9

Infinite Series

9.1. SEQUENCE

A sequence is a function whose domain is the set N of all natural numbers whereas the range may be any set S. In other words, a sequence in a set S is a rule which assigns to each natural number a unique element of S.

9.2. REAL SEQUENCE

A real sequence is a function whose domain is the set N of all natural numbers and range a subset of the set R of real numbers.

Symbolically $f : \mathrm{N} \to \mathrm{R}$ (or $x : \mathrm{N} \to \mathrm{R}$ or $a : \mathrm{N} \to \mathrm{R}$) is a real sequence.

Note. If $x : \mathrm{N} \to \mathrm{R}$ be a sequence, the image of $n \in \mathrm{N}$ instead of denoting it by $x(n)$, we shall generally denote it by x_n. Thus x_1, x_2, x_3 etc. are the real numbers associated to 1, 2, 3, etc. by this mapping. Also, the sequence $x : \mathrm{N} \to \mathrm{R}$ is denoted by $\{x_n\}$ or $< x_n >$.

$x_1, x_2, ...$ are called the first, second ...... terms of the sequence. The m^{th} and n^{th} terms x_m and x_n for $m \neq n$ are treated as distinct event if $x_m = x_n$ *i.e.*, the terms occurring at different positions are treated as distinct terms even if they have the same value.

9.3. RANGE OF A SEQUENCE

The set of all **distinct** terms of a sequence is called its range.

Note. In a sequence $\{x_n\}$, since $n \in \mathrm{N}$ and N is an infinite set, **the number of terms of a sequence is always infinite.** The range of a sequence may be a finite set. *e.g.*, if $x_n = (-1)^n$ then $\{x_n\} = \{-1, 1, -1, 1, ...\}$

The range of sequence $\{x_n\} = \{-1, 1\}$ which is a finite set.

9.4. CONSTANT SEQUENCE

A sequence $\{x_n\}$ defined by $x_n = c \in \mathrm{R} \quad \forall \; n \in \mathrm{N}$ is called a constant sequence.

e.g., $\{x_n\} = \{c, c, c, ...\}$ is a constant sequence with range $= \{c\}$.

9.5. BOUNDED AND UNBOUNDED SEQUENCES

Bounded above sequence. A sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real number K such that $a_n \leq \mathrm{K} \quad \forall \; n \in \mathrm{N}$.

Bounded below sequence. A sequence $\{a_n\}$ is said to be bounded below if $\exists$ a real number K such that $a_n \geq k \quad \forall \; n \in \mathrm{N}$.

Bounded sequence. A sequence $\{a_n\}$ is said to be bounded when it is bounded both above and below.

$\Rightarrow$ A sequence $\{a_n\}$ is bounded if $\exists$ two real numbers k and K ($k \le$ K) such that

$$k \le a_n \le K \quad \forall \; n \in N.$$

Choosing M = maximum $\{ |k|, |K| \}$, we can also define a sequence $\{a_n\}$ to be bounded if $|a_n| \le M \quad \forall \quad n \in N$.

Unbounded sequence. If $\exists$ no real number M such that $|a_n| \le M \; \forall \; n \in N$, then the sequence $\{a_n\}$ is said to be unbounded.

Examples (1). The sequence $\{a_n\}$ defined by $a_n = \dfrac{1}{n}$.

Here $\qquad \{a_n\} = \left\{\dfrac{1}{1}, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\ldots\right\}$

$\because \qquad 0 < a_n \le 1 \quad \forall \quad n \in N$

$\therefore$ $\{a_n\}$ is bounded.

(2) The sequence $\{a_n\}$ defined by $a_n = 2^{n-1}$

Here $\qquad \{a_n\} = \{1, 2, 2^2, 2^3, \ldots\ldots\}$.

Although $a_n \ge 1, \quad \forall \quad n \in N$, $\exists$ no real number K such that $a_n \le$ K.

$\therefore$ The sequence is unbounded above.

9.6. CONVERGENT, DIVERGENT AND OSCILLATING SEQUENCES

Convergent sequence. A sequence $\{a_n\}$ is said to be convergent if $\underset{n \to \infty}{\text{Lt}} a_n$ is finite.

For example, consider the sequence $\dfrac{1}{2}, \dfrac{1}{2^2}, \dfrac{1}{2^3}, \ldots\ldots, \dfrac{1}{2^n} \ldots\ldots$

Here $\quad a_n = \dfrac{1}{2^n}, \underset{n \to \infty}{\text{Lt}} a_n = \underset{n \to \infty}{\text{Lt}} \dfrac{1}{2^n} = 0$, which is finite.

$\Rightarrow$ The sequence $\{a_n\}$ is convergent.

Divergent sequence. A sequence $\{a_n\}$ is said to be divergent if $\underset{n \to \infty}{\text{Lt}} a_n$ is not finite, *i.e.*, if

$$\underset{n \to \infty}{\text{Lt}} a_n = +\infty \text{ or } -\infty.$$

For example

(*i*) Consider the sequence $\{n^2\}$

Here $\quad a_n = n^2, \underset{n \to \infty}{\text{Lt}} a_n = +\infty \quad \Rightarrow \quad$ The sequence $\{n^2\}$ is divergent.

(*ii*) Consider the sequence $\{-2^n\}$.

Here $\qquad a_n = -2^n, \underset{n \to \infty}{\text{Lt}} a_n = \underset{n \to \infty}{\text{Lt}} (-2^n) = -\infty$

$\Rightarrow$ The sequence $\{-2^n\}$ is divergent.

Oscillatory sequence. If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence. Oscillatory sequences are of two types :

(*i*) A bounded sequence which does not converge is said to **oscillate finitely.**

For example, consider the sequence $\{(-1)^n\}$.

Here $\qquad a_n = (-1)^n$

It is a bounded sequence. $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n} = 1$

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n+1} = -1.$$

Thus $\underset{n\to\infty}{\text{Lt}}\ a_n$ does not exist $\Rightarrow$ the sequence does not converge.

Hence this sequence oscillates finitely.

(*ii*) An unbounded sequence which does not diverge is said to **oscillate infinitely.**

For example, consider the sequence $\{(-1)^n\, n\}$.

Here $$a_n = (-1)^n\, n.$$

It is an unbounded sequence.

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n} \,.\, 2n = \underset{n\to\infty}{\text{Lt}}\ 2n = +\infty$$

$$\underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = \underset{n\to\infty}{\text{Lt}}\ (-1)^{2n+1}\,(2n+1) = \underset{n\to\infty}{\text{Lt}}\ -(2n+1) = -\infty.$$

Thus the sequence does not diverge.

Hence this sequence oscillates infinitely.

Note. When we say $\underset{n\to\infty}{\text{Lt}}\ a_n = l$, it means $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = l$

Similarly, $\underset{n\to\infty}{\text{Lt}}\ a_n = +\infty$ means $\underset{n\to\infty}{\text{Lt}}\ a_{2n} = \underset{n\to\infty}{\text{Lt}}\ a_{2n+1} = +\infty.$

9.7. MONOTONIC SEQUENCES

(*i*) A sequence $\{a_n\}$ is said to be **monotonically increasing** if $a_{n+1} \ge a_n \ \ \forall \ \ n \in \mathrm{N}$.

i.e., if $$a_1 \le a_2 \le a_3 \le \ldots\ldots \le a_n \le a_{n+1} \le \ldots\ldots$$

(*ii*) A sequence $\{a_n\}$ is said to be **monotonically decreasing** if $a_{n+1} \le a_n \ \ \forall \ \ n \in \mathrm{N}$.

i.e., if $$a_1 \ge a_2 \ge a_3 \ge \ldots\ldots \ge a_n \ge a_{n+1} \ge \ldots\ldots$$

(*iii*) A sequence $\{a_n\}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

(*iv*) A sequence $\{a_n\}$ is said to be **strictly monotonically increasing** if

$$a_{n+1} > a_n \ \ \forall \ \ n \in \mathrm{N}.$$

(*v*) A sequence $\{a_n\}$ is said to be **strictly monotonically decreasing** if

$$a_{n+1} < a_n \ \ \forall \ \ n \in \mathrm{N}.$$

(*vi*) A sequence $\{a_n\}$ is said to be **strictly monotonic** if it is either strictly monotonically increasing or strictly monotonically decreasing.

9.8. LIMIT OF A SEQUENCE

A sequence $\{a_n\}$ is said to approach the limit l (say) when $n \to \infty$, if for each $\varepsilon > 0$, $\exists$ a + ve integer m (depending upon ε) such that $|\ a_n - l\ | < \varepsilon \ \ \forall \ \ n \ge m$.

In symbols, we write $\underset{n\to\infty}{\text{Lt}}\ a_n = l$.

Note. $|\ a_n - l\ | < \varepsilon \ \ \forall\ n \ge m \ \Rightarrow \ l - \varepsilon < a_n < l + \varepsilon$ for $n = m, m+1, m+2, \ldots\ldots$

9.9. EVERY CONVERGENT SEQUENCE IS BOUNDED

Let the sequence $\{a_n\}$ be convergent. Let it tend to the limit l.

Then given $\varepsilon > 0$, $\exists$ a + ve integer m, such that

$$| a_n - l | < \varepsilon \quad \forall \quad n \ge m$$

$\Rightarrow \qquad l - \varepsilon < a_n < l + \varepsilon \quad \forall \quad n \ge m.$

Let k and K be the least and the greatest of $a_1, a_2, a_3, ..., a_{m-1}, l - \varepsilon, l + \varepsilon$

Then $\qquad k \le a_n \le K \quad \forall \quad n \in N,$

$\Rightarrow$ the sequence $\{a_n\}$ is bounded.

The converse is not always true *i.e.*, a sequence may be bounded, yet it may not be convergent. *e.g.*, consider $a_n = (-1)^n$, then the sequence $\{a_n\}$ is bounded but not convergent since it does not have a unique limiting point.

9.10. CONVERGENCE OF MONOTONIC SEQUENCES

Theorem I. *The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

A monotonic increasing sequence which is bounded above converges.

A monotonic decreasing sequence which is bounded below converges.

Theorem II. *If a monotonic increasing sequence is not bounded above, it diverges to* $+\infty$.

Theorem III. *If a monotonic decreasing sequence is not bounded below, it diverges to* $-\infty$.

Theorem IV. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences, then sequence* $\{a_n + b_n\}$ *is also convergent.*

Or

If $\quad Lt\ a_n = A \quad$ *and* $\quad Lt\ b_n = B, \quad$ *then* $\quad Lt\ (a_n + b_n) = A + B.$

Theorem V. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences such that* $Lt\ a_n = A$ *and* $Lt\ b_n = B$, *then*

(*i*) sequence $\{a_n b_n\}$ is also convergent and converges to AB.

(*ii*) sequence $\left\{\dfrac{a_n}{b_n}\right\}$ is also convergent and converges to $\dfrac{A}{B}$, $(B \ne 0)$.

Theorem VI. *The sequence* $\{| a_n |\}$ *converges to zero if and only if the sequence* $\{a_n\}$ *converges to zero.*

Theorem VII. *If a sequence* $\{a_n\}$ *converges to* a *and* $a_n \ge 0 \quad \forall\ n$, *then* $a \ge 0$.

Theorem VIII. *If* $a_n \to a$, $b_n \to b$ *and* $a_n \le b_n \quad \forall\ n$, *then* $a \le b$.

Theorem IX. *If* $a_n \to l$, $b_n \to l$, *and* $a_n \le c_n \le b_n$, $\forall\ n$, *then* $c_n \to l$. (*Squeeze Principle*)

ILLUSTRATIVE EXAMPLES

Example 1. *Give an example of a monotonic increasing sequence which is (i) convergent, (ii) divergent.*

Sol. (*i*) Consider the sequence $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, ..., \dfrac{n}{n+1}, ...$

Since $\dfrac{1}{2} < \dfrac{2}{3} < \dfrac{3}{4} < ......$ the sequence is monotonic increasing.

$$a_n = \frac{n}{n+1}, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{n+1} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{1+\dfrac{1}{n}} = 1$$

which is finite.

$\therefore$ The sequence is convergent.

(*ii*) Consider the sequence 1, 2, 3, ..., n, ...

Since 1 < 2 < 3 < ......, the sequence is monotonic increasing,

$$a_n = n, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ n = \infty$$

$\therefore$ The sequence diverges to + ∞.

Example 2. *Give an example of a monotonic decreasing sequence which is*

(*i*) *convergent,* (*ii*) *divergent.*

Sol. (*i*) Consider the sequence 1, $\frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}$, ...

Since $1 > \frac{1}{2} > \frac{1}{3} >$ ......, the sequence is monotonic decreasing.

$$a_n = \frac{1}{n}, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n} = 0.$$

$\therefore$ The sequence converges to 0.

(*ii*) Consider the sequence – 1, – 2, – 3, ..., – n, ...

Since – 1 > – 2 > – 3 > ......, the sequence is monotonic decreasing.

$$a_n = -n, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ (-n) = -\infty$$

$\therefore$ The sequence diverges to – ∞.

Example 3. *Discuss the convergence of the sequence* $\{a_n\}$ *where*

(*i*) $a_n = \frac{n+1}{n}$ (*ii*) $a_n = \frac{n}{n^2+1}$ (*iii*) $a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n}$.

Sol. (*i*)

$$a_n = \frac{n+1}{n}$$

$$a_{n+1} - a_n = \frac{n+2}{n+1} - \frac{n+1}{n} = \frac{-1}{n(n+1)} < 0\ \forall\ n$$

$\Rightarrow$ $a_{n+1} < a_n \quad \forall \quad n$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also $a_n = \frac{n+1}{n} = 1 + \frac{1}{n} > 1 \quad \forall \quad n$

$\Rightarrow$ $\{a_n\}$ is bounded below by 1,

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\left(1 + \frac{1}{n}\right) = 1.$$

(*ii*)

$$a_n = \frac{n}{n^2+1}$$

$$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$$

$$= \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0 \quad \forall\ n \quad \Rightarrow \quad a_{n+1} < a_n \quad \forall \quad n$$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also $a_n = \frac{n}{n^2+1} > 0 \quad \forall\ n \quad \Rightarrow \quad \{a_n\}$ is bounded below by 0.

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\, a_n = \underset{n\to\infty}{\text{Lt}}\, \frac{n}{n^2+1} = \underset{n\to\infty}{\text{Lt}}\, \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0.$$

(*iii*)
$$a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n} = \frac{1\left(1-\frac{1}{3^{n+1}}\right)}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right)$$

Now
$$a_{n+1} = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n} + \frac{1}{3^{n+1}}$$

$\therefore$
$$a_{n+1} - a_n = \frac{1}{3^{n+1}} > 0 \quad \forall\, n \quad \Rightarrow \quad a_{n+1} > a_n \quad \forall\, n$$

$\Rightarrow$ $\{a_n\}$ is an increasing sequence.

Also
$$a_n = \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) < \frac{3}{2} \quad \forall\, n \quad \Rightarrow \quad \{a_n\} \text{ is bounded above by } \frac{3}{2}.$$

$\because$ $\{a_n\}$ is increasing and bounded above, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\, a_n = \underset{n\to\infty}{\text{Lt}}\, \frac{3}{2}\left(1-\frac{1}{3^{n+1}}\right) = \frac{3}{2}.$$

9.11. INFINITE SERIES

If $\{u_n\}$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + \ldots\ldots + u_n \ldots\ldots$ [*i.e.*, the sum of the terms of the sequence, which are infinite in number] is called an infinite series.

The infinite series $u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ is usually denoted by $\sum_{n=1}^{\infty} u_n$ or more briefly, by Σu_n.

9.12. SERIES OF POSITIVE TERMS

If all the terms of the series $\Sigma u_n = u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ are positive *i.e.*, if $u_n > 0, \forall\, n$, then the series Σu_n is called a series of positive terms.

9.13. ALTERNATING SERIES

A series in which the terms are alternate positive and negative is called an alternating series. Thus, the series $\Sigma\,(-1)^{n-1}\, u_n = u_1 - u_2 + u_3 - u_4 + \ldots\ldots + (-1)^{n-1}\, u_n + \ldots\ldots$, where $u_n > 0 \;\; \forall\, n$ is an alternating series.

9.14. PARTIAL SUMS

If $\Sigma u_n = u_1 + u_2 + u_3 + \ldots\ldots + u_n + \ldots\ldots$ is an infinite series, where the terms may be + ve or – ve, then

$S_n = u_1 + u_2 + \ldots\ldots + u_n$ is called the n^{th} partial sum of Σu_n. Thus, the n^{th} partial sum of an infinite series is the sum of its first n terms.

S_1, S_2, S_3, ... are the first, second, third, ... partial sums of the series.

Since $n \in N$, $\{S_n\}$ is a sequence called the **sequence of partial sums** of the infinite series Σu_n.

$\therefore$ To every infinite series Σu_n, there corresponds a sequence $\{S_n\}$ of its partial sums.

9.15. BEHAVIOUR OF AN INFINITE SERIES

An infinite series Σu_n converges, diverges or oscillates (finitely or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillates (finitely or infinitely).

(*i*) The series Σu_n converges (or is said to be convergent) if the sequence $\{S_n\}$ of its partial sums converges.

Thus, Σu_n is convergent if $\underset{n \to \infty}{\text{Lt}} S_n$ = finite.

(*ii*) The series Σu_n diverges (or is said to be divergent) if the sequence $\{S_n\}$ of its partial sums diverges.

Thus, Σu_n is divergent if $\underset{n \to \infty}{\text{Lt}} S_n = +\infty$ or $-\infty$.

(*iii*) The series Σu_n oscillates finitely if the sequence $\{S_n\}$ of its partial sums oscillates finitely.

Thus, Σu_n oscillates finitely if $\{S_n\}$ is bounded and neither converges nor diverges.

(*iv*) The series Σu_n oscillates infinitely if the sequence $\{S_n\}$ of its partial sums oscillates infinitely.

Thus, Σu_n oscillates infinitely if $\{S_n\}$ is unbounded and neither converges nor diverges.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence or otherwise of the series*

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots + \frac{1}{n(n+1)} + \ldots\ldots \text{ to } \infty.$$

Sol. Here $\quad u_n = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

Putting $\quad n = 1, 2, 3, \ldots\ldots, n$

$$u_1 = \frac{1}{1} - \frac{1}{2}$$

$$u_2 = \frac{1}{2} - \frac{1}{3}$$

$$u_3 = \frac{1}{3} - \frac{1}{4}$$

....

....

$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding $\quad S_n = 1 - \dfrac{1}{n+1}$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = 1 - 0 = 1$$

$\Rightarrow$ $\{S_n\}$ converges to 1 $\Rightarrow$ Σu_n converges to 1.

Example 2. *Show that the series $1^2 + 2^2 + 3^2 + ... + n^2 + ...$ diverges to $+\infty$.*

Sol.
$$S_n = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = +\infty$$

$\Rightarrow$ $\{S_n\}$ diverges to $+\infty$

$\Rightarrow$ The given series diverges to $+\infty$.

Article 1. *The geometric series $1 + x + x^2 + x^3 + ...$ to ∞*

(*i*) *converges if $-1 < x < 1$ i.e., $|x| < 1$* (*ii*) *diverges if $x \geq 1$*

(*iii*) *oscillates finitely if $x = -1$* (*iv*) *oscillates infinitely if $x < -1$.*

Proof. (*i*) **When $|x| < 1$**

Since $|x| < 1$, $x^n \to 0$ as $n \to \infty$

$$S_n = 1 + x + x^2 + ...\text{ to } n \text{ terms} = \frac{1(1-x^n)}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = \frac{1}{1-x} \quad \Rightarrow \quad \text{the sequence } \{S_n\} \text{ is convergent}$$

$\Rightarrow$ the given series is convergent.

(*ii*) **When $x \geq 1$**

Sub-case I. When $x = 1$

$$S_n = 1 + 1 + 1 + ...\text{ to } n \text{ terms} = n$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = \infty \quad \Rightarrow \quad \text{the sequence } \{S_n\} \text{ diverges to } \infty.$$

$\Rightarrow$ the given series diverges to ∞.

Sub-case II. When $x > 1$, $x^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + ...\text{ to } n \text{ terms} = \frac{1(x^n - 1)}{x-1}$$

$$\underset{n\to\infty}{\text{Lt}}\ S_n = \infty \quad \Rightarrow \quad \text{the sequence } \{S_n\} \text{ diverges to } \infty$$

$\Rightarrow$ the given series diverges to ∞.

(*iii*) **When $x = -1$**

$$S_n = 1 - 1 + 1 - 1 + ...\text{ to } n \text{ terms}$$
$$= 1 \text{ or } 0 \text{ according as } n \text{ is odd or even.}$$

$\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ S_n = 1$ or 0 $\Rightarrow$ the sequence $\{S_n\}$ oscillates finitely.

$\Rightarrow$ the given series oscillates finitely.

(*iv*) **When $x < -1$**

$x < -1 \Rightarrow -x > 1$

Let $r = -x$, then $r > 1$

$\therefore$ $r^n \to \infty$ as $n \to \infty$

$$S_n = 1 + x + x^2 + x^3 + \text{...... to } n \text{ terms} = \frac{1-x^n}{1-x} = \frac{1-(-r)^n}{1+r} \qquad [\because \quad x = -r]$$

$$= \frac{1-r^n}{1+r} \quad \text{or} \quad \frac{1+r^n}{1+r} \text{ according as } n \text{ is even or odd}$$

$$\underset{n\to\infty}{\text{Lt}} S_n = \frac{1-\infty}{1+r} \quad \text{or} \quad \frac{1+\infty}{1+r} = -\infty \text{ or } +\infty$$

$\Rightarrow$ the sequence $\{S_n\}$ oscillates infinitely

$\Rightarrow$ the given series oscillates infinitely.

Article 2. *If a series Σu_n is convergent, then* $\underset{n\to\infty}{Lt}\, u_n = 0.$

Proof. Let S_n denote the n^{th} partial sum of the series Σu_n.

Then Σu_n is convergent $\Rightarrow$ $\{S_n\}$ is convergent

$\Rightarrow$ $\underset{n\to\infty}{\text{Lt}} S_n$ is finite and unique $= s$ (say). $\Rightarrow$ $\underset{n\to\infty}{\text{Lt}} S_{n-1} = s$

Now $\qquad S_n - S_{n-1} = u_n$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} (S_n - S_{n-1}) = \underset{n\to\infty}{\text{Lt}} S_n - \underset{n\to\infty}{\text{Lt}} S_{n-1} = s - s = 0.$$

Hence Σu_n is convergent $\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\, u_n = 0$.

The converse of the above theorem is not always true, *i.e.*, the n^{th} term may tend to zero as $n \to \infty$ even if the series is not convergent.

For example, the series $1 + \frac{1}{2} + \frac{1}{3} + \text{......} + \frac{1}{n} + \text{......}$ diverges, though

$$\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \frac{1}{n} = 0.$$

Note 1. Σu_n is convergent $\Rightarrow$ $\underset{n\to\infty}{\text{Lt}} u_n = 0.$

Note 2. $\underset{n\to\infty}{\text{Lt}} u_n = 0$ $\Rightarrow$ Σu_n may or may not be convergent.

Note 3. $\underset{n\to\infty}{\text{Lt}} u_n \neq 0$ $\Rightarrow$ Σu_n is not convergent.

Article 3. *A positive term series either converges or diverges to $+\infty$.*

Proof. Let Σu_n be a positive term series and S_n be its n^{th} partial sum.

Then $\qquad S_{n+1} = u_1 + u_2 + \text{......} + u_n + u_{n+1} = S_n + u_{n+1}$

$\Rightarrow \qquad S_{n+1} - S_n = u_{n+1} > 0 \quad \forall \quad n \qquad [\because \quad u_n > 0 \;\; \forall \;\; n]$

$\Rightarrow \qquad S_{n+1} > S_n \quad \forall \quad n$

$\Rightarrow$ $\{S_n\}$ is a monotonic increasing sequence.

Two cases arise. The sequence $\{S_n\}$ may be bounded or unbounded above.

Case I. *When $\{S_n\}$ is bounded above.*

Since $\{S_n\}$ is monotonic increasing and bounded above, it is convergent $\Rightarrow$ Σu_n is convergent.

Case II. *When $\{S_n\}$ is not bounded above.*

Since $\{S_n\}$ is monotonic increasing and not bounded above, it diverges to $+\infty$ $\Rightarrow$ Σu_n diverges to $+\infty$.

Hence a positive term series either converges or diverges to $+\infty$.

Cor. *If* $u_n > 0 \quad \forall \quad n$ *and* $\underset{n\to\infty}{Lt}\, u_n \neq 0$, *then the series* Σu_n *diverges to* $+\infty$.

Proof. $u_n > 0 \quad \forall \quad n \quad \Rightarrow \quad \Sigma u_n$ is a series of +ve terms.

$\Rightarrow \quad \Sigma u_n$ either converges or diverges to $+\infty$

Since $$\underset{n\to\infty}{\text{Lt}}\, u_n \neq 0 \quad \text{(given)}$$

$\therefore \quad \Sigma u_n$ does not converge.

Hence Σu_n diverges to $+\infty$.

Article 4. *(a) The necessary and sufficient condition for the convergence of a positive term series* Σu_n *is that the sequence* $\{S_n\}$ *of its partial sums is bounded above.*

Proof. (*i*) Suppose the sequence $\{S_n\}$ is bounded above. Since the series Σu_n is of positive terms, the sequence $\{S_n\}$ is monotonically increasing. Since every monotonically increasing sequence which is bounded above, converges, therefore $\{S_n\}$ and hence Σu_n converges.

(*ii*) Conversely, suppose Σu_n converges. Then the sequence $\{S_n\}$ of its partial sums also converges. Since every convergent sequence is bounded, $\{S_n\}$ is bounded. In particular, $\{S_n\}$ is bounded above.

(*b*) *Cauchy's General Principle of Convergence of Series*

The necessary and sufficient condition for the infinite series $\sum_{n=1}^{\infty} u_n$ to converge is that given $\varepsilon > 0$, however small, there exists a positive integer m such that $|\, u_{m+1} + u_{m+2} + \ldots\ldots + u_n \,| < \varepsilon \quad \forall \quad n > m$.

Example. *Prove with the help of Cauchy's general principle of convergence that the series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} + \ldots\ldots \textit{ does not converge.}$$

Sol. If possible, suppose the given series is convergent.

Take $$\varepsilon = \frac{1}{2}$$

By Cauchy's general principle of convergence, there exists a positive integer m such that

$$\left| \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{n} \right| < \frac{1}{2} \quad \forall \quad n > m$$

or $$\frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{n} < \frac{1}{2} \quad \forall \quad n > m \qquad \ldots(1)$$

By taking $n = 2m$, we observe that $\dfrac{1}{m+1} + \dfrac{1}{m+2} + \ldots\ldots + \dfrac{1}{n}$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{2m} > \frac{1}{2m} + \frac{1}{2m} + \ldots\ldots + \frac{1}{2m} = \left(\frac{m}{2m} = \frac{1}{2}\right)$$

i.e., $$\frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{n} > \frac{1}{2}$$

where $n = 2m > m$. This contradicts (1).

$\Rightarrow$ Our supposition is wrong.

$\Rightarrow$ The given series does not converge.

Article 5. *If m is a given positive integer, then the two series* $u_1 + u_2 + \ldots\ldots + u_{m+1} + u_{m+2} + \ldots\ldots$ *and* $u_{m+1} + u_{m+2} + \ldots\ldots$ *converge or diverge together.*

Proof. Let S_n and s_n denote the nth partial sums of the two series.

Then
$$S_n = u_1 + u_2 + \ldots\ldots + u_n$$
$$s_n = u_{m+1} + u_{m+2} + \ldots\ldots + u_{m+n}$$
$$= (u_1 + u_2 + \ldots\ldots + u_{m+n}) - (u_1 + u_2 + \ldots\ldots + u_m)$$
$$= S_{m+n} - S_m \Rightarrow s_n = S_{m+n} - S_m \quad \ldots(1)$$

S_m being the sum of a finite number of terms of Σu_n is a fixed finite quantity.

(*i*) If $S_{m+n} \to$ a finite limit as $n \to \infty$, then from (1), so does s_n.

(*ii*) If $S_{m+n} \to +\infty$ as $n \to \infty$, so does s_n.

(*iii*) If $S_{m+n} \to -\infty$ as $n \to \infty$, so does s_n.

(*iv*) If S_{m+n} does not tend to any limit (finite or infinite), so does s_n.

$\Rightarrow$ The sequences $\{S_n\}$ and $\{s_n\}$ converge or diverge together.

$\Rightarrow$ The two given series converge or diverge together. Hence the result.

Note. The above theorem shows that the convergence, divergence or oscillation of a series is not affected by addition or omission of a finite number of its terms.

Article 6. *If* Σu_n *and* Σv_n *converge to u and v respectively, then* $\Sigma(u_n + v_n)$ *converges to* $(u + v)$.

Proof. Let
$$U_n = u_1 + u_2 + \ldots\ldots + u_n$$
$$V_n = v_1 + v_2 + \ldots\ldots + v_n$$
and
$$S_n = (u_1 + v_1) + (u_2 + v_2) + \ldots\ldots + (u_n + v_n)$$
Then
$$S_n = (u_1 + u_2 + \ldots\ldots + u_n) + (v_1 + v_2 + \ldots\ldots + v_n) = U_n + V_n.$$

Since Σu_n converges to u, $\underset{n\to\infty}{\text{Lt}}\ U_n = u$

Σv_n converges to v, $\underset{n\to\infty}{\text{Lt}}\ V_n = v$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ S_n = \underset{n\to\infty}{\text{Lt}}\ (U_n + V_n) = \underset{n\to\infty}{\text{Lt}}\ U_n + \underset{n\to\infty}{\text{Lt}}\ V_n = u + v.$$

$\Rightarrow$ $\Sigma(u_n + v_n)$ converges to $(u + v)$.

Article 7. Comparison Tests

Test I. *If* Σu_n *and* Σv_n *are series of positive terms and* Σv_n *is convergent and there is a positive constant k such that* $u_n \le kv_n$, $\forall\ n > m$, *then* Σu_n *is also convergent.*

Proof. Let $U_n = u_1 + u_2 + \ldots\ldots + u_n$ and $V_n = v_1 + v_2 + \ldots\ldots + v_n$

Now
$$u_n \le kv_n \ \forall\ n > m$$
$$\Rightarrow \quad u_{m+1} \le kv_{m+1}$$
$$u_{m+2} \le kv_{m+2}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Adding $u_{m+1} + u_{m+2} + \ldots\ldots + u_n \le k(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$
$$\Rightarrow \quad U_n - U_m \le k\,(V_n - V_m) \quad \forall \quad n > m$$
$$\Rightarrow \quad U_n \le kV_n + (U_m - kV_m) \quad \forall \quad n > m$$
$$\Rightarrow \quad U_n \le kV_n + k_0 \quad \forall \quad n > m \quad \ldots(1)$$

where $k_0 = U_m - kV_m$ is a fixed number. Since Σv_n is convergent, the sequence $\{V_n\}$ is convergent and hence bounded above.

$\therefore$ From (1), the sequence $\{U_n\}$ is bounded above.

$\because$ Σu_n is a series of + ve terms.

$\{U_n\}$ is monotonic increasing.

$\because$ $\{U_n\}$ is monotonic increasing sequence and is bounded above.

$\therefore$ It is convergent.

$\Rightarrow$ Σu_n is convergent.

Test II. *If Σu_n and Σv_n are two series of positive terms and Σv_n is divergent and there is a positive constant k such that $u_n > kv_n$, $\forall\ n > m$, then Σu_n is also divergent.*

Proof. Let $\quad U_n = u_1 + u_2 + \ldots\ldots + u_n$

and $\quad V_n = v_1 + v_2 + \ldots\ldots + v_n$

Now $\quad u_n > kv_n \quad \forall \quad n > m$

$$\Rightarrow \quad u_{m+1} > kv_{m+1}$$
$$u_{m+2} > kv_{m+2}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Adding $u_{m+1} + u_{m+2} + \ldots\ldots. + u_n > k\,(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$

$$\Rightarrow \quad U_n - U_m > k\,(V_n - V_m) \qquad \forall \quad n > m$$

$$\Rightarrow \quad U_n > kV_n + (U_m - kV_m) \qquad \forall \quad n > m$$

$$\Rightarrow \quad U_n > kV_n + k_0 \qquad \forall \quad n > m \qquad \ldots(1)$$

where $k_0 = U_m - kV_m$ is a fixed number.

Since Σv_n is divergent, the sequence $\{V_n\}$ is divergent.

$\Rightarrow$ For each positive real number k_1, however large, there exists a +ve integer m' such that

$$V_n > k_1 \qquad \forall \quad n > m'$$

Let $\quad m^* = \max.\ \{m, m'\}$, then $\quad V_n > k_1 \qquad \forall \quad n > m^*$

From (1), $\quad U_n > kk_1 + k_0 = K \qquad \forall \quad n > m^*$

$\Rightarrow$ $\{U_n\}$ is divergent

$\Rightarrow$ Σu_n is divergent.

Test III. *If Σu_n and Σv_n are two positive term series and there exist two positive constants H and K (independent of n) and a positive integer m such that $H < \dfrac{u_n}{v_n} < K\ \forall\ n > m$ then the two series Σu_n and Σv_n converge or diverge together.*

Proof. Since Σv_n is a series of + ve terms, $v_n > 0, \quad \forall\ n$

$$\therefore \quad H < \frac{u_n}{v_n} < K \qquad \forall\ n > m$$

$$\Rightarrow \quad Hv_n < u_n < Kv_n \qquad \forall\ n > m \qquad \ldots(1)$$

Case I. *When Σv_n is convergent*

From (1), $\quad u_n < Kv_n \qquad \forall\ n > m \quad$ and $\quad \Sigma v_n$ is convergent.

$\Rightarrow$ Σu_n is convergent. [See Test I]

Case II. *When Σv_n is divergent*

From (1), $\quad u_n > Hv_n \qquad \forall\ n > m \quad$ and $\quad \Sigma v_n$ is divergent.

$\Rightarrow$ Σu_n is divergent. [See Test II]

Case III. *When Σu_n is convergent*

From (1), $\quad Hv_n < u_n \qquad \forall\ n > m$

$$\Rightarrow \qquad v_n < \frac{1}{H} u_n \qquad \forall\ n > m \qquad (\because \quad H > 0)$$

Since Σu_n is convergent $\quad\therefore\quad$ Σv_n is convergent. [See Test I]

Case IV. *When Σu_n is divergent*

From (1), $\qquad K v_n > u_n \qquad \forall\ n > m$

$$\Rightarrow \qquad v_n > \frac{1}{K} u_n. \qquad \forall\ n > m \qquad (\because \quad K > 0)$$

Since Σu_n is divergent $\quad\therefore\quad$ Σv_n is divergent. [See Test II]

Particular Case of Test III (When m = 0)

If Σu_n and Σv_n are two positive term series and there exist two positive constants H and K (independent of n) such that $\quad H < \dfrac{u_n}{v_n} < K \qquad \forall\ n,$

then the two series Σu_n and Σv_n converge or diverge together.

Test IV. *Let Σu_n and Σv_n be two positive term series.*

(i) If $\underset{n\to\infty}{Lt} \dfrac{u_n}{v_n} = l$ (finite and non-zero), then Σu_n and Σv_n both converge or diverge together.

(ii) If $\underset{n\to\infty}{Lt} \dfrac{u_n}{v_n} = 0$ and Σv_n converges, then Σu_n also converges.

(iii) If $\underset{n\to\infty}{Lt} \dfrac{u_n}{v_n} = \infty$ and Σv_n diverges, then Σu_n also diverges.

Proof. (*i*) Since $u_n > 0,\ v_n > 0 \quad \therefore \quad \dfrac{u_n}{v_n} > 0$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} \geq 0$$

$$\text{But} \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = l \neq 0 \quad \Rightarrow \quad l > 0$$

$$\text{Now} \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = l$$

$\Rightarrow$ given $\varepsilon > 0$, $\exists$ a +ve integer m such that $\left| \dfrac{u_n}{v_n} - l \right| < \varepsilon \ \ \forall\ \ n > m$

$$\Rightarrow \qquad l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \forall\ n > m$$

$$\Rightarrow \qquad (l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n \qquad \forall\ n > m \qquad (\because \quad v_n > 0)$$

Choose $\varepsilon > 0$ such that $l - \varepsilon > 0$.

Let $\ l - \varepsilon = H,\ l + \varepsilon = K$, where H, K are > 0

$$\therefore \qquad H v_n < u_n < K v_n \qquad \forall\ n > m \qquad \ldots(1)$$

Case I. *When Σu_n is convergent*

From (1), $\qquad H v_n < u_n \qquad \forall\ n > m$

$$\Rightarrow \qquad v_n < \frac{1}{H} u_n \qquad \forall\ n > m \qquad (\because \quad H > 0)$$

Since Σu_n is convergent, Σv_n is also convergent.

Case II. *When Σu_n is divergent.*

From (1), $\quad K v_n > u_n \quad \forall\ n > m$

$$\Rightarrow \quad v_n > \frac{1}{K} u_n \quad \forall\ n > m \qquad (\because\ K > 0)$$

Since Σu_n is divergent, Σv_n is also divergent.

Case III. *When Σv_n is convergent.*

From (1), $\quad u_n < K v_n \quad \forall\ n > m$

Since Σv_n is convergent, Σu_n is also convergent.

Case IV. *When Σv_n is divergent.*

From (1), $\quad u_n > H v_n \quad \forall\ n > m$

Since Σv_n is divergent, Σu_n is also divergent.

Hence Σu_n and Σv_n converge or diverge together.

(*ii*) Here $\displaystyle \underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = 0$

$\therefore$ Given $\varepsilon > 0$, $\exists$ a +ve integer m such that $\left| \dfrac{u_n}{v_n} - 0 \right| < \varepsilon \quad \forall\ n > m$

$$\Rightarrow \quad -\varepsilon < \frac{u_n}{v_n} < \varepsilon \quad \forall\ n > m$$

$$\Rightarrow \quad u_n < \varepsilon\, v_n \quad \forall\ n > m \qquad (\because\ v_n > 0)$$

Since Σv_n is convergent, Σu_n is also convergent.

(*iii*) Here $\displaystyle \underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \infty$

$\therefore$ Given M > 0, however large, $\exists$ a +ve integer m such that $\dfrac{u_n}{v_n} > M \quad \forall\ n > m$

$$\Rightarrow \quad u_n > M v_n \quad \forall\ n > m$$

Since Σv_n is divergent, Σu_n is also divergent.

Test V. *Let Σu_n and Σv_n be two positive term series.*

(*i*) *If $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}} \quad \forall\ n > m$ and Σv_n is convergent, then Σu_n is also convergent.*

(*ii*) *If $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}} \quad \forall\ n > m$ and Σv_n is divergent, then Σu_n is also divergent.*

Proof. (*i*) $\quad \dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}} \quad \forall\ n > m$

$$\Rightarrow \quad \frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}}$$

$$\frac{u_{m+2}}{u_{m+3}} > \frac{v_{m+2}}{v_{m+3}}$$

$$\frac{u_{m+3}}{u_{m+4}} > \frac{v_{m+3}}{v_{m+4}}$$

....

....

$$\frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}$$

Multiplying the corresponding sides of the above inequalities, we have

$$\frac{u_{m+1}}{u_n} > \frac{v_{m+1}}{v_n} \qquad \forall\ n > m$$

$$\Rightarrow \qquad u_n < \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall\ n > m$$

$$\Rightarrow \qquad u_n < kv_n \qquad \forall\ n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is convergent, so is Σu_n.

(*ii*) Using $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}} \qquad \forall\ n > m$

and proceeding as in part (*i*), we have $\dfrac{u_{m+1}}{u_n} < \dfrac{v_{m+1}}{v_n} \qquad \forall\ n > m$

$$\Rightarrow \qquad u_n > \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall\ n > m$$

$$\Rightarrow \qquad u_n > kv_n \qquad \forall\ n > m,$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed +ve quantity.

Since Σv_n is divergent, so is Σu_n.

Article 8. *An Important Test for Comparison.* $\sum \dfrac{1}{n^p}$. *[Hyper harmonic series or p-series]*

The series $\sum \dfrac{1}{n^p} = \dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \ldots\ldots + \dfrac{1}{n^p} + \ldots\ldots$ *to* ∞ *converges if* $p > 1$ *and diverges if* $p \le 1$.

Proof. Case I. When p > 1

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}} \qquad \left[\because\ \frac{1}{3^p} < \frac{1}{2^p}\right]$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$$

$$= \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{(2^{p-1})^2} \qquad \left[\because\ \frac{1}{5^p} < \frac{1}{4^p}, \frac{1}{6^p} < \frac{1}{4^p} \text{ etc.}\right]$$

Similarly, the sum of next eight terms

$$= \frac{1}{8^p} + \frac{1}{9^p} + \ldots\ldots + \frac{1}{15^p} < \frac{1}{8^p} + \frac{1}{8^p} + \ldots\ldots + \frac{1}{8^p} = \frac{8}{8^p}$$

$$= \frac{1}{8^{p-1}} = \frac{1}{(2^{p-1})^3} \text{ and so on.}$$

Now $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$

$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + \ldots\ldots + \frac{1}{15^p}\right) + \ldots\ldots \quad \ldots(1)$$

$\therefore$ Each term of (1) after the first is less than the corresponding term in

$$1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \frac{1}{(2^{p-1})^3} + \ldots\ldots \quad \ldots(2)$$

But (2) is a G.P. whose common ratio $= \frac{1}{2^{p-1}} < 1$ $\quad (\because \quad p > 1)$

$\therefore$ (2) is convergent $\Rightarrow$ (1) is convergent.

Hence the given series is convergent.

Case II. When p = 1

$$\sum \frac{1}{n^p} = \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$$

$$1 + \frac{1}{2} = 1 + \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2} \quad \text{and so on.}$$

Now $\quad \sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n} + \ldots\ldots$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots\ldots \quad \ldots(1)$$

Each term of (1) after the second is greater than the corresponding term in

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots \quad \ldots(2)$$

But after the second term (2) is a G.P. whose common ratio = 1.

$\therefore$ (2) is divergent. $\Rightarrow$ (1) is divergent.

Hence the given series is divergent.

Case III. When p < 1

$$p < 1 \quad \Rightarrow \quad n^p < n \quad \Rightarrow \quad \frac{1}{n^p} > \frac{1}{n} \quad \forall\, n$$

But the series $\sum \frac{1}{n}$ is divergent (Case II).

Hence $\sum \frac{1}{n^p}$ is also divergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of the series:*

(*i*) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+ \ldots\ldots$ to ∞ (*ii*) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\ldots$.

Sol. (*i*) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+ \ldots\ldots$ to ∞

$$= \left(\frac{3}{5}+\frac{3}{5^3}+\ldots\ldots \text{ to } \infty\right)+\left(\frac{4}{5^2}+\frac{4}{5^4}+\ldots\ldots \text{ to } \infty\right) = \Sigma u_n + \Sigma v_n \text{ (say)}$$

Now Σu_n is a G.P. with common ratio $= \frac{1}{5^2}$ which is numerically less than 1,

$\therefore$ Σu_n is convergent.

Σv_n is also a G.P. with common ratio $= \frac{1}{5^2}$ which is numerically less than 1.

$\therefore$ Σv_n is convergent.

$\therefore$ The given series *viz*. $\Sigma(u_n + v_n)$ is also convergent.

(*ii*) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\ldots\ldots$ to $\infty = 1+\frac{1}{(2^2)^{2/3}}+\frac{1}{(3^2)^{2/3}}+\frac{1}{(4^2)^{2/3}}+\ldots\ldots$ to ∞

$$= \frac{1}{1^{4/3}}+\frac{1}{2^{4/3}}+\frac{1}{3^{4/3}}+\frac{1}{4^{4/3}}+\ldots\ldots \text{ to } \infty = \sum \frac{1}{n^{4/3}} = \sum \frac{1}{n^p} \text{ with } p = \frac{4}{3} > 1$$

$\therefore$ By *p*-series test, the given series is convergent.

Example 2. *Test the convergence of the series* : $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots\ldots$

Sol. Here $$u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let us compare Σu_n with Σv_n, where $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2}{(1)(1)} = 2 \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 2 > 1$.

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Example 3. *Test the convergence of the following series:*

(*i*) $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots\ldots$ (*ii*) $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \ldots\ldots + \sqrt{\frac{n}{2(n+1)}} + \ldots\ldots$

Sol. (*i*) Here $$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]}$$

Let us compare Σu_n with Σv_n, where $v_n = \frac{1}{\sqrt{n}}$

$$\frac{u_n}{v_n} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2} \quad \text{which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n^{1/2}}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = \frac{1}{2} < 1$

$\therefore$ Σv_n is divergent $\Rightarrow$ Σu_n is divergent.

(*ii*) Here $$u_n = \sqrt{\frac{n}{2(n+1)}} = \sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}}$$

$$\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \frac{1}{\sqrt{2\left(1+\frac{1}{n}\right)}} = \frac{1}{\sqrt{2}} \neq 0$$

$\Rightarrow$ Σu_n does not converge.

Since the given series is a series of +ve terms, it either converges or diverges. Since it does not converge, it must diverge.

Hence the given series is divergent.

Example 4. *Test the convergence of the series:* $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \ldots\ldots$

Sol. Leaving aside the first term ($\because$ addition or deletion of a finite number of terms does not alter the nature of the series), we have $u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}$

Take $$v_n = \frac{1}{n}.$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{e} \cdot \frac{1}{1} \qquad \left[\because \operatorname*{Lt}_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e\right]$$

$$= \frac{1}{e} \text{ which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma v_n = \Sigma \frac{1}{n}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = 1$

$\therefore$ Σv_n is divergent. $\Rightarrow$ Σu_n is divergent.

Example 5. *Examine the convergence of the series:* $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \ldots\ldots$

Sol. Here
$$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}\right)}{n^3\left[\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}\right]} = \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{n^{5/2}\left[\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}\right]}$$

Take
$$v_n = \frac{1}{n^{5/2}}$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{v_n} = \operatorname*{Lt}_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}} = \frac{\sqrt{1+0}-0}{(1+0)^3-0} = 1 \qquad \text{which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\Sigma u_n = \Sigma \frac{1}{n^{5/2}}$ is of the form $\Sigma \frac{1}{n^p}$ with $p = \frac{5}{2} > 1$.

Σv_n is convergent. $\Rightarrow$ Σu_n is convergent.

Example 6. *Examine the convergence of the series:*

(*i*) $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$ (*ii*) $\sum \left(\sqrt[3]{n^3+1} - n\right)$.

Sol. (*i*) Here
$$u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^p} = \frac{\sqrt{n+1}-\sqrt{n}}{n^p} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \qquad \text{(Rationalising)}$$

$$= \frac{(n+1)-n}{n^p \cdot \sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{n^{p+1/2}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

Take
$$v_n = \frac{1}{n^{p+1/2}}$$

$$\operatorname*{Lt}_{n\to\infty} \frac{u_n}{v_n} = \operatorname*{Lt}_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{1+1} = \frac{1}{2} \qquad \text{which is finite and } \neq 0.$$

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n^{p+1/2}}$ is convergent if $p + \frac{1}{2} > 1$ and divergent if $p + \frac{1}{2} \leq 1$.

i.e., convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$.

$\therefore$ Σu_n is convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$.

(*ii*) Here

$$u_n = (n^3 + 1)^{1/3} - n = \left[n^3\left(1+\frac{1}{n^3}\right)\right]^{1/3} - n$$

$$= n\left(1+\frac{1}{n^3}\right)^{1/3} - n = n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$$

$$= n\left[1+\frac{1}{3}.\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}.\left(\frac{1}{n^3}\right)^2 + \ldots\ldots - 1\right]$$

$$= \frac{n}{n^3}\left[\frac{1}{3}-\frac{1}{9}.\frac{1}{n^3}\ldots\ldots\right] = \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9}.\frac{1}{n^3}\ldots\ldots\right]$$

Take

$$v_n = \frac{1}{n^2}.$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{3}-\frac{1}{9}.\frac{1}{n^3}\ldots\ldots\right) = \frac{1}{3}$$

which is finite and $\neq 0$.

$\therefore$ Σu_n and Σv_n converge or diverge together.

Since $\sum v_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ with $p = 2 > 1$

$\therefore$ Σv_n is convergent $\Rightarrow$ Σu_n is convergent.

Note. Rationalisation is effective only when square roots are involved whereas this method of Binomial Expansion is general.

TEST YOUR KNOWLEDGE

Test the convergence or divergence of the following series :

1. $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4} + \ldots\ldots$ to ∞

2. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4} + \ldots\ldots$ to ∞

3. $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6} + \ldots\ldots$ to ∞

4. $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6} + \ldots\ldots$ to ∞

5. $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7} + \ldots\ldots$ to ∞

6. $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}} + \ldots\ldots$ to ∞

7. $\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p} + \ldots\ldots$ to ∞

8. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p} + \ldots\ldots$ to ∞

9. $\sum_{n=1}^{\infty} \frac{n+1}{n(2n-1)}$

10. $\sum_{n=1}^{\infty} \frac{1}{n^p(n+1)^p}$

11. $\dfrac{2^p}{1^q} + \dfrac{3^p}{2^q} + \dfrac{4^p}{3^q} + \ldots\ldots$ to ∞

(p and q are positive numbers)

12. $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}}{n^2+1}$

13. $\sum \dfrac{2n^3+5}{4n^5+1}$

14. $\sum \dfrac{\sqrt{n^2-1}}{n^3+1}$

15. $\dfrac{1}{5} + \dfrac{\sqrt{2}}{7} + \dfrac{\sqrt{3}}{9} + \dfrac{\sqrt{4}}{11} + \ldots\ldots$ to ∞

16. $\sum \left(\sqrt{n^2+1} - n\right)$

17. $\sum \left(\sqrt{n^3+1} - \sqrt{n^3}\right)$

18. $\sum \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$

19. $\dfrac{\sqrt{2}-\sqrt{1}}{1} + \dfrac{\sqrt{3}-\sqrt{2}}{2} + \dfrac{\sqrt{4}-\sqrt{3}}{3} + \ldots\ldots$

20. $\sum \left(\sqrt[3]{n+1} - \sqrt[3]{n}\right)$

21. $\sum \dfrac{1}{\sqrt{n}} \sin \dfrac{1}{n}$

22. $\sum \cot^{-1} n^2$

23. $\sum \left(\dfrac{1}{n} - \log \dfrac{n+1}{n}\right)$

24. $\sum_{n=2}^{\infty} \dfrac{1}{n \log n}$.

Answers

1. Convergent **2.** Convergent **3.** Convergent **4.** Divergent

5. Divergent **6.** Divergent **7.** Convergent for $p > 1$, divergent for $p \le 1$

8. Convergent for $p > 2$, divergent for $p \le 2$ **9.** Divergent

10. Convergent for $p > \dfrac{1}{2}$, divergent for $p \le \dfrac{1}{2}$ **11.** Convergent for $q > p + 1$, divergent for $q \le p + 1$

12. Convergent **13.** Convergent **14.** Convergent **15.** Divergent

16. Divergent **17.** Convergent **18.** Convergent **19.** Convergent

20. Divergent **21.** Convergent **22.** Convergent **23.** Convergent

24. Divergent.

Article 9. D' Alembert's Ratio Test

Statement. *If Σu_n is a positive term series, and* $\underset{n\to\infty}{Lt} \dfrac{u_{n+1}}{u_n} = l$, *then*

(i) Σu_n is convergent if $l < 1$. *(ii) Σu_n is divergent if $l > 1$.*

Note. If $l = 1$, the test fails, *i.e.*, no conclusion can be drawn about the convergence or divergence of the series.

Proof. Since $\underset{n\to\infty}{\text{Lt}} \dfrac{u_{n+1}}{u_n} = l$,

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$\left| \frac{u_{n+1}}{u_n} - l \right| < \varepsilon \quad \forall\ n \ge m$$

$$\Rightarrow \qquad l - \varepsilon < \frac{u_{n+1}}{u_n} < 1 + \varepsilon \quad \forall\ n \ge m \qquad \ldots(i)$$

(*i*) **When l < 1.**

Choose $\varepsilon > 0$ such that $l < l + \varepsilon < 1$

Put $l + \varepsilon = r$, then $0 < r < 1$.

From (*i*), $$\frac{u_{n+1}}{u_n} < r \quad \forall\, n \geq m \quad \Rightarrow \quad u_{n+1} < ru_n \quad \forall\, n \geq m \qquad (\because \quad u_n > 0)$$

Putting $n = m, m + 1, m + 2, ...$, we get

$$u_{m+1} < ru_m$$
$$u_{m+2} < ru_{m+1} < r^2u_m$$
$$u_{m+3} < ru_{m+2} < r^3u_m \quad \text{and so on.}$$

Adding $u_{m+1} + u_{m+2} + u_{m+3} + ... < u_m\,(r + r^2 + r^3 + ...)$

$\Rightarrow$ each term of the given series Σu_n after leaving the first m terms (*i.e.*, a finite number of terms) is less than the corresponding term of a geometric series which is convergent ($\because$ its common ratio $r < 1$). Hence the given series is also convergent.

(*ii*) **When l > 1.**

Choose $\varepsilon > 0$ such that $l - \varepsilon > 1$

Put $l - \varepsilon = \text{R}$, then $\text{R} > 1$.

From (*i*), $$\frac{u_{n+1}}{u_n} > \text{R} \quad \forall\, n \geq m \quad \Rightarrow \quad u_{n+1} > \text{R}u_n \quad \forall \quad n \geq m \qquad (\because \quad u_n > 0)$$

Putting $n = m, m + 1, m + 2, ...$, we get

$$u_{m+1} > \text{R}u_m$$
$$u_{m+2} > \text{R}u_{m+1} > \text{R}^2u_m$$
$$u_{m+3} > \text{R}u_{m+2} > \text{R}^3u_m \quad \text{and so on.}$$

Adding $u_{m+1} + u_{m+2} + u_{m+3} + ... > u_m\,(\text{R} + \text{R}^2 + \text{R}^3 + ...)$

$\Rightarrow$ each term of the given series Σu_n after leaving the first m terms (*i.e.*, a finite number of terms) is greater than the corresponding term of a geometric series which is divergent ($\because$ its common ratio $\text{R} > 1$). Hence the given series is also divergent.

Practical Form of D'Alembert's Ratio Test

In practice, Ratio Test is used in the following form :

If Σu_n is a positive term series, and $\underset{n \to \infty}{\text{Lt}} \dfrac{u_n}{u_{n+1}} = l$, then

(*i*) Σu_n is convergent if $l > 1$ (*ii*) Σu_n is divergent if $l < 1$.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the following series:*

(*i*) $1 + \dfrac{2^p}{2\,!} + \dfrac{3^p}{3\,!} + \dfrac{4^p}{4\,!} + ..., (p > 0)$ (*ii*) $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{5} + ... + \dfrac{1}{2^{n-1} + 1} + ...$

(*iii*) $\dfrac{1^2 \,.\, 2^2}{1!} + \dfrac{2^2 \,.\, 3^2}{2\,!} + \dfrac{3^2 \,.\, 4^2}{3\,!} + \dfrac{4^2 \,.\, 5^2}{4\,!} + ...$

Sol. (*i*) Here $$u_n = \frac{n^p}{n!} \qquad \left[\because \; 1 = \frac{1^p}{1!}\right]$$

$\therefore$ $$u_{n+1} = \frac{(n+1)^p}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^p}{n!} \frac{(n+1)!}{(n+1)^p} = \frac{n^p \, . (n+1)\, n!}{n!\,(n+1)^p} = \frac{n^p}{(n+1)^{p-1}}$$

$$= \frac{n^p}{n^{p-1}\left(1+\frac{1}{n}\right)^{p-1}} = \frac{n}{\left(1+\frac{1}{n}\right)^{p-1}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n}{\left(1+\frac{1}{n}\right)^{p-1}} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(*ii*) Here $$u_n = \frac{1}{2^{n-1}+1} \quad \therefore \quad u_{n+1} = \frac{1}{2^n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^n+1}{2^{n-1}+1} = \frac{2^n\left(1+\frac{1}{2^n}\right)}{2^{n-1}\left(1+\frac{1}{2^{n-1}}\right)} = 2\,.\frac{1+\frac{1}{2^n}}{1+\frac{1}{2^{n-1}}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\; 2\,.\frac{1+\frac{1}{2^n}}{1+\frac{1}{2^{n-1}}} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(*iii*) Here $$u_n = \frac{n^2(n+1)^2}{n!} \quad \therefore \quad u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2(n+1)^2}{n!} \cdot \frac{(n+1)!}{(n+1)^2(n+2)^2} = \frac{n^2 \, . (n+1)\, n!}{n!\,(n+2)^2}$$

$$= \frac{n^3\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{2}{n}\right)^2} = n\,.\frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\; n\,.\frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2} = \infty > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

Example 2. *Test the convergence of the following series:*

$(i)\ \sum \dfrac{n^3 + a}{2^n + a}$ $\qquad (ii)\ \sum \dfrac{n!\,2^n}{n^n}$.

Sol. (i) Here $u_n = \dfrac{n^3 + a}{2^n + a} \quad \therefore \quad u_{n+1} = \dfrac{(n+1)^3 + a}{2^{n+1} + a}$

$$\frac{u_n}{u_{n+1}} = \frac{n^3 + a}{(n+1)^3 + a} \cdot \frac{2^{n+1} + a}{2^n + a}$$

$$= \frac{n^3\left(1 + \dfrac{a}{n^3}\right)}{n^3\left[\left(1 + \dfrac{1}{n}\right)^3 + \dfrac{a}{n^3}\right]} \cdot \frac{2^{n+1}\left(1 + \dfrac{a}{2^{n+1}}\right)}{2^n\left(1 + \dfrac{a}{2^n}\right)} = \frac{1 + \dfrac{a}{n^3}}{\left(1 + \dfrac{1}{n}\right)^3 + \dfrac{a}{n^3}} \cdot \frac{2\left(1 + \dfrac{a}{2^{n+1}}\right)}{1 + \dfrac{a}{2^n}}$$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1+0}{1+0} \cdot 2 \cdot \frac{1+0}{1+0} = 2 > 1$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

(ii) Here $\qquad u_n = \dfrac{n!\,2^n}{n^n} \quad \therefore \quad u_{n+1} = \dfrac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$

$$\frac{u_n}{u_{n+1}} = \frac{n!\,2^n}{(n+1)!\,2^{n+1}} \cdot \frac{(n+1)^{n+1}}{n^n} = \frac{1}{2(n+1)} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{1}{2} \cdot \frac{(n+1)^n}{n^n} = \frac{1}{2} \cdot \left(\frac{n+1}{n}\right)^n = \frac{1}{2}\left(1 + \frac{1}{n}\right)^n$$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n \to \infty}{\text{Lt}} \frac{1}{2}\left(1 + \frac{1}{n}\right)^n = \frac{e}{2}$$

Now $\qquad 2 < e < 3 \quad \Rightarrow \quad 1 < \dfrac{e}{2} < \dfrac{3}{2}$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{e}{2} > 1 \quad \Rightarrow \quad \Sigma u_n \text{ is convergent.}$$

Example 3. *Discuss the convergence of the series:* $\sum \dfrac{n!}{n^n}$.

Sol. Here $\qquad u_n = \dfrac{n!}{n^n}$

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)\,n!}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n \to \infty}{\text{Lt}} \left(1 + \frac{1}{n}\right)^n = e > 1 \qquad (\because \quad 2 < e < 3)$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n is convergent.

Example 4. *Discuss the convergence of the series:* $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}}\, x^n$.

Sol. Here $$u_n = \sqrt{\frac{n}{n^2+1}}\; x^n$$

$\therefore$ $$u_{n+1} = \sqrt{\frac{n+1}{(n+1)^2+1}}\; .\; x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \sqrt{\frac{n}{n+1} \cdot \frac{n^2+2n+2}{n^2+1}} \cdot \frac{1}{x} = \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \sqrt{\frac{1}{1+\frac{1}{n}} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$

When $x = 1$, the Ratio Test fails.

When $x = 1$, $u_n = \sqrt{\frac{n}{n^2+1}} = \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1+\frac{1}{n^2}}}$

Take $v_n = \frac{1}{\sqrt{n}}$, $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$

which is finite and $\neq 0$.

$\therefore$ By Comparison Test, Σu_n and Σv_n converge or diverge together.

Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is of the form $\sum \frac{1}{n^p}$ with $p = \frac{1}{2}$ (< 1),

Σv_n diverges $\Rightarrow$ Σu_n diverges.

Hence the given series Σu_n converges if $x < 1$ and diverges if $x \geq 1$.

Example 5. *Examine the convergence or divergence of the following series :*

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots\ldots$$

Sol. Here $$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \quad \therefore \quad u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x^2} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}} \cdot \sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x^2} > 1$, *i.e.*, $x^2 < 1$

and diverges if $\frac{1}{x^2} < 1$ *i.e.*, $x^2 > 1$.

When $x^2 = 1$, $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$

Take $v_n = \frac{1}{n^{3/2}}$ By comparison test Σu_n is convergent $(\because \quad p > 1)$

Hence Σu_n is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

Example 6. *Examine the convergence or divergence of the following series:*

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots\ldots + \frac{2^{n+1}-2}{2^{n+1}+1}\,x^n + \ldots\ldots\ (x > 0).$$

Sol. Here, leaving the first term, $u_n = \frac{2^{n+1}-2}{2^{n+1}+1}\,x^n$

$$\therefore \qquad u_{n+1} = \frac{2^{n+2}-2}{2^{n+2}+1}x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^{n+1}-2}{2^{n+1}+1}\cdot\frac{2^{n+2}+1}{2^{n+2}-2}\cdot\frac{1}{x} = \frac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}\cdot\frac{2^{n+2}\left(1+\frac{1}{2^{n+2}}\right)}{2^{n+2}\left(1-\frac{2}{2^{n+2}}\right)}\cdot\frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1-\frac{1}{2^n}}{1+\frac{1}{2^{n+1}}}\cdot\frac{1+\frac{1}{2^{n+2}}}{1-\frac{1}{2^{n+1}}}\cdot\frac{1}{x} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio Test, $\sum u_n$ converges if $\frac{1}{x} > 1$, *i.e.*, $x < 1$

and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

When $x = 1$, $u_n = \frac{2^{n+1}-2}{2^{n+1}+1} = \frac{2^{n+1}\left(1-\frac{2}{2^{n+1}}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+2}}\right)} = \frac{1-\frac{1}{2^n}}{1+\frac{1}{2^{n+1}}}$

$\underset{n\to\infty}{\text{Lt}}\, u_n = 1 \ne 0 \quad \Rightarrow \quad \Sigma u_n$ does not converge. Being a series of + ve terms, it must diverge.

Hence Σu_n is convergent if $x < 1$ and divergent if $x \ge 1$.

Example 7. *Test for convergence the positive term series :*

$$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} + \ldots..$$

Sol. Leaving the first term $u_n = \dfrac{(\alpha+1)(2\alpha+1)\ldots\ldots(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots\ldots(n\beta+1)}$

$\therefore$ $$u_{n+1} = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots(n\alpha+1)[(n+1)\alpha+1]}{(\beta+1)(2\beta+1)\ldots\ldots(n\beta+1)[(n+1)\beta+1]}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)\beta+1}{(n+1)\alpha+1} = \frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}} = \frac{\beta}{\alpha}$$

$\therefore$ By D'Alembert's Ratio Test, Σu_n converges if $\dfrac{\beta}{\alpha} > 1$ *i.e.,* $\beta > \alpha > 0$

and diverges if $\dfrac{\beta}{\alpha} < 1$ *i.e.,* $\beta < \alpha$ or $\alpha > \beta > 0$

When $\alpha = \beta$, the Ratio Test fails.

When $\alpha = \beta$, $u_n = 1$ $\therefore$ $\underset{n\to\infty}{\text{Lt}} u_n = 1 \neq 0$

$\Rightarrow$ Σu_n does not converge. Being a series of +ve terms, it must diverge.

Hence the given series is convergent if $\beta > \alpha > 0$ and divergent if $\alpha \geq \beta > 0$.

TEST YOUR KNOWLEDGE

Discuss the convergence of the following series:

1. $1 + \dfrac{2^2}{2!} + \dfrac{3^2}{3!} + \dfrac{4^2}{4!} + \ldots\ldots$ to ∞

2. $1 + \dfrac{2!}{2^2} + \dfrac{3!}{3^3} + \dfrac{4!}{4^4} + \ldots\ldots$ to ∞

3. $\dfrac{1}{1+2} + \dfrac{2}{1+2^2} + \dfrac{3}{1+2^3} + \ldots\ldots$ to ∞

4. $\dfrac{2}{1} + \dfrac{2.5.8}{1.5.9} + \dfrac{2.5.8.11}{1.5.9.13} + \ldots\ldots$ to ∞

5. $\dfrac{2!}{3} + \dfrac{3!}{3^2} + \dfrac{4!}{3^3} + \ldots\ldots$ to ∞

6. $\left(\dfrac{1}{3}\right)^2 + \left(\dfrac{1.2}{3.5}\right)^2 + \left(\dfrac{1.2.3}{3.5.7}\right)^2 + \left(\dfrac{1.2.3.4}{3.5.7.9}\right)^2 + \ldots\ldots$ to ∞

7. $\sum \dfrac{2^{n-1}}{3^n+1}$

8. $\sum \dfrac{1}{n!}$

9. $\sum \dfrac{n^2(n+1)^2}{n!}$

10. $\sum \dfrac{x^n}{3^n \cdot n^2}$, $x > 0$

11. $\sum \dfrac{x^n}{n}$, $x > 0$

12. $\sum \dfrac{n}{n^2+1} x^n$, $x > 0$

13. $\sum \sqrt{\dfrac{n+1}{n^3+1}} \cdot x^n$, $x > 0$

14. $x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots$ to ∞

15. $1 + \dfrac{x}{2} + \dfrac{x^2}{5} + \dfrac{x^3}{10} + \ldots\ldots \dfrac{x^n}{n^2+1} + \ldots\ldots$ to ∞

16. $\dfrac{x}{1.3} + \dfrac{x^2}{3.5} + \dfrac{x^3}{5.7} + \ldots\ldots$ to ∞

17. $x + \dfrac{3}{5}x^2 + \dfrac{8}{10}x^3 + \dfrac{15}{17}x^4 + \ldots\ldots + \dfrac{n^2-1}{n^2+1}x^n + \ldots\ldots$ to ∞

18. $\dfrac{x}{2\sqrt{3}} + \dfrac{x^2}{3\sqrt{4}} + \dfrac{x^3}{4\sqrt{5}} + \ldots\ldots$ to ∞

19. $\sum \dfrac{3^n - 2}{3^n + 1} \cdot x^{n-1}, x > 0$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$

Answers

1. Convergent	**2.** Convergent	**3.** Convergent	**4.** Convergent
5. Divergent	**6.** Convergent	**7.** Convergent	**8.** Convergent
9. Convergent	**10.** Convergent for $x \le 3$, divergent for $x > 3$.		
11. Convergent for $x < 1$, divergent for $x \ge 1$		**12.** Convergent for $x < 1$, divergent for $x \ge 1$	
13. Convergent for $x < 1$, divergent for $x \ge 1$		**14.** Convergent for $x < 1$, divergent for $x \ge 1$	
15. Convergent for $x \le 1$, divergent for $x > 1$		**16.** Convergent for $x \le 1$, divergent for $x > 1$	
17. Convergent for $x < 1$, divergent for $x \ge 1$		**18.** Convergent for $x \le 1$, divergent for $x > 1$	
19. Convergent for $x < 1$, divergent for $x \ge 1$		**20.** Convergent	**21.** Convergent.

Article 10. Cauchy's Root Test

Statement. *If Σu_n is a positive term series and $\underset{n \to \infty}{Lt} (u_n)^{1/n} = l$, then*

(i) Σu_n is convergent if $l < 1$ *(ii) Σu_n is divergent if $l > 1$.*

Note. If $l = 1$, the test fails *i.e.*, no conclusion can be drawn about the convergence or divergence of the series. The series may converge, it may diverge.

Proof. Since $\underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = l$,

$\therefore$ given $\varepsilon > 0$, however small, there exists a +ve integer m such that

$$| (u_n)^{1/n} - l \,| < \varepsilon \qquad \forall \quad n \ge m$$

$$\Rightarrow \qquad l - \varepsilon < (u_n)^{1/n} < l + \varepsilon \qquad \forall \quad n \ge m$$

$$\Rightarrow \qquad (l - \varepsilon)^n < u_n < (l + \varepsilon)^n \qquad \forall \quad n \ge m \qquad \ldots(1)$$

(*i*) **When $l < 1$**

Choose $\varepsilon > 0$ such that $l < l + \varepsilon < 1$

Put $\quad l + \varepsilon = r$, then $0 < r < 1$

From (1), $\qquad u_n < r^n \qquad \forall \quad n \ge m$

Putting $n = m, m + 1, m + 2, \ldots\ldots$, we get $u_m < r^m$, $u_{m+1} < r^{m+1}$, $u_{m+2} < r^{m+2}$, ... and so on.

Adding $\qquad u_m + u_{m+1} + u_{m+2} + \ldots\ldots + < r^m + r^{m+1} + r^{m+2} + \ldots\ldots$

$\Rightarrow$ each term of the given series Σu_n after leaving the first $(m - 1)$ terms, (*i.e.*, a finite number of terms) is less than the corresponding term of a geometric series which is convergent ($\because$ its common ratio $r < 1$). Hence the given series is also convergent.

(*ii*) **When $l > 1$**

Choose $\varepsilon > 0$ such that $l - \varepsilon > 1$

Put $\quad l - \varepsilon = \text{R}$, then $\text{R} > 1$

From (1), $\qquad u_n > \text{R}^n \qquad \forall \quad n \ge m$

Putting $n = m, m + 1, m + 2, \ldots\ldots$, we get $u_m < \text{R}^m$, $u_{m+1} > \text{R}^{m+1}$, $u_{m+2} < \text{R}^{m+2}$, ... and so on.

Adding $\qquad u_m + u_{m+1} + u_{m+2} + \ldots\ldots > R^m + R^{m+1} + R^{m+2} + \ldots\ldots$

$\Rightarrow$ each term of the series Σu_n after leaving the first $(m-1)$ terms, (*i.e.*, a finite number of terms) is greater than the corresponding term of a geometric series which is divergent.

($\because$ its common ratio $R > 1$). Hence the given series is also divergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the convergence of the following series:*

$$\sum \left(\frac{n}{n+1}\right)^{n^2} \quad or \quad \sum \left(1+\frac{1}{n}\right)^{-n^2}.$$

Sol. Here
$$u_n = \left(\frac{n}{n+1}\right)^{n^2}$$

$$\therefore \qquad (u_n)^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n} = \left(\frac{n}{n+1}\right)^{n} = \left(\frac{n+1}{n}\right)^{-n} = \left[\left(1+\frac{1}{n}\right)^{n}\right]^{-1}$$

$$\operatorname*{Lt}_{n\to\infty} (u_n)^{1/n} = \operatorname*{Lt}_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n}\right]^{-1} = e^{-1} = \frac{1}{e} < 1 \qquad (\because \quad e = 2.7)$$

$\therefore$ By Cauchy's Root Test, the given series Σu_n is convergent.

Example 2. *Examine the convergence of the series:*

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3} + \ldots\ldots$$

Sol. Here
$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

$$(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1} = \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\operatorname*{Lt}_{n\to\infty} (u_n)^{1/n} = \operatorname*{Lt}_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= (e.1-1)^{-1} = \frac{1}{e-1} < 1 \qquad (\because \quad e = 2.7)$$

$\therefore$ By Cauchy's Root Test, Σu_n is convergent.

TEST YOUR KNOWLEDGE

Discuss the convergence of the following series:

1. $\sum \frac{1}{n^n}$ **2.** $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ **3.** $\sum \left(\frac{n+1}{3n}\right)^n$

4. $\sum \frac{(n-\log n)^n}{2^n \cdot n^n}$ **5.** $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ **6.** $\sum \left(\frac{nx}{n+1}\right)^n$

7. $\sum 5^{-n-(-1)^n}$ **8.** $\sum \frac{(1+nx)^n}{n^n}$ **9.** $\sum \frac{(n+1)^n\, x^n}{n^{n+1}}$

10. $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots\ldots$ to ∞.

Answers

1. Convergent **2.** Convergent **3.** Convergent **4.** Convergent
5. Convergent **6.** Convergent for $x < 1$, divergent for $x \geq 1$. **7.** Convergent
8. Convergent for $x < 1$, divergent for $x \geq 1$ **9.** Convergent for $x < 1$, divergent for $x \geq 1$
10. Convergent for $x < 1$, divergent for $x \geq 1$.

Article 11. Raabe's Test

Statement. *If Σu_n is a series of positive terms and $\underset{n\to\infty}{Lt}\, n\left(\frac{u_n}{u_{n+1}}-1\right)=l$, then the series is convergent if $l > 1$ and divergent if $l < 1$.*

Proof. Let us compare the given series Σu_n with an auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^p}$ which we know converges if $p > 1$ and diverges if $p \leq 1$.

Now
$$\frac{v_n}{v_{n+1}} = \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} = \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p = 1+\frac{p}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n^2}+\ldots\ldots$$

Case I. *Let $\Sigma v_n = \Sigma \frac{1}{n^p}$ be convergent, so that $p > 1$.*

Then Σu_n will also converge if $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$

or if
$$\frac{u_n}{u_{n+1}} > 1+\frac{p}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n^2}+\ldots\ldots$$

or if
$$n\left(\frac{u_n}{u_{n+1}}-1\right) > p+\frac{p(p-1)}{2!}\cdot\frac{1}{n}+\ldots\ldots$$

or if
$$\underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) > p \quad \text{or} \quad \text{if } l > p$$

But p is itself greater than 1, $\therefore$ Σu_n is convergent if $l > 1$.

Case II. *Let Σv_n be divergent, so that $p \le 1$.*

Then Σu_n will also diverge if $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$

or if $$\frac{u_n}{u_{n+1}} < 1 + \frac{p}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots\ldots$$

or if $$n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \ldots\ldots$$

or if $$\underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) < p \quad \text{or} \quad \text{if } l < p.$$

But p itself ≤ 1. Thus the given series Σu_n diverges if $l < 1$. This proves the result.

Note 1. If $\underset{n\to\infty}{\text{Lt}}\, n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = 1$, then Raabe's test fails.

Note 2. Raabe's test is used when D'Alembert's Ratio test fails and when in the ratio test, $\dfrac{u_n}{u_{n+1}}$ does not involve the number e. When $\dfrac{u_n}{u_{n+1}}$ involves e, we apply logarithmic test after the ratio test and not Raabe's test.

Article 12. Logarithmic Test

Statement. *A positive term series Σu_n converges or diverges according as:*

$$\underset{n\to\infty}{Lt}\, n \log \frac{u_n}{u_{n+1}} > 1 \text{ or } < 1.$$

Proof. Let us compare the given series Σu_n with an auxiliary series $\Sigma v_n = \Sigma \dfrac{1}{n^p}$ which we know converges if $p > 1$ and diverges if $p \le 1$.

Now $$\frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1 + \frac{1}{n}\right)^p.$$

Case I. *Let Σv_n be convergent, so that $p > 1$.*

Then Σu_n will also be convergent if $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$

or if $$\frac{u_n}{u_{n+1}} > \left(1 + \frac{1}{n}\right)^p$$

or if $$\log \frac{u_n}{u_{n+1}} > \log\left(1 + \frac{1}{n}\right)^p = p \log\left(1 + \frac{1}{n}\right)$$

or if $$\log \frac{u_n}{u_{n+1}} > p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right]$$

or if $$n \log \frac{u_n}{u_{n+1}} > p\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\ldots\right] \quad \text{or} \quad \text{if } \underset{n\to\infty}{\text{Lt}}\, n \log \frac{u_n}{u_{n+1}} > p$$

or if $$\underset{n\to\infty}{\text{Lt}}\, n \log \frac{u_n}{u_{n+1}} > 1 \qquad \mid \because \ p > 1 \text{ (given)}$$

Case II. *Let Σv_n be divergent, so that $p \le 1$.*

Then Σv_n also diverges if $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ or if $\log \dfrac{u_n}{u_{n+1}} < \log \dfrac{v_n}{v_{n+1}}$

or if $$\log \frac{u_n}{u_{n+1}} < \log\left(1+\frac{1}{n}\right)^p = p \log\left(1+\frac{1}{n}\right)$$

or if $$\log \frac{u_n}{u_{n+1}} < p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right)$$

or if $$n \log \frac{u_n}{u_{n+1}} < p\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\ldots\right] \quad \text{or} \quad \text{if } \underset{n\to\infty}{\text{Lt}}\, n \log \frac{u_n}{u_{n+1}} < p$$

or if $$\underset{n\to\infty}{\text{Lt}}\, n \log \frac{u_n}{u_{n+1}} < 1 \qquad \Big|\ \because\ p \le 1$$

Thus the series Σu_n converges or diverges according as $\underset{n\to\infty}{\text{Lt}}\, n \log \dfrac{u_n}{u_{n+1}} > 1$ or < 1.

Note 1. The test fails if $\underset{n\to\infty}{\text{Lt}}\, n \log \dfrac{u_n}{u_{n+1}} = 1$.

Note 2. The test is applied after the failure of Ratio test and generally when in Ratio test, $\dfrac{u_n}{u_{n+1}}$ involves 'e'.

Article 13. Gauss Test

Statement. *If for the series Σu_n of positive terms, $\dfrac{u_n}{u_{n+1}}$ can be expanded in the form*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

then Σu_n converges if $\lambda > 1$ and diverges if $\lambda \le 1$.

Note. The test never fails as we know that the series diverges for $\lambda = 1$. Moreover the test is applied after the failure of Ratio test and when it is possible to expand $\dfrac{u_n}{u_{n+1}}$ in powers of $\dfrac{1}{n}$ by Binomial Theorem or by any other method.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the convergence of the series :* $\dfrac{1}{2} + \dfrac{1.3}{2.4} + \dfrac{1.3.5}{2.4.6} + \ldots\ldots$

Sol. Here $$u_n = \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6.\ldots\ldots 2n}$$

$\therefore$ $$u_{n+1} = \frac{1.3.5\ldots\ldots(2n-1)(2n+1)}{2.4.6.\ldots\ldots(2n)(2n+2)}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{1\,.\,3\,.\,5\,......\,(2n-1)}{2\,.\,4\,.\,6\,......\,2n} \times \frac{2\,.\,4\,.\,6\,......\,2n(2n+2)}{1\,.\,3\,.\,5\,......\,(2n-1)(2n+1)}$$

$$= \frac{2n+2}{2n+1} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}} \to 1 \text{ as } n \to \infty.$$

$\therefore$ D'Alembert's Ratio test fails.

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{2n+2}{2n+1} - 1\right] = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$$\therefore \qquad \operatorname*{Lt}_{n \to \infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \tfrac{1}{2} < 1.$$

$\therefore$ By Raabe's test, Σu_n diverges.

Example 2. *Discuss the convergence of the series* $\dfrac{1^2}{2^2} + \dfrac{1^2\,.\,2^2}{2^2\,.\,4^2} + \dfrac{1^2\,.\,3^2\,.\,5^2}{2^2\,.\,4^2\,.\,6^2} + ...$

Sol. Here
$$u_n = \frac{1^2\,.\,3^2\,.\,5^2\,......\,(2n-1)^2}{2^2\,.\,4^2\,.\,6^2\,......\,(2n)^2}$$

and
$$u_{n+1} = \frac{1^2\,.\,3^2\,.\,5^2\,......\,(2n-1)^2\,(2n+1)^2}{2^2\,.\,4^2\,.\,6^2\,......\,(2n)^2\,(2n+2)^2}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$$\therefore \qquad \operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} = 1.$$

Hence the ratio test fails.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(2n+2)^2}{(2n+1)^2} - 1\right]$$

$$= n\left[\frac{4n^2+8n+4-(4n^2+4n+1)}{(2n+1)^2}\right] = n\,\frac{(4n+3)}{(2n+1)^2} = \frac{4n^2+3n}{(2n+1)^2}$$

$$= \frac{1+\frac{3}{4n}}{\left(1+\frac{1}{2n}\right)^2} \to 1 \text{ as } n \to \infty$$

$\therefore$ Raabe's test also fails.

When D'Alembert ratio test fails, we can directly apply Gauss test. Now,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2} = \left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2}-\ldots\ldots\right) = 1+\frac{1}{n}+\frac{1}{n^2}\left(1-2+\frac{3}{4}\right)+\ldots\ldots$$

$$= 1+\frac{1}{n}-\frac{1}{4n^2}+\ldots\ldots = 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Comparing it with $\dfrac{u_n}{u_{n+1}} = 1+\dfrac{\lambda}{n}+O\left(\dfrac{1}{n^2}\right)$

we have $\lambda = 1$. Thus by Gauss test, the series Σu_n diverges.

Example 3. *Discuss the convergence of the series:*

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots\ldots (x>0).$$

Sol. Neglecting the first term, we have $u_n = \dfrac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}\cdot\dfrac{x^{2n+1}}{2n+1}$

and $$u_{n+1} = \frac{1.3.5.\ldots\ldots(2n-1)(2n+1)}{2.4.6\ldots\ldots(2n)(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\cdot\frac{2n+3}{2n+1}\cdot\frac{1}{x^2} = \frac{2n\left(1+\frac{1}{n}\right).2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{2n}\right).2n\left(1+\frac{1}{2n}\right)}\cdot\frac{1}{x^2}$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2}$$

$$\operatorname*{Lt}_{n\to\infty}\frac{u_n}{u_{n+1}} = \operatorname*{Lt}_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ By Ratio Test, Σu_n is convergent if $\dfrac{1}{x^2} > 1$ *i.e.,* $x^2 < 1$ and divergent if $\dfrac{1}{x^2} < 1$ *i.e.,* $x^2 > 1$.

If $x^2 = 1$, then Ratio Test fails.

When $x^2 = 1$, we have $\dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)(2n+3)}{(2n+1)^2} = \dfrac{4n^2+10n+6}{4n^2+4n+1}$

$$\operatorname*{Lt}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \operatorname*{Lt}_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{6n^2+5n}{4n^2+4n+1} = \underset{n\to\infty}{\text{Lt}} \frac{6+\frac{5}{n}}{4+\frac{4}{n}+\frac{1}{n^2}} = \frac{6}{4} = \frac{3}{2} > 1.$$

$\therefore$ By Raabe's Test, the series converges.

Hence Σu_n is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

Example 4. *Discuss the convergence of the series:* $1 + \frac{x}{2} + \frac{2\,!}{3^2}x^2 + \frac{3\,!}{4^2}x^3 + \frac{4\,!}{5^4}x^4 + \ldots\ldots$

Sol. Neglecting the first term, we have

$$u_n = \frac{n\,!}{(n+1)^n}\,x^n \quad \text{and} \quad u_{n+1} = \frac{(n+1)\,!}{(n+2)^{n+1}}\cdot x^{n+1}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n\,!}{(n+1)^n}\cdot\frac{(n+2)^{n+1}}{(n+1)\,!}\cdot\frac{1}{x}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n^n\left(1+\frac{1}{n}\right)^n}\cdot\frac{n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}{(n+1)}\cdot\frac{1}{x}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{2}{n}\right)^n\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x} = \frac{e^2}{e}\cdot\frac{1}{x} = \frac{e}{x}$$

$\therefore$ By D'Alembert's ratio test, the series converges if $\frac{e}{x} > 1$ or if $x < e$ and diverges if $\frac{e}{x} < 1$ or if $x > e$.

If $x = e$, the ratio test fails, $\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1.$

Now when $x = e$,

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}}\cdot\frac{1}{e}\,.$$

Since the expression $\frac{u_n}{u_{n+1}}$ *involves the number e, so we do not apply Raabe's test but apply logarithmic test.*

$$\therefore \quad \log\frac{u_n}{u_{n+1}} = (n+1)\log\left(1+\frac{2}{n}\right) - (n+1)\log\left(1+\frac{1}{n}\right) - \log e$$

$$= (n+1)\left[\log\left(1+\frac{2}{n}\right) - \log\left(1+\frac{1}{n}\right)\right] - 1$$

$$= (n+1)\left[\left(\frac{2}{n} - \frac{1}{2}\cdot\frac{4}{n^2} + \frac{1}{3}\cdot\frac{8}{n^3} - \ldots\ldots\right) - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\ldots\right)\right] - 1$$

$$= (n+1)\left[\frac{1}{n} - \frac{3}{2n^2} + \ldots\ldots\right] - 1$$

$$= 1 - \frac{3}{2n} + \frac{1}{n} - \frac{3}{2n^2} + \ldots\ldots - 1 = -\frac{1}{2n} - \frac{3}{2n^2} + \ldots\ldots$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ n \log\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\ n\left[-\frac{1}{2n} - \frac{3}{2n^2} + \ldots\ldots\right] = \underset{n\to\infty}{\text{Lt}}\left(-\frac{1}{2} - \frac{3}{2n} + \ldots\ldots\right) = -\tfrac{1}{2} < 1$$

$\therefore$ By logarithmic test, the series *diverges*.

Hence the given series Σu_n converges if $x < e$ and diverges if $x \geq e$.

Example 5. *Discuss the convergence of the series:*

$$1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots\ldots$$

Sol. Neglecting the first term,

$$u_n = \frac{\alpha(\alpha+1)\ldots\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots\ldots(\beta+n-1)}{1.2.3\ldots\ldots n.\gamma(\gamma+1)\ldots\ldots(\gamma+n-1)}\cdot x^n$$

$$u_{n+1} = \frac{\alpha(\alpha+1)\ldots\ldots(\alpha+n-1)(\alpha+n)\,\beta(\beta+1)\ldots\ldots(\beta+n-1)(\beta+n)}{1.2.3\ldots\ldots n(n+1)\,\gamma(\gamma+1)(\gamma+2)\ldots\ldots(\gamma+n-1)(\gamma+n)}\cdot x^{n+1}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\cdot\frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}\cdot\frac{1}{x} \qquad \ldots(1)$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ By D'Alembert's Ratio test, the series Σu_n converges if $\frac{1}{x} > 1$

i.e., if $x < 1$ and diverges if $\frac{1}{x} < 1$ or if $x > 1$.

If $x = 1$, $\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}} = 1$ $\therefore$ Ratio test fails.

Putting $x = 1$ in (1), $\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)}$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1-\frac{\alpha}{n}+\frac{\alpha^2}{n^2}+\ldots\ldots\right)\left(1-\frac{\beta}{n}+\frac{\beta^2}{n^2}\ldots\ldots\right)$$

[Expand by Binomial Theorem]

$$= \left(1+\frac{1}{n}+\frac{\gamma}{n}+\frac{\gamma}{n^2}\right)\left(1-\frac{\alpha}{n}-\frac{\beta}{n}+\frac{\alpha\beta}{n^2}+\frac{\alpha^2}{n^2}+\frac{\beta^2}{n^2}\ldots\ldots\right)$$

$$= 1+\frac{1}{n}\,(1+\gamma-\alpha-\beta)+O\left(\frac{1}{n^2}\right).$$

$\therefore$ By Gauss test, the series Σu_n converges if $1 + \gamma - \alpha - \beta > 1$ *i.e.,* if $\gamma > \alpha + \beta$ and diverges if $1 + \gamma - \alpha - \beta \le 1$ *i.e.,* if $\gamma \le \alpha + \beta$.

Thus the given series *converges if $x < 1$ and diverges if $x > 1$. If $x = 1$, then the series converges if $\gamma > \alpha + \beta$ and diverges if $\gamma \le \alpha + \beta$.*

TEST YOUR KNOWLEDGE

Discuss the convergence of the following series:

1. $1+\frac{2^2}{3^2}+\frac{2^2.4^2}{3^2.5^2}+\frac{2^2.4^2.6^2}{3^2.5^2.7^2}+\ldots$ to ∞

2. $\frac{1^2}{4^2}+\frac{1^2.5^2}{4^2.8^2}+\frac{1^2.5^2.9^2}{4^2.8^2.12^2}+\ldots$ to ∞

3. $1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\ldots$ to ∞

4. $1+\frac{2}{1}.\frac{1}{2}+\frac{2.4}{1.3}.\frac{1}{3}+\frac{2.4.6}{1.3.5}.\frac{1}{4}+\ldots$ to ∞

5. $\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}x+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+\ldots$ to ∞

6. $x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\frac{5^5x^5}{5!}+\ldots$ to ∞

7. $1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\ldots$ to ∞

8. $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\ldots$ to ∞

9. $1^p+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+\ldots$ to ∞

10. $\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\ldots$ to ∞

11. $\frac{a}{b}+\frac{a(a+d)}{b(b+d)}x+\frac{a(a+d)(a+2d)}{b(b+d)(b+2d)}x^2+\ldots$ to ∞ $(a > 0, b > 0, x > 0)$

12. $\sum \frac{n!}{x(x+1)(x+2)\ldots\ldots(x+n-1)}$

13. $1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\ldots$ to ∞ $(x > 0)$

14. $\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}+\ldots$ to ∞ $(x > 0)$

15. $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+\ldots$ to ∞

16. $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\ldots$ to ∞ $(x > 0)$

17. $\sum \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots(2n)}.\frac{x^{2n}}{2n}$

18. $\sum \frac{(n!)^2}{(2n)!}x^{2n}$ $(x > 0)$

19. $\sum \frac{4.7.10\ldots\ldots(3n+1)}{1.2.3\ldots\ldots n}x^n$

20. $1+\frac{(1!)^2}{2!}x^2+\frac{(2!)^2}{4!}x^4+\ldots$ to ∞ $(x > 0)$

21. $x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+\ldots$ to ∞

Answers

1. Divergent **2.** Convergent **3.** Convergent for $x \le 1$, divergent for $x > 1$

4. Divergent **5.** Convergent for $x < 1$, divergent for $x \ge 1$.

6. Convergent for $x < \frac{1}{e}$, divergent for $x \ge \frac{1}{e}$ **7.** Convergent for $x \le \frac{1}{e}$, divergent for $x > \frac{1}{e}$

8. Convergent for $x < \frac{1}{e}$, divergent for $x \geq \frac{1}{e}$

9. Convergent for $p > 2$, divergent for $p \leq 2$

10. Convergent for $b > a + 1$, divergent for $b \leq a + 1$

11. Convergent for $x < 1$ or $x = 1$ and $b > a + d$, divergent for $x > 1$ or $x = 1$ and $b \leq a + d$

12. Convergent for $x > 2$, divergent for $x \leq 2$

13. Convergent for $x < 1$, divergent for $x \geq 1$

14. Convergent for $x \leq 1$, divergent for $x > 1$

15. Convergent for $x \leq 1$, divergent for $x > 1$

16. Convergent for $x < 1$, divergent for $x \geq 1$

17. Convergent for $x^2 \leq 1$, divergent for $x^2 > 1$

18. Convergent for $x^2 < 4$, divergent for $x^2 \geq 4$

19. Convergent for $x < \frac{1}{3}$, divergent for $x \geq \frac{1}{3}$

20. Convergent for $x^2 < 4$, divergent for $x^2 \geq 4$

21. Convergent for $x < 1$, divergent for $x \geq 1$.

Article 14. Cauchy's Integral Test

Statement. *If for $x \geq 1$, $f(x)$ is a non-negative, monotonic decreasing function of x such that $f(n) = u_n$ for all positive integral values of n, then the series Σu_n and the integral $\int_1^\infty f(x)\, dx$ converge or diverge together.*

Proof. Let r be a +ve integer. Choose x such that $r + 1 \geq x \geq r \geq 1$

Since $f(x)$ is a monotonic decreasing function of x.

$$\therefore \quad f(r+1) \leq f(x) \leq f(r) \quad \Rightarrow \quad u_{r+1} \leq f(x) \leq u_r \qquad [\because \; f(n) = u_n,\, n \in \text{N}]$$

$$\Rightarrow \quad \int_r^{r+1} u_{r+1}\, dx \leq \int_r^{r+1} f(x)\, dx \leq \int_r^{r+1} u_r\, dx$$

$$\Rightarrow \quad u_{r+1} \int_r^{r+1} dx \leq \int_r^{r+1} f(x)\, dx \leq u_r \int_r^{r+1} dx$$

$$\Rightarrow \quad u_{r+1} \Big[x \Big]_r^{r+1} \leq \int_r^{r+1} f(x)\, dx \leq u_r \Big[x \Big]_r^{r+1}$$

$$\Rightarrow \quad u_{r+1} \leq \int_r^{r+1} f(x)\, dx \leq u_r \qquad \text{...(1)}$$

Putting $r = 1, 2, 3, \ldots\ldots, n$ in succession in (1), we have

$$u_2 \leq \int_1^2 f(x)\, dx \leq u_1$$

$$u_3 \leq \int_2^3 f(x)\, dx \leq u_2$$

....

....

$$u_{n+1} \leq \int_n^{n+1} f(x)\, dx \leq u_n$$

Adding the above inequalities, we have

$$u_2 + u_3 + \ldots\ldots + u_{n+1} \leq \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \ldots\ldots + \int_n^{n+1} f(x)\, dx \leq u_1 + u_2 + \ldots\ldots + u_n$$

$$\Rightarrow \quad \text{S}_{n+1} - u_1 \leq \int_1^{n+1} f(x)\, dx \leq \text{S}_n \quad \text{where} \quad \text{S}_n = \sum_1^n u_n = u_1 + u_2 + \ldots\ldots + u_n.$$

Proceeding to the limit as $n \to \infty$

$$\underset{n\to\infty}{\text{Lt}}\ S_{n+1} - u_1 \le \underset{n\to\infty}{\text{Lt}} \int_1^{n+1} f(x)\,dx \le \underset{n\to\infty}{\text{Lt}}\ S_n$$

$$\Rightarrow \qquad \underset{n\to\infty}{\text{Lt}}\ S_{n+1} - u_1 \le \int_1^{\infty} f(x)\,dx \le \underset{n\to\infty}{\text{Lt}}\ S_n \qquad ...(2)$$

(*i*) If $\int_1^{\infty} f(x)\,dx$ converges, then $\int_1^{\infty} f(x)\,dx$ = a fixed finite number = I (say).

Then from (2), we have $\underset{n\to\infty}{\text{Lt}}\ S_{n+1} - u_1 \le \text{I}$

$\Rightarrow$ $\underset{n\to\infty}{\text{Lt}}\ S_{n+1} \le \text{I} + u_1$ = a fixed finite number

$\Rightarrow$ $\{S_n\}$ is a convergent sequence

$\Rightarrow$ the series Σu_n is convergent.

(*ii*) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\int_1^{\infty} f(x)\,dx = +\infty$

From (2), $\underset{n\to\infty}{\text{Lt}}\ S_n \ge \underset{n\to\infty}{\text{Lt}} \int_1^{\infty} f(x)\,dx = +\infty$

$\Rightarrow$ $\{S_n\}$ is a divergent sequence

$\Rightarrow$ the series Σu_n is divergent.

Hence Σu_n and $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

Note. If $x \ge k$, then Σu_n and $\int_k^{\infty} f(x)\,dx$ converge or diverge together.

ILLUSTRATIVE EXAMPLES

Example 1. *Test for convergence the series:* $\sum \frac{1}{n^2+1}$.

Sol. Here $\qquad u_n = \frac{1}{n^2+1} = f(n)$

$\therefore \qquad f(x) = \frac{1}{x^2+1}$

For $x \ge 1$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ Cauchy's Integral Test is applicable.

Now $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{x^2+1} = \left[\tan^{-1} x\right]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ = finite

$\Rightarrow$ $\int_1^{\infty} f(x)\,dx$ converges and hence by Integral Test, Σu_n also converges.

Example 2. *Discuss the convergence of the series:* $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, *(p > 0).*

Sol. Here $u_n = \frac{1}{n(\log n)^p} = f(x) \quad \therefore \quad f(x) = \frac{1}{x(\log x)^p}$

For $x \geq 2$, $p > 0$, $f(x)$ is +ve and monotonic decreasing.

$\therefore$ By Cauchy's Integral Test $\sum_{n=2}^{\infty} u_n$ and $\int_2^{\infty} f(x)\,dx$ converge or diverge together.

Case I. When $p \neq 1$

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} (\log x)^{-p} \cdot \frac{1}{x}\,dx = \left[\frac{(\log x)^{-p+1}}{-p+1}\right]_2^{\infty}$$

Sub-Case 1. When $p > 1$, $p - 1$ is +ve, so that $\int_2^{\infty} f(x)\,dx = -\frac{1}{p-1}\left[\frac{1}{(\log x)^{p-1}}\right]_2^{\infty}$

$$= -\frac{1}{p-1}\left[0 - \frac{1}{(\log 2)^{p-1}}\right] = \frac{1}{(p-1)(\log 2)^{p-1}} = \text{finite}$$

$\Rightarrow \int_2^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{n=2}^{\infty} u_n$ converges.

Sub-Case 2. When $p < 1$, $1 - p$ is +ve, so that

$$\int_2^{\infty} f(x)\,dx = \frac{1}{1-p}\left[(\log x)^{1-p}\right]_2^{\infty} = \frac{1}{1-p}\,[\infty - (\log 2)^{1-p}] = \infty$$

$\Rightarrow \int_2^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{n=2}^{\infty} u_n$ diverges.

Case II. When $p = 1$, $f(x) = \frac{1}{x \log x}$

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x \log x} = \int_2^{\infty} \frac{\frac{1}{x}}{\log x}\,dx = \Big[\log \log x\Big]_2^{\infty} = \infty - \log \log 2 = \infty$$

$\Rightarrow \int_2^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{n=2}^{\infty} u_n$ diverges.

Hence Σu_n converges if $p > 1$ and diverges if $0 < p \leq 1$.

TEST YOUR KNOWLEDGE

Using the integral test, discuss the convergence of the following series:

1. $\sum \frac{1}{2n+3}$ **2.** $\sum \frac{1}{n(n+1)}$ **3.** $\sum \frac{1}{\sqrt{n}}$

4. $\sum \dfrac{1}{(n+1)^2}$ **5.** $\sum \dfrac{2n^3}{n^4+3}$ **6.** $\sum \dfrac{n}{(n^2+1)^2}$

7. $\sum_{n=2}^{\infty} \dfrac{1}{n\sqrt{n^2-1}}$ **8.** $\sum n\,e^{-n^2}$ **9.** $\sum_{n=3}^{\infty} \dfrac{1}{n \log n\,(\log \log n)^p}$, $p > 0$.

Answers

1. Divergent **2.** Convergent **3.** Divergent **4.** Convergent
5. Divergent **6.** Convergent **7.** Convergent **8.** Convergent
9. Convergent for $p > 1$, divergent for $p \le 1$.

Article 15. Leibnitz's Test on Alternating Series

Statement. *The alternating series* $\Sigma(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ $(u_n > 0\ \forall\ n)$ *converges if*

(*i*) $u_n > u_{n+1}\ \ \forall\ n$ *and* (*ii*) $\underset{n\to\infty}{\text{Lt}}\ u_n = 0.$

Proof. Let S_n denote the nth partial sum of the series $\Sigma\,(-1)^{n-1} \,.\, u_n$.

$$\begin{aligned}
S_{2n} &= u_1 - u_2 + u_3 - u_4 + u_5 - \ldots\ldots - u_{2n-2} + u_{2n-1} - u_{2n}\\
&= u_1 - (u_2 - u_3) - (u_4 - u_5) \ldots\ldots - (u_{2n-2} - u_{2n-1}) - u_{2n}\\
&= u_1 - [(u_2 - u_3) + (u_4 - u_5) + \ldots\ldots + (u_{2n-2} - u_{2n-1}) + u_{2n}]\\
&< u_1 \qquad [\because\ \ u_n > u_{n+1} \text{ and } u_n > 0 \text{ for all } n]
\end{aligned}$$

$\Rightarrow$ The sequence $\{S_{2n}\}$ is bounded above.

Also $S_{2n+2} = S_{2n} + u_{2n+1} - u_{2n+2}$

$\Rightarrow$ $S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} > 0$ for all n $\Rightarrow$ $S_{2n+2} > S_{2n}$

$\Rightarrow$ The sequence $\{S_{2n}\}$ is monotonically increasing.

Since every monotonically increasing sequence which is bounded above converges, therefore, the sequence $\{S_{2n}\}$ converges. Let it converge to S, then $\underset{n\to\infty}{\text{Lt}}\ S_{2n} = S$

$$\begin{aligned}
\text{Now}\quad \underset{n\to\infty}{\text{Lt}}\ S_{2n+1} &= \underset{n\to\infty}{\text{Lt}}\ (S_{2n} + u_{2n+1}) = \underset{n\to\infty}{\text{Lt}}\ S_{2n} + \underset{n\to\infty}{\text{Lt}}\ u_{2n+1}\\
&= S + 0 \qquad [\because\ \ \underset{n\to\infty}{\text{Lt}}\ u_n = 0]\\
&= S
\end{aligned}$$

$\therefore$ The sequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge to the same real number S.

$\therefore$ Given $\varepsilon > 0$ there exists positive integers m_1 and m_2 such that

$$|\,S_{2n} - S\,| < \varepsilon \quad \forall \quad 2n > m_1 \quad \text{and} \quad |\,S_{2n+1} - S\,| < \varepsilon \quad \forall \quad 2n+1 > m_2$$

Let $m = \max.\{m_1, m_2\}$, then

$$|\,S_{2n} - S\,| < \varepsilon \quad \forall \quad n > m \quad \text{and} \quad |\,S_{2n+1} - S\,| < \varepsilon \quad \forall \quad n > m$$

$\Rightarrow$ $|\,S_n - S\,| < \varepsilon \quad \forall \quad n > m$

$\therefore$ The sequence $\{S_n\}$ converges to S.

Hence the given series is convergent.

Note. The alternating series will not be convergent if any one of the two conditions is not satisfied.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of the series:* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$

Sol. It is an alternating series.

(*i*) $u_n = \frac{1}{n}, \quad u_{n+1} = \frac{1}{n+1}$

$$\because \qquad \frac{1}{n} > \frac{1}{n+1} \quad \forall\ n \quad \therefore \quad u_n > u_{n+1} \quad \forall\ n$$

(*ii*) $\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n} = 0$

$\therefore$ Both the conditions of Leibnitz's Test are satisfied.

Hence the given series is convergent.

Example 2. *Examine the convergence of the series:*

(*a*) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots\ldots$

(*b*) $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots\ldots$

Sol. (*a*) It is an alternating series

(*i*) $u_n = \frac{n+1}{n}, \quad u_{n+1} = \frac{n+2}{n+1}$

$$u_n - u_{n+1} = \frac{n+1}{n} - \frac{n+2}{n+1} = \frac{(n+1)^2 - n(n+2)}{n(n+1)} = \frac{1}{n(n+1)} > 0 \quad \forall\ n$$

$$\Rightarrow \qquad u_n > u_{n+1} \quad \forall\ n$$

(*ii*) $\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n+1}{n} = \underset{n\to\infty}{\text{Lt}}\left(1 + \frac{1}{n}\right) = 1 \neq 0.$

Since the second condition of Leibnitz's Test is not satisfied, the series is not convergent.

(*b*) It is an alternating series.

(*i*) $u_n = \frac{1}{(n+1)^3}\ [1 + 2 + 3 + \ldots\ldots + n] = \frac{1}{(n+1)^3} \cdot \frac{n(n+1)}{2} = \frac{1}{2} \cdot \frac{n}{(n+1)^2}$

$$u_{n+1} = \frac{1}{2} \cdot \frac{n+1}{(n+2)^2}$$

$$u_n - u_{n+1} = \frac{1}{2}\left[\frac{n}{(n+1)^2} - \frac{n+1}{(n+2)^2}\right] = \frac{1}{2}\ \frac{n(n+2)^2 - (n+1)^3}{(n+1)^2(n+2)^2}$$

$$= \frac{1}{2} \cdot \frac{n^2 + n - 1}{(n+1)^2(n+2)^2} > 0 \quad \forall\ n$$

$$\Rightarrow \qquad u_n > u_{n+1} \quad \forall\ n$$

$$(ii)\ \underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{2(n+1)^2} = \underset{n\to\infty}{\text{Lt}}\ \frac{\frac{1}{n}}{2\left(1+\frac{1}{n}\right)^2} = 0.$$

Since both the conditions of Leibnitz's Test are satisfied, the given series is convergent.

Example 3. *Test the convergence of the following series:* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,.\,n}{2n-1}$.

Sol. The given series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,.\,n}{2n-1} = \frac{1}{1} - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \ldots\ldots$

It is an alternating series

(i) $\quad u_n = \frac{n}{2n-1}, \quad u_{n+1} = \frac{n+1}{2n+1}$

$$u_n - u_{n+1} = \frac{1}{4n^2-1} > 0 \quad \forall \quad n \quad \Rightarrow \quad u_n > u_{n+1} \quad \forall \quad n$$

$$(ii)\ \underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{2n-1} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \neq 0.$$

Here the second condition of Leibnitz's Test is not satisfied. Hence the given series is not convergent.

9.16. ABSOLUTE CONVERGENCE OF A SERIES

Def. *If a convergent series whose terms are not all positive, remains convergent when all its terms are made positive, then it is called an* **absolutely convergent series**, *i.e.,*

The series Σu_n is said to be absolutely convergent if $\Sigma\ |\ u_n\ |$ is a convergent series.

A series is said to be conditionally convergent if it is convergent but does not converge absolutely.

Example 4. *Test whether the following series are absolutely convergent or conditionally convergent ?*

(a) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ $\qquad (b)$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

Sol. (a) The series is $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$

(i) In this alternating series, each term is less than the preceeding term numerically.

(ii) Moreover $u_n = \frac{1}{n^2}$ which $\to 0$ as $n \to \infty$.

Hence the series satisfies both the conditions of the test on alternating series and so the given series *converges*.

Again when all the term of the series are made positive, the series becomes

$$\sum\ |\ u_n\ | = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \Sigma\,\frac{1}{n^2}$$

which we know is a convergent series. $\qquad (\because$ here $p = 2 > 1)$

Thus the given series converges absolutely.

(*b*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \Sigma (-1)^{n-1} u_n$ (say)

Putting $n = 1, 2, 3, ...$, the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

The series is clearly an alternating series.

(*i*) The terms go on decreasing numerically and

(*ii*) $\underset{n \to \infty}{\text{Lt}}\, u_n = \underset{n \to \infty}{\text{Lt}}\, \frac{1}{2n-1} = 0$

$\therefore$ By Leibnitz's Test, the series *converges*.

But when all terms are made positive, the series becomes,

$$\sum |u_n| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ...$$

Here $u_n = \frac{1}{2n-1}$. Take $v_n = \frac{1}{n}$.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\, \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}}\, \frac{n}{2n-1} = \underset{n \to \infty}{\text{Lt}} \left[\frac{1}{2 - \frac{1}{n}} \right] = \frac{1}{2} = \text{finite} \neq 0$$

Hence by comparison test series Σu_n and Σv_n behave alike.

But $\Sigma v_n = \Sigma \frac{1}{n}$ is a divergent series ($\because$ here $p = 1$), $\therefore$ Σu_n also diverges.

Hence the given series converges, and the series of absolute terms diverges, therefore the given series converges conditionally.

TEST YOUR KNOWLEDGE

Examine the convergence of the following series :

1. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + ...$ to ∞

2. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...$ to ∞

3. $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + ...$ to ∞

4. $\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + ...$ to ∞ $(a > 0, b > 0)$

5. $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + ...$ to ∞

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-2}$

8. $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{5^n}$

9. $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1} + ...$ to ∞

10. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ **11.** $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$

Answers

1. Convergent	**2.** Convergent	**3.** Convergent	**4.** Convergent
5. Convergent	**6.** Convergent	**7.** Convergent	**8.** Convergent
9. Convergent	**10.** Convergent	**11.** Convergent.	

9.17. EVERY ABSOLUTELY CONVERGENT SERIES IS CONVERGENT

Let $\sum_{n=1}^{\infty} u_n$ be an absolutely convergent series. Then $\sum_{n=1}^{\infty} |u_n|$ is convergent.

By Cauchy's general principle of convergence, given $\varepsilon > 0$, $\exists$ a positive integer m such that

$$|u_{m+1}| + |u_{m+2}| + \ldots\ldots + |u_n| < \varepsilon \quad \forall \quad n > m$$

or
$$|u_{m+1}| + |u_{m+1}| + \ldots\ldots + |u_n| < \varepsilon \quad \forall \quad n > m \qquad \ldots(1)$$

Now, by triangle inequality, we have

$$|u_{m+1} + u_{m+2} + \ldots\ldots + u_n| \le |u_{m+1}| + |u_{m+2}| + \ldots\ldots + |u_n|$$
$$< \varepsilon \quad \forall \quad n > m \qquad \text{[Using (1)]}$$

$\therefore$ By Cauchy's general principle of convergence, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Hence $\sum |u_n|$ is convergent $\Rightarrow$ Σu_n is convergent.

Note 1. Absolute convergence $\Rightarrow$ Convergence, but convergence need not imply absolute convergence *i.e.*, the converse of above theorem need not be true.

For example, consider the series $\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$

It is convergent. [See Example 1 with Leibnitz's Test]

But the series $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ is divergent.

Note 2. The divergence of $\Sigma |u_n|$ does not imply the divergence of Σu_n.

For example, $\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n}$ is divergent whereas $\sum \frac{(-1)^{n-1}}{n}$ is convergent.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that the series* $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots\ldots$ *converges absolutely.*

Sol. The given series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n^3}$

Since $\quad | u_n | = \dfrac{|\sin nx|}{n^3} \le \dfrac{1}{n^3} \ \forall\, n$ and $\sum \dfrac{1}{n^3}$ converges.

$\therefore$ By comparison test, the series $\sum | u_n |$ converges.

$\Rightarrow$ The given series converges absolutely.

Example 2. *Find the interval of convergence of the series* $x - \dfrac{x^2}{\sqrt{2}} + \dfrac{x^3}{\sqrt{3}} - \dfrac{x^4}{\sqrt{4}} + \ldots\ldots$

Sol. The given series is $\displaystyle\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{\sqrt{n}}$

Here $\quad | u_n | = \dfrac{|x^n|}{\sqrt{n}} = \dfrac{|x|^n}{\sqrt{n}}$ and $| u_{n+1} | = \dfrac{|x|^{n+1}}{\sqrt{n+1}}$

$$\therefore \quad \frac{|u_n|}{|u_{n+1}|} = \sqrt{\frac{n+1}{n}} \cdot \frac{1}{|x|} = \sqrt{1+\frac{1}{n}} \cdot \frac{1}{|x|}$$

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \frac{1}{|x|}$$

$\therefore$ By ratio test, the series $\sum | u_n |$ is convergent if $\dfrac{1}{|x|} > 1$ *i.e.,* if $| x | < 1$ *i.e.,* if $-1 < x < 1$ and divergent if $\dfrac{1}{|x|} < 1$ *i.e.,* if $| x | > 1$ *i.e.,* if $x > 1$ or $x < -1$.

Ratio test fails when $| x | = 1$, *i.e.,* when $x = 1$ or -1.

When $x = 1$, the series becomes $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \ldots\ldots$

which is an alternating series and is convergent.

When $x = -1$, the series becomes $-\left(1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \ldots\ldots\right) = -\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

which is divergent by p-series test.

Hence the given series converges for $-1 < x \le 1$.

TEST YOUR KNOWLEDGE

1. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos^2 nx}{n\sqrt{n}}$ converges absolutely.
2. For what values of x are the following series convergent ?

   (*i*) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\ldots$ (*ii*) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots$

(*iii*) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$ (*iv*) $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots\ldots$

Answers

2. (*i*) $-1 < x \le 1$ (*ii*) all x (*iii*) $-1 < x \le 1$ (*iv*) $-1 < x \le 1$.

9.18. UNIFORM CONVERGENCE OF SERIES OF FUNCTIONS

Let $u_n(x)$ be a real valued function defined on an interval I and for each $n \in$ N. Then $u_1(x) + u_2(x) + u_3(x) + \ldots.. = \sum_{n=1}^{\infty} u_n(x)$ is called an infinite series of functions each of which is defined on the interval I.

Let $S_n(x) = u_1(x) + u_2(x) + \ldots\ldots. + u_n(x)$ be the nth partial sum of $\sum u_n(x)$.

Let $\alpha \in$ I and $\lim_{n\to\infty} S_n(\alpha) = S(\alpha)$ then the series $\sum u_n(x)$ is said to converge to $S(\alpha)$ at $x = \alpha$.

Thus, given $\varepsilon > 0$, there exists a positive integer m such that

$$| S_n(\alpha) - S(\alpha) | < \varepsilon \quad \forall \quad n > m.$$

The positive integer m depends on $\alpha \in$ I and the given value of $\varepsilon > 0$, *i.e.*, $m = m(\alpha, \varepsilon)$. It is not always possible to find an m which works for each $x \in$ I. If we can find an m which depends only on ε and not on $x \in$ I, we say $\sum u_n(x)$ is uniformly convergent.

Definition. A series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to a function S(x) if for a given $\varepsilon > 0$, there exists a positive integer m depending only on ε and independent of x such that for every $x \in$ I,

$$| S_n(x) - S(x) | < \varepsilon \quad \forall \quad n > m$$

Note. The method of testing the uniform convergence of a series $\sum u_n(x)$, by definition, involves finding $S_n(x)$ which is not always easy. The following test avoids $S_n(x)$.

Article 16. Weierstrass's M-Test

Statement. *A series* $\sum_{n=1}^{\infty} u_n(x)$ *of functions converges uniformly and absolutely on an interval I if there exists a convergent series* $\sum_{n=1}^{\infty} M_n$ *of positive constants such that*

$$| u_n(x) | \le M_n \quad \forall \quad n \in N \text{ and } \quad \forall \quad x \in I.$$

Proof. Since $\sum_{n=1}^{\infty} M_n$ is convergent, by Cauchy's general principle of convergence, for each $\varepsilon > 0$, there exists a positive integer m such that

$$| M_{m+1} + M_{m+2} + \ldots\ldots + M_n | < \varepsilon \quad \forall \quad n > m$$

or $$M_{m+1} + M_{m+2} + \ldots\ldots + M_n < \varepsilon \;\forall\quad n > m \qquad \ldots(1)$$

Now, for all $x \in I$, $| u_n(x) | \leq M_n$ ...(2)

$$\begin{aligned} \therefore \quad & | u_{m+1}(x) + u_{m+2}(x) + \ldots\ldots + u_n(x) | \\ & \leq | u_{m+1}(x) | + | u_{m+2}(x) | + \ldots\ldots + | u_n(x) | \\ & \leq M_{m+1} + M_{m+2} + \ldots\ldots + M_n && \text{[by (2)]} \\ & < \varepsilon \quad \forall \quad n > m && \text{[by (1)]} \end{aligned}$$

$\Rightarrow$ By Cauchy's criterion, the series $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on I.

Also, $$| u_{m+1}(x) | + | u_{m+2}(x) | + \ldots\ldots + | u_n(x) | < \varepsilon \quad \forall \quad n > m$$

$\Rightarrow$ $$|\, | u_{m+1}(x) | + | u_{m+2}(x) | + \ldots\ldots + | u_n(x) |\, | < \varepsilon \quad \forall \quad n > m$$

$\Rightarrow$ The series $\sum_{n=1}^{\infty} | u_n(x) |$ is uniformly convergent on I.

Hence the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly and absolutely on I.

Example. *Show that the following series are uniformly convergent :*

(*i*) $\sum_{n=1}^{\infty} \frac{\sin(x^2 + nx)}{n(n+2)}$ *for all real x.* (*ii*) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$ *for all real x and p > 1.*

(*iii*) $\sum_{n=1}^{\infty} \frac{1}{n^p + n^q x^2}$ *for all real x and p > 1.*

Sol. (*i*) Here $u_n(x) = \dfrac{\sin(x^2 + nx)}{n(n+2)}$

$$\therefore \quad | u_n(x) | = \left| \frac{\sin(x^2 + nx)}{n(n+2)} \right| = \frac{|\sin(x^2 + nx)|}{n(n+2)} \leq \frac{1}{n(n+2)} < \frac{1}{n^2} \; (= M_n) \quad \forall \quad x \in R$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by M-test, the given series is uniformly convergent for all real x.

(*ii*) Here $$u_n(x) = \frac{\cos nx}{n^p}$$

$$\therefore \quad | u_n(x) | = \left| \frac{\cos nx}{n^p} \right| = \frac{|\cos nx|}{n^p} \leq \frac{1}{n^p} \; (= M_n) \quad \forall \quad x \in R.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$, therefore, by M-test, the given series is uniformly convergent for all real x and $p > 1$.

(*iii*) Here $u_n(x) = \dfrac{1}{n^p + n^q x^2}$

Since $x^2 \geq 0$ for all real x

$\therefore \quad n^q x^2 \geq 0 \Rightarrow n^p + n^q x^2 \geq n^p$

$$\Rightarrow \quad \frac{1}{n^p + n^q x^2} \leq \frac{1}{n^p}$$

$$\therefore \quad |u_n(x)| = \frac{1}{n^p + n^q x^2} \leq \frac{1}{n^p} \; (= \mathrm{M}_n) \quad \forall \quad x \in \mathrm{R}.$$

Since $\sum_{n=1}^{\infty} \mathrm{M}_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$.

therefore, by M-test, the given series is uniformly convergent for all real x and $p > 1$.

TEST YOUR KNOWLEDGE

Test for uniform convergence the series :

1. $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots\ldots$

2. $\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} - \frac{\sin 4x}{4\sqrt{4}} + \ldots\ldots$

3. $\sum_{n=1}^{\infty} \frac{\cos(x^2 + n^2 x)}{n(n^2+2)}$

4. $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^3 x^2}$

5. Show that if $0 < r < 1$, then each of the following series is uniformly convergent on R :

(*i*) $\sum_{n=1}^{\infty} r^n \cos nx$ (*ii*) $\sum_{n=1}^{\infty} r^n \sin nx$

(*iii*) $\sum_{n=1}^{\infty} r^n \cos n^2 x$ (*iv*) $\sum_{n=1}^{\infty} r^n \sin a^n x$.

Answers

1. Uniformly convergent for all real x **2.** Uniformly convergent for all real x

3. Uniformly convergent for all real x **4.** Uniformly convergent for all real x.