Unit - 5.

2 Marks Questions.

Ques 1. Give an example of monotonically increasing sequence which is

a. Convergent.
b. Divergent.

Ans a. The sequence $x_n = \frac{n}{n+1}$ has the terms given by

$$\{x_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \dots\right\}$$

The sequence is monotonoically increasing as $x_1 < x_2 < x_3 < x_4 < \dots$

Also, the sequence $\frac{n}{n+1}$ is bounded below as well as above.

As $n \in N$, $\frac{n}{n+1} > 0$.

and $n < n+1 \Rightarrow \frac{n}{n+1} < 1$.

So, the sequence is bounded below as well as above.

and we know that any monotonically increasing sequence which is bounded above is convergent.

So $x_n = \frac{n}{n+1}$ is a monotonically increasing sequence which is convergent.

b. The sequence $x_n = n$ has the terms given by.

$$\{x_n\} = \{1, 2, 3, 4, 5, \ldots\}.$$

Here $x_1 < x_2 < x_3 < x_4 < \ldots$

So, this is a monotonically increasing sequence but the given sequence is not bounded above.

and we know that a monotonically increasing sequence which is not bounded above is divergent.

Ques 2 Give an example of monotonically decreasing sequence which is

a. Convergent
b. Divergent.

Ans:- a. The sequence $x_n = \frac{1}{n}$ has the terms given by

$$x_n = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

The sequence is monotonically decreasing as $x_1 < x_2 < x_3 < \ldots$

Also, the sequence $x_n = \frac{1}{n}$ is bounded below as well as bounded above.

$$0 < x_n < 1.$$

and we know that a monotonically decreasing sequence which is bounded below is convergent.

So, $x_n = \frac{1}{n}$ is a bounded below sequence and which is monotonically decreasing and hence convergent.

b. The sequence $x_n = -n$ has the terms given by

$$x_n = \{-1, -2, -3, -4, \ldots\}$$

The sequence is monotonically decreasing as $x_1 < x_2 < x_3 < \ldots$

Also the sequence is unbounded below.

and we know that monotonically decreasing sequence which is unbounded below is divergent.

Ques 3 Discuss the nature of the series $2-2+2-2+2-2+2-\ldots$

Ans

$$S_n = \begin{cases} 0 & n \text{ is even} \\ 2 & n \text{ is odd} \end{cases}$$

Here, S_n denotes sequence of partial sums.

Here, S_n does not tend to a unique limit.

Therefore, the given series is oscillating finitely.

Ques 4 Test for convergence of series $1 + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{n}{n+1} + \ldots + \infty$.

Ans Here $u_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)}$$

$$= 1 \neq 0$$

So, the given series is divergent.

Ques5 Examine the convergence of the series

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \text{------}$$

Ans Here $u_n = \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$

$$u_n = \frac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}.$$

Take $v_n = \dfrac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)} * \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

$$= \frac{1}{2} \quad \text{(which is finite)}$$

Then according to Limit Comparison test, Σu_n and Σv_n converge or diverge together.

$\sum v_n = \frac{1}{\sqrt{n}}$ is divergent $\left(p = \frac{1}{2} < 1\right)$.

so $\sum u_n$ is divergent.

6 Marks Questions.

Ques 1 Examine the convergence of the following series

$$\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \text{------}$$

Ans

$$\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \text{-------}$$

$$\Rightarrow \left(\frac{3}{5} + \frac{3}{5^3} + \frac{3}{5^5} + \text{----}\right) + \left(\frac{4}{5^2} + \frac{4}{5^4} + \text{----}\right).$$

$$\Rightarrow \sum u_n + \sum v_n$$

$\sum u_n$ is a geometric series with common ratio less than 1.

$\sum v_n$ is a geometric series with common ratio less than 1.

so, $\sum u_n$ is convergent and also $\sum v_n$ is convergent.

so, $\sum u_n + \sum v_n$ is convergent.

Ques 2 State D' Alembert's Ratio Test for convergence of an infinite series. Using D' Alembert's ratio test, test for convergence of series whose n^{th} term is $\frac{n^2}{2^n}$.

Ans D' Alembert's Ratio Test for Convergence :-

If $\sum u_n$ is a positive term series, and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then

a) $\sum u_n$ is convergent if $l<1$.

b) $\sum u_n$ is divergent if $l>1$.

c) If $l=1$, the test fails.

$$u_n = \frac{n^2}{2^n}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^n}{2^{n+1} \cdot n^2}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{2^n}{2^{n+1}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{2}$$

$$= \frac{1}{2} < 1.$$

Hence, $\sum u_n$ is convergent.

Ques3 Test the following series for convergence

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots\cdots$$

Ans $u_n = \frac{(n+1)}{n^p}$

$$= \frac{n\left(1+\frac{1}{n}\right)}{n^p}$$

$$= \frac{\left(1+\frac{1}{n}\right)}{n^{p-1}}$$

Take $v_n = \frac{1}{n^{p-1}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{n^{p-1}} \frac{n^{p-1}}{* 1}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)$$

$= 1$ (which is finite).

So, Σu_n and Σv_n converges or diverges together.

Σv_n is convergent when $p-1>1$ $(p>2)$ and divergent when $p-1 \le 1$ $(p \le 2)$.

So, Σu_n is convergent when $p>2$ and divergent when $p \le 2$.

Ques 4 Examine the following series for convergence

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} x^n.$$

Ans $u_n = \frac{(n+1)^n}{n^{n+1}} x^n$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{(n+1)^{n/n}}{n^{n+1/n}}\right) \cdot x^{n/n}$$

$$= \lim_{n\to\infty} \frac{n+1}{n \cdot n^{1/n}} \cdot x$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \cdot \frac{1}{n^{1/n}} \cdot x$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \frac{1}{n^{1/n}} \cdot x \qquad \lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{1}$$

$= x.$

$\sum u_n$ is convergent when $x<1$ and divergent when $x>1$.

At $x=1$ $\quad u_n = \dfrac{(n+1)^n}{n^{n+1}}$

$= \left(1+\dfrac{1}{n}\right)^n \cdot \dfrac{1}{n}$

Take $v_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{n} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$= e$ (a finite number).

So, by comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

$\sum v_n$ is divergent ($p=1$).

$\Rightarrow \sum u_n$ is divergent for $x \geq 1$ and convergent for $x<1$.

10 Marks Questions.

Ques 1 Discuss the convergence of the infinite series

$$\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n \qquad (x>0).$$

Ans

$$u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$$

$$u_{n+1} = \frac{\sqrt{(n+1)}}{\sqrt{(n+1)^2+1}} x^{n+1}.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} \cdot \frac{x^{n+1}\sqrt{n^2+1}}{\sqrt{n}\cdot x^n}$$

$$= \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot x \cdot \frac{\cancel{n}\sqrt{1+1/n^2}}{\cancel{n}\sqrt{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot x * \frac{\sqrt{1+\frac{1}{n^2}}}{\sqrt{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}}}$$

$$= x.$$

$\Rightarrow \Sigma u_n$ is convergent if $x<1$ and divergent if $x>1$

At $x=1$. $\quad u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}}$

Take $\quad v_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n^2+1}} * \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1 \text{ (a finite number).}$$

So, Σu_n and Σv_n converge and diverge together (by limit comparison test)

Σv_n is divergent ($p = \frac{1}{2}$; by p test)

Σu_n is divergent at $x=1$.

$\Rightarrow \Sigma u_n$ is convergent if $x<1$ and divergent when $x \geq 1$.

Ques 2 Test the convergence of positive term series.

$$1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\text{--------}$$

Ans Here $u_n=\dfrac{(\alpha+1)(2\alpha+1)\text{----}(n\alpha+1)}{(\beta+1)(2\beta+1)\text{-----}(n\beta+1)}$.

$$u_{n+1}=\frac{(\alpha+1)(2\alpha+1)\text{----}(n\alpha+1)((n+1)\alpha+1)}{(\beta+1)(2\beta+1)\text{---}(n\beta+1)((n+1)\beta+1)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(\alpha+1)(2\alpha+1)\text{---}(n\alpha+1)((n+1)\alpha+1)}{(\beta+1)(2\beta+1)\text{---}(n\beta+1)((n+1)\beta+1)}*\frac{(\beta+1)(2\beta+1)\text{----}(n\beta+1)}{(\alpha+1)(2\alpha+1)\text{----}(n\alpha+1)}$$

$$=\lim_{n\to\infty}\frac{(n+1)\alpha+1}{(n+1)\beta+1}$$

$$=\lim_{n\to\infty}\frac{\cancel{n}\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}}{\cancel{n}\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}$$

$$=\frac{\alpha}{\beta}.$$

Σu_n is convergent when $\frac{\alpha}{\beta}<1$ $(\alpha<\beta)$ and divergent when $\alpha>\beta$.

When $\alpha=\beta$

$$u_n=1+1+1+\text{----}$$
$$=n$$

$$\lim_{n\to\infty}S_n=\infty$$

$\Rightarrow \Sigma u_n$ is convergent when $\alpha<\beta$ and divergent when $\alpha\geq\beta$.

Ques 3 Discuss the convergence of the series

$$\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \text{------}$$

Ans When we neglect first term, we have

$$u_n = \frac{1\cdot 3\cdot 5\text{-----}(2n-1)}{2\cdot 4\cdot 6\text{-----}2n}\cdot\frac{x^{2n+1}}{(2n+1)}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5\text{----}(2n-1)(2n+1)}{2\cdot 4\cdot 6\text{-----}(2n)(2n+2)}\;\frac{x^{2n+3}}{(2n+3)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{1\cdot 3\cdot 5\text{----}(2n-1)(2n+1)}{2\cdot 4\cdot 6\text{-----}(2n)(2n+2)}\;\frac{x^{2n+3}}{(2n+3)}\cdot\frac{2\cdot 4\cdot 6\text{----}2n}{1\cdot 3\cdot 5\text{---}(2n-1)}\;\frac{(2n+1)}{x^{2n+1}}$$

$$= \lim_{n\to\infty}\frac{2n+1}{2n+2}\;x^2\cdot\frac{(2n+1)}{(2n+3)}$$

$$= \lim_{n\to\infty}\frac{n^2}{n^2}\,\frac{(2+1/n)(2+1/n)}{(2+2/n)(2+3/n)}\,x^2$$

$$= x^2.$$

Σu_n is convergent when $x^2<1$ and divergent when $x^2>1$

At $x^2=1$, ratio test fails. So, we apply Rabbe's test.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+6+10n}{4n^2+4n+1}-1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{4n^2+10n+6-4n^2-4n-1}{4n^2+4n+1}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{6n+5}{4n^2+4n+1}\right)$$

$$= \lim_{n\to\infty}\frac{n^2\,(6+5/n)}{n^2\left(4+\frac{4}{n}+\frac{1}{n^2}\right)} = \frac{6}{4} = \frac{3}{2} > 1.$$

Σu_n is convergent by Rabbe's test.

Σu_n is convergent when $x^2 < 1$ and divergent when $x^2 > 1$.

Ques4 Test the convergence of the following series :-

a. $\Sigma \left(\frac{n}{n+1}\right)^{n^2}$.

Ans
$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{1/n}$$
$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}$$
$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^{-1}$$
$$= e^{-1}$$
$$= \frac{1}{e} < 1$$

$\Rightarrow \Sigma u_n$ is convergent.

b. $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\text{--------}$

Ans
$$u_n = \left(\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right)^{-n}$$
$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{1+n}-\frac{n+1}{n}\right]^{-1}$$
$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\cdot\left(1+\frac{1}{n}\right)-\left(1+\frac{1}{n}\right)\right]^{-1}$$
$$= (e-1)^{-1}$$
$$= \frac{1}{e-1} < 1.$$

$\Rightarrow \sum u_n$ is convergent.

Ques 5 Test the convergence of the series.

$$\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \cdots$$

Ans

$$u_n = \frac{x^n}{(2n-1)\,2n}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(2n+1)(2n+2)} \frac{2n\,(2n-1)}{x^n}$$

$$= \lim_{n\to\infty} \frac{2n\cdot 2n \quad x \left(1-\frac{1}{2n}\right)}{2n\cdot 2n \left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}.$$

$$= x.$$

So, $\sum u_n$ is convergent when $x<1$ and divergent when $x>1$.

At $x=1$, ratio test fails.

Now, we apply Rabbe's test.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+6n+2}{4n^2-2n} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{\cancel{4n^2}+6n+2-\cancel{4n^2}+2n}{4n^2-2n}\right)$$

$$= \lim_{n\to\infty} \frac{n}{n^2}\left(\frac{8n+2}{4-2/n}\right)$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2}\left(\frac{8+2/n}{4-2/n}\right)$$

$$= 2.$$

So, $\sum u_n$ is convergent at $x=1$.

Thus $\sum u_n$ is convergent when $x \le 1$ and divergent when $x > 1$.