Ques 1 Find the n^{th} derivative of $y = \sin 2x \cos 3x$.

Ans We know that $\sin(a+b) + \sin(a-b) = 2\sin a \cdot \cos b$.

So $\sin 2x \cos 3x = \frac{1}{2}\left(\sin(2x+3x) + \sin(2x-3x)\right)$.

$= \frac{1}{2}\left(\sin 5x - \sin x\right)$

Also, n^{th} derivative of $\sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$.

So, n^{th} derivative of $\frac{1}{2}\left(\sin 5x - \sin x\right) = \frac{1}{2}\left(5^n \sin\left(5x+\frac{\pi n}{2}\right) - 1^n \sin\left(x+\frac{n\pi}{2}\right)\right)$.

Ques 2 Find the n^{th} derivative of $y = x^2 e^{3x}$.

Ans $y = x^2 e^{3x}$

$u = e^{3x}$ $\quad$ $v = x^2$

$u_1 = 3e^{3x}$ $\quad$ $v_1 = 2x$

$u_2 = 9e^{3x}$ $\quad$ $v_2 = 2$

$v_3 = 0$

$u_n = 3^n e^{3x}$

Now by Leibnitz theorem, we have

$$\frac{d^n}{dx^n}\left(x^2 e^{3x}\right) = {}^nC_0\left(3^n e^{3x}\right) \cdot x^2 + {}^nC_1\left(3^{n-1} e^{3x}\right) \cdot 2x + {}^nC_2\left(3^{n-2} e^{3x}\right) \cdot 2 + 0 -$$

$$= e^{3x}\left[{}^{n}C_0\, 3^{n}\cdot x^{2} + {}^{n}C_1\, 3^{n-1}\, x\cdot 2 + {}^{n}C_2\, 3^{n-2} * 2\right].$$

$$\boxed{\frac{d^{n}(x^{2}e^{3x})}{dx^{n}} = e^{3x}\left[3^{n}x^{2} + 2nx\, 3^{n-1} + n(n-1)\, 3^{n-2}\right].}$$ **Ans**

Ques3 Test the applicability of Rolle's theorem for $f(x) = x^2 - 3x + 4$ on $[0, 2]$.

Ans Rolle's theorem states that if a function $f(x)$ is:

a) continuous on $[a, b]$

b) differentiable on (a, b)

c) $f(a) = f(b)$

Then there exists at least one number $c \in (a, b)$ such that

$f'(c) = 0$.

The given function $f(x) = x^2 - 3x + 4$ is a polynomial function and hence

a) $f(x)$ is continuous on $[0, 2]$

b) $f(x)$ is differentiable on $(0, 2)$

c) $f(0) = 0^2 - 3(0) + 4 = 4$

$f(2) = 4 - 3(2) + 4 = 2$

Here, $f(0) \neq f(2)$

So, Rolle's theorem can't be applied.

Ques4 Test the applicability of ~~Rolle's~~ Lagrange mean value theorem for $f(x) = x^2 - 3x + 4$ on $[0, 2]$.

Ans Lagrange Mean Value theorem states that if a function $f(x)$ is

a) continuous on $[a, b]$

b) differentiable on (a, b).

Then there exist a number 'c' $\in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

The given function $f(x) = x^2-3x+4$ is a polynomial function and hence

a) $f(x)$ is continuous on $[0,2]$.

b) $f(x)$ is differentiable on $(0,2)$.

Hence, Lagrange mean value theorem is applicable on $f(x)$.

Ques 5 Define Leibnitz theorem for successive differentiation.

Ans Leibnitz theorem for successive differentiation states that

If u and v are two function of x having derivatives of n^{th} order, then

$$\frac{d^n}{dx^n}(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_n u v_n.$$

Section B

Ques 1 If $y = a\cos(\ln x) + b\sin(\ln x)$, show that $x^2y_2 + xy_1 + y = 0$ and

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0.$$

Ans given $y = a\cos(\ln x) + b\sin(\ln x)$ —①

Differentiate ① both sides wrt x.

$$\frac{dy}{dx} = -\frac{a\sin(\ln x)}{x} + \frac{b\cos(\ln x)}{x}.$$

$$\Rightarrow xy_1 = -a\sin(\ln x) + b\cos(\ln x) \quad -(2)$$

Differentiate (2) both sides wrt x

$$\Rightarrow xy_2 + y_1 = -\frac{a\cos(\ln x)}{x} - \frac{b\sin(\ln x)}{x}$$

$$\Rightarrow x^2y_2 + xy_1 = -\left[a\cos(\ln x) + b\sin(\ln x)\right]$$

$$\Rightarrow x^2y_2 + xy_1 = -y$$

$$\Rightarrow \boxed{x^2y_2 + xy_1 + y = 0} \quad -(3)$$

Hence Proved

Now Apply Leibnitz theorem on (3)

$$\left(x^2y_2\right)_n + \left(xy_1\right)_n + \left(y\right)_n = 0$$

$$\Rightarrow \left({}^nC_0\, y_{n+2}\, x^2 + {}^nC_1\, y_{n+1}\, 2x + {}^nC_2\, y_n\,(2)\right) + \left({}^nC_0\, y_{n+1}\, x + {}^nC_1\, y_n\right) + y_n = 0.$$

$$\Rightarrow x^2y_{n+2} + 2nx\,y_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n + y_n = 0.$$

$$\Rightarrow x^2y_{n+2} + (2nx + x)y_{n+1} + (n(n-1) + n + 1)y_n = 0.$$

$$\Rightarrow \boxed{x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0}$$

Hence Proved.

Ques 2 If $\ln y = m\cos^{-1}x$, show that $(1-x^2)y_2 - 2xy_1 = m^2 y$ and

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$

Ans Given $\ln y = m\cos^{-1}x$ —(1)

Differentiate both sides wrt x

$$\frac{1}{y}\frac{dy}{dx} = \frac{-m}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\, y_1 = -my \quad -(2)$$

Differentiate (2) both sides wrt x

$$\Rightarrow \sqrt{1-x^2}\, y_2 + y_1 * \frac{-2x}{2\sqrt{1-x^2}} = -my_1$$

$$\Rightarrow \sqrt{1-x^2}\, y_2 - \frac{xy_1}{\sqrt{1-x^2}} = -m * \frac{-my}{\sqrt{1-x^2}}$$

$$\Rightarrow \boxed{(1-x^2)y_2 - xy_1 = m^2 y.} \quad -(3)$$

Hence Proved

Now apply Leibnitz theorem on (3)

$$\left((1-x^2)y_2\right)_n - \left(xy_1\right)_n = \left(m^2 y\right)_n$$

$$\Rightarrow \left((1-x^2)y_{n+2} + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2)\right) - \left(y_{n+1}x + {}^nC_1\, y_n\right) = m^2 y_n$$

$$\Rightarrow \left[(1-x^2)y_{n+2} - 2nx\, y_{n+1} + (-n(n-1)y_n)\right] - \left[xy_{n+1} + ny_n\right] = m^2 y_n$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\, y_{n+1} - (n^2-n+n+m^2)y_n = 0.$$

$$\Rightarrow \boxed{(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0}$$

Ques 3 Verify Rolle's theorem for $f(x) = (x-1)(x+3)e^x$ in $[-3,1]$.

Ans The given function $f(x) = (x-1)(x+3)e^x$ is a polynomial function and hence it is

a) continuous on the interval $[-3,1]$.
b) differentiable on the interval $(-3,1)$
c) Also $f(-3) = 0$ and $f(1) = 0$
$\Rightarrow f(-3) = f(1)$.

So, Rolle's theorem is applicable for given function.

Thus, there exist $c \in (-3,1)$ such that $f'(c) = 0$.

$$f(x) = (x-1)(x+3)e^x$$
$$= e^x(x^2+2x-3).$$

$$f'(x) = e^x(x^2+2x-3) + (2x+2)e^x$$

$$f'(c) = 0$$
$$\Rightarrow e^c(c^2+2c-3+2c+2) = 0$$
$$\Rightarrow e^c(c^2+4c-1) = 0$$
$$\Rightarrow c^2+4c-1 = 0 \quad \text{as } e^c \neq 0$$
$$\Rightarrow c = \frac{-4 \pm \sqrt{16+4}}{2}$$
$$= \frac{-4 \pm 2\sqrt{5}}{2}$$
$$\Rightarrow c = -2 \pm \sqrt{5}.$$

$c = -2+\sqrt{5}$ as $c = -2-\sqrt{5} \notin (-3,1)$.

$$\boxed{c = -2+\sqrt{5}}.$$

Ques 4 Expand $e^x \sin x$ in powers of $(x-1)$ upto third degree term.

Ans According to Taylor's theorem, if we have to expand $f(x)$ in powers of $(x-a)$, then

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Here $a = 1$

$f(x) = e^x \sin x$ $\qquad f(1) = e\sin 1$

$f'(x) = e^x(\sin x + \cos x)$ $\qquad f'(1) = e(\sin 1 + \cos 1)$

$f''(x) = e^x(\cancel{\sin x} + \cos x + \cos x - \cancel{\sin x})$ $\qquad f''(1) = 2e\cos 1$

$= 2e^x \cos x$

$f'''(x) = 2e^x(\cos x - \sin x)$ $\qquad f'''(1) = 2e(\cos 1 - \sin 1)$.

So, $\boxed{f(x) = e\sin 1 + (x-1)e(\sin 1 + \cos 1) + (x-1)^2 e\cos 1 + \frac{(x-1)^3}{3} * e(\cos 1 - \sin 1).}$

Ques 5 Expand $4x^3 - x^2 + 3x - 1$ in powers of $(x+1)$.

Ans According to Taylor's theorem, if we have to expand $f(x)$ in powers of $(x-a)$, then

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \text{------}$$

Here $a = -1$.

$f(x) = 4x^3 - x^2 + 3x - 1$ $\quad\quad$ $f(-1) = -4 - 1 - 3 - 1 = -9$.

$f'(x) = 12x^2 - 2x + 3$ $\quad\quad$ $f'(-1) = 12 + 2 + 3 = 17$

$f''(x) = 24x - 2$ $\quad\quad$ $f''(-1) = -26$.

$f'''(x) = 24$ $\quad\quad$ $f'''(-1) = 24$

$f^{iv}(x) = 0$ $\quad\quad$ $f^{iv}(-1) = 0$.

$$\boxed{f(x) = -9 + (x+1)(17) - 13(x+1)^2 + 4(x+1)^3}$$ Ans

Section C

Ques 1 If $y = x \ln\left(\frac{x-1}{x+1}\right)$, show that $y_n = (-1)^n (n-2)! \left[\frac{(x-n)}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$.

Ans

$$y = x \ln\left(\frac{x-1}{x+1}\right)$$

$$= x\left[\ln(x-1) - \ln(x+1)\right]$$

$$y = x \ln(x-1) - x \ln(x+1). \quad -(1)$$

Differentiate both sides wrt x.

$$\frac{dy}{dx} = \frac{x}{x-1} - \frac{x}{x+1} + \ln(x-1) - \ln(x+1).$$

$$= \frac{x(x+1) - x(x-1)}{(x+1)(x-1)} + \ln(x-1) - \ln(x+1)$$

$$= \frac{2x}{(x+1)(x-1)} + \ln(x-1) - \ln(x+1)$$

$$y_1 = \frac{1}{x+1} + \frac{1}{x-1} + \ln(x-1) - \ln(x+1) \quad \text{—(2)}$$

We know that if $y = \frac{1}{ax+b}$, then $y_n = \frac{(-1)^n \cdot n! \, a^n}{(ax+b)^{n+1}}$

Also if $y = \ln(ax+b)$, then $y_n = \frac{(-1)^{n+1} a^n \cdot (n-1)!}{(ax+b)^n}$

Differentiate (2) $(n-1)$ times.

$$y_n = \frac{(-1)^{n-1}(n-1)!}{(x+1)^{n-1+1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^{n-1+1}} + \frac{(-1)^n \cdot (n-2)!}{(x-1)^{n-1}} - \frac{(-1)^n \cdot (n-2)!}{(x+1)^{n-1}}$$

$$= (-1)^n (n-2)! \left[\frac{1}{(x-1)^{n-1}} - \frac{(n-1)}{(x-1)^{n-1+1}} \right] + (-1)^n (n-2)! \left[\frac{-1}{(x+1)^{n-1}} - \frac{n-1}{(x+1)^n} \right]$$

$$= (-1)^n (n-2)! \left[\frac{x - \cancel{1} - n + \cancel{1}}{(x-1)^n} \right] + (-1)^n (n-2)! \left[\frac{-x - \cancel{1} - n + \cancel{1}}{(x+1)^n} \right]$$

$$\boxed{y_n = (-1)^n (n-2)! \left[\frac{x-n}{(x-1)^n} - \frac{(x+n)}{(x+1)^n} \right].}$$

Hence Proved.

Ques 2 If $y = x^n \ln x$, show that $xy_{n+1} = n!$

Ans $y = x^n \ln x.$

Differentiate both sides wrt x

$$\frac{dy}{dx} = \frac{x^n}{x} + nx^{n-1}\ln x$$

$$y_1 = x^{n-1}(1 + n\ln x) \quad -(1)$$

$$xy_1 = x^n(1 + n\ln x) \quad -(11) \qquad xy_1 = x^n + n(y)$$

Differentiate n times on both sides.

$${}^nC_0\, y_{n+1}\, x + {}^nC_1\, y_n \cdot 1 = n! + n \cdot y_n$$

$$xy_{n+1} + \cancel{ny_n} = n! + \cancel{ny_n}$$

$$\boxed{xy_{n+1} = n!}$$

Hence Proved.

Ques 3 Verify Lagrange's Mean value theorem for $f(x) = x^3 - 6x^2 + 9x + 1$ in $[1, 4]$.

Ans The given function is $f(x) = x^3 - 6x^2 + 9x + 1$.

The function $f(x)$ is a polynomial function and is
a) thus a continuous function on $[1, 4]$.
b) thus a differentiable function on $(1, 4)$.

$$f(1) = 1^3 - 6 + 9 + 1 = 5.$$
$$f(4) = 4^3 - 96 + 36 + 1 = 64 - 60 + 1 = 5.$$

Then, there exist a number $c \in (1, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$\Rightarrow f(x) = x^3 - 6x^2 + 9x + 1$

So $f'(x) = 3x^2 - 12x + 9$

So $f'(c) = 3c^2 - 12c + 9$

$\Rightarrow 3c^2 - 12c + 9 = \frac{5-5}{4-1}$

$\Rightarrow 3c^2 - 12c + 9 = 0$

$\Rightarrow c^2 - 4c + 3 = 0$

$\Rightarrow (c-1)(c-3) = 0$

$\Rightarrow c = 1, \quad c = 3$

$c = 1 \notin (1,4) \qquad c = 3 \in (1,4)$

So, $\boxed{c = 3.}$

Ques 4 Expand $\log(1+x)$ in the powers of x. Hence find the series for $\log\left(\frac{1-x}{1+x}\right)$.

Ans Taylor's theorem is used for expansion of series.

If we want to expand $f(x)$ in power of $(x-a)$, then we can write

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Here $a = 0$

So,

$$f(x) = f(0) + (x-0) f'(0) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-0)^3}{3!} f'''(0) + \cdots$$

$f(x) = \ln(1+x)$. $\qquad f(0) = 0$.

$f'(x) = \frac{1}{(1+x)}$ $\qquad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$f^{iv}(x) = \frac{-6}{(1+x)^4}$ $\qquad f^{iv}(0) = -6$.

and thus

$$f(x) = 0 + x - \frac{x^2}{2} + 2 \cdot \frac{x^3}{3!} - 6 \cdot \frac{x^4}{4!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots \quad -(I)$$

Replace x with $-x$, we get

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \cdots \quad -(II)$$

$$\log\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) \quad = (I) - (II)$$

$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \cdots$$

$$\log\left(\frac{1-x}{1+x}\right) = \log(1-x) - \log(1+x) = (ii) - (i)$$

$$\boxed{\log\left(\frac{1-x}{1+x}\right) = -2x - \frac{2x^3}{3} - \frac{2x^5}{5} - \frac{2x^7}{7} - \cdots}$$

Ques 5 Find Maclaurin's series for $f(x) = e^{\sin x}$ upto powers of x^4.

Maclaurin's series for $f(x)$ is given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$f(x) = e^{\sin x}$ $\qquad f(0) = 1$

$f'(x) = e^{\sin x} \cdot \cos x$ $\qquad f'(0) = 1$

$f''(x) = e^{\sin x}(-\sin x + \cos x \cdot \cos x).$ $\qquad f''(0) = 1$

$$f'''(x) = e^{\sin x}\big(\cos^3 x - \cos^2 x \sin x - \cancel{\sin x +} - \cos x - 2\sin x \cos x\big).$$
$f'''(0) = 0$

$$f^{iv}(x) = \big(\cos^4 x - \cos^3 x \sin x - \cos^2 x - 2\sin x \cos^2 x - 3\cos^2 x \sin x - \cos^3 x + \sin x \sin 2x + \sin x - 2\cos 2x\big)e^{\sin x}$$

$f^{iv}(0) = -3.$

so $f(x) = 1 + x + \frac{x^2}{2} + \frac{0x^3}{3!} - \frac{3x^4}{4!} + \cdots$

$$\boxed{f(x) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots}$$