}, \frac{2}{3}, \dots$

(2) $\{a_n\} = \left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \dots$

(3) $\{a_n\}_{n=1}^{\infty} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots\} = \{\sqrt{n}\}_{n=1}^{\infty} = \{\sqrt{n}\} = \langle \sqrt{n} \rangle$

$\lim_{n\to\infty} a_n = L$ or $\langle a_n \rangle \to L$ (limit of seq)

If no such L exists then we say limit of seq $\langle a_n \rangle$ does not exist or it is not convergent, If limit of a seq exists then it will always be finite & unique.

eg ① $\lim_{n\to\infty} \frac{1}{n} = 0$ cgt (Finite & Unique)

② $\lim_{n\to\infty} a_n = \infty$ (∞ or not finite $\langle a_n \rangle$ is divgt)

$\lim_{n\to\infty} n^2 = \infty$ divgt.

③ $\lim_{n\to\infty} (-1)^n = \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ is even} \end{cases}$ | Finite but not unique. So limit does not exist.

Non dec seq. A seq $\langle a_n \rangle$ is said to be nondec if $a_n \le a_{n+1}$ $\forall n$ (non dec includes constant seq) constant seq is also nondec seq.

$a_{n+1} - a_n \ge 0$

or $\frac{a_{n+1}}{a_n} \ge 1$

eg $\langle a_n \rangle = \langle n \rangle$

$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n} \ge 1 \ \forall n$

→ f(n) Non dec if $f'(n) \ge 0$ $\quad f(x) = \frac{x}{x+1}$

$f'(x) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0 \quad \forall x \ge 1$

$f(x)$ is non dec $\forall x \ge 1$

→ A nondec seq of real no's converges iff it is bounded from above. And it cges to its least upper bound.

→ Non dec seq + cgt ⇒ Bounded above

Nondec seq + Bounded above ⇒ cgt

eg ① Test the convergence / divergence of seq $a_n = \frac{5n+1}{n+1}$ $n=1,2,\ldots$ by using nondec seq thm.

Sol^n $a_n = \frac{5n+1}{n+1}$ $\quad a_{n+1} = \frac{5(n+1)+1}{(n+1)+1} = \frac{5n+6}{n+2}$

$\cancel{\frac{a_{n+1}}{a_n}}$ $\quad a_{n+1} - a_n = \frac{5n+6}{n+2} - \frac{5n+1}{n+1} = \frac{(5n+6)(n+1) - (5n+1)(n+2)}{(n+1)(n+2)}$

$= \frac{\cancel{5n^2} + \cancel{5n} + \cancel{6n} + 6 - \cancel{5n^2} - \cancel{10n} - \cancel{n} - 2}{(n+1)(n+2)} = \frac{4}{(n+1)(n+2)} > 0 \ \forall n \geq 1$

So $\langle a_n \rangle$ is nondec.

<u>To check Bounded above or not</u>

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{5n+1}{n+1} = \lim\limits_{n\to\infty} \frac{\cancel{n}(5+\frac{1}{n})}{\cancel{n}(1+1/n)} = 5$

$\therefore \langle a_n \rangle$ is nondec Also $\lim\limits_{n\to\infty} a_n = 5$ so $a_n < 5 \ \forall n$

Hence $\langle a_n \rangle$ is bdd above by 5.

since its nondec & $\lim\limits_{n\to\infty} a_n = 5$

<u>Since</u> ① $\langle a_n \rangle$ is nondec ② $\langle a_n \rangle$ is bdd above.

So $\langle a_n \rangle$ by nondec seq thm is cgt. & it cges to its least upper bound 5.

<u>Thm</u> If $\langle a_n \rangle$ & $\langle b_n \rangle$ be seq of real no's and let A & B be real no's. s.t. $\langle a_n \rangle \to A$ & $\langle b_n \rangle \to B$ then

① $\lim\limits_{n\to\infty} (a_n \pm b_n) = A \pm B$ ③ $\lim\limits_{n\to\infty} (k.a_n) = k.A$

② $\lim\limits_{n\to\infty} (a_n . b_n) = A.B$ ④ $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = A/B$ $\quad B \neq 0$

Q Check Div / cgt

(i) $\langle a_n \rangle = \left\langle \frac{2^n - 1}{3^n} \right\rangle$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{2^n - 1}{3^n} = \lim\limits_{n\to\infty} \left[\left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n\right] = 0 - 0 = 0$ cgt.

Common Occuring limits

① $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ ② $\lim_{n\to\infty} (n)^{1/n} = 1$ ③ $\lim_{n\to\infty} x^{1/n} = 1 \quad x>0$

④ $\lim_{n\to\infty} x^n = 0 \quad |x|<1$ ⑤ $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ ⑥ $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Q Find the n^{th} term of the seq

① $0, 1, 1, 2, 2, 3, 3 \ldots$ $\quad [\frac{n}{2}]$ $\lfloor \frac{n}{2} \rfloor$ greatest integer f^n

② $0, 0, 1, 1, 1, 2, 2, 2 \ldots$ $\quad [\frac{n}{3}]$ $\lfloor n/3 \rfloor$

③ $1, 0, 1, 0, 1, 0 \ldots$ $\quad \frac{1+(-1)^{n+1}}{2}$ $\quad n = 1, 2, 3 \ldots$

④ $1, 5, 9, 13, 17 \ldots$ $\quad 4n-3$ $\quad n = 1, 2, 3 \ldots$

Bounded & unbounded seq

Bounded above seq A seq $\{a_n\}$ is said to be bounded above if $\exists$ a real number k such that $a_n \leq k \quad \forall n \in \mathbb{N}$

Bounded below seq A seq $\langle a_n \rangle$ is said to be bounded below if $\exists$ a real number k such that $k \leq a_n \quad \forall n \in \mathbb{N}$ $\quad a_n = 2^{n-1} = \{1, 2, 2^2, 2^3 \ldots\}$ bdd below.

Bounded seq A seq $\langle a_n \rangle$ is said to be bounded when it is both bounded both above & below. eg $\langle a_n \rangle = \langle \frac{1}{n} \rangle$ $\quad 0 < a_n \leq 1$

unbdd seq A seq $\langle a_n \rangle$ which is not bounded is known as unbdd seq.

Convergent seq. A seq $\{a_n\}$ is said to be cgt if $\lim_{n\to\infty} a_n$ is finite

eg $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3} \cdots$ $a_n = \frac{1}{2^n}$ $\lim_{n\to\infty} \frac{1}{2^n} = 0$ $\langle a_n \rangle$ is cgt.

Divergent seq A seq $\langle a_n \rangle$ is said to be dvt if $\lim_{n\to\infty} a_n$ is not finite

i.e if $\lim_{n\to\infty} a_n = \infty$ or $-\infty$ if lt does not exist.

eg (i) $a_n = n^2$ $\lim_{n\to\infty} n^2 = \infty$ $\langle n^2 \rangle$ dvt.

(ii) $a_n = \langle -2^n \rangle$ $\lim_{n\to\infty} -2^n = -\infty$ dvt.

Oscillatory seq If a seq $\langle a_n \rangle$ neither cgs to a finite no. nor diverges to $\pm\infty$, it is called an oscillatory seq.

— A bdd seq which does not cgt is said to osillate finitely.

eg $\langle a_n \rangle = \langle (-1)^n \rangle = \{-1, 1, -1, 1, \cdots\}$

$\lim_{n\to\infty} a_n$ is not unique $\therefore$ limit does not exist

$\Rightarrow$ seq does not cgt.

— An unbdd seq which does not cgt is said to oscillate infinitelly eg $\langle a_n \rangle = (-1)^n . n = \{-1, 2, -3, 4, -5, +6 \ldots\}$

$\lim_{n\to\infty} a_{2n} = \infty$ $\lim_{n\to\infty} a_{2n+1} = -\infty$ limit is not unique

The seq does not ~~diverge~~ converge
oscillate infinitely.

Monotonic seq (i) A seq $\{a_n\}$ is said to be monotonically increasing (non dec.) if $a_{n+1} \geq a_n \ \forall n$ i.e $a_1 \leq a_2 \leq a_3 \leq a_4 \cdots$

(ii) A seq $\{a_n\}$ is said to be monotonically dec. if (Non inc)
$a_{n+1} \leq a_n \ \forall n$ i.e $a_1 \geq a_2 \geq a_3 \cdots$

(iii) A seq $\{a_n\}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

(iv) strictly monotonically inc if $a_{n+1} > a_n$ $\forall n \in \mathbb{N}$ (inc)

(v) strictly " dec. if $a_{n+1} < a_n$ $\forall n \in \mathbb{N}$ (dec)

(vi) A seq $\langle a_n \rangle$ is strictly monotonic if it is either st. mon. inc. or st. mon. dec.

Note ① Every cgt seq is bdd But converse is not true.
eg $(-1)^n$ is bdd but not cgt.

② The necessary & sufficient condn for convergence of monotonic seq is that it is bdd.

③ monotonic inc (non dec) seq is not bdd above ⇒ diverges to ∞.

④ monotonic dec (non inc) seq is not bdd below ⇒ div to $-\infty$

⑤ if $\langle a_n \rangle \longrightarrow A$ $\langle b_n \rangle \rightarrow B$ then. $\langle a_n \rangle \rightarrow A$ mean $\lim_{n\to\infty} a_n = A$

-① $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

-② $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

-③ $\lim_{n\to\infty} (k \cdot a_n) = k \cdot A$

-④ $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$ $(B \neq 0)$

⑥ If a seq $\langle a_n \rangle$ ~~cgt~~ converges to a and $a_n \geq 0$ $\forall n$ then $a \geq 0$

⑦ The seq $\langle |a_n| \rangle \rightarrow 0$ iff $\langle a_n \rangle \rightarrow 0$

⑧ If $\langle a_n \rangle \rightarrow a$ & $\langle b_n \rangle \rightarrow b$ ~~then~~ & $a_n \leq b_n$ then $a \leq b$

⑨ If $\langle a_n \rangle \rightarrow l$ & $\langle b_n \rangle \rightarrow l$ & $a_n \leq c_n \leq b_n$ $\forall n \in \mathbb{N}$ then ~~$a_n \rightarrow$~~ $c_n \rightarrow l$ (sandwich thm)

Infinite series Let $\langle a_n \rangle$ be an infinite seq of real no's $\langle a_n \rangle = a_1, a_2, a_3 \ldots a_n, \ldots$ then the expression of the form
$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$ is called ∞ series.
So basically ∞ series is a sum of ∞ seq of Real No
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots = \sum a_n$$

Partial sum Let $\langle a_n \rangle$ be a seq of real no's $a_1, a_2 \ldots a_n \ldots$
Let corresponding series be
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$
Now consider $S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3 \quad \ldots \quad S_n = a_1 + a_2 + \ldots + a_n$
$\vdots$

Then seq $\langle S_n \rangle = S_1, S_2 \ldots S_n \ldots$ is called seq of Partial sums of series $\sum a_n$ and S_n is called n^{th} Partial sum of series $\sum a_n$

Q1 $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots$$

$a_1 = 1 \quad a_2 = \frac{1}{2} \quad a_3 = \frac{1}{2^2} \quad \ldots \quad a_n = \frac{1}{2^{n-1}}, \ldots$

Partial sums
$S_1 = a_1 = 1$
$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$
$S_3 = 1 + \frac{1}{2} + \frac{1}{2^2} = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$
$S_4 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{7}{4} + \frac{1}{8} = \frac{15}{8}$
$\vdots$
$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$

so seq $\langle S_n \rangle = 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \dots$ is called seq of Partial sums.

$\boxed{\text{If } \langle S_n \rangle \to L \text{ then } \Sigma a_n \text{ is cgt } \& \ \Sigma a_n = L}$

Series based concept of Partial sum
→ Geometric series
→ Telescoping series

Geometric series :- $a + ar + ar^2 + \dots + ar^{n-1} + \dots$
$= \sum_{n=1}^{\infty} a r^{n-1}$

sum of 1st n terms of G. series
$= \frac{a(1-r^n)}{1-r} \quad r \neq 1$

sum of ∞ G. series $= \frac{a}{1-r} \quad |r| < 1$ (cgt)

g. series
- cgt if $|r|<1$
- dvt if $|r| \geq 1$
- oscillatory if $r < -1$

G. series diverges if $|r| \geq 1$

Q check convergence of series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n}$

$\Sigma a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n} = \frac{1}{4} - \frac{1}{4^2} + \frac{1}{4^3} - \frac{1}{4^4} + \dots$

$a = \frac{1}{4} \qquad r = -\frac{1}{4} \qquad \because |r| = \frac{1}{4} < 1$

So series Σa_n is cgt
series cges to sum $= \frac{a}{1-r} = \frac{1/4}{1+\frac{1}{4}} = \frac{1}{5}$

Q $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \cancel{\frac{(-1)^{n+1}}{4^n}} (\ln x)^n$ Find value of x for which series cgs Also find $\sum_{n=0}^{\infty} a_n$

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (\ln x)^n = 1 + (\ln x) + (\ln x)^2 + \dots$

clearly G. series $\quad a = 1 \ \& \ r = \ln x$

for cgce $\quad |r| < 1$
$|\ln x| < 1 \qquad -1 < \ln x < 1$

$e^{-1} < x < e^1 \Rightarrow \frac{1}{e} < x < e$

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\ln x}$$

Telescoping series are those series whose partial sums eventually have only fixed. No. of terms after concellation

eg check cgce of series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right) = \sum_{n=1}^{\infty} a_n$$

Sol^n $\displaystyle\sum_{n=1}^{\infty} a_n = \left(1-\cancel{\frac{1}{2^2}}\right)+\left(\cancel{\frac{1}{2^2}}-\cancel{\frac{1}{3^2}}\right)+\left(\cancel{\frac{1}{3^2}}-\frac{1}{4^2}\right)+\dots$

$$S_n = 1-\frac{1}{2^2}+\frac{1}{2^2}-\frac{1}{3^2}+\dots\frac{1}{n^2}-\frac{1}{(n+1)^2} \quad \dots+\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right) \text{ ~~term~~}$$

that terms are getting cancelled

$$S_n = 1-\frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{(n+1)^2}\right) = 1-0 = 1$$

$\langle S_n\rangle \to 1$ so seq of partial sum of series $\sum a_n$ is cgt & it cges to value 1. Hence $\sum a_n$ will also be cgt

$$\& \sum_{n=1}^{\infty} a_n = 1$$

② $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty} a_n$

$$\sum_{n=1}^{\infty} a_n = \cancel{\frac{1}{2}} \; \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\dots$$

$$S_n = \left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\dots+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$= 1-\frac{1}{n+1} \qquad \lim_{n\to\infty} 1-\frac{1}{n+1} = 1$$

Hence $\sum a_n$ is cgt & $\sum a_n = 1$

Q. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \ln n - \ln(n+1)$

$\sum a_n = \ln 1 - \ln 2 + \ln 2 - \ln 3 + \ln 3 - \ln 4 + \dots + \ln n - \ln n \dots + \dots$

$S_n = \ln 1 - \cancel{\ln 2} + \cancel{\ln 2} - \cancel{\ln 3} + \dots + \cancel{\ln n} - \ln n+1$

$S_n = \ln 1 - \ln(n+1)$

$\lim_{n\to\infty} S_n = \ln 1 - \ln\infty = 0 - \infty = -\infty$. Dirgt.

Note
① If $\lim_{n\to\infty} a_n$ fails to exists (not equal to zero) then $\sum a_n$ diverges.

② If $\sum a_n$ cges $\Rightarrow \langle a_n\rangle \to 0$ However if $\langle a_n\rangle \to 0$ then it is not necessary that $\sum a_n$ will be cgt (It can be dirgt also)

③ (i) If $\langle a_n\rangle \not\to 0$ but to some finite value $\sum a_n$ is dirgt.

(ii) If $\lim_{n\to\infty} a_n$ does not exist, $\sum a_n$ dirgt.

④ if $\lim_{n\to\infty} a_n$ does not exist then $\sum a_n$ will be dirgt. By nth term test for divergence. No need to apply any other method.

⑤ if $\lim_{n\to\infty} a_n \not\to 0$ or $\langle a_n\rangle \not\to 0$ Even then $\sum a_n$ will be dirgt by nth term test for dirgce.

If $\lim_{n\to\infty} a_n = 0$ or $\langle a_n\rangle \to 0$ Then series may or may not cgce.

Q1 Check cgce of series $\sum_{n=1}^{\infty} (1-\frac{1}{n})^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1-\frac{1}{n})^n = e^{-1}$$

$\langle a_n \rangle \to \frac{1}{e}$ although seq a_n is cgt but not cgt to 0.

as $\langle a_n \rangle \not\to 0$ $\therefore \sum a_n$ is divgt.

Q2 $\sum_{n=1}^{\infty} \cos(n\pi) = \sum_{n=1}^{\infty} a_n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n\pi) = \lim_{n\to\infty} (-1)^n = \begin{cases} -1 & n \text{ odd} \\ 1 & \text{even} \end{cases}$$

lt is not unique $\therefore \lim_{n\to\infty} a_n$ does not exist. Hence $\sum a_n$ will be divgt by n^{th} term test for divgce.

Q3 $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} (\sqrt{2})^n$ $\qquad \sqrt{2} = 1.4 > 1$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\sqrt{2})^n = \infty$$

or G. series $r = \sqrt{2} > 1$

So $\langle a_n \rangle$ is divgt Hence $\sum a_n$ will also be divgt.

as $\langle a_n \rangle \to \infty$

Behaviour of ∞ series

An ∞ series $\sum a_n$ converges, diverges or oscillates (finitely or infinitely) according as the seq $\{S_n\}$ of it's partial sums converge, diverge or oscillates (finitely or infinitely)

(i) The series $\sum a_n$ converges (or is said to be convergent) if the seq of its partial sums converges. Thus $\sum a_n$ is convergent if $\lim_{n\to\infty} S_n$ = finite & unique

(ii) The series $\sum a_n$ diverges (or is said to be divergent) if the seq of its partial sums diverges. Thus $\sum a_n$ is divergent if $\lim_{n\to\infty} S_n = \pm\infty$ or not unique

(iii) The series $\sum a_n$ oscillates finitely if the seq $\langle S_n \rangle$ of its partial sums oscillates finitely.

Thus $\sum a_n$ oscillates finitely if $\langle S_n \rangle$ is bdd and neither converges nor diverges.

(iv) The series $\sum a_n$ oscillates infinitely if the seq $\langle S_n \rangle$ of its partial sums oscillates infinitely.

Thus $\sum a_n$ oscillates infinitely if $\langle S_n \rangle$ is unbdd and neither converges nor diverges.

whenever we are checking cgce of $\sum a_n$ then we must check the $\lim_{n \to \infty} a_n$ (i.e cgce of $\langle a_n \rangle$)

① $\langle a_n \rangle \not\to 0$ (goes to some finite value other than 0)

$\sum a_n \to$ dvt

② $\langle a_n \rangle \to \pm\infty$ $\quad \sum a_n \to$ div.

$\frac{3}{4} \quad \frac{3}{3} \cdot \frac{1}{5} \quad = \frac{1}{25}$

n^{th} - Term test for Divergence

$\sum_{n\geq 1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from 0.

① $\langle a_n \rangle \not\to 0 \Rightarrow \sum a_n \to$ div. ② $\langle a_n \rangle \to \pm\infty$. $\sum a_n \to$ div.
finite but not 0. If $\langle a_n \rangle \to 0 \not\Rightarrow$ cgt.

Cauchy Integral test

Let $\langle a_n \rangle$ be a seq of +ve terms suppose $a_n = f(n)$ where f is +ve cts and dec. fn of n $\forall n \geq N$ (N is a +ve integer) Then series $\sum_{n=N}^{\infty} a_n$ & $\int_N^{\infty} f(n)\,dn$ both cgs or both div. [i.e if $\int_N^{\infty} f(n)\,dn$ is finite $\sum a_n \to$ cgt.
if $\int_N^{\infty} f(n)\,dn$ is not finite $\sum a_n \to$ dgt.]

① f is +ve $\forall n \geq N$. ② f is cts ③ $f'(n) < 0$ $\forall n \geq N$

p - series test

$\sum_{n\geq 1}^{\infty} \frac{1}{n^p}$ cgt $p > 1$
div $p \leq 1$

Direct comparison test (DCT)

Let $\sum a_n$ be a series with no negative terms
① $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.
② $\sum a_n$ diverges if there is a divergent series of non negative terms $\sum d_n$ with $a_n \geq d_n$ $\forall n > N$, for some N.

Limit Comparison test (LCT) useful for series where a_n is rational fn of (n)

① if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ Then if choosen $\sum b_n$ cgs then $\sum a_n$ also cgs.
if $\sum b_n$ div. then $\sum a_n$ also div.

② If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ if $\sum b_n$ cgs then $\sum a_n$ also cgs.
↳ [In this case if $\sum b_n$ div. then choice of $\sum b_n$ will be wrong.]

(8) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ If $\Sigma b_n \to$ div then $\Sigma a_n \to$ div.

(if Σb_n cgs $\to$ choose some other b_n)

D'Alembert's
Ratio test let Σa_n be a series of +ve terms. [*used when f^n have factorials & exponent]

suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = k$. if $k < 1$ cgt
if $k > 1$ or ∞ divt
If $k = 1$ Test is inconclusive.

Root test (nth root test) let Σa_n be a series of non-ive terms. &
let $\lim_{n\to\infty} (a_n)^{1/n} = k$ if $k < 1$ cgt
if $k > 1$ or ∞ div.
if $k = 1$ inconclusive.

Raabe's Test If Σa_n be a series of positive terms &

8.3
972 $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = k$, then the series is ~~convergent if~~

(i) if $k > 1$ convergent
(ii) if $k < 1$ divergent
(iii) if $k = 1$ test is inconclusive

or
$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = k$
- $k > 1$ cgt
- $k < 1$ div.
- $k = 1$ inc

Note: Raabe's test is applied only when the ratio test fails. (i.e $k = 1$)

Questions on series

Q1 $1^2+2^2+3^2+4^2+\cdots+n^2+\cdots$ diverges to ∞ (Partial sum)

Sol^n $S_n = 1^2+2^2+3^2+\cdots+n^2$ sum of sq of n +ve integers

$$S_n = \frac{n(n+1)(2n+1)}{6} \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6}$$

$$= \infty$$

$\therefore$ Series $1^2+2^2+\cdots$ is divergent.

Q2 $2-2+2-2+\cdots$

$$S_n = \begin{cases} 0 & n \text{ is even} \\ 2 & n \text{ is odd} \end{cases}$$

Hence S_n does not have a unique lt. $\therefore$ series is oscillatory $\therefore$ does not cgt.

(nth term div)

Q3 $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{n}{n+1}+\cdots$

$a_n = \frac{n}{n+1}$ $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+1/n)} = 1 \neq 0$

So the given series is divergent.

(nth term div)

Q4 $\frac{1}{3}+\frac{3}{5}+\frac{8}{10}+\cdots+\frac{2^n-1}{2^n+1}+\cdots$

$a_n = \frac{2^n-1}{2^n+1}$ $\qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n-1}{2^n+1} = \lim_{n\to\infty} \frac{\cancel{2^n}\left(1-\frac{1}{2^n}\right)}{\cancel{2^n}\left(1+\frac{1}{2^n}\right)}$

$= \lim_{n\to\infty} \frac{\left(1-1/2^n\right)}{\left(1+\frac{1}{2^n}\right)} = 1 \neq 0$ $\therefore$ series is divergent.

Q5 $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\cdots$ (geom. series)

$= \left(\frac{3}{5}+\frac{3}{5^3}+\frac{3}{5^5}+\cdots\right) + \left(\frac{4}{5^2}+\frac{4}{5^4}+\cdots\right)$

$= \Sigma U_n + \Sigma V_n$ Both are geometric series with $r = 1/5^2 < 1$

$\because |r|<1$ $\therefore$ both are cgt. So $\Sigma U_n + \Sigma V_n$ is cgt.

(LCT)

Q6 $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots$

Sol^n $a_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n(2-\frac{1}{n})}{n^3(1+\frac{1}{n})(1+\frac{2}{n})} = \frac{1}{n^2}\left(\frac{2-1/n}{(1+\frac{1}{n})(1+\frac{2}{n})}\right)$

$\lim_{n\to\infty} \frac{u_n}{v_n}$ Take $v_n = \frac{1}{n^2}$

$= \lim_{n\to\infty} \frac{(2-\frac{1}{n})}{(1+\frac{1}{n})(1+\frac{2}{n})} = 2$ (finite number)

Σu_n & Σv_n converge or diverge together

$\Sigma v_n = \frac{1}{n^2}$ is convergent $(p = 2 > 1)$

Σu_n is convergent.

(LCT)

Q7 $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots$

Take $v_n = \frac{1}{\sqrt{n}}$

$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\,(1+\sqrt{1+\frac{1}{n}})}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\,(1+\sqrt{1+\frac{1}{n}})}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{2}$ (a finite number)

So Σu_n and Σv_n converge or diverge together.

$\Sigma v_n = \frac{1}{\sqrt{n}}$ is divergent $(p = \frac{1}{2} < 1)$

So Σu_n is divergent.

Q8 $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$ (LCT)

Sol^n $a_n = \frac{n+1}{n^p} = \frac{\cancel{n}(1+1/n)}{\cancel{n}(n^{p-1})} = \frac{1+1/n}{n^{p-1}}$ u_n

Take $v_n = \frac{1}{n^{p-1}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(1+1/n)}{n^{p-1}} \cdot \frac{n^{p-1}}{1} = 1$ (a finite no.)

So Σu_n & Σv_n converges or diverges together.

Σv_n is convergent when $p-1 > 1 \Rightarrow p > 2$ & divergent when $p < 2$

So Σu_n is convergent for $p > 2$ & divergent for $p \leq 2$

Q9 $\sum \frac{n!\,2^n}{n^n}$ (Ratio test)

$a_n = \frac{n!\,2^n}{n^n}$ $a_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n \cdot n!} = \lim_{n\to\infty} \frac{\cancel{(n+1)} \cdot 2\, n^n}{(n+1)^n \cdot \cancel{(n+1)}}$

$= \lim_{n\to\infty} \frac{2\,\cancel{n^n}}{\cancel{n^n}(1+\frac{1}{n})^n} = \lim_{n\to\infty} \frac{2}{(1+\frac{1}{n})^n} = \frac{2}{e} < 1$

$\sum a_n$ converges

Q10 $\frac{n^2}{2^n}$ (Ratio test)

$a_n = \frac{n^2}{2^n}$ $a_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{\cancel{2^n}\cdot 2} \cdot \frac{\cancel{2^n}}{n^2} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} \overset{<1}{\text{conv.}}$

Q11 $\sum \frac{n^2+a}{2^n+a}$ (Ratio test)

$a_n = \frac{n^2+a}{2^n+a}$ $a_{n+1} = \frac{(n+1)^2+a}{2^{n+1}+a}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2+a}{2^{n+1}+a} \cdot \frac{2^n+a}{n^2+a}$

$= \lim_{n\to\infty} \frac{\cancel{n^2}\,\cancel{2^n}\left(\left(1+\frac{1}{n}\right)^2+\frac{a}{n^2}\right)\left(1+\frac{a}{2^n}\right)}{\cancel{n^2}\,\cancel{2^n}\left(2+\frac{a}{2^n}\right)\left(1+\frac{a}{n^2}\right)}$

$= \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} < 1$ convergent.

Q12 $\sum \frac{n!}{n^n}$ (Ratio test)

$a_n = \frac{n!}{n^n}$ $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{\cancel{(n+1)}\, \cancel{n!}\; n^n}{\cancel{(n+1)}\,(n+1)^n\, \cancel{n!}} = \lim_{n\to\infty} \frac{\cancel{n^n}}{\cancel{n^n}(1+\frac{1}{n})^n} = \frac{1}{e} < 1$$

conv.

Q13 $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \dots$ $(p>0)$ (Ratio)

$a_n = \frac{n^p}{n!}$ $\quad a_{n+1} = \frac{(n+1)^p}{(n+1)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \cdot \frac{n!}{n^p} = \lim_{n\to\infty} \frac{\cancel{n^p}(1+\frac{1}{n})^p\, \cancel{n!}}{(n+1)\,\cancel{n!}\,\cdot \cancel{n^p}}$$

$$\lim_{n\to\infty} \frac{1}{(n+1)}\left(1+\frac{1}{n}\right)^p = 0 \cdot 1 = 0 < 1 \text{ conv.}$$

Q14 $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2^{n-1}+1} + \dots$ (Radio)

$a_n = \frac{1}{2^{n-1}+1}$ $\quad a_{n+1} = \frac{1}{2^n+1}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1/2^n+1}{1/2^{n-1}+1} = \lim_{n\to\infty} \frac{1}{2^n+1} \cdot \frac{2^{n-1}+1}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}(2^{-1}+1/2^n)}{\cancel{2^n}(1+\frac{1}{2^n})} = 2^{-1} = \frac{1}{2} < 1 \text{ conv.}$$

Q15 $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n$ (Ratio + LCT)

$a_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n$ $\quad a_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} \cdot x^{n+1}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} \cdot x^{n+1} \cdot \frac{\sqrt{n^2+1}}{\sqrt{n}\cdot x^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}\left(\sqrt{1+\frac{1}{n}}\right) \cdot \cancel{x^n} \cdot x \cdot \cancel{n}\sqrt{1+1/n^2}}{\cancel{n}\sqrt{(1+\frac{1}{n})^2+\frac{1}{n^2}} \cdot \cancel{\sqrt{n}} \cdot \cancel{x^n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\cancel{\frac{1}{n}}^{\,0}} \cdot x \cdot \sqrt{1+\cancel{\frac{1}{n^2}}^{\,0}}}{\sqrt{(1+\cancel{\frac{1}{n}}^{\,0})^2+\cancel{\frac{1}{n^2}}^{\,0}}} = x$$

if $x<1$ $\sum a_n$ is cgt
if $x>1$ dvt

At $x=1$ $a_n = \frac{\sqrt{n}}{\sqrt{n^2+1}}$ u_n

(LCT)

Take $V_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{\sqrt{n}\cdot\sqrt{n}}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n^2}}} = 1$$

Σu_n & ΣV_n converges & diverges together.

ΣV_n is dvt as $p=\frac{1}{2}<1$ (By p-series test)

$\therefore$ Σu_n is cgt when $x<1$ & dvt when $x\geq 1$

Q16 $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$ (Ratio + LCT)

$a_n = \frac{x^{2n}}{(n+1)\sqrt{n}}$ $a_{n+1} = \frac{x^{2(n+1)}}{(n+2)\sqrt{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{2(n+1)}}{(n+2)\sqrt{n+1}} \cdot \frac{(n+1)\sqrt{n}}{x^{2n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{x^{2n}}\cdot x^2\cdot(n+1)\cancel{\sqrt{n}}}{\cancel{x^{2n}}\,(n+2)\cancel{\sqrt{n}}\sqrt{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{\cancel{n}x^2(1+1/n)}{\cancel{n}(1+\frac{2}{n})\sqrt{1+\frac{1}{n}}}$$

$$= x^2$$

Σu_n is convergent when $x^2<1$ & dvgt when $x^2>1$

At $x^2=1$ $a_n = \frac{1}{(n+1)\sqrt{n}}$ taken $V_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)\sqrt{n}}}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\cancel{n^{3/2}}}{\cancel{n^{3/2}}(1+\frac{1}{n})} = 1 \text{ finite no.}$$

$\therefore$ ΣV_n convergent by p-series test

Σu_n is cgt.

Q17 Test the cgce of the term series $1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \cdots$

(neglect 1st term)

(Ratio + nth term)

$$a_n = \frac{(\alpha+1)(2\alpha+1)\cdots(n\alpha+1)}{(\beta+1)(2\beta+1)\cdots(n\beta+1)}$$

$$a_{n+1} = \frac{(\alpha+1)(2\alpha+1)\cdots((n+1)\alpha+1)}{(\beta+1)(2\beta+1)\cdots((n+1)\beta+1)}$$

Q20

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)\alpha+1)}{((n+1)\beta+1)}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\left(\alpha + \frac{1}{n+1}\right)}{\cancel{(n+1)}\left(\beta + \frac{1}{n+1}\right)} = \frac{\alpha}{\beta}$$

$\sum a_n$ is cgt when $\frac{\alpha}{\beta} < 1$ (i.e $\alpha < \beta$) & divergent when $\frac{\alpha}{\beta} > 1$ (i.e $\alpha > \beta$).

When $\alpha = \beta$

$a_n = 1 + 1 + 1 + \dots$

$S_1 = a_1 = 1$, $S_2 = 2$, $S_3 = 3 \dots$ $S_n = n$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$ (Dvt)

$\sum a_n$ is convergent when $\alpha < \beta$ & dvt when $\alpha \geq \beta$

Q18 Test the convergence of series $\sum\left(\frac{n}{n+1}\right)^{n^2}$ or $\sum\left(1+\frac{1}{n}\right)^{-n^2}$

Sol^n $\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{1/n}$ (Root test)

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{-n}\right] = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\sum a_n$ is convergent.

Q19 $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots$ (Root test)

Sol^n $a_n = \left(\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right)^{-n}$

$$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right)^{-1}$$

$$= \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right)^{-1}$$

$$= \lim_{n\to\infty}\left(1+\cancel{\frac{1}{n}}^{0}\right)^{-1}\left(\left(1+\frac{1}{n}\right)^n - 1\right)^{-1} \quad \left[\because \left(1+\frac{x}{n}\right)^n = e^x\right]$$

$$= (e-1)^{-1} = \frac{1}{e-1} < 1 \quad \text{cgt}$$

Q20 $\frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \dots$ (Rabbe's Test)

$$a_n = \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n} \qquad a_{n+1} = \frac{1.3.5 \dots (2n-1).(2(n+1)-1)}{2.4.6 \dots 2n.2(n+1)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{1.3.5 \dots (2n-1)}(2n+1)}{\cancel{2.4.6 \dots 2n}.(2n+2)} \cdot \frac{\cancel{2.4.6 \dots 2n}}{\cancel{1.3.5 \dots 2n-1}}$$

$$= \lim_{n\to\infty} \frac{2n+1}{2n+2} = \lim_{n\to\infty} \frac{\cancel{n}(2+1/n)}{\cancel{n}(2+2/n)} = 1$$ Ratio test fails.

Apply Rabbe's Test

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{\cancel{1.3.5 \dots (2n-1)}}{\cancel{2.4.6 \dots 2n}} \cdot \frac{\cancel{2.4.6 \dots 2n}(2n+2)}{\cancel{1.3.5 \dots (2n-1)}(2n+1)} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1} - 1\right) = \lim_{n\to\infty} n\left(\frac{\cancel{2n}+2-\cancel{2n}-1}{2n+1}\right) = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$$

Σa_n is divergent

Q21 $\frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1\times3}{2\times4}\cdot\frac{x^5}{5} + \frac{1\times3\times5}{2\times4\times6}\cdot\frac{x^7}{7} + \dots$ (Rabbe's Test)

Sol^n neglect first term.

$$a_n = \frac{1\cdot3\cdot5 \dots (2n-1)}{2\cdot4\cdot6 \dots (2n)} \frac{x^{2n+1}}{2n+1}$$

$$a_{n+1} = \frac{1.3.5 \dots (2n-1)(2n+1)}{2.4.6 \dots 2n\ (2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\cancel{1.3.5 \dots (2n-1)}(2n+1)}{\cancel{2.4.6 \dots 2n}(2n+2)} \cdot \frac{\cancel{x^{2n+1}}.x^2}{(2n+3)} \cdot \frac{\cancel{2.4.6 \dots (2n)}(2n+1)}{\cancel{1.3.5 \dots (2n-1)}\cancel{x^{2n+1}}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)^2 x^2}{(2n+2)(2n+3)} = \lim_{n\to\infty} \frac{\cancel{n^2}(2+1/n)^2 x^2}{\cancel{n^2}(2+2/n)(2+3/n)}$$

$$= \frac{\cancel{4}x^2}{\cancel{4}} = x^2$$

Σa_n is convergent when $x^2 < 1$ & divergent when $x^2 > 1$

at $x^2 = 1$ ratio test fails

for this case we will apply Rabbe's Test

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{(2n+2)(2n+3)}{(2n+1)^2 \cdot \underset{=1}{x^2}}-1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{\cancel{4n^2}+10n+6-\cancel{4n^2}-1-4n}{(2n+1)^2}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{6n+5}{(2n+1)^2}\right) = \lim_{n\to\infty} \frac{\cancel{n^2}(6+5/n)}{\cancel{n^2}(2+1/n)^2} = \frac{6}{4} = \frac{3}{2} > 1$$

Σa_n is cgt! by Rabbe's test. if $x^2=1$

Σa_n is cgt when $x^2<1$ & dvt when $x^2>1$

Q22 $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\cdots$ (Rabbe's test)

$a_n = \frac{x^n}{(2n-1)(2n)}$ $\quad a_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(2n+1)(2n+2)} \cdot \frac{(2n-1)(2n)}{x^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{x^n}\cdot x}{(2n+1)(2n+2)} \cdot \frac{(2n-1)(2n)}{\cancel{x^n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\, x\,(2-1/n)\cdot 2}{\cancel{n^2}(2+1/n)(2+2/n)} = \lim_{n\to\infty} \frac{\cancel{4}\cdot x}{\cancel{4}} = x$$

Σa_n is cgt when $x<1$ & divergent when $x>1$

At $x=1$ Ratio test fails so apply Rabbe's test.

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \cancel{\lim_{n\to\infty} n\left(\frac{x\cdot(2n-1)(2n)}{(2n+1)(2n+2)}\right)}$$

$$= \lim_{n\to\infty} n\left(\frac{(2n+1)(2n+2)}{\underset{=1}{x}\,(2n-1)(2n)}-1\right) = \lim_{n\to\infty} n\left(\frac{\cancel{4n^2}+6n+2-\cancel{4n^2}+2n}{(2n-1)(2n)}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{8n+2}{n^2(2-1/n)(2)}\right) = \lim_{n\to\infty} \frac{\cancel{n^2}(8+2/n)}{\cancel{n^2}(2-1/n)\cdot 2} = 2 > 1$$

Σa_n is cgt when $x=1$

Σa_n is cgt when $x<1$ & dvt $x>1$.