66. Let $\delta = 0.2$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |xyz - 0| = |xyz| = |x||y||z| < (0.2)^3 = 0.008 = \epsilon$.

67. Let $\delta = 0.005$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = \left|\frac{x+y+z}{x^2+y^2+z^2+1} - 0\right| = \left|\frac{x+y+z}{x^2+y^2+z^2+1}\right| \le |x+y+z| \le |x| + |y| + |z| < 0.005 + 0.005 + 0.005 = 0.015 = \epsilon$.

68. Let $\delta = \tan^{-1}(0.1)$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |\tan^2 x + \tan^2 y + \tan^2 z| \le |\tan^2 x| + |\tan^2 y| + |\tan^2 z| = \tan^2 x + \tan^2 y + \tan^2 z < \tan^2\delta + \tan^2\delta + \tan^2\delta = 0.01 + 0.01 + 0.01 = 0.03 = \epsilon$.

69. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = \lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} (x+y+z) = x_0 + y_0 + z_0 = f(x_0,y_0,z_0) \Rightarrow$ f is continuous at every $(x_0,y_0,z_0)$

70. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = \lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} (x^2+y^2+z^2) = x_0^2 + y_0^2 + z_0^2 = f(x_0,y_0,z_0) \Rightarrow$ f is continuous at every point $(x_0,y_0,z_0)$

## 12.3 PARTIAL DERIVATIVES

1. $\frac{\partial f}{\partial x} = 4x$, $\frac{\partial f}{\partial y} = -3$

2. $\frac{\partial f}{\partial x} = 2x - y$, $\frac{\partial f}{\partial y} = -x + 2y$

3. $\frac{\partial f}{\partial x} = 2x(y+2)$, $\frac{\partial f}{\partial y} = x^2 - 1$

4. $\frac{\partial f}{\partial x} = 5y - 14x + 3$, $\frac{\partial f}{\partial y} = 5x - 2y - 6$

5. $\frac{\partial f}{\partial x} = 2y(xy-1)$, $\frac{\partial f}{\partial y} = 2x(xy-1)$

6. $\frac{\partial f}{\partial x} = 6(2x-3y)^2$, $\frac{\partial f}{\partial y} = -9(2x-3y)^2$

7. $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

8. $\frac{\partial f}{\partial x} = \frac{2x^2}{\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$, $\frac{\partial f}{\partial y} = \frac{1}{3\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$

9. $\frac{\partial f}{\partial x} = -\frac{1}{(x+y)^2}\cdot\frac{\partial}{\partial x}(x+y) = -\frac{1}{(x+y)^2}$, $\frac{\partial f}{\partial y} = -\frac{1}{(x+y)^2}\cdot\frac{\partial}{\partial y}(x+y) = -\frac{1}{(x+y)^2}$

10. $\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$

11. $\frac{\partial f}{\partial x} = = \frac{(xy-1)(1) - (x+y)(y)}{(xy-1)^2} = \frac{-y^2-1}{(xy-1)^2}$, $\frac{\partial f}{\partial y} = \frac{(xy-1)(1) - (x+y)(x)}{(xy-1)^2} = \frac{-x^2-1}{(xy-1)^2}$

12. $\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]} = -\frac{y}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x\left[1+\left(\frac{y}{x}\right)^2\right]} = \frac{x}{x^2+y^2}$

13. $\frac{\partial f}{\partial x} = e^{(x+y+1)} \cdot \frac{\partial}{\partial x}(x+y+1) = e^{(x+y+1)}$, $\frac{\partial f}{\partial y} = e^{(x+y+1)} \cdot \frac{\partial}{\partial y}(x+y+1) = e^{(x+y+1)}$

14. $\frac{\partial f}{\partial x} = -e^{-x}\sin(x+y) + e^{-x}\cos(x+y)$, $\frac{\partial f}{\partial y} = e^{-x}\cos(x+y)$

15. $\frac{\partial f}{\partial x} = \frac{1}{x+y} \cdot \frac{\partial}{\partial x}(x+y) = \frac{1}{x+y}$, $\frac{\partial f}{\partial y} = \frac{1}{x+y} \cdot \frac{\partial}{\partial y}(x+y) = \frac{1}{x+y}$

16. $\frac{\partial f}{\partial x} = e^{xy} \cdot \frac{\partial}{\partial x}(xy) \cdot \ln y = ye^{xy}\ln y$, $\frac{\partial f}{\partial y} = e^{xy} \cdot \frac{\partial}{\partial y}(xy) \cdot \ln y + e^{xy} \cdot \frac{1}{y} = xe^{xy}\ln y + \frac{e^{xy}}{y}$

17. $\frac{\partial f}{\partial x} = 2\sin(x-3y) \cdot \frac{\partial}{\partial x}\sin(x-3y) = 2\sin(x-3y)\cos(x-3y) \cdot \frac{\partial}{\partial x}(x-3y) = 2\sin(x-3y)\cos(x-3y)$,
$\frac{\partial f}{\partial y} = 2\sin(x-3y) \cdot \frac{\partial}{\partial y}\sin(x-3y) = 2\sin(x-3y)\cos(x-3y) \cdot \frac{\partial}{\partial y}(x-3y) = -6\sin(x-3y)\cos(x-3y)$

18. $\frac{\partial f}{\partial x} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial x}\cos(3x-y^2) = -2\cos(3x-y^2)\sin(3x-y^2) \cdot \frac{\partial}{\partial x}(3x-y^2)$
$= -6\cos(3x-y^2)\sin(3x-y^2)$,
$\frac{\partial f}{\partial y} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial y}\cos(3x-y^2) = -2\cos(3x-y^2)\sin(3x-y^2) \cdot \frac{\partial}{\partial y}(3x-y^2)$
$= 4y\cos(3x-y^2)\sin(3x-y^2)$

19. $\frac{\partial f}{\partial x} = yx^{y-1}$, $\frac{\partial f}{\partial y} = x^y \ln x$

20. $f(x,y) = \frac{\ln x}{\ln y} \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{x\ln y}$ and $\frac{\partial f}{\partial y} = \frac{-\ln x}{y(\ln y)^2}$

21. $\frac{\partial f}{\partial x} = -g(x)$, $\frac{\partial f}{\partial y} = g(y)$

22. $f(x,y) = \sum_{n=0}^{\infty} (xy)^n$, $|xy| < 1 \Rightarrow f(x,y) = \frac{1}{1-xy} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial x}(1-xy) = \frac{y}{(1-xy)^2}$ and
$\frac{\partial f}{\partial y} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial y}(1-xy) = \frac{x}{(1-xy)^2}$

23. $f_x = 1 + y^2$, $f_y = 2xy$, $f_z = -4z$

24. $f_x = y + z$, $f_y = x + z$, $f_z = y + x$

25. $f_x = 1$, $f_y = -\frac{y}{\sqrt{y^2+z^2}}$, $f_z = -\frac{z}{\sqrt{y^2+z^2}}$

26. $f_x = -x(x^2+y^2+z^2)^{-3/2}$, $f_y = -y(x^2+y^2+z^2)^{-3/2}$, $f_z = -z(x^2+y^2+z^2)^{-3/2}$

27. $f_x = \frac{yz}{\sqrt{1-x^2y^2z^2}}$, $f_y = \frac{xz}{\sqrt{1-x^2y^2z^2}}$, $f_z = \frac{xy}{\sqrt{1-x^2y^2z^2}}$

28. $f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}$, $f_y = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}$, $f_z = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}$

29. $f_x = \frac{1}{x+2y+3z}$, $f_y = \frac{2}{x+2y+3z}$, $f_z = \frac{3}{x+2y+3z}$

30. $f_x = yz \cdot \frac{1}{xy} \cdot \frac{\partial}{\partial x}(xy) = \frac{(yz)(y)}{xy} = \frac{yz}{x}$, $f_y = z \ln(xy) + yz \cdot \frac{\partial}{\partial y} \ln(xy) = z \ln(xy) + \frac{yz}{xy} \cdot \frac{\partial}{\partial y}(xy) = z \ln(xy) + z$,
$f_z = y \ln(xy) + yz \cdot \frac{\partial}{\partial z} \ln(xy) = y \ln(xy)$

31. $f_x = -2xe^{-\left(x^2+y^2+z^2\right)}$, $f_y = -2ye^{-\left(x^2+y^2+z^2\right)}$, $f_z = -2ze^{-\left(x^2+y^2+z^2\right)}$

32. $f_x = -yze^{-xyz}$, $f_y = -xze^{-xyz}$, $f_z = -xye^{-xyz}$

33. $f_x = \text{sech}^2(x + 2y + 3z)$, $f_y = 2\ \text{sech}^2(x + 2y + 3z)$, $f_z = 3\ \text{sech}^2(x + 2y + 3z)$

34. $f_x = y \cosh\left(xy - z^2\right)$, $f_y = x \cosh\left(xy - z^2\right)$, $f_z = -2z \cosh\left(xy - z^2\right)$

35. $\frac{\partial f}{\partial t} = -2\pi \sin(2\pi t - \alpha)$, $\frac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)$

36. $\frac{\partial g}{\partial u} = v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial u}\left(\frac{2u}{v}\right) = 2ve^{(2u/v)}$, $\frac{\partial g}{\partial v} = 2ve^{(2u/v)} + v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial v}\left(\frac{2u}{v}\right) = 2ve^{(2u/v)} - 2ue^{(2u/v)}$

37. $\frac{\partial h}{\partial \rho} = \sin\phi \cos\theta$, $\frac{\partial h}{\partial \phi} = \rho \cos\phi \cos\theta$, $\frac{\partial h}{\partial \theta} = -\rho \sin\phi \sin\theta$

38. $\frac{\partial g}{\partial r} = 1 - \cos\theta$, $\frac{\partial g}{\partial \theta} = r \sin\theta$, $\frac{\partial g}{\partial z} = -1$

39. $W_P = V$, $W_V = P + \frac{\delta v^2}{2g}$, $W_\delta = \frac{Vv^2}{2g}$, $W_v = \frac{2V\delta v}{2g} = \frac{V\delta v}{g}$, $W_g = -\frac{V\delta v^2}{2g^2}$

40. $\frac{\partial A}{\partial c} = m$, $\frac{\partial A}{\partial h} = \frac{q}{2}$, $\frac{\partial A}{\partial k} = \frac{m}{q}$, $\frac{\partial A}{\partial m} = \frac{k}{q} + c$, $\frac{\partial A}{\partial q} = -\frac{km}{q^2} + \frac{h}{2}$

41. $\frac{\partial f}{\partial x} = 1 + y$, $\frac{\partial f}{\partial y} = 1 + x$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$

42. $\frac{\partial f}{\partial x} = y \cos xy$, $\frac{\partial f}{\partial y} = x \cos xy$, $\frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy$, $\frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy \sin xy$

43. $\frac{\partial g}{\partial x} = 2xy + y \cos x$, $\frac{\partial g}{\partial y} = x^2 - \sin y + \sin x$, $\frac{\partial^2 g}{\partial x^2} = 2y - y \sin x$, $\frac{\partial^2 g}{\partial y^2} = -\cos y$, $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x$

44. $\frac{\partial h}{\partial x} = e^y$, $\frac{\partial h}{\partial y} = xe^y + 1$, $\frac{\partial^2 h}{\partial x^2} = 0$, $\frac{\partial^2 h}{\partial y^2} = xe^y$, $\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y} = e^y$

45. $\frac{\partial r}{\partial x} = \frac{1}{x+y}$, $\frac{\partial r}{\partial y} = \frac{1}{x+y}$, $\frac{\partial^2 r}{\partial x^2} = \frac{-1}{(x+y)^2}$, $\frac{\partial^2 r}{\partial y^2} = \frac{-1}{(x+y)^2}$, $\frac{\partial^2 r}{\partial y \partial x} = \frac{\partial^2 r}{\partial x \partial y} = \frac{-1}{(x+y)^2}$

46. $\frac{\partial s}{\partial x} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \left(-\frac{y}{x^2}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{-y}{x^2+y^2}$, $\frac{\partial s}{\partial y} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{x}{x^2+y^2}$,

$\frac{\partial^2 s}{\partial x^2} = \frac{y(2x)}{\left(x^2+y^2\right)^2} = \frac{2xy}{\left(x^2+y^2\right)^2}$, $\frac{\partial^2 s}{\partial y^2} = \frac{-x(2y)}{\left(x^2+y^2\right)^2} = -\frac{2xy}{\left(x^2+y^2\right)^2}$,

$\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y} = \frac{\left(x^2+y^2\right)(-1) + y(2y)}{\left(x^2+y^2\right)^2} = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}$

47. $\frac{\partial w}{\partial x} = \frac{2}{2x+3y}$, $\frac{\partial w}{\partial y} = \frac{3}{2x+3y}$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{-6}{(2x+3y)^2}$, and $\frac{\partial^2 w}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$

48. $\frac{\partial w}{\partial x} = e^x + \ln y + \frac{y}{x}$, $\frac{\partial w}{\partial y} = \frac{x}{y} + \ln x$, $\frac{\partial^2 w}{\partial y \partial x} = \ = \frac{1}{y} + \frac{1}{x}$, and $\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{y} + \frac{1}{x}$

49. $\frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4$, $\frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3$, $\frac{\partial^2 w}{\partial y \partial x} = 2y + 6xy^2 + 12x^2y^3$, and $\frac{\partial^2 w}{\partial x \partial y} = 2y + 6xy^2 + 12x^2y^3$

50. $\frac{\partial w}{\partial x} = \sin y + y \cos x + y$, $\frac{\partial w}{\partial y} = x \cos y + \sin x + x$, $\frac{\partial^2 w}{\partial y \partial x} = \cos y + \cos x + 1$, and $\frac{\partial^2 w}{\partial x \partial y} = \cos y + \cos x + 1$

51. (a) x first (b) y first (c) x first (d) x first (e) y first (f) y first

52. (a) y first three times (b) y first three times (c) y first twice (d) x first twice

53. $f_x(1,2) = \lim_{h \to 0} \frac{f(1+h,2) - f(1,2)}{h} = \lim_{h \to 0} \frac{[1 - (1+h) + 2 - 6(1+h)^2] - (2-6)}{h} = \lim_{h \to 0} \frac{-h - 6(1 + 2h + h^2) + 6}{h}$
$= \lim_{h \to 0} \frac{-13h - 6h^2}{h} = \lim_{h \to 0} (-13 - 6h) = -13$,
$f_y(1,2) = \lim_{h \to 0} \frac{f(1,2+h) - f(1,2)}{h} = \lim_{h \to 0} \frac{[1 - 1 + (2+h) - 3(2+h)] - (2-6)}{h} = \lim_{h \to 0} \frac{(2 - 6 - 2h) - (2-6)}{h}$
$= \lim_{h \to 0} (-2) = -2$

54. $f_x(-2,1) = \lim_{h \to 0} \frac{f(-2+h,1) - f(-2,1)}{h} = \lim_{h \to 0} \frac{[4 + 2(-2+h) - 3 - (-2+h)] - (-3+2)}{h}$
$= \lim_{h \to 0} \frac{(2h - 1 - h) + 1}{h} = \lim_{h \to 0} 1 = 1$,
$f_y(-2,1) = \lim_{h \to 0} \frac{f(-2,1+h) - f(-2,1)}{h} = \lim_{h \to 0} \frac{[4 - 4 - 3(1+h) + 2(1+h^2)] - (-3+2)}{h}$
$= \lim_{h \to 0} \frac{(-3 - 3h + 2 + 4h + 2h^2) + 1}{h} = \lim_{h \to 0} \frac{h + 2h^2}{h} = \lim_{h \to 0} (1 + 2h) = 1$

55. $f_z(x_0,y_0,z_0) = \lim_{h \to 0} \frac{f(x_0,y_0,z_0+h) - f(x_0,y_0,z_0)}{h}$;
$f_z(1,2,3) = \lim_{h \to 0} \frac{f(1,2,3+h) - f(1,2,3)}{h} = \lim_{h \to 0} \frac{2(3+h)^2 - 2(9)}{h} = \lim_{h \to 0} \frac{12h + 2h^2}{h} = \lim_{h \to 0} (12 + 2h) = 12$

56. $f_y(x_0,y_0,z_0) = \lim_{h \to 0} \frac{f(x_0,y_0+h,z_0) - f(x_0,y_0,z_0)}{h}$;
$f_y(-1,0,3) = \lim_{h \to 0} \frac{f(-1,h,3) - f(-1,0,3)}{h} = \lim_{h \to 0} \frac{(2h^2 + 9h) - 0}{h} = \lim_{h \to 0} (2h + 9) = 9$