Unit - II

Ques1 Find $\frac{dy}{dx}$ if $y^2 + xy - 3x - 3 = 0$ at $(-1, 1)$.

Ans

$$y^2 + xy - 3x - 3 = 0$$

Differentiate w.r.t x, we get

$$\Rightarrow 2y\frac{dy}{dx} + x\frac{dy}{dx} + y - 3 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3-y}{(2y+x)}$$

At $(-1, 1)$, $\frac{dy}{dx} = \frac{3-1}{2-1} =$

$$\boxed{\frac{dy}{dx} = 2}$$

Ques2 Find $\frac{dy}{dx}$ if $x^2 + xy + y^2 - 7 = 0$ at $(1, 2)$.

Ans

$$x^2 + xy + y^2 - 7 = 0$$

Differentiate both sides w.r.t x, we get

$$\Rightarrow 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(2x+y)}{(x+2y)}$$

At $(1,2)$ $\frac{dy}{dx} = -\frac{(2+2)}{1+4}$

$$\boxed{\frac{dy}{dx} = \frac{-4}{5}}$$

Ques 3 Find jacobian if $x = r\cos\theta$ $y = r\sin\theta$.

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$\boxed{\text{Jacobian} = r.}$$

Ques 4 If $\tan u = \frac{x^3+y^3}{x-y}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Ans Let $z = \tan u = \frac{x^3+y^3}{x-y}$.

$\frac{x^3+y^3}{x-y}$ is a homogeneous function of x and y of order 2

So, we can apply Euler's theorem.

$$\cancel{\frac{\partial z}{\partial}} \; x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\cdot\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= \frac{2\sin u}{\cos u} \cdot \frac{\cos^2 u}{1}$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u}$$

Ques 5 Develop the chain rule for $w = f(x, y, z)$, $x = g(r, s)$ $y = h(r, s)$, $z = k(r, s)$.

Ans $w = f(x, y, z)$ $x = g(r, s)$ $y = h(r, s)$ $z = k(r, s)$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

Ques 6 Find local extreme of the function $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

Ans $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$

$$\frac{\partial f}{\partial x} = 12x - 6x^2 + 6y$$

$$\frac{\partial f}{\partial y} = 6y + 6x$$

$\frac{\partial f}{\partial x} = 0 \quad \Rightarrow 12x - 6x^2 + 6y = 0$

$\frac{\partial f}{\partial y} = 0 \quad \Rightarrow 6y + 6x = 0 \Rightarrow y = -x$

$12x - 6x^2 - 6x = 0$

$\Rightarrow 6x - 6x^2 = 0$

$\Rightarrow 6x(1-x) = 0$

$\Rightarrow x = 0 \quad$ or $\quad x = 1$

So $y = 0$ when $x = 0$

$y = -1$ when $x = 1$

So, critical points are $(0,0)$ and $(1,-1)$.

$\frac{\partial^2 f}{\partial x^2} = 12 - 12x$

$\frac{\partial^2 f}{\partial y^2} = 6$

$\frac{\partial^2 f}{\partial x \partial y} = 6.$

$rt - s^2 = 6(12 - 12x) - 36.$

At $(0,0)$, $r > 0$ $\quad rt - s^2 = 36 > 0$

So, we have local minimum at $(0,0)$

At $(1,-1)$ $\quad r = 0 \quad rt - s^2 = 0$

So, we have saddle point at $(1,-1)$.

Ques 7 Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ upto second degree using Taylor's theorem.

Ans

$$f(x,y) = x^2y + 3y - 2$$

$f_x = 2xy$ $\qquad$ $f_y = x^2 + 3$

$f_{xx} = 2y$ $\qquad$ $f_{yy} = 0$ $\qquad$ $f_{yx} = 2x$ $\qquad$ $f_{yxx} = 2$

$f_{xxx} = 0$ $\qquad$ $f_{xy} = 2y$ $\qquad$ $f_{xyy} = 2$

$$f(1,-2) = 1^2(-2) - 6 - 2 = -10$$

$$f(x,y) = f(1,-2) + \left[(x-1)\frac{\partial f}{\partial x} + (y+2)\frac{\partial f}{\partial y}\right] +$$

$$\frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f}{\partial x^2} + (y+2)^2\frac{\partial^2 f}{\partial y^2} + 2(x-1)(y+2)\frac{\partial^2 f}{\partial x \partial y}\right] +$$

$$\frac{1}{3!}\left[(x-1)^3\frac{\partial^3 f}{\partial x^3} + (y+2)^3\frac{\partial^3 f}{\partial y^3} + 3(x-1)^2(y+2)\frac{\partial^3 f}{\partial x^2 \partial y} + 3(x-1)(y+2)^2\frac{\partial^3 f}{\partial x \partial y^2}\right] + - - - -$$

$$= -10 + \left[(-4)(x-1) + 4(y+2)\right] + \frac{1}{2}\left[-4(x-1)^2 + 4(x-1)(y+2)\right]$$

$$+ \frac{1}{6}\left[3(x-1)^2(y+2)\cdot 2 + 3(x-1)(y+2)^2\cdot 2\right]$$

$$\boxed{f(x,y) = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y-1) + - - -}$$

Ques 8. Use the jacobian to prove that the function

$u = \frac{x-y}{x+y}$, $v = \frac{xy}{(x+y)^2}$ are functionally dependent. Find

that relation between them.

Ans

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x+y-x+y}{(x+y)^2} = \frac{2y}{(x+y)^2} & \frac{-x-y-x+y}{(x+y)^2} = \frac{-2x}{(x+y)^2} \\ \frac{(x+y)^2 \cdot y - xy \cdot 2(x+y)}{(x+y)^4} & \frac{(x+y)^2 \cdot x - xy \cdot 2(x+y)}{(x+y)^4} \end{vmatrix}$$

$$= \left\{\frac{2yx(x+y)^2 - 2y^2x \cdot 2(x+y)}{(x+y)^4}\right\} + \left\{\frac{2xy(x+y)^2 - 4x^2y(x+y)}{(x+y)^4}\right\}$$

$$= \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{y(x+y-2x)}{(x+y)^3} & \frac{x(x+y-2y)}{(x+y)^3} \end{vmatrix}$$

$$= \frac{2xy(x-y)}{(x+y)^3} + \frac{2xy(y-x)}{(x+y)^3}$$

$$= 0.$$

Since $\frac{\partial(u,v)}{\partial(x,y)} = 0 \Rightarrow$ u and v are functionally dependent.

Relation $\Rightarrow$ $1-u^2 = 4v$ Ans.

Ques 8 Find the maximum and minimum value of the function :- $f(x,y) = 3x+4y$ on the circle $x^2+y^2=1$ by using Lagrange method of multipliers.

Ans Maximize | Minimize $f(x,y) = 3x+4y$

such that $x^2+y^2=1$ $\quad g(x,y)$

$$F(x,y,\lambda) = 3x+4y+\lambda(x^2+y^2-1).$$

$$\frac{\partial F}{\partial x} = 3+2\lambda x = 0 \Rightarrow x = \frac{-3}{2\lambda}$$

$$\frac{\partial F}{\partial y} = 4+2\lambda y = 0 \Rightarrow y = \frac{-2}{\lambda}$$

also $x^2+y^2=1$

$$\frac{9}{4\lambda^2} + \frac{2^2}{\lambda^2} = 1$$

$$\Rightarrow 9+16 = 4\lambda^2$$

$$\Rightarrow \lambda = \pm\frac{5}{2}$$

$\lambda = \frac{5}{2}$ $\quad (x,y) = \left(\frac{-3}{5}, \frac{-4}{5}\right)$

$\lambda = -\frac{5}{2}$ $\quad (x,y) = \left(\frac{3}{5}, \frac{4}{5}\right)$

$f(x,y)$ at $\left(\frac{-3}{5}, \frac{-4}{5}\right) = -5 \rightarrow$ Minimum.

$f(x,y)$ at $\left(\frac{3}{5}, \frac{4}{5}\right) = 5 \rightarrow$ Maximum

Ques 9o Write the Maclaurin's series of function

$$f(x,y) = e^x \ln(1+y) \text{ upto second degree.}$$

Ans $f(x,y) = e^x \ln(1+y)$

$f_x = e^x \ln(1+y)$ $\quad f_y = \dfrac{e^x}{(1+y)}$

$f_{xx} = e^x \ln(1+y)$ $\quad f_{yy} = \dfrac{-e^x}{(1+y)^2}$

$f_{xy} = \dfrac{e^x}{1+y}$.

At $(0,0)$ $\quad f_x = 0 \quad f_y = 1$

$f_{xy} = 1$

$f_{xx} = 0 \quad f_{yy} = -1$

$$f(x,y) = 0 + x \cdot 0 + y \cdot 1 + \frac{x^2}{2} \cdot 0 - \frac{y^2}{2} + \frac{2xy}{2}$$

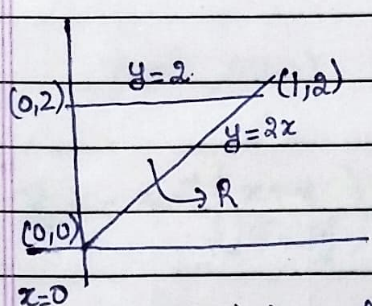
$$\boxed{f(x,y) = y - \frac{y^2}{2} + \frac{2xy}{2}.}$$

* Formula used.

$$f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)\right] + \cdots$$

Ques11: Find the absolute maxima and minima of $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the line $x=0, y=2, y=2x$ in the first quadrant.

Ans



We have to choose the critical points from this region.
and the end points of this region are also critical points.

Maximize $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$

$$\frac{\partial f}{\partial x} = 4x - 4 \qquad \frac{\partial f}{\partial y} = 2y - 4$$

$$\frac{\partial^2 f}{\partial x^2} = 4 \qquad \frac{\partial^2 f}{\partial y^2} = 2.$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$rt - s^2 = 8 > 0.$$

$$\frac{\partial f}{\partial x} = 4x - 4 = 0 \qquad \frac{\partial f}{\partial y} = 2y - 4 = 0$$

$$\Rightarrow x = 1 \qquad \Rightarrow y = 2.$$

(1,2) is critical point and it also lies in given region.

$f(1,2) = 2 - 4 + 4 - 8 + 1$

$\boxed{f(0,0) = 1 \quad \text{(Maximum)}}$

$\boxed{f(1,2) = -5. \quad \text{(Minimum)}}$

$f(0,2) = -3.$

Ques12. Find the relative maximum and minimum points of the function: $f(x,y) = 2(x^2-y^2) - x^4 + y^4$.

Ans

$$f(x,y) = 2x^2 - 2y^2 - x^4 + y^4$$

$$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0$$

$$\frac{\partial f}{\partial y} = -4y + 4y^3 = 0$$

$4x(1-x^2) = 0$ $\qquad$ $4y(y^2-1) = 0$

$\Rightarrow x = 0 \quad x = \pm 1$ $\qquad$ $\Rightarrow y = 0, \ y = \pm 1$.

Critical points :- $(0,0), (0,1), (0,-1), (1,0), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1)$.

$f_{xx} = 4 - 12x^2$ $\qquad$ $f_{yy} = -4 + 12y^2$ $\qquad$ $f_{xy} = 0$.

	$r = 4-12x^2$	$t = -4+12y^2$	$s = 0$	$rt - s^2$
$(0,0)$	4	-4	0	-16
$(0,1)$	4	8	0	32
$(0,-1)$	4	8	0	32
$(1,0)$	-8	$0-4=-4$	0	32
$(1,1)$	-8	8	0	-64
$(1,-1)$	-8	8	0	-64
$(-1,0)$	-8	-4	0	32
$(-1,1)$	-8	8	0	-64
$(-1,-1)$	-8	8	0	-64

$(0,0), (1,1), (1,-1), (-1,1), (-1,-1)$ are critical points.

$(0,1)$ $(0,-1)$ Local minima

$(1,0), (-1,0)$ Local maxima.

Ques13 If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that

1. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

2. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u \cdot \cos 2u}{4\cos^3 u}$.

Ans $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$.

$\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Let $z = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

z is a homogeneous function of order $\frac{1}{2}$

Applying Euler's theorem, we get.

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$.

$\frac{\partial z}{\partial x} = \cos u\frac{\partial u}{\partial x}$ $\frac{\partial z}{\partial y} = \cos u\frac{\partial u}{\partial y}$

$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u}$$

From deductions of Euler's theorem, we have.

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \phi(u)\left[\phi'(u) - 1\right].$$

$$\phi(u) = n\frac{F(u)}{F'(u)}$$

$$\phi(u) = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u.$$

$$\phi'(u) = \frac{\sec^2 u}{2}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\tan u\left(\frac{\sec^2 u}{2} - 1\right).$$

$$= -\frac{1}{2}\frac{\sin u}{\cos u}\left(1 - \frac{\sec^2 u}{2}\right).$$

$$= -\frac{1}{2}\frac{\sin u}{\cos u}\left(\frac{2\cos^2 u - 1}{2\cos^2 u}\right).$$

$$= -\frac{1}{2}\frac{\sin u}{\cos u}\cdot\frac{\cos 2u}{2\cos^2 u}$$

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{4}\frac{\sin u \cdot \cos 2u}{\cos^3 u}}$$

Ques 14 If $u = x(1-r^2)^{-1/2}$ $v = y(1-r^2)^{-1/2}$ $w = z(1-r^2)^{-1/2}$

where $r^2 = x^2+y^2+z^2$, then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (1-r^2)^{-5/2}$$

Ans $r^2 = x^2+y^2+z^2$

$2r\frac{\partial r}{\partial x} = 2x$ $\quad 2r\frac{\partial r}{\partial y} = 2y$ $\quad 2r\frac{\partial r}{\partial z} = 2z$

$\frac{\partial r}{\partial x} = \frac{x}{r}$ $\quad \frac{\partial r}{\partial y} = \frac{y}{r}$ $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$u = x(1-r^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = x * \frac{-1}{\cancel{2}}(1-r^2)^{-3/2} * -\cancel{2r} * \frac{x}{\cancel{r}} + (1-r^2)^{-1/2} * 1$$

$$= x^2(1-r^2)^{-3/2} + (1-r^2)^{-1/2}$$

$$\frac{\partial u}{\partial y} = x\left(\frac{-1}{\cancel{2}}(1-r^2)^{-3/2}\right) * -\cancel{2r} * \frac{y}{\cancel{r}}$$

$$= xy\,(1-r^2)^{-3/2}$$

$$\frac{\partial u}{\partial z} = x\left(\frac{-1}{2}(1-r^2)^{-3/2}\right) * -2r * \frac{z}{r}$$

$$= xz(1-r^2)^{-3/2}$$

$$\frac{\partial v}{\partial x} = y\left(\frac{-1}{2}(1-r^2)^{-3/2} * -2r\right) * \frac{x}{r}$$

$$= xy\,(1-r^2)^{-3/2}$$

$$\frac{\partial v}{\partial y} = (1-r^2)^{-1/2} + y\left(-\frac{1}{2}(1-r^2)^{-3/2} * -2r \cdot \frac{y}{r}\right)$$

$$= (1-r^2)^{-1/2} + y^2(1-r^2)^{-3/2}$$

$$\frac{\partial v}{\partial z} = y\left(-\frac{1}{2}(1-r^2)^{-3/2} * 2r \times \frac{z}{r}\right)$$

$$= yz(1-r^2)^{-3/2}$$

$$\frac{\partial w}{\partial x} = z * -\frac{1}{2}(1-r^2)^{-3/2} * -2r * \frac{x}{r}$$

$$= xz(1-r^2)^{-3/2}$$

$$\frac{\partial w}{\partial y} = z * -\frac{1}{2}(1-r^2)^{-3/2} * -2r * \frac{y}{r}$$

$$= yz(1-r^2)^{-3/2}$$

$$\frac{\partial w}{\partial z} = (1-r^2)^{-1/2} + z\left((1-r^2)^{-3/2} * -\frac{1}{2} * -2r * \frac{z}{r}\right)$$

$$= (1-r^2)^{-1/2} + z^2(1-r^2)^{-3/2}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} (1-r^2)^{-1/2} + x^2(1-r^2)^{-3/2} & xy(1-r^2)^{-3/2} & xz(1-r^2)^{-3/2} \\ xy(1-r^2)^{-3/2} & (1-r^2)^{-1/2} + y^2(1-r^2)^{-3/2} & yz(1-r^2)^{-3/2} \\ xz(1-r^2)^{-3/2} & yz(1-r^2)^{-3/2} & (1-r^2)^{-1/2} + z^2(1-r^2)^{-3/2} \end{vmatrix}$$

Taking $(1-r^2)^{-3/2}$ common from every row.

$$= (1-r^2)^{-3/2} \begin{vmatrix} (1-r^2)+x^2 & xy & xz \\ xy & (1-r^2)+y^2 & yz \\ xz & yz & (1-r^2)+z^2 \end{vmatrix}$$

$$= (1-r^2)^{-9/2}\Big[(1-r^2) + x^2\Big[(1-r^2)^2 + z^2(1-r^2) + y^2(1-r^2) + \cancel{y^2z^2} - \cancel{y^2z^2}\Big]$$

$$- xy\left((1-r^2)xy + \cancel{xyz^2} - \cancel{xyz^2}\right)$$

$$+ xz\left(\cancel{xy^2z} - xz(1-r^2) - \cancel{xy^2z}\right)$$

$$= (1-r^2)^{-9/2}\Big[\left((1-r^2)+x^2\right)\left((1-r^2)^2 + (y^2+z^2)(1-r^2)\right) - x^2y^2(1-r^2) - x^2z^2(1-r^2)\Big]$$

$$= (1-r^2)^{-9/2}\Big[(1-r^2)^3 + (1-r^2)^2(y^2+z^2) + x^2(1-r^2)^2 + \cancel{x^2y^2(1-r^2)} + \cancel{x^2z^2(1-r^2)} - \cancel{x^2y^2(1-r^2)} - \cancel{x^2z^2(1-r^2)}\Big]$$

$$= (1-r^2)^{-9/2}\left[(1-r^2)^3 + (1-r^2)^2(x^2+y^2+z^2)\right]$$

$$= (1-r^2)^{-9/2}\left[(1-r^2)^3 + r^2(1-r^2)^2\right]$$

$$= (1-r^2)^{-9/2}(1-r^2)^2\left[1 - \cancel{r^2} + \cancel{r^2}\right]$$

$$= (1-r^2)^{-9/2}(1-r^2)^2$$

$$= (1-r^2)^{-5/2}$$